ANNALS OF
MATHEMATICS

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SECOND SERIES, VOL. 171, NO. 3

May, 2010

ANMAAH

Annals of Mathematics, **171** (2010), 1401–1477

The global stability of Minkowski space-time in harmonic gauge

By HANS LINDBLAD and IGOR RODNIANSKI

Abstract

We give a new proof of the global stability of Minkowski space originally established in the vacuum case by Christodoulou and Klainerman. The new approach, which relies on the classical harmonic gauge, shows that the Einstein-vacuum and the Einstein-scalar field equations with asymptotically flat initial data satisfying a global smallness condition produce global (causally geodesically complete) solutions asymptotically convergent to the Minkowski space-time.

1. Introduction

In this paper we address the question of stability of Minkowski space-time for the system of the Einstein-scalar field equations[1]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}. \tag{1.1}$$

The equations connect the gravitational tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}$ given in terms of the Ricci $R_{\mu\nu}$ and scalar $R = g^{\mu\nu} R_{\mu\nu}$ curvatures of an unknown Lorentzian metric $g_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$ of a matter field ψ:

$$T_{\mu\nu} = \partial_\mu \psi \, \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha \psi \, \partial_\beta \psi). \tag{1.2}$$

Part of this work was done while H.L. was a Member of the Institute for Advanced Study, Princeton, and was supported by the NSF grant DMS-0111298 to the Institute. H.L. was also partially supported by the NSF Grant DMS-0200226.
Part of this work was done while I.R. was a Clay Mathematics Institute Long-Term Prize Fellow. His work was also partially supported by the NSF grant DMS-01007791.

[1] We use Greek indices $\alpha, \beta, \mu, \nu \dots = 0, \dots, 3$, the summation convention over repeated indices and the notation $\partial_\alpha = \partial/\partial x^\alpha$. The symbol D denotes a covariant Levy-Civita derivative with respect to the metric g.

The Bianchi identities

$$D^\mu G_{\mu\nu} = 0$$

imply that the scalar field ψ satisfies the covariant wave equation

$$\Box_g \psi = \frac{1}{\sqrt{|\det g|}} \partial_\mu \big(g^{\mu\nu} \sqrt{|\det g|} \partial_\nu \psi \big) = 0.$$

The set $(m, \mathbb{R}^{3+1}, 0)$: standard Minkowski metric $g = m = -dt^2 + \sum_{i=1}^3 (dx^i)^2$ on $\mathbb{R}^{3+1}$ and vanishing scalar field $\psi \equiv 0$ describes the Minkowski space-time solution of the system (1.1).

The problem of stability of Minkowski space appears in the *Cauchy formulation of the Einstein equations in which given a* 3*-d manifold* Σ_0 *with a Riemannian metric* g_0, *a symmetric* 2*-tensor* k_0 *and the initial data* (ψ_0, ψ_1) *for the scalar field*, *one needs to find a* 4*-d manifold* $\mathcal{M}$, *with a Lorentzian metric g and a scalar field* ψ *satisfying the Einstein equations* (1.1), *and an imbedding* $\Sigma_0 \subset M$ *such that* g_0 *is the restriction of* g *to* Σ, k_0 *is the second fundamental form of* Σ *and the restriction of* ψ *to* Σ_0 *gives rise to the data* (ψ_0, ψ_1).

The initial value problem is over determined and the data must satisfy the constraint equations

$$R_0 - k_0{}^i_j\, k_0{}^j_i + k_0{}^i_i\, k_0{}^j_j = |\nabla \psi_0|^2 + |\psi_1|^2, \qquad \nabla^j k_{0ij} - \nabla_i\, k_0{}^j_j = \nabla_i \psi_0\, \psi_1.$$

Here R_0 is the scalar curvature of g_0 and ∇ is covariant differentiation with respect to g_0.

The seminal result of Choquet-Bruhat [CB52] followed by the work [CBG69] showed existence and uniqueness (up to a diffeomorphism) of a maximal globally hyperbolic[2] smooth space-time arising from any set of smooth initial data. The work of Choquet-Bruhat used the diffeomorphism invariance of the Einstein equations which allowed her to choose a special *harmonic* (also referred to as a *wave coordinate* or de Donder) gauge, in which the Einstein equations become a system of quasilinear wave equations on the components of the unknown metric $g_{\mu\nu}$

$$\begin{aligned} \widetilde{\Box}_g\, g_{\mu\nu} &= F_{\mu\nu}(g)(\partial g, \partial g) + 2\partial_\mu \psi\, \partial_\nu \psi, \\ \widetilde{\Box}_g \psi &= 0, \quad \text{where } \widetilde{\Box}_g = g^{\alpha\beta} \partial_\alpha \partial_\beta \end{aligned} \tag{1.3}$$

with $F(u)(v,v)$ depending quadratically on v. Wave coordinates $\{x^\mu\}_{\mu=0,\ldots,3}$ are required to be solutions of the wave equations $\Box_g\, x^\mu = 0$, where the geometric wave operator is $\Box_g = D_\alpha D^\alpha = g^{\alpha\beta} \partial_\alpha \partial_\beta + g^{\alpha\beta} \Gamma_{\alpha\beta}^{\ \ \nu} \partial_\nu$. The metric $g_{\mu\nu}$ relative

[2]A space-time is called globally hyperbolic if every inextendable causal curve intersects the initial surface Σ once and only once. A causal curve $x(s)$ is a curve such that $g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \le 0$, where $\dot{x} = dx/ds$. It is future directed if $\dot{x}^0 > 0$.

to wave coordinates $\{x^\mu\}$ satisfies the *wave coordinate condition*

$$g^{\alpha\beta} g_{\nu\mu} \Gamma_{\alpha\beta}^{\ \nu} = g^{\alpha\beta} \partial_\beta g_{\alpha\mu} - \frac{1}{2} g^{\alpha\beta} \partial_\mu g_{\alpha\beta} = 0. \tag{1.4}$$

Under this condition the geometric wave operator $\Box_g$ is equal to the reduced wave operator $\widetilde{\Box}_g$. The use of harmonic gauge goes back to the work of Einstein on post-Newtonian and post-Minkowskian expansions.

In the PDE terminology the result of Choquet-Bruhat corresponds to the *local well-posedness* of the Cauchy problem for the Einstein-vacuum (scalar field) equations with smooth initial data. The term local is appropriate in the sense that the result does not guarantee that the constructed space-time is *causally geodesically complete*[3] and thus could "terminate" in a singularity. Our experience suggests that global results require existence of conserved or more generally monotonic positive quantities. The only known such quantity in the asymptotically flat case is the ADM mass, whose positivity was established by Schoen-Yau [SY79] and Witten [Wit81], is highly supercritical relative to the equations and thus not sufficient to upgrade a local space-time to a global solution. This leaves the questions related to the structure of maximal globally hyperbolic space-times even for generic[4] data firmly in the realm of the outstanding Cosmic Censorship Conjectures of Penrose. Given this state of affairs the problem of stability of special solutions, most importantly the Minkowski space-time, becomes of crucial importance.

STABILITY OF MINKOWSKI SPACE-TIME FOR THE EINSTEIN-VACUUM (SCALAR FIELD) EQUATIONS. *Show the existence of a causally geodesically space-time asymptotically "converging" to the Minkowski space-time for an arbitrary set of smooth asymptotically flat initial data* $(\Sigma_0, g_{0ij}, k_{0ij})$ *with* $\Sigma_0 \approx \mathbb{R}^3$,

$$\begin{aligned} g_{0ij} &= \Big(1 + \frac{M}{r}\Big)\delta_{ij} + o(r^{-1-\alpha}), \\ k_{0ij} &= o(r^{-2-\alpha}), \quad r = |x| \to \infty, \quad \alpha > 0 \end{aligned} \tag{1.5}$$

where $(g_0 - \delta)$ *and* k_0 *satisfy global smallness assumptions. The stability of Minkowski space-time for the Einstein-scalar field equations*: *in addition requires a global smallness assumption on the scalar field data* (ψ_0, ψ_1), *which obey the asymptotic expansion*

$$\psi_0 = o(r^{-1-\alpha}), \qquad \psi_1 = o(r^{-2-\alpha}). \tag{1.6}$$

A positive parameter M in the asymptotic expansion for the metric g_0 is the ADM mass.

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[3] i.e. any causal geodesics $x(s)$, $g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \text{const} \le 0$ can be extended to infinite parameter value $0 \le s < \infty$.

[4] Note that all the known explicit solutions, with exception of the Minkowski space-time, are in fact incomplete.

The stability of Minkowski space for the Einstein-vacuum equations was shown in a remarkable work of Christodoulou-Klainerman for strongly asymptotic initial data (the parameter $\alpha \geq 1/2$ in the asymptotic expansion (1.5).) The approach taken in that work viewed the Einstein-vacuum equations as a system of equations

$$D^{\alpha} W_{\alpha\beta\gamma\delta} = 0, \qquad D^{\alpha} * W_{\alpha\beta\gamma\delta} = 0$$

for the Weyl tensor $W_{\alpha\beta\gamma\delta}$ of the metric $g_{\alpha\beta}$ and used generalized energy inequalities associated with the Bel-Robinson energy-momentum tensor, constructed from components of W, and special geometrically constructed vector fields, designed to mimic the rotation and the conformal Morawetz vector fields of the Minkowski space-time, that is, "almost conformally Killing" vector fields of the unknown metric g. The proof was manifestly invariant, in particular it did not use the wave coordinate gauge. This approach was later extended to the Einstein-Maxwell equations by N. Zipser, [Zip00].

Nevertheless the PDE appeal (for the other motivations see the discussion below) of the harmonic gauge for the proof of stability of Minkowski space-time lies in the fact that the latter can be simply viewed[5] as a *small data global existence result* for the quasilinear system (1.3). However, usefulness of the harmonic gauge in this context was questioned earlier and it was suspected that wave coordinates are "unstable in the large", [CB73]. The conclusion is suggested from the analysis of the iteration scheme for the system (1.3):

$$g_{\mu\nu} = m_{\mu\nu} + \varepsilon g^{(1)}_{\mu\nu} + \varepsilon^2 g^{(2)}_{\mu\nu} + \dots,$$

where $g^{(1)}_{\mu\nu}, g^{(2)}_{\mu\nu}$ satisfy respectively homogeneous and inhomogeneous wave equations on Minkowski background

$$\Box\, g^{(1)}_{\mu\nu} = 0, \quad \Box\, g^{(2)}_{\mu\nu} = F(m_{\mu\nu})\big(\partial g^{(1)}_{\mu\nu}, \partial g^{(1)}_{\mu\nu}\big), \quad \text{where } \Box = m^{\alpha\beta}\partial_\alpha\partial_\beta. \tag{1.7}$$

As a solution of the homogeneous $3+1$-d wave equation with smooth decaying initial data the functions $g^{(1)}_{\mu\nu} \approx C\varepsilon t^{-1}$ in the so-called wave zone $t \approx r$ as $t \to \infty$. Integrating the second equation implies that $g^{(2)}_{\mu\nu} \approx C'\varepsilon t^{-1} \ln t$. This suggests that already the asymptotic behavior of the second iterate deviates from the one of the free waves in Minkowski space-time and plants the seeds of doubt about global stability of such a scheme. We should note that this formal iteration procedure is termed the post-Minkowskian expansion and plays an important role in the study of gravitational radiation from isolated sources; see e.g. [Bla02], [BD92].

[5] This statement requires additional care since *a priori* there is no guarantee that obtained "global in time" solution $g_{\mu\nu}$ defines a causally geodesically complete metric. However, the latter can be established provided one has good control on the difference between $g_{\mu\nu}$ and the Minkowski metric $m_{\mu\nu}$ (see [LR05]).

To understand some of the difficulties in establishing a small data global existence result for the system (1.3) let us consider a generic quasilinear system of the form

$$\square\,\phi_i = \sum b_i^{jk\alpha\beta}\,\partial_\alpha\phi_j\,\partial_\beta\phi_k + \sum c_i^{jk\alpha\beta}\,\phi_j\,\partial_\alpha\partial_\beta\phi_k + \text{cubic terms.} \tag{1.8}$$

The influence of cubic terms is negligible while the quadratic terms are of two types, the *semilinear terms* and the *quasilinear terms*, each of which present their own problems. The semilinear terms can cause blow-up in finite time for smooth arbitrarily small initial data, as was shown by John [Joh81], for the equation $\square\,\phi = (\partial_t\phi)^2$. D. Christodoulou [Chr86] and S. Klainerman [Kla86] showed global existence for systems of the form (1.8) if the semilinear terms satisfy the *null condition* and the quasilinear terms are absent. The null condition, first introduced by S. Klainerman in [Kla84], was designed to detect systems for which solutions are asymptotically free and decay like solutions of a linear equation. It requires special algebraic cancellations in the coefficients $b_i^{jk\alpha\beta}$, e.g. $\square\,\phi = (\partial_t\phi)^2 - |\nabla_x\phi|^2$. However, the semilinear terms for the Einstein equations do not satisfy the null condition; see [CB00]. The quasilinear terms is another source of trouble. The only nontrivial example of a quasilinear equation of the type (1.8), for which the small data global existence result holds, is the model equation $\square\,\phi = \phi\,\Delta\phi$, as shown in [Lin92] (radial case) and [Ali03] (general case). However the solutions only decay like $\phi \sim \varepsilon t^{-1+C\varepsilon}$. This in particular implies that the characteristic surfaces of the associated Lorentzian metric $g = -dt^2 + (1+\phi)\sum_{i=1}^3 (dx^i)^2$ diverge distance $\sim t^{C\varepsilon}$ from the Minkowski cones.

In our previous work [LR03] we identified criteria under which it is more likely that a quasilinear system of the form (2.9) has global solutions.[6] We said that a system of the form (2.9) satisfies the *weak null condition* if the corresponding *asymptotic system* (cf. [Hör87], [Hör97]) has global solutions.[7] We showed that the Einstein equations in wave coordinates satisfy the weak null condition. In addition there is some additional cancellation for the Einstein equations in wave coordinates that makes it better than a general system satisfying the weak null condition. The system decouples to leading order, when decomposed relative to the Minkowski *null frame*. An approximate model that describes the semilinear terms has the form

$$\square\,\phi_2 = (\partial_t\phi_1)^2, \qquad \square\,\phi_1 = 0.$$

While every solution of this system is global in time, the system fails to satisfy the classical null condition and solutions are not asymptotically free: $\phi_2 \sim \varepsilon t^{-1}\ln|t|$.

[6] At this point, it is unclear whether this criterion is sufficient for establishing a "small data global existence" result for a *general* system of quasilinear hyperbolic equations.

[7] The condition trivially holds for the class of equations satisfying the standard null condition. For more discussion of the weak null condition and a general intuition behind the proof, see §12.

The semilinear terms in Einstein's equations can be shown to either satisfy the classical null condition or decouple in the above fashion when expressed in a null frame. The quasilinear terms also decouple but in a more subtle way. The influence of quasilinear terms can be detected via asymptotic behavior of the characteristic surfaces of metric g. It turns out that the main features of the characteristic surfaces at infinity are determined by a particular *null* component of the metric. The asymptotic flatness of the initial data and the wave coordinate condition (1.4) give good control of this particular component, i.e., $\sim M/r$, which in turn implies that the light cones associated with the metric g diverge only logarithmically $\sim M \ln t$ from the Minkowski cones.

In [LR05] we were able to establish the "small data global existence" result for the Einstein-vacuum equations in harmonic gauge for special restricted type of initial data. In addition to the standard constraint compatibility and smallness conditions the initial data was assumed to coincide with the Schwarzschild data

$$g_{0ij} = \Big(1 + \frac{M}{r}\Big)\delta_{ij}, \qquad k_{0ij} = 0$$

in the complement of the ball of radius one centered at the origin. The existence of such data was recently demonstrated in [Cor00], [CD02]. The stability of Minkowski space-time for the Einstein-vacuum equations for such data of course also follows from the work of Christodoulou-Klainerman and yet another approach of Friedrich [Fri86]. This choice of data allowed us to completely ignore the problem of the long range effect of the mass and the exterior existence.

In this paper we prove stability of Minkowski space-time for the Einstein-vacuum and the Einstein-scalar field equations in harmonic gauge for general asymptotically flat initial data (any $\alpha > 0$ in (1.5)–(1.6)) close to the data for the Minkowski solution.

The asymptotic behavior of null components of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ of metric g — the so-called "peeling estimates" — was discussed in the works of Bondi, Sachs and Penrose and becomes important in the framework of asymptotically simple space-times (roughly speaking, space-times which can be conformally compactified); see also the paper of Christodoulou [Chr02] for further discussion of such space-times. The work of [CK93] provided very precise, although not entirely consistent with peeling estimates, analysis of the asymptotic behavior of constructed global solutions. However, global solutions obtained by Klainerman-Nicolò [KN03a] in the problem of exterior[8] stability of Minkowski space were shown to possess peeling estimates for special initial data, [KN03b].

Our work is less precise about the asymptotic behavior and is focused more on developing a relatively technically simple approach allowing us to prove stability

[8]Outside of the domain of dependence of a compact set.

of Minkowski space in a physically interesting wave coordinate gauge, for general asymptotically flat data, and simultaneously treating the case of the Einstein equations coupled to a scalar field.

THEOREM 1.1. *Let $(\Sigma, g_0, k_0, \psi_0, \psi_1)$ be initial data for the Einstein-scalar field equations. Assume that the initial time slice Σ is diffeomorphic to $\mathbb{R}^3$ and admits a global coordinate chart relative to which the data is close to the initial data for the Minkowski space-time. More precisely, we assume that the data $(g_0, k_0, \psi_0, \psi_1)$ is smooth asymptotically flat in the sense of* (1.5)–(1.6) *with mass M and impose the following smallness assumption: Let*

$$g_0 = \delta + h^0_0 + h^1_0, \qquad \text{where} \quad h^0_{ij\,0} = \chi(r)\frac{M}{r}\,\delta_{ij},$$

where $\chi(s) \in C^\infty$ is 1 *when $s \geq 3/4$ and* 0 *when $s \leq 1/2$. Set*

$$E_N(0) = \sum_{0\leq|I|\leq N} \|(1+r)^{1/2+\gamma+|I|}\nabla\,\nabla^I h^1_0\|_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\nabla^I k_0\|_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\nabla\,\nabla^I \psi_0\|_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\nabla^I \psi_1\|_{L^2}. \tag{1.9}$$

There is a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and initial data verifying the condition

$$E_N(0) + M \leq \varepsilon \tag{1.10}$$

for some $\gamma > \gamma_0(\varepsilon_0)$ with $\gamma_0(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$ and $N \geq 6$, the Einstein-scalar field equations possess a future causally geodesically complete solution (g, ψ) asymptotically converging to Minkowski space-time. More precisely, there exists a global system of coordinates (t, x^1, x^2, x^3) in which the energy

$$\begin{aligned} \mathcal{E}_N(t) &= \sum_{|I|\leq N} \|w^{1/2}\partial Z^I h^1(t,\cdot)\|_{L^2} + \|w^{1/2}\partial Z^I \psi(t,\cdot)\|_{L^2}, \\ w^{1/2} &= \begin{cases} (1+|r-t|)^{1/2+\gamma}, & r > t, \\ 1, & r \leq t, \end{cases} \end{aligned} \tag{1.11}$$

with Minkowski vector fields $Z \in \{\partial_\alpha, x_\alpha\partial_\beta - x_\beta\partial_\alpha, x^\alpha\partial_\alpha\}$ of the solution

$$g(t) = m + h^0(t) + h^1(t), \qquad h^0_{\alpha\beta} = \chi(r/t)\chi(r)\frac{M}{r}\delta_{\alpha\beta}$$

obeys the estimate

$$\mathcal{E}_N(t) \leq C_N\varepsilon\,(1+t)^{C_N\varepsilon}. \tag{1.12}$$

Moreover,

$$\begin{aligned} &\partial Z^I h^1| + |\partial Z^I \psi| \\ &\quad\leq \begin{cases} C'_N\varepsilon(1+t+r)^{-1+C_N\varepsilon}(1+|t-r|)^{-1-\gamma}, & r > t, \\ C'_N\varepsilon(1+t+r)^{-1+C_N\varepsilon}(1+|t-r|)^{-1/2}, & r \leq t, \end{cases} \quad |I| \leq N-2 \end{aligned} \tag{1.13}$$

and

$$(1.14)\quad Z^I h^1| + |Z^I\psi| \\ \le \begin{cases} C''_N\varepsilon(1+t+r)^{-1+C_N\varepsilon}(1+|t-r|)^{-\gamma}, & r\ge t, \quad |I|\le N-2\\ C''_N\varepsilon(1+t+r)^{-1+2C_N\varepsilon}, & r\le t, \quad |I|\le N-3.\end{cases}$$

For $|I|=0$ we prove the stronger bound $C\varepsilon t^{-1}\ln|t|$ and in fact show that all but one component of h^1, expressed relative to a null frame, can be bounded by $C\varepsilon t^{-1}$. This more precise information is needed in the proof.

As we will show in a future paper, the decay estimates above are sufficient to prove the peeling properties of the curvature tensor up to order t^{-3} along any forward light cone, but peeling properties of higher order require stronger decay of initial data.

2. Strategy of the proof and outline of the paper

2.1. *Notation and conventions.* Coordinates:

- $\{x^\alpha\}_{\alpha=0,..,3}=(t,x)$ with $t=x^0$ and $x=(x_1,x_2,x_3),\ r=|x|$ are the standard space-time coordinates
- $s=r+t, q=r-t$ are the null coordinates

Derivatives:

- $\nabla=(\partial_1,\partial_2,\partial_3)$ denotes spatial derivatives
- $\partial=(\partial_t,\nabla)$ denotes space-time derivatives
- $\not\partial_i$ denotes spatial angular components of the derivatives ∂_i
- $\bar\partial=(\partial_t+\partial_r,\not\partial)$ denotes derivatives tangent to the light cones $t-r=$ constant
- $\partial_s=\frac12(\partial_r+\partial_t),\ \partial_q=\frac12(\partial_r-\partial_t)$ denote the null derivatives

Metrics:

- $m=-dt^2+\sum_{i=1}^3(dx^i)^2$ denotes the standard Minkowski metric on $\mathbb{R}^{3+1}$
- g denotes a Lorentzian metric solution of the Einstein equations
- Raising and lowering of indices in this paper is always done with respect to the metric m, i.e., for an arbitrary n-tensor $\Pi_{\alpha_1,\alpha_2..,\alpha_n}$ we define $\Pi^{\alpha_1}{}_{\alpha_2,..,\alpha_n}=m^{\alpha_1\beta}\Pi_{\beta,\alpha_2,..,\alpha_n}$. The only exception is made for the tensor $g^{\alpha\beta}$ which stands for the inverse of the metric $g_{\alpha\beta}$.

Null frame:

- $L=\partial_t+\partial_r$ denotes the vector field generating the forward Minkowski light cones $t-r=$ constant
- $\underline{L}=\partial_t-\partial_r$ denotes the vector field transversal to the light cones $t-r=$ constant

- S_1, S_2 denotes orthonormal vector fields spanning the tangent space of the spheres t =constant, r = constant
- The collection $\mathcal{T} = \{L, S_1, S_2\}$ denotes the frame vector fields tangent to the light cones
- The collection $\mathcal{U} = \{L, \underline{L}, S_1, S_2\}$ denotes the full null frame

Null forms:

- $Q_{\alpha\beta}(\partial\phi, \partial\psi) = \partial_\alpha\phi\partial_\beta\psi - \partial_\alpha\phi\partial_\alpha\psi$, $Q_0(\partial\phi, \partial\psi) = m^{\alpha\beta}\partial_\phi\partial_\beta\psi$ are the standard null forms

Null frame decompositions:

- For an arbitrary vector field X and frame vector U, $X_U = X_\alpha U^\alpha$, where $X_\alpha = m_{\alpha\beta}X^\beta$
- For an arbitrary vector field $X = X^\alpha\partial_a = X^L L + X^{\underline{L}}\underline{L} + X^{S_1}S_1 + X^{S_2}S_2$, where $X^L = -X_{\underline{L}}/2$, $X^{\underline{L}} = -X_L/2$, $X^{S_i} = X_{S_i}$
- For an arbitrary pair of vector fields X, Y: $X^\alpha Y_\alpha = -X_L Y_{\underline{L}}/2 - X_{\underline{L}}Y_L/2 + X_{S_1}Y_{S_1} + X_{S_2}Y_{S_2}$
- Similar identities hold for higher order tensors

Minkowski vector fields $\{Z\} = \mathcal{Z}$:

- $\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha, S = x^\alpha\partial_a\}$
- Commutation properties: $[\Box, \partial_\alpha] = [\Box, \Omega_{\alpha\beta}] = 0$, $[\Box, S] = 2\Box$, $c_Z\Box := [Z, \Box]$.

2.2. *Strategy of the proof.* Our approach to constructing solutions of the Einstein (Einstein-scalar field) equations is to solve the corresponding system of the reduced Einstein equations (1.3)

$$\widetilde{\Box}_g\, g_{\mu\nu} = \tilde{F}_{\mu\nu}(g)(\partial g, \partial g) + 2\partial_\mu\psi\,\partial_\nu\psi, \qquad \widetilde{\Box}_g\psi = 0. \tag{2.1}$$

It is well known that any solution (g, ψ) of the full system of Einstein equations, written in wave coordinates, will satisfy (2.1). The wave coordinate condition can be recast as a requirement that relative to a coordinate system $\{x^\mu\}_{\mu=0,\dots,3}$ the tensor $g_{\mu\nu}$ verifies

$$g^{\alpha\beta}\partial_\beta g_{\alpha\mu} = \frac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}. \tag{2.2}$$

Conversely, any solution of the system (2.1) with initial data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0})$ and $(\psi|_{t=0}, \partial_t\psi|_{t=0})$ compatible with (2.2) and constraint equations, gives rise to a solution of the full Einstein system. In addition, thus constructed tensor $g_{\mu\nu}$ will satisfy the wave coordinate condition (2.2) for all times. Verification of the above statements is straightforward and can be found in e.g. [Wal84].

Construction of initial data. The assumptions of the theorem assert that the initial data $(g_{0ij}, k_{0ij}, \psi_0, \psi_1)$ verifies the global smallness condition (1.9). We define the initial data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0}, \psi|_{t=0}, \partial_t\psi|_{t=0})$ for the system (2.1) as follows:

$$g_{ij}|_{t=0} = g_{0ij}, \quad g_{00}|_{t=0} = -a^2, \quad g_{0i}|_{t=0} = 0, \quad \psi|_{t=0} = \psi_0 \tag{2.3}$$

$$\partial_t g_{ij}|_{t=0} = -2ak_{0ij}, \quad \partial_t g_{00}|_{t=0} = 2a^3 g_0^{ij} k_{0ij}, \quad \partial_t\psi|_{t=0} = a\psi_1, \tag{2.4}$$

$$\partial_t g_{0\ell} = a^2 g_0^{ij}\partial_j\, g_{0i\ell} - \frac{1}{2}a^2 g_0^{ij}\partial_\ell\, g_{0ij} - a\partial_\ell\, a. \tag{2.5}$$

The lapse function $a^2 = (1 - M\chi(r)r^{-1})$. The data constructed above is compatible with the constraint equations and satisfies the wave coordinate condition (2.2). In fact, it simply corresponds to a choice of a local system of coordinates $\{x^\mu\}_{\mu=0,\dots,3}$ satisfying the wave coordinate condition $\Box_g x^\mu = 0$ at $x^0 = t = 0$. This procedure is standard and its slightly simpler version can be found e.g. in [Wal84]; see also [LR05].

The smallness condition $E_N(0) + M \le \varepsilon$ for the initial data $(g_{0ij}, k_{0ij}, \psi_0, \psi_1)$ implies the smallness condition

$$\mathcal{E}_N(0) + M \le \varepsilon$$

for the energy $\mathcal{E}_N(0)$:

$$\mathcal{E}_N(0) = \sum_{0\le|I|\le N} \|(1+r)^{1/2+\gamma+|I|}\partial\,\partial^I h_0^1\|_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\partial\,\partial^I \psi_0\|_{L^2}$$

defined for a tensor $h^1_{0\mu\nu}$, which appears in the decomposition

$$g|_{t=0} = m + h_0^0 + h_0^1, \qquad \text{where} \quad h^0{}_{0\mu\nu} = \chi(r)\frac{M}{r}\,\delta_{\mu\nu}.$$

This allows us to reformulate the problem of global stability of Minkowski space as a small data global existence problem for the system

$$\widetilde{\Box}_g\, h^1_{\mu\nu} = F_{\mu\nu}(g)(\partial h, \partial h) + 2\partial_\mu\psi\,\partial_\nu\psi - \widetilde{\Box}_g h^0, \quad \widetilde{\Box}_g\psi = 0 \tag{2.6}$$

with $h^0_{\mu\nu} = \chi(r/t)\chi(r)\dfrac{M}{r}\delta_{\mu\nu}$. Arguing as in Section 4 of [LR05] we can show that the tensor $g_{\mu\nu}(t) = m + h^0_{\mu\nu}(t) + h^1_{\mu\nu}(t)$ obtained by solving (2.6) with initial data given in (2.3)–(2.5) defines a solution of the Einstein-scalar field equations. Moreover, the wave coordinate condition (2.2) is propagated in time.

The above discussion leads to the following result:

THEOREM 2.1. *A global in time solution of* (2.6) *obeying the energy bound* $\mathcal{E}_N(t) \le C\varepsilon(1+t)^\delta$ *for some sufficiently small* $\delta > 0$ *gives rise to a future causally geodesically complete solution of the Einstein equations* (1.1) *converging to the Minkowski space-time.*

The proof of the geodesic completeness can be established by arguments identical to those in [LR05] and will not be repeated here.

