A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver

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Abstract

We extend the notion of a $p$-local finite group (defined in [BLO03]) to the notion of a $p$-local group. We define morphisms of $p$-local groups, obtaining thereby a category, and we introduce the notion of a representation of a $p$-local group via signalizer functors associated with groups. We construct a chain $G = (G_0 \to G_1 \to \cdots)$ of 2-local finite groups, via a representation of a chain $G^* = (G_0 \to G_1 \to \cdots)$ of groups, such that $G_0$ is the 2-local finite group of the third Conway sporadic group $Co_3$, and for $n > 0$, $G_n$ is one of the 2-local finite groups constructed by Levi and Oliver in [LO02]. We show that the direct limit $G$ exists in the category of 2-local groups, and that it is the 2-local group of the union of the chain $G^*$. The 2-completed classifying space of $G$ is shown to be the classifying space $B D I(4)$ of the exotic 2-compact group of Dwyer and Wilkerson [DW93].

Introduction

In [BLO03], Broto, Levi, and Oliver introduced the notion of a $p$-local finite group $\mathcal{G}$, consisting of a finite $p$-group $S$ and a pair of categories $\mathcal{F}$ and $\mathcal{L}$ (the fusion system and the centric linking system) whose objects are subgroups of $S$, and which satisfy axioms which encode much of the structure that one expects from a finite group having $S$ as a Sylow $p$-subgroup. If indeed $G$ is a finite group with Sylow $p$-subgroup $S$, then there is a canonical construction which associates to $G$ a $p$-local finite group $\mathcal{G} = \mathcal{G}_S(G)$, such that the $p$-completed nerve of $\mathcal{L}$ is homotopically equivalent to the $p$-completed classifying space of $G$. A $p$-local finite group $\mathcal{G}$ is said to be exotic if $\mathcal{G}$ is not equal to $\mathcal{G}_S(G)$ for any finite group $G$ with Sylow group $S$.

From the work of various authors (cf. [BLO03, §9]), it has begun to appear that for $p$ odd, exotic $p$-local finite groups are plentiful. On the other hand, exotic

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2-local finite groups are – as things stand at this date – quite exceptional. In fact, the known examples of exotic 2-local finite groups fall into a single family \( \mathcal{G}_{\text{Sol}}(q) \), \( q \) an odd prime power, constructed by Ran Levi and Bob Oliver [LO02]. With hindsight, the work of Ron Solomon [Sol74] in the early 1970’s may be thought of as a proof that \( \mathcal{G}_{\text{Sol}}(q) \neq \mathcal{G}_S(G) \) for any finite simple group \( G \) with Sylow group \( S \).

Solomon considered finite simple groups \( G \) having a Sylow 2-subgroup isomorphic to that of \( \text{Co}_3 \) (the smallest of the three sporadic groups discovered by John Conway), and he showed that any such \( G \) is isomorphic to \( \text{Co}_3 \). While proving this, he was also led to consider the situation in which \( G \) has a single conjugacy class \( z^G \) of involutions, and \( C_G(z) \) has a subgroup \( H \) with the following properties:

\[
H \cong \text{Spin}_7(q), \quad q = r^n, \quad q \equiv 3 \text{ or } 5 \mod 8, \quad \text{and} \quad C_G(z) = O(C_G(z))H.
\]

Here \( \text{Spin}_7(q) \) is a perfect central extension of the simple orthogonal group \( \Omega_7(q) \) by a group of order 2, and for any group \( X \), \( O(X) \) denotes the largest normal subgroup of \( X \) all of whose elements are of odd order.

Solomon showed that there is no finite simple group \( G \) which satisfies the above conditions – but he was not able to do this by means of “2-local analysis” (i.e. the study of the normalizers of 2-subgroups of \( G \)). Indeed a potential counterexample possessed a rich and internally consistent 2-local structure. It was only after turning from 2-local subgroups to local subgroups for the prime \( r \) that a contradiction was reached.

One of the achievements of [LO02] is to suggest that the single “sporadic” object \( \text{Co}_3 \) in the category of groups is a member of an infinite family of exceptional objects in the category of 2-local groups. But in addition, [LO02] establishes a special relationship between the \( \mathcal{G}_{\text{Sol}}(q) \)’s and the exotic 2-adic finite loop space \( DI(4) \) of Dwyer and Wilkerson [DW93]. Namely, in [LO02] it is shown that the classifying space \( BD\text{I}(4) \) is homotopy equivalent to the 2-completion of the nerve of a union of subcategories of the linking systems \( L_{\text{Sol}}(q^n) \), with the union taken for any fixed \( q \) as \( n \) goes to infinity. (This result was prefigured in, and motivated by, work of David Benson [Ben94]. Benson showed, first, that the 2-cohomology ring \( H^*(BD\text{I}(4); 2) \) is finitely generated over \( H^*(\text{Co}_3; 2) \), and second, that the 2-cohomology of the space of fixed points in \( BD\text{I}(4) \) of an unstable Adams operation \( \psi_q \), would be that of the “Solomon groups”, if such groups existed.) Moreover [BLO05] introduces the notion of a “\( p \)-local compact group”, and Theorem 9.8 in [BLO05] shows that each \( p \)-compact group supports the structure of a \( p \)-local compact group. As a special case, \( DI(4) \) supports such a structure. We give here an alternate, constructive proof of this fact.

Our paper is built around an alternate construction (Theorem A) of the 2-local finite groups

\[
\mathcal{G}_k = \mathcal{G}_{k,r} = \mathcal{G}_{\text{Sol}}(r^{2^k+1}) = (S_k, \mathcal{F}_k, \mathcal{L}_k)
\]
where \( r \) is a prime congruent to 3 or 5 mod 8. The construction is based on a notion of the “representation” of a \( p \)-local finite group \( \mathcal{G} = (S, \mathcal{F}, \mathcal{L}) \) as the \( p \)-local group of a not necessarily finite group \( G \), by means of a “signalizer functor” \( \theta \). This means, first of all, that \( S \) is a Sylow \( p \)-subgroup of \( G \) (in a sense which we shall make precise), and that the fusion system \( \mathcal{F} \) may be identified with the fusion system \( \mathcal{F}_S(G) \) consisting of all the maps between subgroups of \( S \) that are induced by conjugation by elements of \( G \). Second, it means that whenever \( P \) is a subgroup of \( S \) which contains every \( p \)-element of its \( G \)-centralizer (i.e. whenever \( P \) is centric in \( \mathcal{F} \)), there is a direct-product factorization

\[
C_G(P) = Z(P) \times \theta(P)
\]

where the operator \( \theta \) is inclusion-reversing and conjugation-equivariant. Then \( \theta \) gives rise to a centric linking system \( \mathcal{L}_\theta \) associated with \( \mathcal{F} \), with the property that

\[
\text{Aut}_\mathcal{F}(P) = N_G(P)/\theta(P)
\]

for any \( \mathcal{F} \)-centric subgroup \( P \) of \( S \). One says that \( \mathcal{G} \) is represented in \( G \) via \( \theta \) if the \( p \)-local groups \( \mathcal{G} \) and \( (S, \mathcal{F}, \mathcal{L}_\theta) \) are isomorphic.

The notions of \( p \)-local finite group and of representation via a signalizer functor can be generalized to obtain a representation of the 2-local compact group of DI(4), by allowing \( S \) to be an infinite 2-group. We also introduce a notion of morphism, to obtain a category of \( p \)-local groups, having \( p \)-local finite groups as a full subcategory. As an application we show in Theorem B that the 2-local finite group \( \mathcal{G}_0 \) associated with \( \text{Co}_3 \) is a “subgroup” of each \( \mathcal{G}_k \).

In the final section of this paper we introduce a notion of direct limit of a directed system of embeddings of \( p \)-local groups, and in this way obtain (Theorem C) a 2-local compact group \( \mathcal{G}_{\varnothing} \) which is the direct limit of a directed system \( \mathcal{G} = (\mathcal{G}_k \rightarrow \mathcal{G}_{k+1})_{k \geq 0} \) of embeddings of 2-local finite groups. The identification of \( \mathcal{G}_{\varnothing} \) with the 2-local compact group of DI(4) (Theorem D) is a corollary of results in [LO02], obtained by setting up a homotopy equivalence between the nerve of our direct limit and the nerve of a category \( \mathcal{F}_{\text{Sol}}(p^\infty) \) constructed in [LO02]. The 2-completion of the latter category is shown in [LO02] to be homotopy equivalent to \( B \text{ DI}(4) \).

In view of the length of this paper, the reader may find the following outline helpful. The first three sections are concerned with general principles and supporting results. Then in Section 4, which provides information on certain spin groups, the argument actually begins to take shape.

Let \( p \) be an odd prime. For reasons which will not be immediately apparent, it will be necessary to take \( p \) to be congruent to 3 or 5 mod 8. Let \( \mathcal{F} \) be an algebraic closure of the field of \( p \) elements. There is a subfield \( \mathcal{F} \) of \( \overline{\mathcal{F}} \), obtained as the union of the tower of subfields of \( \overline{\mathcal{F}} \) of order \( p^{2^n} \), \( n \geq 0 \). Take \( \overline{H} \) to be the group \( \text{Spin}_7(\overline{\mathcal{F}}) \)
– the universal covering group of the simple orthogonal group $\Omega_7(\mathbb{F})$. Let $\psi$ be an endomorphism of $\tilde{H}$ such that $C_{\tilde{H}}(\psi) \cong \text{Spin}_7(p)$, and set

$$H = \bigcup_{n \geq 0} C_{\tilde{H}}(\psi^{2n}).$$

Then $H$ is a group of $\mathbb{F}$-rational points of $\tilde{H}$. One finds that all fours groups in $H$ containing $Z(H)$ are conjugate, and that if $U$ is such a fours group then the identity component $B^0$ of the group $B = N_H(U)$ is a commuting product of three copies of $\text{SL}_2(\mathbb{F})$.

In Section 5 we show that there is an automorphism $y$ of $B^0$ of order 3, which transitively permutes the three $\text{SL}_2(\mathbb{F})$ components of $B$, and which when chosen carefully, interacts in a special way (to be described shortly) with the normalizer in $H$ of a maximal torus $T$ of $B$. It is at this point, in choosing an appropriate automorphism $y$, that we require that $p$ be congruent to 3 or 5 mod 8. Once $y$ has been fixed in the appropriate way, we form a group $K = \langle B, y \rangle$ which is isomorphic to a split extension of $B^0$ by the symmetric group of degree 3. We then form the amalgam $\mathcal{A} = (H > B < K)$, and its associated free amalgamated product

$$G = H \ast_B K.$$  

This is the group which informs and guides our investigation.

We need the following notion of “Sylow 2-subgroup”: A subgroup $S$ of $G$ is a Sylow 2-subgroup of $G$ if every element of $S$ has order a power of 2 (i.e. $S$ is a 2-group), $S$ is maximal with respect to inclusion among the 2-subgroups of $G$, and every finite 2-subgroup of $G$ is conjugate to a subgroup of $S$. It turns out that the normalizer in $H$ of a maximal torus $T$ of $B^0$ contains a Sylow 2-subgroup $S$ of $G$. Moreover, if $T$ is chosen to be $y$-invariant then $S$ is a Sylow 2-subgroup of each of the groups $H$, $B$, and $K$. The special way in which $y$ interacts with $S$ may be summarized as follows: for the Sylow 2-subgroup $S_\infty = S \cap T$ of $T$, we have

(*) \hspace{1cm} N_G(S_\infty) = N_H(S_\infty) \ast_{N_B(S_\infty)} N_K(S_\infty), \quad \text{and}

(**) \hspace{1cm} \text{Aut}_G(S_\infty) := N_G(S_\infty)/C_G(S_\infty) \cong \text{GL}(3, 2) \times C_2.

The effect of (**) is that $S/S_\infty$ may be identified with a Sylow 2-subgroup of $\text{Aut}_G(S_\infty)$, and it is this property which, as is made clear in [LO02], turns out to be the key to fulfilling the axioms for “saturation” (defined in 1.5, below).

One feature of our treatment is the use of amalgams (cf. §3) to keep track of the various fusion systems which can be constructed from $H$, $B$, and $K$, and to distinguish the system with property (**). We prove in Theorem 5.2 that the amalgam $\mathcal{A}$ with property (**) is unique. Then we carry out the remainder of our analysis in the universal completion $G$ of $\mathcal{A}$, using the “standard tree” of $G$. 

as a source of geometric intuition, and as the basis for geometric arguments. The formalization by means of the amalgam $A$ and its free amalgamated product $G$ provides, at the very least, a useful system of bookkeeping. For example, the language of amalgams provides a conceptual framework within which one can rigorously consider the question of which of the fusion systems constructed from $H$, $B$, and $K$ is the “right” system. We mention that amalgams have also been used in recent work of G. Robinson [Rob07], and of Ian Leary and Radu Stancu [LS] as a tool for studying abstract fusion systems.

Setting $Z = Z(H)$ one has $|Z| = 2$, and $C_G(Z)$ is in fact a rather complicated subgroup of $G$, properly containing $H$. Our proof that the fusion system $\mathcal{F}_S(G)$ is saturated is modeled on the proof of saturation in [LO02] for the fusion systems $\mathcal{F}_{\text{Sol}}(q)$ defined over finite 2-groups. Thus, the main step is to establish that $H$ controls $C_G(Z)$-fusion in $S$. That is, the fusion system $\mathcal{F}_S(H)$ is equal to the a priori larger system $\mathcal{F}_S(C_G(Z))$.

The proof of saturation in $G$ is accompanied by the construction of a linking system by means of a signalizer functor. These steps require information on fusion among the centric subgroups of $S$, obtained in Sections 6 through 8. After this, in order to prepare the way for the construction of morphisms, we determine in complete detail the radical centric subgroups of $S$ and of $S$, where $\sigma$ is an automorphism of $G$ which fixes $S$ and which induces a Frobenius endomorphism of $H$. Here a subgroup $P$ of $S$ is defined to be radical if $\text{Inn}(P) = O_2(\text{Aut}_G(P))$.

One of the radical centric subgroups of $S$ is an elementary abelian group $A$ of order 16 which has the property, as in (*) , that

$$N_G(A) = N_H(A) * N_B(A) N_K(A).$$

Here one can do better than to determine $\text{Aut}_G(A)$ in analogy with (**). Indeed, there is a surjective homomorphism

$$\phi_A: N_G(A) \longrightarrow L,$$

with $C_G(A) = A \times \ker(\phi_A)$, where $L$ is a maximal subgroup of the sporadic group $\text{Co}_3$, isomorphic to a nonsplit extension of $A$ by $\text{Aut}(A)$. We then define a normal subset $X$ of $G$ by

$$X = \bigcup_{g \in G} \ker(\phi_A)^g.$$

For any centric subgroup $P$ of $S$ we define a subset $\theta(P)$ of $C_G(P)$ by

$$\theta(P) = C_X(P) O(C_G(P)).$$

It turns out that $\theta(P)$ is a subgroup of $C_G(P)$ and that $\theta$ is a signalizer functor (cf. Theorem 8.8 below).
We next consider the groups $G_\sigma$ of fixed points of automorphisms $\sigma$ of $G$, such that $\sigma$ fixes both $H$ and $K$, and such that the restriction of $\sigma$ to $H$ is a Frobenius map with $H_\sigma \cong \text{Spin}_7(\mathbf{F}_q)$, $q = p^{2^n}$. The groups $G_\sigma$, for $n \geq 0$, provide representations of the 2-local finite groups of $[LO02]$. In particular, for each such $\sigma$ (chosen so that $S_\sigma$ is a Sylow 2-subgroup of $G_\sigma$), the fusion system $\mathcal{F}_{S_\sigma}(G_\sigma)$ is saturated, and the signalizer functor $\theta_\sigma$ given by

$$\theta_\sigma(P) = C_{X_\sigma}(P)O(C_{G_\sigma}(P)),$$

for centric subgroups $P$ of $S_\sigma$, defines a centric linking system $\mathcal{L}_{\theta_\sigma}(S_\sigma)$ associated with $\mathcal{F}_{S_\sigma}(G_\sigma)$.

This completes our outline of the proof of Theorem A. One aim of this paper is thus to suggest the possibility that many $p$-local finite groups may best be studied via a representation in terms of free amalgamated products and signalizer functors. For example, to study the fusion system $\mathcal{F}$ on a $p$-group $S$ generated by systems $\mathcal{F}_S(G_i)$ for some family $\mathcal{G} = (G_i \mid i \in I)$ of finite groups with Sylow group $S$, perhaps one should study the various amalgams $\mathcal{A}$ obtained from $\mathcal{G}$, and the corresponding free amalgamated products $G = G(\mathcal{A})$. If $\mathcal{A}$ is well chosen, then $S$ is Sylow in $G$ and $\mathcal{F} = \mathcal{F}_S(G)$ is saturated. Then one can consider suitable overamalgams $\mathcal{B}$ of $\mathcal{A}$, and the kernels of surjections from subgroups $G(\mathcal{B})$ of $G$ onto suitable finite groups and use these kernels to construct a signalizer functor $\theta$ and the corresponding $p$-local finite group from $\mathcal{F}$. If $\mathcal{A}$ is the amalgam of some family of subgroups generating a finite group $\hat{G}$, then the kernel of the surjection $G \to \hat{G}$ will be $\{\theta(P)^g \mid P \in \mathcal{F}, g \in G\}$, whence $\mathcal{F} \cong \mathcal{F}_S(\hat{G})$ (cf. Example 2.13, below). But in other cases one may hope for exotic $p$-local finite groups, such as $\mathcal{L}_{\text{sol}}(q)$.

