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Abstract

This is one of a series of papers examining the interplay between differentiation theory for Lipschitz maps $X \to V$ and bi-Lipschitz nonembeddability, where X is a metric measure space and V is a Banach space. Here, we consider the case $V = L^1$, where differentiability fails. We establish another kind of differentiability for certain X, including $\mathbb{R}^n$ and $\mathbb{H}$, the Heisenberg group with its Carnot-Carathéodory metric. It follows that $\mathbb{H}$ does not bi-Lipschitz embed into L^1, as conjectured by J. Lee and A. Naor. When combined with their work, this provides a natural counterexample to the Goemans-Linial conjecture in theoretical computer science; the first such counterexample was found by Khot-Vishnoi [KV05].

A key ingredient in the proof of our main theorem is a new connection between Lipschitz maps to L^1 and functions of bounded variation, which permits us to exploit results on the structure of BV functions on the Heisenberg group [FSSC01].

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1. Introduction

1.1. *Overview.* The interplay between differentiability and bi-Lipschitz nonembeddability is the common theme of this paper and [CK06b], [CK08b], [CK06c]; see also [CK06a], [CK08a].

Specifically, we are concerned with the case in which the target is an infinite-dimensional Banach space and the domain is a complete metric measure space for which the measure satisfies a doubling condition and a Poincaré inequality holds in the sense of upper gradients; see [HK96]. Such metric measure spaces will be referred to as PI *spaces*.

In [CK08b], the differentiability theorem for real-valued Lipschitz functions on all PI spaces and the resulting bi-Lipschitz nonembedding theorem for certain PI spaces, proved in [Che99] for finite-dimensional Banach space targets, are extended to the class of infinite-dimensional Banach space targets with the Radon-Nikodym Property. A Banach space V is said to have the *Radon-Nikodym Property* if every Lipschitz map from the real line to V is differentiable almost everywhere; see [BL00, Ch. 5]. Hence, the differentiation result of [CK08b] is optimal. Domains covered by the nonembedding theorem include Bourdon-Pajot spaces, Laakso spaces (which are PI spaces of topological dimension 1, for which the Hausdorff dimension can be any real number > 1, [Laa00]) and the Heisenberg group, $\mathbb{H}$, with its Carnot-Carathéodory metric $d^{\mathbb{H}}$. (For earlier but less general differentiation and nonembedding results, for a certain class of targets, shown in [CK08a] to consist precisely of separable dual spaces, see [CK06b].)

In the present paper, we examine maps $X \to L^1 = L^1(\mathbb{R}, \mathscr{L})$ where $\mathscr{L}$ denotes Lebesgue measure on the real numbers $\mathbb{R}$. This target does *not* possess the Radon-Nikodym Property. We show that for a special class of PI spaces, including $\mathbb{R}^k$ and $(\mathbb{H}, d^{\mathbb{H}}, \mathscr{L})$, despite the failure of the usual form of differentiability for Lipschitz maps to L^1, a novel form of differentiability does in fact hold. As a direct consequence, it follows that $(\mathbb{H}, d^{\mathbb{H}})$ does not bi-Lipschitz embed in L^1. This proves a conjecture of J. Lee and A. Naor, which provided the motivation for our work. The significance of this conjecture in the context of theoretical computer science is briefly indicated below.

Since L^1 does not have the Radon-Nikodym Property, the nonembedding theorem of [CK08b] does not apply, and as it turns out, the conclusion does not always hold. In [CK06c], it is shown that members of a class of PI spaces, including Laakso spaces, which satisfy the assumptions of the nonembedding theorem of [CK08b] do embed in L^1.

1.2. *Rademacher's theorem and its descendents.* Rademacher's differentiation theorem states that a Lipschitz map $f : \mathbb{R}^k \to \mathbb{R}^l$ is differentiable almost everywhere. Hence, the geometry of such Lipschitz maps becomes rigid at small scale. Specifically, in the limit under suitable rescaling, the map becomes linear. The literature contains numerous extensions of this result, in which either the domain, the target, or the class of maps is generalized. Classical examples include almost everywhere approximate differentiability of Sobolev functions [Zie89], analysis of and on rectifiable sets, almost everywhere differentiability of quasiconformal homeomorphisms between domains in $\mathbb{R}^k$, and almost everywhere differentiability of Lipschitz maps from domains in $\mathbb{R}^k$, to Banach spaces which have the Radon-Nikodym Property.

In many of the recent results in this vein, a significant part of the achievement is to make sense of differentiation in a context where some component of the classical setting is absent, e.g. the infinitesimal affine structure on the domain or target, or a good measure on the domain:

- Pansu's differentiation theorem [Pan89] for Lipschitz maps between graded nilpotent Lie groups. Here, one cannot use Euclidean rescaling; the rescaling procedure has to be adapted to the grading on the groups.
- Differentiation theory for real-valued Lipschitz functions on Banach spaces with a separable dual [LP01]. This requires replacing the classical notion of "almost everywhere" by something else.
- Metric differentiation [Kir94], [Pau01]. The target is an arbitrary metric space with no linear structure, and differentiability refers to a property of the pull-back of the distance function from the target.
- The differentiation theory developed in [Che99] for Lipschitz functions on PI spaces. Typically, these carry no infinitesimal affine structure.

1.3. *Differentiation and bi-Lipschitz nonembeddability.* Since differentiation theorems assert that the small-scale structure of maps is very restricted, one can use them to show that certain mapping problems have no solution. For instance, it was observed by Semmes, [Sem96], that Pansu's differentiation theorem implies that a Lipschitz map $f : U \to \mathbb{R}^k$ where U is an open subset of $(\mathbb{H}, d^{\mathbb{H}})$, cannot be bi-Lipschitz. In another instance, the differentiation theory of [Che99] was applied to give a unified proof of bi-Lipschitz nonembeddability in $\mathbb{R}^k$ of several families of spaces ([Che99, §14]) including Carnot groups such as $(\mathbb{H}, d^{\mathbb{H}})$, Laakso spaces, and the Bourdon-Pajot spaces (of [BP99]).

The paper [CK08b] extends the differentiation theory of [Che99] to Lipschitz maps $X \to V$, where (X, d^X, μ) is a PI space and V is a Banach space with the

Radon-Nikodym Property. As a consequence, the statement and proof of the bi-Lipschitz nonembedding theorem of [Che99, §14], extend verbatim to targets with the Radon-Nikodym Property. This (or even the less general result of [CK06b]) covers Lipschitz maps into arbitrary reflexive spaces (separable or not), such as the L^p-space of an arbitrary measure space (Y, ν), for $1 < p < \infty$, and also maps into the space ℓ^1 of absolutely summable sequences.

In light of the above, the existence of bi-Lipschitz embeddings for certain domains and targets, implies the nonexistence of a differentiation theorem. For example, every metric space X admits a canonical isometric embedding into the space $L^\infty(X)$, namely the Kuratowski embedding, which assigns to $x \in X$, the function $d^X(x, \cdot) - d^X(x_0, \cdot)$, where $x_0 \in X$ is a basepoint. Hence, there cannot be *any* relevant differentiation theorem for maps into L^∞. In particular, the procedure employed in the present paper to circumvent the failure of the standard differentiation theorem for L^1-targets is useless when the target is L^∞; see Remark 4.1.

1.4. *Failure of differentiability for Lipschitz maps to L^1.* For the target $L^1(\mathbb{R})$ the failure of differentiation theory is well-known, and is illustrated by the "moving characteristic function" (cf. [Aro76]):

$$f : [0,1] \to L^1(\mathbb{R}), \quad f(t) := \chi_{[0,t]},$$

where χ_A denotes the characteristic function of a subset $A \subset \mathbb{R}$. Note that f is actually an isometric embedding. For this map, the difference quotients at $t \in \mathbb{R}$ do not converge in L^1, but rather, when regarded as measures, convergence weakly to the delta function $\delta_t \notin L^1$ concentrated at t.

1.5. *Bi-Lipschitz embedding of Laakso spaces in L^1.* The main result of [CK06c] states that members of a class of spaces, which includes the PI spaces of Laakso (as well as other interesting PI spaces) admit bi-Lipschitz embeddings into L^1. According to [CK06b] (or [CK08b]) these spaces do not bi-Lipschitz embed in ℓ^1. In particular, for these domains, the differentiation results cannot be extended to L^1-targets in any form that is relevant to bi-Lipschitz nonembeddability. To our knowledge, these spaces are the first examples of doubling metric spaces which bi-Lipschitz embed in L^1, but do not bi-Lipschitz embed in ℓ^1.

1.6. *Bi-Lipschitz nonembedding in L^1; the Heisenberg group.* The Heisenberg group $(\mathbb{H}, d^H, \mathscr{L})$, where $\mathscr{L}$ denotes Lebesgue measure, is a PI space. The motivation for [CK06c] and for the present paper came from the following conjecture of J. Lee and A. Naor which is proved here.

CONJECTURE 1.1 (Lee-Naor). $(\mathbb{H}, d^{\mathbb{H}})$, *the Heisenberg group equipped with its Carnot-Carathéodory metric, does not admit a bi-Lipschitz embedding into L^1.*

This conjecture arose from [LN06], in which it is shown that the nonexistence of such an embedding would provide a natural counterexample to the Goemans-Linial conjecture of theoretical computer science; for the first such counterexample, see [KV05]. Very roughly, the point is that certain basic questions in algorithm design, such as the sparsest cut problem, could be solved up to a universal constant factor in polynomial time, if it were possible to embed a certain class of finite metric spaces (those with metrics of negative type) into ℓ^1 with universally bounded bi-Lipschitz distortion i.e. distortion independent of the particular metric and the cardinality. (A metric space (Y, d) is said to have *negative type* if $(Y, d^{\frac{1}{2}})$ embeds isometrically in Hilbert space.)

We now state a simplified version of our differentiation theorem. Let $e \in \mathbb{H}$ denote the identity element.

THEOREM 1.2 (Center collapse). *If $U \subset \mathbb{H}$ is an open subset, and $f : U \to L^1$ is a Lipschitz map, then for almost every point $x \in \mathbb{H}$, the map collapses in the direction of the center of $\mathbb{H}$; i.e.,*

$$\lim_{g \to e} \frac{\| f(gx) - f(x)\|_{L^1}}{d^{\mathbb{H}}(gx, x)} = 0\,, \qquad g \in \mathrm{Center}(\mathbb{H})\,. \tag{1.3}$$

Theorem 1.2 implies that f cannot be a bi-Lipschitz embedding, thus proving Conjecture 1.1. In particular:

COROLLARY 1.4. *There is a compact doubling metric space which does not bi-Lipschitz embed in L^1.*

To our knowledge, the Heisenberg group provides the first example of a metric space with property stated in Corollary 1.4.

Two metric spaces W_1, W_2 are called *quasi-isometric* if for some $D < \infty$, there exist D-dense subsets, $\Lambda_i \subset W_i$, and a bi-Lipschitz homeomorphism, $\Lambda_1 \to \Lambda_2$. If (W, d^W) is quasi-isometric to $(\mathbb{H}, d^{\mathbb{H}})$, then the rescaled sequence $(W, i^{-1} d^W)$ converges in the (pointed) Gromov-Haudorff sense to a metric space bi-Lipschitz homeomorphic to $(\mathbb{H}, d^{\mathbb{H}})$.

COROLLARY 1.5. *A metric space, W, that is quasi-isometric to $\mathbb{H}$ does not admit a bi-Lipschitz embedding in L^1.*

Corollary 1.5 follows from Theorem 1.2 by applying a general limiting argument [HM82], [BL00]. If the statement were false, then by the theory of ultralimits, $(\mathbb{H}, d^{\mathbb{H}})$ would bi-Lipschitz embed in some Banach space V which is an ultralimit of L^1-spaces. Then from Kakutani's abstract characterization of L^1-spaces, [Kak39], it follows that V is itself an L^1-space; this contradicts Theorem 1.2.

The canonical example of a space W to which Corollary 1.5 applies is a Cayley graph W for the integer Heisenberg group, i.e. the subgroup of $\mathbb{H}$ for which a, b, c of (2.4) below are integers. Recall that a Cayley graph W for a group G is obtained by choosing a finite generating set $\Sigma \subset G$ and declaring that two elements $g, g' \in G$ span an edge in W if and only if $g = g'\sigma$ for some $\sigma \in \Sigma$. We equip W with a G-invariant path metric.