The mass problem. A first glance at the system (2.1) suggests that a natural approach to recast the problem of global stability of Minkowski space as a small data global existence question for a system of quasilinear wave equations is to rewrite (2.1) as an equation for the tensor $h = g - m$:

$$\widetilde{\Box}_g\, h_{\mu\nu} = F_{\mu\nu}(\partial h, \partial h) + 2\partial_\mu \psi\, \partial_\nu \psi, \qquad \widetilde{\Box}_g \psi = 0. \tag{2.7}$$

The initial data for h possess the asymptotic expansion as $r \to \infty$:

$$h_{\mu\nu}|_{t=0} = \frac{M}{r}\delta_{\mu\nu} + O(r^{-1-\alpha}), \quad \partial_t h_{\mu\nu}|_{t=0} = O(r^{-2-\alpha}) \tag{2.8}$$

for some positive $\alpha > 0$. While the data appears to be "small" it does not have sufficient decay rate in r at infinity due to the presence of the term with positive mass M. This could potentially lead to a "long range effect" problem, i.e., the decay of the solution at the time-like infinity $t \to \infty$ is affected by the slow fall-off at the space-like infinity $r \to \infty$. In the work [CK93] this problem was resolved (albeit in a different language) by taking advantage of the fact that the long range term is spherically symmetric and the ADM mass M is conserved along the Einstein flow. Thus differentiating the solution (in [CK93] this means the Weyl field corresponding to the conformally invariant part of the Riemann curvature tensor of metric g) with respect to properly defined (non-Minkowskian) angular momentum and time-like vector fields, one obtains a new field still approximately satisfying the Einstein field equations but with considerably better decay properties at space-like infinity. We however pursue a different approach by taking an educated "guess" about the contribution of the long range term $M\delta_{\mu\nu}/r$ to the solution. Thus we set

$$h^0_{\mu\nu} = \chi\Big(\frac{r}{t}\Big)\chi(r)\frac{M}{r}\delta_{\mu\nu},$$

split the tensor $h = h^1 + h^0$ and write the equation for the new unknown h^1. The important cancellation occurring in a new inhomogeneous term $\widetilde{\Box}_g h^0$ is the vanishing of $\Box \dfrac{M}{r} = 0$ away from $r = 0$. The cut-off function $\chi(r/t)$ ensures that the essential contribution of $\widetilde{\Box}_g h^0$ comes from the "good" interior region $1/2t \le r \le 3/4t$ — the support of the derivatives of $\chi(r/t)$. Finally, observe that in the presence of the term containing the mass M the tensor $h = h^0 + h^1$ has infinite energy $\mathcal{E}_N(0)$.

The system (2.6) is a system of quasilinear wave equations. The results of [Chr86], [Kla86] show that a sufficient condition for a system of quasilinear wave equations to have global solutions for all smooth sufficiently small data is the null condition. However quasilinear problems where the metric depends on the solution, rather than its derivatives, as in the problems arising in elasticity, do not satisfy the null condition. Moreover, as shown in [CB52], [CB73], even the semilinear terms $F_{\mu\nu}(\partial h, \partial h)$ violate the standard null condition.

Connection with the weak null condition. Consider a general system of quasilinear wave equations:

$$\Box\phi_I = \sum_{|\alpha|\le|\beta|\le 2,\ |\beta|\ge 1} A^{JK}_{I,\alpha\beta}\, \partial^\alpha\phi_J\, \partial^\beta\phi_K + \text{cubic terms}. \tag{2.9}$$

The weak null condition, introduced in [LR03] requires that the asymptotic system for $\Phi_I = r\phi_I$ corresponding to (2.9):

$$(\partial_t+\partial_r)(\partial_t-\partial_r)\Phi_I \sim r^{-1} \sum_{n\le m\le 2,\ m\ge 1} A^{JK}_{I,nm}(\partial_t-\partial_r)^n\Phi_J\ (\partial_t-\partial_r)^m\Phi_K \tag{2.10}$$

has global solutions for all small data. Here, the tensor

$$A^{JK}_{I,nm}(\omega) := (-2)^{-m-n} \sum_{|\alpha|=n,\,|\beta|=m} A^{JK}_{I,\alpha\beta}\,\hat\omega^\alpha\hat\omega^\beta, \quad \hat\omega = (-1,\omega),\ \omega\in\mathbf{S}^2.$$

The standard null condition, which guarantees a small data global existence result for the system (2.9), is that $A^{JK}_{I,nm}(\omega)\equiv 0$ and clearly is included in the weak null condition, since in that case the corresponding asymptotic system (2.10) is represented by a *linear* equation. Asymptotic systems were introduced by Hörmander (see [Hör87], [Hör97]) as a tool to find the exact blow-up time of solutions of scalar equations violating the standard null condition with quadratic terms independent of ϕ, i.e., $|\alpha|\ge 1$ in (2.9). This program was completed by Alinhac; see e.g. [Ali00]. It was observed in [Lin90b] that the asymptotic systems for quasilinear equations of the form $\Box\,\phi = \sum c^{\alpha\beta}\phi\ \partial_\alpha\partial_\beta\phi$ in fact have global solutions. In other words these equations satisfy the weak null condition. It was therefore conjectured that these wave equations should have global solutions for small data. This has been so far only proven for the equation

$$\Box\phi = \phi\,\Delta\phi, \tag{2.11}$$

[Lin90b] (radial case), [Ali03] (general case). Note that the case $|\alpha|=|\beta|=0$, in (2.9) is excluded. The asymptotic system only predicts the behavior of the solution close to the light cones, and for the case $\Box\phi=\phi^2$ the blow-up occurs in the interior and much sooner; see [Joh85], [Lin90a].

The asymptotic system (2.10) is obtained from the system (2.9) by neglecting derivatives tangential to the outgoing Minkowski light cones and cubic terms, that can be expected to decay faster. In particular,

$$\Box\phi = r^{-1}(\partial_t+\partial_r)(\partial_t-\partial_r)(r\phi) + \text{angular derivatives},$$
$$\partial_\mu = -\frac{1}{2}\hat\omega_\mu(\partial_t-\partial_r) + \text{tangential derivatives}.$$

Recall that for solutions of linear wave homogeneous wave equation derivatives tangential to the forward light cone $t = r$ decay faster: $\Box\phi = 0$ implies that $|\partial\phi| \le C/t$ while $|\bar{\partial}\phi| \le C/t^2$.

A simple example of a system satisfying the weak null condition, violating the standard null condition and yet possessing global solutions is

$$\Box\phi_1 = \phi_3 \cdot \partial^2\phi_1 + (\partial\phi_2)^2, \qquad \Box\phi_2 = 0, \qquad \Box\phi_3 = 0. \tag{2.12}$$

Another, far less trivial example is provided by equation (2.11).

The asymptotic system for Einstein equations can be modeled by that of (2.12). To see this we introduce a null-frame $\{L, \underline{L}, S_1, S_2\}$ decomposition of Einstein's equation. With $h=g-m$ we have

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h) + 2\partial_\mu\psi\,\partial_\nu\partial\psi, \qquad \widetilde{\Box}_g\psi = 0 \tag{2.13}$$

where

$$F_{\mu\nu}(h)(\partial h, \partial h) = \frac{1}{4}\partial_\mu h_\alpha^\alpha \partial_\nu h_\beta^\beta - \frac{1}{2}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$

with $Q_{\mu\nu}$-linear combinations of the standard null-forms and $G_{\mu\nu}(h)(\partial h, \partial h)$ contains only cubic terms.

The asymptotic system for the reduced Einstein equations has the following form:

(2.14)

$$\begin{aligned}(\partial_t + \partial_r)\partial_q D_{\mu\nu} &= H_{LL}\partial_q^2 D_{\mu\nu}\\ &\qquad - \frac{1}{4r}\hat{\omega}_\mu\hat{\omega}_\nu\big(P(\partial_q D, \partial_q D) + 2\partial_q\Phi\,\partial_q\Phi\big), \quad \partial_q = \partial_t - \partial_r,\end{aligned}$$

(2.15)

$$\begin{aligned}(\partial_t + \partial_r)\partial_q\Phi = H_{LL}\partial_q^2\Phi, \quad D_{\mu\nu} \sim r h_{\mu\nu}, \quad \Phi \sim r\psi,\\ H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}, \quad H_{LL} = H_{\alpha\beta}L^\alpha L^\beta, \quad L^\alpha\partial_\alpha = \partial_t + \partial_r.\end{aligned}$$

Here

$$P(D, E) = \frac{1}{4}D_\alpha{}^\alpha E_\beta{}^\beta - \frac{1}{2}D^{\alpha\beta}E_{\alpha\beta}. \tag{2.16}$$

On the other hand the asymptotic form of the wave coordinate condition (2.2) is

$$\partial_q D_{LT} \sim 0, \qquad T \in \mathcal{T} = \{L, S_1, S_2\}. \tag{2.17}$$

Observe that (2.17) combined with the initial asymptotic expansion (2.8) of the metric g suggests that asymptotically, as $t \to \infty$ and $|q| = |t - r| \le C$,

$$h_{L\alpha} \sim H_{L\alpha} \sim \frac{M}{t}c_\alpha, \qquad c_\alpha = \begin{cases}0, & \alpha = 0,\\ x_i/|x|, & \alpha = i.\end{cases} \tag{2.18}$$

Decomposing the system with the help of the null frame $\{L, S_1, S_2\} = \mathcal{T}$ and $\underline{L}:\ \underline{L}^\alpha \partial_\alpha = \partial_r - \partial_t$ and using that $L^\mu \hat{\omega}_\mu = A^\mu \hat{\omega}_\mu = 0$, we obtain that

$$(\partial_t + \partial_r)\partial_q D_{\underline{L}\,\underline{L}} = H_{LL}\partial_q^2 D_{\underline{L}\underline{L}} - 2r^{-1}P(\partial_q D, \partial_q D) - r^{-1}\partial_q \Phi\, \partial_q \Phi \tag{2.19}$$

$$(\partial_t + \partial_r)\partial_q D_{TU} = H_{LL}\partial_q^2 D_{TU}, \qquad \begin{array}{l} T \in \mathcal{T} = \{L, S_1, S_2\}, \\ U \in \mathcal{U} = \{L, \underline{L}, S_1, S_2\}, \end{array} \tag{2.20}$$

$$(\partial_t + \partial_r)\partial_q \Phi = H_{LL}\partial_q^2 \Phi. \tag{2.21}$$

In view of (2.18) the equations (2.20) allow us to find the asymptotic behavior of the components $\partial_q D_{TU} \sim$ const and consequently $h_{TU} \sim t^{-1}$. The asymptotic behavior $\Phi \sim t^{-1}$ can be similarly determined from (2.21). On the other hand,

$$P(\partial_q D, \partial_q D) = \frac{1}{4}\partial_q D_{LL}\partial_q D_{\underline{LL}} + \partial_q D_{TU}\cdot \partial_q D_{TU}.$$

It is the absence of the quadratic term $(\partial_q D_{\underline{LL}})^2$ and the boundedness of the components $\partial_q D_{TU}$ that allows us to solve equation (2.19) for $D_{\underline{LL}}$ although the suggested asymptotic behavior is different $h_{\underline{LL}} \sim H_{\underline{LL}} \sim t^{-1}\ln t$.

It turns out that the asymptotic system indeed correctly predicts the asymptotic behavior of the tensor $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}$-solution of the Einstein equations in wave coordinates

Before proceeding to explain the strategy of the proof of our result we review the ingredients of a proof of a typical small data global existence for quasilinear wave equations in dimensions $n \ge 4$ or equations satisfying the standard null condition. For simplicity we consider a semilinear equation

$$\Box\phi = N(\partial\phi, \partial\phi) \tag{2.22}$$

with a quadratic nonlinearity N. We first note that almost every small data global existence result was established under the assumption of compactly supported (or rapidly decaying) data, which by finite speed of propagation ensures that a solution is supported in the interior of a light cone $r = t + C$ for some sufficiently large constant C. The proof is based on generalized energy estimates

$$E_N(t) \le E_N(0) + \sum_{Z\in\mathcal{Z}, |I|\le N} \int_0^t \|Z^I F(\tau)\|_{L^2} E_N^{\frac{1}{2}}(\tau)\, d\tau \tag{2.23}$$

for solutions of an inhomogeneous wave equation $\Box\phi = F$. The generalized energy

$$E_N(t) = \sum_{Z\in\mathcal{Z}, |I|\le N} \|\partial Z^I \phi(t)\|_{L^2}^2,$$

with vector fields $\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha\beta} = -x_\alpha\partial_\beta + x_\beta\partial_\alpha, S = x^\alpha\partial_\alpha\}$, of a solution of (2.22) can be shown to satisfy the inequality

$$E_N(t) \le \exp\big[C \sup_{Z\in\mathcal{Z}, |I|\le N/2+2} \int_0^t \|\partial Z^I\phi(\tau)\|_{L^\infty_x}\big] E_N(0). \tag{2.24}$$

The other crucial component is the Klainerman-Sobolev inequality which asserts that for an arbitrary smooth function ψ

$$|\partial\psi(t,x)| \le C(1+t+r)^{-\frac{n-1}{2}}(1+|q|)^{-\frac{1}{2}}\ \ E_4(t), \qquad r=|x|,\ \ q=r-t. \tag{2.25}$$

Combining (2.25) with the energy inequality leads to the proof of the small data global existence result for a generic semilinear wave equation in dimension $n \ge 4$. It also elucidates the difficulty of proving such a result in dimension $n=3$. In fact, as was shown in [Joh81], the result can be false in dimension $n=3$, e.g., the equation $\Box\phi = \phi_t^2$ admits small data solutions with finite time of existence. It is interesting to note that the corresponding asymptotic system for such equation leads to a Riccati type ODE.

In the case when the semilinear terms $N(\partial\phi, \partial\phi)$ obey the standard null condition, i.e., it is a linear combination of the quadratic null forms $Q_{\alpha\beta}, Q_0$, it is possible to refine the energy inequality (2.24) so that a combination of the energy and Klainerman-Sobolev inequalities still yields a small data global existence result in dimension $n=3$. This can be traced to the following pointwise estimate on a null form Q:

$$|Q(\partial\phi, \partial\phi)| \le C(1+t+r)^{-1}\sum_{Z\in\mathcal{Z}} |\partial\phi|\,|Z\phi|.$$

In the absence of the standard null condition in dimension $n=3$ the inequalities (2.24)–(2.25) are just barely insufficient for the desired result. An illuminating example is provided by a semilinear version of the system (2.12)

$$\Box\phi_1 = (\partial\phi_2)^2, \qquad \Box\phi_2 = 0.$$

A combination of (2.24)–(2.25) applied to the vector $\phi = (\phi_1, \phi_2)$ would incorrectly suggest a possible finite time blow-up. This analysis however reflects the following phenomena: *small data solutions of the system above have a polynomially growing energy $E_N(T) \sim t^\delta$ and the asymptotic behavior $\partial\phi(t) \sim t^{-1}\ln t$.* Of course the small data global existence result in this case by applying (2.23)–(2.25) *separately* to each of the components of $\phi = (\phi_1, \phi_2)$. Recall however that the system (2.12) models the Einstein equations only after the latter is decomposed relative to its null frame components $h_{LL}, h_{\underline{L}\,\underline{L}}\ldots$. Such decompositions do not commute with the wave equation and thus prevent one from deriving separate energy estimates for each of the null components of h. In this paper we are able to solve this problem by adding another ingredient: *an "independent" decay estimate*. The discussion below will be focused on the tensor $h^1 = g - m - h^0$ obtained from the original metric g by subtracting its "Schwarzschild part". We reluctantly allow for the fact that the energy $E_N(t)$ of the tensor h^1 might be growing with the rate of $\varepsilon^2 t^\delta$ as $t \to \infty$ for some small constant $\delta > 0$ dependent on the smallness of the initial data. The Klainerman-Sobolev inequality (2.25) would then imply that

$$\sup_{Z\in\mathcal{Z},|I|\le N-3} |\partial Z^I h^1(t,x)| \le C\varepsilon(1+t+r)^{-1+\delta}(1+|q|)^{-\frac{1}{2}}. \tag{2.26}$$

In order to close the argument, i.e., verify that the energy $E_N(t)$ indeed grows at the rate of at most $\varepsilon^2 t^\delta$ we must, according to (2.24), upgrade the decay estimates (2.26) to the decay rate of εt^{-1}. To do that we invoke the asymptotic system (2.14)–(2.15). It turns out that merely assuming that the energy $E_N(t) \le \varepsilon(1+t)^\delta$ and consequently (2.26) allows us to show that the asymptotic system provides an effective approximation of the full nonlinear system, i.e., the discarded terms containing tangential derivatives do not influence the asymptotic behavior of the field h^1. The crucial property of the asymptotic system (2.14)–(2.15) is that, as opposed to the full nonlinear equation, it does commute with the null frame decomposition of h thus leading to the system (2.19)–(2.21). Once the validity of the asymptotic system is established the asymptotic behavior of its solutions provides a new decay estimate and, as we described above, shows that for $\mathcal{T} = \{L, S_1, S_2\}$, $\mathcal{U} = \{L, \underline{L}, S_1, S_2\}$

$$|\partial h|_{\mathcal{T}\mathcal{U}} \le C\varepsilon t^{-1}, \qquad |\partial h|_{\underline{\mathcal{L}}\underline{\mathcal{L}}} \le C\varepsilon t^{-1}\ln t, \qquad |\partial\psi| \le C\varepsilon t^{-1}.$$

The decay rate of $\varepsilon t^{-1}\ln t$ can be potentially disastrous since (2.24) would imply that the energy $E_N(t)$ could grow at the super-polynomial rate of $\varepsilon^2 \exp[C\varepsilon \ln^2 t]$. However the remarkable structure of the Einstein equations comes to the rescue. The analysis of the semilinear terms $F_{\mu\nu}(\partial h, \partial h)$ in the equation

$$\widetilde{\Box}_g h^1_{\mu\nu} = F_{\mu\nu}(\partial h, \partial h) - \widetilde{\Box}_g h^0_{\mu\nu} + 2\partial_\mu\psi\,\partial_\nu\psi$$

shows that the sharp t^{-1} decay is only required for the $\partial h_{\mathcal{T}\mathcal{U}}$ components in order for the energy estimate $E_N(t) \le C\varepsilon^2 t^\delta$ to hold. A glimpse of this structure has already appeared in our discussion of the weak null condition. Recall that

$$F_{\mu\nu}(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h).$$

The cubic term G and the quadratic form Q satisfying the standard null condition are consistent with the uniform boundedness of the energy $E_N(t)$ and thus irrelevant for this discussion. The quadratic form $P(\partial_\mu h, \partial_\nu h)$ can be decomposed relative to the null frame and obeys the following estimate

$$|P(\partial_\mu h, \partial_\nu h)| \le |\partial h|_{\mathcal{T}\mathcal{U}}|\partial h|_{\mathcal{U}\mathcal{U}},$$

i.e., the most dangerous term $|\partial h|^2_{\underline{\mathcal{L}\mathcal{L}}}$, which would lead to the damaging estimate

$$E_0(t) \le \exp[C \int_0^t |\partial h(\tau)|_{\underline{\mathcal{L}\mathcal{L}}}] E_0(0),$$

is absent!

The picture painted above is clearly somewhat simplified: we have only indicated the argument implying that the lowest order energy $E_0(t) \le C\varepsilon^2 t^\delta$ and we have not even begun to address the effect of the quasilinear terms. In what follows we describe the building blocks of our result and explain challenges of the quasilinear structure.

Energy inequality with weights.

$$\int_{\Sigma_T} |\partial\phi|^2 w + \int_0^T \int_{\Sigma_t} |\bar{\partial}\phi|^2 w' \le 8 \int_{\Sigma_0} |\partial\phi|^2 w + C\varepsilon \int_0^T \int_{\Sigma_t} \frac{|\partial\phi|^2 w}{1+t} + 16 \int_0^T \int_{\Sigma_t} |\widetilde{\Box}_g \phi||\partial_t \phi| w, \tag{2.27}$$

where $\bar{\partial}$ denotes derivatives tangential to the outgoing Minkowski light cones $q = r - t$. The weight function

$$w(q) = \begin{cases} 1 + (1+|q|)^{1+2\gamma}, & q \ge 0, \\ 1 + (1+|q|)^{-2\mu}, & q < 0, \end{cases}$$

with $\mu \ge 0, \gamma \ge -1/2$ serves a double purpose: it generates an additional *positive* space-time integral giving an *a priori* control of the tangential derivatives $\bar{\partial}\phi$ (we mention in passing that the use of such space-time norms leads to a very simple proof of the small data global existence result for semilinear equations $\Box\phi = Q(\partial\phi, \partial\phi)$ satisfying the standard null condition. To our knowledge this argument has not appeared in the literature before); it provides the means to establish additional decay in q via the Klainerman-Sobolev inequality. The energy estimate (2.27) is established under very weak general assumptions on the background metric $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$, e.g.,

$$|\partial H| \le C\varepsilon(1+t)^{-\frac{1}{2}}(1+|q|)^{-\frac{1}{2}}(1+q_-)^{-\mu}, \qquad q_- = |\min(q,0)|,$$

which are consistent with our expectations that if the energy $E_N(t) \le \varepsilon^2(1+t)^\delta$ then the tensor h and consequently $H = -h + O(h^2)$ decay with a rate of at least $t^{-1+\delta}$. However, as is the case with virtually every estimate in this paper, special stronger conditions are required for the H_{LL} component of the metric:

$$|\partial H|_{LL} \le C\varepsilon(1+t)^{-1}(1+|q|)^{-1}.$$

Once again we note that such decay is consistent with the behavior predicted by the asymptotic system for the Einstein equations, yet even better control for this component is provided by the wave coordinate condition, which we will turn to momentarily.

The interior estimate (2.27), i.e., with $w(q) \equiv 0$ for $q \geq 0$, in the constant coefficient case basically follows by averaging the energy estimates on light cones used e.g. in [Shu91]. We also note that the interior energy estimates with space-time quantities involving special derivatives of a solution were also considered and used in the work of Alinhac; see e.g. [Ali01], [Ali03]. In [LR05] we proved the interior estimate (2.27) under natural assumptions on the metric g following in particular from the wave coordinate condition. The use of the weights $w(q)$ in the exterior $q \geq 0$ in energy estimates for the space part $\int_{\Sigma_T} |\partial\phi|^2 w$ originates in [KN03b].

Motivated by the energy inequality (2.27) we define the energy $E_N(t)$ associated with the tensor

$$h^1_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu} - h^0_{\mu\nu}, \qquad h^0_{\mu\nu} = \chi\Big(\frac{r}{t}\Big)\chi(r)\frac{M}{r}\delta_{\mu\nu}$$

solution of the reduced Einstein equations $\widetilde{\Box}_g h^1_{\mu\nu} = F_{\mu\nu} - \widetilde{\Box}_g h^0 + \partial_\mu\psi\,\partial_\nu\psi$ and a collection $\mathscr{Z} = \{\partial_\alpha, \Omega_{\alpha\beta} = -x_\alpha\partial_\beta + x_\beta\partial_\alpha, S = x^\alpha\partial_\alpha\}$ of commuting Minkowski vector fields, as follows:

$$E_N(t) = \sum_{Z\in\mathscr{Z}, |I|\leq N} \Big(\|w^{\frac12}\,\partial Z^I h^1(t,\cdot)\|^2_{L^2} + \|w^{\frac12}\,\partial Z^I\psi(t,\cdot)\|^2_{L^2}\Big).$$

The Klainerman-Sobolev inequalities with weights.

$$w^{1/2}\Big(|\partial Z^I h^1(t,x)| + |\partial Z^I\psi(t,x)|\Big) \leq \frac{CE_N(t)}{(1+t+r)(1+|q|)^{1/2}}, \qquad |I| \leq N-2$$

is the fundamental tool allowing us to derive first preliminary, sometimes referred to as weak, decay estimates.

Decay estimate. The following decay estimate

$$\varpi(q)(1+t)|\partial\phi(t,x)| \lesssim \int_0^t (1+\tau)\|\varpi\widetilde{\Box}_g\phi(\tau,\cdot)\|_{L^\infty}d\tau + \sum_{|I|\leq 2}\int_0^t \|\varpi Z^I\phi(\tau,\cdot)\|_{L^\infty}\frac{d\tau}{1+\tau} \tag{2.28}$$

is the additional "independent" estimate designed to boost the weak decay estimates derived via the Klainerman-Sobolev inequality. The weight function

$$\varpi(q) = \begin{cases} 1 + (1+|q|)^{1+\gamma'}, & q \geq 0,\\ 1 + (1+|q|)^{\frac12-\mu'}, & q < 0,\end{cases}$$

with $\gamma' \geq -1$, $\mu' \leq 1/2$ is chosen in harmony with the weight function $w(q)$ of the energy estimates. The unweighted version of this estimate was used in [Lin90b] in the constant coefficient case. We generalize this estimate to the variable coefficient operator $\widetilde{\square}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta$ under very weak general assumptions on the metric $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$, e.g.,

$$\int_0^t \|H(\tau,\cdot)\|_{L^\infty} \frac{d\tau}{1+\tau} \leq \varepsilon.$$

However, once again the $H_{\mathcal{L}\mathcal{T}}$ components are required to obey the stronger condition

$$|\partial H|_{\mathcal{L}\mathcal{T}} \leq C\varepsilon(1+t)^{-1}(1+|q|)^{-1}.$$

The estimate above is closely connected with the asymptotic equation in the sense that it is obtained by treating the angular derivatives as lower order and integrating the equation

$$(\partial_t + \partial_r)\partial_q(r\phi) = r\square\phi + \frac{1}{r}\triangle_\omega\phi.$$

An important virtue of (2.28) is that when applied to systems $\widetilde{\square}_g\phi_{\mu\nu} = F_{\mu\nu}$ it can be derived for each *null* component of ϕ. This property is indispensable in view of the fact that the weak null condition for the system of reduced Einstein equations becomes transparent only after decomposing the tensor $h_{\mu\nu}$ relative to a null frame $\{L, \underline{L}, A, B\}$. In particular, (2.28) will lead to the estimates

$$|\partial\psi| + |\partial h|_{\mathcal{T}\mathcal{U}} \leq C\varepsilon(1+t+|q|)^{-1}, \qquad |\partial h| \leq C\varepsilon(1+t)^{-1}\ln(1+t).$$

Commutators. As was mentioned above the combination: "energy estimate — Klainerman-Sobolev inequality — decay estimate", results in sharp decay estimates for the first derivatives of the tensor h^1 and merely polynomial growth of the energy $E_0(t)$. However, both the Klainerman-Sobolev and the additional decay estimate require control of the higher energy norms $E_N(t)$ involving the collection $\mathcal{Z}$ of Minkowski vector fields. To obtain that control one needs to commute vector fields $Z \in \mathcal{Z}$ through the equation $\widetilde{\square}_g h^1 = F - \widetilde{\square}_g h^0$ and reapply the energy estimate (2.27). This is by far the most difficult task in dealing with quasilinear equations. The collection $\mathcal{Z}$ enjoys good commutation properties with the wave operator $\square$ of Minkowski space: $[Z, \square] = -c_Z\square$, where $c_Z = 2$ for $Z = S$ and is zero otherwise, however its commutator with the wave operator $\widetilde{\square}_g$ can create undesirable terms. It is exactly for this reason that both in the work [CK93] on stability of Minkowski space and [Ali03] on small data global existence for the equation $\square\phi = \phi\Delta\phi$ a different collection of *geometrically modified* vector fields $\mathcal{Z}$ was used. The new vector fields are adapted to the true characteristic surfaces of the metric g (this originates and is especially manifest in the beautiful construction of [CK93]) and better suited for commuting through $\widetilde{\square}_g$; this construction however adds another complicated layer to the proof.