Now here are the main theorems.

**Theorem A.** Let $p$ be a prime, $p \equiv 3$ or $5 \mod 8$, let $\overline{F}$ be an algebraic closure of the field $\mathbf{F}_p$ of $p$ elements, and let $F$ be the union of the subfields of $\overline{F}$ of order $q_n = p^{2^n}$, $n \geq 0$. Then there is a group $G = G(p)$, an automorphism $\psi_0$ of $G$, a Sylow 2-subgroup $S$ of $G$, and a $\psi_0$-invariant normal subset $X$ of $G$ such that, for any power $\sigma$ of $\psi_0$ of the form $\psi_0^{2^n}$, we have the following.

1. $G = H \star_B K$ is the free amalgamated product of an amalgam

$$\mathcal{A} = (H \leftarrow B \rightarrow K),$$

where $H$ is a group of $F$-rational points in $\text{Spin}_7(\overline{F})$, $B$ is the normalizer in $H$ of a fours group $U$ of $H$ containing $Z(H)$, and $K$ is a group which contains $B$ as a subgroup of index 3 where $K$ has the property that $\text{Aut}_K(U) \cong \text{GL}(2,2)$.

2. $\psi_0$ leaves invariant each of the subgroups $H$, $K$, and $B$ of $G$; the restriction of $\psi_0$ to $H$ is the restriction of a Frobenius automorphism of $\text{Spin}_7(F)$, and $C_H(\psi_0) \cong \text{Spin}_7(p)$. 

(3) The group $S_\sigma = C_S(\sigma)$ is a finite Sylow 2-subgroup of the group $G_\sigma = C_G(\sigma)$, and there exists a unique choice of the amalgam $\mathcal{A}$ such that, for all $\sigma$, the fusion system $\mathcal{F}_\sigma = \mathcal{F}_{S_\sigma}(G_\sigma)$ is isomorphic to the fusion system $\mathcal{F}_{S_\sigma}(G_\sigma)$ of [LO02], $(q = p^{2^n})$.

(4) For any $\mathcal{F}_\sigma$-centric subgroup $P$ of $S_\sigma$, the set
$$\theta_\sigma(P) := C_X\cap G_\sigma(P)O(C_{G_\sigma}(P))$$

is a group, and is a complement to $Z(P)$ in $C_{G_\sigma}(P)$. Moreover, $\theta_\sigma$ defines a 2-local finite group $G_{\sigma} = (S_{\sigma}, \mathcal{F}_{\sigma}, \mathcal{L}_{\sigma})$ isomorphic to the 2-local finite group $S_{\sigma}$ of [LO02].

(5) The order of a maximal elementary abelian 2-subgroup $A$ of $S$ is 16, and all maximal elementary abelian 2-subgroups of $G$ are conjugate in $G$. Moreover, $C_G(A) = A \times C_X(A)$, where $C_X(A)$ is a free normal subgroup of $C_G(A)$, $N_G(A)/C_X(A)$ is isomorphic to a nonsplit extension of $A$ by $\mathrm{Aut}(A)$, and $X$ is the union of the conjugates of $C_X(A)$ in $G$.

**THEOREM B.** Let $p, \mathcal{A}, G, \psi_0$, and $X$ be as in Theorem A. Then there exist subgroups $H_0, K_0$, and $B_0 = H_0 \cap K_0$ of $H_{\psi_0}, K_{\psi_0},$ and $B_{\psi_0}$, respectively, such that the following hold.

1. $H_0$ is isomorphic to a perfect central extension of $\mathrm{Sp}(6,2)$ by $\mathbb{Z}_2$, $K_0$ is a group of order $2^{10}3^3$, and $B_0$ is of index 3 in $K_0$.
2. Setting $G_0 = \langle H_0, K_0 \rangle$, we have
   (a) $X \cap H_0 = X \cap K_0 = \{1\}$, and
   (b) $G_0/(X \cap G_0)$ is isomorphic to the colimit of the amalgam $\mathcal{M}$ of maximal subgroups of $G_0$ containing a fixed Sylow 2-subgroup of $G_0$.

3. Let $S_0$ be a Sylow 2-subgroup of $G_0$ and $S_0$ a Sylow 2-subgroup of $B_0$. Then there is an isomorphism of 2-local finite groups
$$\mathcal{G}_{S_0}(G_0) \cong \mathcal{G}_{S_0}(G_0).$$

**THEOREM C.** For any positive integer $i$, let $\mathcal{G}_i$ be the 2-local finite group $\mathcal{G}_{\psi_0, i-1}$ of Theorem A, and let $\mathcal{G}_0$ be the 2-local finite group associated with $G_0$ as in Theorem B. Let $\mathcal{G}$ be the 2-local group $(S, \mathcal{F}, \mathcal{L})$ associated with $G$ via the fusion system $\mathcal{F} = \mathcal{F}_{S}(G)$ and via the signalizer functor $\theta$ defined by the subset $X$ of $G$. Then there exists a directed system
$$\mathcal{G} = (\beta_{i,j} : \mathcal{G}_i \rightarrow \mathcal{G}_j)_{0 \leq i \leq j}$$
of embeddings of 2-local finite groups, possessing a limit $\mathcal{G}_\varnothing$ which is canonically isomorphic to the 2-local group $\mathcal{G}$.

The 2-local group $\mathcal{G}(G)$ is a 2-local compact group, as defined in [BLO05].
It will be proved in [COS06] that the exotic fusion systems \( \mathcal{F}_{\text{Sol}}(q) \) of Levi and Oliver, defined over 2-groups \( S_q \), are determined by the isomorphism type of \( S_q \). This implies that for any odd prime power \( q \), and any prime \( p \equiv 3 \) or \( 5 \mod 8 \), there is a unique \( \sigma \) such that the Levi-Oliver fusion system \( \mathcal{F}_{\text{Sol}}(q) \) is isomorphic to \( \mathcal{F}_{S_\sigma}(G_\sigma) \), where \( G = G(p) \) is the group in characteristic \( p \) constructed here. This is needed for the proof of the following result.

**Theorem D.** Let \( \mathcal{L} := \mathcal{L}_{S_\theta}(G) \) be the centric linking system over \( \mathcal{F} \) as given in Theorem C. Then the 2-completed nerve \( |\mathcal{L}|_2^\wedge \) is homotopy equivalent to \( B \text{D I}(4) \). In particular, \( \text{D I}(4) \) may be given the structure of the 2-local group \( \mathcal{G} \), and \( \text{D I}(4) \) is then a 2-local compact group.

We are grateful to Bob Oliver for many helpful conversations about the 2-local finite groups \( \mathcal{G}_{\text{Sol}}(q) \) which he and Ran Levi constructed, and for his help in understanding the space \( B \text{D I}(4) \) of Dwyer and Wilkerson. The proof of Theorem D was communicated to us by Levi and Oliver. We would also like to thank Ron Solomon and the other members of his seminar at Ohio State, for suggesting improvements to an earlier version of this manuscript.

**Remarks and questions.**

1. One might imagine that the normal subgroup \( \langle X \rangle \) of \( G \) leads to an interesting factor group \( G/\langle X \rangle \). But the fact is that \( \langle X \rangle = G \). Moreover, \( G_\sigma/\langle X_\sigma \rangle = 1 \) for any automorphism \( \sigma \) of \( G \) as in Theorem A, while \( G_0/\langle X \cap G_0 \rangle \) is in fact isomorphic to \( \text{Co}_3 \). These results will appear in [COS06].

2. To what extent can our method of construction of the Levi-Oliver fusion and linking systems be carried out in a characteristic 0 context? For example, one might consider a subring \( \mathbb{C} \) of the field of complex numbers, and ask whether there is a 7-dimensional quadratic space over \( \mathbb{C} \), yielding a group \( H_\mathbb{C} = \text{Spin}_7(\mathbb{C}) \), from which to build up a suitable free amalgamated product and linking system as we do here in characteristic \( p \). One requires \( 1/2 \in \mathbb{C} \) in order to have an isomorphism of \( \text{PSL}_2(\mathbb{C}) \) with a suitable 3-dimensional orthogonal group. The rings
   \[
   \mathbb{C}_m = \mathbb{Z}[\omega/2],
   \]
   where \( \omega \) is a primitive \( 2^m \)-th root of unity, are possible candidates for this, and there may be others.

3. The sporadic group \( O'N \) (or rather, the 2-local finite group associated with \( O'N \)) can be shown to occur as a subgroup of some of the 2-local finite groups constructed here. Since \( O'N \) and its subgroup \( J_1 \) are “pariahs”, i.e. are not among the twenty sporadic simple groups which are involved in the Monster, it is of some interest to have a context in which these groups, and \( \text{Co}_3 \) (which is not a pariah) can live together in harmony. This will be the subject of another paper.
1. Fusion systems and Sylow subgroups

We shall need to consider fusion systems both over finite \( p \)-groups, and over certain infinite \( p \)-groups. In the finite case the definitions are due first of all to Lluis Puig [Pui06], and then to Broto, Levi, and Oliver [BLO03]. The latter three authors also consider a class of infinite \( p \)-groups which they call discrete \( p \)-toral groups, in [BLO05], and this class includes all of the \( p \)-groups that will be studied here. For reasons of exposition, however, we shall present the definitions in a somewhat more general context – but we emphasize that the main concepts, and the proofs of the basic lemmas, come from the above-cited works.

We follow the practice, peculiar to finite group theory, of using right-hand notation for conjugation within a group, and for group homomorphisms. But we use left-hand notation for functors, and for auxiliary mappings associated with some of our functors. It may also be worth mentioning that if \( X \) is a set admitting action by a group \( G \), and \( g \) is an element of \( G \), then the set of fixed points for \( g \) in \( X \) is denoted \( X^g \), rather than the topologist’s \( X_g \).

If \( G \) is a group, \( g \) an element of \( G \), and \( X \) a subset or an element of \( G \), we write \( X^g \) for the image of \( X \) under the conjugation automorphism

\[
c_g : G \to G, \quad (c_g : x \mapsto x^g := g^{-1}xg \quad \text{for all } x \in G).
\]

We also write \( c_g : P \to Q \) for the mapping of \( P \) into \( Q \) given by \( g \)-conjugation, whenever \( P \) and \( Q \) are subgroups of \( G \) with \( P^g \leq Q \). The transporter of \( P \) into \( Q \) is the set

\[
N_G(P, Q) := \{ g \in G \mid P^g \leq Q \},
\]
and we define

\[
\text{Hom}_G(P, Q) := \{ c_g : P \to Q \mid g \in N_G(P, Q) \}.
\]

Denote by \( \text{Inj}(P, Q) \) the set of all injective homomorphisms of \( P \) into \( Q \). If \( \alpha : P \to Q \) is an isomorphism, write \( \alpha^* \) for the isomorphism from \( \text{Aut}(P) \) to \( \text{Aut}(Q) \) defined by \( \alpha^* : \beta \mapsto \alpha^{-1}\beta\alpha \).

**Definition 1.1.** A fusion system \( \mathcal{F} \) over a group \( S \) is a category whose objects are the subgroups of \( S \), and whose morphism-sets \( \text{Hom}_\mathcal{F}(P, Q) \) satisfy the following two conditions.

1. \( \text{Hom}_\mathcal{F}(P, Q) \subseteq \text{Hom}_G(P, Q) \subseteq \text{Inj}(P, Q) \).
2. If \( \alpha \in \text{Hom}_\mathcal{F}(P, Q) \) then the isomorphisms \( \alpha : P \to P\alpha \) and \( \alpha^{-1} : P\alpha \to P \) are morphisms in \( \mathcal{F} \).

**Example 1.2.** Let \( G \) be a group and \( S \) a subgroup of \( G \). For subgroups \( P \) and \( Q \) of \( S \), set

\[
\text{Hom}_\mathcal{F}(P, Q) = \text{Hom}_G(P, Q).
\]

Then \( \mathcal{F} \) is a fusion system over \( S \), denoted \( \mathcal{F}_S(G) \).
Let $\mathcal{F}$ be a fusion system over $S$, let $P$ be a subgroup of $S$, and let $\alpha \in \text{Aut}_\mathcal{F}(P)$. Set 

$$N_\alpha = \{ x \in N_S(P) \mid (c_x)\alpha^* \in \text{Aut}_S(P) \}.$$ 

Thus, $N_\alpha$ is the largest subgroup $R$ of $N_S(P)$ having the property that, in the group $\text{Aut}_\mathcal{F}(P)$, the conjugation map $c_\alpha$ carries $\text{Aut}_R(P)$ into $\text{Aut}_S(P)$.

**Lemma 1.3.** Let $S$ be a subgroup of a group $G$, and let $P$ be a subgroup of $S$. Set $\mathcal{F} = \mathcal{F}_S(G)$, let $g \in N_G(P)$, and set $\alpha = c_g \in \text{Aut}_\mathcal{F}(P)$.

(a) $(N_\alpha)^g = S^g \cap (C_G(P)N_S(P))$.

(b) If $S$ is a $p$-group, and every $p$-subgroup of $C_G(P)N_S(P)$ is conjugate via $C_G(P)$ to a subgroup of $N_S(P)$, then there exists $\tilde{\alpha} \in \text{Aut}_{\mathcal{F}}(N_\alpha, N_S(P))$ extending $\alpha$.

**Proof.** Set $R = N_\alpha$. By definition, $R^g$ consists of those $x \in N_G(P)$ such that $x \in S^g$ and $c_xP \in \text{Aut}_S(P)$. But $c_xP \in \text{Aut}_S(P)$ if and only if $x \in C_G(P)N_S(P)$, and thus (a) holds.

Assume the hypothesis of (b). Then $R^g$ is a $p$-subgroup of $C_G(P)N_S(P)$, by (a), so that by the hypothesis of (b) there exists $h \in C_G(P)$ such that $R^h \leq N_S(P)$. Now $\alpha$ is the restriction to $P$ of $\tilde{\alpha} = c_{gh} : R \rightarrow N_S(P)$, and we have (b). $\square$

**Definition 1.4.** Let $p$ be a prime. A group $S$ is a $p$-group if for every $x \in S$, the order of $x$ is a power of $p$. A $p$-subgroup $S$ of a group $G$ is a Sylow $p$-subgroup of $G$ if

1. $S$ is maximal (with respect to inclusion) among all $p$-subgroups of $G$, and
2. $S$ contains a conjugate of every finite $p$-subgroup of $G$.

The set of all Sylow $p$-subgroups of $G$ is denoted $\text{Syl}_p(G)$. The group generated by the set of normal $p$-subgroups of $G$ is itself a normal $p$-subgroup of $G$, and is denoted $O_p(G)$.

**Definition 1.5.** Let $p$ be a prime, let $S$ be a $p$-group, and let $\mathcal{F}$ be a fusion system over $S$. A subgroup $P$ of $S$ is fully normalized in $\mathcal{F}$ if, for every $\phi \in \text{Hom}_{\mathcal{F}}(P, S)$, there exists $\psi \in \text{Hom}_{\mathcal{F}}(N_S(P\phi), N_S(P))$ such that $\psi$ maps $P\phi$ to $P$. We say that $\mathcal{F}$ is saturated if the following two conditions hold for every subgroup $P$ of $S$.

(I) There exists $\phi \in \text{Hom}_{\mathcal{F}}(P, S)$ such that $P\phi$ is fully normalized in $\mathcal{F}$.

(II) If $P$ is fully normalized in $\mathcal{F}$ then:

(A) $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(P))$, and

(B) each $\alpha \in \text{Aut}_{\mathcal{F}}(P)$ extends to a member of $\text{Hom}_{\mathcal{F}}(N_\alpha, S)$. 


The preceding definition of saturation is formulated so as to make no mention of \(|N_S(P)|\) for \(P \leq S\), and it is equivalent to the standard definition (cf. [BLO03]) in the case that \(S\) is finite. This follows from [BCG+05, Lemma 2.3], and it is then easy to check that our definition of “fully normalized” is equivalent to the usual one, in a saturated fusion system over a finite \(p\)-group.

**Lemma 1.6.** Let \(p\) be a prime and \(G\) a group. Let \(\mathcal{Y}\) be the set of subgroups \(Y\) of \(G\) such that the set \(\mathcal{F}(Y)\) of maximal \(p\)-subgroups of \(Y\) is nonempty, and such that \(Y\) is transitive on \(\mathcal{F}(Y)\) by conjugation. Assume for each \(p\)-subgroup \(P\) of \(G\) that:

1. \(N_G(P), C_G(P)P,\) and \(C_G(P)T\) are in \(\mathcal{Y}\) for \(T \in \mathcal{F}(N_G(P))\).
2. \(\text{Out}_G(P)\) is finite.
3. \(P\) is Artinian (i.e., any descending chain of subgroups of \(P\) stabilizes).
4. If \(P \neq 1\) then \(N_G(P)\) is locally finite.

Then

(a) \(G\) has a Sylow \(p\)-subgroup \(S\).
(b) A subgroup \(P\) of \(S\) is fully normalized in \(\mathcal{F}_S(G)\) if and only if \(N_S(P)\) is a Sylow \(p\)-subgroup of \(N_G(P)\).
(c) \(\mathcal{F}_S(G)\) is saturated.

