ANNALS OF MATHEMATICS

Boundary rigidity and filling volume minimality of metrics close to a flat one

By Dmitri Burago and Sergei Ivanov

SECOND SERIES, VOL. 171, NO. 2

March, 2010

ANMAAH

Annals of Mathematics, **171** (2010), 1183–1211

Boundary rigidity and filling volume minimality of metrics close to a flat one

By DMITRI BURAGO and SERGEI IVANOV

Abstract

We say that a Riemannian manifold (M, g) with a non-empty boundary ∂M is a *minimal orientable filling* if, for every compact orientable $(\widetilde{M}, \tilde{g})$ with $\partial\widetilde{M} = \partial M$, the inequality $d_{\tilde{g}}(x, y) \geq d_g(x, y)$ for all $x, y \in \partial M$ implies $\operatorname{vol}(\widetilde{M}, \tilde{g}) \geq \operatorname{vol}(M, g)$. We show that if a metric g on a region $M \subset \mathbf{R}^n$ with a connected boundary is sufficiently C^2-close to a Euclidean one, then it is a minimal filling. By studying the equality case $\operatorname{vol}(\widetilde{M}, \tilde{g}) = \operatorname{vol}(M, g)$ we show that if $d_{\tilde{g}}(x, y) = d_g(x, y)$ for all $x, y \in \partial M$ then (M, g) is isometric to $(\widetilde{M}, \tilde{g})$. This gives the first known open class of boundary rigid manifolds in dimensions higher than two and makes a step towards a proof of Michel's conjecture.

1. **Introduction**

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M. Its *boundary distance function* is the restriction of the Riemannian distance d_g to $\partial M \times \partial M$. The term "boundary rigidity" means that the metric is uniquely determined by its boundary distance function. More precisely,

Definition 1.1. (M, g) is *boundary rigid* if every compact Riemannian manifold $(\widetilde{M}, \tilde{g})$ with the same boundary and the same boundary distance function is isometric to (M, g) via a boundary-preserving isometry.

It is easy to construct metrics that are not boundary rigid. For example, consider a metric on a disc with a "big bump" around a point p, such that the distance from p to the boundary is greater than the diameter of the boundary. Since no minimal geodesic between boundary points passes through p, a perturbation of the metric near p does not change the boundary distance function.

The first author is partially supported by NSF Grant DMS-0412166. The second author is partially supported by the RFBR grant 05-01-00939 and the Dynasty Foundation.

Thus one has to impose restrictions on the metric in order to make the boundary rigidity problem sensible. One natural restriction is the following: a Riemannian manifold (M, g) is called *simple* if the boundary ∂M is strictly convex, every two points $x, y \in M$ are connected by a unique geodesic, and geodesics have no conjugate points (cf. [15]). A more general condition called SGM ("strong geodesic minimizing") was introduced in [9] in order to allow nonconvex boundaries. Note that if (M, g) is simple, then M is a topological disc. The simplicity of (M, g) can be seen from the boundary distance function. The convexity of ∂M is equivalent to a (local) inequality between boundary distances and intrinsic distances of ∂M. The uniqueness of geodesics is equivalent to smoothness of the boundary distances. Thus if two Riemannian manifolds have the same boundary and the same boundary distance functions, then either both are simple or both are not.

CONJECTURE 1.2 (Michel [15]). *All simple manifolds are boundary rigid.*

Pestov and Uhlmann [17] proved this conjecture in dimension 2. In higher dimensions, few examples of boundary rigid metrics are known. They are: regions in $\mathbf{R}^n$ [12], in the open hemisphere [15], in symmetric spaces of negative curvature (follows from the main result of [3]), and in products of domains without conjugate points with $\mathbf{R}$ ([11]). We refer the reader to [8] and [17] for a survey of boundary rigidity, other inverse problems, and their applications.

One of the main results of this paper asserts that if (M, g) is C^2-close to a region in the Euclidean space, then (M, g) is rigid. For instance, to the best of our knowledge, this is the first known example of boundary rigid metrics in higher dimensions which do not have a special curvature tensor and, in particular, the first known open set of boundary rigid matrices. Our result also requires only C^2-smoothness, so that even in dimension 2 it is not completely covered by Pestov-Uhlmann's 2-dimensional theorem [17].

Our approach to boundary rigidity grew from [6] and [7], where we study minimality of flats in normed spaces, asymptotic volume of Finsler tori, and ellipticity of surface-area functionals. Even though our proof is not directly based on Finsler geometry, it is strongly motivated by Finsler considerations. Boundary rigidity here is treated as the equality case of the minimal filling problem discussed in [6] and [14].

Definition 1.3. (M, g) is a *minimal filling* if, for every compact $(\widetilde{M}, \tilde{g})$ with $\partial \widetilde{M} = \partial M$, the inequality

$$d_{\tilde{g}}(x, y) \geq d_g(x, y) \quad \text{for all } x, y \in \partial M$$

implies

$$\operatorname{vol}(\widetilde{M}, \tilde{g}) \geq \operatorname{vol}(M, g).$$

We say that (M, g) is a *minimal orientable filling* if the same holds under the additional assumption that $(\widetilde{M}, \tilde{g})$ is orientable.

CONJECTURE 1.4. *Every simple manifold is a minimal filling.*

If (M, g) is simple, then $\mathrm{vol}(M, g)$ is uniquely determined by d_g, namely there is an integral formula expressing $\mathrm{vol}(M, g)$ via d_g and its first order derivatives (the Santaló formula, [18]). It is not clear though whether the formula is monotone in d_g.

Our approach to Michel's Conjecture is to prove Conjecture 1.4 and then to obtain Michel's Conjecture by studying the equality case. So far we were able to carry out this plan for metrics close to a Euclidean one.

The main result of this paper is the following theorem:

THEOREM 1. *Let $M \subset \mathbf{R}^n$ be a compact region with a smooth boundary. There exists a C^2-neighborhood U of the Euclidean metric g_E on M such that, every $g \in U$ is a minimal orientable filling and is boundary rigid.*

One can check that actually we show that there is a $c(n) > 0$ such that, if g is a Riemannian metric in $\mathbf{R}^n$ satisfying $g = g_E$ outside $B_R(0)$ and $|K_\sigma| < \frac{c(n)}{R^2}$, then for any $\Omega \subset B_R(0)$, the space (Ω, g) is a minimal orientable filling and is boundary rigid. We do not know if the orientability assumption can be removed; this seems to be a rather intriguing question.

Known higher-dimensional examples of minimal fillings form a subset of known examples of rigid metrics: regions in $\mathbf{R}^n$ (follows from the Besikovitch inequality [2]) and regions in symmetric spaces of negative curvature [3].

There are many more examples of *locally* rigid metrics: for instance, simple almost nonpositively curved metrics and simple analytic metrics are locally rigid [10], [19]. The manifold (M, g) is said to be *locally (boundary) rigid* if every compact Riemannian manifold $(\widetilde{M}, \tilde{g})$ with the same boundary and the same boundary distance function is isometric to (M, g) via a boundary preserving isometry provided that g and $\tilde{g}$ are *a priori* sufficiently close. We want to emphasize that in Theorem 1 we do not impose any restrictions on $\widetilde{M}$.

All 2-dimensional simple manifolds are minimal fillings in a *restricted sense*: they are minimal only within the class of fillings homeomorphic to the disc [14]. In general (when $\widetilde{M}$ from Definition 1.3 may have handles), it is not known even if the standard hemisphere is a minimal orientable filling. That is, the filling volume (in the sense of M. Gromov) of the standard circle is not known.

However, it has been noticed by M. Gromov [12] that if $n \geq 3$, then one can assume that $\widetilde{M} \simeq D^n$ without loss of generality (i.e., the orientable filling volume can be realized by topological discs).

Remark 1.5. The Finsler case was very important for motivating our argument. Little is known about minimality of Finsler metrics, even though the Santaló formula still yields the normalized symplectic volume of the unit cotangent bundle (the Holmes-Thompson volume). This work originated from our study of minimality of flat Finsler metrics. However, there is no rigidity in the Finsler case. Here is a simple example.

Example. Let (M, g) be a simple Riemannian manifold. For every $p \in \partial M$ define a function $f_p : M \to \mathbf{R}$ by

$$f_p(x) = \mathrm{dist}_g(p, x).$$

Let $\{\tilde{f}_p\}$ be a C^3 perturbation of $\{f_p\}$ in the interior of M. Then $\{\tilde{f}_p\}$ is a family of distance functions of a Finsler metric with the same boundary distances (this metric is possibly nonsymmetric, but it can be made symmetric with some additional work). This Finsler metric is defined by

$$\|v\|_x = \sup_p \{d f_p(v)\}, \qquad x \in M,\ v \in T_x M.$$

We obtain Theorem 1 as a corollary of the following (more technical and more general):

THEOREM 2. *Let $M \subset \mathbf{R}^n$ be a compact region with a smooth boundary. There exists a C^2-neighborhood U of the Euclidean metric g_E on M such that for every $g \in U$ the following holds.*

If $(\widetilde{M}, \tilde{g})$ is an orientable piecewise C^0 Riemannian manifold such that $\partial \widetilde{M} = \partial M$ and the respective Riemannian distance functions d and $\tilde{d}$ satisfy

$$\tilde{d}(x, y) \geq d(x, y) \qquad \text{for all } x, y \in \partial M,$$

then

1. $\mathrm{vol}(\widetilde{M}, \tilde{g}) \geq \mathrm{vol}(M, g)$;
2. *If* $\mathrm{vol}(\widetilde{M}, \tilde{g}) = \mathrm{vol}(M, g)$ *then* $(\widetilde{M}, \tilde{g})$ *is isometric to* (M, g) *via a boundary-preserving isometry.*

Here by a piecewise C^0 Riemannian manifold we mean a smooth manifold, possibly with boundary, triangulated into simplices such that each simplex is C^1-diffeomorphic to the standard one and equipped with a continuous Riemannian structure. The Riemannian metrics on simplices do not have to agree on their common faces.

Deducing Theorem 1 *from Theorem* 2. To deduce Theorem 1 from Theorem 2 it suffices to check the following two facts.

1. The equality $\tilde{d}(x, y) = d(x, y)$ for all $x, y \in \partial M$ implies $\mathrm{vol}(\widetilde{M}, \tilde{g}) = \mathrm{vol}(M, g)$. Indeed, if M is convex (and hence simple), this immediately follows from the Santaló formula. Since we do not assume convexity, M may fail to be simple. However, it is easy to check that it still satisfies the SGM (Strong Geodesic

Minimizing) condition introduced by C. Croke [9]. Then Lemma 5.1 from [9] implies the desired equality $\mathrm{vol}(\widetilde{M}, \tilde{g}) = \mathrm{vol}(M, g)$.

2. The equality $\tilde{d}(x, y) = d(x, y)$ for all $x, y \in \partial M$ also implies that $\widetilde{M}$ is orientable. In fact, $\widetilde{M}$ is homeomorphic to M. Again, if M is convex, it is easy to show that both M and $\widetilde{M}$ are homeomorphic to a disc. For a general region $M \subset \mathbf{R}^N$ satisfying the conditions of Theorem 1 this is the contents of Remark 5.2 in the above mentioned paper [9].