In this work we employ the collection $\mathcal{Z}$ of original Minkowski vector fields and argue that the bad commutation properties of $\mathcal{Z}$ with $\widetilde{\Box}_g$ can be improved if one takes into account the wave coordinate condition

$$\partial_\mu\Big(g^{\mu\nu}\sqrt{|\det g|}\Big) = 0 \tag{2.29}$$

satisfied by the metric g in a coordinate system $\{x^\mu\}_{\mu=0,\dots,3}$. To explain this phenomenon consider the commutator $[Z, \widetilde{\Box}_g]$ with one of the Minkowski vector fields $Z \in \mathcal{Z}$. For simplicity assume that $Z \neq S$ so that $[Z, \Box] = 0$. Then

$$[Z, \widetilde{\Box}_g] = [Z, H^{\alpha\beta}\partial_\alpha\partial_\beta] = \big(ZH^{\alpha\beta} + H_Z^{\alpha\beta}\big)\partial_\alpha\partial_\beta, \quad H_Z^{\alpha\beta} := H^{\alpha\gamma}c_\gamma{}^\beta + H^{\gamma\beta}c_\gamma{}^\alpha$$

and the coefficients $c_\alpha{}^\beta = \partial_\alpha Z^\beta$. It turns out that since Z is either Killing or conformally Killing vector field ($Z = S$) vector field of Minkowski space the coefficients $c_\alpha{}^\beta$ have the property that $c_{\underline{L}}{}^{\underline{L}} = 0$, which implies that $|H_Z^{\underline{L}\underline{L}}| \le |H|_{\mathcal{L}\mathcal{T}}$. In accordance with the usual arguments the worst term generated by the commutator $[Z, \widetilde{\Box}_g]$ contains two derivatives ∂_q^2 transversal to the light cones $q = r - t$ and any modification to the vector fields Z targets to eliminate such a term. In our case this term comes with a coefficient $(ZH^{\underline{L}\underline{L}} + H_Z^{\underline{L}\underline{L}})$ and thus can be estimated by

$$\frac{|ZH|_{\mathcal{L}\mathcal{L}} + |H|_{\mathcal{L}\mathcal{T}}}{1+|q|}|\partial Z h^1|.$$

Examining the equation

$$\widetilde{\Box}_g Z h^1 = [\widetilde{\Box}_g, Z]h^1 + ZF - Z\widetilde{\Box}_g h^0 + Z(\partial\psi\,\partial\psi)$$

it is not too difficult to see that in order to control (i.e., establish a polynomial bound) the energy $E_1(t)$ one needs to at least show that

$$|ZH|_{\mathcal{L}\mathcal{L}} + |H|_{\mathcal{L}\mathcal{T}} \le C\varepsilon(1+t)^{-1}(1+|q|).$$

To explain the difficulty of such estimate we note that in the region $|q| \le C$ this estimate is saturated by the Schwarzschild part $\chi(r/t)\chi(r)M\delta_{\mu\nu}/r$ of H. While the desired estimate for $H_{\mathcal{L}\mathcal{T}}$ can be obtained from the sharp decay estimates on the first derivatives of h^1, the estimate for ZH_{LL} is a much more subtle issue. An attempt to return to the asymptotic system for the Einstein equations or alternatively the decay estimate (2.28) and show that ZH_{LL} still satisfies the sharp decay estimate requires once again a commutator argument, this time in the context of the decay estimates, and fails. In fact, we can only show that the higher Z derivatives of h^1 decay at the rate of $C\varepsilon(1+t)^{-1+C\varepsilon}$. It is at this point that we recall that the tensor $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$ verifies the wave coordinate condition (2.29) which after simple manipulations implies that

$$\partial_q H_{\mathcal{L}\mathcal{T}} = \bar{\partial} H + O(H \cdot \partial H). \tag{2.30}$$

Here $\bar\partial$ stands for a tangential derivative, which in particular means that $|\bar\partial H| \le (1+t)^{-1}|ZH|$. Thus, if we expect H and its higher Z derivatives to decay with the rate $t^{-1+\delta}$ consistent with the t^δ energy growth and the Klainerman-Sobolev inequality, we can conclude that $\partial_q H_{\mathcal{L}\mathcal{T}}$ will decay with the rate of at least $t^{-2+\delta}$. Integrating with respect to the $q = r - t$ variable from the initial data at $t = 0$ we can get the desired rate of decay for $H_{\mathcal{L}\mathcal{T}}$. We should note that, in order to exploit the additional decay in t, we need to keep track and use the decay in $|q|$. This, in particular directly applies to the integration described above. The necessity of utilizing the decay in q explains our desire to work with weighted energy and decay estimates. Amazingly (2.30) is preserved after commuting with a Z vector field, but only for the H_{LL} component, i.e.,

$$\partial_q Z H_{\mathcal{L}\mathcal{L}} = \bar\partial Z H + Z O(H \cdot \partial H).$$

Moreover, further commutations would destroy this precise structure. Luckily, the control of higher energies $E_k(t)$ still only requires the estimate

$$|ZH|_{\mathcal{L}\mathcal{L}} + |H|_{\mathcal{L}\mathcal{T}} \le C\varepsilon(1+t)^{-1}(1+|q|),$$

since the principal term in the commutator $[Z^I, \widetilde{\Box}_g]$ is still $(ZH^{\underline{LL}} + H^{\underline{LL}}_{Z})\partial_q^2 Z^I$. We should also mention that from the point of view of the energy estimates the commutator $[Z^I, \widetilde{\Box}_g]h^1$ contains another dangerous term:

$$Z^I H_{\mathcal{L}\mathcal{L}} \partial_q^2 h^1.$$

Since most of the Z derivatives fall on the first factor one is forced to take it in the energy norm and use the decay estimates for $\partial_q^2 h^1$. Note the two essential problems: the term $Z^I H_{\mathcal{L}\mathcal{L}}$ does not contain a derivative ∂, which is required in order to identify with the energy norm, and the decay of $\partial_q^2 h^1$ falls short of the needed decay rate of t^{-1}. We resolve these problems as follows. First, a Hardy-type inequality allows to convert a weighted L^2 norm of $Z^I H_{\mathcal{L}\mathcal{L}}$ into a weighted L^2 norm of its derivatives. We then use the wave coordinate condition, which even after commutation of Z^I preserves some of its strength:

$$|\partial Z^I|_{\mathcal{L}\mathcal{T}} \le \sum_{|J|\le|I|} |\bar\partial Z^I H| + \sum_{|J|<|I|} |\partial Z^J H| + Z^I O(H \cdot \partial H).$$

The crucial fact here is that the principal term $\bar\partial Z^I H$ contains tangential derivatives $\bar\partial$ and thus can be compared to the positive space-time integral on the left-hand side of the energy inequality (2.27), which in this case will be

$$\int_0^t \int_{\Sigma_\tau} |\bar\partial Z^I h^1|^2 w',$$

which means that we no longer have to be concerned about the decay of $\partial_q^2 h^1$ in t but rather trade it for the decay in q, also needed in the Hardy-type inequality, which we always have in abundance.

Scalar field. The scalar field ψ satisfies the wave equation $\tilde{\Box}_g\psi = 0$ and contributes the quadratic term of the form $\partial_\mu\psi\,\partial_\nu\psi$ to the right-hand side of the equation for the tensor $h_{\mu\nu}$. Its asymptotic behavior is determined by the decay rate $|\partial\psi| \le C\varepsilon(1+t+|q|)^{-1}$. All of these properties indicate that ψ is very similar to the "good" components $h_{\mathcal{T}\mathcal{U}}$ of the tensor h. The only substantial difference, which in fact works to the advantage of ψ, is that unlike $h_{\mathcal{T}\mathcal{U}}$ function ψ satisfies its own wave equation and thus admits its own independent energy estimates. This discussion indicates that in the problem of stability of Minkowski space in harmonic gauge, coupling the scalar field to the Einstein equations does not lead to fundamental changes in the structure of the equations and requires only superficial modification of our analysis in the vacuum problem. As a consequence, starting from Section 9, the proof will be given only for the vacuum case with $\psi \equiv 0$.

We conclude this section by giving the plan of the paper. For the sake of simplicity and clarity of the exposition we will only deal with the case of a vanishing scalar field. The generalization to the case of nonvanishing scalar field is at each point immediate. Theorem 1.1 for the case of vanishing scalar field is stated in Section 9, where the actual proof of the theorem starts.

In Section 3 we write the Einstein equations as a system of quasilinear wave equations for the tensor $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$ relative to a system of wave coordinates $\{x^\mu\}_{\mu=0,\dots,3}$.

In Section 4 we define the Minkowski null frame $\{L, \underline{L}, S_1, S_2\}$ and describe null frame decompositions. We estimate relevant tensorial quantities, including the $\widetilde{\Box}_g$ and the quadratic form P appearing on the right-hand side of the reduced Einstein equations, in terms of their null decompositions.

Section 5 introduces the collection of Minkowski vector fields $\mathcal{Z}$ and records relations between the standard derivatives ∂ and vector fields Z. It contains an important proposition giving the estimate for the commutator between $\widetilde{\Box}_g$ and the powers Z^I. More details are contained in the Appendix A.

In Section 6 we derive our basic weighted energy estimate.

Section 7 deals with the decay estimates for solutions of an inhomogeneous wave equation $\widetilde{\Box}_g\phi = F$. The key result is contained in Corollary 7.2.

In Section 8 we discuss the wave coordinate condition. In particular, we state the estimates for the $H_{\mathcal{L}\mathcal{L}}$ and $H_{\mathcal{L}\mathcal{T}}$ components of the tensor $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ and its Z derivatives. Some of the details are provided in Appendix D.

Sections 9–11 contain the statement and the proof of the small data global existence result for the system of reduced Einstein equations.

In Section 9 state our main result and set up our inductive argument. We assume that the energy $\mathcal{E}_N(t)$ of the tensor $h^1_{\mu\nu} = h_{\mu\nu} - h^0_{\mu\nu}$ obeys the estimate $E_N(t) \le C\varepsilon^2(1+t)^\delta$ and on its basis derive the weak decay estimates for h and h^1 and estimate the inhomogeneous terms $F_{\mu\nu}$, $F^0_{\mu\nu} = \widetilde{\Box}_g h^0_{\mu\nu}$ appearing on the right-hand side of the equation for h^1. The weak decay estimates are obtained by means of the Klainerman-Sobolev inequality proved in Appendix C.

In Section 10 we use the decay estimates derived in Section 7 to upgrade the weak decay estimate of Section 9.

Section 11 uses the energy estimate of Section 6 to verify the inductive assumption on the energy $\mathcal{E}_N(t)$. The results of Section 11.3 heavily rely on Hardy-type inequalities established in Appendix B.

3. **The Einstein equations in wave coordinates**

For a Lorentzian metric g and a system of coordinates $\{x^\mu\}_{\mu=0,..,3}$, we denote

$$\Gamma_{\mu\ \nu}^{\ \lambda} = \frac{1}{2} g^{\lambda\delta}\big(\partial_\mu g_{\delta\nu} + \partial_\nu g_{\delta\mu} - \partial_\delta g_{\mu\nu}\big) \tag{3.1}$$

the Christoffel symbols of g with respect to $\{x^\mu\}$. We recall that

$$R_{\mu\ \nu\delta}^{\ \lambda} = \partial_\delta \Gamma_{\mu\ \nu}^{\ \lambda} - \partial_\nu \Gamma_{\mu\ \delta}^{\ \lambda} + \Gamma_{\rho\ \delta}^{\ \lambda}\Gamma_{\mu\ \nu}^{\ \rho} - \Gamma_{\rho\ \nu}^{\ \lambda}\Gamma_{\mu\ \delta}^{\ \rho} \tag{3.2}$$

is the Riemann curvature tensor of g and $R_{\mu\nu} = R_{\mu\ \nu\alpha}^{\ \alpha}$ is the Ricci tensor.

We assume that the metric g together with a field ψ is a solution of the Einstein-scalar field equations

$$R_{\mu\nu} = \partial_\mu\psi\,\partial_\nu\psi. \tag{3.3}$$

A system of coordinates $\{x^\mu\}$ is called the wave coordinates if and only if the Christoffel symbols Γ verify the condition:

$$g^{\alpha\beta}\,\Gamma_{\alpha\ \beta}^{\ \lambda} = 0, \qquad \forall\lambda = 0,\ldots,3. \tag{3.4}$$

Each of the following three equations is equivalent to (3.4), which we will refer to as the *wave coordinate condition*,

$$\partial_\alpha\big(g^{\alpha\beta}\sqrt{|g|}\big) = 0, \quad g^{\alpha\beta}\partial_\alpha g_{\beta\mu} = \frac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}, \quad \partial_\alpha g^{\alpha\nu} = \frac{1}{2}g_{\alpha\beta}g^{\nu\mu}\partial_\mu g^{\alpha\beta}. \tag{3.5}$$

The covariant wave operator $\Box_g$ in wave coordinates, in view of (3.5), coincides with the reduced wave operator $\widetilde{\Box}_g = g^{\alpha\beta}\partial^2_{\alpha\beta}$,

$$\widetilde{\Box}_g = \Box_g = \frac{1}{\sqrt{|g|}}\partial_\alpha g^{\alpha\beta}\sqrt{|g|}\partial_\beta. \tag{3.6}$$

Define a 2-tensor h according to the decomposition

$$g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu},$$

where m is the standard Minkowski metric on $\mathbb{R}^{3+1}$. Let $m^{\mu\nu}$ be the inverse of $m_{\mu\nu}$. Then for small h

$$H^{\mu\nu} := g^{\mu\nu} - m^{\mu\nu} = -h^{\mu\nu} + O^{\mu\nu}(h^2), \quad \text{where } h^{\mu\nu} = m^{\mu\mu'}m^{\nu\nu'}h_{\mu'\nu'}$$

and $O^{\mu\nu}(h^2)$ vanishes to second order at $h = 0$. Recall that, according to our conventions, the indices of tensors $h_{\mu\nu}$, $H^{\mu\nu}$ are raised/lowered with respect to the metric m. The following proposition was established in the vacuum case in [LR03] and amounts to a rather tedious calculation.

PROPOSITION 3.1. *Let (g, ψ) be a solution of the Einstein-scalar field equations then relative to the wave coordinates the tensor h and ψ solve a system of wave equations*

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h) + 2\partial_\mu\psi\,\partial_\nu\psi, \qquad \widetilde{\Box}_g\psi = 0. \tag{3.7}$$

The inhomogeneous term $F_{\mu\nu}$ has the following structure:

$$F_{\mu\nu}(h)(\partial h, \partial h)7 = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h) \tag{3.8}$$

$$P(\partial_\mu h, \partial_\nu h) = \frac{1}{4}m^{\alpha\alpha'}\partial_\mu h_{\alpha\alpha'}\,m^{\beta\beta'}\partial_\nu h_{\beta\beta'} - \frac{1}{2}m^{\alpha\alpha'}m^{\beta\beta'}\partial_\mu h_{\alpha\beta}\,\partial_\nu h_{\alpha'\beta'} \tag{3.9}$$

quadratic form $Q_{\mu\nu}(U, V)$ for each (μ, ν) is a linear combination of the standard null forms $m^{\alpha\beta}\partial_\alpha U\partial_\beta V$ and $\partial_\alpha U\partial_\beta V - \partial_\beta U\partial_\alpha V$ and $G_{\mu\nu}(h)(\partial h, \partial h)$ is a quadratic form in ∂h with coefficients smoothly dependent on h and vanishing for $h = 0$: $G_{\mu\nu}(0)(\partial h, \partial h) = 0$, i.e., G is a cubic term.

Observe that the quadratic terms in (3.9) do not satisfy the classical null condition. However the trace $\mathrm{tr}h = m^{\mu\nu}h_{\mu\nu}$ satisfies a nonlinear wave equation with semilinear terms obeying the null condition:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta m^{\mu\nu}h_{\mu\nu} = Q(\partial h, \partial h) + G(h)(\partial h, \partial h) + Q(\partial\psi, \partial\psi).$$

4. **The null-frame**

At each point (t, x) we introduce a pair of null vectors $(L, \underline{L})$

$$L^0 = 1, \;\; L^i = x^i/|x|, \;\; i = 1, 2, 3, \;\; \text{and} \;\; \underline{L}^0 = 1, \;\; \underline{L}^i = -x^i/|x|, \;\; i = 1, 2, 3.$$

Let S_1 and S_2 be two orthonormal smooth tangent vector fields to the sphere $\mathbf{S}^2$. S_1, S_2 are then orthogonal to the normal $\omega = x/|x|$ to $\mathbf{S}^2$ and $(L, \underline{L}, S_1, S_2)$ form a nullframe.

Remark 4.1. The null frame described above is defined only locally. Replacing orthonormal vector fields S_1, S_2 with the projections

$$\bar{\partial}_i = \partial_i - \omega_i\omega^j\partial_j, \qquad \omega = \frac{x}{|x|} \tag{4.1}$$

of the standard coordinate vector fields ∂_i would define a global frame. Set $\bar\partial_0 = L^\alpha \partial_\alpha$ and $\bar\partial_i = \bar\partial_i$, for $i = 1, 2, 3$. Then $\{\bar\partial_0, \dots, \bar\partial_3\}$ span the tangent space of the forward light cone.

We raise and lower indices with respect to the Minkowski metric; $X_\alpha = m_{\alpha\beta} X^\beta$. For a vector field/one-form X we define its components relative to a null frame according to

$$X^\alpha = X^L L^\alpha + X^{\underline{L}} \underline{L}^\alpha + X^A A^\alpha, \qquad X_\alpha = X^L L_\alpha + X^{\underline{L}} \underline{L}_\alpha + X^A A_\alpha.$$

Here and in what follows $A, B, C \dots$ denotes any of the vectors S_1, S_2, and we used the summation convention

$$X^A A^\alpha = X^{S_1} S_1^\alpha + X^{S_2} S_2^\alpha.$$

The components can be calculated using the following formulas

$$X^L = -\frac{1}{2} X_{\underline{L}}, \quad X^{\underline{L}} = -\frac{1}{2} X_L, \quad X^A = X_A,$$

where

$$X_Y = X_\alpha Y^\alpha.$$

Similarly for a two-form π and two vector fields X and Y we define

$$\pi_{XY} = \pi_{\alpha\beta} X^\alpha Y^\beta.$$

The Minkowski metric m has the following form relative to a null frame:

$$m_{LL} = m_{\underline{L}\underline{L}} = m_{LA} = m_{\underline{L}A} = 0, \qquad m_{L\underline{L}} = m_{\underline{L}L} = -2, \qquad m_{AB} = \delta_{AB},$$

i.e., $m_{\alpha\beta} X^\alpha Y^\beta = -2(X^L Y^{\underline{L}} + X^{\underline{L}} Y^L) + \delta_{AB} X^A Y^B$, where $\delta_{AB} X^A Y^B = X^{S_1} Y^{S_1} + X^{S_2} Y^{S_2}$. The inverse of the metric has the form

$$m^{LL} = m^{\underline{L}\underline{L}} = m^{LA} = m^{\underline{L}A} = 0, \;\; m^{L\underline{L}} = m^{\underline{L}L} = -1/2, \;\; m^{AB} = \delta^{AB}.$$

We also define the tangential trace

$$\overline{\mathrm{tr}}\, k = \delta^{AB} k_{AB} \tag{4.2}$$

and record the identity

$$\pi^{\alpha\beta} \partial_\alpha = \pi^{L\beta} L + \pi^{\underline{L}\beta} \underline{L} + \pi^{A\beta} A = -\frac{1}{2} \pi_{\underline{L}}{}^{\beta} L - \frac{1}{2} \pi_L{}^{\beta} \underline{L} + \pi_A{}^{\beta} A.$$

We introduce the following notation. Let $\mathcal{T} = \{L, S_1, S_2\}$, $\mathcal{U} = \{\underline{L}, L, S_1, S_2\}$, $\mathcal{L} = \{L\}$ and $\mathcal{S} = \{S_1, S_2\}$. For any two of these families $\mathcal{V}$ and $\mathcal{W}$ and an arbitrary two-tensor p we define

$$|p|_{\mathcal{V}\mathcal{W}} = \sum_{V\in\mathcal{V}, W\in\mathcal{W},} |p_{\beta\gamma} V^\beta W^\gamma|, \tag{4.3}$$

$$|\partial p|_{\mathcal{V}\mathcal{W}} = \sum_{U\in\mathcal{U}, V\in\mathcal{V}, W\in\mathcal{W},} |(\partial_\alpha p_{\beta\gamma}) U^\alpha V^\beta W^\gamma|, \tag{4.4}$$

$$|\bar{\partial} p|_{\mathcal{V}\mathcal{W}} = \sum_{T\in\mathcal{T}, V\in\mathcal{V}, W\in\mathcal{W},} |(\partial_\alpha p_{\beta\gamma}) T^\alpha V^\beta W^\gamma|. \tag{4.5}$$

Note that the contractions with the frame are outside the differentiation so they are not differentiated.

Let Q be one of the null quadratic form, i.e. $Q_{\alpha\beta}(\partial\phi, \partial\psi) = \partial_\alpha\phi\, \partial_\beta\psi - \partial_\beta\phi\, \partial_\alpha\psi$ if $\alpha \neq \beta$ and $Q_0(\partial\phi, \partial\psi) = m^{\alpha\beta}\partial_\alpha\phi\, \partial_\beta\psi$. Motivated by (3.9) we define the quadratic form P

$$P(\pi, \theta) = \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \pi_{\alpha\beta}\theta_{\alpha'\beta'} - \frac{1}{4} m^{\alpha\beta} m^{\alpha'\beta'} \pi_{\alpha\beta}\theta_{\alpha'\beta'}. \tag{4.6}$$

The proof of the following result is a simple exercise and we leave it to the reader.

LEMMA 4.2. *For an arbitrary* 2*-tensor* π *and a scalar function* ϕ

$$|\pi^{\alpha\beta}\partial_\alpha\phi\, \partial_\beta\phi| \lesssim \big(|\pi|_{\mathcal{L}\mathcal{L}}|\partial\phi|^2 + |\pi|\, |\bar{\partial}\phi||\partial\phi|\big), \tag{4.7}$$

$$|L_\alpha \pi^{\alpha\beta}\partial_\beta\phi| \lesssim \big(|\pi|_{\mathcal{L}\mathcal{L}}|\partial\phi| + |\pi|\, |\bar{\partial}\phi|\big), \tag{4.8}$$

$$|(\partial_\alpha\pi^{\alpha\beta})\partial_\beta\phi| \lesssim \big(|\partial\pi|_{\mathcal{L}\mathcal{L}} + |\bar{\partial}\pi|\big)\, |\partial\phi| + |\partial\pi|\, |\bar{\partial}\phi| \tag{4.9}$$

$$|\pi^{\alpha\beta}\partial_\alpha\partial_\beta\phi| \lesssim \big(|\pi|_{\mathcal{L}\mathcal{L}}|\partial^2\phi| + |\pi|\, |\bar{\partial}\partial\phi|\big). \tag{4.10}$$

For the quadratic form P *defined in* (4.6) *and a null form* Q

$$|P(\pi, \theta)| \lesssim |\pi\, |_{\mathcal{T}\mathcal{U}}|\theta|_{\mathcal{T}\mathcal{U}} + |\pi\, |_{\mathcal{L}\mathcal{L}}|\theta| + |\pi\, |\, |\theta|_{\mathcal{L}\mathcal{L}}, \tag{4.11}$$

$$|Q(\partial\phi, \partial\psi)| \lesssim |\bar{\partial}\phi||\partial\phi| + |\partial\phi||\bar{\partial}\psi|. \tag{4.12}$$

For a function ϕ, $|\bar{\partial}\phi| = \sum_{\alpha=0}^3 |\bar{\partial}_\alpha\phi|$, where $\bar{\partial}_\alpha$ the tangential derivatives defined after (4.1), is equivalent to $\sum_{T\in\mathcal{T}} |T^\alpha\partial_\alpha\phi|$. However

$$|\bar{\partial}^2\phi| = \sum_{\alpha,\beta=0}^{3} |\bar{\partial}_\alpha\bar{\partial}_\beta\phi|$$

is not equivalent to $\sum_{S,T\in\mathcal{T}} |T^\alpha S^\beta\partial_\alpha\partial_\beta\phi|$, the difference is of order $|\partial\phi|/r$ which is a main term. Furthermore $|\bar{\partial}^2\phi|$ need not even be equivalent to

$$\sum_{S,T\in\mathcal{T}} |T^\alpha\partial_\alpha(S^\beta\partial_\beta\phi)|,$$

the difference is however bounded by $|\bar{\partial}\phi|/r$ which is a lower order term.

LEMMA 4.3. *For an arbitrary symmetric 2-tensor $\pi^{\alpha\beta}$ and a function ϕ*

$$\big|\pi^{\alpha\beta}\partial_\alpha\partial_\beta\phi-\pi_{LL}\partial_q^2\phi-2\pi_{L\underline{L}}\partial_s\partial_q\phi-r^{-1}\,\overline{\operatorname{tr}}\,\pi\;\partial_q\phi\big| \lesssim |\pi|_{\mathcal{L}\mathcal{T}}|\bar\partial\partial\phi|+|\pi|\,\big(|\bar\partial^2\phi|+r^{-1}|\bar\partial\phi|\big). \tag{4.13}$$

Proof. Using that ∂_q,∂_s derivatives commute with the frame $\{L,\underline{L},S_1,S_2\}$, we obtain that

$$\pi^{\alpha\beta}\partial_\alpha\partial_\beta\phi-\pi_{LL}\partial_q^2\phi-2\pi_{L\underline{L}}\partial_s\partial_q\phi = \pi_{\underline{L}\underline{L}}\partial_s^2\phi+2\pi_{\underline{L}A}\partial_s(A^i\partial_i\phi)-\pi_{LA}\underline{L}^\alpha A^i\partial_i\partial_\alpha\phi+\pi_{AB}A^iB^j\partial_i\partial_j\phi.$$

Furthermore,

$$\pi_{AB}A^iB^j\partial_i\partial_j\,\phi=\pi_{AB}A^i\partial_i(B^j\partial_j\,\phi)-\pi_{AB}(A^i\partial_iB^j)\partial_j\,\phi.$$

Decomposing with respect to the null frame we obtain

$$(A^i\partial_iB^j)\partial_j=(A^i\partial_iB)^LL+(A^i\partial_iB)^{\underline{L}}\underline{L}+(A^i\partial_iB)^CC.$$

Note that $|A^i\partial_iB^j|\le Cr^{-1}$, since B^j are smooth functions of $\omega=x/|x|\in\mathbf{S}^2$. Now, since $B^j\omega_j=0$,

$$\begin{aligned}2(A^i\partial_iB)^{\underline{L}}&=-(A^i\partial_iB^j)\,\omega_j=A^iB^j\partial_i\,\omega_j\\ &=A^iB^j\frac{1}{r}(\delta_{ij}-\omega_i\,\omega_j)=\frac{1}{r}\delta_{ij}\,A^iB^j=\frac{1}{r}\delta_{AB}\end{aligned}$$

is precisely the null second fundamental form of the outgoing null cone $t-r=\text{const}$. The lemma now follows since as pointed out before the lemma

$$\sum_{S,T\in\mathcal{T}}|T^\alpha\partial_\alpha(S^\beta\partial_\beta\phi)|\le C\big(|\bar\partial^2\phi|+r^{-1}|\bar\partial\phi|\big). \qquad \square$$

COROLLARY 4.4. *Let ϕ be a solution of the reduced wave equation $\widetilde{\square}_g\phi=F$ with a metric g such that $H^{\alpha\beta}=g^{\alpha\beta}-m^{\alpha\beta}$ satisfies the condition that $|H^{L\underline{L}}|<\frac14$. Then*

$$\Big|\Big(4\partial_s-\frac{H_{LL}}{2g^{L\underline{L}}}\partial_q-\frac{\overline{\operatorname{tr}}\,H+H_{L\underline{L}}}{2g^{L\underline{L}}\,r}\Big)\partial_q(r\phi)+\frac{rF}{2g^{L\underline{L}}}\Big| \lesssim r|\triangle_\omega\phi|+|H|_{L\mathcal{T}}\big(r\,|\bar\partial\partial\phi|+|\partial\phi|\big)+|H|\,\big(r\,|\bar\partial^2\phi|+|\bar\partial\phi|+r^{-1}|\phi|\big), \tag{4.14}$$

where $\triangle_\omega=\bar\triangle=\delta^{ij}\bar\partial_i\bar\partial_j$.

Proof. Define the new metric

$$\tilde g^{\alpha\beta}=\frac{g^{\alpha\beta}}{-2g^{L\underline{L}}}.$$

The equation $g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = F$ then takes the form

$$\tilde{g}^{\alpha\beta}\partial_\alpha\partial_\beta\phi = \frac{F}{-2g^{L\underline{L}}},$$

which can also be written as

$$\Box\phi + (\tilde{g}^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\partial_\beta\phi = \frac{F}{-2g^{L\underline{L}}}.$$

Let $\pi^{\alpha\beta}$ be the tensor $\pi^{\alpha\beta} = (\tilde{g}^{\alpha\beta} - m^{\alpha\beta})$ Observe that

$$\begin{aligned}\pi^{\alpha\beta} = (-2g^{L\underline{L}})^{-1}\big(g^{\alpha\beta} + 2m^{\alpha\beta}g^{L\underline{L}}\big) &= (-2g^{L\underline{L}})^{-1}\big(H^{\alpha\beta} + m^{\alpha\beta}(2g^{L\underline{L}}+1)\big)\\ &= (-2g^{L\underline{L}})^{-1}\big(H^{\alpha\beta} + 2m^{\alpha\beta}H^{L\underline{L}}\big).\end{aligned}$$

Thus,

$$\pi_{L\underline{L}} = 0, \quad \pi_{L\mathcal{T}} = (-2g^{L\underline{L}})^{-1}H_{L\mathcal{T}}, \quad \overline{\operatorname{tr}}\,\pi = (-2g^{L\underline{L}})^{-1}\big(\overline{\operatorname{tr}}\,H + H_{L\underline{L}}\big). \tag{4.15}$$

Moreover, $|\pi| \lesssim |H|$, since $g^{L\underline{L}} = H^{L\underline{L}} - \frac{1}{2}$, and, by the assumptions of the corollary, $|H^{L\underline{L}}| < \frac{1}{4}$.

Now using (4.13) of Lemma 4.3, with the condition that $\pi_{L\underline{L}} = 0$, together with the decomposition

$$\Box\phi = -\partial_t^2\phi + \triangle\phi = \frac{1}{r}(\partial_t + \partial_r)(\partial_r - \partial_t)r\phi + \triangle_\omega\phi = \frac{4}{r}\partial_s\partial_q r\phi + \triangle_\omega\phi.$$

we find that the identity $\Box\phi + \pi^{\alpha\beta}\partial_\alpha\partial_\beta\phi = (-2g^{L\underline{L}})^{-1}F$ leads to the inequality

$$\begin{aligned}&\big|4\partial_s\partial_q r\phi + r\pi_{LL}\partial_q^2\phi + \overline{\operatorname{tr}}\,\pi\;\partial_q\phi + (2g^{L\underline{L}})^{-1}rF\big|\\ &\qquad\lesssim r|\triangle_\omega\phi| + r|\pi|_{L\mathcal{T}}|\bar{\partial}\partial\phi| + |\pi|\,\big(r\,|\bar{\partial}^2\phi| + |\bar{\partial}\phi|\big).\end{aligned}$$

Finally, identity (4.15) and a crude estimate $|\pi| \lesssim |H|$ yield the desired result. $\Box$

5. Vector fields

The family of vector fields

$$\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha\beta} = -x_\alpha\partial_\beta + x_\beta\partial_\alpha, S = t\partial_t + r\partial_r\}$$

plays a special role in the study of the wave equation in Minkowski space-time. We denote the above vector fields by Z^ι with an 11-dimensional integer index $\iota = (0,\dots 1,..,0)$. Let $I = (\iota_1,\dots,\iota_k)$, where $|\iota_i| = 1$, be a multi-index of length $|I| = k$ and let $Z^I = Z^{\iota_1}\cdots Z^{\iota_k}$ denote a product of k vector fields from the family $\mathcal{Z}$. By a sum $I_1 + I_2 = I$ we mean a sum over all possible order preserving partitions of the multi-index I into two multi-indices I_1 and I_2, i.e., if $I = (\iota_1,\dots,\iota_k)$, then $I_1 = (\iota_{i_1},\dots,\iota_{i_n})$ and $I_2 = (\iota_{i_{n+1}},\dots,\iota_{i_k})$, where $i_1,\dots,i_k$ is any reordering of the integers $1,\dots,k$ such that $i_1 < \dots < i_n$ and $i_{n+1} < \dots < i_k$. With this convention Leibnitz rule becomes $Z^I(fg) = \sum_{I_1+I_2=I}(Z^{I_1}f)(Z^{I_2}g)$.