**Proof.** We have \(G \in \mathcal{Y}\), by (1) as applied to \(P = 1\). Thus \(\mathcal{F}(G) = Syl_p(G)\), and (a) holds.

Let \(P \leq S\) and set \(L = N_G(P), K = C_G(P),\) and \(T = N_S(P)\). Applying (1) to \(P\), we obtain \(T \leq X\) for some \(X \in Syl_p(L)\). Let \(Q \leq S\) and \(g \in G\) with \(Q^g = P\). Then \(N_S(Q)^g\) is a \(p\)-subgroup of \(L\), and since \(L \in \mathcal{Y}\) there exists \(l \in L\) with \(N_S(Q)^g \leq X\). We conclude that \(P\) is fully normalized if \(T = X\). Further, as \(G \in \mathcal{Y}\) there exists \(h \in G\) with \(X^h \leq S\), and so \(X^h \in Syl_p(L^h)\) and hence \(P^h\) is fully normalized. This verifies axiom (1) in the definition 1.5 of saturation.

Assume that \(P\) is fully normalized. Then there exists \(y \in G\) with \(P^{hy} = P\) and with \(X^{hy} \leq T \leq X\). Then \(X^{(hy)^n} \leq X\) for all \(n > 0\), and it follows from (3) that \(X = X^{hy}\), and then that \(X = T\). This completes the proof of (b).

Set \(\bar{L} = L/PK\). We have \(T \in Syl_p(L)\) by (b), and \(\bar{L}\) is finite by (2). Let \(Y\) be the pre-image in \(L\) of a Sylow \(p\)-subgroup of \(\bar{L}\) containing \(\bar{T}\). By (4) there is a finite subgroup \(U\) of \(Y\) with \(\bar{Y} = \bar{U}\). As \(\bar{U}\) is a \(p\)-group we have \(\bar{U} = \bar{V}\) for some \(V \in Syl_p(U)\). As \(L \in \mathcal{Y}\) there exists \(a \in L\) with \(V^a \leq T\). Then \(|\bar{Y}| = |\bar{V}| \leq |\bar{T}| \leq |\bar{Y}|\), so that \(\bar{Y} = \bar{T}\), and we have verified axiom (IIA) in 1.5.

Finally, set \(\mathcal{F} = \mathcal{F}_S(G)\) and let \(\alpha \in \text{Aut}_{\mathcal{F}}(P)\). As \(TK \in \mathcal{Y}\), by (1), we conclude from 1.3(b) that \(\alpha\) extends to an element of \(\text{Hom}_{\mathcal{F}}(N_\alpha, S)\), verifying axiom (II) for saturation, and completing the proof of (c).
Notice that if $G$ is a finite group then the hypotheses of 1.6 are satisfied by $G$. Thus, it is a corollary of 1.6 that for any finite group $G$ and $S \in \text{Syl}_p(G)$, the fusion system $\mathcal{F}_S(G)$ is saturated.

Let $\mathcal{F}_i$, $i = 1, 2$, be fusion systems over subgroups $S_i$ of a group $S$. We say that $\mathcal{F}_1$ is a fusion subsystem of $\mathcal{F}_2$ (and write $\mathcal{F}_1 \leq \mathcal{F}_2$) if $S_1 \leq S_2$ and $\text{Hom}_{\mathcal{F}_1}(P, Q) \subseteq \text{Hom}_{\mathcal{F}_2}(P, Q)$ for all $P, Q \leq S_1$.

Given a set $\mathcal{F}$ of fusion systems over $S$, there is a largest fusion system

$$\mathcal{F}_\mathcal{F} := \bigcap_{\mathcal{F} \in \mathcal{F}} \mathcal{F}$$

which is a subsystem of each $\mathcal{F} \in \mathcal{F}$. Thus

$$\text{Hom}_{\mathcal{F}_\mathcal{F}}(P, Q) = \bigcap_{\mathcal{F} \in \mathcal{F}} \text{Hom}_{\mathcal{F}}(P, Q).$$

Given a set $\mathcal{E}$ of fusion systems, each of which is defined over a subgroup of $S$, define $\langle \mathcal{E} \rangle$ — the fusion system generated by $\mathcal{E}$ — to be the fusion system $\mathcal{F}_\mathcal{F}$, where $\mathcal{F}$ is the set of all fusion systems over $S$ which contain each member of $\mathcal{E}$. The proof of the following result is straightforward:

**Lemma 1.7.** Let $S$ be a group and let $(S_i : i \in I)$ be a collection of subgroups of $S$. For each $i \in I$, let $\mathcal{F}_i$ be a fusion system over $S_i$, and set $F = (\mathcal{F}_i \mid i \in I)$. Assume that $S_i = S$ for at least one index $i$, and define a fusion system $\mathcal{G}$ on $S$ by taking $\text{Hom}_{\mathcal{G}}(P, Q)$ to consist of the maps $\alpha_0 \ldots \alpha_r$ such that, for each $0 \leq j \leq r$, there exists $i(j) \in I$, $P_j \leq S_{i(j)}$, and $\alpha_j \in \text{Hom}_{\mathcal{F}_{i(j)}}(P_j, S_{i(j)})$, such that $P_{j+1} = P_j \alpha_j$, $P_0 = P$, and $P_{r+1} = Q$. Then $\langle F \rangle = \mathcal{G}$. □

**Lemma 1.8.** Let $G$ be a group, $S \in \text{Syl}_p(G)$, and let $X$ be a normal subgroup of $G$ of index $p$ in $S$, such that $S / X \cong N_{G / X}(S / X)$. Then $\mathcal{F}_S(G) = \langle \mathcal{F}_S(S), \mathcal{F}_X(G) \rangle$.

**Proof.** Set $G^* = G / X$ and $\mathcal{E} = \langle \mathcal{F}_S(S), \mathcal{F}_X(G) \rangle$. As $\mathcal{F}_S(S)$ and $\mathcal{F}_X(G)$ are contained in $\mathcal{F} = \mathcal{F}_S(G)$, we have $\mathcal{E} \subseteq \mathcal{F}$ by definition of $\mathcal{E}$, and it remains to establish the opposite inclusion. Let $P, Q \leq S$ and $\alpha \in \text{Hom}_{\mathcal{F}}(P, Q)$. If $P \not\subseteq X$ then as $S^*$ is of order $p$ and is equal to its normalizer in $G^*$ we get $Q \not\subseteq X$ and $\alpha \in \text{Hom}_{\mathcal{F}_S(S)}(P, Q)$. Similarly if $P \subseteq X$ but $Q \not\subseteq X$ then $\alpha = \beta \gamma$ where $\beta \in \text{Hom}_{\mathcal{F}}(P, P\beta)$, $P\beta \subseteq X$, and $\gamma : P\beta \to Q$ is the inclusion map. Thus it remains to show that $\text{Hom}_{\mathcal{F}}(P, Q) \subseteq \text{Hom}_{\mathcal{E}}(P, Q)$ for $P, Q \leq X$, which follows since $\mathcal{F}_X(G) \subseteq \mathcal{E}$. □

The next lemma states a weak form of the Alperin-Goldschmidt fusion theorem [Gol70], in the language of fusion systems. This result will be of use in the proof of Theorem B.

**Lemma 1.9.** Let $G$ be a finite group, $S \in \text{Syl}_p(G)$, and denote by $N$ the set of subgroups $N$ of $G$ having the following properties.
(1) \( N = N_G(O_p(N)) \),
(2) \( S \cap N \in \text{Syl}_p(N) \), and
(3) \( C_S(O_p(N)) \leq O_p(N) \).

Let \( S_0 \) be the set of minimal members of \( S \), with respect to inclusion. Then
\[ \mathcal{F}_S(G) = (\mathcal{F}_{S \cap N}(N) \mid N \in S_0) \).

We end this section with a generalization of a well-known result concerning
finite \( p \)-groups.

**Lemma 1.10.** Let \( P \) be a \( p \)-group, set \( A = \text{Aut}(P) \), and let
\[ \mathcal{C} = (P = P_0 \geq P_1 \geq \cdots \geq P_k = 1) \]
be a chain of normal subgroups of \( P \). Let \( \Lambda \) be a subgroup of the group
\[ C_A(\mathcal{C}) = \{ \alpha \in A \mid [P_i, \alpha] \leq P_{i+1} \quad \text{for all } i, 0 \leq i < k \}, \]
and assume that either \( \Lambda \) is a torsion group or that \( P_1 \) is of bounded exponent.
Then \( \Lambda \) is a \( p \)-group.

**Proof.** Apply induction on \( k \). The lemma is trivial when \( k = 1 \), so take \( k > 1 \)
and set \( P^* = P/P_{k-1} \). Then \( \Lambda \) centralizes the chain \( \mathcal{C}^* = (P_0^* \geq \cdots \geq P_{k-1}^* = 1) \).
By induction, \( \text{Aut}_A(\mathcal{C}^*) \) is a \( p \)-group, so it remains to show that \( C_A(P^*) \) is a
\( p \)-group. Thus, we may take \( k = 2 \). Let \( \alpha \in \Lambda, x \in P \), and set \( c = [x, \alpha] \). Then
\( c \in P_1 \leq C_P(\alpha) \), so that \( x^{\alpha^n} = xc^n \), and hence \( |\alpha| |x| = |c| \). If \( \alpha \) is of finite order,
or \( P_1 \) is of bounded exponent, we conclude that
\[ |\alpha| = \text{lcm}\{|[x, \alpha]| \mid x \in P\} \]
is a power of \( p \). \( \square \)

2. **Linking systems, signalizer functors, and \( p \)-local groups**

Let \( \mathcal{F} \) be a fusion system over a \( p \)-group \( S \). A subgroup \( P \) of \( S \) is \( \mathcal{F} \)-centric if
\( C_S(P\phi) = Z(P\phi) \) for every \( \phi \in \text{Hom}_\mathcal{F}(P, S) \), and \( P \) is \( \mathcal{F} \)-radical if \( O_p(\text{Aut}_\mathcal{F}(P)) = \text{Inn}(P) \). Write \( \mathcal{F}^c \) for the set of \( \mathcal{F} \)-centric subgroups of \( S \), \( \mathcal{F}^r \) for the set of \( \mathcal{F} \)-radical subgroups of \( S \), and \( \mathcal{F}^{c^r} \) for \( \mathcal{F}^c \cap \mathcal{F}^r \).

**Lemma 2.1.** Let \( S \) be a Sylow \( p \)-subgroup of a group \( G \), set \( \mathcal{F} = \mathcal{F}_S(G) \),
and let \( P \leq S \).

(a) If \( P \in \mathcal{F}^c \), and \( g \in G \) with \( P \leq S^g \), then \( C_{S^g}(P) \leq P \).
(b) If \( P \) contains every finite \( p \)-subgroup of \( C_G(P) \) then \( P \in \mathcal{F}^c \).
(c) If \( P \) is finite and \( P \in \mathcal{F}^c \), then \( Z(P) \) contains every \( p \)-subgroup of \( C_G(P) \).
Proof. Part (a) is immediate from the definition of $\mathcal{F}^c$. Now suppose that $Z(P)$ contains every finite $p$-subgroup of $C_G(P)$, and let $g \in N_G(P, S)$. Then every element of $C_S(P^g)$ is contained in $Z(P^g)$, and thus $P \in \mathcal{F}^c$, proving (b).

Finally, assume that $P$ is finite and that $P \in \mathcal{F}^c$, and let $R$ be a finite $P$-subgroup of $C_G(P)$. Then $RP$ is a finite $p$-subgroup of $G$. Since $S$ is a Sylow $p$-subgroup of $G$, there exists $g \in G$ with $(RP)^g \leq S$. As $P \in \mathcal{F}^c$ we have $C_S(P^g) = Z(P^g)$, and so $R \leq Z(P)$. Thus $Z(P)$ contains every finite $p$-subgroup of $C_G(P)$, and since every $p$-group is the union of its finite subgroups, we obtain (c). \qed

Lemma 2.2. Let $S$ be a Sylow $p$-subgroup of a group $G$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Let $P \in \mathcal{F}^c$ such that $P$ contains every $p$-element of $C_G(P)$, and let

$\mathcal{C} = \{P = P_0 \geq P_1 \geq \cdots \geq P_k = 1\}$

be a chain of $N_G(P)$-invariant subgroups of $P$. Suppose that $P_1$ is of bounded exponent. Then $P$ contains every finite $P$-invariant $p$-subgroup $R$ of $C_G(\mathcal{C})$.

Proof. First, let $R_0$ be a $p$-subgroup of $C_G(P)P$. Then $R_0P = C_{R_0P}(P)P$ by the Dedekind Lemma. As $P \leq C_G(P)P$, $R_0P$ is a $p$-group, and then $C_{R_0P}(P) \leq P$ by hypothesis. Thus

(*) \[ R_0 \leq R_0P \leq C_{R_0P}(P)P \leq P. \]

Now let $R$ be a $p$-subgroup of $C_G(\mathcal{C})$. Set $\Lambda = C_{\text{Aut}_G(P)}(\mathcal{C})$. Then $\Lambda \trianglelefteq \text{Aut}_G(P)$, and $\Lambda$ is a $p$-group by 1.10. As $P$ is $\mathcal{F}$-radical, it follows that $\text{Aut}_R(P) \leq \text{Inn}(P)$, and thus $R \leq C_G(P)P$. Then $R \leq P$ by (*). \qed

A set $\mathcal{F}_0$ of objects in a fusion system $\mathcal{F}$ is closed under $\mathcal{F}$-conjugation if $P\phi \in \mathcal{F}_0$ for all $P \in \mathcal{F}_0$ and all morphisms $\phi \in \mathcal{F}$ defined on $P$.

Lemma 2.3. Let $\mathcal{F}$ be a fusion system on $S$. Then $\mathcal{F}^c$, $\mathcal{F}^r$, and $\mathcal{F}^{rc}$ are closed under $\mathcal{F}$-conjugation.

Proof. Let $P \leq S$ and let $\phi \in \text{Hom}_{\mathcal{F}}(P, S)$. Then $P \in \mathcal{F}^c$ if and only if $P\phi \in \mathcal{F}^c$, by definition. Now let $P \in \mathcal{F}^r$. The natural map $\phi^* : \text{Aut}_{\mathcal{F}}(P) \to \text{Aut}_{\mathcal{F}}(P\phi)$ is an isomorphism, and since $\text{Inn}(P) = O_p(\text{Aut}_{\mathcal{F}}(P))$, it follows that $\text{Inn}(P\phi) = O_p(\text{Aut}_{\mathcal{F}}(P\phi))$. Thus, $P \in \mathcal{F}^r$ if and only if $P\phi \in \mathcal{F}^r$. \qed

Definition 2.4. Let $S$ be a $p$-group, let $\mathcal{F}$ be a fusion system over $S$, and let $\mathcal{C}$ be a subset of $\mathcal{F}^c$. An $\mathcal{C}$-linking system (or linking system on $\mathcal{C}$), consists of

1. a category $\mathcal{L}$ with $\text{Obj}(\mathcal{L}) = \mathcal{C}$ and composition $\cdot$ (read from left to right),
2. a functor $\pi : \mathcal{L} \to \mathcal{F}$, for which the associated map of objects induces the identity map $\text{Obj}(\mathcal{L}) \to \mathcal{C}$, and
3. a collection $\delta = \{\delta_P : P \to \text{Aut}_{\mathcal{F}}(P) \mid P \in \mathcal{C}\}$ of injective group homomorphisms,

such that the following three conditions hold for any $P$ and $Q$ in $\mathcal{C}$. 4
(A) The left action of $Z(P)$ on $\text{Mor}_\mathcal{F}(P, Q)$ given by
$$z \cdot \phi \mapsto \delta_P(z^{-1}) \cdot \phi$$
is free (i.e. $\text{Mor}_\mathcal{F}(P, Q)$ is a union of regular orbits for $Z(P)$), and the map $Z(P) \phi \mapsto \pi(\phi)$ is a bijection of $Z(P) \setminus \text{Mor}_\mathcal{F}(P, Q)$ with $\text{Hom}_\mathcal{F}(P, Q)$. In particular, $\pi$ is surjective on morphism sets, and $\pi(\mathcal{L})$ is a full subcategory of $\mathcal{F}$.

(B) For all $g \in P$,
$$\pi(\delta_P(g)) = c_g \in \text{Aut}_\mathcal{F}(P).$$

(C) For each $\psi \in \text{Mor}_\mathcal{F}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc}
P & \xrightarrow{\psi} & Q \\
\delta_P(g) \downarrow & & \downarrow \delta_Q(g \pi(\psi)) \\
P & \xrightarrow{\psi} & Q.
\end{array}$$

A centric linking system on $\mathcal{F}$ is a linking system on $\mathcal{F}^c$. A pre-local group consists of a triple $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ where $S$ is a $p$-group, $\mathcal{F}$ is a fusion system over $S$, and $\mathcal{L} = (\mathcal{L}, \pi, \delta)$ is a linking system on a subset $\mathcal{C}$ of $\mathcal{F}^c$. If $\mathcal{F}$ is saturated, and $\mathcal{L}$ is a centric linking system, then $\mathcal{G}$ is a $p$-local group. A $p$-local finite group is a $p$-local group $\mathcal{G}$ in which $S$ is finite.

We are following the notational conventions in [BLO03] in writing $\text{Mor}_\mathcal{F}(P, Q)$ (rather than $\text{Hom}_\mathcal{F}(P, Q)$) for the set of morphisms in $\mathcal{L}$ from $P$ to $Q$, in order to emphasize that in general, $\mathcal{L}$-morphisms are not mappings.