2. **Plan of the proof**

In the "ideal world", the proof of boundary rigidity *should* go as follows: It is well-known that every compact metric space X can be embedded into $L^\infty(X)$ isometrically by sending x to $d(x, \cdot)$. By attaching appropriate collars, one can assume that both boundaries $\partial M = \partial \widetilde{M} = S$, where S is a standard sphere in $\mathbf{R}^n$, and that both metrics d and $\tilde{d}$ are extended by the standard Euclidean metric to the outside of S. Denote by $T_\alpha S$ the supporting hyperplane to S at $\alpha \in S$. One can see that since (M, g) is simple, the map ϕ from M to $\mathcal{L} = L^\infty(S)$ sending x to $\phi_x : S \to \mathbf{R} : \phi_x(\alpha) = d(x, T_p S)$ is also an isometry (in the strongest possible sense: it is a distance preserving map). Thus it is very tempting to think of this embedding as a "minimal surface" in $\mathcal{L}$. Applying the same construction to $\widetilde{M}$ one gets a Lipschitz-1 (and hence an area-nonincreasing) map $\tilde{\phi}$. Since M and $\widetilde{M}$ have the same boundary distance function, the embeddings ϕ and $\tilde{\phi}$ coincide on the common boundary $S = \partial M = \partial \widetilde{M}$. Furthermore, if d is a flat metric, then ϕ is a linear embedding. Hence our assumption that d is close to a Euclidean metric tells us that ϕ is close to a linear embedding. Then all we would need to conclude the "proof" is an infinite-dimensional analog of a well-known theorem (for instance, see Theorem 3 and Remark 3.1 of [16]) that a minimal surface close to an affine plane of the same dimension is the unique area-minimizer among all surfaces with the same boundary.

However, this approach encounters a number of difficulties:

1. When we speak about minimal surfaces, we need to define surface area. This is a major question. The space $\mathcal{L}$ naturally carries the structure of a normed space, and there are many different notions of surface area in normed spaces. It is very convenient to work with symplectic (the Holmes-Thompson, [13], [20]) surface area; however, there are too many minimal surfaces with respect to this surfaces area. We will fix this by introducing a surface area induced by a family of L^2-structure on $\mathcal{L}$.

2. We need to prove that ϕ is indeed a minimal surface. The fact that it is totally geodesic does not imply by itself minimality for nonstandard surface areas (e.g., see [1]). We verify minimality by means of a rather straightforward but cumbersome computation.

3. We need a very "robust" argument for the uniqueness of minimal surfaces close to affine planes. Our proof models a co-dimension-one argument showing that two co-dimension-one minimal surfaces with the same boundary coincide provided that both of them are graphs of functions (with respect to the same coordinates). Indeed, if the surfaces are graphs of f and g, consider a function $v(t) = \text{area}(\text{Graph}(tf + (1-t)g)$. We have $v'(0) = v'(1) = 0$ by minimality of f and g. By the Cauchy inequality v is convex on $t \in [0, 1]$. Furthermore, it is strictly convex unless $f = g$, and this implies that $f = g$. We will generalize this argument to higher co-dimensions (using the assumption that one of the surfaces is close to a plane).

3. **Attaching a collar**

This is a purely technical section. Its purpose is to reduce the problem to a special case when M is a Euclidean disc of radius 1, and g coincides with the standard Euclidean metric outside the ball of radius $\frac{1}{10n}$.

PROPOSITION 3.1. *Theorem* 2 *follows from its special case when*

(i) *M is a unit disc $D = B_1(0) \subset \mathbf{R}^n$ and g coincides with the standard Euclidean metric g_E on the "collar" $N = B_1(0) \setminus B_{1/10n}(0)$;*

(ii) *$\widetilde{M}$ contains N (with $\partial\widetilde{M} = \partial N$) and $\tilde{g} = g$ on N;*

(iii) *the distance functions d_g and $d_{\tilde{g}}$ satisfy the inequality $d_{\tilde{g}}(x, y) \geq d_g(x, y)$ for all $x, y \in N$.*

Proof. Let (M, g) and $(\widetilde{M}, \tilde{g})$ be as in Theorem 2. By means of re-scaling we assume that M is contained in the ball $B_{1/20n}(0) \subset \mathbf{R}^n$. We extend g to a smooth metric on the whole $\mathbf{R}^n$ so that g remains C^2-close to g_E and $g = g_E$ outside the ball $B_{1/10n}(0)$. (The extended metric is denoted by the same letter g.)

Let $M^+ = (D, g)$. We can think of M^+ as the result of attaching another "collar" $N' = D \setminus M$ to M. Attaching the same collar (N', g) to $(\widetilde{M}, \tilde{g})$ we obtain a manifold $\widetilde{M}^+ = \widetilde{M} \cup N'$ with a piecewise C^0 Riemannian metric (which we will also denote by $\tilde{g}$). Note that $N \subset N'$, so that $\tilde{g} = g = g_E$ on N.

The new spaces (M^+, g) and $(\widetilde{M}^+, \tilde{g})$ satisfy the conditions (i)–(iii). The first two are obvious. To verify (iii), consider $x, y \in N$ and observe that the length distance $d_{(\widetilde{M}^+,\tilde{g})}(x, y)$ depends only on $g|_N$ and $d_{(\widetilde{M},\tilde{g})}|_{\partial M\times\partial M}$ and the latter dependency is monotonous. Since $d_{(\widetilde{M},\tilde{g})} \geq d_{(M,g)}$ on ∂M, it follows that $d_{(\widetilde{M}^+,\tilde{g})}(x, y) \geq d_{(M^+,g)}(x, y)$.

It remains to note that the conclusion of Theorem 2 for (M^+, g) and $(\widetilde{M}^+, \tilde{g})$ implies the conclusion for (M, g) and $(M, \tilde{g})$. $\square$

Convention. From now we assume that (M, g) and $(\widetilde{M}, \tilde{g})$ from Theorem 2 satisfy the additional assumptions from Proposition 3.1.

4. **Distance-preserving embedding into L^∞**

We fix the following notation: $S = \partial M = \partial \widetilde{M} = S^{n-1}$ (recall that $M = D$ by the convention from the previous section); $\mathcal{L} = L^\infty(S)$.

The goal of this section is to construct Lipschitz-1 maps Φ_E, Φ and $\widetilde{\Phi}$ from $(\mathbf{R}^n, g_E)$, (M, g) and $(\widetilde{M}, \tilde{g})$ resp., to $\mathcal{L}$. When we speak about maps to $\mathcal{L}$, we always keep in mind the following construction.

Definition 4.1. Given a (measurable) family $\{F_\alpha\}_{\alpha\in S}$, of uniformly locally bounded functions $F_\alpha : M \to \mathbf{R}$, one can think of this family as a map $F : M \to \mathcal{L}$ where $F(x)(\alpha) = F_\alpha(x)$ for $x \in M$, $\alpha \in S$. We say that F_α are *coordinate functions* of F.

Note that a family $\{F_\alpha\}$ defining a given map F is not unique and may be defined only for almost every α.

LEMMA 4.2. *If $F : M \to \mathcal{L}$ is defined by a family $\{F_\alpha\}$ of coordinate functions and every F_α is Lipschitz*-1, *then so is F.*

Proof. This is immediate from the definition of the distance in $\mathcal{L} = L^\infty(S)$. □

Conversely, every Lipschitz-1 map $\Phi : M \to \mathcal{L}$ can be represented by Lipschitz-1 coordinate functions. We prove this in the next section; cf. Lemma 5.1.

Definition 4.3. Define $\Phi_E : \mathbf{R}^n \to \mathcal{L}$ by

$$\Phi_E(x)(\alpha) = \langle x, \alpha\rangle, \quad x \in \mathbf{R}^n, \alpha \in S$$

where $\langle , \rangle$ is the standard scalar product in $\mathbf{R}^n$.

Obviously Φ_E is a linear map. For $\alpha \in S$, the corresponding coordinate function $\Phi_{E\alpha} : \mathbf{R}^n \to \mathbf{R}$ is the scalar multiplication by α. Since α is a unit vector (recall that $S = \partial D$ is the unit sphere in $\mathbf{R}^n$), $\Phi_{E\alpha}$ is a Lipschitz-1 function. Then so is Φ_E. Moreover Φ_E is an isometric embedding. Indeed,

$$\|\Phi_E(x)\| = \sup_{\alpha\in S} \langle x, \alpha\rangle = |x|.$$

Definition 4.4. Let $\Phi : M \to \mathcal{L}$ be a map whose coordinate functions $\{\Phi_\alpha\}_{\alpha\in S}$ are given by

$$\Phi_\alpha(x) = 1 - \operatorname{dist}_g(x, H_\alpha)$$

where H_α is the hyperplane tangent to S at α, and dist_g is the distance with respect to g (assuming that $g = g_E$ outside M; recall that this is a smooth extension).

Observe that if this definition is applied to the Euclidean metric g_E in place of g, it yields the map $\Phi_E|_M$. Indeed, the Euclidean distance from H_α to $x \in M$ equals $1 - \langle x, \alpha\rangle$.

Since the metric g is C^2-close to g_E, the hyperplanes H_α have no focal points in M, hence the functions Φ_α are smooth distance-like functions. The Riemannian gradient of Φ_α at $x \in M$ is the initial velocity vector of the unique minimal geodesic connecting x to H_α.

Definition 4.5. Define a map $G : M \times S \to UTM$ by

$$G(x, \alpha) = \operatorname{grad} \Phi_\alpha(x)$$

where the gradient is taken with respect to the metric g.

We denote by G_E the similar function for g_E in place of g. Then

$$g_E(x, \alpha) = (x, \alpha) \in \mathbf{R}^n \times S \cong UT\mathbf{R}^n$$

(recall that S is the unit sphere in $\mathbf{R}^n$).

PROPOSITION 4.6. 1. $\Phi : (M, g) \to \mathcal{L}$ *is a distance-preserving map.*
2. Φ *is* C^1 *smooth.*
3. *The map* $G : M \times S \to UTM$ *is a diffeomorphism.*
4. Φ *is* C^1*-close to* Φ_E; G *is* C^1*-close to* G_E.

Proof. 1. Every Φ_α is Lipschitz-1, and so is Φ (by Lemma 4.2). It remains to show that $\|\Phi(x) - \Phi(y)\| \geq d_g(x, y)$, for all $x, y \in M$. Since $\Phi_\alpha(x)$ is continuous in α, we have

$$\|\Phi(x) - \Phi(y)\| = \sup_{\alpha \in S} |\Phi_\alpha(x) - \Phi_\alpha(y)|.$$

Let γ be a geodesic from x through y ($x = \gamma(0)$, $y = \gamma(t_1)$). It is close to a straight line while in M and coincides with a straight line after it leaves M. Eventually γ hits orthogonally one of the hyperplanes H_α; that is, $\gamma(t_2) \in H_\alpha$ and $\gamma'(t_2) \perp H_\alpha$ for some $\alpha \in S$ and $t_2 > t_1$. Since H_α has no focal points in M, we have $\operatorname{dist}_g(x, H_\alpha) = t_2$ and $\operatorname{dist}_g(y, H_\alpha) = t_2 - t_1$. Then

$$|\Phi_\alpha(x) - \Phi_\alpha(y)| = |\operatorname{dist}_g(x, H_\alpha) - \operatorname{dist}_g(y, H_\alpha)| = t_1 = d_g(x, y)$$

and the desired inequality follows.

2–4. Since g is C^2-close to g_E, the geodesic flow of g is C^1-close to that of g_E. In particular, the hyperplanes have no focal points in M. Then the distance functions of the hyperplanes and their gradients are recovered from the union of the hyperplanes' normal geodesic flows via the implicit function theorem, and they are C^1 close to their Euclidean counterparts. □

Remark 4.7. The assumption that g is close to g_E is needed only for the last statement of the proposition. The first three would follow for any simple metric g if we defined $\Phi_\alpha(x) = \operatorname{dist}_g(x, \alpha)$.