We recall that the family $\mathcal{Z}$ possesses special commutation properties $\mathcal{Z}$: for any vector field $Z \in \mathcal{Z}$, we have $[Z, \Box] = -c_Z \Box$, where the constant c_Z is different from zero only in the case of the scaling vector field $c_S = 2$. We also record the following expressions for the coordinate vector fields:

$$\partial_t = \frac{tS - x^i \Omega_{0i}}{t^2 - r^2}, \tag{5.1}$$

$$\partial_r = \omega^i \partial_i = \frac{t\omega^i \Omega_{0i} - rS}{t^2 - r^2}, \tag{5.2}$$

$$\partial_i = \frac{-x^j \Omega_{ij} + t\Omega_{0i} - x_i S}{t^2 - r^2} = -\frac{x_i S}{t^2 - r^2} + \frac{x_i x^j \Omega_{0j}}{t(t^2 - r^2)} + \frac{\Omega_{0i}}{t}. \tag{5.3}$$

In particular,

$$\begin{aligned} \bar\partial_0 = \partial_s = \frac{1}{2}\big(\partial_t + \partial_r\big) &= \frac{S + \omega^i \Omega_{0i}}{2(t+r)}, \\ \bar\partial_i = \partial_i - \omega_i \partial_r = \frac{\omega^j \Omega_{ij}}{r} &= \frac{-\omega_i \omega^j \Omega_{0j} + \Omega_{0i}}{t}. \end{aligned} \tag{5.4}$$

Recall that $\bar\partial$ denotes the tangential derivatives, i.e.,

$$\mathrm{Span}\{\bar\partial_0, \bar\partial_1, \bar\partial_2, \bar\partial_3\} = \mathrm{Span}\{\partial_s, S_1, S_2\}.$$

LEMMA 5.1. *For any function ϕ and a symmetric 2-tensor π*

$$(1 + t + |q|)|\bar\partial\phi| + (1 + |q|)|\partial\phi| \lesssim C \sum_{|I|=1} |Z^I \phi|, \tag{5.5}$$

$$|\bar\partial^2 \phi| + r^{-1}|\bar\partial\phi| \lesssim \frac{C}{r} \sum_{|I|\le 2} \frac{|Z^I\phi|}{1 + t + |q|}, \tag{5.6}$$

$$\text{where } |\bar\partial^2\phi|^2 = \sum_{\alpha,\beta=0,1,2,3} |\bar\partial_\alpha \bar\partial_\beta \phi|^2,$$

$$|\pi^{\alpha\beta}\partial_\alpha\partial_\beta\phi| \le C\Big(\frac{|\pi|}{1 + t + |q|} + \frac{|\pi|_{\mathcal{L}\mathcal{L}}}{1 + |q|}\Big) \sum_{|I|\le 1} |\partial Z^I \phi|. \tag{5.7}$$

Proof. First we note that if $r + t \le 1$ then (5.5) holds since the standard derivatives ∂_α are included in the sum on the right. The inequality for $|\bar\partial\phi|$ in (5.5) follows directly from (5.4). The inequality for $|\partial f|$ in (5.5) follows from (5.1) and the first identity in (5.3).

The proof of (5.6) follows immediately from (5.4) and the inequality $|\partial_i \omega_j| \le C r^{-1}$. The inequality (5.7) follows from Lemma 4.3, (5.5), and the commutator identity $[Z, \partial_i] = c_i^\alpha \partial_\alpha$, which holds with constants c_i^α. $\Box$

Next we state the result following from the above lemma and Corollary 4.4.

LEMMA 5.2. *Let ϕ be a solution of the equation $\widetilde{\Box}_g\phi = F$ with $g^{\alpha\beta} = m^{\alpha\beta} + H^{\alpha\beta}$. Then*

$$\Big|\Big(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q - \frac{\overline{\operatorname{tr}}\, H\ + H_{L\underline{L}}}{2g^{L\underline{L}}\ r}\Big)\partial_q(r\phi) + \frac{rF}{2g^{L\underline{L}}}\Big| \lesssim \Big(1 + \frac{r\,|H|_{\mathcal{L}\mathcal{T}}}{1+|q|} + |H|\Big) r^{-1}\sum_{|I|\le 2}|Z^I\phi|. \tag{5.8}$$

Proof. By Corollary 4.4

$$\Big|\Big(4\partial_s - \frac{H_{LL}}{2g^{L\underline{L}}}\partial_q - \frac{\overline{\operatorname{tr}}\, H\ + H_{L\underline{L}}}{2g^{L\underline{L}}\ r}\Big)\partial_q(r\phi) + \frac{rF}{2g^{L\underline{L}}}\Big| \lesssim r|\triangle_\omega\phi| + |H|_{L\mathcal{T}}\big(r|\bar\partial\partial\phi| + |\partial\phi|\big) + |H|\big(r\,|\bar\partial^2\phi| + |\bar\partial\phi| + r^{-1}|\phi|\big)$$

where $\triangle_\omega = \delta^{ij}\bar\partial_i\bar\partial_j$. Here all the derivatives can be re-expressed in terms of the vector fields Z and ∂_q using Lemma 5.1, yielding the expression (5.8). Note that

$$|\bar\partial\partial\phi| \lesssim \frac{\sum_{|I|=1}|Z^I\partial\phi|}{1+t+|q|} \lesssim \frac{\sum_{|I|\le 1}|\partial Z^I\phi|}{1+t+|q|} \lesssim \frac{\sum_{|I|\le 2}|Z^I\phi|}{(1+|q|)(1+t+|q|)}. \qquad \Box$$

The last result of this section records an important statement concerning commutation between the reduced wave operator $\widetilde{\Box}_g$ and the family of Minkowski vector fields $\mathcal{Z}$. As we already explained in Section 2 our small data global existence result for the system of reduced Einstein equations $\widetilde{\Box}_g h^1 = F - \widetilde{\Box}_g h^0$ is based on controlling the energy and pointwise norms of the quantities $Z^I h^1$ with vector fields $Z \in \mathcal{Z}$ — the family of *Minkowski vector fields*. The above control is achieved via the energy and decay estimates for solutions of the inhomogeneous wave equation

$$\widetilde{\Box}_g Z^I h^1 = \hat Z^I\widetilde{\Box}_g h^1 - \widetilde{\Box}_g Z^I h^1 + \hat Z^I F - \hat Z^I\widetilde{\Box}_g h^0, \qquad \hat Z = Z + c_Z$$

and therefore requires good estimates on the commutator $\hat Z^I\widetilde{\Box}_g - \widetilde{\Box}_g Z^I$. To underscore the importance of this commutator we recall that the only two known examples of quasilinear hyperbolic systems, with the metric dependent on the solution rather than its derivatives, possessing small data global solutions required the use of *modified* vector fields primarily due to the lack of good estimates for the commutator, [Ali03], [CK93]. The proof of the proposition below together with other commutator related statements is contained in Appendix A.

PROPOSITION 5.3. *Let $\widetilde{\Box}_g = \Box + H^{\alpha\beta}\partial_\alpha\partial_\beta$. Then for any vector field $Z \in \mathcal{Z}$ we have with $\hat Z = Z + c_Z$*

(5.9)

$$|\widetilde{\Box}_g Z^I\phi - \widehat{Z}^I\widetilde{\Box}_g\phi| \lesssim \frac{1}{1+t+|q|}\sum_{|K|\le|I|,}\ \sum_{|J|+(|K|-1)_+\le|I|}|Z^J H|\,|\partial Z^K\phi|$$
$$+\frac{1}{1+|q|}\sum_{|K|\le|I|}\Bigg(\sum_{|J|+(|K|-1)_+\le|I|}|Z^J H|_{LL}+\sum_{|J'|+(|K|-1)_+\le|I|-1}|Z^{J'}H|_{L\mathcal{T}}$$
$$+\sum_{|J''|+(|K|-1)_+\le|I|-2}|Z^{J''}H|\Bigg)|\partial Z^K\phi|,$$

where $(|K|-1)_+ = |K|-1$ *if* $|K|\ge 1$ *and* $(|K|-1)_+ = 0$ *if* $|K|=0$.

6. **Energy estimates in curved space-time**

In this section we establish basic weighted energy identities and estimates for solutions of the inhomogeneous wave equation

$$\widetilde{\Box}_g\phi = F. \tag{6.1}$$

The weights under consideration will depend only on the distance $q=r-t$ to the light cone $t=r$ and will be defined by a function w

$$w = w(q)\ge 0, \quad w'(q)\ge 0.$$

The weights will serve a two-fold purpose: 1) to provide additional decay of ϕ in the exterior region $q\ge 0$, which will follow from the energy estimates via a Klainerman-Sobolev type inequality, and 2) to establish an additional *a priori* bound on a *space-time* integral involving tangential derivatives $\bar\partial\phi$ of the solution.

LEMMA 6.1. *Let* ϕ *be a solution of equation* (6.1) *decaying sufficiently fast as* $|x|\to\infty$. *Assume that the background metric* g *is such that the tensor* $H^{\alpha\beta} = g^{\alpha\beta}-m^{\alpha\beta}$ *satisfies* $|H|\le\frac12$. *Then with* $\omega = x/|x|$

$$\int_{\Sigma_{t_2}}\big(|\partial_t\phi|^2+|\nabla\phi|^2\big)w(q)dx$$
$$+2\int_{t_1}^{t_2}\int_{\Sigma_\tau}|\bar\partial\phi|^2 w'(q)dxdt \le 4\int_{\Sigma_{t_1}}\big(|\partial_t\phi|^2+|\nabla\phi|^2\big)w(q)\,dx$$
$$+2\int_{t_1}^{t_2}\int_{\Sigma_\tau}\big|(2\partial_\alpha H^{\alpha\beta})\partial_\beta\phi\partial_t\phi-(\partial_t H^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi+2F\partial_t\phi\big|w(q)dxdt$$
$$+2\int_{t_1}^{t_2}\int_{\Sigma_\tau}\big|H^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+2(\omega_i H^{i\beta}-H^{0\beta})\partial_\beta\phi\partial_t\phi\big|w'(q)\,dxdt. \tag{6.2}$$

Proof. Let $\phi_i=\partial_i\phi$, $i=1,2,3$, and $\phi_t=\partial_t\phi$. Differentiating under the integral sign and integrating by parts we get

$$\begin{aligned}
(6.3)\quad & \frac{d}{dt}\int \big(-g^{00}\phi_t^2+g^{ij}\phi_i\phi_j\big)w(q)\,dx-\int 2\partial_j\big(g^{0j}\phi_t^2 w(q)\big)\,dx\\
&=2\int w(q)\big(-g^{00}\phi_t\phi_{tt}+g^{ij}\phi_i\phi_{tj}-2g^{0j}\phi_t\phi_{tj}\big)\,dx\}\\
&\quad+\int w(q)\big(-(\partial_t g^{00})\phi_t^2+(\partial_t g^{ij})\phi_i\phi_j-2(\partial_j g^{0j})\phi_t^2\big)\\
&\quad-w'(q)\big(-g^{00}\phi_t^2+g^{ij}\phi_i\phi_j+2\omega_j g^{0j}\phi_t^2\big)\,dx\\
&=2\int w(q)\big(-g^{00}\phi_t\phi_{tt}-g^{ij}\phi_t\phi_{ij}-2g^{0j}\phi_t\phi_{tj}\big)\,dx\\
&\quad+\int w(q)\big(-(\partial_t g^{00})\phi_t^2+(\partial_t g^{ij})\phi_i\phi_j-2(\partial_j g^{0j})\phi_t^2-2(\partial_i g^{ij})\phi_t\phi_j\big)\,dx\\
&\quad-\int w'(q)\big(-g^{00}\phi_t^2+g^{ij}\phi_i\phi_j+2\omega_j g^{0j}\phi_t^2+2\omega_i g^{ij}\phi_t\phi_j\big)\,dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
(6.4)\qquad & \frac{d}{dt}\int \big(-g^{00}\phi_t^2+g^{ij}\phi_i\phi_j\big)w(q)\,dx\\
&=-\int w(q)\big(2\phi_t\,\Box_g\phi-(\partial_t g^{\alpha\beta})\phi_\alpha\phi_\beta+2(\partial_\alpha g^{\alpha\beta})\phi_\beta\phi_t\big)\,dx\\
&\quad-\int w'(q)\big(g^{\alpha\beta}\phi_\alpha\phi_\beta+2\big(\omega_i g^{i\alpha}-g^{0\alpha}\big)\phi_t\phi_\alpha\,dx.
\end{aligned}$$

Furthermore, with $\phi_r=\omega^i\phi_i=\partial_r\phi$ and $\overline{\phi}_i=\phi_i-\omega_i\phi_r=\overline{\partial}_i\phi$

$$\begin{aligned}
m^{\alpha\beta}\phi_\alpha\phi_\beta+2\phi_t(\omega_i m^{i\alpha}-m^{0\alpha})\phi_\alpha&=-\phi_t^2+\delta^{ij}\phi_i\phi_j+2\phi_t(\omega^i\phi_i+\phi_t)\\
&=(\phi_t+\phi_r)^2+\delta^{ij}\overline{\phi}_i\overline{\phi}_j=|\overline{\partial}\phi|^2.
\end{aligned}$$

Since $|H|<1/2$ we also have that

$$\frac{1}{2}(\phi_t^2+\delta^{ij}\phi_i\phi_j)\le -g^{00}\phi_t^2+g^{ij}\phi_i\phi_j\le 2(\phi_t^2+\delta^{ij}\phi_i\phi_j).$$

The lemma follows. $\square$

We now consider the following weight function:

$$(6.5)\qquad w=w(q)=\begin{cases}1+(1+|q|)^{1+2\gamma}, & \text{when } q>0,\\ 1+(1+|q|)^{-2\mu}, & \text{when } q<0\end{cases}$$

for some $\mu\ge 0$, $\gamma\ge -1$. We clearly have

$$(6.6)\qquad w'\le 4w(1+|q|)^{-1}\le 16\gamma^{-1}w'(1+q_-)^{2\mu},$$

where $q_-=|q|$ for $q\le 0$ and $q_-=0$ otherwise.

PROPOSITION 6.2. *Let* ϕ *be a solution of the wave equation* (6.1) *with the metric* g *such that for* $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$;

$$(1+|q|)^{-1}|H|_{LL} + |\partial H|_{LL} + |\bar\partial H| \le C\varepsilon'(1+t)^{-1}, \tag{6.7}$$
$$(1+|q|)^{-1}\,|H| + |\partial H| \le C\varepsilon'(1+t)^{-\frac12}(1+|q|)^{-\frac12}(1+q_-)^{-\mu}.$$

Then for any $0 < \gamma \le 1$, *and* $0 < \varepsilon' \le \gamma/C_1$, *we have*

$$\int_{\Sigma_t} |\partial\phi|^2\, w + \int_0^t \int_{\Sigma_\tau} |\bar\partial\phi|^2\, w' \le 8\int_{\Sigma_0} |\partial\phi|^2\, w + 16\int_0^t \int_{\Sigma_t} \Big(\frac{C\varepsilon\,|\partial\phi|^2}{1+t} + |F|\,|\partial\phi|\Big)\, w. \tag{6.8}$$

Remark 6.3. Observe that by the Gronwall inequality the energy estimate of the above proposition implies $t^{\varepsilon'}$ growth of the energy.

Remark 6.4. We recall again that the interior estimate (6.8), i.e., with $w(q) \equiv 0$ for $q \ge 0$, in the constant coefficient case basically follows by averaging the energy estimates on light cones used e.g. in [Shu91]. The interior energy estimates with space-time quantities involving special derivatives of a solution were also considered and used in the work of Alinhac; see e.g. [Ali01], [Ali03]. In [LR05] we proved the interior version of (6.8). The use of the weights $w(q)$ in energy estimates in the exterior $q \ge 0$ for the space part $\int_{\Sigma_T} |\partial\phi|^2 w$ originates in [KN03b].

Proof. The proof of the proposition relies on the energy estimate obtained in Lemma 6.1. We decompose the terms on the right-hand side of (6.2) with respect to the null frame, using Lemma 4.2

$$|(\partial_\alpha H^{\alpha\beta})\partial_\beta\phi\partial_t\phi| + |(\partial_t H^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi| \le \big(|(\partial H)_{LL}| + |\bar\partial H|\big)\,|\partial\phi|^2 + |\partial H|\,|\bar\partial\phi|\,|\partial\phi|.$$

Therefore, using the assumptions (6.7) on the metric g, we obtain that

$$|2(\partial_\alpha H^{\alpha\beta})\partial_\beta\phi\partial_t\phi - (\partial_t H^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi| \lesssim \frac{\varepsilon'}{1+t}|\partial\phi|^2 + \frac{\varepsilon'}{(1+|q|)(1+q_-)^{2\mu}}|\bar\partial\phi|^2. \tag{6.9}$$

Decomposing the remaining terms we infer that

$$|H^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi| + |L_\alpha H^{\alpha\beta}(\partial_t\phi)\partial_\beta\phi| \le \big|H|_{LL}|\partial\phi|^2 + |H||\bar\partial\phi|\,|\partial\phi|.$$

Once again, using the assumptions (6.7), we have

$$|2H^{\alpha\beta}L_\alpha\partial_\beta\phi\partial_t\phi + H^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi| \lesssim \varepsilon'\frac{1+|q|}{1+t}|\partial\phi|^2 + \varepsilon'\frac{|\bar\partial\phi|^2}{(1+q_-)^{2\mu}}. \tag{6.10}$$

Thus, with the help of (6.6),

$$\int_{\Sigma_t} |\partial\phi|^2 w + \int_0^t \int_{\Sigma_\tau} |\bar\partial\phi|^2 w' \lesssim \int_{\Sigma_0} |\partial\phi|^2 w + \varepsilon \int_0^t \int_{\Sigma_\tau} \Big(\frac{|\partial\phi|^2}{1+t} w + |\bar\partial\phi|^2 \frac{w'}{\gamma}\Big) + \int_0^t \int_{\Sigma_\tau} |F|\,|\partial\phi| w$$

and the desired estimate follows. □

7. Decay estimates in curved space-time

In this section we derive L^∞-estimates for the first derivatives of solutions of the equation

$$\widetilde{\Box}_g \phi = F. \tag{7.1}$$

These estimates are complimentary to the global Sobolev inequalities derived in Appendix C and will provide a way to improve upon the decay estimates derived from the weighted energy estimates via global Sobolev inequalities. Estimates of these type were obtained in [Lin90b] in the case of the wave equation in Minkowski space-time, i.e., constant coefficient metric g, and [LR03] for variable coefficients. Here, however, we introduce a weighted version of the L^∞-estimates deigned to capture additional decay in $|q| = |r-t|$.

For $\gamma' \ge -1$, $\mu' \le 1/2$ define the weight

$$\varpi = \varpi(q) = \begin{cases} (1+|q|)^{1+\gamma'}, & \text{when } q > 0, \\ (1+|q|)^{1/2-\mu'}, & \text{when } q < 0. \end{cases} \tag{7.2}$$

LEMMA 7.1. *Let ϕ be a solution of the reduced scalar wave equation* (7.1) *on a curved background with metric g. Assume that the tensor $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ obeys the estimates*

$$|H| \le \varepsilon', \qquad \int_0^\infty \|H(t,\cdot)\|_{L^\infty(D_t)} \frac{dt}{1+t} \le \frac{1}{4}, \qquad |H|_{L\mathcal{T}} \le \varepsilon' \frac{|q|+1}{1+t+|x|}, \tag{7.3}$$

in the region $D_t = \{x : t/2 < |x| < 2t\}$. Then for $\alpha = \max(1+\gamma', 1/2-\mu')$

$$\begin{aligned}(1+t)\varpi(q)\,|\partial\phi(t,x)| \le C \sup_{0\le\tau\le t} \sum_{|I|\le 1} \|\varpi(q)\, Z^I\phi(\tau,\cdot)\|_{L^\infty} \\ + C\int_0^t \Big(\varepsilon'\alpha \|\varpi(q)\,\partial\phi(\tau,\cdot)\|_{L^\infty} + (1+\tau)\|\varpi(q)\, F(\tau,\cdot)\|_{L^\infty(D_\tau)} \\ + \sum_{|I|\le 2} (1+\tau)^{-1} \|\varpi(q)\, Z^I\phi(\tau,\cdot)\|_{L^\infty(D_\tau)}\Big)\, d\tau.\end{aligned} \tag{7.4}$$

Proof. Since by Lemma 5.1

$$(1+|t-r|)|\partial\phi|+(1+t+r)|\bar\partial\phi|\le C\sum_{|I|=1}|Z^I\phi|,\qquad r=|x|, \tag{7.5}$$

the inequality (7.4) holds when $r<t/2+1/2$ or $r>2t-1$.

By Lemma 5.2 with $\partial_s=1/2(\partial_t+\partial_r)$ and $\partial_q=1/2(\partial_r-\partial_t)$,

$$\begin{aligned}\big|(4\partial_s-\frac{H_{LL}}{2g^{L\underline{L}}}\partial_q)\partial_q(r\phi)\big|\\ \lesssim\Big(1+\frac{r\,|H|_{\mathcal{L}\mathcal{T}}}{1+|q|}+|H|\Big)r^{-1}\sum_{|I|\le 2}|Z^I\phi|+|H|\,r^{-1}\,|\partial_q(r\phi)|+r|F|.\end{aligned} \tag{7.6}$$

Multiplying by the weight $\varpi(q)$ and using that $\varpi'(q)\le C\,\varpi(q)/(1+|q|)$ along with the assumptions (7.3) we obtain

$$\begin{aligned}\big|(4\partial_s-\frac{H_{LL}}{2g^{L\underline{L}}}\partial_q)\varpi(q)\partial_q(r\phi)\big|\lesssim\Big(\frac{|H|}{1+t}+\alpha\frac{H_{LL}}{1+|q|}\Big)\varpi(q)|\partial_q(r\phi)|\\ +\sum_{|I|\le 2}\frac{\varpi(q)|Z^I\phi|}{1+t}+C(t+1)\varpi(q)|F|\end{aligned} \tag{7.7}$$

in the region $t/2+1/2<r<2t-1$. Let $(\tau,x(\tau))$ be the integral curve of the vector field $\partial_s+H^{\underline{L}\underline{L}}(2g^{L\underline{L}})^{-1}\partial_q$ passing through a given point (t,x) contained in the region $t/2+1/2\le r\le 2t-1$. Observe that, by the smallness assumption on H, any such curve has to intersect the boundary of the set $t/2+1/2\le r\le 2t-1$ at (τ,y) such that $|y|=\tau/2+1/2$ or $|y|=2\,\tau-1$.

Then along such a curve the function $\psi:=\varpi(q)\partial_q(r\phi)$ satisfies the following equation:

$$\Big|\frac{d}{dt}\psi\Big|\le\hat h|\psi|+f \tag{7.8}$$

where

$$\hat h=C\frac{|H|}{1+t},\quad f=\frac{\varepsilon'\alpha}{1+t}\varpi(q)|\partial_q r\phi|+C(1+t)\varpi(q)|F|+C\sum_{|I|\le 2}\frac{\varpi(q)|Z^I\phi|}{1+t}.$$

Thus using the integrating factor $e^{-\hat H}$ with $\hat H=\int\hat h(s)\,ds$ and integrating along the integral curve $(\tau,x(\tau))$ from any point (t,x) in the set $t/2+1/2\le r\le 2t-1$ to the first point of intersection (t_0,x_0) with the boundary of the set $t/2+1/2\le r\le 2t-1$

we obtain

$$|\psi(t,x)| \le \exp\Big(\int_\tau^t \|\hat{h}(\sigma,\cdot)\|_{L^\infty}\, d\sigma\Big)|\psi(t_0,x_0)| \\ + \int_\tau^t \exp\Big(\int_{\tau'}^t \|\hat{h}(\sigma,\cdot)\|_{L^\infty}\, d\sigma\Big)\|f(\tau',\cdot)\|_{L^\infty}\, d\tau',$$

with the L^∞ norms are taken over the set $t/1+1/2 \le r \le 2t-1$.

For the points (t_0, x_0) such that $|x_0| = t_0/2+1/2$ or $|x_0| = 2t_0-1$ we have by (7.5) that

$$|\psi(t_0,x_0)| \le C r\varpi(q)|\partial_q\phi| + C\varpi(q)|\phi| \le C \sum_{|I|\le 1} \varpi(q)|Z^I\phi|.$$

The desired inequality now follows from (7.3), which implies that

$$\int_0^\infty \|\hat{h}(\sigma,\cdot)\|_{L^\infty}\, d\sigma \le \frac{1}{4}$$

and the inequality

$$(1+t+r)|\partial\phi| \le C \sum_{|I|\le 1} |Z^I\phi| + C|\partial_q(r\phi)|. \qquad \square$$

We now state similar estimates for a system

$$\widetilde{\square}\phi_{\mu\nu} = F_{\mu\nu}. \tag{7.9}$$

While it is trivial to extend the estimates of Lemma 7.1 to each of the components of $\phi_{\mu\nu}$ our interest lies in the estimates for the *null* components of ϕ. Contracting (7.9) with the vector fields $\{L, \underline{L}, A, B\}$ is far from straightforward. We instead exploit that the null derivatives ∂_s, ∂_q commute with any of the vector fields of the null frame. We assume that the weight function $\varpi(q)$ is as in (7.2) and $\alpha = \max(1+\gamma', 1/2-\mu')$.

COROLLARY 7.2. *Let $\phi_{\mu\nu}$ be a solution of the reduced wave equation system* (7.9) *on a curved background with a metric g. Assume that $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ satisfies*

$$|H| \le \frac{\varepsilon'}{4}, \qquad \int_0^\infty \| H(t,\cdot)\|_{L^\infty(D_t)} \frac{dt}{1+t} \le \frac{\varepsilon'}{4}, \qquad |H|_{L\mathcal{T}} \le \frac{\varepsilon'}{4}\, \frac{|q|+1}{1+t+|x|} \tag{7.10}$$

in the region $D_t = \{x \in \mathbf{R}^3;\ t/2 \le |x| \le 2t\}$. Then for any $U, V \in \{L, \underline{L}, A, B\}$ and an arbitrary point $x \in D_t$:

(7.11)

$$(1+t+|x|)|\varpi(q)\partial\phi(t,x)|_{UV} \lesssim \sup_{0\le\tau\le t}\sum_{|I|\le 1}\|\,\varpi(q)Z^I\phi(\tau,\cdot)\|_{L^\infty}$$
$$+\int_0^t\Big(\varepsilon'\alpha\|\varpi(q)|\partial\phi(t,\cdot)|_{UV}\|_{L^\infty}+(1+\tau)\|\varpi(q)|F(\tau,\cdot)|_{UV}\|_{L^\infty(D_\tau)}$$
$$+\sum_{|I|\le 2}(1+\tau)^{-1}\|\varpi(q)Z^I\phi(\tau,\cdot)\|_{L^\infty(D_\tau)}\Big)\,d\tau.$$

Proof. By Lemma 5.2 for each component we have the estimate

$$\Big|\Big(4\partial_s-\frac{H_{LL}}{2g^{L\underline{L}}}\partial_q-\frac{\overline{\operatorname{tr}}\,H\,+H_{L\underline{L}}}{2g^{L\underline{L}}\,r}\Big)\partial_q(r\phi_{\mu\nu})+\frac{rF_{\mu\nu}}{2g^{L\underline{L}}}\Big| \lesssim \Big(1+\frac{r\,|H|_{\mathcal{L}\mathcal{T}}}{1+|q|}+|H|\Big)r^{-1}\sum_{|I|\le 2}|Z^I\phi_{\mu\nu}| \tag{7.12}$$

and, since ∂_s and ∂_q commute with contraction with the frame vectors, we get

$$\Big|\Big(4\partial_s-\frac{H_{LL}}{2g^{L\underline{L}}}\partial_q-\frac{\overline{\operatorname{tr}}\,H\,+H_{L\underline{L}}}{2g^{L\underline{L}}\,r}\Big)\partial_q(r\phi_{UV})+\frac{rF_{UV}}{2g^{L\underline{L}}}\Big| \lesssim \Big(1+\frac{r\,|H|_{\mathcal{L}\mathcal{T}}}{1+|q|}+|H|\Big)r^{-1}\sum_{|I|\le 2}|Z^I\phi|. \tag{7.13}$$

The proof now proceeds as in Lemma 7.1. □

8. **The wave coordinate condition**

The results of previous sections underscore the special role played by the H_{LL} components of the tensor $H^{\alpha\beta}=g^{\alpha\beta}-m^{\alpha\beta}$ in the energy and decay estimates for solutions of the wave equation $\widetilde{\Box}_g\phi=F$. In this section we explain how the wave coordinate condition on the tensor $g_{\mu\nu}$ provides additional information about H_{LL}.

Recall that the wave coordinate condition for metric g in a coordinate system $\{x^\mu\}_{\mu=0,\dots,3}$ takes the form

$$\partial_\mu\Big(g^{\mu\nu}\sqrt{|\det g|}\Big)=0. \tag{8.1}$$

Expressing $g^{\mu\nu}$ in terms of the tensor $H^{\mu\nu}$ we obtain

$$g^{\mu\nu}\sqrt{|\det g|}=\big(m^{\mu\nu}+H^{\mu\nu}\big)\big(1-\frac{1}{2}\mathrm{tr}H+O(H^2)\big).$$

Therefore,

(8.2)

$$\partial_\mu\Big(H^{\mu\nu}-\frac{1}{2}m^{\mu\nu}\operatorname{tr}\,H+O^{\mu\nu}(H^2)\Big)=0,\qquad\text{where}\quad O^{\mu\nu}(H^2)=O(|H|^2).$$

Recall also the family of vector fields $\mathcal{T} = \{L, A, B\}$, tangent to the outgoing Minkowski light cones.