**Definition 2.5.** Let $G$ be a group, let $S$ be a Sylow $p$-subgroup of $G$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Let $\mathcal{F}(G)$ be the set of all subgroups of $G$. An $\mathcal{F}$-signalizer functor is a mapping
$$\theta: \mathcal{F}^c \rightarrow \mathcal{F}(G)$$
satisfying the following three conditions:

1. $\theta(P)$ is a complement to $Z(P)$ in $C_G(P)$.
2. $\theta(P^g) = \theta(P)^g$ for all $g \in N_G(P, S)$.
3. $\theta(Q) \leq \theta(P)$ for all $Q$ with $P \leq Q \leq S$.

**Remark.** Signalizer functors are bona fide contravariant functors. Namely, in 2.5, view $\mathcal{F} = \mathcal{F}(G)$ as a category whose morphism sets are given by
$$\text{Mor}_\mathcal{F}(X, Y) = N_G(X, Y),$$
for any subgroups $X$ and $Y$ of $G$, and where composition is given by multiplication in $G$. Let $\mathcal{F}^c$ be the full subcategory of $\mathcal{F}$ whose set of objects is $\mathcal{F}^c$. Condition (2)
in 2.5 implies that an $\mathcal{F}$-signalizer functor $\theta$ is a contravariant functor from $\mathcal{F}^c$ to $\mathcal{F}$, if we define

$$\theta: \text{Mor}_{\mathcal{F}^c}(P, Q) \to \text{Mor}_{\mathcal{F}}(\theta(P), \theta(Q))$$

by $\theta(g) = g^{-1}$. $\square$

Given the setup of 2.5, define $\mathcal{L} = \mathcal{L}_\theta$ to be the category whose objects are the $\mathcal{F}$-centric subgroups of $S$, with morphisms

$$\text{Mor}_{\mathcal{F}}(P, Q) = \theta(P) \setminus N_G(P, Q).$$

The composition of morphisms is defined by

$$\theta(P)g \cdot \theta(Q)h = \theta(P)gh,$$

for $g \in N_G(P, Q)$ and $h \in N_G(Q, R)$. In fact, this composition is no more than ordinary multiplication of subsets of $G$. To see this, notice that if $P^g \leq Q$, then the signalizer functor axioms (2) and (3) yield $\theta(Q) \leq \theta(P^g) = \theta(P)^g$. Thus $\theta(Q)^g^{-1} \leq \theta(P)$, and so

$$(\theta(P)g)(\theta(Q)h) = \theta(P)\theta(Q)^g^{-1}gh = \theta(P)gh.$$

Next, define a functor

$$\pi = \pi_\theta: \mathcal{L}_\theta \to \mathcal{F}^c$$

by $\pi(P) = P$ and by $\pi(\theta(P)g) = c_g$ for $g \in N_G(P, Q)$. Finally, define a family $\delta = \delta_\theta = \{\delta_P \mid P \in \mathcal{F}^c\}$ of monomorphisms

$$\delta_P = \delta_{P, \theta}: P \to \text{Aut}_{\mathcal{F}}(P)$$

by $\delta_P(g) = \theta(P)g$, for $P \in \mathcal{F}^c$.

**Lemma 2.6.** Let $S$ be a Sylow $p$-subgroup of a group $G$, set $\mathcal{F} = \mathcal{F}_S(G)$, and let $\theta$ be an $\mathcal{F}$-signalizer functor. Then

(a) $(\mathcal{L}_\theta, \pi_\theta, \delta_\theta)$ is a centric linking system on $\mathcal{F}$.

(b) If $\mathcal{F}$ is saturated then $\mathcal{G}_{S, \theta}(G) := (S, \mathcal{F}_S(G), \mathcal{L}_\theta)$ is a $p$-local group.

**Proof.** Let $P, Q \in \mathcal{F}^c$, let $z$ be a nonidentity element of $Z(P)$, and let $g \in N_G(P, Q)$. Then $\theta(P)g \in \text{Mor}_{\mathcal{F}}(P, Q)$, and

$$\delta_P(z) \cdot \theta(P)g = \theta(P)z \cdot \theta(P)g = \theta(P)zg.$$

Here $\theta(P)zg \neq \theta(P)g$ since $\theta(P) \cap Z(P) = 1$. Thus, $Z(P)$ acts freely on $\text{Hom}_{\mathcal{F}}(P, Q)$. Similarly, since $C_G(P) = \theta(P) \times Z(P)$, the map

$$Z(P)\theta(P)g \mapsto c_g = \pi(\theta(P)g)$$

is a bijection from $\text{Hom}_{\mathcal{F}}(P, Q)/Z(P)$ to $\text{Hom}_{\mathcal{F}}(P, Q)$. Thus, condition (A) in Definition 2.4 is satisfied in our setup.
Now suppose that \( g \in P \). By construction, we then have
\[
\pi(\delta_P(g)) = \pi(\theta(P)g) = c_g,
\]
and so condition (B) is satisfied. Finally, let \( f = \theta(P)x \in \text{Mor}_F(P, Q) \). We then have \( \pi(f) = c_x: P \rightarrow Q \), and \( g\pi(f) = g^x \). Since also \( \delta_Q(g^x) = \theta(Q)g^x \), we obtain
\[
f \cdot \delta_Q((g\pi(f)) = \theta(P)x \cdot \theta(Q)g^x = \theta(P)xg^x = \theta(P)gx = \theta(P)g \cdot \theta(P)x = \delta_P(g) \cdot f.
\]
Thus, condition (C) is satisfied, and (a) is proved. Part (b) follows from (a), by the definition of \( p \)-local group. \( \square \)

**Proposition 2.7.** Let \( S \) be a Sylow \( p \)-subgroup of a group \( G \) and set \( \mathcal{F} = \mathcal{F}_S(G) \). Suppose that \( N_G(P) \) is finite for every \( P \in \mathcal{F} \).

(a) There is a unique \( \mathcal{F} \)-signalizer functor \( \theta \) given by \( \theta(P) = O^P(C_G(P)) \).

(b) Set \( \mathcal{L}_S(G) = \mathcal{L}_\theta \), and suppose that \( \mathcal{F} \) is saturated. Then
\[
\mathcal{G}_S(G) := (S, \mathcal{F}_S(G), \mathcal{L}_S(G))
\]
is a \( p \)-local finite group.

**Proof.** Part (a) follows from 2.1(c), and then (b) follows from (a) and from 2.6(b). \( \square \)

The following lemma is intended as a remark, to point out the connection between Definition 2.5 and the usual notion of “balanced signalizer functor” in finite group theory. It will not be used in the sequel.

**Lemma 2.8.** Let \( \theta \) be an \( \mathcal{F} \)-signalizer functor, where \( \mathcal{F} \) is a fusion system over a finite \( p \)-group. Then, for any \( P, Q \in \mathcal{F} \) with \( P \leq Q \),

(a) \( Z(Q) \leq Z(P) \), and

(b) \( \theta(Q) = C_{\theta(P)}(Q) \).

**Proof.** Part (a) is immediate from 2.1(c). By definition,
\[
\theta(Q) \leq C_{\theta(P)}(Q) \leq C_G(Q) = \theta(Q) \times Z(Q),
\]
so that \( C_{\theta(P)}(Q) = \theta(Q) \times (Z(Q) \cap \theta(P)) \). Since \( Z(Q) \leq Z(P) \), we obtain (b). \( \square \)

**Definition 2.9.** Let \( \mathcal{F} \) and \( \mathcal{F} \) be fusion systems over the groups \( S \) and \( \tilde{S} \), respectively. A morphism \( \alpha: \mathcal{F} \rightarrow \tilde{\mathcal{F}} \) of fusion systems consists of a functor \( \alpha_1: \mathcal{F} \rightarrow \tilde{\mathcal{F}} \), and a homomorphism \( \alpha_0: S \rightarrow \tilde{S} \) of groups, satisfying the following two conditions.

(MF1) For every subgroup \( P \) of \( S \), \( \alpha_1(P) = \alpha_0(P) \), and

(MF2) for each \( \phi \in \text{Hom}_\mathcal{F}(P, Q) \), we have \( \alpha_0 \circ \alpha(\phi) = \phi \circ \alpha_0 \) (in right-hand notation).
We most often write $\alpha$ for both $\alpha_0$ and $\alpha_1$. In the case that $\alpha_0$ is given by inclusion of $S$ into $\tilde{S}$, and $\alpha_1$ is given by inclusion of $\text{Hom}_{\tilde{F}}(P, Q)$ into $\text{Hom}_{\tilde{F}}(P, Q)$ for all $P, Q \in \mathcal{F}$, we say that $\alpha$ is an embedding. We reserve the symbol $\iota$ to denote an embedding of fusion systems.

In general, if $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are fusion systems over finite $p$-groups $S$ and $\tilde{S}$, respectively, then a morphism $\alpha: \mathcal{F} \to \tilde{\mathcal{F}}$ of fusion systems need not send $\mathcal{F}$-centric subgroups of $S$ to $\tilde{\mathcal{F}}$-centric subgroups of $\tilde{S}$. For example, let $G$ be a finite group, take $\tilde{G}$ to be the direct product of $G$ with a nonidentity $p$-group $R$, let $\tilde{S}$ be a Sylow $p$-subgroup of $\tilde{G}$, and take $S = \tilde{S} \cap G$. Then the inclusion map $\alpha: \mathcal{F}(G) \to \tilde{\mathcal{F}}(\tilde{G})$ carries no centric subgroup to a centric subgroup.

Recall that if $\mathcal{F}$ is a saturated fusion system over a $p$-group $S$, then $\mathcal{F}^{\text{rc}}$ denotes the set of subgroups of $S$ which are both $\mathcal{F}$-centric and $\mathcal{F}$-radical. Given a $p$-local group $G = (S, \mathcal{F}, L)$, denote by $L^{\text{rc}}$ the full subcategory of $L$ whose objects are the objects of $\mathcal{F}^{\text{rc}}$. Thus $L^{\text{rc}}$ is a linking system on $\mathcal{F}^{\text{rc}}$.

By [BCG +05, Th. B], the classifying spaces $|L|$ and $|L^{\text{rc}}|$ are homotopy equivalent in the case that $S$ is finite. This provides some justification for the following definition of morphism of $p$-local groups.

**Definition 2.10.** Let $\mathcal{G} = (S, \mathcal{F}, L)$ and $\tilde{\mathcal{G}} = (\tilde{S}, \tilde{\mathcal{F}}, \tilde{L})$ be pre-local groups. A morphism of pre-local groups from $\mathcal{G}$ to $\tilde{\mathcal{G}}$ is a pair $(\alpha, \beta)$, where $\alpha: \mathcal{F} \to \tilde{\mathcal{F}}$ is a morphism of fusion systems, and $\beta: L \to \tilde{L}$ is a functor which, for each pair $P, Q$ of objects of $L$, satisfies the following conditions.

1. **(MG1)** $\alpha(P) \leq \beta(P)$.
2. **(MG2)** For each $\psi \in \text{Mor}_\mathcal{F}(P, Q)$, the restriction of $\tilde{\alpha}(\beta(\psi))$ to $\alpha(P)$ maps $\alpha(P)$ into $\alpha(Q)$, and $\alpha(\pi(\psi)) = \tilde{\alpha}(\beta(\psi)) |_{\alpha(P)}$.
3. **(MG3)** $\delta_P \circ \delta_P = \delta_{\beta(P)} \circ \delta_{\alpha_0}$.

We say that the morphism $(\alpha, \beta)$ is an embedding if $\alpha$ is an embedding of fusion systems and $\beta: \text{Mor}_\mathcal{F}(P, Q) \to \text{Mor}_{\tilde{\mathcal{F}}}(\alpha(P), \alpha(Q))$ is an injection for all $P, Q \in L$. We say that $\mathcal{G}$ is a pre-subgroup of $\tilde{\mathcal{G}}$ if there is an embedding $(\iota, \beta)$ of $\mathcal{G}$ into $\tilde{\mathcal{G}}$, and in this case one may say simply that $\beta: \mathcal{G} \to \tilde{\mathcal{G}}$ is an embedding. If $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are $p$-local groups, then a morphism of $p$-local groups from $\mathcal{G}$ to $\tilde{\mathcal{G}}$ is a morphism of pre-local groups

$$(\alpha, \beta): (S, \mathcal{F}, L^{\text{rc}}) \longrightarrow (\tilde{S}, \tilde{\mathcal{F}}, \tilde{L}^{\text{rc}}).$$

Such a morphism is an embedding of $p$-local groups if it is an embedding of pre-local groups, and if $\alpha$ is given by inclusion, we say that $\mathcal{G}$ is a subgroup of $\tilde{\mathcal{G}}$.

The next two results provide tools for carrying out the construction of morphisms, and particularly of embeddings, of $p$-local groups.
PROPOSITION 2.11. Let $G_1$ be a subgroup of a group $G_2$, and assume that there are Sylow $p$-subgroups $S_i$ of $G_i$ with $S_1 = G_1 \cap S_2$. Assume that for each $i$, the fusion system $\mathcal{F}_i := \mathcal{F}_{S_i}(G_i)$ is saturated, and that we are given an $\mathcal{F}_i$-signalizer functor $\theta_i$. In addition, assume given a mapping $\beta: \mathcal{F}_1^{rc} \to \mathcal{F}_2^{rc}$ such that the following conditions hold for every $P \in \mathcal{F}_1^{rc}$.

1. $P \leq \beta(P)$.
2. For each $g \in N_{G_1}(P, S_1)$ we have $\beta(P^g) = \beta(P)^g$.
3. For each $Q \in \mathcal{F}_1^{rc}$ with $P \leq Q$ we have $\beta(P) \leq \beta(Q)$.
4. $\theta_2(\beta(P)) \cap G_1 = \theta_1(P)$.

Let $\iota: \mathcal{F}_1 \to \mathcal{F}_2$ be the inclusion functor, and write $\mathcal{G}_i = (S_i, \mathcal{F}_i, \mathcal{L}_i)$ for the $p$-local group which is canonically associated with $\mathcal{F}_i$ and $\theta_i$ (cf. 2.6). Then for any $P, Q \in \mathcal{F}_1^{rc}$ there is a mapping

\[ \beta_{P, Q}: \text{Mor}_{\mathcal{F}_1}(P, Q) \to \text{Mor}_{\mathcal{F}_2}(P, Q) \]

given by $\theta_1(P)g \mapsto \theta_2(\beta(P))g$; and $(\iota, \beta)$ is an embedding of $\mathcal{G}_1$ into $\mathcal{G}_2$. That is, $\mathcal{G}_1$ is a $p$-local subgroup of $\mathcal{G}_2$.

Proof. Let $P, Q \in \mathcal{F}_1^{rc}$ and let $g \in N_{G_1}(P, Q)$. Then $P^g \in \mathcal{F}_1^{rc}$ by 2.3. Since $P^g \leq Q$, (3) yields $\beta(P^g) \leq \beta(Q)$. Then $\beta(P)^g \leq \beta(Q)$ by (2), and so $g \in N_{G_2}(\beta(P), \beta(Q))$. We have $\theta_1(P)g = \theta_1(P)h$ if and only if $hg^{-1} \in \theta_1(P)$, while by (4), $hg^{-1} \in \theta_1(P)$ if and only if $hg^{-1} \in \theta_2(\beta(P)) \cap G_1$, which holds if and only if $\theta_2(\beta(P))g = \theta_2(\beta(P))h$. This shows that the mappings $\beta_{P, Q}$ in (*) are well-defined injections. Visibly, $\beta$ preserves composition, so that $\beta$ is a functor from $\mathcal{L}_1$ to $\mathcal{L}_2$.

Axiom (MG1) in 2.10 is an immediate consequence of (1). Next,

\[ \alpha(\pi_1(\theta_1(P)g)) = \alpha(c_{gP}) = c_{gP} = \pi_2(\theta_2(\beta(P))g)_{P} = \pi_2(\beta(\theta_1(P)g))_{P}, \]

so that (MG2) holds. Finally, for any $g \in P$,

\[ \beta(\delta_{1,P}(g)) = \beta(\theta_1(P)g) = \theta_2(\beta(P))g = \delta_{1,\beta(P)}(g) = \delta_{1,\beta(P)}(\alpha(g)), \]

and so (MG3) holds. \qed

PROPOSITION 2.12. Let $G_1$ be a subgroup of a group $G_2$, assume that there are Sylow $p$-subgroups $S_i$ of $G_i$ with $S_1 = G_1 \cap S_2$, and let $\theta_i$ be a signalizer functor on the fusion system $\mathcal{F}_i := \mathcal{F}_{S_i}(G_i)$. Assume also that each $\mathcal{F}_i$ is saturated. For $P \in \mathcal{F}_1^{rc}$, denote by $\mathcal{B}(P)$ the set of $N_{G_1}(P)$-invariant $p$-subgroups of $G_2$, and set $\beta(P) = (\mathcal{B}(P))$. Assume that the following two conditions hold for each $P \in \mathcal{F}_1^{rc}$.

1. $\beta(P) \in \mathcal{F}_2^{rc}$. (In particular, $\beta(P) \leq S_2$.)
2. $\theta_1(P) = \theta_2(\beta(P)) \cap G_1$. 