Let W denote some Cayley graph for the integer Heisenberg group, and for $k \geq 0$, let

$$W_k := B_k(e) \subset W$$

denote the combinatorial k-ball in W.

COROLLARY 1.6. *The sequence $\{W_k\}$ is a sequence of uniformly doubling finite graphs with uniformly bounded valence, which do not admit embeddings into L^1 with uniformly bounded bi-Lipschitz distortion.*

1.7. *Indication of proof.* Our approach to differentiating maps to L^1 begins with the equivalence between (pseudo-)metrics d_f, on a set X which are induced by a map $f : X \to L^1$, and metrics which are representable as so-called "cut metrics".

For now, the term *cut* just means *subset*. A cut $E \subset X$ defines an "elementary cut metric" d_E, for which x_1, x_2 have distance 1 if either both points lie in E or neither point lies in E, and distance 0 otherwise. A *cut metric* d_Σ is a superposition of elementary cut metrics, with respect to a measure, Σ, on the power set 2^X;

$$d_\Sigma(x_1, x_2) = \int_{2^X} d_E(x_1, x_2)\, d\Sigma\,.$$

The measure, Σ, is called a *cut measure*.

The basic fact (see e.g. Lemma 4.2.5 of [DL97]) is that *any* metric d_f induced by a map f from X to an L^1-space, can be realized as a cut metric d_{Σ_f} relative to a cut measure Σ_f canonically associated to f;

$$d_f(x_1, x_2) = \int_{2^X} d_E(x_1, x_2)\, d\Sigma_f\,. \tag{1.7}$$

In actuality, the set theoretic framework just described is not adequate for our subsequent purposes and we will require a variant in which X carries a σ-finite measure μ; see Section 3. However, for the remainder of this subsection we will ignore this point.

Our main new observation is that if X is a PI space and the map f is Lipschitz, or more generally of bounded variation, then the cut measure Σ will be supported on a very special subsets of 2^X, namely on those $E \subset 2^X$ with *finite perimeter*; see Section 2 for the definition and some basic properties of sets of finite perimeter.

Let $U \subset \mathbb{H}$ be open and let $E \subset \mathbb{H}$ have finite perimeter in U. Let $\mathrm{Per}(E, U) \subset$ Radon(U) denote the perimeter measure of E in U. By a recent structure theorem in geometric measure theory, for $\mathrm{Per}(E, U)$ almost every point $x \in U$, when one blows up E at x, the resulting sequence of characteristic functions converges in

L^1_{loc} to a half-space; [Amb01], [Amb02], [FSSC01], [FSSC03]. Here a *half-space* in the Heisenberg group is a subset of the form $p^{-1}(\overline{K})$, where $p : \mathbb{H} \to \mathbb{R}^{2n}$ is the quotient of $\mathbb{H}$ by its center and $\overline{K} \subset \mathbb{R}^{2n}$ is a half-space in the usual sense. (The corresponding result for subsets of finite perimeter in $\mathbb{R}^n$ is classical and due to DiGiorgi; [DG55].)

Note that for a subset of the form $E = p^{-1}(\overline{K})$, the associated elementary cut metric assigns distance 0 to any pair of points which lie on the same coset of the center Z. In view of (1.7), this strongly suggests that under blow up, a cut metric which is supported on sets of finite perimeter should become degenerate in the direction of the center. Most of our technical work consists of making this simple idea rigorous.

1.8. *Metric differentiation and monotonicity.* Here we discuss some results related to our main theorem, which will appear elsewhere.

There is an alternative approach to the main theorem which is based on metric differentiation and monotonicity. We recall that [Kir94], [Pau01] showed that any Lipschitz map $f : X \longrightarrow Y$ from a Carnot group X to an arbitrary metric space Y has a full measure set of points of metric differentiability. This implies that blow ups of f at almost every point of X yield limit maps $f_\omega : X \longrightarrow Y_\omega$, where Y_ω is an ultralimit of rescalings of Y. When $Y = L^1$ then the ultralimit Y_ω is also an L^1-space. One can show that the cut measure associated with f_ω is supported by *monotone* subsets of X; these are measurable subsets $E \subset X$ such that for almost every horizontal geodesic, $\gamma \subset X$, the intersection $E \cap \gamma$ is — modulo a set of zero 1-dimensional Hausdorff measure — either a ray, the empty set, or γ itself. An analysis of the structure of monotone subsets of the Heisenberg group eventually leads to another proof of Theorem 1.2. The details of this will appear in [CK08c], together with other applications of the same circle of ideas.

Theorem 1.2 implies that any Lipschitz map from a ball $U \subset \mathbb{H}$ into L^1 cannot be bi-Lipschitz. By a compactness argument, it follows that if $f : U \to L^1$ is an 1-Lipschitz map, then the quantity

$$\eta(r) := \inf \left\{ \frac{d(f(x), f(y))}{d(x,y)} \;\middle|\; x, y \in U,\ d(x,y) \geq r \right\}$$

can be bounded above by a function $\hat{\eta} : [0, \infty) \to \mathbb{R}$ which is independent of the particular map f and which satisfies $\lim_{r \to 0} \hat{\eta}(r) = 0$. In a paper, [CKN], with Assaf Naor we will show that for some (computable) constant $a > 1$,

$$\eta(r) \leq \frac{1}{(-\log r)^{\frac{1}{a}}}\,; \tag{1.8}$$

see [CKN] for a more precise result. This is interest in computer science, in particular, in connection with the failure of the Goemans-Linial conjecture.

1.9. *Organization of the paper.* The remaining sections of the paper are organized as follows.

In Section 2, we collect some background material on PI spaces, BV functions and sets of finite perimeter, which is used in subsequent sections.

In Section 3, under the additional assumption that X carries a measure, μ, we give alternative characterizations of L^1-maps $f : X \to L^1(Y, \nu)$. We also discuss in this setting, the equivalence between metrics induced by maps to L^1 and cut metrics.

In Section 4, assuming in addition that X is a PI space, we show the equivalence between metrics induced by BV maps to L^1 and cut measures which are supported on sets of finite perimeter (FP cut measures). This equivalence is the basic new conceptual idea in this paper.

In Section 5, and for the remainder of the paper, we consider an FP cut measure Σ. We construct the *total perimeter measure*, $\lambda \in \mathrm{Radon}(X)$, associated to Σ.

In Section 6, and in the sections which follow, we specialize to the Heisenberg group $\mathbb{H}$. We specify the bad part of λ, taking into account location and scale. Here the "bad part" means the part carried by those cuts which are not close to a half-space. Getting suitable bounds on the bad part of λ is the key to proving our main differentiation theorem.

In Section 7, we prove a parametrized version of the main result of [FSSC01]; see Theorem 7.1. This result is of crucial importance, and it is the only place where we appeal to [FSSC01]. From Theorem 7.1 and a straightforward argument based on measure differentiation, we derive the required bounds on the bad part of the perimeter measure.

In Section 8, we introduce collections, $\mathcal{G}, \mathcal{B}$, of good and bad cuts, taking into account location and scale. Then we translate the estimates of Section 7 into estimates on $\mathcal{G}$ and $\mathcal{B}$.

In Section 9, we construct an FP cut measure, $\widehat{\Sigma}$, associated to Σ, which is supported on cuts which are half-spaces. In constructing $\widehat{\Sigma}$, we approximate cuts in $\mathcal{G}$ by cuts which are true half-spaces.

In Section 10, we prove Theorem 10.2, our main differentiation theorem. Namely, we show that at most locations, the normalized L^1-distance between the distance functions induced by Σ and $\widehat{\Sigma}$ can be made as small as we like, provided we go to a sufficiently small scale. The preceding sections have been organized in such a way that the proof uses only the estimates of Sections 8, 9, and the Poincaré inequality.

2. **Preliminaries**

In this section, we collect some relevant background material on PI spaces, BV functions and sets of finite perimeter.

2.1. PI *spaces.* A PI *space* is a metric measure space (X, d^X, μ) for which the metric is complete, and the doubling condition and Poincaré inequality hold.

The doubling condition on the measure μ states that for some $\beta(R) < \infty$,

$$\mu(B_{2r}(x)) \leq \beta(R) \cdot \mu(B_r(x)) \qquad r \leq R\,. \tag{2.1}$$

A Borel measurable function $g : X \to [0, \infty]$ is called an *upper gradient* for f if for every rectifiable curve $c : [0, \ell] \to X$ parametrized by arclength s,

$$|f(c(\ell)) - f(c(0))| \leq \int_0^\ell g(c(s))\, ds\,.$$

Put

$$f_{x,r} = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f\, d\mu\,.$$

The $(1,1)$-Poincaré inequality is the condition that for some $\tau(R), \lambda < \infty$,

$$\int_{B_r(x)} |f - f_{x,r}|\, d\mu \leq r \cdot \tau(R) \int_{B_{\lambda r}(x)} g\, d\mu\,, \tag{2.2}$$

where g is any upper gradient of f. An equivalent form of the Poincaré inequality is

$$\int_{B_r(x) \times B_r(x)} |f(x_1) - f(x_2)|\, d\mu \times d\mu \leq r \cdot \tau'(R) \int_{B_{\lambda r}(x)} g\, d\mu\,. \tag{2.3}$$

For definiteness and without essential loss of generality, in the sequel, we will assume $\lambda = 2$. For our present purposes, it is enough to consider say $r \leq 1$. Thus, $\kappa(1) = \kappa$, $\tau(1) = \tau$, $\tau'(1) = \tau'$.

2.2. *The Heisenberg group.* The Heisenberg group $\mathbb{H}$ is a 2-step nilpotent Lie group diffeomorphic to $\mathbb{R}^{2n+1}$. When equipped with the Carnot-Carathéodory metric, its Hausdorff dimension is $2n + 2$. We will recall the definition in dimension 3. For an extended discussion, see [Gro96].

The 3-dimensional Heisenberg group $\mathbb{H} \subset \mathrm{GL}(3)$ consists of matrices,

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.4}$$

$a, b, c \in \mathbb{R}$. In particular, $\mathbb{H}$ is diffeomorphic to $\mathbb{R}^3$.

As a vector space, the Lie algebra of $\mathbb{H}$ is $\mathbb{R}^3 = (a, b, c)$, realized as the space of matrices,

$$\begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix},$$

Let $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $Z = (0, 0, 1)$.

We have the commutation relations for the Lie algebra, $[P, Q] = Z$, $[P, Z] = [Q, Z] = 0$. In particular, Z is a basis for the center of the Lie algebra and the center of $\mathbb{H}$ is the 1-parameter subgroup

$$\mathrm{Center}(\mathbb{H}) = \{\exp(tZ)\}_{t\in\mathbb{R}} .$$

To define the Carnot-Carathéodory metric, view P, Q, Z as orthonormal left-invariant vector fields on $\mathbb{H}$ and denote by Δ, the *horizontal distribution*, i.e. the 2-dimensional distribution on $\mathbb{H}$ spanned by the left-invariant vector fields P, Q. The Carnot-Carathéodory distance $d^{\mathbb{H}}(x_1, x_2)$ between two points $x_1, x_2 \in \mathbb{H}$, is defined as the infimum of the length of paths $\gamma : [0,1] \to \mathbb{H}$ such that γ joins x_1 to x_2 and the velocity vector of γ is everywhere tangent to Δ. When equipped with the Carnot-Carathéodory metric and Lebesgue measure, $\mathcal{L}$ (equivalently, Haar measure), the metric measure space $(\mathbb{H}, d^{\mathbb{H}}, \mathcal{L})$ is a PI space.

2.3. *Functions of bounded variation on metric measure spaces.* Let (X, d^X), (W, d^W) denote metric spaces.

Given a Lipschitz function $f \in \mathrm{Lip}(X, W)$, the Lipschitz constant LIP f is

$$\mathrm{LIP}\ f := \sup_{x,x'} \frac{d^W(f(x), f(x'))}{d^X(x, x')} .$$

The *pointwise Lipschitz constant*, Lip f, is

$$\mathrm{Lip}(f(x)) := \liminf_{r\to 0} \sup_{d^X(x,x')<r} \frac{d^W(f(x), f(x'))}{r} . \tag{2.5}$$

Note that Lip f is an upper gradient for f.

Now assume in addition, that X is a PI space and let $U \subset X$ denote an open set. Let V denote a Banach space.