The divergence of a vector field can be expressed relative to the null frame as follows:

$$\partial_\mu F^\mu = L_\mu \partial_q F^\mu - \underline{L}_\mu \partial_s F^\mu + A_\mu \partial_A F^\mu. \tag{8.3}$$

We can now easily prove

LEMMA 8.1. *Assume that* $|H| \le 1/4$. *Then*

$$|\partial H|_{\mathcal{L}\mathcal{T}} \lesssim |\bar\partial H| + |H|\,|\partial H|. \tag{8.4}$$

Proof. It follows from (8.2) and (8.3) that

$$\big|L_\mu \partial\big(H^{\mu\nu} - \frac{1}{2} m^{\mu\nu} \operatorname{tr} H\big)\big| \le |\bar\partial H| + |H||\partial H|. \tag{8.5}$$

Contracting with $T \in \mathcal{T}$ and using that $m_{TL} = 0$ give the desired result. □

We now state a generalization of the above result containing estimates for the quantities $Z^I H_{LT}$ with vector fields $Z \in \mathcal{Z}$ our family of Minkowski Killing and conformally Killing vector fields. The result is a rather tedious consequence of commuting vector fields Z through the wave coordinate condition (8.1) and we postpone the details of the proof until Appendix D.

PROPOSITION 8.2. *Let g be a Lorentzian metric satisfying the wave coordinate condition* (8.1) *relative to a coordinate system* $\{x^\mu\}_{\mu=0,\dots,3}$. *Let I be a multi-index and assume that the tensor $H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$ verifies the condition*

$$|Z^J H| \le C, \qquad \forall |J| \le |I|/2, \quad \forall Z \in \mathcal{Z}.$$

Then for some constant C'

$$|\partial Z^I H|_{L\mathcal{T}} \le C'\Big(\sum_{|J|\le|I|} |\bar\partial Z^J H| + \sum_{|J|\le|I|-1} |\partial Z^J H| + \sum_{|I_1|+||I_2|\le|I|} |Z^{I_2} H||\partial Z^{I_1} H|\Big) \tag{8.6}$$

$$|\partial Z^I H|_{LL} \lesssim C'\Big(\sum_{|J|\le|I|} |\bar\partial Z^J H| + \sum_{|J|\le|I|-2} |\partial Z^J H| + \sum_{|I_1|+|I_2|\le|I|,\, m\ge2} |Z^{I_2} H||\partial Z^{I_1} H|\Big). \tag{8.7}$$

Similar estimates hold for the tensor $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$.

9. Statement of the main theorem and beginning of the proof

We consider the initial data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0})$ for the Einstein-scalar field equations $R_{\mu\nu} = \partial_\mu\psi\,\partial_\nu\psi$, constructed in (2.3)–(2.5). The spatial part g_{0ij} of $g_{\mu\nu}|_{t=0}$ together with the second fundamental form $k_{0ij} = -1/2\partial_t g_{ij}|_{t=0}$ satisfy the constraint equations

$$R_0 - |k_0|^2 + (\mathrm{tr} k_0)^2 = \psi_1^2 + |\nabla\psi_0|^2, \qquad \nabla^j k_{0ij} - \nabla_i \mathrm{tr} k_0 = \psi_1\,\partial_i\psi_0.$$

By construction the initial data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0})$ satisfies the wave coordinate condition

$$\partial_\mu(g^{\mu\nu}\sqrt{|\det g|}) = 0.$$

The reduced Einstein equations, written relative to a 2-tensor $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$ take the form

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu} + 2\partial_\mu\psi\,\partial_\nu\psi, \quad \widetilde{\Box}_g\psi = 0, \tag{9.1}$$

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h), \tag{9.2}$$

$$P(\partial_\mu h, \partial_\nu h) := \frac{1}{4}\partial_\mu \mathrm{tr} h\,\partial_\nu \mathrm{tr} h - \frac{1}{2}\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}, \tag{9.3}$$

where Q is a linear combination of standard quadratic null forms and G is a quadratic term in ∂h vanishing in h at $h = 0$. We assume that the initial data for $h(t)$ are such that at $t = 0$, $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}$ verifies the constraint equations and the wave coordinate condition.

Define 2-tensor $h^1_{\mu\nu}(t)$

$$h^1_{\mu\nu}(t) := h_{\mu\nu}(t) - h^0_{\mu\nu}(t), \quad \text{where } h^0_{\mu\nu}(t) = \chi(r/t)\chi(r)\frac{M}{r}\delta_{\mu\nu} \tag{9.4}$$

where $\chi(s) \in C^\infty$ is 1 when $s \geq 3/4$ and 0 when $s \leq 1/2$. It is clear that we can reinterpret (9.1) as the equation for a new unknown — 2-tensor h^1:

$$\widetilde{\Box}_g h^1_{\mu\nu} = F_{\mu\nu} - \widetilde{\Box}_g h^0_{\mu\nu} + 2\partial_\mu\psi\,\partial_\nu\psi. \tag{9.5}$$

Set

$$\mathcal{E}_N(t) = \sum_{|I|\leq N, Z\in\mathcal{Z}} \Big(\|w^{1/2}\partial Z^I h^1(t,\cdot)\|_{L^2} + \|w^{1/2}\partial Z^I\psi(t,\cdot)\|_{L^2}\Big), \tag{9.6}$$

where

$$w = \begin{cases} 1 + (1+|q|)^{1+2\gamma}, & q > 0, \\ 1 + (1+|q|)^{-2\mu}, & q < 0, \end{cases}$$

where $q = r - t$ and $\gamma, \mu > 0$. We recall that the initial data for ψ is given by $\psi|_{t=0} = \psi_0$ and $\partial_t\psi|_{t=0} = a\psi_1$ with $a^2 = (1 - \chi(r)Mr^{-1})$.

We now state the main result.

THEOREM 9.1. *There exist a constant* $\varepsilon_0 > 0$ *such that if* $\varepsilon \le \varepsilon_0$ *and the initial data* $(h^1|_{t=0}, \partial_t h^1|_{t=0}, \psi_0, \psi_1)$ *are smooth, and obey* $\mathcal{E}_N(0) + M \le \varepsilon$ *together with the condition*

$$\liminf_{|x|\to\infty} \big(|h^1(0,x)| + |\psi_0(x)|\big) \to 0, \tag{9.7}$$

then the solution $h(t)$ *of the reduced Einstein equations* (9.1) *can be extended to a global smooth solution satisfying*

$$\mathcal{E}_N(t) \le C_N \varepsilon\, (1+t)^{c\varepsilon} \tag{9.8}$$

where C_N *is a constant depending only on* N *and* c *is independent* ε.

Remark 9.2. Recall that if the initial data for $(g_{\mu\nu}, \psi)$ verifies the constraint equations and the wave coordinate condition then there exists a local in time classical solution $(h_{\mu\nu}(t), \psi(t))$ of (9.1) such that $g_{\mu\nu}(t) = m_{\mu\nu}(t) + h_{\mu\nu}(t)$ obeys the wave coordinate condition for any time t in a maximum interval of existence.

Therefore, in what follows we shall assume that a local in time solution $g_{\mu\nu}(t) = m_{\mu\nu} + h_{\mu\nu}(t)$ *obeys the wave coordinate condition*

$$\partial_\mu\Big(g^{\mu\nu}(t)\sqrt{|\det g(t)|}\Big) = 0 \tag{9.9}$$

for any $0 \le t \le T_0$.

We also note that the maximum time of existence T_0 can be characterized by the blow-up of the energy $\mathcal{E}_N(t) \to \infty$ as $t \to T_0^-$.

Remark 9.3. As was explained in Section 2 the proof of the result for the Einstein-scalar field problem requires only superficial modifications as compared to the vacuum case.

Therefore, in what follows all the arguments will be provided only for the vacuum problem with $\psi \equiv 0$.

For the proof we let δ be any fixed number $0 < \delta < 1/4$ and $\delta < \gamma$. We define the time $T < T_0$ to be the maximal time such that the inequality

$$\mathcal{E}_N(t) \le 2C_N \varepsilon (1+t)^{\delta} \tag{9.10}$$

holds for all $0 \le t \le T$. Note that by the assumptions of the theorem $T > 0$. We will show that if $\varepsilon > 0$ is sufficiently small then this inequality implies the same inequality with $2C_N$ replaced by C_N for $t \le T$. Since the quantity is continuous this contradicts the maximality of T and it follows that the inequality holds for all $T \le T_0$. Moreover, since the energy $\mathcal{E}_N(t)$ is now finite at $t = T_0$ we can extend the solution beyond T_0 thus contradicting maximality of T_0 and showing that $T_0 = +\infty$.

The first step is to derive the preliminary decay estimates for h^1 under the assumption (9.10). The estimate (9.10) can be explicitly written in the form

$$\sum_{Z\in\mathcal{Z},|I|\le N}\|w(q)^{1/2}\partial Z^I h^1(t,\cdot)\|_{L^2}\le C\varepsilon(1+t)^{\delta},\qquad 0<\delta<\gamma \tag{9.11}$$

with some sufficiently large constant C. The following result is a consequence of weighted global Sobolev inequalities proved in Appendix C.

COROLLARY 9.4. *Let h^1 verify* (9.11) *and h^0 be as in* (9.4). *Then for $i=0,1$,*

$$|\partial Z^I h^i(t,x)|\le\begin{cases}C\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{-1-\delta'}, & q>0,\\ C\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{-1/2}, & q<0\end{cases}\qquad |I|\le N-2. \tag{9.12}$$

where $\delta'=\delta$, if $i=0$ and $\delta'=\gamma>\delta$ if $i=1$. Furthermore,

$$|Z^I h^i(t,x)|\le\begin{cases}C\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{-\delta'}, & q>0,\\ C\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{1/2}, & q<0,\end{cases}\qquad |I|\le N-2, \tag{9.13}$$

and

$$|\bar\partial Z^I h^i(t,x)|\le\begin{cases}C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{-\delta'}, & q>0,\\ C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{1/2}, & q<0,\end{cases}\qquad |I|\le N-3. \tag{9.14}$$

Proof. We will only prove the estimates for $i=1$ since the estimates for $i=0$ follow by a direct calculation from the form of h^0. Estimate (9.12) follows from the weighted Sobolev inequality of Proposition 14.1. We claim that

$$\lim_{|x|\to\infty}|Z^I h^1(0,x)|\to 0,\qquad |I|\le N-2. \tag{9.15}$$

In fact, if $|I|=0$ this follows from (9.7) and (9.12), and if $|I|\ge 1$ this follows from (9.12), since $|Z\phi|\le C(1+t+|x|)|\partial\phi|$. Estimate (9.13) for $t=0$ follows by integrating (9.12) from space-like infinity, where (9.15) holds.

Remark 9.5. The weighted Sobolev inequality of Proposition 14.1 in fact implies the estimate

$$|\partial Z^I h^1(t,x)|\le\begin{cases}C\varepsilon(1+t+|q|)^{-1}(1+|q|)^{-1-\gamma}(1+t)^{\delta}, & q>0,\\ C\varepsilon(1+t+|q|)^{-1}(1+|q|)^{-1/2}(1+t)^{\delta}, & q<0,\end{cases}\quad |I|\le N-2.$$

In particular,

$$|\partial Z^I h^1(0,x)|\le C\varepsilon(1+|x|)^{-2-\gamma}, \tag{9.16}$$

$$|Z^I h^1(0,x)|\le C\varepsilon(1+|x|)^{-1-\gamma}. \tag{9.17}$$

The estimate (9.13) for $r > t$ follows by integrating (9.12) from the hyperplane $t = 0$ along the lines with $t + r$ and $\omega = x/|x|$ fixed:

(9.18)
$$\begin{aligned}|Z^I h^1(t, r\omega)| &\le \int_r^{t+r} |\partial Z^I h^1(t + r - \rho, \rho\omega)|\, d\rho + |Z^I h^1\big(0, (t + r)\omega\big)|\\ &\le C\varepsilon \int_r^{\infty} \frac{d\rho}{(1 + t + r)^{1-\delta}(1 + |t - \rho|)^{1+\gamma}} + \frac{C\varepsilon}{(1 + t + r)^{1+\gamma-\delta}}\\ &\le \frac{C_\gamma \varepsilon}{(1 + t + r)^{1-\delta}(1 + |t - r|)^{\gamma}}.\end{aligned}$$

A similar argument yields (9.13) for $r < t$. Inequality (9.14) follows from (9.13) using that $|\bar\partial f| \lesssim \sum_{|I|=1} |Z^I f|/(1 + t + |q|)$. □

The next subsection is devoted to the preparational estimates for the inhomogeneous terms $F_{\mu\nu}$ and $F^0_{\mu\nu} = \widetilde{\Box}_g h^0_{\mu\nu}$ arising in equation (9.5) for the tensor h^1.

9.1. *Estimates for the inhomogeneous terms.* These estimates will play a key role in the derivation of the improved decay and energy estimates in the following two sections. As we have mentioned above the quadratic terms in $F_{\mu\nu}$ in (9.1) do not satisfy the standard null condition. Nevertheless, a special tensorial structure and the wave coordinate condition will allow us to obtain favorable estimates.

Recall that the inhomogeneous term $F_{\mu\nu}$ has the following structure:

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h), \tag{9.19}$$

$$P(\partial_\mu \pi, \partial_\nu \theta) := \frac{1}{2}\partial_\mu \pi^{\alpha\beta}\partial_\nu \theta_{\alpha\beta} - \frac{1}{4}\partial_\mu \mathrm{tr}\pi\, \partial_\nu \mathrm{tr}\theta. \tag{9.20}$$

The quadratic term $Q\mu\nu$ is a linear combination of the null-forms and $G_{\mu\nu}(h)(\partial h, \partial h)$ is a quadratic form in ∂h with the coefficients — a smooth function of h vanishing at $h = 0$. The following result is an immediate consequence of (9.20); see also Lemma 4.2.

LEMMA 9.6. *The quadratic form P satisfies the following pointwise estimate*:

$$|P(\partial\pi, \partial\theta)|_{\mathcal{T}\mathcal{U}} \lesssim |\bar\partial\pi\,||\partial\theta| + |\partial\pi\,||\bar\partial\theta|, \tag{9.21}$$

$$|P(\partial\pi, \partial\theta)| \lesssim |\partial\pi\,|_{\mathcal{T}\mathcal{U}}|\partial\theta|_{\mathcal{T}\mathcal{U}} + |\partial\pi\,|_{\mathcal{L}\mathcal{L}}|\partial\theta| + |\partial\pi\,||\partial\theta|_{\mathcal{L}\mathcal{L}}. \tag{9.22}$$

Using the additional estimates on the h_{LL} component, derived in Proposition 8.2 under the assumption that the wave coordinate condition holds, we obtain the following:

COROLLARY 9.7. *Let metric g satisfy the wave coordinate condition* (8.1) *relative to coordinates $\{x^\mu\}$, the quadratic form P obeys the following estimate on* *a 2-tensor* $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$:

$$|P(\partial h, \partial h)|_{\mathcal{T}\mathcal{U}} \lesssim |\bar\partial h|\,|\partial h|, \tag{9.23}$$

$$|P(\partial h, \partial h)| \lesssim |\partial h|^2_{\mathcal{T}\mathcal{U}} + |\bar\partial h|\,|\partial h| + |h|\,|\partial h|^2. \tag{9.24}$$

In addition, assuming that $|Z^J h| \le C$ *for all multi-indices* $|J| \le |I|$ *and vector fields* $Z \in \mathcal{Z}$,

$$\begin{aligned}
|Z^I P(\partial h, \partial h)| \lesssim &\sum_{|J|+|K|\le|I|} \big(|\partial Z^J h|_{\mathcal{T}\mathcal{U}}\,|\partial Z^K h|_{\mathcal{T}\mathcal{U}} + |\bar\partial Z^J h|\,|\partial Z^K h|\big)\\
&+ \sum_{|J|+|K|\le|I|-1} |\partial Z^J h|_{L\mathcal{T}}|\partial Z^K h + \sum_{|J|+|K|\le|I|-2} |\partial Z^J h|\,|\partial Z^K h||\\
&+ \sum_{|J_1|+|J_2|+|J_3|\le|I|} |Z^{J_3} h|\,|\partial Z^{J_2} h||\partial Z^{J_1} h|.
\end{aligned}$$

Proof. The estimate (9.23) follows directly from (9.21). To prove (9.24) we use (9.22) and that by the wave coordinate condition $|\partial h|_{LL} \lesssim |\bar\partial h| + |h|\,|\partial h|$.

Noting that $Z^I P(\partial_\mu h, \partial_\nu h)$ is a sum of terms of the form $P(\partial_\alpha Z^J h, \partial_\beta Z^K h)$ for some α, β and $|J| + |K| \le I$, we see that

$$|Z^I P(\partial h, \partial h)| \le C \sum_{|J|+|K|\le|I|} |P(\partial Z^J h, \partial Z^K h)|.$$

It follows from (9.22) and Proposition 8.2 that

$$\begin{aligned}
&\sum_{|J|+|K|\le|I|} |P(\partial Z^J h, \partial Z^K h)| \\
&\quad\lesssim \sum_{|J|+|K|\le|I|} |\partial Z^J h|_{\mathcal{T}\mathcal{U}}|\partial Z^K h|_{\mathcal{T}\mathcal{U}} + |\partial Z^J h|_{LL}\,|\partial Z^K h| \\
&\quad\lesssim \sum_{|J|+|K|\le|I|} |\bar\partial Z^J h|\,|Z^K h| + |\partial Z^J h|_{\mathcal{T}\mathcal{U}}\,|\partial Z^K h|_{\mathcal{T}\mathcal{U}}
\end{aligned} \tag{9.25}$$

$$\begin{aligned}
+ \sum_{|J|+|K|\le|I|} \Big(&\sum_{|J'|\le|J|-1} |\partial Z^{J'} h|_{\mathcal{L}\mathcal{T}} + \sum_{|J''|\le|J|-2} |\partial Z^{J''} h| \\
&+ \sum_{|J_1|+|J_2|\le|J|} |Z^{J_2} h|\,|\partial Z^{J_1} h|\Big)|\partial Z^K h|
\end{aligned}$$

which proves the result. □

We now state the complete estimates for the inhomogeneous term $F_{\mu\nu}$.

PROPOSITION 9.8. *Let* $F_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h)$ *be as in* (9.19) *and assume that the wave coordinate condition* (8.1) *holds for the metric* $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}$

relative to coordinates $\{x^\mu\}$. *Then*

$$|F|_{\mathcal{T}\mathcal{U}} \lesssim |\bar\partial h|\,|\partial h| + |h|\,|\partial h|^2, \tag{9.26}$$

$$|F| \lesssim |\partial h|^2_{\mathcal{T}\mathcal{U}} + |\bar\partial h|\,|\partial h| + |h|\,|\partial h|^2. \tag{9.27}$$

In addition, assuming that $|Z^J h| \le C$ *for all multi-indices* $|J| \le |I|$ *and vector fields* $Z \in \mathcal{Z}$,

(9.28)
$$\begin{aligned}|Z^I F| \lesssim &\sum_{|J|+|K|\le|I|} \big(|\partial Z^J h|_{\mathcal{T}\mathcal{U}}\,|\partial Z^K h|_{\mathcal{T}\mathcal{U}} + |\bar\partial Z^J h|\,|\partial Z^K h|\big)\\ &+ \sum_{|J|+|K|\le|I|-1} |\partial Z^J h|_{L\mathcal{T}}|\partial Z^K h| + \sum_{|J|+|K|\le|I|-2} |\partial Z^J h|\,|\partial Z^K h|\\ &+ \sum_{|J_1|+|J_2|+|J_3|\le|I|} |Z^{J_3} h|\,|\partial Z^{J_2} h||\partial Z^{J_1} h|.\end{aligned}$$

Proof. First

$$|Z^I G_{\mu\nu}(h)(\partial h, \partial h)| \le C \sum_{|I_1|+\cdots+|I_k|\le|I|,\,k\ge 3} |Z^{I_k} h|\cdots|Z^{I_3} h|\,|\partial Z^{I_2} h|\,|\partial Z^{I_1} h|.$$

Since $ZQ(\partial u, \partial v) = Q(\partial u, \partial Z v) + Q(\partial Z u, \partial v) + a^{ij} Q_{ij}(\partial u, \partial v)$, and

$$|Q_{\mu\nu}(\partial h, \partial k)| \le |\partial h|\,|\bar\partial k| + |\partial k|\,|\bar\partial h|,$$

it follows that

$$\begin{aligned}|Z^I Q_{\mu\nu}(\partial h, \partial h)| &\le C \sum_{|J|+|k|\le|I|} |Q_{\mu\nu}(\partial Z^J h, \partial Z^K h)|\\ &\le C \sum_{|J|+|k|\le|I|} |\partial Z^J h|\,|\bar\partial Z^K h|\end{aligned}$$

and the proposition follows. □

The next step is to estimate the extra inhomogeneous term $\widetilde{\square}_g h^0$ appearing in the wave equation $\widetilde{\square}_g h^1_{\mu\nu} = F_{\mu\nu} - \widetilde{\square}_g h^0_{\mu\nu}$ via the decomposition $h = h^1 + h^0$.

LEMMA 9.9. *Let*

$$F^0_{\mu\nu} = \widetilde{\square}_g h^0_{\mu\nu}, \qquad h^0_{\mu\nu} = \chi(r)\chi(t/r)\frac{M}{r}\,\delta_{\mu\nu}.$$

Then, *if the weak decay estimates in Corollary* 9.4 *hold*, *we have*

$$|Z^I F^0| \le \begin{cases} C\varepsilon^2(t+|q|+1)^{-4+\delta}(1+|q|)^{-\delta}, & q>0,\\ C\varepsilon(t+|q|+1)^{-3}, & q<0,\end{cases} \qquad |I| \le N-2.$$

More generally,

$$|Z^I F^0| \le \begin{cases} C_N \varepsilon^2 (t+|q|+1)^{-4}, & q>0,\\ C_N \varepsilon (t+|q|+1)^{-3}, & q<0,\end{cases}$$
$$+ \frac{C_N \varepsilon}{(t+|q|+1)^3} \sum_{|J|\le |I|} |Z^J h^1|, \quad |I|\le N.$$

Proof. Set $F^0 = \widetilde{\Box}_g h^0 = F^{00} + F^{01}$, $F^{00} = \Box h^0$,$F^{01} = H^{\alpha\beta}\partial_\alpha\partial_\beta h^0$. One can easily check that

$$|Z^I F^{00}| \le \frac{C_N \varepsilon}{(t+|q|+1)^3}, \quad \text{and} \quad F^{00}\equiv 0, \quad \text{for } r<t/2 \text{ or } r>3t/4,$$
$$|Z^I F^{01}| \lesssim \frac{C_N \varepsilon}{(t+|q|+1)^3} \sum_{|J|\le|I|} |Z^J H|.$$

On the other hand, with the help of (9.13) for h^0,

$$\sum_{|J|\le k} |Z^J H| \lesssim \frac{C_N\varepsilon}{t+|q|+1} + \sum_{|J_1|+\dots+|J_n|\le k} |Z^{J_1} h^1|\cdots|Z^{J_n} h^1|,$$

since $H = -h + O(h^2)$, $h = h^0 + h^1$. Using (9.13), this time for h^1, we obtain

$$|Z^I F^{01}| \le \begin{cases} C_N\varepsilon^2 (t+|q|+1)^{-4+\delta}(1+|q|)^{-\delta}, & q>0,\\ C_N\varepsilon^2(t+|q|+1)^{-4+\delta}(1+|q|)^{1/2}, & q<0,\end{cases} \qquad |I|\le N-2. \qquad \Box$$

10. **Decay estimates for the Einstein equations**

In this section we put to use the wave coordinate condition (9.9) for the metric $g(t)$ in coordinates $\{x^\mu\}_{\mu=0,\dots,3}$, which, as explained in the Remark 9.2, is satisfied on the maximum interval of existence $[0, T_0]$, and the decay estimates for solutions of the wave equation $\widetilde{\Box}_g \phi_{\mu\nu} = W_{\mu\nu}$, derived in Corollary 7.2, to upgrade pointwise estimates of Corollary 9.4.

PROPOSITION 10.1 (Estimates for h). *Let $h = h^1 + h^0$ be a solution of the reduced Einstein equations* (9.1). *Assume that h^1 verifies the energy estimate* (9.10) *on the time interval* $[0,T]$. *Then for any $t\in[0,T]$,*

$$|\partial h|_{L\mathcal{T}} + |\partial Z h|_{LL} \le \begin{cases} C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{-\delta}, & q>0,\\ C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{1/2}, & q<0,\end{cases} \tag{10.1}$$

$$|h|_{L\mathcal{T}} + |Zh|_{LL} \le \begin{cases} C\varepsilon(1+t+|q|)^{-1}, & q>0,\\ C\varepsilon(1+t+|q|)^{-1}(1+|q|)^{1/2+\delta}, & q<0.\end{cases} \tag{10.2}$$

Furthermore,

$$|\partial h|_{\mathcal{T}\mathcal{U}} \le C\varepsilon(1+t+|q|)^{-1}, \tag{10.3}$$

$$|\partial h| \le C\varepsilon t^{-1}\ln t. \tag{10.4}$$

PROPOSITION 10.2 (Estimates for h^1). *Under the same assumptions let* $\gamma' < \gamma-\delta$ *and* $\mu' > \delta > 0$ *be fixed. Then there exist constants* M_k, C_k *and* ε_k, *depending on* (γ', μ', δ), *such that*

$$|\partial Z^I h^1| \le \begin{cases} C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1-\gamma'}, & q>0,\\ C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1/2+\mu'}, & q<0,\end{cases} \tag{10.5}$$

for $|I| = k \le N/2+2$, *and*

$$|Z^I h^1| \le \begin{cases} C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-\gamma'}, & q>0,\\ C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{1/2+\mu'}, & q<0,\end{cases} \tag{10.6}$$

for $|I| = k \le N/2+2$. *The same estimates hold for* h^0 *if we replace* γ' *by* $M_k\varepsilon$.

Remark 10.3. Note the difference between the estimates (10.3), (10.4) for h and (10.5) with $|I| = 0$ for h^1.

The estimates (10.1)–(10.2) follow from the wave coordinate condition combined with the weak decay estimates of Corollary 9.4. In the derivation of the sharp decay estimates (10.3)–(10.6) we will use decay estimates of Corollary 7.2 with various weights.

10.1. *Proof of* (10.1)–(10.2). We rely on Proposition 8.2 to establish an even more general version of the desired estimates. We remind the reader that under the assumptions of Proposition 10.1 both the wave coordinate condition (9.9) for tensor $g_{\mu\nu}$ and the weak decay estimates of Corollary 9.4 hold true.

LEMMA 10.4. *Under the assumptions of Proposition* 10.1

$$\sum_{|I|\le k} |\partial Z^I h|_{LL} + \sum_{|J|\le k-1} |\partial Z^J h|_{\mathcal{L}\mathcal{T}} \lesssim \sum_{|K|\le k-2} |\partial Z^K h| + \begin{cases} \varepsilon(1+t+|q|)^{-2+2\delta}(1+|q|)^{-2\delta}, & q>0,\\ \varepsilon(1+t+|q|)^{-2+2\delta}(1+|q|)^{1/2-\delta}, & q<0,\end{cases} \tag{10.7}$$

$$\sum_{|I|\le k} |Z^I h|_{LL} + \sum_{|J|\le k-1} |Z^J h|_{\mathcal{L}\mathcal{T}} \lesssim \sum_{|K|\le k-2} \int_{s,\omega=\text{const}} |\partial Z^K h| + \begin{cases} \varepsilon(1+t+|q|)^{-1}, & q>0,\\ \varepsilon(1+t+|q|)^{-1}(1+|q|)^{1/2+\delta}, & q<0.\end{cases} \tag{10.8}$$

Here the sums over $k-2$ *are absent if* $k \le 1$, *the sums over* $k-1$ *are absent if* $k=0$ *and* $\int\limits_{s,\omega=\text{const}}$ *stands for an integral along the segment* $\tau+|y|, y/|y| = \text{const}$ *connecting a given point* (t,x) *with a hyperplane* $t=0$.

Proof. The proof follows immediately from Proposition 8.2 and the estimates of Corollary 9.4 since

$$
\begin{aligned}
(10.9)\qquad & \sum_{|I|\le k} |\partial Z^I H|_{LL} + \sum_{|J|\le k-1} |\partial Z^J H|_{L\mathcal{T}} \\
&\lesssim \sum_{|J|\le |I|} |\bar\partial Z^J H| + \sum_{|K|\le k-2} |\partial Z^K H| + \sum_{I_1+I_2=I} |Z^{I_2} H|\,|\partial Z^{I_1} H|.
\end{aligned}
$$

Integrating (10.7) along the lines on which the angle $\omega = y/|y|$ and the null coordinate $s=\tau+|y|$ are constant, as in the proof of Corollary 9.4, and using (9.13) at $t=0$, yields (10.8). □

10.2. *Proof of* (10.3)–(10.4). We apply the L^∞ estimates, derived in Corollary 7.2, for the reduced wave equation

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}$$

with $F_{\mu\nu}$ given in (9.2) or (9.19)–(9.20). Observe that the weak decay estimates of Corollary 9.4 guarantee that the tensor $H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$ verifies the required assumptions of Corollary 7.2.

First we derive the L^∞ estimates for $F_{\mu\nu}$.