Now we are in a position to define a "surface" $\widetilde{\Phi} : \widetilde{M} \to \mathscr{L}$ spanning the same boundary as Φ. All we need is a Lipschitz-1 extension $\widetilde{\Phi}_\alpha$ of $\Phi_\alpha|_{\partial M}$ from $\partial M = \partial \widetilde{M}$ to $\widetilde{M}$. Such an extension exists due to the fact that $\Phi_\alpha|_{\partial \widetilde{M}}$ is Lipschitz-1 with respect to $d_{\tilde{g}}$. Indeed, it is Lipschitz-1 with respect to d_g and $d_{\tilde{g}} \geq d_g$ on $\partial \widetilde{M}$. (This is the only point where we use this key assumption of Theorem 2.) In order to ensure that the family $\{\widetilde{\Phi}_\alpha\}$ is measurable (in fact, continuous), we define an extension by an explicit formula. We also want $\widetilde{\Phi}$ to be reasonably close to Φ, and so we cut off too large values of the functions.

Definition 4.8. Let $\widetilde{\Phi} : \widetilde{M} \to \mathscr{L}$ be a map whose coordinate functions $\{\widetilde{\Phi}_\alpha\}_{\alpha \in S}$ are given by

$$\widetilde{\Phi}_\alpha(x) = \operatorname{cutoff}\Big(\inf_{y \in N} \{\Phi_\alpha(y) + d_{\tilde{g}}(x, y)\},\ \frac{2}{10n} + \operatorname{dist}_{\tilde{g}}(x, \widetilde{M} \setminus N)\Big)$$

where

$$\operatorname{cutoff}(a, b) = \min\{b, \max\{-b, a\}\}.$$

Recall that N is the "collar" (cf. Proposition 3.1).

PROPOSITION 4.9. 1. *$\widetilde{\Phi} : (\widetilde{M}, \tilde{g}) \to \mathscr{L}$ is a Lipschitz-*1 *map.*
2. $\Phi|_N = \widetilde{\Phi}|_N$.
3. $\widetilde{\Phi}(\widetilde{M} \setminus N)$ *is contained in the ball of radius* $\frac{2}{10n}$ *centered at the origin of* $\mathscr{L}$.

Proof. 1. Every $\widetilde{\Phi}_\alpha$ is Lipschitz-1 since it is obtained from a family of Lipschitz-1 functions by means of suprema and infima. Then by Lemma 4.2 $\widetilde{\Phi}$ is Lipschitz-1.

2. Since Φ is close to a linear isometry Φ_E and $M \setminus N$ is the disc of radius $\frac{1}{10n}$, we have $\sup_{M \setminus N} |\Phi_\alpha| \leq \frac{2}{10n}$. Let $x \in N$. Then

$$|\Phi_\alpha(x)| \leq \sup_{M \setminus N} |\Phi_\alpha| + \operatorname{dist}_g(x, \widetilde{M} \setminus N) \leq \frac{2}{10n} + \operatorname{dist}_{\tilde{g}}(x, \widetilde{M} \setminus N);$$

hence the cutoff does not apply. Furthermore,

$$\Phi_\alpha(x) \leq \Phi_\alpha(y) + d_g(x, y) \leq \Phi_\alpha(y) + d_{\tilde{g}}(x, y)$$

for all $y \in N$. (The inequalities follow from the facts that Φ_α is Lipschitz-1 with respect to g and $d_g \leq d_{\tilde{g}}$ on N.) Then the infimum in the definition of $\widetilde{\Phi}_\alpha$ is attained at $y = x$ and $\widetilde{\Phi}_\alpha(x) = \Phi_\alpha(x)$.

3. If $x \in \widetilde{M} \setminus N$, then $|\Phi_\alpha(x)| \leq \frac{2}{10n}$ due to cutoff, hence $\|\widetilde{\Phi}(x)\| \leq \frac{2}{10n}$. $\square$

5. Coordinates and derivatives

This section is technical. Its purpose is to validate our view of $\mathscr{L}$ as a "coordinate space" and $\widetilde{\Phi}$ as a "surface" (with tangent planes) in this space.

In this section M denotes an arbitrary Riemannian manifold while $S = S^{n-1}$ and $\mathscr{L} = L^\infty(S)$ are the same as in the previous section. Recall that a family $\{F_\alpha\}$ of functions on M defines a map $F : M \to \mathscr{L}$ (cf. Definition 4.1). The converse is more complicated since a point in $\mathscr{L}$ is a "function defined a.e." whose individual values do not make sense.

LEMMA 5.1. 1. *Every Lipschitz map $F : M \to \mathscr{L}$ can be represented by a family $\{F_\alpha\}_{\alpha\in S}$ of coordinate functions so that every $F_\alpha : M \to \mathbf{R}$ is Lipschitz with the same Lipschitz constant.*

2. *If $\{F_\alpha\}$ and $\{F'_\alpha\}$ are two such representations, then for almost every $\alpha \in S$, $F_\alpha = F'_\alpha$ everywhere on M.*

3. *If, in addition, M is a vector space and F is linear, then F_α is linear for almost every α.*

Proof. 1. Let X be a countable dense subset of M. For every $x \in X$, pick a function $f_x : S \to \mathbf{R}$ representing $F(x) \in L^\infty(S)$. Then for every $x, y \in X$,

$$|f_x(\alpha) - f_y(\alpha)| \le C|xy| \quad \text{for a.e. } \alpha \in S$$

where C is the Lipschitz constant of F and $|xy|$ is the distance in M. Since X is countable, we can redefine $f_x(\alpha)$ to be zero whenever the above inequality fails for at least one $y \in X$. Then $|f_x(\alpha) - f_y(\alpha)| \le C|xy|$ for all $x, y \in X$ and $\alpha \in S$, and we get a family of Lipschitz functions $F_\alpha : X \to M$. Every F_α admits a unique Lipschitz extension to the whole M, also denoted by F_α. It remains to note that for every $z \in M$, the function $\alpha \mapsto F_\alpha(z)$ represents $F(z)$ in $L^\infty(S)$. Indeed, if $f_z : S \to \mathbf{R}$ represents $F(z)$, then for almost every α the inequality $|f_z(\alpha) - f_x(\alpha)| \le C|zx|$ holds for all $x \in X$, and this property uniquely determines $f_z(\alpha) = F_\alpha(z)$.

2. For every $x \in M$, we have $F_\alpha(x) = F'_\alpha(x)$ for almost all α. Then by Fubini, for almost every α, the relation $F_\alpha(x) = F'_\alpha(x)$ holds for almost all $x \in M$, and hence for all $x \in M$ by continuity of F_α and F'_α.

3. Similarly, for almost every α, the relation $F_\alpha(x + y) = F_\alpha(x) + F_\alpha(y)$ holds for almost all pairs (x, y), and hence for all x, y. □

Definition 5.2. We say that a Lipschitz map $F : M \to \mathscr{L}$ is *weakly differentiable* at $x \in M$ if the coordinate function F_α is differentiable at x for almost every α. If so, we define the derivative $d_x F : T_x M \to \mathscr{L}$ to be the map whose coordinate functions are $d_x F_\alpha$.

We need the following version of Rademacher's Theorem:

LEMMA 5.3. *Let $F : M \to \mathscr{L}$ be a Lipschitz function. Then*

1. *F is weakly differentiable almost everywhere*;

2. *If F is weakly differentiable at $x \in M$, then the derivative $d_x F : T_x M \to \mathcal{L}$ is a Lipschitz linear map with the same Lipschitz constant.*

Proof. Every coordinate function F_α is Lipschitz and hence differentiable a.e. (by Rademacher's Theorem). Then by Fubini almost every $x \in M$ satisfies the following: for almost all α, F_α is differentiable at x. Furthermore, $\|d_x F_\alpha\| \le C$ where C is a Lipschitz constant for F. Then Lemmas 4.2 and 5.1, imply that $d_x F : T_x M \to \mathcal{L}$ is correctly defined and Lipschitz with the same constant. □

The map $d_x F$ introduced above is not a derivative in any traditional sense. We will use only a limited set of features of this "derivative", namely the following chain rule.

LEMMA 5.4. *Let $F : M \to \mathcal{L}$ be a Lipschitz function weakly differentiable at $x \in M$, and let μ be a continuous finite measure on S (that is, a measure with an L^1 density). Then*

1. *If $L : \mathcal{L} \to \mathbf{R}$ is a linear function of the form*

$$L(f) = \int_S f \, d\mu$$

then $L \circ F$ is differentiable at x and

$$d_x(L \circ F) = L \circ d_x F.$$

2. *If W is a finite-dimensional subspace of $\mathcal{L}$ and $P : \mathcal{L} \to W$ is the orthogonal projection with respect to the L^2 structure defined by μ, then $P \circ F$ is differentiable at x and*

$$d_x(L \circ P) = L \circ d_x P.$$

Proof. 1. Since the functions F_α are uniformly Lipschitz, the lemma follows immediately by differentiation under the symbol of integration.

2. The first part of the lemma implies that for every $w \in W$, the function $f \mapsto \langle f, w \rangle$ on $\mathcal{L}$ commutes with differentiation. Applying this to every w from a basis of W yields the second part. □

6. A Riemannian structure on $\mathcal{L}$

Definition 6.1. Let μ be a probability measure on S. We define a scalar product $\langle , \rangle_\mu$ on $\mathcal{L}$ by

$$\langle f, g \rangle_\mu = n \int_S fg \, d\mu.$$

We denote the space $\mathcal{L}$ equipped with this scalar product by $\mathcal{L}_\mu$, and the identical map $\mathrm{id}_{\mathcal{L}}$ regarded as a map from $\mathcal{L}$ to $\mathcal{L}_\mu$ by i_μ. Obviously i_μ is a Lipschitz map with Lipschitz constant n.

The normalizing factor n in the definition is introduced for the following reason: The integral of the square of a linear function of norm one against the normalized surface area over the unit sphere is equal to $\frac{1}{n}$.

LEMMA 6.2. *Let $A : \mathbf{R}^n \to \mathcal{L}$ be a Lipschitz-1 linear map. Then the composition $i_\mu \circ A : \mathbf{R}^n \to \mathcal{L}_\mu$ is area-nonexpanding. Furthermore, if $i_\mu \circ A$ is an area-preserving map then A and $i_\mu \circ A$ are linear isometries.*

Proof. Let $\{A_\alpha\}_{\alpha\in S}$ be the coordinate functions of A and $g_\mu = A^*(\langle,\rangle_\mu)$ be the pull-back of the scalar product in $\mathcal{L}_\mu$. Then

$$g_\mu(v,v) = n\int_S A_\alpha(v)^2 d\mu(\alpha).$$

Hence

$$\operatorname{trace}(g_\mu) = n\int_S \operatorname{trace}\left(A_\alpha^2\right) d\mu(\alpha) \le n$$

since trace $A_\alpha^2 = \|A_\alpha\|^2 \le 1$. Since g_μ is a positive definite symmetric matrix, we conclude the proof of the inequality by applying the inequality

$$\det(g_\mu) \le \left(\frac{1}{n}\operatorname{trace}(g_\mu)\right)^{n/2}.$$

The equality case obviously follows from the equality case in the above inequality. $\square$

Recall that there is a diffeomorphism $G : M\times S \to UTM$ with $G(x,\alpha) \in UT_xM$ (cf. Definition 4.5 and Proposition 4.6). Then for every $x\in M$, the map $G(x,\cdot) : S \to UT_xM$ is a diffeomorphism.

Definition 6.3. Let $x \in M$. We denote the inverse of $G(x,\cdot)$ by ω_x; that is, we define a map $\omega_x : UT_xM \to S$ by

$$\omega_x(G(x,\alpha)) = \alpha$$

for all $\alpha \in S$.