Definition 2.6. The map $h \in L^1(U, V)$ has *bounded variation*,

$$h \in \mathrm{BV}(U, d^X, \mu, V),$$

if there exists a sequence of locally Lipschitz functions $\{h_i\}$ such that $h_i \xrightarrow{L^1_{\mathrm{loc}}} h$, and

$$\liminf_{i\to\infty} \int_U \mathrm{Lip}\ h_i \, d\mu < \infty ;$$

see Definition 2.11 and Remark 2.16 of [Che99], and [Amb01], [Amb02].

As usual, we just write $f \in \mathrm{BV}(U, V)$ and $f \in \mathrm{BV}(U)$ if $V = \mathbb{R}$.

Remark 2.7. Definition 2.6 makes sense when the target is an arbitrary metric space.

The *variation* of $h \in \mathrm{BV}(U, V)$ is

$$\mathrm{VAR}(h, U) := \inf_{h_i} \liminf_{i\to\infty} \int_U \mathrm{Lip}\ h_i\, d\mu\,. \tag{2.8}$$

When there is no danger of confusion about the domain, in place of $\mathrm{VAR}(h, U)$, we sometimes write $\mathrm{VAR}(h)$.

If $U' \subset U$ is an open subset, there is a natural variation decreasing restriction map $\mathrm{BV}(U) \to \mathrm{BV}(U')$. In fact, given $f \in \mathrm{BV}(U)$, there is a canonically associated Radon measure $\mathrm{Var}(h, U)$ on U (sometimes denoted $\mathrm{Var}(h)$) the *variation measure* of h, whose value on any open set, $U' \subset U$ is

$$\mathrm{VAR}(h, U') = \mathrm{Var}(h, U)(U')\,; \tag{2.9}$$

see [Mir03]. In particular, for $h \in \mathrm{BV}(U)$,

$$\mathrm{VAR}(h, U) = \mathrm{Mass}(\mathrm{Var}(h, U))\,,$$

where by definition, for θ a measure on U,

$$\mathrm{Mass}(\theta) = \theta^{\mathrm{pos}}(U) - \theta^{\mathrm{neg}}(U)\,. \tag{2.10}$$

(θ^{pos}, θ^{neg} denote the positive and negative parts of θ, relative to the Hahn decomposition.)

The measure $\mathrm{Var}(h)$ can also be constructed in a manner analogous to the construction of the minimal upper gradient in [Che99]. Note that the measure $\mathrm{Var}(h)$ need *not* be absolutely continuous with respect to ν; e.g., if, as considered below, h is a characteristic function χ_E.

It is immediate that the variation is lower semicontinuous under L^1 convergence. The variation measure satisfies an analogous weak lower semicontinuity property under L^1 convergence; compare Proposition 5.6.

By a diagonal argument, there exists a sequence of locally Lipschitz functions, $h_i \xrightarrow{L^1_{\mathrm{loc}}} h$, with

$$\lim_{i\to\infty} \int_U \mathrm{Lip}\ h_i\, d\mu = \mathrm{Var}(h)(U)\,. \tag{2.11}$$

Note also that if $U \subset X$, $\mu(U) < \infty$, and $f : U \to V$ is Lipschitz, then $f \in \mathrm{BV}(U, V)$.

Remark 2.12. In defining *real-valued* BV functions on $\mathbb{H}$ with its Carnot-Carathéodory metric, it is equivalent to assume $h_i \in C^1$, and replace $\mathrm{Lip}\ h$, in (2.8) by the norm of the horizontal derivative — the restriction of the classical differential to the horizontal subspace Δ; see [DG55], [Giu84] for the classical theory of BV functions on $\mathbb{R}^n$ and [Amb01], [Amb02], [FSSC01], [FSSC03] for the Heisenberg case.

Remark 2.13. It is clear that if $f \in \mathrm{BV}(X)$ then (2.2), (2.3), hold, with the integral of g replaced by $\mathrm{Var}(f)(B_{\lambda r}(x))$ on the right-hand sides.

2.4. *Sets of finite perimeter.* The *perimeter measure* of a measurable set $E \subset U$ is the variation measure of the characteristic function χ_E,

$$\mathrm{Per}(E, U) := \mathrm{Var}(\chi_E, U)\,. \tag{2.14}$$

The *perimeter* of E is the mass of $\mathrm{Per}(E)$,

$$\mathrm{PER}(E, U) := \mathrm{Mass}(\mathrm{Per}(E, U))\,.$$

The measurable set E has *finite perimeter in* U if

$$\mathrm{PER}(E, U) = \mathrm{VAR}(\chi_E, U) < \infty\,.$$

As usual, below we tend to suppress the dependence on U.

As above, the perimeter and perimeter measure are lower semicontinuous (respectively weakly lower semicontinuous) under L^1 convergence of characteristic functions.

The *coarea formula* for $h \in BV(U)$ functions asserts

$$\mathrm{Var}(h)(U) = \int_{\mathbb{R}} \mathrm{PER}(\{h \ge t\} \cap U)\, d\mathscr{L}(t)\,; \tag{2.15}$$

see [AMP04].

3. L^1-maps into L^1-spaces

In this section, (X, μ), (Y, ν) will denote σ-finite measure spaces. Here we show that an L^1-map $f : X \to L^1(Y)$ gives rise to an L^1-function on the product $X \times Y$, and an L^1-map $g : Y \to L^1(X)$. We also show that the metric, d_f, induced by such a map, f, has a cut metric representation, i.e. it is a superposition of elementary cut metrics,

$$d_f(x_1, x_2) = \int_{\mathrm{Cut}(X)} d_E(x_1, x_2)\, d\Sigma_f(E)\,.$$

Here $\mathrm{Cut}(X)$, d_E, and Σ_f are the L^1 versions of the objects in Section 1.7.

We begin with some general remarks and notation.

3.1. *L^1-maps to Banach spaces.* Denote by $L^1(X, \mu, V)$, the L^1-space of (X, μ), with values in the Banach space V. If the second argument is omitted, we understand $V = \mathbb{R}$. Recall that elements of $L^1(X, \mu, V)$ are equivalence classes of of Borel measurable maps $f : X \to V$, for which the norm, $|f| : X \to \mathbb{R}$, is an integrable function on X. We will often write $f \in L^1(X, \mu, V)$ when we mean that f is such an equivalence class and refer to the L^1-function, f, when we mean that f is a representative of such an equivalence class.

Given $f \in L^1(X, \mu, V)$, there is a well-defined pushforward measure, $f_*(\mu)$, which is a Borel measure on V, with associated L^1-space, $L^1(X, f_*(\mu))$. The induced map (in the opposite direction) on real-valued functions gives rise to a map, f^* on L^1-spaces, which is an isometric embedding,

$$f^* : L^1(V, f_*(\mu)) \longrightarrow L^1(X, \mu) .$$

We may also use f to pullback the distance function from V, thereby obtaining a well-defined equivalence class of measurable functions, i.e.

$$d_f : X \times X \longrightarrow \mathbb{R} . \tag{3.1}$$

Note that the restriction

$$d_f \big|_{S \times S} : S \times S \longrightarrow \mathbb{R} \tag{3.2}$$

is integrable when $\mu(S) < \infty$.

In general, we use the term L^1_{loc}*-distance function* to refer to equivalence classes of measurable distance functions on $X \times X$ which are integrable on subsets of the form $S \times S$, where $\mu(S) < \infty$. Note that it makes sense to integrate a map from a measure space (Z, ζ) taking values in the space of L^1_{loc}-distance functions, provided it becomes an L^1-map,

$$(Z, \zeta) \longrightarrow L^1(S \times S), \tag{3.3}$$

when the distance functions are restricted to $S \times S$, for any finite measure subset $S \subset X$.

3.2. L^1*-targets; a variant of Fubini's theorem.* From now on, we will usually write $L^1(X)$, $L^1(Y)$ for $L^1(X, \mu)$, $L^1(Y, \nu)$ respectively and write $L^1(X \times Y)$ for $L^1(X \times Y, \mu \times \nu)$, suppressing the dependence on the measures.

Given a measurable function $f : X \to L^1(Y)$ representing an L^1-map, we obtain an element $f(x) \in L^1(Y)$ for each $x \in X$; this is itself an equivalence class of measurable functions on Y. The main technical point of the next result is that one may choose representatives of these equivalence classes in a measurably varying fashion.

PROPOSITION 3.4. *The spaces* $L^1(X, L^1(Y))$, $L^1(X \times Y)$, *and* $L^1(Y, L^1(X))$ *are canonically isometric. In particular*:

1) *Given* $f \in L^1(X, L^1(Y))$, *there exists* $H \in L^1(X \times Y)$, *such that for a.e.* $x \in X$,

$$H(x, y) = f(x) \quad in \quad L^1(Y) . \tag{3.5}$$

Alternatively, by Fubini's theorem, if we view H as an integrable measurable function on $X \times Y$, then for $\mu \times \nu$ a.e. $(x, y) \in X \times Y$,

$$H(x, y) = f(x)(y). \tag{3.6}$$

2) *If $H \in L^1(X \times Y)$, then for ν a.e. $y \in Y$, the function*

$$g(y) := \begin{cases} H(x, y) & H(x, y) \in L^1(X), \\ 0 \in L^1(X) & \text{otherwise} \end{cases}$$

defines an element of $L^1(X, L^1(Y))$.

Proof. 1) By definition, there is a sequence of integrable simple maps,

$$f_k : X \to L^1(Y),$$

such that

$$\lim_{k \to \infty} \| f - f_k \|_{L^1} = 0. \tag{3.7}$$

Without loss of generality, we may assume that in addition, the sequence has bounded variation in $L^1(X, L^1(Y))$; i.e.,

$$\sum_k \| f_{k+1} - f_k \|_{L^1} < \infty. \tag{3.8}$$

For each k, the map f_k takes finitely many values; for each of these we pick a measurable representing function, and thereby get a function $H_k : X \times Y \to \mathbb{R}$, which is clearly measurable. By (3.8), for μ a.e. $x \in X$, the sequence of integrable functions, $H_k(x, \cdot)$, has bounded variation in $L^1(Y)$:

$$\sum_k \| H_{k+1}(x, \cdot) - H_k(x, \cdot) \|_{L^1} < \infty.$$

Therefore, the sequence, H_k, converges pointwise $\mu \times \nu$ almost everywhere. Thus, we get a measurable function,

$$H := \liminf_{k \to \infty} H_k,$$

which is integrable by Fubini's theorem, and as a consequence of (3.7), satisfies (3.5).

2) This follows by approximating the positive (respectively negative) part of H by a monotone nondecreasing (respectively decreasing) sequence of functions H_k, where each H_k is a finite linear combination of characteristic functions of rectangles in $X \times Y$.

It is clear that the constructions in 1) and 2) above define isometries which, by Fubini's theorem, are inverses of one another. $\square$

3.3. *Borel measures on L^1- and tautological maps.* By Proposition 3.4, an L^1-map, $f : X \to L^1(Y)$, induces an L^1-map $g : Y \to L^1(X)$.