Then $\beta$ satisfies conditions (1) through (4) of 2.11, and defines an embedding $(1, \beta): \mathcal{G}_1 \rightarrow \mathcal{G}_2$ of $p$-local groups.

**Proof.** By (1'), $\beta$ is indeed a map from $\mathcal{T}_{1c}$ to $\mathcal{T}_{2c}$. Let $P \in \mathcal{T}_{1c}$. As $P \in \mathcal{B}(P)$, condition (1) of 2.11 holds. We have $\mathcal{B}(P)^g = \mathcal{B}(g)$ for any $g \in N_{G_1}(P, S_1)$, so condition (2) holds.

Let $Q \in \mathcal{T}_{1c}$ with $P \leq Q$. For any $x \in N_{G_1}(Q)$ we have $P^x \leq S_1$. Then $P^x \in \mathcal{T}_{1c}$, and so $\beta(P)^x = \beta(P^x) \leq S_2$ by (2). Thus $\langle \beta(P)^{N_{G_1}(Q)} \rangle$ is a $p$-group, invariant under $N_{G_1}(Q)$, and hence contained in $\beta(Q)$. Thus condition (3) holds. Condition (4) is equivalent to (2'), and so the proof is complete. □

**Example 2.13.** Let $G$ be a group and $K \triangleleft G$ such that $\bar{G} := G/K$ is finite. Let $S$ be a $p$-subgroup of $G$ such that $S \cap K = 1$, $S \in \operatorname{Syl}_p(KS)$, and $\bar{S} \in \operatorname{Syl}_p(\bar{G})$. Set $\mathcal{F} := \mathcal{F}_S(G)$, and for $P \in \mathcal{F}$ set $\hat{\theta}(P) = O_{P'}(C_{\bar{G}}(P))$. Let $\theta(P)$ be the preimage in $C_G(P)$ of $\hat{\theta}(P)$.

Set $\mathcal{F} = \mathcal{F}_S(\bar{G})$. Then $\hat{\theta}$ is an $\mathcal{F}$-signalizer functor, and $\mathcal{G} := \mathcal{G}_{S, \hat{\theta}}(\bar{G})$ is the natural $p$-local group $\mathcal{G}_{S, \hat{\theta}}(\bar{G})$. Let $\alpha_0: S \rightarrow \bar{S}$ be the restriction to $S$ of the quotient map $G \rightarrow \bar{G}$. For $P \leq S$ define $\alpha_1$ on $\operatorname{Hom}_G(P, S)$ by $\alpha_1: c_g \mapsto c_{\bar{g}}$.

Observe that for $P \leq S$, $\alpha_0: N_G(P, S) \rightarrow N_{\bar{G}}(\bar{P}, \bar{S})$ is a surjection. Namely, if $g \in G$ with $\bar{P} \bar{g} \leq \bar{S}$ then, as $S \in \operatorname{Syl}_p(KS)$, there is $k \in K$ with $P g^k \leq S$, and we have $\bar{g} = gk$. Similarly $C_{\bar{G}}(\bar{P}) = \bar{C}_G(P)$. Also, if $g, h \in N_G(P, S)$ with $\alpha_1(c_g) = \alpha_1(c_h)$ then $P^g \leq S \geq P^h$ with $gh^{-1}$ centralizing $\bar{P}$, so $gh^{-1} \in N_{G}(P)$ with $[P, gh^{-1}] \leq P \cap K = 1$. Thus $\alpha_1: \operatorname{Hom}_{\mathcal{F}}(P, S) \rightarrow \operatorname{Hom}_{\mathcal{F}}(\bar{P}, \bar{S})$ is a bijection.

Therefore $(\alpha_0, \alpha_1)$ is an isomorphism of $\mathcal{F}$ with $\mathcal{F}$, and since $\mathcal{F}$ is saturated, so is $\mathcal{F}$. It is now easy to check that $\theta$ is an $\mathcal{F}$-signalizer functor, and then $\mathcal{G} := \mathcal{G}_{S, \theta}(G)$ is a $p$-local finite group by 2.7(b).

Define $\beta: \mathcal{F} \rightarrow \mathcal{F}$ by $\beta(P) = \bar{P}$ (on objects), and by $\beta: \theta(P)g \mapsto \hat{\theta}(\bar{P})g$ (on morphisms). One may now check that $(\alpha, \beta): \mathcal{G} \rightarrow \mathcal{G}$ is an isomorphism of $p$-local finite groups.

The hypothesis that $S \in \operatorname{Syl}_p(KS)$ was used only to verify that the maps $\alpha_0: N_G(P, S) \rightarrow N_{\bar{G}}(\bar{P}, \bar{S})$ are surjective, and that $\hat{\theta}(\bar{P}) \leq \bar{C}_G(P)$ for each $P \in \mathcal{F}_{1c}$. Thus, that hypothesis may be replaced by the hypothesis that $\alpha_0: N_G(P, S) \rightarrow N_{\bar{G}}(\bar{P}, \bar{S})$ is surjective and $\hat{\theta}(\bar{P}) = 1$ for each $\bar{P} \in \mathcal{F}_{1c}$.

3. **Amalgams**

In this section, an *amalgam* of groups will always mean a pair

$$\mathcal{A} = (A_1 \leftarrow A_{1,2} \rightarrow A_2)$$

of injective group homomorphisms. A *morphism* from the amalgam $\mathcal{A}$ to an amalgam $\mathcal{B} = (B_1 \leftarrow B_{1,2} \rightarrow B_2)$ is a triple $\gamma = (\gamma_J \mid \emptyset \neq J \subseteq \{1, 2\})$ of injective group homomorphisms $\gamma_J: A_J \rightarrow B_J$, such that $\alpha_i \gamma_i = \gamma_{1,2} \beta_i$ for $i = 1, 2$. 
For example, if $G$ is a group with subgroups $G_1$ and $G_2$, and $G_{1,2}$ is a subgroup of $G_1 \cap G_2$, then there is a *subgroup amalgam* $\mathcal{A}$ given by the inclusion maps $\alpha_i : G_{1,2} \to G_i$. A *completion* of an amalgam $\mathcal{A}$ is an isomorphism $\gamma : \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ with a subgroup amalgam $\mathcal{B}$ in a group $G$, such that $G = (G_1, G_2)$. One often abuses the terminology and says simply that $G$ is a completion of $\mathcal{A}$.

Let $\mathcal{A} = (G_1 \leftarrow B \rightarrow G_2)$ be an amalgam and let $G = G_1 *_B G_2$ be the associated free amalgamated product. Then $G$ is a completion of $\mathcal{A}$, and indeed the universal completion of $\mathcal{A}$. We identify $\mathcal{A}$ with the subgroup amalgam of $G$ which is the image of $\mathcal{A}$ under this completion, and in particular regard $G_1$, $G_2$ and $B$ as subgroups of $G$ with $G_1 \cap G_2 = B$.

For any subgroup $X$ of $G$, denote by $X \setminus G$ the set of right cosets of $X$ in $G$. Set $\Gamma_i = G_i \setminus G$. Then $\Gamma = \Gamma(\mathcal{A})$ is the graph whose vertex set is the disjoint union $V(\Gamma) = \Gamma_1 \coprod \Gamma_2$, and whose set of edges is the set $E(\Gamma)$ of 2-subsets $\{G_1 x, G_2 x\}$ with $x \in G$. We call $\Gamma = \Gamma(\mathcal{A})$ the *standard tree* associated with $\mathcal{A}$ and with $G$, and we refer to [Se] for the fact that $\Gamma$ really is a tree. Observe that $G$ is represented as a group of automorphisms of $\Gamma$ via right multiplication, and that the kernel of this representation is the largest normal subgroup of $G$ which is contained in $B$. Evidently, $G$ acts transitively on $E(\Gamma)$, while $\Gamma_1$ and $\Gamma_2$ are the (distinct) orbits for $G$ on $V(\Gamma)$. It is also evident that $G$ is *locally transitive* on $\Gamma$; that is the stabilizer $G_\delta$ of any vertex $\delta$ acts transitively on the set $\Gamma(\delta)$ defined by

$$\Gamma(\delta) = \{ \gamma \in \Gamma \mid \{ \delta, \gamma \} \in E(\Gamma) \}.$$  

For any subgroup or element $X$ of $G$, write $\Gamma_X$ for the subgraph of $\Gamma$ induced on the set of vertices which are fixed by $X$. For any connected graph $\Delta$, and vertices $\alpha$ and $\beta$ of $\Delta$, the length of the shortest geodesic path from $\alpha$ to $\beta$ in $\Delta$ is denoted $d(\alpha, \beta)$. Then $(\Delta, d)$ is a discrete metric space, and automorphisms of $\Delta$ are isometries. An isometry of a tree is said to be *hyperbolic* if it fixes no vertices or edges. The following two results, noticed first by J. Tits [Tit70], are elementary.

**Lemma 3.1.** Let $h$ be an automorphism of a tree $\Gamma$, and denote by $\Lambda(h)$ the intersection of all the $h$-invariant subtrees of $\Gamma$. Suppose that there exists an edge $\{\gamma, \delta\}$ of $\Gamma$ such that

1. $d(\gamma, \gamma h) = d(\delta, \delta h) \neq 0$ and
2. $\{\gamma, \delta\}$ is not fixed by $h$.

Then $h$ is hyperbolic, and the set of all edges $\{\gamma, \delta\}$ which satisfy condition (1) is the edge set of $\Lambda(h)$.  

□
Lemma 3.2. Let $\Gamma$ be a tree, let $g$ be a hyperbolic isometry of $\Gamma$, and define $\Lambda = \Lambda(g)$ as in 3.1. Set $d = \min\{d(\delta, \delta^g) \mid \delta \in V(\Gamma)\}$. Then:
(a) $\Lambda$ is isomorphic to the graph $\tilde{Z}$ whose vertex set is the set $\mathbb{Z}$ of integers, and whose edges are the pairs $\{n, n + 1\}$ for $n \in \mathbb{Z}$.
(b) There is an isomorphism $\psi: \Lambda \to \tilde{Z}$ such that $\psi^{-1} g \psi: n \mapsto n + d$ for all $n \in \mathbb{Z}$.
(c) For any vertex $\gamma$ of $\Gamma$, the geodesic in $\Gamma$ from $\gamma$ to $\gamma^g$ has length $d + 2e$, where $e$ is the minimal distance from $\gamma$ to a vertex of $\Lambda$.

Lemma 3.3. Let $\Gamma$ be a tree, and let $x, y \in \text{Aut}(\Gamma)$, $\delta \in \Gamma_x$, $\gamma \in \Gamma_y$, and $(\alpha_0, \ldots, \alpha_d)$ the geodesic from $\gamma$ to $\delta$. Suppose $x$ does not fix $\alpha = \alpha_{d-1}$ and $y$ does not fix $\alpha_1$. Then $xy$ is hyperbolic.

Proof. Observe first of all that $(\alpha_0y, \ldots, \alpha_dy)$ is the geodesic from $\gamma = \alpha_0y$ to $\delta y$, so that the path $\rho = (\alpha_dy, \ldots, \alpha_0y, \alpha_1, \ldots, \alpha_d)$ contains a geodesic $g$ from $\delta y$ to $\delta$. Indeed as $\alpha_1 \neq \alpha_1y$, $g = \rho$ is of length $2d$. Then $g' = (\alpha_{d-1}y, \ldots, \alpha_0y, \alpha_1, \ldots, \alpha_d)$ is the geodesic from $\alpha y$ to $\delta$, and since $\beta = \alpha x^{-1} \neq \alpha$, it follows that $g'\beta$ is the geodesic of length $2d$ from $\alpha y$ to $\beta$. But $\delta y = \delta xy$ and $\beta xy = \alpha y$, and so $d(\delta, \delta xy) = d(\beta, \beta xy)$. The lemma now follows from 3.1.

We will work under the following hypothesis for the remainder of this section.

Hypothesis 3.4. $G = G_1 \ast_B G_2$ is the free amalgamated product associated with an amalgam

$$\mathcal{A} = (G_1 \leftarrow B \rightarrow G_2),$$

and $G_1, G_2,$ and $B$ are regarded as subgroups of $G$ in the canonical way. Let $\Gamma$ be the standard tree associated with $\mathcal{A}$ and $G$, and let $\Gamma_i$ be the subset $G_i \backslash G$ of the vertex set of $\Gamma$. Assume

1. There is a subgroup $S$ of $G$ such that $S$ is a (possibly infinite) Sylow $p$-subgroup of each of the groups $G_1, G_2,$ and $B$.
2. $N_{G_i}(S) \leq B \neq G_i$, for $i = 1$ and $2$.

The vertex $G_i$ of $\Gamma$ will most often be denoted $\gamma_i$.

Lemma 3.5. Assume Hypothesis 3.4, and assume also that

\begin{equation}
\{S^g \mid g \in G_1 \cup G_2, \ S^g \leq B\} = S^B.
\end{equation}

Then:

(a) $\Gamma_S = \{\gamma_1, \gamma_2\}$, and
(b) $S \in \text{Syl}_p(G)$.

Proof. It follows from (*) and from [Asc86, 5.21] that $N_{G_i}(S)$ is transitive on $\Gamma_S(\gamma_i)$. Since $N_{G_i}(S) \leq B$, by 3.4, we obtain (a).

Let $S^*$ be a $p$-subgroup of $G$ containing $S$ and let $x \in S^*$. Then $|x|$ is finite, so that 3.2 implies that $\Gamma_x \neq \emptyset$. Choose $\delta \in \Gamma_x$ and $\gamma \in \Gamma_S$ with $d := d(\delta, \gamma)$
Moreover. Thus $q \in G_y$, and there exists $y \in S - G_y$. As $|xy|$ is finite, this contradicts 3.3, and so we conclude that $d = 0$. Then (a) yields $S^* \leq G_i$ for some $i$. Since $S$ is a maximal $p$-subgroup of $G_i$ we then have $S^* = S$, and thus $S$ is a maximal $p$-subgroup of $G$.

Let $P$ be a finite $p$-subgroup of $G$, and let $P_0$ be a subgroup of $P$ which fixes a vertex of $\Gamma$, and which is maximal for this condition. Then $N_P(P_0)$ acts on the tree $\Gamma_0 = \Gamma_{P_0}$, and no element of $N_P(P_0) - P_0$ fixes a vertex of $\Gamma_0$. Now, $N_P(P_0) - P_0$ consists of hyperbolic isometries on $\Gamma_0$, by 3.2, and hence $N_P(P_0) = P_0 = P$. Since $G$ is edge-transitive on $\Gamma$, it follows that $P$ is conjugate to a subgroup of $G_1$ or $G_2$. Then $P$ is conjugate to a subgroup of $S$, by 3.4(1), and $S$ is a Sylow $p$-subgroup of $G$. □

**Lemma 3.6.** Assume Hypothesis 3.4, let $P$ be a finite subgroup of $S$, and let $g \in N_G(P, S)$. Set $F_0 = P$. Then there exists a positive integer $n$, elements $g_1, \ldots, g_n$ of $G_1$, elements $h_1, \ldots, h_n$ of $G_2$, and subgroups $E_i$ and $F_i$ of $S$, $1 \leq i \leq n$, such that the following conditions hold.

1. $g_i \notin B$ for $1 \leq i \leq n$, and $h_i \notin B$ for $1 \leq i < n$.
2. $E_i = F_{i-1}^{g_i}$ and $F_i = E_i^{h_i}$ for all $i$ with $1 \leq i \leq n$.
3. $g = g_1 h_1 \cdots g_n h_n$.

Moreover, the minimal length of $g$ as a word in the generating set $G_1 \cup G_2$ is $2n - 2$ if $g_1, h_n \in B$, $2n - 1$ if exactly one of $g_1$ and $h_n$ is in $B$, and $2n$ if neither $g_1$ nor $h_n$ is in $B$.

**Proof.** Since $G_1$ and $G_2$ generate $G$, we may choose elements $g_i \in G_1$ and $h_i \in G_2$, satisfying the conditions in (a) and (c). Set $\alpha_0 = \gamma_2$, $\beta_0 = \gamma_1 h_n$, and for $1 \leq i \leq n$ set $w_i = g_{n-i+1} h_{n-i+1} \cdots g_n h_n$, $\alpha_i = \gamma_2 w_i$, and $\beta_i = \gamma_1 h_{n-i} w_i$. Then $q = (\alpha_0, \beta_0, \ldots, \beta_{n-1})$ is a path in $\Gamma$ with $\beta_i \neq \beta_{i+1}$ and $\alpha_i \neq \alpha_{i+1}$ for $i < n - 1$. Thus $q$ is a geodesic from $\alpha_0$ to $\beta_{n-1}$, and if $g_1 \notin B$ then also the path $q\alpha_n$ is a geodesic. In particular if $n > 1$ or $g_1 \notin B$ then $d(a_0, a_n) > 0$, and $w_n \notin B$.