LEMMA 10.5. *Suppose that the assumptions of Proposition* 10.1 *hold and let* $F_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h)$ *be as in* (9.2). *Then*

$$(10.10)\qquad |F|_{\mathcal{T}\mathcal{U}} \le C\varepsilon t^{-3/2+\delta} |\partial h|,$$

$$(10.11)\qquad |F| \le C\varepsilon t^{-3/2+\delta}|\partial h| + C|\partial h|^2_{\mathcal{T}\mathcal{U}}.$$

Proof. This follows from Proposition 9.8 and the weak decay estimates of Corollary 9.4. □

The 2-tensor h is a solution of the wave equation $\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}$. We now recall that according to Corollary 7.2 with a trivial weight $\varpi(q)=1$, i.e., $\alpha=0$, we have that for any $U,V \in \{L, \underline{L}, A, B\}$,

$$
\begin{aligned}
(10.12)\qquad & (1+t+|x|)\,|\partial h(t,x)|_{UV} \lesssim \sup_{0\le\tau\le t} \sum_{|I|\le 1} \| Z^I h(\tau,\cdot)\|_{L^\infty} \\
&+ \int_0^t \Big((1+\tau)\| \, |F(\tau,\cdot)|_{UV} \|_{L^\infty(D_\tau)} + \sum_{|I|\le 2} (1+\tau)^{-1} \| Z^I h(\tau,\cdot)\|_{L^\infty(D_\tau)} \Big)\, d\tau.
\end{aligned}
$$

Then with the help of Lemma 10.8 and the weak decay estimates of Corollary 9.4 we obtain

LEMMA 10.6. *With a constant depending on $\gamma > 0$ we have*

$$(1+t)\|\,|\partial h|_{\mathcal{T}\mathcal{U}}(t,\cdot)\|_{L^\infty} \le C\varepsilon + C\varepsilon \int_0^t (1+\tau)^{\delta-1/2}\|\partial h(\tau,\cdot)\|_{L^\infty}\,d\tau,$$

$$(1+t)\|\partial h(t,\cdot)\|_{L^\infty} \le C\varepsilon + C\int_0^t \Big(\varepsilon(1+\tau)^{\delta-1/2}\|\partial h(\tau,\cdot)\|_{L^\infty} + (1+\tau)\|\,|\partial h|_{\mathcal{T}\mathcal{U}}(\tau,\cdot)\|_{L^\infty}^2\Big)\,d\tau.$$

Here, by assumption $\delta < 1/4$ so $\delta - 1/2 < -1/4$. The estimates (10.3) and (10.4) are now consequences of the above lemma and the following technical result applied to the functions $b(t) := (1+t)\|\,|\partial h|_{\mathcal{T}\mathcal{U}}(t,\cdot)\|_{L^\infty}$ and $c(t) := (1+t)\|\partial h(t,\cdot)\|_{L^\infty}$:

LEMMA 10.7. *Assume that the functions $b(t) \ge 0$ and $c(t) \ge 0$ satisfy*

$$b(t) \le C\varepsilon\Big(\int_0^t (1+s)^{-1-a}c(s)\,ds + 1\Big) \tag{10.13}$$

$$c(t) \le C\varepsilon\Big(\int_0^t (1+s)^{-1-a}c(s)\,ds + 1\Big) + C\int_0^t (1+s)^{-1}b^2(s)\,ds \tag{10.14}$$

for some positive constants such that $a \ge C^2\varepsilon$ and $a \ge 4C\varepsilon/(1-2C\varepsilon)$. Then

$$b(t) \le 2C\varepsilon, \qquad \text{and} \qquad c(t) \le 2C\varepsilon\big(1 + a\ln(1+t)\big). \tag{10.15}$$

Proof. Let τ_0 be the largest time such that holds. Substituting these bounds into (10.13)–(10.14) and taking into account that

$$\int_0^\infty (1+s)^{-1-a}\big(1 + a\ln(1+s)\big)\,ds \le a^{-1}\int_0^\infty (1+\tau)\,e^{-\tau}\,d\tau = 2a^{-1} + 1$$

we obtain that for any $0 \le t \le \tau_0$

$$b(t) \le C\varepsilon(2C\varepsilon(1+2a^{-1}) + 1) < 2C\varepsilon,$$
$$c(t) \le C\varepsilon\big(2C\varepsilon(1+2a^{-1}) + C^2\varepsilon\ln(1+t)\big) < 2C\varepsilon\big(1 + a\ln(1+t)\big),$$

which implies that $\tau_0 = \infty$, as desired. □

10.3. *Proof of* (10.5)–(10.6). The proof of the estimates for the tensor h^1 proceeds by induction. We assume that (10.5)–(10.6) hold for all values of multi-index $|I| \le k$ and prove the estimate for $|I| = k+1$. The argument below will also apply unconditionally to the base of the induction $k = 0$.

Once again the first step is to establish L^∞ estimates for $Z^I F$.

LEMMA 10.8. *Suppose that the assumptions of Proposition* 10.1 *hold and let* $F_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h)$ *be as in* (9.2). *Then*
(10.16)
$$|Z^I F| \le C\varepsilon \sum_{|K|\le|I|} \frac{|\partial Z^K h|}{1+t} + C \sum_{|J|+|K|\le|I|,\,|J|\le|K|<|I|} |\partial Z^J h||\partial Z^K h|.$$

Proof. The result follows from Proposition 9.8 with the help of (9.12)–(9.14) and (10.3). □

Recall now that the 2-tensor $h^1 = g - m - h^0$ is a solution of the reduced wave equation

$$\widetilde{\Box}_g h^1_{\mu\nu} = F^1_{\mu\nu} = F_{\mu\nu} - F^0_{\mu\nu}, \qquad \text{where} \qquad F^0_{\mu\nu} = \widetilde{\Box}_g h^0_{\mu\nu} \tag{10.17}$$

and that the terms $F_{\mu\nu}$ and $F^0_{\mu\nu}$ have been treated in Proposition 9.8 and Lemma 9.9 respectively.

To prove the estimates for $Z^I h^1$ with vector fields $Z \in \mathscr{Z}$ we commute equation (10.17) with Z^I. By Proposition 5.3

$$\begin{aligned}|\widetilde{\Box}_g Z^I h^1| \lesssim |\hat{Z}^I F^1| &+ (1+t)^{-1} \sum_{|K|\le|I|,} \ \sum_{|J|+(|K|-1)_+\le|I|} |Z^J H||\partial Z^K h^1| \\ &+ \frac{C}{1+|q|} \sum_{|K|\le|I|} \Big(\sum_{|J|+(|K|-1)_+\le|I|} |Z^J H|_{LL} + \sum_{|J'|+(|K|-1)_+\le|I|-1} |Z^{J'} H|_{L\mathscr{T}} \\ &\qquad + \sum_{|J''|+(|K|-1)_+\le|I|-2} |Z^{J''} H| \Big) |\partial Z^K h^1|\end{aligned} \tag{10.18}$$

where $(|K|-1)_+ = |K|-1$, if $|K|\ge 1$, and 0, if $|K|=0$. Using Lemma 10.4 and (10.5), which is inductively assumed to be true for $|I|\le k$, we get

$$\begin{aligned}&(1+|q|)^{-1} \sum_{|J|\le k,\,|J'|\le k-1,\,|J''|\le k-2} \Big(|Z^J H|_{LL} + |Z^{J'} H|_{L\mathscr{T}} + |Z^{J''} H| \Big) \\ &\lesssim \begin{cases} C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1-M_k\varepsilon} + \varepsilon(1+t+|q|)^{-1}(1+|q|)^{-1}, & q>0, \\ C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1/2+\mu'} + \varepsilon(1+t+|q|)^{-1}(1+|q|)^{-1/2+\delta}, & q<0,\end{cases}\end{aligned} \tag{10.19}$$

while contribution of the terms in (10.18) coupled to the highest order term — $\partial Z^K h^1$, $|K| = |I|$, according to (10.2) amounts to

$$\begin{aligned}&(1+|q|)^{-1} \sum_{|J|\le 1} \Big(|Z^J H|_{LL} + |H|_{L\mathscr{T}} \Big) \\ &\qquad \lesssim \begin{cases} \varepsilon(1+t+|q|)^{-1}(1+|q|)^{-1}, & q>0, \\ \varepsilon(1+t+|q|)^{-1}(1+|q|)^{-1/2+\delta}, & q<0.\end{cases}\end{aligned} \tag{10.20}$$

Therefore splitting the sum in (10.18) according to whether $|K| < |I|$ or $|K| = |I|$ and using the inductive assumption (10.5) for $|K| < |I| = k+1$, we obtain

(10.21)
$$|\widetilde{\Box}_g Z^I h^1| \le C\varepsilon \sum_{|K|\le|I|} \frac{|\partial Z^K h^1|}{1+t} + |\hat{Z}^I F^0| + \begin{cases} C_k^2\varepsilon^2(1+t+|q|)^{-2+2M_k\varepsilon}(1+|q|)^{-2-2M_k\varepsilon}, & q>0,\\ C_k^2\varepsilon^2(1+t+|q|)^{-2+2M_k\varepsilon}(1+|q|)^{-1+2\mu'}, & q<0.\end{cases}$$

Now by Lemma 9.9

$$|Z^I F^0| \le \begin{cases} C\varepsilon^2(t+|q|+1)^{-4+\delta}(1+|q|)^{-\delta}, & q>0,\\ C\varepsilon(t+|q|+1)^{-3}, & q<0,\end{cases} \qquad |I|\le N-2.$$

Set

$$n_{k+1}(t) = (1+t)\sum_{|I|\le k+1} \|\varpi(q)\partial Z^I h^1(t,\cdot)\|_{L^\infty}, \tag{10.22}$$
$$\varpi(q) = \begin{cases} (1+|q|)^{1+\gamma'}, & q>0,\\ (1+|q|)^{1/2-\mu'}, & q<0,\end{cases}$$

where $\mu' > \delta$ and $\gamma' < \gamma - \delta$. Then we have established that for $|I| = k+1$:

(10.23)
$$\varpi(q)|\widetilde{\Box}_g Z^I h^1| \lesssim (1+t)^{-2}\big(\varepsilon n_{k+1}(t) + \varepsilon^2 C_k^2 (1+t)^{2M_k\varepsilon} + C\varepsilon(1+t)^{-1/2-\mu'}\big).$$

The weak decay estimates of Corollary 9.4 imply that

$$\varpi(q)|Z^I h^1(t,x)| \lesssim \begin{cases} \varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{-\gamma}(1+|q|)^{1+\gamma'}, & q>0,\\ \varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{1/2}(1+|q|)^{1/2-\mu'}, & q<0\end{cases}$$
$$\lesssim \varepsilon(1+t)^{-a}, a>0$$

provided that $a = \min(\mu'-\delta, \gamma-\delta-\gamma') > 0$. The decay estimate proved in Corollary 7.2 therefore shows that for some constant $C = C(k)$ we have the inequality

(10.24)
$$n_{k+1}(t) \le C\varepsilon + C\int_0^t (1+\tau)^{-1}\big(\varepsilon n_{k+1}(\tau) + \varepsilon^2(1+\tau)^{C\varepsilon} + C\varepsilon(1+t)^{-1/2-\mu'}\big)\,d\tau.$$

The bound $n_{k+1}(t) \le 2C\varepsilon(1+t)^{2C\varepsilon}$ follows from the Gronwall inequality. This proves (10.5). The estimate (10.6) then follows by integrating (10.5) along the line $\omega = y/|y|, \tau + |y| = \text{const}$ from the hyperplane $t = 0$ using (9.17).

11. **Energy estimates for Einstein equations**

Recall the definition of the weighted energy

$$\mathcal{E}_N(t) = \sup_{0\le\tau\le t} \sum_{|I|\le N} \int_{\Sigma_t} |\partial Z^I h^1|^2 w(q), \tag{11.1}$$

where

$$w = \begin{cases} 1+(1+|q|)^{1+2\gamma}, & q>0,\\ 1+(1+|q|)^{-2\mu}, & q<0,\end{cases} \qquad w' = \begin{cases} (1+2\gamma)(1+|q|)^{2\gamma}, & q>0,\\ 2\mu(1+|q|)^{1-2\mu}, & q<0.\end{cases} \tag{11.2}$$

Recall also our decomposition

$$g_{\mu\nu}(t) = m_{\mu\nu} + h_{\mu\nu}(t) = m_{\mu\nu} + h^1_{\mu\nu}(t) + h^0_{\mu\nu}(t), \quad h^0_{\mu\nu}(t) = \chi\big(\tfrac{r}{t}\big)\chi(r)\frac{M}{r}\delta_{\mu\nu}$$

of a local in time smooth solution $g_{\mu\nu}(t)$ of the reduced Einstein equations, and the definition of the tensor $H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$.

In this section we prove the following result.

THEOREM 11.1. *Let $g_{\mu\nu}(t) = h_{\mu\nu}(t) + m_{\mu\nu}$ be a local in time solution of the reduced Einstein equations* (9.1) *satisfying the wave coordinate condition* (9.9) *on the interval* $[0, T)$. *Suppose also that for some* $0 < \mu' < 1/2$ *and* $0 < \gamma < 1/2$ *we have the following estimates for* $0 \le t \le T$, *all multi-indices* $|I| \le N/2 + 2$ *and the collections* $\mathcal{T} = \{L, S_1, S_2\}$, $\mathcal{U} = \{L, \underline{L}, S_1, S_2\}$:

$$|\partial H|_{\mathcal{T}\mathcal{U}} + (1+|q|)^{-1}|H|_{\mathcal{T}\mathcal{L}} + (1+|q|)^{-1}|ZH|_{\mathcal{L}\mathcal{L}} \le C\varepsilon(1+t)^{-1}, \tag{11.3}$$

$$|\partial Z^I h| + \frac{|Z^I h|}{1+|q|} + \frac{1+t+|q|}{1+|q|}|\bar\partial Z^I h| \lesssim \begin{cases} C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1-C\varepsilon}, & q>0,\\ C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'}, & q<0,\end{cases} \tag{11.4}$$

$$E_N(0) + M^2 \le \varepsilon^2. \tag{11.5}$$

Then there is a positive constant c independent of T such that if $\varepsilon \le c^{-2}$ we have the energy estimate

$$\mathcal{E}_N(t) \le C_N\, \varepsilon^2 (1+t)^{c\varepsilon}, \tag{11.6}$$

for $0 \le t \le T$. *Here C_N is a constant that depends only on N.*

Assuming the conclusions of Theorem 11.1 for a moment we finish the proof of the main Theorem 9.1.

11.1. *End of the proof of Theorem* 9.1. Recall that T was defined as the maximal time with the property that the bound

$$\mathscr{E}_N(t) \le 2C_N\varepsilon(1+t)^\delta$$

holds for all $0 \le t \le T$. Assuming the energy bound above we have established in Propositions 10.1–10.2 the decay estimates for the tensors $h = h^0 + h^1$ and h^1 respectively. Direct check shows that the estimates of Propositions 10.1–10.2 imply the assumptions (11.3)–(11.4). The conclusion of Theorem 11.1 states that the energy

$$\mathscr{E}_N(t) \le C_N\varepsilon^2(1+t)^{c\varepsilon}, \qquad \forall 0 \le t \le T.$$

Thus choosing a sufficiently small $\varepsilon > 0$ we can show that $\mathscr{E}_N(t) \le C_N\varepsilon(1+t)^\delta$ thus contracting the maximality of T and consequently proving that $g_{\mu\nu}$ is a global solution. It therefore remains to prove Theorem 11.1.

11.2. *Proof of Theorem* 11.1.

Proof. Recall that the components of the tensor $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$ satisfy the wave equations:

$$\widetilde{\Box}_g h_{\mu\nu} = F_{\mu\nu}, \quad F_{\mu\nu} = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h), \tag{11.7}$$

$$P(\partial_\mu h, \partial_\nu h) = \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta}\, \partial_\nu h_{\alpha'\beta'} - \frac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'}\, m^{\beta\beta'} \partial_\nu h_{\beta\beta'}. \tag{11.8}$$

Our goal is to compute the energy norms of $Z^I h^1$, where $Z \in \mathscr{Z}$ and h^1 is a solution of the problem

$$\widetilde{\Box}_g\, h^1_{\mu\nu} = F^1_{\mu\nu}, \qquad \text{where} \qquad F^1 = F - F^0, \qquad F^0 = \widetilde{\Box}_g h^0. \tag{11.9}$$

Commuting with the vector fields $\widehat{Z}^I$, $\widehat{Z} = Z + c_Z$ we get

$$\widetilde{\Box}_g\, Z^I h^1_{\mu\nu} = F^{1\,I}_{\mu\nu}, \tag{11.10}$$

where

$$F^{1I} = \widehat{Z}^I F - \widehat{Z}^I F^0 - D^I, \qquad D^I = \big(\widehat{Z}^I \widetilde{\Box}_g h^1 - \widetilde{\Box}_g Z^I h^1\big). \tag{11.11}$$

We base our argument on the energy estimate (6.8) for a solution ϕ of the wave equation $\widetilde{\Box}_g \phi = F$ of Proposition 6.2. Observe that the conditions of our theorem on the tensor $H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$ imply that the assumptions of Proposition 6.2 for the metric g hold true. In particular, we have

$$\begin{aligned} &\int_{\Sigma_t} |\partial\phi|^2\, w + \int_0^t \int_{\Sigma_\tau} |\bar{\partial}\phi|^2\, w' \\ &\qquad \le 8 \int_{\Sigma_0} |\partial\phi|^2\, w + 16 \int_0^t \int_{\Sigma_t} \Big(\frac{C\varepsilon\, |\partial\phi|^2}{1+t} + |\widetilde{\Box}_g\phi|\, |\partial\phi| \Big)\, w. \end{aligned} \tag{11.12}$$

This applied to (11.10) gives

$$
\begin{aligned}
(11.13)\quad & \int_{\Sigma_t} |\partial Z^I h^1|^2\, w + \int_0^t \int_{\Sigma_\tau} |\bar\partial Z^I h^1|^2\, w' \\
&\le 8 \int_{\Sigma_0} |\partial h^1|^2\, w + 16 \int_0^t \int_{\Sigma_t} \Big(\frac{C\varepsilon\, |\partial Z^I h^1|^2}{1+t} + |F_1^I|\, |\partial Z^I h^1|\Big)\, w \\
&\le 8 \int_{\Sigma_0} |\partial h^1|^2\, w + 16 \int_0^t \int_{\Sigma_t} \Big(\frac{C\varepsilon\, |\partial Z^I h^1|^2}{1+t}\, w \\
&\quad + \varepsilon^{-1}\big(|\hat{Z}^I F|^2 + \big|D^I\big|^2\big)\,(1+t)\, w\ + |Z^I F^0|\, |\partial Z^I h^1|\, w\Big).
\end{aligned}
$$

We begin with the following estimate on the inhomogeneous term F:

LEMMA 11.2. *Under the assumptions of Theorem* 11.1

(11.14)

$$
\begin{aligned}
|Z^I F| \lesssim & \sum_{|J|\le|I|} \Big(\frac{\varepsilon|\partial Z^J h_1|}{1+t} + \frac{\varepsilon(1+|q|)^{\mu'-1/2}}{(1+t+|q|)^{1-C\varepsilon}}\, |\bar\partial Z^J h_1| + \frac{\varepsilon^2}{1+t+|q|}\, \frac{|Z^J h_1|}{1+|q|}\Big) \\
& + \sum_{|J|\le|I|-1} \frac{\varepsilon\, |\partial Z^J h_1|}{(1+t)^{1-C\varepsilon}} + \frac{\varepsilon^2}{(1+t+|q|)^4}.
\end{aligned}
$$

Proof. According to Proposition 9.8

$$
\begin{aligned}
(11.15)\quad |Z^I F| \lesssim & \sum_{|J|+|K|\le|I|} \big(|\partial Z^J h|_{\mathcal{T}\mathcal{U}}\, |\partial Z^K h|_{\mathcal{T}\mathcal{U}} + |\bar\partial Z^J h|\, |\partial Z^K h|\big) \\
& + \sum_{|J|+|K|\le|I|-1} |\partial Z^J h|\, |\partial Z^K h| + \sum_{|J_1|+|J_2|+|J_3|\le|I|} |Z^{J_3} h|\, |\partial Z^{J_2} h| |\partial Z^{J_1} h|.
\end{aligned}
$$

Since $h = h^1 + h^0$ and h^0 obeys the estimates $|\partial Z^J h^0| \lesssim \varepsilon(1+t+|q|)^{-2}$ and $|Z^I h^0| \lesssim \varepsilon(1+t+|q|)^{-1}$ for all $|I| \le N$, it follows that both h^0 and h^1 obey the estimates (11.3) and (11.4) of the theorem. Substituting $h = h^0 + h^1$ into (11.15) and expanding the products we obtain terms in which either all factors contain h^0, in which case we can estimate them by $\varepsilon^2(1+t+|q|)^{-4}$, or at least one factor contains h^1, in which case we can simply estimate the other factors by (11.3)–(11.4). It follows that (11.15) leads to the estimate

$$
\begin{aligned}
|Z^I F| \lesssim & \sum_{|J|\le|I|} \Big(\frac{\varepsilon|\partial Z^J h^1|}{1+t} + \frac{\varepsilon(1+|q|)^{\mu'-1/2}}{(1+t+|q|)^{1-C\varepsilon}}\, |\bar\partial Z^J h^1| + \frac{\varepsilon^2}{1+t+|q|}\, \frac{|Z^J h^1|}{1+|q|}\Big) \\
& + \sum_{|J|\le|I|-1} \frac{\varepsilon\, |\partial Z^J h^1|}{(1+t)^{1-C\varepsilon}} + \frac{\varepsilon^2}{(1+t+|q|)^4}. \qquad \square
\end{aligned}
$$

LEMMA 11.3. *Under the assumptions of Theorem* 11.1

$$\varepsilon^{-1}\int_0^T\int |Z^I F|^2\,(1+t)\,w\,dxdt$$
$$\lesssim \sum_{|J|\le|I|}\int_0^T\int \varepsilon\Big(\frac{|\partial Z^J h^1|^2}{1+t}\,w+|\bar\partial Z^J h^1|^2\,w'\Big)dxdt$$
$$+\sum_{|J|\le|I|-1}\int_0^T\int \varepsilon\frac{|\partial Z^J h^1|^2}{(1+t)^{1-2C\varepsilon}}\,w\,dxdt+\varepsilon^3.$$

Proof. The estimate is a straightforward application of Lemma 11.2. We took into account that $w\le w'(1+|q|)(1+q_-)^{2\mu}$ and the inequality $\mu<1-2\mu'$. The estimate

$$\int\frac{1}{1+t+|q|}\,\frac{|Z^J h^1|^2}{(1+|q|)^2}\,w\,dx\lesssim\int\frac{|\partial Z^J h^1|^2}{(1+t+|q|)}\,w\,dx$$

is the Hardy-type inequality established in Corollary 13.3 of Appendix B. □

Next we estimate $F^0=\widetilde{\Box}_g h^0$:

LEMMA 11.4. *The following inequality holds true*:

$$\int_0^T\int |Z^I F^0|\,|\partial Z^I h^1| w\,dxdt\le C_N\varepsilon\sum_{|J|\le|I|}\Bigg(\int_0^T\int\frac{|\partial Z^J h^1|^2}{(1+t)^2}w\,dxdt$$
$$+\int_0^T\Big(\int|\partial Z^J h^1|^2 w\,dx\Big)^{1/2}\frac{dt}{(1+t)^{3/2}}\Bigg),$$

where C_N denotes a constant that depends only on N.

Proof. We start with the estimate

$$\int_0^T\int |Z^I F^0|\,|\partial Z^I h^1| w\,dxdt$$
$$\le\int_0^T\Big(\int|Z^I F^0|^2 w\,dx\Big)^{\frac12}\Big(\int|\partial Z^I h^1|^2 w\,dx\Big)^{\frac12}dt.$$

Now by Lemma 9.9

$$|Z^I F^0|\le\begin{cases}C_N\varepsilon^2(t+|q|+1)^{-4}, & q>0,\\ C_N\varepsilon(t+|q|+1)^{-3}, & q<0,\end{cases}+\frac{C_N\varepsilon}{(t+|q|+1)^3}\sum_{|J|\le|I|}|Z^J h^1|.$$

Therefore, once again using a Hardy-type inequality of Corollary 13.3, we obtain

$$\begin{aligned}\int |Z^I F^0|^2 w\,dx &\le C_N \int_0^t \frac{\varepsilon^2 r^2 dr}{(1+t)^6} + C_N \int_t^\infty \frac{\varepsilon^4 r^2 (1+r)^{1+2\gamma}\,dr}{(1+r)^8}\\ &\quad + \frac{C_N\varepsilon^2}{(1+t)^4}\int \frac{|Z^I h_1|^2}{(1+|q|)^2} w\,dx\\ &\le \frac{C_N\varepsilon^2}{(1+t)^3} + \frac{C_N\varepsilon^2}{(1+t)^4}\int |\partial Z^I h^1|^2 w\,dx. \qquad \square\end{aligned}$$

The estimate for the term containing the commutator term $D^I = \widetilde{\Box}_g Z^I h^1 - \widehat{Z}^I \widetilde{\Box}_g h^1$ follows from the following

LEMMA 11.5. *Under the assumptions of Theorem* 11.1,

$$\begin{aligned}\varepsilon^{-1}\int_0^T \int \left|\widetilde{\Box}_g Z^I h^1 - \widehat{Z}^I \widetilde{\Box}_g h^1\right|^2 (1+t)\,w\,dx\,dt &\\ \lesssim \varepsilon \sum_{|J|\le|I|}\int_0^T \int \left(\frac{|\partial Z^J h^1|^2}{1+t} w + |\bar\partial Z^J h^1|^2 w'\right) dxdt &\\ + \varepsilon \sum_{|J|\le|I|-1}\int_0^T \int \frac{|\partial Z^J h^1|^2}{(1+t)^{1-2C\varepsilon}} w\,dxdt + \varepsilon^3.&\end{aligned}\tag{11.16}$$

We postpone the proof of Lemma 11.5 for a moment and finish the proof of Theorem 11.1.

Using (11.13) together with Lemmas 11.3–11.5 yields

(11.17)

$$\begin{aligned}&\int_{\Sigma_t} |\partial Z^I h^1|^2 w + \int_0^t \int_{\Sigma_\tau} |\bar\partial Z^I h^1|^2 w'\\ &\quad\le 8\int_{\Sigma_0} |\partial h^1|^2 w + C_N\varepsilon \sum_{|J|\le|I|}\int_0^T \frac{1}{(1+t)^{3/2}}\Big(\int |\partial Z^J h^1|^2 w\,dx\Big)^{1/2} dt\\ &\qquad + C\varepsilon \sum_{|J|\le|I|}\int_0^T \int \Big(\frac{|\partial Z^J h^1|^2}{1+t} w + |\bar\partial Z^J h^1|^2 w'\Big) dxdt\\ &\qquad + C\varepsilon \sum_{|J|\le|I|-1}\int_0^T \int \frac{|\partial Z^J h^1|^2}{(1+t)^{1-2C\varepsilon}} w\,dxdt + C\varepsilon^3\end{aligned}$$

where C_N depends only on $|I|\le N$. As before we denote

$$\mathcal{E}_k(t) = \sup_{0\le\tau\le t} \sum_{Z\in\mathcal{Z}, |I|\le k} \int_{\Sigma_\tau} |\partial Z^I h^1|^2 w\,dx$$

and let

$$S_k(t) := \sum_{Z\in\mathcal{Z}, |I|\le k} \int_0^t \int_{\Sigma_\tau} |\bar\partial Z^I h^1|^2 w'\, dx.$$

It therefore follows that

$$E_k(t) + S_k(t) \le 8E_k(0) + C\varepsilon S_k(t) + \int_0^t \frac{C\varepsilon E_k(\tau)}{1+\tau} d\tau + \int_0^t \frac{C_N\varepsilon E_k(\tau)^{1/2}}{(1+\tau)^{3/2}} d\tau + C\varepsilon^3 + \int_0^t \frac{C\varepsilon E_{k-1}(\tau)}{(1+\tau)^{1-C\varepsilon}} d\tau. \tag{11.18}$$

For $C\varepsilon$ sufficiently small we can absorb the space-time integral $S_k(t)$ into the one on the left-hand side at the expense of at most doubling all the constants on the right-hand side. Similarly, since $16C_N\varepsilon 1/4E_k^{1/2} \le E_k + 64^2C_N^2\varepsilon^2$ and $E_k(t)$ is increasing, we can absorb $1/4\int_0^t E_k(\tau)\,(1+\tau)^{-3/2} \le 1/2E_k(t)$. If we also use the assumption $E_N(0) \le \varepsilon^2$ we obtain for $\varepsilon > 0$ sufficiently small

$$E_k(t) + S_k(t) \le C_N\varepsilon^2 + \int_0^t \frac{C\varepsilon E_k(\tau)}{1+\tau} d\tau + \int_0^t \frac{C\varepsilon E_{k-1}(\tau)}{(1+\tau)^{1-C\varepsilon}} d\tau \tag{11.19}$$

where the last term is absent if $k = 0$ and C_N is a constant that depends only on N.

For $k = 0$ this yields the estimate

$$E_0(t) \le C_N\varepsilon^2 + \int_0^t \frac{c_0\varepsilon E_0(\tau)}{1+\tau} d\tau$$

and the Gronwall inequality gives the bound

$$E_0(t) \le C_N(1+t)^{c_0\varepsilon}$$

which prove (11.6) for $k = 0$.

Assuming (11.6) for k replaced by $k-1$ we get from (11.19)

$$E_k(t) \le C_N\varepsilon^2 + \int_0^t \frac{c\varepsilon E_k(\tau)}{1+\tau} d\tau + \int_0^t \frac{c\varepsilon^3\, d\tau}{(1+\tau)^{1-c\varepsilon}} \tag{11.20}$$

which leads to the bound

$$E_k(t) \le C_N\varepsilon^2(1+t)^{2c\varepsilon}.$$

This concludes the induction and the proof of the theorem. □

11.3. *Proof of Lemma* 11.5.