Definition 3.9. The Borel measure, $\mathcal{T}_f$, on $L^1(X)$ is the measure

$$\mathcal{T}_f := g_*(\nu) .$$

By Fubini's theorem, we have

$$\int_{L^1(X)} \|u\|_{L^1}\, d\mathcal{T}_f(u) = \|f\|_{L^1} = \|g\|_{L^1} < \infty . \tag{3.10}$$

More generally, let $\mathcal{T}$ denote an arbitrary Borel measure on $L^1(X)$ satisfying the integrability condition

$$\int_{L^1(X)} \|u\|_{L^1}\, d\mathcal{T}(u) < \infty . \tag{3.11}$$

The identity map

$$(L^1(X), \mathcal{T}) \longrightarrow L^1(X, \mu),$$

where the domain is viewed as a measure space, and the target is viewed as an L^1-space, satisfies the hypotheses of Proposition 3.4, where $(L^1(X), \mathcal{T})$ plays the role of (X, μ), and $L^1(X, \mu)$ plays the role of $L^1(Y, \nu)$. This yields:

COROLLARY 3.12. *There is an L^1-function*

$$\Lambda \in L^1(L^1(X) \times X, \mathcal{T} \times \mu) \tag{3.13}$$

and an L^1-map

$$\mathrm{Taut}_{\mathcal{T}} : X \to L^1(L^1(X), \mathcal{T}) \tag{3.14}$$

such that:

1) *For any representative of* Λ, *we have*

$$\Lambda(u, x) = u(x) \quad \text{for } \mathcal{T} \times \mu \text{ a.e. } (u, x) \in L^1(X) \times X . \tag{3.15}$$

2) *For any representative of* Λ, *we obtain a representative of* $\mathrm{Taut}_{\mathcal{T}}$ *by the formula*

$$\mathrm{Taut}_{\mathcal{T}}(x) = \Lambda(\cdot\,, x). \tag{3.16}$$

Note that

$$\|\,\mathrm{Taut}_{\mathcal{T}}\,\|_{L^1} = \int_{L^1(X)} \|u\|_{L^1} d\mathcal{T}(u).$$

Proof. Observe that $\mathcal{T}$ is a σ-finite measure, since the function

$$\|\cdot\| : L^1(X) \to \mathbb{R}$$

is integrable with respect to $\mathscr{T}$. Also, the identity map

$$L^1(X,\mu) \longrightarrow L^1(X,\mu)$$

is Borel measurable, and by (3.11), determines an L^1-map

$$(L^1(X),\mathscr{T}) \longrightarrow L^1(X).$$

We now apply Proposition 3.4 with

$$f = \mathrm{id}_{L^1(X)},$$

and set $\Lambda := H$. The lemma follows. $\square$

LEMMA 3.17. *Let $f : X \to L^1(Y)$ be an L^1-map, and*

$$\mathrm{Taut}_{\mathscr{T}_f} : X \longrightarrow L^1(L^1(X),\mathscr{T}_f)$$

be the map of Corollary 3.12. Then

1) $f = g^* \circ \mathrm{Taut}_{\mathscr{T}_f}$, *where*

$$X \xrightarrow{\mathrm{Taut}_{\mathscr{T}_f}} L^1(L^1(X),\mathscr{T}_f) \xrightarrow{g^*} L^1(Y,\nu).$$

2) *The distance functions induced by f and* $\mathrm{Taut}_{\mathscr{T}_f}$ *coincide.* (*Recall that as in* (3.1) *these are equivalence classes of measurable functions on $X \times X$.*)

Proof. Let Λ be as in Corollary 3.12, so the map

$$x \mapsto \Lambda(\cdot, x)$$

is a representative of $\mathrm{Taut}_{\mathscr{T}_f}$, and

$$\Lambda(u,x) = u(x) \quad \text{for} \quad \mathscr{T}_f \times \mu \ \text{ a.e.} \quad (u,x) \in L^1 \times X. \tag{3.18}$$

Let $H : X \times Y \to \mathbb{R}$ be as in Proposition 3.4, so

$$H(x,y) = g(y)(x) = f(x)(y) \tag{3.19}$$

for $\mu \times \nu$ a.e. $(x,y) \in X \times Y$. Since $\mathscr{T}_f = g_*(\nu)$, (3.18) and (3.19) imply that

$$\Lambda(H(\cdot,y),x) = H(x,y) \tag{3.20}$$

for μ a.e. x and ν a.e. y. Therefore for μ a.e. $x \in X$ and ν a.e. $y \in Y$,

$$\begin{aligned} \big(g^* \circ \mathrm{Taut}_{\mathscr{T}_f}(x)\big)(y) &= g^*(\Lambda(\cdot,x))\,(y) \\ &= \Lambda(H(\cdot,y),x) \\ &= H(x,y) \quad \text{by (3.20)} \\ &= f(x)(y), \end{aligned} \tag{3.21}$$

which implies 1).

Assertion 2) follows immediately from 1) because g^* is an isometric embedding. □

3.4. *Cut measures.*

Definition 3.22. A *cut* in X is an equivalence class of finite measure subsets of X.

We denote the set of cuts in X by $\mathrm{Cut}(X)$, and identify it with the set of elements of $L^1(X)$ which can be represented by characteristic functions. This is a closed subset of $L^1(X)$. In particular, $\mathrm{Cut}(X)$ inherits a metric from $L^1(X)$.

Definition 3.23. A Borel measure Σ on $\mathrm{Cut}(X)$ is a *cut measure* if

$$\int_{\mathrm{Cut}(X)} \|\cdot\|_{L^1}\, d\Sigma < \infty.$$

Since $\mathrm{Cut}(X)$ is a closed subset of $L^1(X)$, we may view Σ as a measure satisfying (3.11). Therefore by Corollary 3.12 we obtain a tautological map

$$\mathrm{Taut}_\Sigma : X \to L^1(\mathrm{Cut}(X), \Sigma), \tag{3.24}$$

where we have used the fact that $L^1(L^1(X), \Sigma)$ is isometric to $L^1(\mathrm{Cut}(X), \Sigma)$.

Next, using slices, we show how a measure $\mathcal{T}$ on $L^1(X)$ satisfying (3.11) gives rise to an associated cut measure $\Sigma_{\mathcal{T}}$.

LEMMA 3.25. *Let* Slice *denote the map*

$$\mathrm{Slice} : L^1(X) \times \mathbb{R} \longrightarrow \mathrm{Cut}(X) \tag{3.26}$$

be given by

$$\mathrm{Slice}(u,t) := \begin{cases} \{u \geq t\} & \text{when } t > 0\,, \\ \varnothing & \text{when } t = 0\,, \\ \{u \leq t\} & \text{when } t < 0\,. \end{cases} \tag{3.27}$$

Then

1) Slice *is well-defined.*
2) Slice *has a set-theoretic semicontinuity property: if* $(u_k, t_k) \in L^1(X) \times \mathbb{R}$ *is a sequence converging to* (u,t), *then*

$$\mu(\mathrm{Slice}(u_k\,,\, t_k) \setminus \mathrm{Slice}(u\,, t)) \to 0 \text{ as } k \to \infty. \tag{3.28}$$

3) Slice *is Borel measurable.*

Proof. It suffices to consider the case when $t \geq 0$, and the functions are nonnegative.

The map Slice is well-defined, because if two measurable functions u, v represent the same element of $L^1(X)$, then for every $t \in \mathbb{R}$, the symmetric difference

$$\{u \geq t\} \,\Delta\, \{v \geq t\} \tag{3.29}$$

has measure zero, and hence the two sets determine the same element of $\mathrm{Cut}(X)$.

We now prove 2). Pick $\delta > 0$. Then

$$\|u_k - u\|_{L^1} = \int_X |u_k - u| d\mu \geq (\delta + t_k - t)\, \mu(\{u_k \geq t_k\} \setminus \{u \geq t - \delta\}), \tag{3.30}$$

which forces

$$\mu(\{u_k \geq t_k\} \setminus \{u \geq t - \delta\}) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.31}$$

Since

$$\mu(\{u \geq t - \delta\} \setminus \{u \geq t\}) \to 0 \quad \text{as} \quad \delta \to 0, \tag{3.32}$$

this implies 2).

Borel measurability of Slice follows from the fact that the collection of open sets

$$U(E, \varepsilon) := \{E' \in \mathrm{Cut}(X) \mid \mu(E' \setminus E) < \varepsilon\}$$

generates the full Borel σ-algebra, and by assertion 2), $\mathrm{Slice}^{-1}(U(E, r))$ is open in $L^1(X) \times \mathbb{R}$ for all $E \in \mathrm{Cut}(X)$, $r > 0$. $\square$

Given a Borel measure $\mathcal{T}$ on $L^1(X)$ satisfying (3.11), we obtain a cut measure

$$\Sigma_{\mathcal{T}} = \mathrm{Slice}_*(\mathcal{T} \times \mathcal{L})\,. \tag{3.33}$$

Definition 3.34. The *cut measure associated with an L^1-map $f : X \to L^1(Y)$* is the Borel measure $\Sigma_{\mathcal{T}_f}$, where $\mathcal{T}_f$ is as in 3.9.

3.5. *The cut metric representation.*

Definition 3.35. The *elementary cut metric d_E* associated with a cut $E \in \mathrm{Cut}(X)$ is the L^1_{loc}-distance function given by

$$d_E(x_1, x_2) = |\chi_E(x_1) - \chi_E(x_2)|.$$

The *cut metric d_Σ* associated with a cut measure Σ is the corresponding superposition of elementary cut metrics:

$$d_\Sigma \;=\; \int_{\mathrm{Cut}(X)} d_E \; d\Sigma(E). \tag{3.36}$$

Here we view the integration on the right-hand side as taking place in the space of L^1_{loc}-distance functions, as in (3.3).

PROPOSITION 3.37. *Let Σ be a cut measure, and $\mathcal{T}$ be a measure satisfying* (3.11). *Then*:

1) *The distance function induced by the tautological map*
$$\mathrm{Taut}_\Sigma : X \longrightarrow L^1(L^1(X), \Sigma)$$
coincides with the cut metric d_Σ.

2) *If $\Sigma = \Sigma_{\mathcal{T}}$, then*
$$d_{\mathrm{Taut}_{\mathcal{T}}} = d_{\mathrm{Taut}_\Sigma} = d_\Sigma.$$

Proof. Let
$$\Lambda_{\mathcal{T}} : L^1(X) \times X \longrightarrow \mathbb{R}, \quad \Lambda_\Sigma : \mathrm{Cut}(X) \times X \longrightarrow \mathbb{R}$$
be measurable functions as in Corollary 3.12. Then for $\mu \times \mu$ a.e. $(x_1, x_2) \in X \times X$,

$$\begin{aligned} d_{\mathrm{Taut}_\Sigma}(x_1, x_2) &= \int_{\mathrm{Cut}(X)} |\Lambda_\Sigma(E, x_1) - \Lambda_\Sigma(E, x_2)|\, d\Sigma(E) \\ &= \int_{\mathrm{Cut}(X)} |\chi_E(x_1) - \chi_E(x_2)|\, d\Sigma(E) \\ &= \int_{\mathrm{Cut}(X)} d_E \, d\Sigma(E) \\ &= d_\Sigma\,. \end{aligned} \tag{3.38}$$

If $\Sigma = \Sigma_{\mathcal{T}}$, then by the definition of $\Sigma_{\mathcal{T}}$ as the pushforward of $\mathcal{T} \times \mathbb{R}$ under Slice, we may continue the calculation:

$$\begin{aligned} d_{\mathrm{Taut}_\Sigma} &= \int_{\mathrm{Cut}(X)} |\chi_E(x_1) - \chi_E(x_2)|\, d\Sigma(E) \\ &= \int_{L^1(X)\times\mathbb{R}} |\chi_{\mathrm{Slice}(u,t)}(x_1) - \chi_{\mathrm{Slice}(u,t)}(x_2)| d(\mathcal{T} \times \mathcal{L})(u, t) \\ &= \int_{L^1(X)} \int_{\mathbb{R}} |\chi_{\mathrm{Slice}(u,t)}(x_1) - \chi_{\mathrm{Slice}(u,t)}(x_2)| d\mathcal{L}(t)\, d\mathcal{T}(u) \\ &= \int_{L^1(X)} |u(x_1) - u(x_2)| d\mathcal{T}(u) \\ &= d_{\mathrm{Taut}_{\Sigma_{\mathcal{T}}}}\,. \end{aligned} \tag{3.39}$$

This concludes the proof of Proposition 3.37. □

PROPOSITION 3.40. *The distance function induced by an L^1-map*
$$f : X \longrightarrow L^1(Y)$$
is the same as that induced by the tautological map
$$\mathrm{Taut}_{\Sigma_f} : X \longrightarrow L^1(\mathrm{Cut}(X), \Sigma_f).$$

Proof. By Lemma 3.17 and Proposition 3.37, we have
$$d_f = d_{\mathrm{Taut}_{\mathcal{T}_f}} = d_{\mathrm{Taut}_{\Sigma_f}}\,. \tag{3.41}$$
□

4. BV **maps to** L^1 **and** FP **cut measures**

We retain our notation from the previous section. Thus (X, μ) and (Y, ν) will be σ-finite measure spaces. However, we assume in addition that X carries a metric, d^X, such that (X, d^X, μ) is a PI space, i.e. μ satisfies a doubling condition and a $(1,1)$-Poincaré inequality; we let κ and τ be as in Section 2.

The key new observation of this paper can be summarized as follows: Suppose $f : X \to L^1(Y)$ is a map of bounded variation; for instance f could be any Lipschitz map, provided $\mu(X) < \infty$. Let $g : Y \to L^1(X)$ be the L^1-map provided by Proposition 3.4. Now, the roles of X, f and Y, g are no longer symmetrical.