Let μ_x be the push-forward by ω_x of the normalized standard $(n-1)$-volume on the unit sphere UT_xM. For brevity, we denote $\mathcal{L}_{\mu_x}$ by $\mathcal{L}_x$ and similarly i_{μ_x} by i_x.

LEMMA 6.4. *In the above notation, $i_x \circ d_x\Phi : T_xM \to \mathcal{L}_x$ is a linear isometric embedding for every $x \in M$.*

Proof. Denote $U = UT_xM$. For every $v\in U$,

$$\begin{aligned}\|d_x\Phi(v)\|^2_{\mathcal{L}_x} &= n\int_S |d_x\Phi_\alpha(v)|^2\, d\mu_x(\alpha)\\ &= n\int_S \langle v, \omega_x^{-1}(\alpha)\rangle^2\, d\mu_x(\alpha) = n\int_U \langle v,u\rangle^2\, du = 1,\end{aligned}$$

where du denotes the normalized $(n-1)$-volume on U. The second equality follows from the definitions of G and ω_x: $\operatorname{grad}\Phi_\alpha(x) = G(x,\alpha) = \omega_x^{-1}(\alpha)$. The last integral equals $\frac{1}{n}$ since it does not depend on $v \in U$ (due to the symmetry of the measure), and if v ranges over an orthonormal basis of T_xM, the sum of the corresponding functions under the integral is the constant 1. $\square$

Recall that our surface $\Phi(M)$ is close to an n-dimensional linear subspace $\Phi_E(\mathbf{R}^n)$. We want to think of this surface as a graph of a map from this subspace to its "orthogonal complement" denoted by Q (see below). Then we extend our family of scalar products $\{\langle,\rangle_x\}_{x\in M}$ to a Riemannian structure on the whole $\mathscr{L}$. This Riemannian structure equals $\langle,\rangle_x$ at $\Phi(x)$ and is constant along subspaces parallel to Q. Then Lemmas 6.4 and 6.2 imply that Φ is an isometric embedding and $\widetilde{\Phi}$ is area-nonexpanding with respect to this Riemannian structure. We are going to prove the main theorem by comparing the areas of surfaces $\Phi(M)$ and $\widetilde{\Phi}(\widetilde{M})$ in the resulting infinite-dimensional Riemannian space.

To avoid unnecessary technical details, we do not refer directly to the Riemannian structure in $\mathscr{L}$. Instead, we consider a projection of $\widetilde{M}$ to M corresponding to the projection of $\widetilde{\Phi}(\widetilde{M})$ to $\Phi(M)$ along Q, and define "areas" in terms of scalar products $\langle,\rangle_x$.

Definition 6.5. Let H be a Euclidean space (not necessarily finite-dimensional) and $\varepsilon > 0$. We say that linear subspaces W_1 and W_2 of H are *ε-orthogonal* if $\angle(w_1, w_2) \geq \frac{\pi}{2} - \varepsilon$ for all nonzero vectors $w_1 \in W_1$, $w_2 \in W_2$.

PROPOSITION 6.6. *There are a codimension n linear subspace $Q \subset \mathscr{L}$ and a Lipschitz map $\pi : \widetilde{M} \to M$ satisfying the following*:

1. *For every $x \in M$, Q is ε-orthogonal to the image of $d_x\Phi$ in $\mathscr{L}_x$ for a small $\varepsilon > 0$.*

2. *For every $x \in \widetilde{M}$, $\Phi(\pi(x)) - \widetilde{\Phi}(x) \in Q$.*

3. *If $\widetilde{\Phi}$ is weakly differentiable at an $x \in \widetilde{M}$, then π is differentiable at x and $d_x(\Phi\circ\pi - \widetilde{\Phi})(v) \in Q$ for all $v \in T_x\widetilde{M}$.*

Proof. If M is Euclidean (that is, $g = g_E$) then μ_x is independent of x and coincides with the standard normalized $(n-1)$-volume ν on S. Since the map G is close to its Euclidean counterpart (cf. Proposition 4.6), the measures μ_x are absolutely continuous with respect to ν and have densities close to one. Thus every scalar product $\langle,\rangle_x$, $x \in M$, is close to the "flat" L^2 structure $\langle,\rangle_\nu$.

Let Q be the orthogonal complement to $W = \Phi_E(\mathbf{R}^n)$ with respect to $\langle,\rangle_\nu$. Since every scalar product $\langle,\rangle_x$ is close to $\langle,\rangle_\nu$, the first assertion of the proposition follows. Let $P : \mathscr{L} \to W$ be the orthogonal projection with respect to $\langle,\rangle_\nu$. Since Φ is C^1 close to Φ_E, the map $P\circ\Phi$ is a diffeomorphism of M to a region $\Omega \subset W$, and Ω is close to the unit ball in W.

Recall that (by Proposition 4.9) $\widetilde{\Phi}$ coincides with Φ on the “collar” N, and $\widetilde{\Phi}(\widetilde{M} \setminus N)$ is contained within the ball of radius $\frac{2}{10n}$ in $\mathscr{L}$, and hence within the ball of radius $\frac{2}{10}$ in $\mathscr{L}_\nu$. Therefore $P \circ \widetilde{\Phi}(\widetilde{M}) \subset \Omega$, and we can define $\pi : \widetilde{M} \to M$ by

$$\pi = (P \circ \Phi)^{-1} \circ (P \circ \widetilde{\Phi}).$$

The second assertion of the proposition follows immediately. If $\widetilde{\Phi}$ is weakly differentiable at x, then by the second part of Lemma 5.4 the map $P \circ \widetilde{\Phi}$ is differentiable at x and $d_x(P \circ \widetilde{\Phi}) = P \circ d_x\widetilde{\Phi}$. Then the last assertion follows since $P \circ \Phi$ is a diffeomorphism and Φ is smooth. □

Notation 6.7. We fix the notation π introduced in Proposition 6.6 for the rest of the paper, and introduce $\Phi^\pi = \Phi \circ \pi$ and $\mathscr{V} = \widetilde{\Phi} - \Phi^\pi$.

Definition 6.8. If $\widetilde{\Phi}$ is weakly differentiable at an $x \in \widetilde{M}$, denote by $J_x\widetilde{\Phi}$ the Jacobian (that is, the area-expansion coefficient) of $d_x\widetilde{\Phi}$ as a map from $T_x\widetilde{M}$ to $\mathscr{L}_{\pi(x)}$. By Lemma 5.3, $J_x\widetilde{\Phi}$ is defined for a.e. $x \in \widetilde{M}$. Then define

$$\operatorname{Area}(\widetilde{\Phi}) = \int_{\widetilde{M}} J_x\widetilde{\Phi}\, dx$$

where the integral is taken with respect to the Riemannian volume on $(\widetilde{M}, \tilde{g})$.

Now Lemma 6.2 implies

LEMMA 6.9. $\operatorname{Area}(\widetilde{\Phi}) \le \operatorname{vol}(\widetilde{M}, \tilde{g})$. *The equality in this inequality implies that* $J_x\widetilde{\Phi} = 1$ *for a.e.* $x \in \widetilde{M}$ *and* $d_x\widetilde{\Phi}$ *is a linear isometry.* □

7. First variation of surface area

The maps Φ^π and $\widetilde{\Phi}$ can be connected by a linear family of maps $\{\Phi_t\}_{t\in[0,1]}$ from $\widetilde{M}$ to $\mathscr{L}$ defined by $\Phi_t = \Phi^\pi + t\mathscr{V}$. We think of $\mathscr{V}$ as a vector field of variation of a surface Φ^π and introduce a quantity $\delta A(\Phi^\pi, \mathscr{V})$ which we call the first variation of surface area.

Definition 7.1. Let H be a (possibly infinite-dimensional) Euclidean space, and W an oriented n-dimensional linear subspace of H. Let P_W denote the orthogonal projection to W.

For an oriented Euclidean n-space X and a linear map $L : X \to H$, let $J_W(L)$ denote the Jacobian determinant of $P_W \circ L$ (which takes into account the orientation of X and W). We also think of $J_W(L)$ as an element of $\Lambda^n X^*$ (i.e., an exterior n-form on X), using the natural identification $\Lambda^n X^* = \mathbf{R}$. In this interpretation, $J_W(L)$ does not depend on the Euclidean structure of X.

For linear maps $L, V : X \to H$ introduce

$$\delta J_W(L, V) = \frac{d}{dt}\Big|_{t=0} J_W(L + tV).$$

Now define

$$\delta A(\Phi^\pi, \mathcal{V}) = \int_{\widetilde{M}} \delta J_{W_{\pi(x)}}(d_x\Phi^\pi, d_x\mathcal{V})\, dx, \tag{1}$$

where $W_{\pi(x)} = d_{\pi(x)}\Phi(T_{\pi(x)}M)$ is the tangent space to $\Phi(M)$ at $\Phi^\pi(x)$ regarded as a subspace of $\mathcal{L}_{\pi(x)}$, so that the term $J_{W_{\pi(x)}}$ is computed with respect to the scalar product $\langle , \rangle_{\pi(x)}$. The quantity $\delta A(\Phi^\pi, \mathcal{V})$ is well-defined since both $d_x\Phi^\pi$ and $d_x\mathcal{V}$ are defined a.e. The orientation of $W_{\pi(x)}$ is defined so that the map $d_{\pi(x)}\Phi : T_{\pi(x)}M \to W_{\pi(x)}$ is orientation-preserving.

Formula (1) can be read in two equivalent ways. First, it is an integral of a real-valued function against the Riemannian volume dx on $\widetilde{M}$. Second, the integrand can be regarded as an exterior n-form on T_xM (independent of the Riemannian structure), thus defining a (measurable) differential n-form on $\widetilde{M}$, and δA is the integral of this n-form over $\widetilde{M}$. In this section we use the latter meaning.

One can check that if π is a diffeomorphism, then $\delta A(\Phi^\pi, \mathcal{V})$ is the derivative at $t = 0$ of the n-dimensional surface area of $\Phi_t = \Phi^\pi + t\mathcal{V}$. Since we will not use this fact, we do not prove it here. We need a more complicated formula to handle the case of noninjective and singular π.

We think of Φ as a minimal surface, and therefore it is natural to expect that the first variation of surface area is zero. Indeed, this is the case, and the rest of this section is devoted to a proof of the following key proposition:

PROPOSITION 7.2. $\delta A(\Phi^\pi, \mathcal{V}) = 0$.

The proof consists of two parts. First, we compute the integrand of (1) at a point $x \in \widetilde{M}$. The result is written in terms of derivatives of π and the coordinate functions $\{\mathcal{V}_\alpha\}_{\alpha \in S}$ of $\mathcal{V}$.

Second, we represent the resulting expression as a differential form in a suitable manifold and integrate it using Stokes' formula. While this computation is probably valid for functions of so low regularity as we have, we do not verify this for every formula. Instead, we perform the computation assuming that the maps π and $\mathcal{V}$ are smooth. Then the general case follows by approximation. Indeed, we do not use any specific properties of our maps except that $\Phi^\pi = \Phi \circ \pi$ and that $\pi : \widetilde{M} \to M$ is a Lipschitz map, so that the computation proves the identity $\delta A(\Phi^\pi, \mathcal{V}) = 0$ for arbitrary smooth maps $\pi : \widetilde{M} \to M$ and $\mathcal{V} : \widetilde{M} \to \mathcal{L}$. The identity then follows for all Lipschitz maps since the integrand of (1) is expressed in terms of the first-order derivatives.

In addition, note that $\delta A(\Phi^\pi, \mathcal{V})$ is independent of the Riemannian metric on $\widetilde{M}$, so the fact that it is only piecewise C^0 does not play any role.