Proof. Define the tensor

$$H_1^{\mu\nu} := H^{\mu\nu} - H_0^{\mu\nu}, \qquad H_0^{\mu\nu} = -\chi\Big(\frac{r}{t}\Big)\chi(r)\frac{M}{r}\delta^{\mu\nu}.$$

Observe that H_0 coincides with the tensor $-h^0$. Define also the wave operator $\widetilde{\Box}_1 = \Box + H_1^{\alpha\beta}\partial_\alpha\partial_\beta$. According to (5.9) of Proposition 5.3

$$
\begin{aligned}
(11.21)\quad & \big|\widetilde{\Box}_1 Z^I h^1 - \hat{Z}^I \widetilde{\Box}_1 h^1\big| \\
&\lesssim \sum_{|K|\le|I|}\ \sum_{|J|+(|K|-1)_+\le|I|} \Big(\frac{|Z^J H_1|}{1+t+|q|} + \frac{|Z^J H_1|_{\mathcal{L}\mathcal{L}}}{1+|q|}\Big)|\partial Z^K h^1| \\
&\quad + \sum_{|K|\le|I|}\Big(\sum_{|J|+(|K|-1)_+\le|I|-1} \frac{|Z^J H_1|_{\mathcal{L}\mathcal{T}}}{1+|q|} \\
&\qquad\qquad + \sum_{|J|+(|K|-1)_+\le|I|-2} \frac{|Z^J H_1|}{1+|q|}\Big)|\partial Z^K h^1|.
\end{aligned}
$$

Our goal is to obtain the estimate for the quantity

$$
\sum_{|I|\le N}\int_0^T\!\!\int \big|\widetilde{\Box}_1 Z^I h^1 - \hat{Z}^I \widetilde{\Box}_1 h^1\big|^2 (1+t)w\, dxdt.
$$

Let us first deal with the terms in (11.21) with $|K|\le N/2+1$. In this case we use the decay estimates (11.4)

$$
\begin{aligned}
|\partial Z^K h| &\le C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'} \\
&\le C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-C\varepsilon-\mu}
\end{aligned}
$$

guaranteed by the assumptions of Theorem 11.1, provided that $\mu < 1/2-\mu'$. It is clear that in this case it suffices to consider the expression

$$
\begin{aligned}
(11.22)\quad & \int_0^T\!\!\int \Big(\frac{|Z^J H_1|^2}{(1+t+|q|)^2} + \frac{|Z^J H_1|^2_{\mathcal{L}\mathcal{L}} + |Z^{J'} H_1|^2_{\mathcal{L}\mathcal{T}} + |Z^K H_1|^2}{(1+|q|)^2}\Big) \\
&\qquad\qquad \times \frac{\varepsilon^2(1+|q|)^{-2C\varepsilon}}{(1+t+|q|)^{1-2C\varepsilon}}\frac{w\, dxdt}{(1+q_-)^{2\mu}} \\
&\lesssim \int_0^T\!\!\int \frac{|Z^J H_1|^2}{(1+|q|)^2}\frac{\varepsilon^2\, w\, dxdt}{1+t} + \int_0^T\!\!\int \frac{|Z^K H_1|^2}{(1+|q|)^2}\frac{\varepsilon^2 w\, dxdt}{(1+t)^{1-2C\varepsilon}} \\
&\quad + \int_0^T\!\!\int \frac{|Z^J H_1|^2_{\mathcal{L}\mathcal{L}} + |Z^{J'} H_1|^2_{\mathcal{L}\mathcal{T}}}{(1+|q|)^2}\frac{\varepsilon^2(1+|q|)^{-2C\varepsilon}}{(1+t+|q|)^{1-2C\varepsilon}}\frac{w\, dxdt}{(1+q_-)^{2\mu}}
\end{aligned}
$$

with $|J|\le k$, $|J'|\le k-1$, $|K|\le k-2$, where $k=|I|$. After applying the Hardy-type inequalities of Corollary 13.3 the above expression is bounded by

$$\varepsilon^2\Big(\int_0^T\int\frac{|\partial Z^J H_1|^2}{1+t}\,w\,dxdt+\int_0^T\int\frac{|\partial Z^K H_1|^2}{(1+t)^{1-2C\varepsilon}}\,w\,dxdt + \int_0^T\int(|\partial Z^J H_1|^2_{\mathcal{L}\mathcal{L}}+|\partial Z^{J'} H_1|^2_{\mathcal{L}\mathcal{T}})\,\widetilde{w}\,dxdt\Big),$$

where

$$\widetilde{w}=\min\Big(w',\frac{w}{(1+t+|q|)^{1-2C\varepsilon}}\Big).$$

Ignoring the difference, which we shall comment on at the end of the proof, between the tensors $H_1^{\mu\nu}=g^{\mu\nu}-m^{\mu\nu}-H_0^{\mu\nu}$ and $h^1_{\mu\nu}=g_{\mu\nu}-m_{\nu\mu}-h^0_{\mu\nu}$, we see that the first two terms are as claimed in the statement of the lemma. We now recall that according to Lemma 15.4 of Appendix D

$$\sum_{|J|\le k}\big|\partial Z^J H_1\big|_{\mathcal{L}\mathcal{L}}+\sum_{|J|\le k-1}\big|\partial Z^J H_1\big|_{\mathcal{L}\mathcal{T}}\lesssim\sum_{|J|\le k}|\bar\partial Z^J H_1| + \sum_{|J'|\le k-1}|\partial Z^{J'}H_1|+\varepsilon\frac{1}{1+t+|q|}\sum_{|J|\le k}\Big(|\partial Z^J H_1|+\frac{|Z^J H_1|}{1+t+|q|}\Big) + \sum_{|J_1|+|J_2|\le k}|Z^{J_1}H_1||\partial Z^{J_2}H_1|+\frac{C\varepsilon\chi_0(1/2<r/t<3/4)}{(1+t+|q|)^2}+\frac{C\varepsilon^2}{(1+|t|+|q|)^3}.$$

It is clear that in the sum $\sum_{|J_1|+|J_2|\le k}$ above at least one of the indices is $\le N/2$ and therefore we can use the decay estimates (11.4): for $|J'|\le N/2+2$

$$\frac{|Z^{J'}h|}{1+|q|}+|\partial Z^{J'}h|\le C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'} \le C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-C\varepsilon}.$$

Thus, with $|J|\le k, |J'|\le k-1$,

(11.23)
$$\int_0^T\int\Big(|\partial Z^J H_1|^2_{\mathcal{L}\mathcal{L}}+|\partial Z^{J'}H_1|^2_{\mathcal{L}\mathcal{T}}\Big)\widetilde{w}\,dxdt\lesssim\int_0^T\int\sum_{|J|\le k}|\bar\partial Z^J H_1|^2 w'\,dxdt + \varepsilon^2\int_0^T\int\sum_{|J|\le k}\Big(|\partial Z^J H_1|^2+\frac{|Z^J H_1|^2}{(1+|q|)^2}\Big)\Big(\frac{1+|q|}{1+t+|q|}\Big)^{2-2C\varepsilon}w'\,dxdt + \sum_{|J'|\le k-1}\int_0^T\int\frac{|\partial Z^{J'}H_1|^2}{(1+t)^{1-2C\varepsilon}}w\,dxdt + \int_0^T\int\Big(\frac{C\varepsilon^2\chi_0(1/2<r/t<3/4)}{(1+t+|q|)^4}+\frac{C\varepsilon^4}{(1+|t|+|q|)^6}\Big)\,w'\,dxdt$$

where $\chi_0(1/2 < r/t < 3/4)$ is the characteristic function of the set where $t/2 < r < 3t/4$. Using the properties of the function w', in particular that $w' \lesssim w/(1+|q|)$ we obtain that the above has a bound

$$
\begin{aligned}
(11.23) \lesssim \sum_{|J|\le k} \int_0^T \int |\bar\partial Z^J H_1|^2 w' \, dx dt \\
&+ \varepsilon^2 \int_0^T \int \sum_{|J|\le k} \Big(|\partial Z^J H_1|^2 + \frac{|Z^J H_1|^2}{(1+|q|)^2}\Big) \frac{w \, dx dt}{1+t} \\
&+ \sum_{|J'|\le k-1} \int_0^T \int \frac{|\partial Z^{J'} H_1|^2}{(1+t)^{1-2C\varepsilon}} w \, dx dt \\
&+ \int_0^T \int \frac{C\varepsilon^2 \chi_0^2(1/2 < r/t < 3/4)}{(1+t+|q|)^4 (1+|q|)^{1+2\mu}} \, dx dt \\
&+ \int_0^T \frac{C\varepsilon^4}{(1+|t|+|q|)^6} (1+|q|)^{1+2\gamma} \, dx dt.
\end{aligned}
$$

Once again we use the Hardy-type inequality of Corollary 13.3 with $a = 0$ to conclude that

$$
\begin{aligned}
\int_0^T \int \Big(|\partial Z^I H_1|^2_{\mathcal{L}\mathcal{L}} + |\partial Z^J H_1|^2_{\mathcal{L}\mathcal{T}} \Big) w' \, dx dt \\
\lesssim \sum_{|J|\le k} \int_0^T \int \Big(|\bar\partial Z^J H_1|^2 w' + \frac{|\partial Z^J H_1|^2}{1+t} w \Big) dx dt \\
+ \sum_{|J'|\le k-1} \int_0^T \int \frac{|\partial Z^{J'} H_1|^2}{(1+t)^{1-2C\varepsilon}} w \, dx dt + C\varepsilon^2
\end{aligned}
$$

as desired, modulo the difference between H_1 and h^1, in Lemma 11.5. To finish the proof of the lemma for the case $|K| \le N/2+1$ it remains address the difference between the tensors

$$
H_1^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu} + h^{0\mu\nu} \quad \text{and} \quad h^1_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu} - h^0{}_{\mu\nu}
$$

The above expressions imply that

$$
(m + h^0 + h^1)^{-1}_{\mu\nu} = (m - h^0 + H_1)^{\mu\nu}.
$$

Therefore, $H_1^{\mu\nu} = -h^{1\mu\nu} + O^{\mu\nu}\Big((h^0+h^1)^2\Big)$ and it follows that

$$
\begin{aligned}
|\partial Z^J H_1| \lesssim |\partial Z^J h^1| + \sum_{|J_1|+|J_2|\le |J|} \Big(|Z^{J_1} h^0| |\partial Z^{J_2} h^1| + |Z^{J_1} h^1| |\partial Z^{J_2} h^1| \\
+ |\partial Z^{J_1} h^0| |Z^{J_2} h^1| + |Z^{J_1} h^0| |\partial Z^{J_2} h^0| \Big),
\end{aligned}
$$

$$|\bar\partial Z^J H_1| \lesssim |\bar\partial Z^J h^1| + \sum_{|J_1|+|J_2|\le |J|} \Big(|Z^{J_1} h^0||\partial Z^{J_2} h^1| + |Z^{J_1} h^1||\partial Z^{J_2} h^1| \\ + |\partial Z^{J_1} h^0||Z^{J_2} h^1| + |Z^{J_1} h^0||\partial Z^{J_2} h^0|\Big).$$

Taking into account that for $i = 0, 1$ and $|J'| \le N/2 + 2$

$$\frac{|Z^{J'} h^i|}{1+|q|} + |\partial Z^{J'} h| \le C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'} \\ \le C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-C\varepsilon},$$

we obtain that

$$|\partial Z^J H_1| \lesssim |\partial Z^J h^1| + \Big(\frac{1+|q|}{1+t+|q|}\Big)^{1-C\varepsilon} \\ \times \sum_{|K|\le |J|} \Big(|\partial Z^K h^1| + \frac{|Z^K h^1|}{1+|q|}\Big) + \frac{\varepsilon^2}{(1+t+|q|)^3},$$

$$|\bar\partial Z^J H_1| \lesssim |\bar\partial Z^J h^1| + \Big(\frac{1+|q|}{1+t+|q|}\Big)^{1-C\varepsilon} \\ \times \sum_{|K|\le |J|} \Big(|\partial Z^K h^1| + \frac{|Z^K h^1|}{1+|q|}\Big) + \frac{\varepsilon^2}{(1+t+|q|)^3}.$$

Thus, with the help of the inequality $w' \le w/(1+|q|)$,

$$\sum_{|J|\le k, |K|\le k-1} \int_0^T \int \Big(\frac{|\partial Z^J H_1|^2}{1+t}\, w\, dx dt + \frac{|\partial Z^K H_1|^2}{(1+t)^{1-2C\varepsilon}}\Big) w\, dx dt \\ + \int_0^T \int |\bar\partial Z^J H_1|^2\, w'\, dx dt \\ \lesssim \sum_{|J|\le k, |K|\le k-1} \int_0^T \int \Big(\frac{|\partial Z^J h^1|^2}{1+t}\, w\, dx dt + \frac{|\partial Z^K h^1|^2}{(1+t)^{1-2C\varepsilon}}\Big) w\, dx dtt \\ + \int_0^T \int |\bar\partial Z^J h^1|^2\, w'\, dx dt + \int_0^T \int \frac{1}{(1+t)} \frac{|Z^J h^1|^2}{(1+|q|)^2}\, w\, dx dtt \\ + \int_0^T \int \frac{1}{(1+t)^{1-2C\varepsilon}} \frac{|Z^K h^1|^2}{(1+|q|)^2}\, w\, dx dt + \varepsilon^2 \\ \lesssim \sum_{|J|\le k, |K|\le k-1} \int_0^T \int \Big(\frac{|\partial Z^J h^1|^2}{1+t}\, w\, dx dt + \frac{|\partial Z^K h^1|^2}{(1+t)^{1-2C\varepsilon}}\Big) w\, dx dtt \\ + \int_0^T \int |\bar\partial Z^J h^1|^2\, w'\, dx dt + \varepsilon^2,$$

where to pass to the last inequality we once again used the Hardy-type inequality of Corollary 13.3.

Returning to (11.21) we now deal with the case $|K| \ge N/2$, which implies that $|J| \le N/2 + 1$ and allows us to use the decay estimates (11.3)–(11.4) for $H_1 = -h^1 + O(h^2)$. Therefore, the contribution of the terms with $|K| \ge N/2$ to $\big|\widetilde{\Box}_1 Z^I h^1 - \widehat{Z}^I \widetilde{\Box}_1 h^1\big|$ can be bounded by

$$\sum_{|K|=|I|}\sum_{|J|=1}\Big(\frac{|Z^J H_1|}{1+t+|q|} + \frac{|Z^J H_1|_{\mathcal{L}\mathcal{L}} + |H_1|_{\mathcal{L}\mathcal{T}}}{1+|q|}\Big)|\partial Z^K h^1| \\ + \sum_{|K|<|I|} \frac{|Z^J H_1|}{1+|q|}|\partial Z^K h^1| \lesssim \sum_{|K|=|I|}\frac{|\partial Z^K h^1|}{1+t} + \sum_{|K|<|I|}\frac{|\partial Z^K h^1|}{(1+t)^{1-C\varepsilon}}$$

and the desired result follows.

To estimate the commutator $\widetilde{\Box}_g Z^I h^1 - \widehat{Z}^I \widetilde{\Box}_g h^1$ it remains to address the term

$$|H_0^{\alpha\beta}\partial_\alpha\partial_\beta Z^I h^1 - \widehat{Z}^I\big(H_0^{\alpha\beta}\partial_\alpha\partial_\beta Z^I h^1\big)| \le \varepsilon\frac{1}{1+t+|q|}\sum_{|J|\le|I|} C_J\frac{|\partial Z^J h^1|}{1+|q|}.$$

Therefore,

$$\int_0^T\int |H_0^{\alpha\beta}\partial_\alpha\partial_\beta Z^I h^1 - \widehat{Z}^I\big(H_0^{\alpha\beta}\partial_\alpha\partial_\beta Z^I h^1\big)|^2(1+t)\,w\,dxdt \\ \lesssim \varepsilon^2\sum_{|J|\le|I|}\int_0^T\int\frac{|\partial Z^J h^1|^2}{1+t}\,w\,dxdt. \qquad \Box$$

12. **Appendix A. Commutators**

Recall the family of vector fields

$$\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha\beta} = -x_\alpha\partial_b + x_\beta\partial_\alpha, S = t\partial_t + r\partial_r\}.$$

In this section we address commutation properties of the family $\mathcal{Z}$ with various differential structures. Recall that for any $Z \in \mathcal{Z}$ we have $[Z, \Box] = -c_Z\Box$, where c_Z is different from zero only for the scaling vector field S, $c_S = 2$.

LEMMA 12.1. *Let $Z \in \mathcal{Z}$ and let the constants c_α^μ be defined by*

$$[\partial_\alpha, Z] = c_\alpha{}^\mu\partial_\mu, \qquad c_\alpha{}^\mu = \partial_\alpha Z^\mu.$$

Then $c_{LL} = c^{\underline{LL}} = 0$. In addition, if Q is a null form, then

$$ZQ(\partial\phi,\partial\psi) = Q(\partial\phi,\partial Z\psi) + Q(\partial Z\phi,\partial\psi) + \tilde{Q}(\partial\phi,\partial\psi) \tag{12.1}$$

for some null form $\tilde{Q}$ on the right-hand side.

Proof. Since $Z = Z^\alpha \partial_\alpha$ is a Killing or conformally Killing vector field we have

$$\partial_\alpha Z_\beta + \partial_\beta Z_\alpha = f m_{\alpha\beta} \tag{12.2}$$

where $Z_\alpha = m_{\alpha\beta} Z^\beta$. In fact, for the vector fields above, $f = 0$ unless $Z = S$ in which case $f = 2$. In particular,

$$L^\alpha L^\beta \partial_\alpha Z_\beta = 0.$$

If ${c_\alpha}^\mu$ is as defined above and $c_{\alpha\beta} = {c_\alpha}^\mu m_{\mu\beta} = \partial_\alpha Z_\beta$, then the above simply means that $c_{LL} = c^{\underline{LL}} = 0$, which proves the first part of the lemma. To verify (12.1) we first consider the null form $Q = Q_{\alpha\beta}$ We have

$$\begin{aligned} ZQ_{\alpha\beta}(\partial\phi, \partial\psi) &= Q_{\alpha\beta}(\partial Z\phi, \partial\psi) + Q_{\alpha\beta}(\partial\phi, \partial Z\psi) + [Z, \partial_\alpha]\phi\partial_\beta\psi \\ &\quad - \partial_\beta\phi[Z, \partial_\alpha]\psi + [Z, \partial_\beta]\phi\partial_\alpha\psi - \partial_\alpha\phi[Z, \partial_\beta]\psi \\ &= Q_{\alpha\beta}(\partial Z\phi, \partial\psi) + Q_{\alpha\beta}(\partial\phi, \partial Z\psi) \\ &\quad - c_\alpha^\mu(\partial_\mu\phi\partial_\beta\psi - \partial_\beta\phi\partial_\mu\psi) - c_\beta^\mu(\partial_\mu\phi\partial_\alpha\psi - \partial_\alpha\phi\partial_\mu\psi) \\ &= Q_{\alpha\beta}(\partial Z\phi, \partial\psi) + Q_{\alpha\beta}(\partial\phi, \partial Z\psi) \\ &\quad - c_\alpha^\mu Q_{\mu\beta}(\partial\phi, \partial\psi) - c_\beta^\mu Q_{\mu\alpha}(\partial\phi, \partial\psi). \end{aligned}$$

The calculation for the null form $Q_0(\partial\phi, \partial\psi) = m^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi$ is similar and we leave it to the reader. □

For any symmetric 2-tensor π and a vector field $Z \in \mathcal{Z}$ define

$$\pi_Z^{\alpha\beta}\partial_\alpha\partial_\beta = \pi^{\alpha\beta}[\partial_\alpha\partial_\beta, Z], \qquad \text{i.e.,} \qquad \pi_Z^{\alpha\beta} := \pi^{\alpha\gamma}{c_\gamma}^\beta + \pi^{\gamma\beta}{c_\gamma}^\alpha. \tag{12.3}$$

LEMMA 12.2. *The tensor coefficients $\pi_Z^{\alpha\beta}$ verify the following estimate*:

$$|\pi_Z|_{LL} \le 2|\pi|_{L\mathcal{T}}. \tag{12.4}$$

In general,

(12.5)

$$\begin{aligned} [\pi^{\alpha\beta}\partial_\alpha\partial_\beta, Z^I] &= \sum_{I_1+I_2=I,\, |I_2|<|I|} \pi^{I_1\alpha\beta}\partial_\alpha\partial_\beta Z^{I_2}, \\ \pi^{J\alpha\beta} &:= \sum_{|K|\le|J|} c_{K\mu\nu}^{J\alpha\beta} Z^K(\pi^{\mu\nu}) \\ &= -Z^J(\pi^{\alpha\beta}) - \sum_{K+Z=J} Z^K \pi_Z^{\alpha\beta} + \sum_{|K|\le|J|-2} d_{K\mu\nu}^{J\alpha\beta} Z^K(\pi^{\mu\nu}) \end{aligned}$$

for some constants $c_{M\mu\nu}^{J\alpha\beta}$ *and* $d_{M\mu\nu}^{J\alpha\beta}$. *Here the sum* (12.5) *means the sum over all possible order preserving partitions of the multi-index* I *into multi-indices* I_1, I_2.

Proof. First observe that since the vector fields Z are linear in t and x we have

$$[\partial^2_{\alpha\beta}, Z] = [\partial_\beta, Z]\partial_\alpha + [\partial_\alpha, Z]\partial_\beta = c_\beta{}^\gamma \partial_\gamma\partial_\alpha + c_\alpha{}^\gamma\partial_\gamma\partial_\beta,$$

which proves the first statement, while the second follows since $c_L{}^L = 0$.

To prove (12.5) we first write

$$Z^I\big(\pi^{\alpha\beta}\partial_\alpha\partial_\beta\phi\big) = \sum_{K+J=I} (Z^K\pi^{\alpha\beta})Z^J\big(\partial_\alpha\partial_\beta\phi\big).$$

Then we observe that

$$\text{(12.6)}\quad Z^J\partial_\alpha\partial_\beta\phi = \sum_{J_1+J_2=J,\, J_1=(\iota_{\ell_1},\dots,\iota_{\ell_n})} \times\Big[Z^{\iota_{\ell_1}},\Big[Z^{\iota_{\ell_2}},\Big[\dots,\Big[Z^{\iota_{\ell_{n-1}}},[Z^{\iota_{\ell_n}},\partial^2_{\alpha\beta}]\Big]\dots\Big]\Big]\Big]Z^{J_2}\phi,$$

where the sum is over all order preserving partitions of $(1,\dots,k)$ into two ordered sequences $(\ell_1,\dots,\ell_n)$ and $(\ell_{n+1},\dots,\ell_k)$ such that $J_2 = (\iota_{\ell_{n+1}},\dots,\iota_{\ell_k})$. It therefore follows that

$$\pi^{J\alpha\beta} = -\sum_{K+L=J,\, L=(\iota_1,\dots,\iota_l)} \times(Z^K\pi^{\alpha\beta})\Big[Z^{\iota_1},\Big[Z^{\iota_2},\Big[\dots,\Big[Z^{\iota_{l-1}},[Z^{\iota_l},\partial^2_{\alpha\beta}]\Big]\dots\Big]\Big]\Big].$$

The desired representation follows after taking into account that

$$(Z^K\pi^{\alpha\beta})[Z,\partial^2_{\alpha\beta}] = -(Z^K\pi_Z^{\alpha\beta})\partial_\alpha\partial_\beta. \qquad \square$$

For a symmetric 2-tensor H and a vector field $Z\in\mathcal{Z}$ we set $\widehat{Z} = Z + c_Z$, where c_Z is the constant in the commutator $[Z,\square] = -c_Z\square$, and

$$\text{(12.7)}\quad \hat{H}^{J\alpha\beta} = \sum_{|M|\le|J|} c_{M\mu\nu}^{J\alpha\beta}\widehat{Z}^M H^{\mu\nu} = -\widehat{Z}^J H^{\alpha\beta} - \sum_{M+Z=J}\widehat{Z}^M H_Z^{\alpha\beta} + \sum_{|M|\le|J|-2} d_{M\mu\nu}^{J\alpha\beta}\widehat{Z}^M H^{\mu\nu}.$$

COROLLARY 12.3. *Let* $\widetilde{\square}_g = \square + H^{\alpha\beta}\partial_\alpha\partial_\beta$. *Then*

$$\text{(12.8)}\quad \widetilde{\square}_g Z\phi - \widehat{Z}\widetilde{\square}_g\phi = -(\widehat{Z}H^{\alpha\beta} + H_Z^{\alpha\beta})\partial_\alpha\partial_\beta\phi,$$

$$\text{(12.9)}\quad \big|\widetilde{\square}_g Z\phi - \widehat{Z}\widetilde{\square}_g\phi\big| \lesssim \bigg(\frac{|ZH|+|H|}{1+t+|q|} + \frac{|ZH|_{LL}+|H|_{L\mathcal{T}}}{1+|q|}\bigg)\sum_{|I|\le 1}|\partial Z^I\phi|.$$

In general,

$$\widetilde{\Box}_g Z^I\phi - \widehat{Z}^I\widetilde{\Box}_g\phi = -\sum_{I_1+I_2=I,\,|I_2|<|I|} \hat{H}^{I_1\alpha\beta}\partial_\alpha\partial_\beta Z^{I_2}\phi, \tag{12.10}$$

$$\begin{aligned}|\widetilde{\Box}_g Z^I\phi - \widehat{Z}^I\widetilde{\Box}_g\phi| \lesssim{}& \frac{1}{1+t+|q|}\sum_{|K|\le|I|,}\ \sum_{|J|+(|K|-1)_+\le|I|} |Z^J H|\,|\partial Z^K\phi| \\ &+\frac{1}{1+|q|}\sum_{|K|\le|I|}\Bigg(\sum_{|J|+(|K|-1)_+\le|I|}|Z^J H|_{LL} + \sum_{|J'|+(|K|-1)_+\le|I|-1}|Z^{J'}H|_{L\mathcal{T}} \\ &+\sum_{|J''|+(|K|-1)_+\le|I|-2}|Z^{J''}H|\Bigg)|\partial Z^K\phi|,\end{aligned} \tag{12.11}$$

where $(|K|-1)_+ = |K|-1$ *if* $|K|\ge 1$ *and* $(|K|-1)_+ = 0$ *if* $|K|=0$.

Proof. First observe that

$$\begin{aligned}\widehat{Z}\widetilde{\Box}_g\phi &= (Z+c_Z)\Box\phi + (Z+c_Z)H^{\alpha\beta}\partial^2_{\alpha\beta}\phi \\ &= \Box Z\phi + H^{\alpha\beta}\partial^2_{\alpha\beta}Z\phi + (ZH^{\alpha\beta})\partial^2_{\alpha\beta}\phi + (H_Z^{\alpha\beta}+c_Z H^{\alpha\beta})\partial^2_{\alpha\beta}\phi \\ &= \widetilde{\Box}_g Z\phi + (ZH^{\alpha\beta})\partial^2_{\alpha\beta}\phi + (H_Z^{\alpha\beta}+c_Z H^{\alpha\beta})\partial^2_{\alpha\beta}\phi.\end{aligned}$$

Recall now that the constant c_Z is different from 0 only in the case of the scaling vector field S. Moreover, in that case

$$H_S^{\alpha\beta} + c_S H^{\alpha\beta} = 0.$$

The inequality (12.9) now follows from (12.8), (12.4) and the estimate (5.7). The general commutation formula (12.10) follows from the following calculation, similar to the one in Lemma 12.2. We have

$$\widehat{Z}^I\widetilde{\Box}_g\phi = \widehat{Z}^I\Box\phi + \widehat{Z}^I H^{\alpha\beta}\partial^2_{\alpha\beta}\phi = \Box Z^I\phi + \sum_{J+K=I}\hat{Z}^J H^{\alpha\beta} Z^K\partial^2_{\alpha\beta}\phi.$$

If we now use (12.6) we get (12.10) as in the proof of Lemma 12.2. The inequality (12.11) now follows from (12.10), (12.4) and the estimate (5.7). □

13. Appendix B. Hardy-type inequality

In this section we prove a version of the classical three dimensional Hardy inequality

$$\int_{\mathbb{R}^3}\frac{|f(x)|^2}{|x|^2}\,dx \le 4\int_{\mathbb{R}^3}|\nabla f(x)|^2 dx.$$

The Hardy inequality converts the weighted L^2 norm of the function f into the L^2 norm of its gradient. These type of estimates prove to be useful in the context of energy estimates for solutions of a quasilinear wave equation $\Box_{g(\phi)}\phi = F$, where the energy norms contain only the derivatives of ϕ while the error terms, generated by the metric $g(\phi)$ also depend on the solution ϕ itself. The disadvantage of the classical Hardy inequality in this context is that it requires a costly weight r^{-2}. The "cost" here refers to the rate of decay of the weight in the wave zone $r \approx t$. Therefore we seek a version of the Hardy inequality with the weight dependent on the distance to the cone $r = t$ rather than the origin $r = 0$.

LEMMA 13.1. *Let $0 \le \alpha \le 2$, $1 + \mu > 0$ and $\gamma > 0$. Then for any function $u \in C_0^1([0,\infty))$ and an arbitrary $t \ge 0$ there is a constant C, depending on a lower bound for $\gamma > 0$ and $1 + \mu > 0$, such that*

(13.1)
$$\int_0^t \frac{u^2}{(1+|r-t|)^{2+\mu}} \frac{r^2\,dr}{(1+t+r)^\alpha} + \int_t^\infty \frac{u^2}{(1+|r-t|)^{1-\gamma}} \frac{r^2\,dr}{(1+t+r)^\alpha}$$
$$\le C\int_0^t \frac{|\partial_r u|^2}{(1+|r-t|)^{\mu}} \frac{r^2}{(1+t+r)^\alpha}\,dr + C\int_t^\infty |\partial_r u|^2 \frac{(1+|r-t|)^{1+\gamma}}{(1+t+r)^\alpha} r^2\,dr.$$

Remark 13.2. The inequality (13.1) with $\alpha = 0$ appears to be the precise analogue of the classical (spherically-symmetric) Hardy inequality. The presence of the additional growing weight $(1+|t-r|)^{1+\gamma}$ seems to be necessary and in fact fits perfectly in the context of our weighted energy estimates. In the previous work ([LR05]) the estimates used to convert a weighted norm of a solution into a norm of its derivative, e.g.,

$$\int_0^\infty \frac{u^2}{(1+|r-t|)^{2+\mu}} r^2 dr \le C\Big((1+t)^2|u(1+t)|^2 + \int_0^\infty \frac{|\partial_r u|^2}{(1+|r-t|)^{2+\mu}} r^2 dr\Big)$$

were more reminiscent of the classical Poincaré inequality.