Although the regularity of the map g is worse than that of f — it is typically only measurable whereas f is BV — the typical function, $g(y) \in L^1(X)$, has better regularity than the typical function $f(x) \in L^1(Y)$: $g(y)$ has *bounded variation*, $\mathrm{VAR}(g(y)) < \infty$, and the the integral over Y of the function, $\mathrm{VAR}(g(y))$, is *finite*. In fact, these conditions provide a characterization of BV maps to L^1; see Theorem 4.4.

We also give a second and, in a sense, more directly relevant characterization of BV maps to L^1, in terms of what we call "FP cut measures" (where FP stands for finite perimeter). We show that $f \in \mathrm{BV}(U, L^1(Y))$ if and only if the cut measure, Σ_f, is an FP cut measure. Essentially, this follows from the previous characterization via the coarea formula.

Remark 4.1. By way of contrast with the case of L^1-targets, note that for the Kuratowski embedding of (X, d^X) into $L^\infty(X, d^X)$, we have $X = Y$, $f = g$, and nothing is gained. On the other hand, our present point of view may be useful when studying other function space targets.

4.1. *Characterizing* BV *maps to* L^1 *by variation.* Let $U \subset X$ denote an open subset. Let $f \in L^1(U, L^1(Y))$ and let H, g denote the maps in Proposition 3.4.

Note that since $\mathrm{VAR}(\cdot)$ is a lower semicontinuous function on $L^1(U)$, the integral

$$\int_Y \mathrm{VAR}(g(y), U)\, d\nu$$

is a well-defined extended real number.

Definition 4.2. The map, $f \in L^1(U, L^1(Y))$ has *finite total variation*, if $g(y) \in \mathrm{BV}(U)$, for ν a.e. $y \in Y$ and

$$\int_Y \mathrm{VAR}(g(y), U)\, d\nu < \infty\,. \tag{4.3}$$

The quantity in (4.3) is the *total variation* of f.

The following theorem shows that the total variation, which is defined only for L^1-targets, is comparable to the variation defined in (2.8), which is defined for arbitrary Banach space targets.

THEOREM 4.4. *$f \in \mathrm{BV}(U, L^1(Y))$ if and only if f has finite total variation. Moreover, there is a constant, $c = c(\kappa, \tau) > 0$, such that*

$$c^{-1} \cdot \mathrm{VAR}(f, U) \le \int_Y \mathrm{VAR}(g(y), U)\, d\nu \le c \cdot \mathrm{VAR}(f, U)\,. \tag{4.5}$$

Proof. Assume $f \in \mathrm{BV}(U, L^1(Y))$. Since $\mathrm{VAR}(g(y), U) < \infty$ implies that $\mathrm{Var}(g(y), U)$ is Borel regular, by the monotone convergence theorem, it suffices to consider an open set $U' \subset U$ with compact closure in U, and to establish the inequalities in (4.5) with $\mathrm{VAR}(g(y), U)$ calculated on U' rather than on U.

By (2.11), there exists a sequence, $f_i \in \mathrm{Lip}(U, L^1(Y))$, with $f_i \xrightarrow{L^1_{\mathrm{loc}}} f$, such that

$$\lim_{i\to\infty} \int_U \mathrm{Lip}\ f_i\, d\mu = \mathrm{VAR}(f, U)\,.$$

Fix an open set $U' \subset U$ with compact closure in U. We will construct a sequence, $f_{i,j} \in \mathrm{Lip}(U', L^1(Y))$, with $f_{i,j} \xrightarrow{C^0} f_i$ on U', such that for $c = c(\kappa, \tau)$,

$$\int_{U'} \mathrm{Lip}\ f_{i,j}\, d\mu \le c \cdot \int_{U'} \mathrm{Lip}\ f_i\, d\mu\,, \tag{4.6}$$

$$\int_Y \left(\int_{U'} \mathrm{Lip}\ g_{i,j}(x, y)\, d\mu \right) d\nu \le c \cdot \int_{U'} \mathrm{Lip}\ f_{i,j}\, d\mu\,. \tag{4.7}$$

Then the claim follows by a diagonal argument, together with Fatou's lemma.

Let $\{x_{i,j,k}\}$ denote a maximal j^{-1} separated set in U. By a standard lemma, the multiplicity of the covering, $\{B_{2j^{-1}}(x_{i,j,k})\}$, is bounded by $N = N(\kappa)$. Also, by using distance functions from the points, $x_{i,j,k}$, we can construct in standard fashion, a partition of unity, $\{\phi_{i,j,k}\}$, subordinate to $\{B_{2j^{-1}}(x_{i,j,k})\}$, with

$$\mathrm{LIP}(\phi_{i,j,k}) \le c(\kappa) \cdot j\,. \tag{4.8}$$

Define the *regularization* $f_{i,j}$ of f_i by

$$f_{i,j} = \sum_k \bar{f}_{i,j,k} \cdot \phi_{i,j,k}\,, \tag{4.9}$$

where

$$\bar{f}_{i,j,k} = \frac{1}{\mu(B_{2j^{-1}}(x_{i,j,k}))} \int_{B_{2j^{-1}}(x_{i,j,k})} f_i\, d\mu\,.$$

Since, f_i is Lipschitz, it follows that $f_{i,j} \xrightarrow{C^0} f_i$ in precompact subset U'.

From now on, we only consider j so large that if $\mathrm{Supp}(\phi_{i,j,k}) \cap U' \neq \emptyset$, then $B_{8j^{-1}}(x_{i,j,k}) \subset U$.

Let ℓ denote a linear functional of norm 1 on $L^1(Y)$. Then

$$\ell(\overline{f}_{i,j,k}) = \int_{B_{2j^{-1}}(x_{i,j,k})} \ell(f_i)\, d\mu\,.$$

By applying the Poincaré inequality on $B_{8j^{-1}}(x_{i,j,k}) \subset U$ to the Lipschitz function $\ell \circ f_i$ for all such ℓ, and using the Hahn-Banach theorem, we conclude that for all $x_{i,j,k}, x_{i,j,k'}$, with $d^X(x_{i,j,k}, x_{i,j,k'}) \leq 4j^{-1}$,

$$\|\overline{f}_{i,j,k} - \overline{f}_{i,j,k'}\|_{L^1} \leq c \cdot j^{-1} \frac{1}{\mu(B_{8j^{-1}}(x_{i,j,k}))} \cdot \int_{B_{8j^{-1}}(x_{i,j,k})} \mathrm{Lip}\ f_i\, d\mu\,, \tag{4.10}$$

where $c = c(\kappa, \tau)$.

For any fixed index, k^*, we can write

$$f_{i,j} = \overline{f}_{i,j,k^*} + \sum_k (\overline{f}_{i,j,k} - \overline{f}_{i,j,k^*}) \cdot \phi_{i,j,k}\,. \tag{4.11}$$

Since $\mathrm{Lip}\ f_i \leq \mathrm{LIP}\ f_i < \infty$, from (4.8), (4.10), and the Lebesgue differentiation theorem applied to $\mathrm{Lip}\ f_i$, we easily get (4.6). Since $\overline{f}_{i,j,k} \in L^1(Y)$, relation (4.7) follows from (4.11), and a straightforward argument based on Fubini's theorem.

Now assume, conversely, that $g(y) \in \mathrm{BV}(U)$, for ν a.e. $y \in Y$ and that (4.3) holds. By an exhaustion argument it is easily checked that it suffices to assume that $\nu(Y) < \infty$. Similarly, by a truncation argument, one can assume that $H(x, y)$ is bounded.

Define the regularization, f_j, of f as in (4.9). Then f_j and H_j are Lipschitz and for ν a.e. $y \in Y$, the function, H_j, is equal to the corresponding regularization $g_j(y)$ of $g(y)$. Moreover, $f_j \xrightarrow{L^1_{\mathrm{loc}}} f$.

By arguing as above (compare the verification of (4.6), (4.7)) and using Fubini's theorem, we have

$$\begin{aligned}\infty > c \cdot \int_Y \mathrm{Var}(g(y), U)\, d\nu &\geq \int_Y \left(\int_{U'} \mathrm{Lip}\ H_j(x, y)\, d\mu\right) d\nu \\ &= \int_{U'} \left(\int_Y \mathrm{Lip}\ H_j(x, y)\, d\nu\right) d\mu \geq \int_{U'} \mathrm{Lip}\ f_k\,.\end{aligned}$$

This suffices to complete the proof. □

Remark 4.12. In actuality, a metric measure space with the doubling property satisfies a Poincaré inequality for real-valued functions if and only if it satisfies a Poincaré inequality for functions with values in an arbitrary Banach space; see [HKST01]. In justifying (4.10) above, rather than using this result, we appealed directly to the Hahn-Banach theorem.

4.2. BV *maps to* L^1 *and* FP *cut measures.* Let

$$\mathrm{PER}_U : \mathrm{Cut}(U) \to [0, \infty]$$

be given by

$$E \mapsto \mathrm{PER}(E, U)\,.$$

Definition 4.13. A cut measure Σ is defined as an FP *cut measure* if $\mathrm{PER}_U \in L^1(\mathrm{Cut}(U), \Sigma)$:

$$\int_{\mathrm{Cut}(U)} \mathrm{PER}(E, U)\, d\Sigma < \infty. \tag{4.14}$$

The quantity in (4.14) is the *total perimeter* of Σ.

Definition 4.15. $E \in \mathrm{Cut}(U)$ is an FP *cut* if $\mathrm{PER}(E, U) < \infty$.

Let $\mathrm{FP}(X) \subset \mathrm{Cut}(U)$ denote the collection of FP cuts. Since $\mathrm{PER}_U \equiv \infty$ on $\mathrm{Cut}(U) \setminus \mathrm{FP}(U)$, it follows that

$$\Sigma\,(\mathrm{Cut}(U) \setminus \mathrm{FP}(U)) = 0\,.$$

Let $\mathrm{Taut}_\Sigma : U \to L^1(\mathrm{Cut}(U), \Sigma))$ be as in (3.24).

PROPOSITION 4.16. *A cut measure* Σ *is an* FP *cut measure if and only if* $\mathrm{Taut}_\Sigma \in \mathrm{BV}(U, L^1(\mathrm{Cut}(U), \Sigma))$.

Proof. The map, $g : \mathrm{Cut}(U) \to L^1(U)$, associated with

$$\mathrm{Taut}_\Sigma : U \to L^1(\mathrm{Cut}(X), \Sigma)$$

is given by $g(E) = \chi_E$. Therefore, by Theorem 4.4, the map Taut_Σ is BV if and only if

$$\infty > \int_{\mathrm{Cut}(U)} \mathrm{VAR}(\chi_E)\, d\Sigma = \int_{\mathrm{Cut}(U)} \mathrm{PER}(E, U)\, d\Sigma\,,$$

if and only if Σ is an FP cut measure. □

The next proposition asserts the equality of the total variation of f and total perimeter of Σ_f.

PROPOSITION 4.17. *If* $f \in L^1(U, L^1(Y))$, *then* Σ_f *is an* FP *cut measure if and only if* $f \in \mathrm{BV}(U, L^1(Y))$. *Moreover,*

$$\int_{\mathrm{Cut}(U)} \mathrm{PER}(E, U)\, d\Sigma_f = \int_Y \mathrm{VAR}(g(y, U))\, d\nu\,. \tag{4.18}$$

Proof. We define

$$S : \mathbb{R} \times Y \longrightarrow \mathrm{Cut}(X)$$

by

$$S(t, y) := \mathrm{Slice}(g(y), t),$$

where

$$g : Y \longrightarrow L^1(U)$$

is the map of Proposition 3.4. By the definition of the cut measure Σ_f, Fubini's theorem, and (2.15), we have

$$\begin{aligned}\int_{\mathrm{Cut}(U)} \mathrm{PER}(E,U)\, d\Sigma_f &= \int_{Y\times\mathbb{R}} \mathrm{PER}(S(t,y),U)\, d(\mathcal{L}\times\nu) \\ &= \int_Y \left(\int_{\mathbb{R}} \mathrm{PER}(S(t,y),U)\, d\nu\right) d\mathcal{L} \\ &= \int_Y \mathrm{VAR}(g(y),U)\, d\nu\,. \end{aligned} \tag{4.19}$$

Therefore by Theorem 4.4, the map f is BV if and only if Σ is an FP cut measure. □

5. The total perimeter measure

We retain the notation of the preceding section. In this section we will associate to each FP cut measure, Σ, a Radon measure, $\lambda_\Sigma \in \mathrm{Radon}(U)$, called the total perimeter measure of Σ, whose mass is the total perimeter of Σ. The measure λ_Σ is obtained by integrating the measure-valued function,

$$\mathrm{Per}(E,U) : \mathrm{Cut}(U) \to \mathrm{Radon}(U),$$

with respect to Σ.