Take (b) as the definition of the groups $E_i$ and $F_i$ for $i > 0$. We now show that these groups are contained in $B$. Let $x \in F_0$, set $y_0 = x$, and for $1 \leq i \leq n$ define $x_i$ and $y_i$ recursively, by

$$x_i = y_{i-1}^{g_i} \quad \text{and} \quad y_i = x_i^{h_i}.$$ 

Suppose that for some $j$, either $x_j$ or $y_j$ is not in $B$, and let $j$ be the smallest such index. Suppose that $x_j \notin B$. Then $y_{j-1} \in B$, and so $x_j = y_{j-1}^{g_j} \in G_1 - B$. Then

$$x^g = h_n^{-1} g_n^{-1} \cdots h_j^{-1} x_j h_j \cdots g_n h_n$$
is an alternating product of elements of $G_2$ and $G_1$, in which none of the factors lies in $B$ except possibly for the first and the last. It follows from paragraph one that $x^g \not\in B$, whereas $x^g \in S \leq B$. A similar argument shows $y_j \in B$. Therefore $x_i$ and $y_i$ are in $B$ for all $i$, and so each of the groups $E_i$ and $F_i$ is contained in $B$. Since $S$ is a Sylow $p$-subgroup of $B$, we may adjust our choices of the elements $g_i$ and $h_i$, via right multiplication by elements of $B$, to ensure that $E_i$ and $F_i$ are in $S$ for all $i$.

It remains to prove the final statement in the lemma. This follows since, by paragraph one, the length $\ell(g)$ of $g$ as a word in the generating set $G_1 \cup G_2$ for $G$ is equal to the shortest distance in the tree $\Gamma(S)$ from a vertex in $\{\gamma_1 g, \gamma_2 g\}$ to a vertex in $\{\gamma_1, \gamma_2\}$.

We have the following immediate consequence of 3.6.

**Corollary 3.7.** Assume Hypothesis 3.4. Then

$$\mathcal{F}_S(G) = \langle \mathcal{F}_S(G_1), \mathcal{F}_S(G_2) \rangle.$$  \hfill \Box

For any subgroup $X$ of $G$, and any elementary abelian $p$-subgroup $A$ of $X$, denote by $\mathcal{E}_n(X, A)$ the set of elementary abelian $p$-subgroups of $X$ which have order $p^n$ and which contain $A$. Write $\mathcal{E}_n(X)$ for $\mathcal{E}_n(X, 1)$.

In the remainder of this section we assume the following hypothesis.

**Hypothesis 3.8.** Hypothesis 3.4 holds, and so do the following conditions.

1. There is a normal subgroup $Z$ of $G_1$ of order $p$, and $Z$ is the unique subgroup of order $p$ in $Z(S)$.
2. There exists $U \in \mathcal{E}_2(G_2, Z)$ with $U \unlhd G_2$, and $G_2$ acts transitively on $\mathcal{E}_1(U)$.
3. $B = N_{G_1}(U) = N_{G_2}(Z)$.
4. For each $X \in \{H, K, B\}$, $X$ is transitive on its set of maximal $p$-subgroups.

**Lemma 3.9.** Let $P$ be a subgroup of $S$ and let $X$ be a subgroup of $Z(P)$ of order $p$.

(a) Let $g \in G_2$ with $P^g \leq S$. Then one of the following holds.
   (i) $g \in B$, and neither $P$ nor $P^g$ is contained in $C_G(U)$.
   (ii) $P \leq C_G(U)$.
(b) If $X \not= Z$ and $X \leq U$ then there exists $g \in G_2$ with $X^g = Z$ and with $P^g \leq S$.

**Proof.** Let $g$ be as in (a), and set $Q = P^g$. Since $U \unlhd G_2$, we have $P \leq C_G(U)$ if and only if $Q \leq C_G(U)$. Since conclusion (ii) of (a) does not hold, neither $P$ nor $Q$ is contained in $C_G(U)$. Set $\bar{G}_2 = G_2/C_{G_2}(U)$. Then $\bar{P}$ and $\bar{Q}$ are nontrivial $p$-subgroups of $\bar{S}$. Hence by 3.8(2), $\bar{G}_2$ is isomorphic to a subgroup of $\text{GL}_2(p)$ containing $\text{SL}_2(p)$, and $B = N_{G_2}(Z)$ has index $p + 1$ in $G_2$. Thus $\bar{P} = \bar{Q} = \bar{S}$,
and $\overline{B} = N_{G_2}(\overline{S})$. Since $P^g = Q$, we then have $\overline{g} \in \overline{B}$. Thus $g \in B$ and (a) is proved.

Suppose that $Z \neq X \leq U$. By 3.8(2) there exists $g \in G_2$ with $X^g = Z$. Since $X \leq Z(P)$, we have $P^g \leq C_{G_2}(Z) \leq B$. By 3.8(4) there exists $h \in B$ with $P^{gh} \leq S$. Replacing $g$ with $gh$, we obtain (b).

For $\delta \in \Gamma$ and $g \in G$ with $\delta = \gamma_i g$ for some $i$, $Z_\delta$ will denote $Z^g$ if $i = 1$, and $U^g$ if $i = 2$. This notation is well defined as a consequence of 3.8(3).

**Lemma 3.10.** Let $\Sigma$ be a subtree of $\Gamma$ and let $\gamma \in \Sigma$. Set $Y = \langle Z_\delta \mid \delta \in \Sigma \rangle$. Then $C_{G_\gamma}(Y)$ fixes $\Sigma$ vertex-wise.

**Proof.** Let $\gamma \in \Sigma$, set $X = C_{G_\gamma}(Y)$, and assume that $X \neq G_\Sigma$. Among all pairs $(\delta, x)$ with $\delta \in \Sigma$ and $x \in X$ with $\delta x \neq \delta$, choose $(\delta, x)$ so that $d := d(\delta, \delta x)$ is minimal. Let $\alpha \in \Sigma(\delta)$ be of distance $d - 1$ from $\gamma$. Then $X$ fixes $\alpha$ and centralizes $Y$, so $X$ centralizes $Z_\delta$. Thus $X \leq C_{G_\alpha}(Z_\delta) \leq G_\delta$ by 3.8(3), and contrary to our choice of $\delta$.

**Lemma 3.11.** Let $\delta$ and $\gamma$ be distinct vertices in $\Gamma_1$ with $Z_\delta = Z_\gamma$. Then $d(\delta, \gamma) \geq 6$. Moreover, the following hold.

(a) Let $\alpha, \beta \in \Gamma_1$ with $d(\alpha, \beta) = 2$, and let $X$ be a subgroup of $G$ fixing $\beta$ and centralizing $Z_\alpha$ and $Z_\beta$. Then $X$ fixes $\alpha$.

(b) Let $\alpha_0, \alpha_4 \in \Gamma_1$ with $d(\alpha_0, \alpha_4) = 4$, and such that $Z_{\alpha_4}$ centralizes $Z_{\alpha_0}$. Then $Z_{\alpha_4}$ fixes $\alpha_0$.

**Proof.** Suppose $d := d(\delta, \gamma) < 6$. Then $d = 2$ or 4. If $d = 2$, and $(\delta, \beta, \gamma)$ is the geodesic from $\delta$ to $\gamma$, then $Z_\beta = Z_\delta Z_\gamma$ is of order $p$, which is not the case. Thus $d = 4$. Write $(\delta, \beta, \delta', \beta', \gamma)$ for the geodesic from $\delta$ to $\gamma$. Then $Z_\beta = Z_\delta Z_{\delta'} = Z_\gamma Z_{\delta'} = Z_{\beta'}$.

By edge-transitivity, we may take $\delta' = \gamma_1$ and $\beta' = \gamma_2$. Then $U = Z_{\beta'} = Z_\beta$, and by local transitivity there exists $g \in G_1$ with $\beta^g = \beta'$. Then $U = U^g$, so $g \in B$, and $\beta = \beta'$. Then $d < 4$, and we have a contradiction.

Assume the hypothesis of (a). Without loss, $\beta = \gamma_1$ and $\{\gamma_2\} = \Gamma(\alpha) \cap \Gamma(\beta)$. As $X$ fixes $\beta$ and centralizes $Z_{\alpha}$ and $Z_{\beta}$, we obtain $X \leq C_H(Z_{\alpha} Z_{\beta}) = C_H(U) \leq B \leq G_\alpha$, establishing (a).

Now assume the hypothesis of (b), and take $X$ to be $Z_{\alpha_4}$. Let $(\alpha_0, \ldots, \alpha_4)$ be the geodesic from $\alpha_0$ to $\alpha_4$. Then $X \leq Z_{\alpha_3} = Z_{\alpha_2} Z_{\alpha_4} \leq C_G(Z_{\alpha_2})$, and so $X$ fixes $\alpha_2$ by (a). Then, since $\text{dist}(\alpha_0, \alpha_2) = 2$ and $X$ centralizes $Z_{\alpha_i}$ for $i = 0, 2$, $X$ fixes $\alpha_0$ by another application of (a).

**Hypothesis 3.12.** Hypothesis 3.8 holds, and every subgroup of $G_1$ of order $p$ is conjugate in $G_1$ to a subgroup of $U$. 

**Lemma 3.13.** Assume Hypothesis 3.12. Let \( P \) be a subgroup of \( S \) and let \( X \) be a subgroup of \( Z(P) \) of order \( p \).

(a) If \( X \neq U \) then there exists \( g \in G_1 \) with \( X^g \leq U \) and with \( P^g \leq S \).

(b) If \( X \neq Z \) then there exists \( g \in G \) with \( X^g = Z \), \((XZ)^g = U \), \((PU)^g \leq S \), and \( g = g_1g_2 \) where \( g_i \in G_i \) and \( P^{g_i} \leq S \).

*Proof.* Suppose that \( X \not\leq U \). By 3.12 there exists \( g \in G_1 \) with \( X^g \leq U \). Set \( Y = X^g \). Then \( YZ = U \), so that \( P^g \leq C_{G_1}(U) \leq B \), and by 3.8(4) there exists \( h \in B \) such that \( P^{gh} \leq C_S(Y) \). Replacing \( g \) with \( gh \), we obtain (a).

Now assume that \( X \not\leq Z \). If \( X \leq U \) we appeal to 3.9(b), with \( PU \) in the role of \( P \), in order to obtain \( g_2 \in G_2 \) with \( X^{g_2} = Z \) and \((PU)^{g_2} \leq S \). With \( g_1 = 1 \), (b) holds in this case. So assume that \( X \not\leq U \). If \( U \) centralizes \( X \) we apply (a) to \( PU \), obtaining \( g_1 \in G_1 \) such that \( X^{g_1} \leq U \) and \((PU)^{g_1} \leq S \). Then \((XZ)^{g_1} = U \), and by 3.9(b) there exists \( g_2 \in G_2 \) with \( X^{g_1g_2} = Z \) and \((PU)^{g_1g_2} \leq S \). Thus (b) also holds in this case, and we are reduced to the case where \( U \) does not centralize \( X \). Since \( P \leq S \), 3.8(3) implies that \([U, X] = [U, S] = Z \). Then \( XZ \leq PU \). By 3.12 there exists \( g_1 \in G_1 \) with \( X^{g_1} \leq U \). Then \((XZ)^{g_1} = U \), so that \((PU)^{g_1} \leq N_{G_1}(U) = B \). By 3.8(4) we may assume that \( g_1 \) was chosen so that \((PU)^{g_1} \leq S \). Then (a) applies, and completes the proof of (b). \( \square \)

The next result amounts to a re-working of [LO02, Lemma 1.4] in our tree-theoretic setup. The formulation given here is different in several respects from the one in [LO02], but the main idea of the proof has not been altered. We remark that we shall only use part (c) of 3.14, and this will occur only once, in the proof of 9.2.

**Proposition 3.14.** Assume Hypothesis 3.12. Set \( D = N_G(Z) \) and assume that \( \Phi_S(D) \neq \Phi_S(G_1) \). Denote by \( \Delta \) the set of all pairs \((P, g)\) such that \( P \) is a finite subgroup of \( S \), \( g \in N_D(P, S) \), and \( c_g \notin \text{Hom}_{G_1}(P, S) \). Set

\[
\mathcal{D} = \{ P \mid (P, g) \in \Delta \text{ for some } g \in D \},
\]

and let \( P \in \mathcal{D} \). Choose \((P, g) \in \Delta \) so that the length \( \ell(g) \) of \( g \), as a word in the set \( G_1 \cup G_2 \) of generators of \( G \), is minimal. Then \( [P, U] = 1 \), \((PU, g) \in \Delta \), and upon replacing \( P \) with a suitable subgroup of \( C_S(P) \) containing \( P \), we have

(a) \( \ell(g) = 5 \), and \( g = g_1g_2g_3g_4g_5 \) where \( g_i \in G_2 \) for \( i \) odd, and \( g_i \in G_1 \) for \( i \) even.

(b) The elements \( g_1 \) through \( g_5 \) in (a) may be chosen so that \( U \leq Z(P^{g_1 \cdots g_i}) \), and \( P^{g_1 \cdots g_i} \leq S \) for all \( i \), \( 1 \leq i \leq 5 \).

(c) There exists \( E \in \mathcal{E}_3(Z(P), U) \) such that \( U \leq E^g \), \( C_B(E)^g \leq B \), and \( C_B(E^g) \leq B^g \).
Proof. Set \( Q = P^g \). Since \( g \in D \), also \( g \in N_D(PZ, QZ) \), so that we may assume \( Z \leq P \).

By 3.6 we have \( g = g_1 \ldots g_n \), where each \( g_i \) is in \( G_1 \cup G_2 \), and where \( P^{g_1 \ldots g_i} \leq S \) for all \( i, 1 \leq i \leq n \). Moreover, the sequence \( (g_1, \ldots, g_n) \) may be chosen so that \( \ell(g) = n \).

Set \( P_0 = P, Z_0 = Z \), and for \( 1 \leq i \leq n \) set \( P_i = P^{g_i}_{i-1} \) and \( Z_i = Z^{g_i}_{i-1} \). If \( Z_i = Z \) for some \( i \) with \( 0 < i < n \), then the minimality of \( n \) implies that, for \( x = g_1 \ldots g_i \) and \( y = g_{i+1} \ldots g_n \), we have \( c_x \in \text{Hom}_{G_1}(P, P_i) \) and \( c_y \in \text{Hom}_{G_1}(P_i, Q) \). But in that case we get \( c_g = c_x c_y \in \text{Hom}_{G_1}(P, Q) \), contrary to hypothesis. Thus:

1. \( Z_i = Z \) if and only if \( i = 0 \) or \( n \).

By 3.13(b) there exist elements \( v_1 \) through \( v_{n-1} \) of \( G_1 G_2 \), such that

2. \( Z_i^{v_i} = Z, (ZZ_i)^{v_i} = U \), and \((P_i U)^{v_i} \leq S \).

Set \( v_0 = v_n = 1 \), and for each \( i, 0 \leq i \leq n \), choose \( r_i \in G_1 \) and \( s_i \in G_2 \) with \( v_i = r_i s_i \). Set \( k_i = v_i^{-1} g_i v_i \) for \( i \geq 1 \), and set \( P_i' = P_i^{v_i} \) for \( i \geq 0 \). Then \( P_i' \leq S \) for all \( i \), by (2), and \((P_i')^{k_i} = P_i' \) for \( i \geq 1 \). Notice that

\[
g_1 \ldots g_n = k_1 \ldots k_n.
\]

Suppose that, for all \( i \geq 1 \), we have \( c_{k_i} \in \text{Hom}_{G_1}(P_i'_{i-1}, P_i') \). Choose \( t_i \in G_1 \) so that \( k_i t_i^{-1} \in C_G(P_i'_{i-1}) \), and set \( t = t_1 \ldots t_n \). Then \( g t^{-1} \in C_G(P) \) by (\(*\)), so that \( c_g \in \text{Hom}_{G_1}(P, Q) \), contrary to hypothesis. Thus, \( c_{k_i} \notin \text{Hom}_{G_1}(P_i'_{i-1}, P_i') \) for some \( i \geq 1 \). Since \( k_i = (r_i s_i^{-1})^{-1} g_i r_i s_i \) is of length 5, it follows from the minimality of \( n \) that \( n \leq 5 \).

Set \( \alpha = \alpha_0 = \gamma_1 \), and define vertices \( \alpha_1 \) through \( \alpha_n \) by \( \alpha_i = \alpha_{i-1} g_i \). Let \( \Sigma \) be the subtree of \( \Gamma \) generated by \( \{ \alpha_0, \ldots, \alpha_n \} \), and let \( \Sigma_i \) be the subtree of \( \Sigma \) generated by \( \alpha_0 \) and \( \alpha_i \). Thus \( \Sigma \) is the union of its subtrees \( \Sigma_i \), and \( \Sigma_i \) is the geodesic from \( \alpha \) to \( \alpha_i \) in \( \Gamma \).

Minimality of \( n \) implies that \( g_1 \in G_2 - G_1 \), as otherwise we may replace \((P, g)\) with \((P^{g_1}, g_2 \ldots g_n)\). Then \( g_i \) is in \( G_2 - G_1 \) for \( i \) odd, and in \( G_1 - G_2 \) for \( i \) even. Observe that

\[
d(\alpha, \alpha_i) = \begin{cases} 
  i + 1 & \text{if } i \text{ is odd}, \\
  i & \text{if } i \text{ is even}.
\end{cases}
\]

Notice that \( Z_i = Z_{\alpha_i} \). Then 3.11 implies that \( Z_\delta \neq Z \) for any \( \delta \in V(\Sigma) \) with \( d(\alpha, \delta) \leq 5 \). Since \( n \leq 5 \), it now follows from (\(\star\)) that \( n = 5 \).