Notation. We denote by λ the oriented Riemannian volume form of (M, g). That is, if $y \in M$ and $v_1, \dots, v_n \in T_yM$, then $\lambda(v_1, \dots, v_n)$ is the oriented volume of the parallelotope spanned by $v_1, \dots, v_n$.

If ξ is an exterior k-form on a vector space X and $v \in X$, then $u \neg \xi$ denotes the $(k-1)$-form on X defined by

$$(v \neg \xi)(v_1, \dots, v_{n-1}) = \xi(v, v_1, \dots, v_{n-1})$$

for all $v_1, \dots, v_{n-1} \in X$. If ξ is a differential form and v is a vector field, the notation is applied point-wise.

Point-wise computation. Fix $x \in \widetilde{M}$ and denote $y = \pi(x) \in M$. To avoid cumbersome formulas, we introduce the following temporary notation: $U = UT_yM$, $W = W_y = d_y\Phi(T_yM)$. We regard W as a subspace of the Euclidean space $\mathscr{L}_y$ with the scalar product $\langle , \rangle_y$.

Recall that the unit sphere U with the standard normalized volume du is identified with (S, μ_y) via a map $\omega_y : U \to S$ (cf. Definition 6.3). Then we can "change coordinates" in $\mathscr{L}$ by identifying it with $L^\infty(U)$; this way $\langle , \rangle_y$ becomes the standard scalar product in $L^2(U, du)$.

LEMMA 7.3. *Let $L : T_x\widetilde{M} \to \mathscr{L}$ be a linear map with coordinate functions $\{L_\alpha\}_{\alpha \in S}$; then*

$$J_W(L) = \frac{n^n}{n!} \int_{U^n} \lambda(u_1, \dots, u_n)\, l_{u_1} \wedge l_{u_2} \wedge \dots \wedge l_{u_n}\, du_1 \dots du_n, \tag{2}$$

where $l_u = L_{\omega_y(u)}$.

Proof. Let $\{e_1, e_2, \dots e_n\}$ be an orthonormal positively oriented basis in T_yM. Then

$$J_W(L) = P_1 \wedge P_2 \dots \wedge P_n, \tag{3}$$

where P_i is a linear function on $T_x\widetilde{M}$ defined by

$$P_i(v) = \langle L(v), d_y\Phi(e_i)\rangle_y.$$

Indeed, $d_y\Phi$ is an isometric embedding of T_yM to $\mathscr{L}_y$ (cf. Lemma 6.4) and P_i is a composition of L and the orthogonal projection to the image of e_i. Then by definition of the scalar product in $\mathscr{L}_y$,

$$P_i(v) = n \int_S L_\alpha(v) d_y\Phi_\alpha(e_i)\, d\mu_y(\alpha) = n \int_S L_\alpha(v)\langle G(y, \alpha), e_i\rangle\, d\mu_y(\alpha)$$

(recall that $G(y, \alpha) = \operatorname{grad} \Phi_\alpha(y)$). Using the definition of μ_y (cf. 6.3) we rewrite the formula as

$$P_i(v) = n \int_U l_u(v)\langle u, e_i\rangle\, du.$$

Then (3) takes the form

$$J_W(L) = n^n \int_{U^n} l_{u_1} \wedge l_{u_2} \wedge \dots \wedge l_{u_n} \langle u_1, e_1\rangle\langle u_2, e_2\rangle \dots \langle u_n, e_n\rangle\, du_1 \dots du_n.$$

Note that if we replaced the basis $\{e_i\}$ by another one obtained by permuting the vectors $e_1, e_2, \dots, e_n$, the same formula holds for positive permutations, and it acquires a minus sign for negative ones. Adding these formulas for all permutations of $\{e_1, e_2, \dots e_n\}$, we get

$$n! J_W(L) = n^n \int_{U^n} l_{u_1} \wedge l_{u_2} \wedge \dots \wedge l_{u_n} \det(\langle u_i, e_j \rangle)_{i,j=1}^n \, du_1 \dots du_n.$$

We complete the proof of the lemma by noting that the determinant of the matrix $(\langle u_i, e_j \rangle)$ is just the oriented volume of the parallelotope spanned by $u_1, u_2, \dots, u_n$. □

LEMMA 7.4. *If $L = d_x \Phi^\pi$ and $V : T_x \widetilde{M} \to \mathscr{L}$ is a linear map with coordinates $\{V_\alpha\}_{\alpha \in S}$, then*

$$\delta J_W(L, V) = c(n) \int_U v_u \wedge \pi^*(u \neg \lambda) \, du \tag{4}$$

where $v_u = V_{\omega_y(u)}$ and π^* denotes the pull-back of a form under (the derivative of) π.

Proof. As in Lemma 7.3, define $l_u = L_{\omega_y(u)}$ where $\{L_\alpha\}_{\alpha \in S}$ are coordinate functions of L. Then for $\xi \in T_x \widetilde{M}$, $u \in U$ and $\alpha = \omega_y(u)$, we have

$$l_u(\xi) = L_\alpha(\xi) = d_y \Phi_\alpha(d_x \pi(\xi)) = \langle G(y, \alpha), d_x \pi(\xi) \rangle = \langle u, d_x \pi(\xi) \rangle.$$

Introducing a co-vector $u^\circ \in T_y^* M$ by $u^\circ = \langle u, \cdot \rangle$, we rewrite this formula as

$$l_u = \pi^*(u^\circ). \tag{5}$$

To compute $\delta J_W(L, V) = \frac{d}{dt}\big|_{t=0} J_W(L + tV)$, we plug $l_u + t v_u$ for l_u in (2) and differentiate it with respect to t. We get

$$\delta J_W(L, V) = \frac{n^n}{n!} \int_{U^n} \sum_{k=1}^n (-1)^{k-1} \lambda(\mathbf{u}) \, v_{u_k} \wedge \Big(\bigwedge_{i \neq k} l_{u_i} \Big) \, d\mathbf{u}$$

where $\mathbf{u}$ stands for $(u_1, \dots, u_n)$ and $d\mathbf{u}$ for $du_1 \dots du_n$. Using the symmetry of the formula with respect to permuting u_i's, we rewrite it as

$$\delta J_W(L, V) = \frac{n^{n+1}}{n!} \int_{U^n} \lambda(\mathbf{u}) \, v_{u_1} \wedge \Big(\bigwedge_{i=2}^n l_{u_i} \Big) \, d\mathbf{u} = \frac{n^{n+1}}{n!} \int_U v_u \wedge A(u) \, du, \tag{6}$$

where $A(u)$ is an $(n-1)$-form on $T_x \widetilde{M}$ given by

$$A(u) = \int_{U^{n-1}} \Big(\lambda(u, u_1, \dots, u_{n-1}) \bigwedge_{i=1}^{n-1} l_{u_i} \Big) \, du_1 \dots du_{n-1}.$$

From (5) we have $l_{u_i} = \pi^*(u_i^\circ)$; then

$$A(u) = \pi^*(B(u))$$

where

$$B(u) = \int_{U^{n-1}} \left(\lambda(u, u_1, \dots, u_{n-1}) \bigwedge_{i=1}^{n-1} u_i^\circ \right) du_1 \dots du_{n-1}.$$

Observe that $B(u)$ depends only on u and the Euclidean structure of $T_y M$, in particular, it is equivariant under the action of the orthogonal group. Such an $(n-1)$-form is unique up to a constant factor, and $u \neg \lambda$ is an example of such a form. Therefore $B(u) = c_1(n) u \neg \lambda$, $A(u) = c_1(n) \pi^*(u \neg \lambda)$ and the lemma follows by plugging this into (6). □

Changing the variable u to $\alpha = \omega_y(u)$ under the integral in (4), we get

$$\delta J_W(L, V) = c(n) \int_S V_\alpha \wedge \pi^*(G(y, \alpha) \neg \lambda)\, d\mu_y(\alpha).$$

(recall that $G(y, \alpha) = \omega_y^{-1}(\alpha)$). This finishes the point-wise computation for which we needed temporary notation. Substituting the definitions of L, y and U, we get

$$\delta J_{W_{\pi(x)}}(d_x \Phi^\pi, V) = c(n) \int_S V_\alpha \wedge \pi^*(G(\pi(x), \alpha) \neg \lambda)\, d\mu_y(\alpha).$$

Substitution of $d_x \mathcal{V}$ for $\mathcal{V}$ (assuming that $\mathcal{V}$ is weakly differentiable at x) yields

$$(7) \qquad \delta J_{W_{\pi(x)}}(d_x \Phi^\pi, d_x \mathcal{V}) = c(n) \int_S d_x \mathcal{V}_\alpha \wedge \pi^*(G(\pi(x), \alpha) \neg \lambda)\, d\mu_y(\alpha),$$

where $\{\mathcal{V}_\alpha\}_{\alpha \in S}$ are the coordinate functions of $\mathcal{V}$.

Integration of the form. Note that the expression in (7) (as a function of x) is a differential n-form on $\widetilde{M}$, and $\delta A(\Phi^\pi, \mathcal{V})$ is the integral of this form over $\widetilde{M}$. We are going to rewrite this as an integral of a differential $(2n-1)$-form over $\widetilde{M} \times S$. Define a map $P : \widetilde{M} \times S \to M \times S$ by

$$P(x, \alpha) = (\pi(x), \alpha), \qquad x \in \widetilde{M}, \alpha \in S.$$

We need $(n-1)$-forms σ and $\tilde{\sigma}$ on $M \times S$ and $\widetilde{M} \times S$ to represent integration over the family of measures μ_y, $y \in M$. Namely, define

$$\sigma(y, \alpha) = P_2^* \mu_y(\alpha), \qquad y \in M, \alpha \in S,$$

where $P_2 : M \times S \to S$ is the coordinate projection and μ_y is regarded as an $(n-1)$-form on S. Similarly define

$$\tilde{\sigma}(x, \alpha) = \widetilde{P}_2^* \mu_{\pi(x)}(\alpha), \qquad x \in \widetilde{M}, \alpha \in S,$$

where $\widetilde{P}_2$ is the coordinate projection $\widetilde{M} \times S \to S$. Note that $\tilde{\sigma} = P^*(\sigma)$.

We say that a differential form ξ on $M\times S$ represent a family of forms $\{\xi_\alpha\}_{\alpha\in S}$ on M if for every $\alpha\in S$, $\xi_\alpha=\xi|_{M\times\{\alpha\}}$; more precisely, $\xi_\alpha=i_\alpha^*(\xi)$ where $i_\alpha: M\to M\times S$ is defined by $i_\alpha(x)=(x,\alpha)$. One easily checks the following properties:

1. If forms ξ and η represent families $\{\xi_\alpha\}_{\alpha\in S}$ and $\{\eta_\alpha\}_{\alpha\in S}$, then $\xi\wedge\eta$ represents $\{\xi_\alpha\wedge\eta_\alpha\}_{\alpha\in S}$.

2. If a form ξ on $M\times S$ represents a family $\{\xi_\alpha\}_{\alpha\in S}$ of forms on M, then the form $P^*\xi$ on $\widetilde{M}\times S$ represents the family $\{\pi^*\xi\}$ of forms on $\widetilde{M}$.