Proof. Set

$$m(q) = \begin{cases} (1+q)^\gamma, & q > 0,\\ (1-q)^{-1-\mu}, & q \le 0.\end{cases}$$

Then

$$m'(q) = \begin{cases} \gamma(1+q)^{-1+\gamma}, & q \ge 0,\\ (1+\mu)(1-q)^{-2-\mu}, & q \le 0.\end{cases}$$

Since $\alpha \le 2$

$$\partial_r\Big(r^2(1+t+r)^{-\alpha} m(r-t)\Big)$$
$$= \Big(\frac{2}{r} - \frac{\alpha}{1+t+r} + \frac{m'(r-t)}{m(r-t)}\Big)\frac{r^2 m(r-t)}{(1+t+r)^\alpha} \ge \frac{r^2\, m'(r-t)}{(1+t+r)^\alpha}.$$

Hence

$$\partial_r\Big(r^2(1+t+r)^{-\alpha}m(r-t)\phi^2\Big) \geq m'(r-t)r^2(1+t+r)^{-\alpha}\phi^2 \\ + 2r^2(1+t+r)^{-\alpha}m(r-t)\phi\partial_r\phi.$$

If we integrate the above from 0 to ∞ and use that ϕ has compact support we see that

$$\int_0^\infty m'(r-t)\phi^2\,\frac{r^2dr}{(1+t+r)^\alpha} \leq 2\int_0^\infty m(r-t)\phi\partial_r\phi\frac{r^2dr}{(1+t+r)^\alpha}.$$

Since $m' > 0$ and $m \geq 0$ it follows from Cauchy-Schwarz inequality that

$$\int_0^\infty m'(r-t)\phi^2\,\frac{r^2dr}{(1+t+r)^\alpha} \leq \sqrt{2}\int_0^\infty \frac{m(r-t)^2}{m'(r-t)}|\partial_r\phi|^2\frac{r^2dr}{(1+t+r)^\alpha}$$

from which the lemma follows. □

COROLLARY 13.3. *Let $\gamma > 0$ and $\mu > 0$ and set, for $q = r - t$,*

$$w(q) = \begin{cases} 1 + (1+|q|)^{1+2\gamma}, & q > 0, \\ 1 + (1+|q|)^{-2\mu}, & q < 0. \end{cases}$$

Then for any $-1 \leq a \leq 1$ and any $\phi \in C_0^\infty(\mathbb{R}^3)$

$$\int \frac{|\phi|^2}{(1+|q|)^2}\,\frac{w\,dx}{(1+t+|q|)^{1-a}} \lesssim \int |\partial\phi|^2\,\frac{w\,dx}{(1+t+|q|)^{1-a}}.$$

If, in addition, $a < 2\min(\gamma,\mu)$,

$$\int\frac{|\phi|^2}{(1+|q|)^2}\,\frac{(1+|q|)^{-a}}{(1+t+|q|)^{1-a}}\,\frac{w\,dx}{(1+q_-)^{2\mu}} \lesssim \int |\partial\phi|^2 \min\big(w', \frac{w}{(1+t+|q|)^{1-a}}\big)\,dx$$

where $q_- = |q|$, when $q < 0$ and $q_- = 0$, when $q > 0$.

14. Appendix C. Weighted Klainerman-Sobolev inequalities

In this section we provide a straightforward generalization of the Klainerman-Sobolev inequalities, expressing pointwise decay in terms of the bounds on L^2 norms involving vector field $Z \in \mathcal{Z}$. We consider energy norms with the following weight function

$$w = w(q) = \begin{cases} 1 + (1+|q|)^{1+2\gamma}, & \text{when } q > 0, \\ 1 + (1+|q|)^{-2\mu}, & \text{when } q < 0 \end{cases}$$

for some $0 < \gamma < 1$. Therefore,

$$w' =: w'(q) = \begin{cases} (1+2\gamma)(1+|q|)^{2\gamma}, & \text{when } q > 0, \\ 2\mu(1+|q|)^{-1-2\mu}, & \text{when } \quad q < 0, \end{cases}$$
$$w' \le 4w(1+|q|)^{-1} \le 16\gamma^{-1} w'(1+q_-)^{2\mu}.$$

We have the following global Sobolev inequality

PROPOSITION 14.1. *For any function $\phi \in C_0^\infty(\mathbb{R}^3)$ and an arbitrary (t,x),*

$$|\phi(t,x)|(1+t+|q|)\big[(1+|q|)w(q)\big]^{1/2} \le C \sum_{|I|\le 3} \|w^{1/2} Z^I \phi(t,\cdot)\|_{L^2}, \quad q = t-r.$$

Proof. First note that it is sufficient to consider two cases when the support of ϕ is in the sets $r \le t/2$ and $t/4 \le r$ respectively. We argue as follows. Let $\chi(\tau)$ be a smooth cut-off function such $\chi(\tau) = 1$ when $\tau \le -3/5$, and $\chi(\tau) = 0$ for $\tau \ge -1/3$. Define

$$\phi_1(t,x) = \chi\big(\frac{r-t}{r+t}\big)\phi(t,x), \qquad \phi_2(t,x) = \Big(1 - \chi\big(\frac{r-t}{r+t}\big)\Big)\phi(t,x).$$

The supports of functions ϕ_1, ϕ_2 then belong to the desired regions and

$$\sum_{|I|\le 3} |Z^I \phi_i| \le C \sum_{|I|\le 3} |Z^I \phi|,$$

since $|Z^I \chi_i| \le C$.

First for $r \le t/2$ consider the rescaled function $c\phi_t(x) = \phi(t, tx)$, where $|x| \le 1/2$ in the support. By the standard Klainerman-Sobolev estimate

$$\|\phi_t(x)\|_{L^\infty_x} \le C \sum_{|\alpha|\le 2} \|\partial^\alpha \phi_t(x)\|_{L^2_x} \le C \sum_{|\alpha|\le 2} \|(t^{|\alpha|}\partial^\alpha \phi)(t, tx)\|_{L^2_x}.$$

Since

$$\sum_{|\alpha|\le k} |t-r|^{|\alpha|} |\partial^\alpha \phi(t,x)| \le C \sum_{|I|\le k} |Z^I \phi(t,x)|,$$

we obtain that

$$\|\phi(t,x)\|_{L^\infty_x} \le C t^{-3/2} \sum_{|I|\le k} \|Z^I \phi(t,x)\|_{L^2_x}.$$

Multiplying both sides of the above inequality by $[w(-t)]^{1/2}$, which is $\approx [w(q)]^{1/2}$ when $r \le t/2$, proves the proposition in the case when ϕ is supported in the region $r \le t/2$.

When $r \geq t/4$ we can estimate the L^∞ norm by the L^1 norms of derivatives:

$$|\phi(t,x)|^2(1+t+|q|)^2 w(q)(1+|q|) \\ \leq C \sum_{|\alpha|\leq 2} \int_{S^2} \int \Big|\partial_\omega^\alpha \partial_q \Big(w(q)(1+|q|)(t+q)^2 \phi\big(t,(t+q)\omega\big)^2\Big)\Big| \, dq\, dS(\omega).$$

Since $|w'(q)|(1+|q|) \leq C w(q)$ and $|q| \leq C|t+q| = Cr$ on the support of ϕ, it follows that we have the bound

$$\sum_{|\alpha|\leq 2, k=0,1} \int_{S^2} \int w(q)(t+q)^2 \Big| (q\partial_q)^k \partial_\omega^\alpha \phi\big(t,(t+q)\omega\big)\Big|^2 \, dq\, dS(\omega) \\ \leq C \sum_{|I|\leq 3} \int_{\mathbf{R}^3} w(q)|Z^I \phi(t,x)|^2 \, dx. \qquad \square$$

15. Appendix D. Wave coordinate condition

In this section we provide the details on the estimates following from the wave coordinate condition

$$\partial_\mu\Big(g^{\mu\nu}\sqrt{|\det g|}\Big) = 0 \tag{15.1}$$

for a Lorentzian metric g in a coordinate system $\{x^\mu\}_{\mu=0,\ldots,3}$, leading up to the proof of Proposition 8.2. We recall the definition of the tensor $H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$ and below state the consequences of (15.1) in terms of estimates for H.

We first observe that for any vector field X and a collection of our special vector fields $Z \in \mathcal{Z}$ we have that

$$Z^I \partial_\alpha X^\alpha = \partial_\alpha\Big(Z^I X^\alpha + \sum_{|J|<|I|} c^{I}_{J}{}^{\alpha}_{\gamma} Z^J X^\gamma\Big) = \partial_\alpha\Big(\sum_{|J\leq|I|} c^{I}_{J}{}^{\alpha}_{\gamma} Z^J X^\gamma\Big), \tag{15.2}$$

where $c_{J}{}^{\alpha}_{\gamma}$ are constants such that

$$c^{I}_{J}{}^{\alpha}_{\gamma} = \delta^\alpha_\gamma, \quad \text{for} \quad |J| = |I| \qquad \text{and} \qquad c^{I}_{J}{}^{\underline{L}}_{L} = 0, \quad \text{for} \quad |J| = |I|-1.$$

The last identity is a consequence of the relation between $c^{I}_{J}{}^{\gamma}_{\alpha}$ and the commutator constants $c_{\alpha\beta} = [\partial_\alpha, Z]_\beta$ for which we have established in Lemma 12.1 that $c_{\underline{L}L} = 0$. It therefore follows from (15.1) and (15.2) that

$$H^{[I]}_{\mu\nu} := Z^I \widetilde{H}_{\mu\nu} + \sum_{|J|<|I|} c^{I}_{J}{}^{\gamma}_{\mu} Z^J \widetilde{H}_{\gamma\nu}, \qquad \text{with} \quad \widetilde{H}_{\mu\nu} := H_{\mu\nu} - \frac{m_{\mu\nu}}{2}\operatorname{tr} H, \tag{15.3}$$

satisfies

$$\partial_\mu H^{[I]\mu\nu} + Z^I \partial_\mu O^{\mu\nu}(H^2) = 0, \qquad \text{where} \quad O^{\mu\nu}(H^2) = O(|H|^2). \tag{15.4}$$

LEMMA 15.1. *Let H be a 2-tensor and let $H^{[I]}$ be defined by* (15.3). *Then, for $j=0,1$;*

$$\sum_{|I|\le k} |\partial^j Z^I H|_{LL} + \sum_{|J|\le k-1} |\partial^j Z^J H|_{L\mathcal{T}} \lesssim \sum_{|K|\le k-2} |\partial^j Z^K H| + \sum_{|I|\le k} |\partial^j H^{[I]}|_{L\mathcal{T}}. \tag{15.5}$$

Proof. It is easy to see that (15.3) implies that for any α

$$\sum_{|I|\le k} |\partial^\alpha Z^I H|_{LL} \lesssim \sum_{|J|\le k-1} |\partial^\alpha Z^J H|_{L\mathcal{T}} + \sum_{|K|\le k-2} |\partial^\alpha Z^K H| + \sum_{|I|\le k} |\partial^\alpha H^{[I]}|_{LL},$$

$$\sum_{|J|\le k-1} |\partial^\alpha Z^J H|_{L\mathcal{T}} \le \sum_{|K|\le k-2} |\partial^\alpha Z^K H| + \sum_{|J|\le k-1} |\partial^\alpha H^{[J]}|_{L\mathcal{T}}$$

and (15.5) follows. □

LEMMA 15.2. *Let Z represent Minkowski Killing or conformally Killing vector fields from our family $\mathfrak{Z}$ and assume the tensor H satisfies the wave coordinate condition* (15.1). *Then for any multi-index I tensor $H^{[I]}$, defined in* (15.3), *satisfies the following estimate*

$$\big|\partial H^{[I]}\big|_{L\mathcal{T}} \lesssim \sum_{|J|\le|I|} |\bar\partial Z^J H| + \sum_{I_1+\cdots+I_k=I,\, k\ge 2} |Z^{I_k} H|\cdots|Z^{I_2} H|\,|\partial Z^{I_1} H|.$$

Proof. The wave coordinate condition (8.1) can be written in the form

$$\partial_\mu\big(\widetilde G^{\mu\nu}\big)=0, \qquad \text{where} \quad \widetilde G^{\mu\nu}=(m^{\mu\nu}+H^{\mu\nu})\sqrt{|\det g|}.$$

It follows from (15.2) that

$$\partial_\mu\Big(\sum_{|J|\le|I|} c_J^{I\ \mu\gamma} Z^J \widetilde G_{\gamma\nu}\Big)=0.$$

Decomposing relative to the null frame $(L,\underline{L},A,B)$ we obtain

$$\partial_q\Big(\sum_{|J|\le|I|} c_J^{I\ \underline{L}\gamma} Z^J \widetilde G_{\gamma\nu}\Big)=\partial_s\Big(\sum_{|J|\le|I|} c_J^{I\ L\gamma} Z^J \widetilde G_{\gamma\nu}\Big) - A_\mu\bar\partial_A\Big(\sum_{|J|\le|I|} c_J^{I\ \mu\gamma} Z^J \widetilde G_{\gamma\nu}\Big).$$

We now contract the above identity with one of the tangential vector fields T^ν, $T \in \{L, A, B\}$ to obtain

$$\Big|L^\gamma T^\nu \partial_q Z^I \widetilde{G}_{\gamma\nu} + \sum_{|J|<|I|} c_J^I{}^{L\gamma} T^\nu \partial_q Z^J \widetilde{G}_{\gamma\nu}\Big| \lesssim \sum_{|J|\le|I|} \big|\bar\partial Z^I \tilde{G}\big|.$$

We now examine the expression

$$\begin{aligned} L^\gamma T^\nu Z^J \partial_q \widetilde{G}_{\gamma\nu} &= L^\gamma T^\nu \partial_q Z^J \Big((m_{\gamma\nu} + H_{\gamma\nu})\sqrt{|\det g|}\Big) \\ &= \sum_{J_1+J_2=J} L^\gamma T^\nu \partial_q \Big((Z^{J_1} H_{\gamma\nu}) Z^{J_2} \sqrt{|\det g|}\Big) \end{aligned}$$

since $m_{LT} = L^\gamma T^\nu m_{\gamma\nu} = 0$. The desired estimate now follows from the identity $\sqrt{|\det g|} = 1 + f(H)$, which holds with a smooth function $f(H)$ such that $f(0) = 0$ and $f(H) = -\mathrm{tr}H/2 + O(H^2)$. □

We now summarize the above results in the following

LEMMA 15.3. *Let g be a Lorentzian metric satisfying the wave coordinate condition* (15.1) *relative to a coordinate system $\{x^\mu\}_{\mu=0,\dots,3}$. Then the following estimates for the tensor $H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$ and a multi-index I hold true under the assumption that $|Z^J H| \le C$, for all $|J| \le |I|/2$:*

$$(15.6)\quad |\partial Z^I H|_{L\mathcal{T}} \lesssim \Big(\sum_{|J|\le|I|} |\bar\partial Z^J H| + \sum_{|J|\le|I|-1} |\partial Z^J H| + \sum_{|I_1|+||I_2|\le|I|} |Z^{I_2} H||\partial Z^{I_1} H| \Big),$$

$$(15.7)\quad |\partial Z^I H|_{LL} \lesssim \Big(\sum_{|J|\le|I|} |\bar\partial Z^J H| + \sum_{|J|\le|I|-2} |\partial Z^J H| + \sum_{|I_1|+|I_2|\le|I|} |Z^{I_2} H||\partial Z^{I_1} H| \Big).$$

We now establish that the tensor

$$(15.8)\qquad H_1^{\mu\nu} = H^{\mu\nu} - H_0^{\mu\nu}, \qquad H_0^{\mu\nu} := -\chi\Big(\frac{r}{t}\Big)\chi(r)\frac{M}{r}\delta^{\mu\nu}$$

obtained by subtracting the "Schwarzschild part" H_0 from H obeys similar structure estimates to those of H.

LEMMA 15.4. *Let g be a Lorentzian metric satisfying the wave coordinate condition* (15.1) *relative to a coordinate system $\{x^\mu\}_{\mu=0,\dots,3}$. For a given integer $k \ge 0$ we have the following estimates for the tensor $H_1^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu} - H_0^{\mu\nu}$,* *under the assumption that* $|Z^J H| \le C$, $\forall |J| \le k/2$,

$$\sum_{|I|\le k} \big|\partial Z^I H_1\big|_{\mathcal{L}\mathcal{L}} + \sum_{|I|\le k-1} \big|\partial Z^I H_1\big|_{\mathcal{L}\mathcal{T}} \lesssim \sum_{|I|\le k} |\bar\partial Z^I H_1| + \sum_{|I|\le k-2} |\partial Z^I H_1| + \varepsilon \frac{1}{1+t+|q|} \sum_{|I|\le k} \Big(|\partial Z^I H_1| + \frac{|Z^I H_1|}{1+t+|q|} \Big) + \sum_{|I|+|J|\le k} |Z^I H_1||\partial Z^J H_1| + \frac{C\varepsilon \chi_0(1/2<r/t<3/4)}{(1+t+|q|)^2} + \frac{C\varepsilon^2}{(1+|t|+|q|)^3}, \tag{15.9}$$

where $\chi_0(1/2 < r/t < 3/4)$ *is the characteristic function of the set where* $t/2 < r < 3t/4$.

Proof. We define the tensors

$$H^{[I]}_{\mu\nu} = Z^I \widetilde{H}_{\mu\nu} + \sum_{|J|<|I|} c^{I}_{J}{}^{\gamma}_{\mu} Z^J \widetilde{H}_{\gamma\nu}, \qquad \widetilde{H}_{\mu\nu} = H_{\mu\nu} - \frac{m_{\mu\nu}}{2}\mathrm{tr} H,$$

$$H_0{}^{[I]}_{\mu\nu} = Z^I \widetilde{H_0}_{\mu\nu} + \sum_{|J|<|I|} c^{I}_{J}{}^{\gamma}_{\mu} Z^J \widetilde{H_0}_{\gamma\nu}, \qquad \widetilde{H_0}_{\mu\nu} = H_{0\mu\nu} - \frac{m_{\mu\nu}}{2}\mathrm{tr} H_0,$$

$$H_1{}^{[I]}_{\mu\nu} = Z^I \widetilde{H_1}_{\mu\nu} + \sum_{|J|<|I|} c^{I}_{J}{}^{\gamma}_{\mu} Z^J \widetilde{H_1}_{\gamma\nu}, \qquad \widetilde{H_1}_{\mu\nu} = H_{1\mu\nu} - \frac{m_{\mu\nu}}{2}\mathrm{tr} H_1$$

with the constants $c^{I}_{J}{}^{\gamma}_{\mu}$ such that $c^{I}_{J}{}^{\underline{L}}_{L} = 0$, if $|J| = |I| - 1$. Observe that the tensor $H_0{}^{[I]}_{\mu\nu}$ is defined in such a way that

$$\partial^\mu H_0{}^{[I]}_{\mu\nu} = Z^I \Big(\partial^\mu \widetilde{H_0}_{\mu\nu} \Big).$$

Since the tensor $g^{\mu\nu}$ satisfies the wave coordinate condition it follows that

$$\partial^\mu H^{[I]}_{\mu\nu} + Z^I \partial_\mu O^{\mu\nu}(H^2) = 0, \qquad \text{where} \quad O^{\mu\nu}(H^2) = O(|H|^2). \tag{15.10}$$

On the other hand, calculating using the definition of H_0,

$$\partial^\mu \widetilde{H_0}_{\mu\nu} = 2\chi'(r/t)\chi(r)M/t^2 \delta_{\nu 0}.$$

Therefore, since $H_{\mu\nu} = H_{0\mu\nu} + H_{1\mu\nu}$ and $H^{[I]}_{\mu\nu} = H_0{}^{[I]}_{\mu\nu} + H_1{}^{[I]}_{\mu\nu}$ we obtain

$$\partial^\mu H_1{}^{[I]}_{\mu\nu} = -Z^I \partial_\mu O^{\mu\nu}(H^2) + 2Z^I \big(\chi'(r/t)\chi(r)M/t^2\big)\delta_{\nu 0}.$$

Arguing as in the proof of Lemma 15.2 we derive

$$
\begin{aligned}
(15.11)\qquad &\sum_{|I|\le k} \big|\partial Z^I H_1\big|_{\mathcal{L}\mathcal{L}} + \sum_{|I|\le k-1} \big|\partial Z^I H_1\big|_{\mathcal{L}\mathcal{T}} \\
&\lesssim \sum_{|I|\le k} \big|\partial H_1^{[I]}\big|_{L\mathcal{T}} + \sum_{|I|\le k-2} |\partial Z^I H_1| \\
&\lesssim \sum_{|I|\le k} |\bar\partial Z^I H_1| + \sum_{|I|\le k}\ \sum_{I_1+\cdots+I_n=I,\, n\ge 2} |Z^{I_n} H|\cdots|Z^{I_2}H|\,|\partial Z^{I_1} H| \\
&\quad + \sum_{|I|\le k-2} |\partial Z^I H_1| + \frac{C\varepsilon\chi_0(1/2<r/t<3/4)}{(1+t+|q|)^2},
\end{aligned}
$$

where $\chi_0(1/2<r/t<3/4)$ is the characteristic function of the set where $t/2 < r < 3t/4$. Using the condition $|Z^I H|\lesssim 1$, $\forall |I|\le k/2$, and the estimates

$$
|Z^I H_0| + (1+t+|q|)|\partial Z^I H_0| \le \frac{\varepsilon}{1+t+|q|}
$$

we obtain

(15.12)

$$
\begin{aligned}
&\sum_{I_1+\cdots+I_n=I,\, n\ge 2} |Z^{I_n} H|\cdots|Z^{I_2}H|\,|\partial Z^{I_1} H| \lesssim \sum_{|J|+|K|\le |I|} |Z^J H|\,|\partial Z^K H| \\
&\lesssim \sum_{|J|+|K|\le |I|} |Z^J H_0|\,|\partial Z^K H_1| + |Z^J H_1|\,|\partial Z^K H_0| \\
&\qquad\qquad + |Z^J H_1|\,|\partial Z^K H_1| + |Z^J H_0|\,|\partial Z^K H_0| \\
&\lesssim \sum_{|J|\le |I|} \varepsilon\frac{1}{1+t+|q|}\Big(|\partial Z^J H_1| + \frac{|Z^J H_1|}{1+t+|q|}\Big) + \frac{\varepsilon^2}{(1+t+|q|)^3}
\end{aligned}
$$

and the lemma follows. □

16. Appendix E. $L^1 - L^\infty$-estimates and additional decay

The bounds

(16.1)

$$
\mathcal{E}_N(t) = \sum_{|I|\le N} \|w^{1/2}\partial Z^I h^1(t,\cdot)\|_{L^2} + \|w^{1/2}\partial Z^I \psi(t,\cdot)\|_{L^2} \le C_N(1+t)^{C_N\varepsilon},
$$

with

$$
(16.2)\qquad w^{1/2} = \begin{cases} (1+|r-t|)^{1/2+\gamma}, & r>t,\\ 1, & r\le t \end{cases}
$$

imply, with the help of the weighted Klainerman-Sobolev inequality, that

$$
(16.3)\quad |\partial Z^I h^1| + |\partial Z^I \psi| \\
\le \begin{cases} C_N' \varepsilon (1+t+r)^{-1+C_N\varepsilon}(1+|t-r|)^{-1-\gamma}, & r>t, \\ C_N' \varepsilon (1+t+r)^{-1+C_N\varepsilon}(1+|t-r|)^{-1/2}, & r\le t, \end{cases} \qquad |I|\le N-2.
$$

The exponent in the interior decay estimate (16.3) in $|t-r|$ can be improved from $-1/2$ to -1 using the estimate

$$
(16.4)\qquad |Z^I h^1| + |Z^I \psi| \le C_N'' \varepsilon (1+t)^{-1+2C_N\varepsilon}, \qquad |I| \le N-3
$$

that we will now prove. We will divide the solution of Einstein's equations into a linear and a nonlinear part with vanishing initial data $h^1_{\mu\nu} = v_{\mu\nu} + w_{\mu\nu}$. The estimate for the linear part follows from:

LEMMA 16.1. *If v is the solution of*

$$
\Box\, v = 0, \qquad v\big|_{t=0} = v_0, \quad \partial_t v\big|_{t=0} = v_1,
$$

then for any $\gamma > 0$;

$$
(16.5)\\
(1+t)|v(t,x)| \le C \sup_x \big((1+|x|)^{2+\gamma}(|v_1(x)| + |\partial v_0(x)|) + (1+|x|)^{1+\gamma}|v_0(x)| \big).
$$

Proof. The proof is an immediate consequence of the Kirchhoff's formula

$$
v(t,x) = t\int_{|\omega|=1} \big(v_1(x+t\omega) + \langle v_0'(x+t\omega),\omega\rangle \big)\, dS(\omega) \\
+ \int_{|\omega|=1} v_0(x+t\omega)\, dS(\omega),
$$

where $dS(\omega)$ is the normalized surface measure on S^2. Suppose that $x = r\mathbf{e}_1$, where $\mathbf{e}_1 = (1,0,0)$. Then for $k = 1,2$ we must estimate

$$
(16.6)\quad \int \frac{dS(\omega)}{1+|r\mathbf{e}_1 + t\omega|^{k+\gamma}} = \int_{-1}^{1} \frac{C\, d\omega_1}{1+\big((r-t\omega_1)^2 + t^2(1-\omega_1^2)\big)^{(k+\gamma)/2}} \\
\le \int_0^2 \frac{C\, ds}{1+\big((r-t+ts)^2 + t^2 s\big)^{(k+\gamma)/2}}.
$$

If $k = 2$ we make the change of variables $t^2 s = \tau$ to get an integral bounded by Ct^{-2} and if $k = 1$, we make the change of variables $ts = \tau$ to get an integral bounded by t^{-1}. $\Box$

To estimate the nonlinear part we use Hörmander's $L^1 - L^\infty$-estimates for the fundamental solution of $\Box$, see [Hör87], [Lin90b]:

PROPOSITION 16.2. *If v be the solution of*

$$\square\, w = g, \qquad w\big|_{t=0} = \partial_t w\big|_{t=0} = 0,$$

then

$$|w(t,x)|(1+t+|x|) \le C \sum_{|I|\le 2} \int_0^t \int_{\mathbf{R}^3} \frac{|Z^I g(s,y)|}{1+s+|y|}\, dy\, ds. \tag{16.7}$$

Let $h^1_{\mu\nu} = v_{\mu\ nu} + w_{\mu\nu}$ where,

$$\square w_{\mu\nu} = -H^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + F_{\mu\nu}(h)(\partial h, \partial h), \qquad w_{\mu\nu}\big|_{t=0} = \partial_t w_{\mu\nu}\big|_{t=0} = 0, \tag{16.8}$$

and

$$\square v_{\mu\nu} = 0, \qquad v_{\mu\nu}\big|_{t=0} = h^1_{\mu\nu}\big|_{t=0}, \qquad \partial_t v_{\mu\nu}\big|_{t=0} = \partial_t h^1_{\mu\nu}\big|_{t=0}. \tag{16.9}$$

We have

$$|Z^I F_{\mu\nu}(h)(\partial h, \partial h)| \le C \sum_{|J|+|K|\le|I|} |\partial Z^J h|\, |\partial Z^K h| + C \sum_{|J|+|K|\le|I|} \frac{|Z^J h|}{1+|q|}\, |\partial Z^K h|, \tag{16.10}$$

and since $H^{\alpha\beta} = -h_{\alpha\beta} + O(h^2)$,

$$\big|Z^I\big(H^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}\big)\big| \le C \sum_{|J|+|K|\le|I|+1,\, |J|\le|I|} \frac{|Z^J h|}{1+|q|}\, |\partial Z^K h|. \tag{16.11}$$

Now

$$\int |\partial Z^J h|\, |\partial Z^K h|(s,y)\, dy \le \sum_{|I|\le N} \|\partial Z^I h(s,\cdot)\|^2_{L^2} \le C(1+s)^{2C_N\varepsilon}. \tag{16.12}$$

We write $h = h^0 + h^1$ and estimate

$$\int \frac{|h^0(s,y)|^2}{(1+|q|)^2}\, dy \le M^2 \int_0^\infty \frac{r^2\, dr}{(1+|t+r|)^2(1+|t-r|)^2} \le CM^2 \tag{16.13}$$

and by Corollary 13.3

$$\int \frac{|h^1(s,y)|^2}{(1+|q|)^2}\, dy \le C \int |\partial h^1(s,y)|^2 w(q)\, dy \le C_N^2 \varepsilon^2 (1+t)^{2C_N\varepsilon}, \tag{16.14}$$

where w is as in (16.2). Hence

$$\int \frac{|Z^J h|}{1+|q|}\, |\partial Z^K h|(s,y)\, dy \le C\varepsilon^2(1+t)^{2C_N\varepsilon}. \tag{16.15}$$

It now follows from Proposition 16.2 that

$$|w_{\mu\nu}(t,x)|(1+t+|x|) \leq \int_0^t \frac{\varepsilon^2\, ds}{(1+s)^{1-2C_N\varepsilon}} \leq C\varepsilon(1+s)^{2C_N\varepsilon} \tag{16.16}$$

which proves (16.4).

Acknowledgments. The authors would like to thank Sergiu Klainerman for valuable suggestions and discussions.

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(Received December 20, 2004)

E-mail address: lindblad@math.ucsd.edu
UNIVERSITY OF CALIFORNIA, SAN DIEGO, DEPARTMENT OF MATHEMATICS,
LA JOLLA CA 92093-0112, UNITED STATES

E-mail address: irod@math.princeton.edu
PRINCETON UNIVERSITY, DEPARTMENT OF MATHEMATICS, FINE HALL–WASHINGTON ROAD,
PRINCETON NJ 08544, UNITED STATES