The reader may find it convenient to have a “picture” of \( \Sigma \), at this point, as a visual reference for the remainder of the argument.
Since $P_i \leq S \leq G_{\alpha_0}$, $P_i$ fixes every vertex of $\Sigma_i$, and then 3.10 implies that $[Z_\delta, P_i] = 1$ for every $\delta \in \Sigma_i$. Set $\beta = \gamma_2$. Then from paragraph one of the proof of 3.6, for $i$ odd, $\Sigma_i$ is the tree induced on the geodesic $(\alpha, \beta, \alpha g_1, \beta g_{i-1} g_i, \ldots, \alpha g_1 \ldots g_i)$. In particular, $\beta$ is a vertex of $\Sigma_i$ for $i$ odd. Since $Z_\beta = U$, it follows that $[U, P_i] = 1$ for $i$ odd. But for $i$ odd we also have $g_i \in G_2 \leq N_G(U)$, and then since $P_i = P_{i-1}^g$, we conclude that:

(3) $[U, P_i] = 1$ for all $i$, $0 \leq i \leq 5$.

From the description of $\Sigma_5$ above, $\beta g$ is the vertex of $\Sigma_5$ at distance 5 from $\alpha$, and $\alpha g \neq \alpha_1 g \in \Gamma(\beta g)$. Set $\beta' = \beta g$, let $\alpha'$ be the vertex in $\Gamma_{\beta'}$ at distance 4 from $\alpha$, and let $\alpha''$ be the vertex of $\Sigma_5$ at distance 2 from $\alpha$. By (3), $U$ centralizes $Z_{\alpha''}$, so as $Z_{\beta'} = ZZ_{\alpha'} \geq Z_1^g$, we get $Z_1^g \leq C_G(U) \leq C_G(Z_{\alpha''})$. Then $Z_1^g$ fixes $\alpha''$, by 3.11(b). Also $[Z_1^g, Z] = 1$, so that 3.11(a) implies $Z_1^g$ fixes $\alpha$. Since $U^g = (ZZ_1)^g = ZZ_1^g$, we conclude that $U^g \leq G_\alpha \cap G_\beta$. That is, $U^g \leq B$, and so $QU^g$ is a $p$-subgroup of $B$. By 3.8(4) there exists $h \in B$ with $(QU^g)^h \leq S$, and we may replace $(P, g)$ with $(PU, gh)$, without increasing the length $n$. Thus, we may assume henceforth that $U \leq P$. By symmetry between $(P, g)$ and $(Q, g^{-1})$, we may assume also that $U \leq Q$. Then since $g_1 \in N_G(U)$, also $U \leq P_1$. Since $g_5 \in N_G(U)$ we then obtain $U \leq P_4$.

As $g_2, g_4 \in G_1$ and $U \leq P_1 \cap P_4$, we have $Z \leq P_2 \cap P_3$. Since $g_3 \in N_G(U)$, we have $U \leq P_2$ if and only if $U \leq P_3$. Suppose that $U \not\leq P_2$. Then

$$Z^{g_3} = (U \cap P_2)^{g_3} = U \cap P_3 = Z,$$

contrary to $g_3 \in G_2 \backslash G_1$. Therefore $U \leq P_i$ for all $i$. Then by (3):

(4) $U \leq Z(P_i)$ for all $i$, $0 \leq i \leq 5$. 

Set $U_0 = U = U_{-1}$, and for $1 \leq i \leq 5$, let $U_i = U_{g_1 \cdots g_i}$. For $0 \leq i \leq 5$ set $E_i = U_{i-1} U_i$. As $g_1 \in G_2$, we have $U = U_1 \leq E_2$, and so

$$Z = Z^g \leq U^g = U^{g_3 g_4} \leq E_2^{g_3 g_4} = E_4.$$  

Also $g_5 \in G_2 - G_1$, so that $Z \neq Z^{g_5} = E_5^{-1} \leq E_4$. Then $U = ZZ^{g_5} = E_4$. Since $g_5 \in G_2$, also $U \leq E_5$.

Set $F = C_B(E_0)$. Then $F^g = C_{B^g}(E_5)$, and $B^g$ is the stabilizer in $G$ of the edge $\{a_5, b_5\}$ of $\Sigma_5$. Next $U^g \leq E_0^g = E_5$, and $U^g = Z_{b_5}$, so that $F^g$ centralizes $Z_{b_5}$. Then $F^g$ fixes the vertex $a'$ of $\Sigma_5$, by 3.11(a). Denote by $\beta''$ the vertex of $\Sigma_5$ at distance 3 from $a$, and hence adjacent to $a'$. From an earlier remark, we have $\beta'' = \beta g_4 g_5$, and so

$$Z_{\beta''} = U^{g_4 g_5} = U^{g_3 g_4 g_5} \leq E_2^{g_3 g_4 g_5} = E_5.$$  

Thus $F^g$ centralizes $Z_{\beta''}$, and so $F^g$ fixes every vertex in $\Gamma(\beta'')$ by 3.11(a). In particular, $F^g$ fixes $\alpha''$. Since $[U, F^g] = 1$, $F^g$ fixes every vertex in $\Gamma(\beta)$ by 3.11(a). Thus, $F^g$ fixes $\alpha$ and $\beta$, and so $F^g \leq B$. This yields the first part of (c), with $E_0$ in the role of $E$. Since $Z_{\beta''} Z_{\beta'} = E_5$, 3.11(a) yields $C_B(E_5) \leq B^g$, and thus all parts of 3.14 have been established. \hfill $\square$

4. Spin$_7(F)$

Let $p$ be an odd prime, let $\overline{F}$ be an algebraic closure of the field of $p$ elements, let $\overline{V}$ be a vector space over $\overline{F}$ (of finite dimension $d$), and let $f$ be a symmetric, nondegenerate bilinear form on $\overline{V}$. The form $f$ is essentially unique, as $\overline{V}$ has an orthonormal basis with respect to $f$. The isometry group $O(\overline{V}, f)$ will be denoted also $O(\overline{V})$ (and $O_d(\overline{F})$). The identity component of $O(\overline{V})$ is denoted $\Omega(\overline{V})$, and has index 2 in $O(\overline{V})$. Indeed, we have $O(\overline{V}) = \Omega(\overline{V}) \langle \tau \rangle$, where $\tau$ is a reflection on $\overline{V}$. In the case that $d$ is odd, we have $O(\overline{V}) = \Omega(\overline{V}) \times \{\pm I\}$. The universal covering group of $\Omega(\overline{V})$ is denoted Spin$(\overline{V})$ (or Spin$_d(\overline{F})$).

There is a rational representation $\phi: \text{Spin}(\overline{V}) \rightarrow \Omega(\overline{V})$, with kernel contained in $Z(\text{Spin}(\overline{V}))$. From [C], one has $|\ker(\phi)| = 2$, and $\ker(\phi) = Z(\text{Spin}(\overline{V}))$ if $d$ is odd.

For any subset or element $D$ of Spin$(\overline{V})$, we write $C_D(\overline{V})$ and $[\overline{V}, D]$ for $C_{\overline{V}}(D \phi)$ and $[\overline{V}, D \phi]$, respectively.

Let $\overline{T}$ be a maximal torus of Spin$(\overline{V})$. By a weight of $\overline{T}$ on $\overline{V}$ we mean a homomorphism $\lambda: \overline{T} \rightarrow \overline{F}^\times$ such that the space $\overline{V}_\lambda = \{v \in \overline{V} \mid v a = \lambda(a) v \text{ for all } a \in \overline{T}\}$ is nonzero. The set of such weights is denoted $\Lambda(\overline{T})$.

A hyperbolic line in $\overline{V}$ is a nondegenerate subspace $\ell$ of $\overline{V}$ of dimension 2. Such a subspace has exactly two 1-dimensional singular subspaces (or points), and
from this one may easily deduce that $\Omega(\ell) \cong \overline{\mathbb{F}}^\times$ and that $O(\ell) = \Omega(\ell)t$, where $t$ is an involution which interchanges the singular points of $\ell$.

The following result is well known, and its proof is elementary.

**Lemma 4.1.** Let $(\tilde{V}, f)$ be a nondegenerate orthogonal space over $\overline{\mathbb{F}}$ of dimension $d$, and let $\tilde{T}$ be a maximal torus of $\Omega(\tilde{V})$. Then there exists a set $\ell(\tilde{T}) = \{\ell_1, \ldots, \ell_k\}$ of $\tilde{T}$-invariant, pairwise orthogonal hyperbolic lines in $\tilde{V}$, such that the following hold.

(a) $d = 2k$ or $d = 2k + 1$.

(b) $[\tilde{V}, \tilde{T}] = \ell_1 + \cdots + \ell_k$, and either $C_{\tilde{V}}(T) = 0$ or $C_{\tilde{V}}(\tilde{T})$ is a nonsingular 1-space, orthogonal to $[\tilde{V}, \tilde{T}]$.

(c) Each $\ell_i$ is a sum of two weight spaces $\tilde{V}_\lambda$ and $\tilde{V}_{\lambda^{-1}}$, where $\lambda \neq \lambda^{-1}$. These weight spaces are the singular points of $\ell_i$.

The set $\ell(\tilde{T})$ is uniquely determined by the conditions (a) and (c). Conversely, for any maximal set $\mathfrak{u}$ of pairwise orthogonal, hyperbolic lines in $\tilde{V}$, there is a unique maximal torus $\tilde{T}$ in $\Omega(\tilde{V})$ with $\ell(\tilde{T}) = \mathfrak{u}$.

From now on, let $\tilde{V}$ be a vector space of dimension 7 over $\overline{\mathbb{F}}$, and set $\tilde{H} = \text{Spin}(\tilde{V})$. Write $Z$ for the kernel of $\phi$. Then $Z = \langle z \rangle$ where $z$ is of order 2.

It is well known that an involution $t$ in an orthogonal group (over a field of characteristic different from 2) lifts to an involution in the corresponding spin group if and only if the dimension of the commutator space of $t$ is divisible by 4. This implies the following result.

**Lemma 4.2.** Let $x \in \tilde{H}$ with $|\phi(x)| = 2$. Then $|x| = 2$ if and only if $\dim([\tilde{V}, x]) = 4$.

Let $\tilde{T}$ be a maximal torus of $\tilde{H}$. By 4.1, the commutator space $[\tilde{V}, \tilde{T}]$ is the orthogonal direct sum of three hyperbolic lines $\ell_1, \ell_2,$ and $\ell_3$, where each $\ell_i$ is a sum of two weight spaces for $\tilde{T}$, with weights $\lambda_i$ and $\lambda_i^{-1}$. For $i = 1, 2, 3$, fix a basis $\{x_{2i-1}, x_{2i}\}$ for $\ell_i$ of singular vectors, with $f(x_{2i-1}, x_{2i}) = 1$. Then 4.1 yields

$$[\tilde{V}, \tilde{T}] = \ell_1 + \ell_2 + \ell_3.$$  

Let $x_7 \in C_{\tilde{V}}(\tilde{T})$, with $f(x_7, x_7) = 1$. Then

$$C_{\tilde{V}}(\tilde{T}) = [\tilde{V}, \tilde{T}]^\perp = \overline{\mathbb{F}}x_7.$$  

Identify $\tilde{V}$ with $\overline{\mathbb{F}}^{(7)}$, via the ordered basis $(x_1, \ldots, x_7)$.

The semidirect product $\tilde{V} \tilde{H}$ is an algebraic group in which $\tilde{T}$ is a maximal torus, and in which $\tilde{V}$ is the unipotent radical. Let $\zeta$ be a Frobenius endomorphism of $\tilde{V} \tilde{H}$ which induces the $p^th$-power map on $\tilde{T}$. Then $\zeta$ fixes $\ell(\tilde{T})$ pointwise, and fixes the vectors $x_1$ through $x_7$. 
Denote also by $\zeta$ the $p^{th}$-power automorphism of $\overline{F}$, and set

$$F = \bigcup_{k \geq 0} C_{\overline{F}}(\zeta^{2^k}).$$

Evidently, $F$ is a subfield of $\overline{F}$. Denote by $V$ the $F$-span of $\{x_1, \ldots, x_7\}$ in $\overline{V}$, and by $H$ the group of $F$-rational points in $\overline{H}$, with respect to the matrix representation given by the chosen basis for $\overline{V}$. The restriction of $f$ to $V \times V$ defines an orthogonal form on $V$, and $\phi(H)$ is contained in $O(V)$. Set $T = \overline{T} \cap H$, and set $E = \{x \in T \mid x^2 = 1\}$.

Since $O(\overline{V}) = \Omega(\overline{V})\langle t \rangle$ for any reflection $t \in O(\overline{V})$, it follows that $\Omega(\overline{V})$ is the group $SO(\overline{V})$ of determinant 1 isometries. Any element $x$ of $O([\overline{V}, \overline{T}])$ extends to an element of $\Omega(\overline{V})$, since we are free to adjust the action of $x$ on $C_{\overline{V}}(\overline{T})$ by $\pm 1$. In particular, there exists an element $w_0$ of $\overline{H}$ such that $w_0$ acts on $[\overline{V}, \overline{T}]$ by the permutation $(x_1 x_2)(x_3 x_4)(x_5 x_6)$ of the basis vectors; and then $x_7 w_0 = -x_7$. Evidently $w_0$ commutes with $\zeta$, so $w_0 \in H$. By 4.1, $w_0 \in N_H(T)$, and one can check that $w_0$ acts on $\phi(T)$ by inversion. Since every element of $F$ is a square, it follows that $w_0$ acts on $T$ by inversion.

Similarly we choose elements $w_1, w_2, w_3, w$, and $\rho$ of $N_H(T)$ so that:

$$\phi(w) = (x_1 x_3 x_5)(x_2 x_4 x_6),$$
$$\phi(\rho) = (x_1 x_3)(x_2 x_4), \quad \text{and}$$
$$\phi(w_i) = (x_{2i-1} x_{2i}), \quad 1 \leq i \leq 3.$$ 

Thus, we may take $w_0 = w_1 w_2 w_3$.

(4.2.2) Fix $n \geq 0$ and set $q = p^{2^n}$. Denote by $\omega$ both the inner automorphism of $\overline{H}$ induced by $w_0$, and the identity map on $\overline{F}$. For any $k \geq 0$, let $\psi_k$ be the automorphism of $\overline{H}$ defined by

$$\psi_k = \begin{cases} 
(\omega \zeta)^{2^k} & \text{if } p \equiv 3 \mod 4, \quad \text{and} \\
\zeta^{2^k} & \text{if } p \equiv 1 \mod 4.
\end{cases}$$

We now fix $n$, and set

$$\sigma = \psi_n.$$ 

Notice that since $\omega^2 = 1$, we in fact have $\sigma = \zeta^{2^n}$ unless $n = 0$ and $p \equiv 3 \mod 4$.

The restriction of $\sigma$ to $H$ will again be denoted $\sigma$. For any subgroup $D$ of $H$, write $D_\sigma$ for $C_D(\sigma)$, and write $F_\sigma$ for $C_F(\sigma)$.

**Lemma 4.3.** Set $W = \langle z, w_1, w_2, w_3, w, \rho \rangle$. Then $W \leq H_\sigma$, and the following hold.

(a) $T = C_H(T)$, and $w_0$ acts on $T$ as inversion.
(b) \( N_H(T) = WT \).
(c) \( \text{Aut}_H(T) \cong \text{Sym}(4) \times C_2 \).
(d) Set \( E = \{ x \in T \mid x^2 = 1 \} \). Then \( E \) is an elementary abelian subgroup of \( T_\sigma \) of rank 3, \( C_H(E) = T \langle w_0 \rangle \), \( N_{\tilde{H}^*}(E) = \tilde{T} W \), and \( N_H(E) = N_H(T) \).

Proof. Set \( \mathcal{L} = \{ \ell_1, \ell_2, \ell_3 \} \), and denote by \( \tilde{T}^* \) the pointwise stabilizer of \( \mathcal{L} \) in \( \tilde{H} \). Thus \( T \leq \tilde{T}^* \), and \( \phi(\tilde{T}^*) \) is contained in the direct product of the orthogonal groups \( O(\ell_i) \), so that \( \tilde{T}^* = \tilde{T} \langle w_1, w_2, w_3 \rangle \).

By 4.1, \( C_{\tilde{H}^*}(\tilde{T}) \leq \tilde{T}^* \) and \( N_{\tilde{H}^*}(\tilde{T}) \) permutes \( \mathcal{L} \). Since \( \text{Aut}_{\tilde{T}}(\ell_i) \) contains its centralizer in \( \text{GL}(\ell_i) \), we have \( \tilde{T} = C_{\tilde{H}^*}(\tilde{T}) \). Similarly \( C_H(T) = T \).

Evidently, each of the elements \( w_i, w \), and \( \rho \) commutes with both \( \sigma \) and \( w_0 \), and so \( W \leq N_{H_{\sigma}}(T) \). From the definitions of these elements, we obtain \( \phi(w_0) \in Z(\phi(W)) \), \( \phi((w_1, w_2, w_3)) \) is elementary abelian of order 8, \( \phi((w, \rho)) \cong \text{Sym}(3) \), and \( \langle w, \rho \rangle \) acts naturally on \( \phi((w_1, w_2, w_3)) \) and on \( \mathcal{L} \). We conclude that \( WT \) contains \( H \cap T^* \), \( WT = N_H(T) \), that \( WT/T \cong \text{Sym}(4) \times C_2 \), and that \( \langle w_0 \rangle T/T = Z(WT/T) \). As \( w_0 \) inverts \( \tilde{T} \), it inverts \( T \). Thus, parts (a) through (c) hold.