3. If ξ is an n-form on $\widetilde{M}\times S$ representing a family $\{\xi_\alpha\}_{\alpha\in S}$, then

$$\int_{\widetilde{M}}\left(\int_S \xi_\alpha(x)\,d\mu_{\pi(x)}(\alpha)\right)dx=\int_{\widetilde{M}\times S}\xi\wedge\tilde{\sigma}.$$

Combining this with (7) we get

$$\delta A(\Phi^\pi,\mathcal{V})=\int_{\widetilde{M}}\delta J_{W_{\pi(x)}}(d_x\Phi^\pi,d_x\mathcal{V})\,dx=c(n)\int_{\widetilde{M}\times S}\xi\wedge P^*\eta\wedge\tilde{\sigma} \tag{8}$$

where ξ is any 1-form on $\widetilde{M}\times S$ representing the family $\{d_x\mathcal{V}_\alpha\}_{\alpha\in S}$ of 1-forms on $\widetilde{M}$, η is an $(n-1)$-form on $M\times S$ representing the family $\{G_\alpha\neg\lambda\}_{\alpha\in S}$ of $(n-1)$-forms on M. Here G_α is a vector field on M defined by $G_\alpha(x)=G(x,\alpha)$.

We have to specify ξ and η in (8). First define $\xi=dF$ where the function $F:\widetilde{M}\times S\to\mathbf{R}$ is given by

$$F(x,\alpha)=\mathcal{V}_\alpha(x). \tag{9}$$

Obviously $\xi=dF$ represents the family $\{d_x\mathcal{V}_\alpha\}_{\alpha\in S}$.

To define η, introduce a vector field γ on $M\times S$ so that for every $(y,\alpha)\in M\times S$ the projection of the vector $\gamma(y,\alpha)$ to M equals $G_\alpha(y)$ and the projection to S is zero. Let λ_0 denote the n-form on $M\times S$ computing the oriented Riemannian volume of the projection to M. Note that λ_0 is the pull-back of λ under the coordinate projection $M\times S\to M$. Now define

$$\eta=\gamma\neg\lambda_0.$$

The definitions imply that η represents the family $\{G_\alpha\neg\lambda\}_{\alpha\in S}$.

Plugging $\xi=dF$ into (8), we get

$$\delta A(\Phi^\pi,\mathcal{V})=c(n)\int_{\widetilde{M}\times S}dF\wedge P^*\eta\wedge\tilde{\sigma}.$$

Using the identity $\tilde{\sigma}=P^*\sigma$, we rewrite this as follows:

$$\delta A(\Phi^\pi,\mathcal{V})=c(n)\int_{\widetilde{M}\times S}dF\wedge P^*(\eta\wedge\sigma). \tag{10}$$

Recall that $G: UTM\to M\times S$ is a diffeomorphism, and the measure $d\mu_y dy$ on $M\times S$ (where du is the Riemannian volume on M) is the pull-back of the

Liouville measure on UTM under G. Denote by μ the differential $(2n-1)$-form on $M \times S$ corresponding to this measure. Then

$$\mu = \lambda_0 \wedge \sigma$$

by the definitions of λ_0 and σ. Observe that $\gamma \neg \sigma = 0$ since γ is tangent to the fibers $M \times \{\alpha\}$ and these fibers annulate σ. Hence

$$\eta \wedge \sigma = (\gamma \neg \lambda_0) \wedge \sigma = \gamma \neg (\lambda_0 \wedge \sigma) = \gamma \neg \mu.$$

Then (10) takes the form

$$\delta A(\Phi^\pi, \mathcal{V}) = c(n) \int_{\widetilde{M} \times S} dF \wedge P^*(\gamma \neg \mu). \tag{11}$$

For every $\alpha \in S$, the vector field γ on a $M \times \{\alpha\}$ projects to the vector field $G_\alpha = \operatorname{grad} \Phi_\alpha$ on M. The trajectories of G_α are geodesics since Φ_α is a distance function. Hence the flow on $M \times S$ generated by γ is mapped by G to the geodesic flow on UTM. Since the geodesic flow preserves the Liouville measure, the flow generated by γ preserves μ. This implies that $\gamma \neg \mu$ is a closed form. Then $P^*(\gamma \neg \mu)$ is closed: $d(P^*(\gamma \neg \mu)) = 0$. Therefore

$$dF \wedge P^*(\gamma \neg \mu) = d(F \cdot P^*(\gamma \neg \mu)).$$

Then from (11),

$$\delta A(\Phi^\pi, \mathcal{V}) = c(n) \int_{\widetilde{M} \times S} d(F \cdot P^*(\gamma \neg \mu)) = c(n) \int_{\partial \widetilde{M} \times S} F \cdot P^*(\gamma \neg \mu))$$

by Stokes' formula. The last integral is zero since F vanishes on the boundary of $\widetilde{M} \times S$ (cf. (9)). This finishes the proof of Proposition 7.2.

8. **An estimate on δJ**

Let H be a (possibly infinite-dimensional) Euclidean space and X an oriented Euclidean n-space. For a linear map $L : X \to H$ we denote by $J(L)$ the (nonnegative) Jacobian of L.

Let W be an oriented n-dimensional subspace of H. We use the notation $J_W(L)$ and $\delta J_W(L, V)$ from Definition 7.1 for linear maps $L, V : X \to H$.

PROPOSITION 8.1. *There exists a constant $\varepsilon = \varepsilon(n) > 0$ such that the following holds. In the above notation, if $L(X) \subset W$ and $V(X) \subset Q$ where $Q \subset H$ is a codimension n linear subspace and Q is ε-orthogonal to W (cf. Definition* 6.5), *then*

$$J(L + V) \geq J_W(L) + \delta J_W(L, V), \tag{12}$$

and the equality implies that either $V = 0$ or both L and $L + V$ are degenerate (have ranks less than n), and in either case $J(L + V) = J_W(L)$.

Proof. The images of maps L, V and $L+V$ are contained in the subspace $W+L(X)$ of dimension at most $2n$. Therefore it suffices to prove the proposition in the case when $\dim H = 2n$. Then $\dim W = \dim Q = n$.

Introduce a family of linear maps $L_t : X \to H$, $t \in [0,1]$ by $L_t = L + t \cdot V$. Then by definition,

$$\delta J_W(L,V) = \frac{d}{dt}\Big|_{t=0} J_W(L_t).$$

We will show that

$$J(L_t) \geq J_W(L) + t \cdot \delta J_W(L,V) \tag{13}$$

for all $t \geq 0$; then (12) follows by substitution of $t = 1$.

If $\alpha \in \Lambda^n(H)$ is a decomposable n-vector $\alpha = v_1 \wedge v_2 \wedge \cdots \wedge v_n$, we denote by $\|\alpha\|$ the n-volume of the parallelotope spanned by $v_1, v_2, \ldots, v_n$. Note that the scalar product $\langle , \rangle$ in H canonically determines a scalar product in $\Lambda^n(H)$. We also denote this scalar product by $\langle , \rangle$. Then $\|\cdot\|$ is a Euclidean norm on $\Lambda^n(H)$ corresponding to this scalar product.

Denote $\Lambda_k = \Lambda^k(W) \wedge \Lambda^{n-k}(Q)$. The assumption that Q and W are almost orthogonal implies that Λ_i and Λ_j $(i \neq j)$ are almost orthogonal. Namely, if $\xi \in \Lambda_i$ and $\eta \in \Lambda_j$ $(i \neq j)$ then

$$\langle \xi, \eta \rangle \leq \varepsilon_1 \, \|\xi\| \, \|\eta\| \tag{14}$$

for some $\varepsilon_1 = \varepsilon_1(\varepsilon, n)$, $\varepsilon_1 \to 0$ as $\varepsilon \to 0$.

Let $\alpha(t) \in \Lambda^n(H)$ denote the image of the unit positively oriented n-vector in $\Lambda^n(X) \simeq \mathbf{R}$ under $(L_t)_*$. In other words,

$$\alpha(t) = L_t(e_1) \wedge L_t(e_2) \wedge \cdots \wedge L_t(e_n)$$

where $e_1, e_2, \ldots, e_n$ is a positive orthonormal basis of X. Then $J(L_t) = \|\alpha(t)\|$. Obviously $\alpha(t)$ is a polynomial of the form

$$\alpha(t) = \sum_{i=0}^{n} \alpha_i t^i, \tag{15}$$

where $\alpha_i \in \Lambda_i$.

LEMMA 8.2. *Assuming that ε is sufficiently small, there exists a constant $c(n)$ such that*

$$\|\alpha_0\| \, \|\alpha_k\| \leq c(n) \, \|\alpha_1\| \, \|\alpha_{k-1}\| \tag{16}$$

where α_i are as defined by (15).

Proof. Since Q and W are ε-orthogonal, application of a linear transformation making them orthogonal changes all norms in the exterior algebra by factors close to 1. Thus we can assume that Q and W are orthogonal, and identify $H = W \oplus Q$ with $\mathbf{R}^n \times \mathbf{R}^n$.

If L_0 is degenerate then the left-hand side of (16) is zero, and the inequality is obvious. Otherwise we can choose a basis in X so that the matrix $\{L_{ij}, i = 1, 2 \dots 2n, j = 1, 2 \dots n\}$ of L_0 consists of two blocks: the identity matrix $\{L_{ij}, i = 1, 2 \dots n, j = 1, 2 \dots n\}$ (corresponding to the projection to W) and the zero matrix $\{L_{ij}, i = n+1, n+2 \dots 2n, j = 1, 2 \dots n\}$ (corresponding to the projection to Q). Then the first block of L_t remains the identity matrix for all t (by the definition of the family $\{L_t\}$, and the second block is tB, where B is some (fixed) matrix. Even though the norms on exterior powers depend on the choice of a basis, both parts of (16) are multiplied by the same constant. Hence changing coordinates in X is an admissible procedure.

Note that $\|\alpha_k\|^2$ is the sum of the squares of all $n \times n$-minors of (the matrix of) L_1 such that exactly k rows are chosen in the lower half of the matrix (that is, in B). Since the upper-half of L_t is the identity matrix, every such minor is equal to a $k \times k$-minor of B. Hence $\|\alpha_k\|^2$ is the binomial coefficient times the sum of the squares of all $k \times k$-minors of B.

In our coordinates, $\alpha_0 = 1$. Since every $k \times k$-minor is a sum of products of $(k-1) \times (k-1)$-minors and 1×1-minors, the lemma follows. $\square$

Let σ denote the unit positively oriented n-vector in $\Lambda^n W \simeq \mathbf{R}$. Note that $J_W(\beta) = \langle \sigma, \beta \rangle$ for every $\beta \in \Lambda^n(H)$. Hence $\delta J_W(L, V) = \langle \alpha_1, \sigma \rangle$ and $J_W(L_0) = \langle \alpha_0, \sigma \rangle$. Thus (13) takes the form

$$\|\alpha(t)\| \geq \langle \alpha_0, \sigma \rangle + t \langle \alpha_1, \sigma \rangle,$$

or, after squaring (note that the left-hand side is nonnegative),

$$\|\alpha(t)\|^2 \geq \langle \alpha_0, \sigma \rangle^2 + 2t \langle \alpha_0, \sigma \rangle \langle \alpha_1, \sigma \rangle + t^2 \langle \alpha_1, \sigma \rangle^2.$$

Since α_0 is proportional to σ and $\|\sigma\| = 1$, we have $|\langle \alpha_0, \sigma \rangle| = \|\alpha_0\|$ and $\langle \alpha_0, \sigma \rangle \langle \alpha_1, \sigma \rangle = \langle \alpha_0, \alpha_1 \rangle$. Thus the desired inequality takes the form

$$\|\alpha(t)\|^2 \geq \|\alpha_0\|^2 + 2t \langle \alpha_0, \alpha_1 \rangle + t^2 \langle \alpha_1, \sigma \rangle^2.$$

We will actually prove the following stronger inequality:

$$\|\alpha(t)\|^2 \geq \|\alpha_0\|^2 + 2t \langle \alpha_0, \alpha_1 \rangle + t^2 \langle \alpha_1, \sigma \rangle^2 + \frac{1}{10} \|\alpha(t) - \alpha_0\|^2. \tag{17}$$

The additional term $\frac{1}{10} \|\alpha(t) - \alpha_0\|^2$ in the right-hand side of this inequality will help us to analyze the equality case in (13).