In the main result on the Heisenberg group, an essential point is to suitably control the "bad part" of λ_Σ; see Sections 6–10.

5.1. *Integrating measure-valued functions.* Let (Z,ζ) denote a measure space. Let L denote a locally compact Hausdorff space and $C_c(L)$ the space of continuous functions of compact support, equipped with the sup norm.

A map,

$$\Psi : (Z,\zeta) \to \mathrm{Radon}(L)\,,$$

is *weakly measurable* if for every $\phi \in C_c(L)$,

$$z \mapsto \int_L \phi\, d\Psi \tag{5.1}$$

is a measurable function on Z.

The map Ψ is *weakly* L^1 if it is weakly measurable and there exists $C < \infty$, such that for all $\phi \in C_c(L)$,

$$\int_Z \int_L \phi\; d\Psi\, d\zeta \le C \cdot \|\phi\|_{L^\infty}\,. \tag{5.2}$$

According to the next proposition, a weakly L^1-map into Radon(L) can be integrated to obtain a Radon measure.

PROPOSITION 5.3. *Let* $\Psi : (Z, \zeta) \to \text{Radon}(L)$ *denote a weakly* L^1*-map. Then there is a measure*, $\eta \in \text{Radon}(L)$, *such that for every Borel set* $A \subset L$,

$$\eta(A) = \int_Z \Psi(z)(A)\, d\zeta\,.$$

If the measure $\Psi(z)$ *is nonnegative for* ζ *a.e.* $z \in Z$, *then*

$$\text{Mass}(\eta) = \int_Z \text{Mass}(\Psi(z))\, d\zeta. \tag{5.4}$$

Proof. Since Ψ is weakly L^1, it follows that the formula

$$\phi \mapsto \int_Z \left(\int_L \phi(x)\, d(\Psi(z))(x) \right) d\zeta(z) \tag{5.5}$$

defines a bounded linear functional on $C_c(L)$. Thus, the proposition follows from the Riesz representation theorem. □

5.2. *Constructing the total perimeter measure* λ_Σ.

PROPOSITION 5.6. *Given an* FP *cut measure* Σ, *the map given by*

$$E \mapsto \text{Per}(E, U)$$

defines a weakly L^1*-map*,

$$(\text{Cut}(U), \Sigma) \to \text{Radon}(U)\,.$$

Proof. By essentially the same observation as that which shows that $\text{Per}(E, U)$ is lower semicontinous under L^1 convergence of characteristic functions, it follows that the map in (5.1) is the difference of two lower semicontinuous functions (corresponding to the nonnegative and nonpositive parts of the function ϕ). It is then clear that (5.2) holds. □

Definition 5.7. The *total perimeter measure* $\lambda_\Sigma \in \text{Radon}(U)$ of the FP cut measure Σ is the measure obtained by integrating the weakly L^1-map $E \to \text{Per}(E, U)$.

Remark 5.8. Note that by (5.4),

$$\text{Mass}(\lambda_\Sigma) = \int_{\text{Cut}(U)} \text{PER}(E, U)\, d\Sigma\,; \tag{5.9}$$

see Definition 4.13. In case $\Sigma = \Sigma_f$ for some some $f \in \text{BV}(U, L^1(Y))$, the total perimeter of Σ_f is equal to the total variation of f; see Definition 4.2 and (4.18) of Proposition 4.17.

5.3. *Lipschitz maps to L^1.*

PROPOSITION 5.10. *There is a constant, $C < \infty$, depending only on the constants β, λ and τ, with the following property. A* BV *map $f : U \to L^1(Y)$ admits an L-Lipschitz representative if and only if for every ball, $B_r(x)$,*

$$\frac{\lambda_{\Sigma_f}(B_r(x))}{\mu(B_r(x))} \leq C \cdot L \,. \tag{5.11}$$

Proof. Since, Proposition 5.10 is not required in the sequel, we will be very brief. The necessity of (5.11) follows from the argument used in proving Theorem 4.4.

The sufficiency follows from an application of the "telescope estimate" as in the proof of the standard estimate, (4.19), of [Che99]. Here is a sketch of a variant of that argument. One considers a pair of points $x, x' \in U$, and for small $r > 0$, a suitably chosen sequence of points

$$x = x_1, \ldots, x_k = x'$$

where $d(x_i, x_{i+1}) < \frac{r}{2}$, and

$$k \leq \text{const}\, \frac{d(x, x')}{r}.$$

The Poincaré inequality and (5.11) imply that there is a constant $C = C(\beta, \lambda, \tau)$, such that for all $1 \leq i < k$, the average of f over $B_r(x_i)$ differs by at most CLr from its average over $B_r(x_{i+1})$. So

$$\left| \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, d\mu - \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, d\mu \right| \leq CLkr < C'Ld(x, x'),$$

where $C' = C'(\beta, \lambda, \tau)$. Since this estimate is independent of r, it follows that f has a $C'L$-Lipschitz representative. □

6. The total bad perimeter measure

We retain the notation of the preceding sections, except that we will just write $\mathrm{Per}(E)$ in place of $\mathrm{Per}(E, U)$, suppressing the dependence on U.

In the remaining sections, we are concerned with properties of sets of finite perimeter which are not valid in general PI spaces. For this reason, from now on, X will be either $\mathbb{R}^n$ or the Heisenberg group $\mathbb{H}$ with its Carnot-Carathéodory metric, μ will denote Lebesgue measure (or equivalently Hausdorff measure) and U will denote a ball in X. In actuality, the discussion has a direct extension to the case in which X is replaced by any 2-step nilpotent Lie group.

We call a subset $E \subset X$ a *half-space* if either $X = \mathbb{R}^n$ and E is a half-space in the usual sense, or $X = \mathbb{H}$ and E is the inverse image of a Euclidean half-space under the quotient homomorphism, $\mathbb{H} \to \mathbb{H}/Z(\mathbb{H}) \simeq \mathbb{R}^{2n}$.

We will begin by introducing a quantity, α, which measures how close $E \in \mathrm{FP}(U)$ is to being a half-space, taking into account location and scale. For our purposes, not being close to a half-space is "bad".

Given a finite perimeter cut measure, Σ, we define the corresponding the bad part, $\lambda^{\mathrm{Bad}}_{\varepsilon,R}$, of the total perimeter measure, $\lambda = \lambda_\Sigma$, where the parameters, ε, R specify the degree of badness and the scale respectively. Control on $\lambda^{\mathrm{Bad}}_{\varepsilon,R}$, which is the key to proving our main result, is obtained in Section 7. Theorem 10.2 is proved by translating the bounds on $\lambda^{\mathrm{Bad}}_{\varepsilon,R}$ into bounds on the cut measure Σ.

6.1. *Measuring closeness of* FP *cuts to half-spaces.* We denote the collection of all half-spaces in X by HS, and let

$$\mathrm{HS}_x := \{E \in \mathrm{HS} \mid \partial E \text{ contains } x\}\,.$$

Definition 6.1. Define

$$\alpha : \mathrm{FP}(U) \times U \times (0, \infty) \to \mathbb{R}_+$$

to be the normalized L^1-distance between E and HS_x in the ball $B_r(x)$:

$$\alpha(E, x, r) = \min_{H \in \mathrm{HS}_x} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |\chi_E - \chi_H|\, d\mu\,.$$

LEMMA 6.2. *α is a locally Lipschitz function of all three variables.*

Proof. Changing r and x slightly only adds or subtracts a small amount of measure. Locally Lipschitz dependence on E is clear. □

For ε, $R > 0$, and $E \in \mathrm{FP}(U)$, let

$$\mathrm{Bad}_{\varepsilon,R}(E) := \{x \in U \mid d(x, X \setminus U) < R \text{ or } \alpha(E, x, r) > \varepsilon \text{ for some } r \in (0, R]\}\,,$$

$$\mathrm{Good}_{\varepsilon,R}(E) := \{x \in U \mid d(x, X \setminus U) \le R \text{ and } al(E, x, r) \le \varepsilon \text{ for all } r \in (0, R]\}\,.$$

Thus,

$$\mathrm{Good}_{\varepsilon,R}(E) = U \setminus \mathrm{Bad}_{\varepsilon,R}(E)\,.$$

Also, put

$$\mathrm{Bad}_{\varepsilon,R} := \{(E, x) \in \mathrm{FP}(U) \times U \mid x \in \mathrm{Bad}_{\varepsilon,R}(E)\}\,,$$

$$\mathrm{Good}_{\varepsilon,R} := \{(E, x) \in \mathrm{FP}(U) \times U \mid x \in \mathrm{Good}_{\varepsilon,R}(E)\}\,.$$

LEMMA 6.3. $\mathrm{Bad}_{\varepsilon,R}$ *is an open subset of* $\mathrm{FP}(U) \times U$.

Proof. The set $\mathrm{Bad}_{\varepsilon,R}$ is the image of the open set

$$\{(E, x, r) \subset \mathrm{FP}(U) \times U \times (0, R] \mid d(x, X \setminus U) < R \quad \text{or} \quad \alpha(E, x, r) > \varepsilon\}$$

under the open projection map,

$$\mathrm{FP}(U) \times U \times (0, R] \to \mathrm{FP}(U) \times U\,.$$

The conclusion follows. □

6.2. *The total bad perimeter measure.* As in Section 5, let Σ denote an FP cut measure and $\lambda = \lambda_\Sigma$ the associated total perimeter measure.

Given a measure ζ on Z, and a measurable subset $A \subset Z$, let $\zeta \llcorner A$ denote the measure given by

$$\zeta \llcorner A(F) = \zeta(A \cap F)\,. \tag{6.4}$$

Let the map,

$$\mathrm{Per} \llcorner \mathrm{Bad}_{\varepsilon,R} : \mathrm{FP}(U) \longrightarrow \mathrm{Radon}(U)\,,$$

be given by

$$\mathrm{Per} \llcorner \mathrm{Bad}_{\varepsilon,R}(E) = \mathrm{Per}(E) \llcorner \mathrm{Bad}_{\varepsilon,R}(E)\,. \tag{6.5}$$

Recall the notion of weakly L^1-map; see (5.2).

LEMMA 6.6. *The map*

$$\mathrm{Per} \llcorner \mathrm{Bad}_{\varepsilon,R} : \mathrm{FP}(U) \longrightarrow \mathrm{Radon}(U)$$

is weakly L^1.

Proof. For all k, let

$$\Phi_k : \mathrm{FP}(U) \times U \to \mathbb{R}$$

denote a continuous function satisfying:

1. $0 \le \Phi_k \le 1$.
2. $\Phi_k \equiv 1$ on
$$\left\{ (E, x) \mid d\left((E, x), \mathrm{Good}_{\varepsilon,R}\right) \ge \frac{1}{k} \right\}\,,$$
where the distance on the product is the sum of the factor distances.
3. $\Phi_k \equiv 0$ on
$$\left\{ (E, x) \mid d\left((E, x), \mathrm{Good}_{\varepsilon,R}\right) \le \frac{1}{k+1} \right\}\,.$$

Fix $\phi \in C_c(U)$ and define

$$\Psi_k : \mathrm{FP}(U) \to \mathbb{R}$$

by

$$\Psi_k(E) = \int_U \phi\, \Phi_k(E, \cdot)\, d\,\mathrm{Per}(E)\,.$$

The map Ψ_k is Borel measurable, since it is the pointwise limit of a sequence of measurable functions $\{\Psi_{k,l}\}$ obtained by approximating the map

$$E \to \phi(\cdot)\, \Phi_k(E, \cdot)$$

by simple functions, and each of the $\Psi_{k,l}$'s is measurable.

For fixed E, the compact subsets,

$$\mathrm{Supp}(\Phi_k(E,\cdot)) \subset \mathrm{Bad}_{\varepsilon,R}(E),$$

exhaust the open set $\mathrm{Bad}_{\varepsilon,R}(E)$. To see this note that for each compact set, $K \subset \mathrm{Bad}_{\varepsilon,R}(E)$, the subset,

$$\{E\} \times K \subset \mathrm{FP} \times U\,,$$

has positive distance from the closed set $\mathrm{Good}_{\varepsilon,R}(E)$, and is therefore contained in $\mathrm{Supp}(\Phi_k(E,\cdot))$ for sufficiently large k.

It follows from the above that the mass of the difference measure

$$\mathrm{Per}(E) \llcorner \mathrm{Bad}_{\varepsilon,R}(E) - \Phi_k(E,x)\,\mathrm{Per}(E)$$

tends to 0 as $k \to \infty$. Thus, for each $E \in \mathrm{FP}(U)$, the integrals,

$$\Psi_k(E) = \int_U \phi\, \Phi_k(E,x)\, d\,\mathrm{Per}(E)\,,$$

converge as $k \to \infty$, to

$$\int_U \phi\, d\left(\mathrm{Per}(E) \llcorner \mathrm{Bad}_{\varepsilon,R}(E)\right).$$

The map

$$E \to \int_U \phi\, d\left(\mathrm{Per}(E) \llcorner \mathrm{Bad}_{\varepsilon,R}(E)\right)$$

is a pointwise limit of Borel measurable functions and is therefore Borel measurable. Since ϕ is arbitrary, it follows that the map,

$$\mathrm{Per} \llcorner \mathrm{Bad}_{\varepsilon,R} : \mathrm{FP}(U) \to \mathrm{Radon}(U)$$

is weakly measurable.

Now (4.14) implies that $\mathrm{Per} \llcorner \mathrm{Bad}_{\varepsilon,R}$ is weakly L^1. □

Definition 6.7. The *total bad perimeter measure* is the Radon measure

$$\lambda^{\mathrm{Bad}}_{\varepsilon,R} \in \mathrm{Radon}(U)$$

obtained by applying Proposition 5.3 to the weakly L^1 map

$$\mathrm{Per} \llcorner \mathrm{Bad}_{\varepsilon,R} : \mathrm{FP}(X) \to \mathrm{Radon}(U)\,.$$

7. **Controlling the total bad perimeter measure**

Recall that from now on (X,μ) will denote either $\mathbb{R}^n$ or $\mathbb{H}$, and $U \subset X$ will denote a ball. Also, Σ will denote an FP cut measure on $\mathrm{Cut}(U)$, with associated total perimeter measure λ, and associated good and bad measures $\lambda^{\mathrm{Good}}_{\varepsilon,R}$, $\lambda^{\mathrm{Bad}}_{\varepsilon,R}$.

One of the main results of [FSSC01] (see [DG55] for the $\mathbb{R}^n$ case) states that if E is a set of finite perimeter in U, then for $\mathrm{Per}(E)$ a.e. $x \in U$, the blow ups of E at x converge in L^1_{loc} to a half-space. In this section we give a version of the above result in the parametrized setting. Namely, given an FP cut measure Σ with total perimeter measure, $\lambda = \lambda_\Sigma$, we show in Theorem 7.1, that for any fixed ε, the mass of $\lambda^{\mathrm{Bad}}_{\varepsilon,R}$, goes to zero as $R \to 0$. Theorem 7.1 is of crucial importance; its proof constitutes the one and only place where we explicitly appeal to [FSSC01].

From Theorem 7.1 and a straightforward differentiation argument, it follows that for any $\varepsilon > 0$ we can find a set with almost full measure on which

$$\frac{\lambda^{\mathrm{Bad}}_{\varepsilon,R}(B_r(x))}{\mu(B_r(x))}$$

is as small as we like, provided we take R sufficiently small; see Proposition 7.5.

THEOREM 7.1. *For all* $\varepsilon > 0$,

$$\lim_{R\to 0} \mathrm{Mass}(\lambda^{\mathrm{Bad}}_{\varepsilon,R}) = 0\,. \tag{7.2}$$

Proof. By (5.4), (6.5),

$$\mathrm{Mass}(\lambda^{\mathrm{Bad}}_{\varepsilon,R}) = \int_{\mathrm{Cut}(U)} \mathrm{Mass}\left(\mathrm{Per}(E) \llcorner (\mathrm{Bad}_{\varepsilon,R}(E)\right) \, d\Sigma\,. \tag{7.3}$$

By the main result of [FSSC01], for fixed $\varepsilon > 0$, $E \in \mathrm{FP}$, we have

$$\lim_{R\to 0} \mathrm{Mass}(\mathrm{Per}(E) \llcorner (\mathrm{Bad}_{\varepsilon,R}(E)) = 0\,. \tag{7.4}$$

(For equivalent ways of expressing this, compare (2.10) and (6.4).)

Since

$$\mathrm{Mass}(\mathrm{Per}(E) \llcorner (\mathrm{Bad}_{\varepsilon,R}(E)) \le \mathrm{Mass}(\mathrm{Per}(E))\,,$$

and $\mathrm{Per} \in L^1(\mathrm{Cut}(U), \Sigma)$ (see Definition 4.15 and Proposition 4.17) the claim follows from (7.3) and the dominated convergence theorem. □

PROPOSITION 7.5. *For all* $\delta > 0$, $\varepsilon > 0$, *there exists* $r_0(\delta, \Sigma) > 0$, $r_1(\delta, \varepsilon, \Sigma) > 0$, $R_0 = R_0(\delta, \varepsilon, \Sigma) > 0$, *and a subset*, $U_{\delta,\varepsilon}(\Sigma) \subset X$, *such that*

$$\mu(U \setminus U_{\delta,\varepsilon}) < 2\delta\,(1 + \mathrm{Mass}(\lambda))\,, \tag{7.6}$$

$$\frac{\lambda(B_r(x))}{\mu(B_r(x))} < \delta^{-1}\,, \qquad \text{if } x \in U_{\delta,\varepsilon},\ r \le r_0(\delta, \Sigma)\,, \tag{7.7}$$

$$\frac{\lambda^{\mathrm{Bad}}_{\varepsilon,R_0}(B_r(x))}{\mu(B_r(x))} < \varepsilon\,, \qquad \text{if } x \in U_{\delta,\varepsilon}\,,\ r \le r_1(\delta, \varepsilon, \Sigma)\,. \tag{7.8}$$

Proof. Given Theorem 7.1, this is a straightforward application of measure differentiation. By the Lebesgue decomposition theorem, there exists, $U' \subset U$, with $\mu(U \setminus U') = 0$, such that λ is absolutely continuous with respect to μ on U'.

Since

$$\int_{U'} \frac{d\lambda}{d\mu}\, d\mu \le \mathrm{Mass}(\lambda) < \infty\,,$$

there exists $U_1 \subset U$ such that

$$\mu(U \setminus U_1) < 2\delta\,\mathrm{Mass}(\lambda)\,,$$
$$\frac{d\lambda}{d\mu} < \frac{\delta^{-1}}{2} \quad \text{on} \quad U_1\,.$$

By measure differentiation, for μ a.e. $x \in U_1$,

$$\lim_{r\to\infty} \frac{\lambda(B_r(x))}{\mu(B_r(x))} = \frac{d\lambda}{d\mu}(x)\,.$$

Therefore, there exists $r_0(\delta,\Sigma) > 0$ and $U_2 \subseteq U_1$, such that for all $0 < r \le r_1$, $x \in U_2$,

$$\mu(U_1 \setminus U_2) < \delta\,,$$

and

$$\frac{\lambda(B_r(x))}{\mu(B_r(x))} < \delta^{-1}\,, \qquad \text{if } x \in U_2,\ r \le r_0(\delta,\Sigma)\,.$$

Since by (7.2),

$$\lim_{R\to 0} \mathrm{Mass}(\lambda^{\mathrm{Bad}}_{\varepsilon,R}) = 0\,,$$

there exists $U_3 \subset U$, with

$$\mu(U \setminus U_3) < \frac{\delta}{2}\,,$$

and $R_0(\delta,\varepsilon,\Sigma) > 0$, such that

$$\frac{d\lambda^{\mathrm{Bad}}_{\varepsilon,R_0}}{d\mu} < \frac{\varepsilon}{2} \qquad \text{on } U_3\,.$$

As above, by using measure differentiation, there exists $U_4 \subset U_3$ and $r_1(\delta,\varepsilon,\Sigma) > 0$, such that

$$\mu(U_3 \setminus U_4) < \frac{\delta}{2}\,,$$

and

$$\frac{\lambda^{\mathrm{Bad}}_{\varepsilon,R_0}(B_r(x))}{\mu(B_r(x))} < \varepsilon\,, \qquad \text{if } x \in U_4,\ r \le r_1(\delta,\varepsilon,\Sigma)\,.$$

Now take $U_{\delta,\varepsilon} := U_2 \cap U_4$. □

8. Collections of good and bad cuts

In this section, given an FP cut measure, Σ, we introduce sets of good and bad cuts, $\mathcal{G}$, $\mathcal{B}$, where as usual, we take into account location and scale. Estimates on $\mathcal{G}$ and $\mathcal{B}$, are derived from Proposition 7.5.

In Section 9, using the set $\mathcal{G}$, we will construct a measure, $\widehat{\Sigma}$, which is supported on half-spaces. In Section 10, our main theorem is established by proving that, for μ a.e. $x \in U$, in the limit as $r \to 0$, the normalized L^1-distance between, d_Σ and $d_{\hat{\Sigma}}$ converges to zero.

For $\delta > 0$, $\varepsilon > 0$, let $r_0(\delta, \Sigma)$, $r_1(\delta, \varepsilon, \Sigma)$, $R_0(\delta, \varepsilon, \Sigma)$, $U_{\delta,\varepsilon}$, be as in Proposition 7.5. For all $x \in U$, $r > 0$, we define

$$\mathcal{G}(x, \delta, \varepsilon, r, \Sigma) \subset \mathrm{FP}(U)\,,$$
$$\mathcal{B}(x, \delta, \varepsilon, r, \Sigma) \subset \mathrm{FP}(U)\,,$$

by

$$\begin{aligned} \mathcal{G}(x, \delta, \varepsilon, r, \Sigma) &= \{E \in \mathrm{FP}(U) \mid \overline{B_r(x)} \cap \mathrm{Good}_{\varepsilon, R_0}(E) \neq \varnothing\}\,, \\ \mathcal{B}(x, \delta, \varepsilon, r, \Sigma) &= \mathrm{FP}(U) \setminus \mathcal{G}(x, \delta, \varepsilon, r, \Sigma)\,. \end{aligned} \tag{8.1}$$

Note that $\mathcal{B}$ is an open subset of $\mathrm{FP}(U)$. In particular, $\mathcal{G}$ and $\mathcal{B}$ are both Borel sets.

PROPOSITION 8.2. *Pick $\delta > 0$, $\varepsilon > 0$. If $x \in U_{\delta,\varepsilon}$ and*

$$r < \min\left(\frac{R_0}{2}, r_1\right)\,, \tag{8.3}$$

then the total perimeter of $\mathcal{B}(x, \delta, \varepsilon, r, \Sigma) \subset \mathrm{FP}(U)$ in $B_r(x)$ is bounded by $\delta \cdot \mu(B_r(x))$:

$$\frac{1}{\mu(B_r(x))} \int_{\mathcal{B}} \mathrm{Per}(E)(B_r(x))\, d\Sigma < \delta\,. \tag{8.4}$$

Proof. The definition of $\mathcal{B}$ together with (8.3) implies that if $E \in \mathcal{B}$, then $\overline{B_r(x)} \subset \mathrm{Bad}_{\varepsilon, R_0}(E)$. Hence,

$$\mathrm{Per}(E)(B_r(x)) = \big(\mathrm{Per}(E) \llcorner \mathrm{Bad}_{\varepsilon, R_0}(E)\big)\, (B_r(x)). \tag{8.5}$$

Therefore,

$$\begin{aligned} \int_{\mathcal{B}} \mathrm{Per}(E)(B_r(x))\, d\Sigma &= \int_{\mathcal{B}} (\mathrm{Per}(E) \llcorner \mathrm{Bad}_{\varepsilon, R_0}(E))\, d\Sigma \\ &\le \int_{\mathrm{Cut}(U)} \big(\mathrm{Per}(E) \llcorner \mathrm{Bad}_{\varepsilon, R_0}(E)\big)\, d\Sigma \\ &= \lambda^{\mathrm{Bad}}_{\varepsilon, R_0}(B_r(x)) < \delta \cdot \mu(B_r(x))\,, \end{aligned}$$

where the last inequality follows from (7.8). (Actually, we only used $x \in U_4 \supset U_{\delta,\varepsilon}$, for U_4 as in the proof of Proposition 7.5.) $\square$

For the next lemma, we need a standard fact concerning $\mathbb{H}$. Namely, there exists $c > 0$ such that if $H \in \mathrm{HS}_{x'}$ for some $x' \in B_r(x)$, then

$$c \cdot r^{-1} \mu(B_{2r}(x)) \le \mathrm{Per}(H)(B_{2r}(x))\,. \tag{8.6}$$

This is easy to see, for example, by employing the coarea formula.