Denote $\beta(t) = t^2 \alpha_2 \cdots + t^n \alpha_n$; then $\alpha(t) = \alpha_0 + t\alpha_1 + \beta(t)$ and

$$\|\alpha(t)\|^2 = \|\alpha_0\|^2 + 2t \langle \alpha_0, \alpha_1 \rangle + t^2 \|\alpha_1\|^2 + 2 \langle \alpha_0, \beta(t) \rangle + 2t \langle \alpha_1, \beta(t) \rangle + \|\beta(t)\|^2.$$

Since $\alpha_1 \in \Lambda_1$ is ε_1-orthogonal to $\sigma \in \Lambda_0$, we have $\|\alpha_1\|^2 \geq 10\langle\alpha_1, \sigma\rangle^2$, so it suffices to prove that

$$\frac{9}{10}t^2\|\alpha_1\|^2 + 2\langle\alpha_0, \beta(t)\rangle + 2t\langle\alpha_1, \beta(t)\rangle + \|\beta(t)\|^2 \geq \frac{1}{10}\|\alpha(t) - \alpha_0\|^2.$$

Since α_1 is ε_1-orthogonal to each Λ_i, $i > 1$, which in their turn are also almost orthogonal, one can easily see that α_1 is, say, $2\sqrt{n}\varepsilon_1$-orthogonal to $\beta(t) \in \Lambda_2 \oplus \cdots \oplus \Lambda_n$ (provided that ε_1 is small enough). Then we have

$$\frac{1}{10}t^2\|\alpha_1\|^2 + 2t\langle\alpha_1, \beta(t)\rangle + \frac{1}{10}\|\beta(t)\|^2 \geq 0.$$

It remains to prove that

$$\frac{8}{10}t^2\|\alpha_1\|^2 + 2\langle\alpha_0, \beta(t)\rangle + \frac{9}{10}\|\beta(t)\|^2 \geq \frac{1}{10}\|\alpha(t) - \alpha_0\|^2.$$

Observe that

$$\frac{1}{10}\|\alpha(t) - \alpha_0\|^2 = \frac{1}{10}\|t\alpha_1 + \beta(t)\|^2 \leq \frac{2}{10}\left(t^2\|\alpha_1\|^2 + \|\beta(t)\|^2\right);$$

hence it suffices to prove that

$$\frac{6}{10}t^2\|\alpha_1\|^2 + 2\langle\alpha_0, \beta(t)\rangle + \frac{7}{10}\|\beta(t)\|^2 \geq 0. \tag{18}$$

Combining the triangle inequality with (14) and (16), we get

$$|\langle\alpha_0, \beta(t)\rangle| \leq \sum_{i=2}^{n} |\langle\alpha_0, t^i\alpha_i\rangle| \leq \varepsilon_1 \sum_{i=2}^{n} t^i \|\alpha_0\|\|\alpha_i\| \leq \varepsilon_1 c(n) \sum_{i=2}^{n} t^i \|\alpha_1\|\|\alpha_{i-1}\|.$$

We may assume that $\varepsilon_1 c(n) < \frac{1}{10}$. Then, separating the first term, we get

$$|\langle\alpha_0, \beta(t)\rangle| \leq \frac{1}{10}t^2\|\alpha_1\|^2 + \varepsilon_1 c(n) \sum_{i=3}^{n} t^i \|\alpha_1\|\|\alpha_{i-1}\|.$$

Using the above inequality, one sees that, to prove (18) it suffices to show that

$$\frac{4}{10}t^2\|\alpha_1\|^2 - 2\varepsilon_1 c(n) \sum_{i=3}^{n} t^i \|\alpha_1\|\|\alpha_{i-1}\| + \frac{7}{10}\|\beta(t)\|^2 \geq 0. \tag{19}$$

Recall that $\beta(t) = \sum_{i=2}^{n} t^i\alpha_i$, and the terms $t^i\alpha_i$ are mutually ε_1-orthogonal. Hence

$$\|\beta(t)\|^2 \geq \frac{3}{4}\sum_{i=2}^{n} t^{2i}\|\alpha_i\|^2$$

provided that ε_1 small enough (and $\frac{3}{4}$ is just a number smaller than 1). Then (19) follows from

$$\frac{4}{10}t^2\|\alpha_1\|^2 - 2\varepsilon_1 c(n)\sum_{i=2}^{n-1} t^{i+1}\|\alpha_1\|\|\alpha_i\| + \frac{4}{10}\sum_{i=2}^{n} t^{2i}\|\alpha_i\|^2 \ge 0. \tag{20}$$

We assume that ε is so small that $\varepsilon_1 c(n) < \frac{1}{10n}$. Then

$$\frac{1}{10n}t^2\|\alpha_1\|^2 - 2\varepsilon_1 c(n)t^{i+1}\|\alpha_1\|\|\alpha_i\| + \frac{1}{10}t^{2i}\|\alpha_i\|^2 \ge 0,$$

for all $i = 2, 3, \dots, n-1$, and the desired inequality (20) follows by adding them.

Now let us consider the equality case in (12), or, equivalently, in (13) for $t = 1$. Since we proved a stronger inequality (17), the equality implies that $\|\alpha(1)-\alpha_0\|=0$. Hence the images of L and $L_1 = L + V$ either coincide or degenerate (of dimension less than n). Furthermore, since the image of L is almost orthogonal to the image of L, this implies that $V = 0$ unless L has rank smaller than n, in which case V has rank smaller than n as well. Since $\alpha(1) - \alpha_0 = \alpha_1 + \alpha_2 + \dots + \alpha_n$ and the terms α_i belong to the respective components Λ_i of the direct sum $\Lambda^n(H) = \bigoplus \Lambda_i$, it follows that $\alpha_i = 0$ for all $i \ge 1$. Then $\delta J_W(L, V) = \langle \alpha_1, \sigma\rangle = 0$, hence $J(L + V) = J_W(L)$. □

9. **Proof of Theorem** 2

Let $x \in \widetilde{M}$ be such that $\widetilde{\Phi}$ is weakly differentiable at x. Consider $X = T_x\widetilde{M}$, $H = \mathcal{L}_{\pi(x)}$, $L = d_x\Phi^\pi : X \to H$, $V = d_x\mathcal{V} : X \to H$ (cf. Notation 6.7) and $W = W_{\pi(x)}$ (cf. Definition 7.1). Note that $L(X) \subset W$. By Proposition 6.6, $L(V) \subset Q$ where Q is ε-orthogonal to W for a small ε. Then Proposition 8.1 applies, and we have

$$J_x(\widetilde{\Phi}) \ge J_{W_{\pi(x)}}(d_x\Phi^\pi) + \delta J_{W_{\pi(x)}}(d_x\Phi^\pi, d_x\mathcal{V}). \tag{21}$$

By means of integration we get

$$\mathrm{Area}(\widetilde{\Phi}) \ge \int_{\widetilde{M}} J_{W_{\pi(x)}}(d_x\widetilde{\Phi}^\pi)\,dx + \delta A(\Phi^\pi, \mathcal{V}).$$

(cf. Definitions 6.8 and 7.1). By Proposition 7.2, the last term is zero; thus

$$\mathrm{Area}(\widetilde{\Phi}) \ge \int_{\widetilde{M}} J_{W_{\pi(x)}}(d_x\Phi^\pi). \tag{22}$$

Recall that $\Phi^\pi = \Phi \circ \pi$ and hence $d_x\Phi^\pi = d_{\pi(x)}\Phi \circ d_x\pi$. By Definition 7.1 and Lemma 6.9, $d_{\pi(x)}\Phi$ is an orientation-preserving isometry from $T_{\pi(x)}M$ to $W_{\pi(x)}$. Hence the integrand $J_{W_{\pi(x)}}(d_x\Phi^\pi)$ of (22) is nothing but the signed Jacobian of the map $\pi : \widetilde{M} \to M$ at x. Then the right-hand part of (22) equals the volume of (M, g); thus

$$\mathrm{Area}(\widetilde{\Phi}) \ge \mathrm{vol}(M, g).$$

By Lemma 6.9 we have $\mathrm{Area}(\widetilde{\Phi}) \le \mathrm{vol}(\widetilde{M}, \tilde{g})$, and the inequality part of the theorem follows.

To analyze the equality case, note that all the above inequalities have to turn into equalities almost everywhere on $\widetilde{M}$. The equality part of Lemma 6.9 implies that $J_x(\widetilde{\Phi}) = 1$ for a.e. $x \in \widetilde{M}$. Then by Proposition 8.1, the equality in (21) implies that

$$J_{W_{\pi(x)}}(d_x \Phi^\pi) = J_x(\widetilde{\Phi}) = 1$$

for a.e. $x \in \widetilde{M}$. Hence by the equality case of Proposition 8.1, we conclude that $d_x \mathcal{V} = 0$ (that is, the tangent spaces to the images of Φ and $\widetilde{\Phi}$ at corresponding points are parallel). Again observe that $J_{W_{\pi(x)}}(d_x \Phi^\pi)$ equals the signed Jacobian of π at x, and thus we get that $d_x \pi$ is an orientation-preserving linear isometry from $T_x \widetilde{M}$ to $T_{\pi(x)} M$ for almost all $x \in \widetilde{M}$.

Now the theorem follows from the next lemma (compare with Sublemma for Theorem 1 of [5] and Appendix C of [3]) :

LEMMA 9.1. *Let $\widetilde{M}$ be a piece-wise C^0 Riemannian manifold and M a smooth Riemannian manifold and* $\mathrm{vol}(\widetilde{M}) = \mathrm{vol}(M)$. *Let $\pi : \widetilde{M} \to M$ be a surjective Lipschitz map such that the differential $d_x \pi$ is a linear isometry for almost all x, and $\pi(\partial \widetilde{M}) \subset \partial M$. Then π is an isometry.*

Proof. Since $d_x \pi$ is a linear isometry for almost all $x \in \widetilde{M}$, π is a Lipschitz-1 map. Hence it is volume-nonexpanding. Then the assumption $\mathrm{vol}(\widetilde{M}) = \mathrm{vol}(M) = \mathrm{vol}(\pi(\widetilde{M}))$ implies that π is volume-preserving: $\mathrm{vol}(\pi(U)) = \mathrm{vol}(U)$ for every measurable set $U \subset \widetilde{M}$.

Recall that $\widetilde{M}$ is triangulated into n-dimensional simplices with C^0 Riemannian metrics. Let Σ be the union of $\partial \widetilde{M}$ and the $(n-2)$-skeleton of the triangulation.

For an $x \in \widetilde{M}$, we denote by C_x the tangent cone of $\widetilde{M}$ at x. By definition, it is a length space identified with the vector space $T_x \widetilde{M}$ (or half-space if $x \in \partial \widetilde{M}$) and split into a number of polyhedral cones corresponding to simplices adjacent to x. Each cone carries a flat metric defined by the Riemannian tensor of the corresponding simplex at x, and the whole metric of C_x is obtained by gluing these Euclidean metrics together in the usual length metric sense.

It is easy to see that the volume of a small metric ball centered at $x \in \widetilde{M}$ is approximately equal to that of a similar ball in C_x. More precisely,

$$\mathrm{vol}(B_\varepsilon(x)) = \mathrm{vol}(B)\varepsilon^n + o(\varepsilon^n), \quad \varepsilon \to 0,$$

where B is a unit metric ball in C_x centered at the origin. (Note that B may be larger than the union of balls in the polyhedral cones that form C_x since nonisometric gluing can decrease distances). If $x \in \widetilde{M} \setminus \Sigma$, then the tangent cone is a Euclidean space or the result of gluing of two half-spaces along a linear map between their boundaries. Hence

$$\operatorname{vol}(B_\varepsilon(x)) \geq \omega_n \varepsilon^n + o(\varepsilon^n), \quad \varepsilon \to 0,$$

where ω_n is the volume of the standard Euclidean n-ball.