PROPOSITION 8.7. *There are constants* $\varepsilon_0 > 0$, $c_0 < \infty$, *such that if*

$$\varepsilon < \varepsilon_0 \,, \tag{8.8}$$

$$r < \min\left(\frac{r_0}{2}, \frac{R_0}{2}\right) , \tag{8.9}$$

then,

$$\Sigma(\mathcal{G}) \le c_0 r \delta^{-1} \,. \tag{8.10}$$

Proof. By the definition of $E \in \mathcal{G}$, there exists $x' \in \overline{B_r(x)} \cap \mathrm{Good}_{\varepsilon,R_0}$. By (8.9), this implies that for α as in Definition 6.1, we have

$$\alpha(E, x', 2r) \le \varepsilon \,.$$

Therefore, for some half-space, $H \in \mathrm{HS}_{x'}$,

$$\frac{1}{\mu(B_{2r}(x'))} \int_{B_{2r}(x')} |\chi_H - \chi_E| \, d\mu \le \varepsilon \,.$$

Thus, by (8.6) and the lower semicontinuity of perimeter with respect to L^1 convergence, there exists $c_1 > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,

$$c_1(n) r^{-1} \mu(B_{2r}(x)) \le \mathrm{Per}(E)(B_{2r}(x)) \,.$$

Therefore,

$$\lambda(B_{2r}(x)) \ge \int_{\mathcal{G}} \mathrm{Per}(E)(B_{2r}(x)) \, d\Sigma > c_1 r^{-1} \mu(B_{2r}(x)) \Sigma(\mathcal{G}).$$

Hence,

$$\begin{aligned} \Sigma(\mathcal{G}) &< \frac{\lambda(B_{2r}(x))}{c_1 r^{-1} \mu(B_{2r}(x))} \\ &< \delta^{-1} \frac{\mu(B_{2r}(x))}{c_1 r^{-1} \mu(B_{2r}(x))} \\ &= c_0 \delta^{-1} r \,, \end{aligned}$$

where the second inequality is a consequence of (7.7). □

9. The approximating cut measure supported on half-spaces

We retain the notation from the preceding section: Σ denotes an FP cut measure on a ball $U \subset X$, $x \in U$, $\delta > 0$, $\varepsilon > 0$, $r > 0$, and $\mathcal{G} = \mathcal{G}(x, \delta, \varepsilon, r, \Sigma)$, $\mathcal{B} = \mathcal{B}(x, \delta, \varepsilon, r, \Sigma)$ denote the corresponding sets of good and bad cuts.

We now construct a cut measure, $\widehat{\Sigma}(x, \delta, \varepsilon, r, \Sigma)$, supported on the collection of half-spaces HS. The measure, $\widehat{\Sigma}$, will be constructed by "straightening" the cuts in $\mathcal{G}$. If $x \in U_{\delta,\varepsilon}$ and r satisfies the smallness conditions (8.3), (8.9) in Propositions 8.2, 8.7, then the cut metric $d_{\hat{\Sigma}}$ will be close to d_Σ in the normalized L^1 metric on $B_r(x)$.

LEMMA 9.1. *There is a Borel map,*

$$\gamma : \mathcal{G} \to \mathrm{HS} \subset \mathrm{FP}(U)\,,$$

sending each $E \in \mathcal{G}$ *to a half-space,* $\gamma(E) \in \mathrm{HS}(\mathbb{H})$, *such that for some* $x' \in \overline{B_r(x)}$,

$$\frac{1}{\mu(B_{2r}(x'))} \int_{B_{2r}(x')} |\chi_E - \chi_{\gamma(E)}|\, d\mu < 2\varepsilon\,. \tag{9.2}$$

Proof. Let

$$W := \{(E, x', H) \in \mathrm{FP}(U) \times \overline{B_r(x)} \times \mathrm{HS} \mid H \in \mathrm{HS}_{x'}\}\,.$$

The collection of elements, $(E, x', H) \in W$, which satisfy

$$\frac{1}{\mu(B_{2r}(x'))} \int_{B_{2r}(x')} |\chi_E - \chi_H|\, d\mu < 2\varepsilon \tag{9.3}$$

is open in W, and as a consequence of its definition, maps to $\mathcal{G}$ under the projection, $\mathrm{FP}(U) \times \overline{B_r(x)} \times \mathrm{HS} \to \mathrm{FP}(U)$. Therefore, we can construct a Borel section of this projection over $\mathcal{G}$ as follows.

Each $E \in \mathcal{G}$ lies in an open set, $U_E \subset \mathcal{G}$, over which one has a section $\sigma_E : U_E \to W$ whose HS component is constant. A countable collection of these will cover $\mathcal{G}$. Hence, there is a countable disjoint cover by Borel sets, $\{V_i\}$, such that each V_i lies in a U_E. We define $\sigma : \mathcal{G} \to W$ by declaring that its restriction to V_i agrees with σ_E restricted to V_i. □

We define the Borel measure, $\hat{\Sigma}(x, \delta, \varepsilon, r, \Sigma)$, to be the pushforward under γ of the measure $\Sigma \llcorner \mathcal{G}$, where $\mathcal{G} = \mathcal{G}(x, \delta, \varepsilon, r, \Sigma)$:

$$\hat{\Sigma}(x, \delta, \varepsilon, r, \Sigma) := \gamma_*(\Sigma \llcorner \mathcal{G})\,. \tag{9.4}$$

It follows immediately that $\hat{\Sigma}$ is supported on $\mathrm{HS} \subset \mathrm{FP}(U)$. Since each $E \in \mathrm{HS}$ contributes uniformly bounded measure and uniformly bounded perimeter in U, it follows from (8.10) that $\hat{\Sigma}$ is an FP cut measure.

10. Proof of the main theorem

Here we prove Theorem 10.2, the main differentiation assertion for FP cut measures. Theorem 10.2 immediately implies Theorem 1.2. For convenience, we will assume U is a ball in $\mathbb{H}$. The argument also applies, mutatis mutandis, if $\mathbb{H}$ is replaced by $\mathbb{R}^k$. We retain the notation from Section 9.

For all $r > 0$, let $S_r : \mathbb{H} \to \mathbb{H}$ denote an automorphism which scales by r, and let $S_{x,r} : \mathbb{H} \to \mathbb{H}$ be the composition

$$\mathbb{H} \xrightarrow{S_r} \mathbb{H} \xrightarrow{l_x} \mathbb{H},$$

where $l_x : \mathbb{H} \longrightarrow \mathbb{H}$ denotes left translation by $x \in \mathbb{H}$. The pullback of a distance, d, under $S_{x,r}$ is denoted $S^*_{x,r}(d)$.

As in (3.36), let d_Σ denote the distance on U associated to a cut measure Σ. The L^1-distance between metrics, d, d', on $A \subset U$ is

$$\|d - d'\|_{L^1} := \int_{A\times A} |d(x_1, x_2) - d'(x_1, x_2)|\, d\mu \times d\mu\,. \tag{10.1}$$

We let $\mathcal{D}_{\mathrm{HS}}$ denote the collection of FP cut measures which are supported on half-spaces.

THEOREM 10.2. *Given an* FP *cut measure*, Σ, *there is a subset $U_0 \subset U$ of full Lebesgue measure such that if $x \in U_0$, then*

$$\lim_{r\to 0}\ \inf_{\bar\Sigma\in\mathcal{D}_{\mathrm{HS}}}\ \|\frac{1}{r} S^*_{x,r}(d_\Sigma) - d_{\bar\Sigma}\|_{L^1} = 0\,, \tag{10.3}$$

where the L^1-norm is taken on the unit ball $B_1(e)$. In particular, if $\Sigma = \Sigma_f$ is the cut measure corresponding to a BV *map*, $f : \mathbb{H} \to L^1(Y)$, *then* (10.3) *holds.*

Proof. Let x, r satisfy the hypotheses of Propositions 8.2, 8.7 and Lemma 9.1. We will show that on $B_r(x)$,

$$\|d_\Sigma - d_{\widehat\Sigma}\|_{L^1} \le r(4c_0\varepsilon\delta^{-1} + \tau'\delta)(\mu(B_r(x)))^2\,. \tag{10.4}$$

Here τ' denotes the constant in the Poincaré inequality (2.3). The theorem follows by letting $\varepsilon \to 0$ (which requires $r \to 0$), and then $\delta \to 0$.

Let $\widehat\Sigma$ denote the FP cut measure supported on half-spaces defined in (9.4). On $B_r(x)$, the triangle inequality gives

$$\begin{aligned}\|d_\Sigma - d_{\widehat\Sigma}\|_{L^1} &\le \|d_{\Sigma\llcorner\mathcal{G}} - d_{\widehat\Sigma}\|_{L^1} + \|d_{\Sigma\llcorner\mathcal{G}} - d_\Sigma\|_{L^1}\\ &= \|d_{\Sigma\llcorner\mathcal{G}} - d_{\widehat\Sigma}\|_{L^1} + \|d_{\Sigma\llcorner\mathcal{B}}\|_{L^1}\,.\end{aligned} \tag{10.5}$$

To complete the proof, we estimate each term on the right-hand side on the second line of (10.5).

The estimate from Proposition 8.7, which bounds the good cut measure, enters in the proof of the next lemma in a crucial way. Without it we would only be able to estimate the L^1 discrepancy between individual good cuts and their half-space approximations, but would be unable to estimate the aggregate effect on the cut metric of this discrepancy.

LEMMA 10.6. *On $B_r(x)$,*

$$\|d_{\Sigma\llcorner\mathcal{G}} - d_{\widehat\Sigma}\|_{L^1} \le 4c_0 r\varepsilon\delta^{-1}\,. \tag{10.7}$$

Proof. On $B_r(x)$, the left-hand side of (10.7) is equal to

$$\int_{B(x,r)\times B_r(x)} |d_{\Sigma\llcorner\mathscr{G}}(x_1,x_2) - d_{\hat{\Sigma}}(x_1,x_2)|\, d\mu\times d\mu$$
$$\leq \int_{B_r(x)\times B_r(x)} \left(\int_{\mathscr{G}} \big||\chi_E(x_1)-\chi_E(x_2)| - |\chi_{\gamma(E)}(x_1)-\chi_{\gamma(E)}(x_2)|\big|\, d\Sigma \right) d\mu\times d\mu$$
$$\leq \int_{\mathscr{G}} \left(\int_{B_r(x)} 2\mu(B_r(x))|\chi_E - \chi_{\gamma(E)}|\, d\mu \right) d\Sigma$$
$$\leq 4\int_{\mathscr{G}} \mu(B_r(x))(\varepsilon\mu(B_r(x)))\, d\Sigma \quad \text{(by (9.2))}$$
$$\leq 4\varepsilon\Sigma(\mathscr{G})(\mu(B_r(x)))^2$$
$$\leq 4c_0 r\varepsilon\delta^{-1} \quad \text{(by (8.10))}\,.$$

This concludes the proof. □

LEMMA 10.8. *On $B_r(x)$,*

$$\|d_{\Sigma\llcorner\mathscr{B}}\|_{L^1} \leq r\tau\delta(\mu(B_r(x)))^2\,. \tag{10.9}$$

Proof.

$$\|d_{\Sigma\llcorner\mathscr{B}}\|_{L^1} = \int_{\mathscr{B}} \left(\int_{B_r(x)\times B_r(x)} d_E(x,y)\, d\mu(x)\times d\mu(y) \right) d\Sigma \tag{10.10}$$
$$\leq cr\mu(B_r(x)) \int_{\mathscr{B}} \mathrm{Per}(E)(B_r(x))\, d\Sigma\;,$$

where the last inequality follows from the Poincaré inequality (2.3). From (10.10) and (8.4), we get (10.9). □

Combining (10.5), (10.7) and (10.9) gives (10.4), which suffices to complete the proof. □

Remark 10.11. A refinement of the proof of Theorem 10.2 yields the following stronger statement: For μ a.e. $x\in U$, blow ups of the FP cut measure converge to a translation invariant cut measure which is supported on half-spaces.

Remark 10.12. The proof presented here works for any Carnot group G for which the blow up result of [FSSC01] is valid; in particular, it holds for an arbitrary 2-step nilpotent Lie group. It seems almost certain that their result will hold for general nilpotent Lie groups.

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