We prove the lemma in a number of steps.

1. *The map $\pi_1 := \pi|_{\widetilde{M}\setminus\Sigma}$ is injective and its image is contained in $M \setminus \partial M$.*

Suppose that $\pi(x) = \pi(y)$ for some $x, y \in \widetilde{M} \setminus \Sigma$, $x \neq y$. For a sufficiently small $\varepsilon > 0$, the balls $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint and contained in $\widetilde{M} \setminus \Sigma$. Since C_x is either a Euclidean space or a union of two half-spaces, we have

$$\operatorname{vol}(B_\varepsilon(x)) \geq \omega_n \varepsilon^n + o(\varepsilon^n), \quad \varepsilon \to 0,$$

and similarly for y,

$$\operatorname{vol}(B_\varepsilon(x) \cup B_\varepsilon(y)) \geq 2\omega_n \varepsilon^n + o(\varepsilon^n), \quad \varepsilon \to 0.$$

Since π is Lipschitz-1, the images of balls $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are contained in the ε-ball centered at $\pi(x) = \pi(y)$. On the other hand,

$$\operatorname{vol}(B_\varepsilon(\pi(x))) = \omega_n \varepsilon^n + o(\varepsilon^n) < \operatorname{vol}(B_\varepsilon(x) \cup B_\varepsilon(y))$$

contrary to the fact that π is volume-preserving. Thus π_1 is injective.

The second statement follows similarly: if $x \in \widetilde{M} \setminus \Sigma$ and $\pi(x) \in \partial M$, then

$$\operatorname{vol}(B_\varepsilon(\pi(x))) = \frac{1}{2}\omega_n \varepsilon^n + o(\varepsilon^n) < \operatorname{vol}(B_\varepsilon(x)),$$

a contradiction.

2. *The metrics of the adjacent simplices of the triangulation of $\widetilde{M}$ agree on the $(n-1)$-dimensional faces.*

Let $x \in \widetilde{M} \setminus \Sigma$. The tangent cone C_x is obtained by gluing together two Euclidean half-spaces H_1 and H_2. We have to show that the metrics of H_1 and H_2 agree on their common hyperplane. Suppose the contrary. Then some points are closer to the origin in C_x than they would be in the disjoint union of H_1 and H_2. Hence the unit ball in C_x is strictly larger that the union of two Euclidean half-balls in H_1 and H_2; therefore the volume of the ball is greater than ω_n. Thus

$$\operatorname{vol}(B_\varepsilon(x)) = C\omega_n \varepsilon^n + o(\varepsilon^n)$$

for some $C > 1$. This leads to a contradiction as in Step 1.

3. *The map $\pi_1 = \pi|_{\widetilde{M}\setminus\Sigma}$ is a locally bi-Lipschitz homeomorphism onto an open subset of $M \setminus \partial M$.*

Since $\widetilde{M} \setminus \Sigma$ and $M \setminus \partial M$ are n-dimensional manifolds without boundaries, by the Brouwer Invariance of the Domain Theorem ([4]), the injectivity implies that π_1 is an open map; hence its inverse π_1^{-1} is continuous.

Since the metrics agree on the $(n-1)$-dimensional faces of $\widetilde{M}$, we may regard $\widetilde{M} \setminus \Sigma$ as a manifold (with some differential structure) with a C^0 Riemannian metric. Note that the continuity of metric coefficients implies that the relation

$$\mathrm{vol}(B_\varepsilon(x)) = \omega_n \varepsilon^n + o(\varepsilon^n), \quad \varepsilon \to 0,$$

is uniform in x on any compact subset of $\widetilde{M} \setminus \Sigma$, and similarly in $M \setminus \partial M$. Fix an $x \in \widetilde{M} \setminus \Sigma$, let y be sufficiently close to x, and suppose that $\varepsilon := |\pi(x)\pi(y)| < \frac{1}{2}|xy|$. Then the balls $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint; therefore

$$\mathrm{vol}(\pi(B_\varepsilon(x) \cup B_\varepsilon(y))) = \mathrm{vol}(B_\varepsilon(x) \cup B_\varepsilon(y)) = 2\omega_n \varepsilon^n + o(\varepsilon^n).$$

On the other hand, $\pi(B_\varepsilon(x) \cup B_\varepsilon(y)) \subset B_\varepsilon(\pi(x)) \cup B_\varepsilon(\pi(y))$ since π is Lipschitz-1, but the balls $B_\varepsilon(\pi(x))$ and $B_\varepsilon(\pi(y))$ contain a ball of radius $\varepsilon/2$ in their intersection; therefore

$$\mathrm{vol}(B_\varepsilon(\pi(x)) \cup B_\varepsilon(\pi(y))) \le \big(2 - 1/2^n\big)\omega_n \varepsilon^n + o(\varepsilon^n) < \mathrm{vol}(\pi(B_\varepsilon(x) \cup B_\varepsilon(y)))$$

if ε is small enough. This contradiction shows that $|\pi(x)\pi(y)| \ge \frac{1}{2}|xy|$ if y is sufficiently close to x. It follows that π_1^{-1} is locally Lipschitz at $\pi(x)$.

4. *π is an isometry.*

First observe that π_1 is an isometry of length spaces $\widetilde{M} \setminus \Sigma$ and $\pi(\widetilde{M} \setminus \Sigma)$. Indeed, since π_1^{-1} is Lipschitz, it is differentiable a.e., and its differential, wherever defined, is the inverse of that of π. Then $d_y(\pi_1^{-1})$ is a linear isometry for almost all $y \in \pi(\widetilde{M} \setminus \Sigma)$. It follows that π_1^{-1} is Lipschitz-1 (with respect to the induced length distances). Since both π_1 and π_1^{-1} are Lipschitz-1, π_1 is an isometry (of induced length metrics).

It remains to show that the induced length metrics on $\widetilde{M} \setminus \Sigma$ and $\pi(\widetilde{M} \setminus \Sigma)$ coincide with the restrictions of the ambient metrics of $\widetilde{M}$ and M. This follows from the fact that the sets Σ in $\widetilde{M}$ and $\pi(\Sigma)$ are "small": each of them consists of a subset of a boundary and a set of Hausdorff dimension at most $n-2$. Every piecewise curve can be perturbed so as to avoid such a set while almost preserving the length; so removing these sets does not change the length distances. □

Acknowledgements. The authors are grateful to Y. Kurylev and G. Uhlmann for introducing them to and giving them a taste of Inverse Problems and for many useful remarks. The first author is grateful to the organizers of the conference "Perspectives in Inverse Problems", 2004, Helsinki. To a large extent, the very idea to undertake this project was ignited by this conference. The authors are grateful to the anonymous referee for an extraordinary detailed and useful report and, in particular, for pointing out a few inaccuracies in the preliminary version of the paper.

References

[1] J. C. ALVAREZ-PAIVA and G. BERCK, What is wrong with the Hausdorff measure in Finsler spaces, *Adv. Math.* **204** (2006), 647–663. MR 2007g:53079 Zbl 1099.53018

[2] A. S. BESICOVITCH, On two problems of Loewner, *J. London Math. Soc.* **27** (1952), 141–144. MR 13,831d Zbl 0046.05304

[3] G. BESSON, G. COURTOIS, and S. GALLOT, Entropies et rigidités des espaces localement symétriques de courbure strictement négative, *Geom. Funct. Anal.* **5** (1995), 731–799. MR 96i:58136 Zbl 0851.53032

[4] L. E. J. BROUWER, Beweis der invarianz des n-dimensionalen gebiets, *Math. Ann.* **71** (1911), 305–313. MR 1511658 JFM 42.0418.01

[5] D. BURAGO and S. IVANOV, On asymptotic volume of tori, *Geom. Funct. Anal.* **5** (1995), 800–808. MR 96h:53041 Zbl 0846.53043

[6] ———, On asymptotic volume of Finsler tori, minimal surfaces in normed spaces, and symplectic filling volume, *Ann. of Math.* **156** (2002), 891–914. MR 2003k:53088 Zbl 1040.53088

[7] ———, Gaussian images of surfaces and ellipticity of surface area functionals, *Geom. Funct. Anal.* **14** (2004), 469–490. MR 2006b:53099 Zbl 1067.53042

[8] C. B. CROKE, I. LASIECKA, G. UHLMANN, and M. S. VOGELIUS (eds.), *Geometric Methods in Inverse Problems and* PDE *Control*, *The* IMA *Volumes in Mathematics and its Applications* **137**, Springer-Verlag, New York, 2004. MR 2006d:00010 Zbl 1051.35003

[9] C. B. CROKE, Rigidity and the distance between boundary points, *J. Differential Geom.* **33** (1991), 445–464. MR 92a:53053 Zbl 0729.53043

[10] C. B. CROKE, N. S. DAIRBEKOV, and V. A. SHARAFUTDINOV, Local boundary rigidity of a compact Riemannian manifold with curvature bounded above, *Trans. Amer. Math. Soc.* **352** (2000), 3937–3956. MR 2000m:53054 Zbl 0958.53027

[11] C. B. CROKE and B. KLEINER, A rigidity theorem for simply connected manifolds without conjugate points, *Ergodic Theory and Dynam. Systems* **18** (1998), 807–812. MR 99j:53035 Zbl 0924.53027

[12] M. GROMOV, Filling Riemannian manifolds, *J. Differential Geom.* **18** (1983), 1–147. MR 85h:53029 Zbl 0515.53037

[13] R. D. HOLMES and A. C. THOMPSON, n-dimensional area and content in Minkowski spaces, *Pacific J. Math.* **85** (1979), 77–110. MR 81k:52023 Zbl 0467.51007

[14] S. V. IVANOV, On two-dimensional minimal fillings, *Algebra i Analiz* **13** (2001), 26–38. MR 2002b:58016 Zbl 0995.58010

[15] R. MICHEL, Sur la rigidité imposée par la longueur des géodésiques, *Invent. Math.* **65** (1981), 71–83. MR 83d:58021 Zbl 0471.53030

[16] F. MORGAN, Examples of unoriented area-minimizing surfaces, *Trans. Amer. Math. Soc.* **283** (1984), 225–237. MR 85b:49079 Zbl 0509.53007

[17] L. PESTOV and G. UHLMANN, Two dimensional compact simple Riemannian manifolds are boundary distance rigid, *Ann. of Math.* **161** (2005), 1093–1110. MR 2006c:53038 Zbl 1076.53044

[18] L. A. SANTALÓ, *Integral Geometry and Geometric Probability*, Addison-Wesley Publ. Co., Reading, MA, 1976. MR 55 #6340 Zbl 0342.53049

[19] P. STEFANOV and G. UHLMANN, Boundary rigidity and stability for generic simple metrics, *J. Amer. Math. Soc.* **18** (2005), 975–1003. MR 2006h:53031 Zbl 1079.53061

[20] A. C. THOMPSON, *Minkowski Geometry*, *Encycl. Math. Appl.* **63**, Cambridge Univ. Press, Cambridge, 1996. MR 97f:52001 Zbl 0868.52001

(Received June 29, 2006)

E-mail address: burago@math.psu.edu
DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY,
STATE COLLEGE, PA 16802

E-mail address: svivanov@pdmi.ras.ru
V. A. STEKLOV MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES,
FONTANKA 27, 191023 ST. PETERSBURG, RUSSIA

http://eimi.imi.ras.ru/eng/perso/svivanov.php