Notice that \( C_H([V, T]) = Z \) since \( \phi(H) \) contains no reflections. As \( \omega \) inverts \( T \) and \( \zeta \) induces a power map on \( \tilde{T} \), \( T_\sigma \) contains the group \( E = \{ t \in T \mid t^2 = 1 \} \). From 4.2, \( \phi(E) \) is a fours group and \( E \) is elementary abelian of order 8. The lines \( \ell_i \) are the fixed point spaces for the three involutions in \( \phi(E) \) on \([\tilde{V}, E]) \), so that \( N_{\tilde{H}^*}(E) = N_{\tilde{H}}(\tilde{T}) \), and hence \( N_H(E) = N_H(T) \). Since \( w_0 \) inverts \( T \), \( w_0 \in C_H(E) \).

Since \( C_H(E) \leq \tilde{T}^* \), we have \( C_H(E) = T \langle w_1, w_2, w_3 \rangle \). By 4.2, for \( \{ i, j, k \} = \{ 1, 2, 3 \} \) and \( x \in E \) such that \( [\tilde{V}, x] = \ell_j + \ell_k \), each of \( w_i, w_i x, w_j w_k, w_j w_k x \) is of order 4. Then \( x \) does not centralize \( w_i \) or \( w_j w_k \), and \( C_H(E) = T \langle w_0 \rangle \), completing the proof of (d).

We may choose \( z_1 \in \tilde{T} \) so that \( z_1 \) acts as the scalar \(-1 \) on \( \ell_1 + \ell_2 \), and as \( 1 \) on \( \ell_3 \). Then \( z_1 \) centralizes \( x_7 \). Set \( U = \langle z, z_1 \rangle \), \( \tilde{B} = N_{\tilde{H}}(U) \), \( B = \tilde{B} \cap H \), and denote the identity component of \( \tilde{B} \) by \( \tilde{B}^0 \). Set \( B^0 = \tilde{B}^0 \cap H \).

**Lemma 4.4.** The following hold.

(a) \( \tilde{V} \) is the orthogonal direct sum of \([\tilde{V}, U]\) and \( C_{\tilde{V}}(U) \), of dimensions 4 and 3, respectively, over \( \tilde{F} \).

(b) \( \tilde{B} \) is the stabilizer in \( \tilde{H} \) of \([\tilde{V}, U]\) and of \( C_{\tilde{V}}(U) \).

(c) \( \tilde{B}^0 = C_{\tilde{H}}(U) = \tilde{L}_1 \tilde{L}_2 \tilde{L}_3 \), where \( \tilde{L}_i \cong \text{SL}_2(\tilde{F}) \) and where \( \tilde{L}_i \tilde{L}_j \cong \text{SL}_2(\tilde{F}) \times \text{SL}_2(\tilde{F}) \) for all distinct \( i \) and \( j \). Moreover, the indexing may be chosen so that

(i) \( \tilde{L}_3 \) centralizes \([\tilde{V}, U]\) and \( \tilde{L}_3 \phi = \Omega(C_{\tilde{V}}(U)) \).

(ii) \( \tilde{L}_1 \tilde{L}_2 \) centralizes \( C_{\tilde{V}}(U) \) and \( \tilde{L}_1 \tilde{L}_2 \phi = \Omega([\tilde{V}, U]) \).
(iii) The maximal singular subspaces of \([\hat{V}, U]\) spanned by \(\{x_1, x_4\}\) and \(\{x_2, x_3\}\) over \(\text{F}\) are natural \(\text{SL}_2(\text{F})\)-modules for \(\tilde{L}_1\), and the maximal singular subspaces spanned by \(\{x_1, x_3\}\) and \(\{x_2, x_4\}\) are natural \(\text{SL}_2(\text{F})\)-modules for \(\tilde{L}_2\).

(d) \(U = Z(\tilde{B}^0)\), and notation may be chosen so that \(z_1 \in \tilde{L}_1\). When \(z_i\) is the involution in \(\tilde{L}_i\), then

\[
z = z_1z_2 = z_3.
\]

(e) \(\tilde{B} = \tilde{B}^0(w_1) = \tilde{B}^0(w_2)\), where both \(w_1\) and \(w_2\) interchange \(\tilde{L}_1\) and \(\tilde{L}_2\) by conjugation.

Proof. As \([\hat{V}, U] = [\hat{V}, z_1]\) and \(C_{\hat{V}}(U) = C_{\hat{V}}(z_1)\), part (a) is immediate from our choice of \(z_1\). The stabilizer in \(\hat{H}\) of \([\hat{V}, U]\) normalizes the unique subgroup \(U\) of \(\hat{H}\) containing \(Z\) which acts as \(-I\) on \([\hat{V}, U]\) and as \(I\) on \([\hat{V}, U]^\perp\). Similarly, the stabilizer in \(\hat{H}\) of \(C_{\hat{V}}(U)\) normalizes \(U\), establishing (b).

Set \(\tilde{K} = C_{\hat{H}}(C_{\hat{V}}(U))^0\) and \(\tilde{L}_i = C_{\hat{H}}([\hat{V}, U])^0\). From the Steinberg relations, \(\tilde{K} = \tilde{L}_1 \times \tilde{L}_2\), and \(\tilde{L}_i \cong \text{SL}_2(\text{F})\) for \(i = 1, 2, 3\). Thus \(\tilde{B}^0\) is a commuting product of these three copies of \(\text{SL}_2(\text{F})\), and \(U \leq Z(\tilde{B}^0)\). Here \([\hat{V}, U]\) is a natural \(\Omega_4(\text{F})\)-module for \(\tilde{L}_1\tilde{L}_2\), and is therefore a direct sum of two natural \(\text{SL}_2(\text{F})\)-modules for each of \(\tilde{L}_1\) and \(\tilde{L}_2\). Observe that \(\tilde{T} \leq \tilde{B}^0\) since \(\tilde{T}\) is connected. Then \(\tilde{T}\) is a maximal torus of \(\tilde{B}^0\), and hence \(\tilde{T} = \tilde{T}_1 \tilde{T}_2 \tilde{T}_3\), where \(\tilde{T}_i := \tilde{T} \cap \tilde{L}_i\). Since \([\tilde{L}_1, \tilde{L}_2\tilde{L}_3] = 1\), the irreducible \(\tilde{L}_1\tilde{T}\)-submodules of \([\hat{V}, U]\) are weight spaces for \(\tilde{T}_1\tilde{T}_2\tilde{T}_3\). Since these irreducible \(\tilde{L}_1\tilde{T}\)-submodules are also maximal singular subspaces of \([\hat{V}, U]\), the only possibilities are that the two irreducible \(\tilde{L}_1\tilde{T}\)-submodules of \([\hat{V}, U]\) are

\[
\{(x_1, x_3), (x_2, x_4)\} \quad \text{or} \quad \{(x_1, x_4), (x_2, x_3)\}.
\]

We may therefore choose the indexing so that (c)(iii) holds.

To complete the proof of (c), it remains to show that \(\tilde{B}^0 = C_{\hat{H}}(U)\). This will follow from (e), and since (d) is immediate from the parts of (c) which have already been established, we now need only prove (e).

The group \(O([\hat{V}, U])\) is generated by \(\Omega([\hat{V}, U])\) together with a reflection interchanging \(\phi(\tilde{L}_1)\) and \(\phi(\tilde{L}_2)\). Similarly, \(O(C_{\hat{V}}(U))\) is generated by \(\Omega(C_{\hat{V}}(U))\) together with a reflection on \(C_{\hat{V}}(U)\). Since \(\phi(\hat{H})\) contains no reflections on \(\hat{V}\), we have \(|\tilde{B}: \tilde{B}^0| = 2\), and then (e) follows from the definitions of \(w_1\) and \(w_2\).

Notice that each of the groups \(\tilde{L}_i\) is \(\zeta\)-invariant. Set \(L_i = \tilde{L}_i \cap H, 1 \leq i \leq 3\). The following result should then be evident:

**Lemma 4.5.** All parts of 4.4 hold, with \(B, B^0, \text{F}\), and \(L_i\) in place of \(\tilde{B}, \tilde{B}^0, \text{F}\), and \(\tilde{L}_i\). 
\(\square\)
Given a basis \( v_1, \ldots, v_m \) of an \( F \)-space \( V \), and elements \( d_i \) of \( F \), \( 1 \leq i \leq m \), we write \( \partial(d_1, d_2, \ldots, d_m) \) for the diagonal map \( v_i \mapsto d_i v_i \) for each \( i \).

**Lemma 4.6.** Let \( V \) be a 2-dimensional \( F \)-space with basis \( \mathcal{B} = \{ v_1, v_2 \} \), set \( L = \text{SL}(V) \), and let \( T \) be the maximal torus \( \{ \partial(a, a^{-1}) \mid a \in F \} \) of \( L \) determined by \( \mathcal{B} \). Set \( X = L \times L \times L \), set \( [[X]] = X / \langle (-I, -I, -I) \rangle \), and write \( [[a, b, c]] \) for the image of \( (a, b, c) \in X \) under the canonical surjection \( X \to [[X]] \). Finally, let \( \gamma_1, \delta_1, \gamma_2, \) and \( \delta_2 \) be maps from \( \mathcal{B} \) into \( V \) which send the ordered basis \( (v_1, v_2) \) to the pairs \( (x_1, x_4), (x_3, x_2), (x_1, x_3), \) and \( (x_4, x_2) \), respectively.

(a) There are isomorphisms \( \alpha_i : L \to L_i \), \( i = 1, 2, 3 \), such that

(i) \( \alpha_1, \gamma_1 \) and \( \alpha_1, \delta_1 \) are quasi-equivalences of the representation of \( L \) on \( V \) with the representations of \( L_1 \) on \( \langle x_1, x_4 \rangle \) and \( \langle x_3, x_2 \rangle \), respectively.

(ii) \( \alpha_2, \gamma_2 \) and \( \alpha_2, \delta_2 \) are quasi-equivalences of the representation of \( L \) on \( V \) with the representations of \( L_2 \) on \( \langle x_1, x_3 \rangle \) and \( \langle x_4, x_2 \rangle \), respectively.

(iii) The map \( \alpha_3 \) is the 3-dimensional orthogonal representation of \( L \) in which \( \partial(c, c^{-1}) \) acts as \( \partial(c^2, 1, c^{-2}) \) with respect to the ordered basis \( \langle x_5, x_7, x_6 \rangle \) of \( C_V(U) \).

(b) The map \( \alpha_1 \times \alpha_2 \times \alpha_3 : X \to B^0 \) given by

\[
(a, b, c) \mapsto (a \alpha_1)(b \alpha_2)(c \alpha_3)
\]

induces an isomorphism of \( [[X]] \) with \( B^0 \).

(c) \( (T \cap L_i) \alpha_i^{-1} \) is the set of diagonal maps in \( L \). For each \( i \), let \( \beta_i : F \to T \cap L_i \) be the composition of \( \partial \) with \( \alpha_i \). Set \( Y = F \times F \times F \), and \( [Y] = Y / \langle (-1, -1, -1) \rangle \), with \( [a, b, c] \) the image of \( (a, b, c) \in Y \) in \( [Y] \). Then the map \( \beta_1 \times \beta_2 \times \beta_2 : Y \to T \) induces an isomorphism \( [a, b, c] \mapsto (a \beta_1)(b \beta_2)(c \beta_3) \) of \( [Y] \) with \( T \).

**Proof.** This is straightforward, given 4.4. \( \square \)

From now on we use 4.6 to identify \( B^0 \) with the set of equivalence classes

\[
[[a, b, c]] = [[-a, b, c]], \quad a, b, c \in \text{SL}_2(F),
\]

and identify \( T \) with the set of equivalence classes

\[
[a, b, c] = [-a, b, c] \quad a, b, c \in F^\times.
\]

**Lemma 4.7.** (a) The element \([a, b, c]\) of \( T \) acts diagonally as

\[
\partial(ab, a^{-1}b^{-1}, ab^{-1}, a^{-1}b, c^2, c^{-2}, 1)
\]

with respect to the ordered basis \( (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \) of \( V \).
(b) The action of $W$ on $T$ is given as follows.

$$w_1:[a, b, c] \mapsto [b^{-1}, a^{-1}, c],$$

$$w_2:[a, b, c] \mapsto [b, a, c],$$

$$w_3:[a, b, c] \mapsto [a, b, c^{-1}],$$

$$w:[a^2, b^2, c^2] \mapsto [abc^2, a^{-1}b^{-1}c^2, ab^{-1}],$$

$$\rho:[a, b, c] \mapsto [a, b^{-1}, c].$$

**Proof.** Again, this is straightforward, given 4.6. \qed

Denote by $S_\infty$ the set of elements of $T$ whose order is a power of 2. Set $W_S = \langle w_1, w_2, w_3, \rho \rangle$, and set $S = S_\infty W_S$. Also, for any $k \geq 1$ set

$$T_k = \{t \in T \mid t^{2^k} = 1\}.$$

Thus $T_k$ is a subgroup of $S_\infty$, and $T_1$ is the group $E$ appearing in 4.2.

We shall henceforth take $p$ to be congruent to 3 or 5 mod 8. One reason for this choice is that it allows us to keep track of the structure of Sylow 2-subgroups of the groups $H$ and $H_a$, as in the following two lemmas.

**Lemma 4.8.** Let $k$ be a nonnegative integer, and set $\psi = \psi_k$. Then $C_{S_\infty}(\psi) = T_{k+2}$ is homocyclic abelian of rank 3 and exponent $2^{k+2}$.

**Proof.** For any integer $m$, and any $k \geq 1$,

$$m^{2^k} - 1 = (m^{2^k-1} + 1)(m^{2^k-1} - 1).$$

A straightforward induction argument then yields the following fact.

(*) For any integer $m$ with $m \equiv 5 \mod 8$, and for any nonnegative integer $k$, we have

$$m^{2^k} \equiv 1 + 2^{k+2} \mod 2^{k+3}.$$

Set $D = \{d \in F \mid d^{2^{k+2}} = 1\}$, set $D' = \{f \in F \mid f^{2^{k+3}} = 1\}$, and fix $f \in D' - D$. Then $D' = D \cup Df$. Set $Q_k = \{[a, b, c] \in T \mid a, b, c \in D\}$ and $R_k = T_{k+2}$. That is $R_k = \{x \in T \mid x^{2^{k+2}} = 1\}$. As $[a, b, c]^{2^k} = 1$ in $T$ if and only if $a^{2^k} = b^{2^k} = c^{2^k} = \pm 1$, it follows that $R_k = Q_k \cup Q_k[f, f, f]$. Thus $Q_k$ has index 2 in $R_k$. Let $A$ be a homocyclic abelian group of rank 3 and exponent $2^{k+2}$. Since $[a, b, c] = [-a, -b, -c]$, there is an exact sequence

$$1 \longrightarrow C_2 \longrightarrow A \longrightarrow Q_k \longrightarrow 1,$$

and thus $|A| = 2|Q_k| = |R_k|$. Since $R_k$ is abelian of rank 3, exponent $2^{k+2}$, and order $|A|$, it follows from the fundamental theorem of finite abelian groups that $R_k \cong A$.  

Suppose that \( p \equiv 3 \mod 8 \) and that \( k = 0 \). For \( [a, b, c] \in T \),
\[
[a, b, c] \psi = [a^{-p}, b^{-p}, c^{-p}],
\]
so that \([a, b, c] \in T_\psi\) if and only if \( a^{p+1} = b^{p+1} = c^{p+1} = \pm 1 \). As \( p \equiv 3 \mod 8 \), 4
is the largest power of 2 dividing \( p + 1 \), and it follows from the preceding paragraph
that \( C_{S_\infty}(\psi) = R_k \) in this case. On the other hand, suppose that \( p \equiv 5 \mod 8 \) or
that \( k > 0 \). Then
\[
[a, b, c] \psi = [a^{p^{2k}}, b^{p^{2k}}, c^{p^{2k}}].
\]
Notice that if \( p \equiv 3 \mod 8 \) then \(-p \equiv 5 \mod 8 \), while for \( k > 0 \) we have \( p^{2k} = (-p)^{2k} \). Now (*) shows that, in any case, we have \( C_{S_\infty}(\psi) = R_k \). This yields
the lemma. \( \square \)

**Lemma 4.9.** The following hold.

(a) \( S_\sigma = C_{S_\infty}(\sigma)W_S \), and \( S_\sigma \) is a Sylow 2-subgroup of \( H_\sigma \).

(b) \( S \) is a Sylow 2-subgroup of every subgroup \( X \) of \( H \) which contains \( S \).

(c) \( T_2 \) is the unique homocyclic abelian subgroup of \( S \) of rank 3 and exponent 4.
Moreover, we have \( T = C_H(T_2) \), and \( T_2 \leq T_\sigma = C_{H_\sigma}(T_2) \).

(d) \( S^B \) is the set of maximal 2-subgroups of \( B \) containing a subgroup isomorphic to \( T_2 \).

**Proof.** For any subgroup \( P \) of \( S \), denote by \( \mathcal{A}(P) \) the set of homocyclic abelian subgroups of \( P \) of rank 3 and exponent 4. Let \( \psi \) and \( R_k \) be defined as in
the preceding lemma, set \( Q = W_SR_k \), and set \( Q_0 = (w_0)R_k \).

Suppose first that that there exists \( A \in \mathcal{A}(Q) \) with \( A \neq T_2 \). Then \( A \not\unlhd T_2 \).
Suppose that \( A \cap Q_0 \) has rank 3 and exponent 4. Since \( w_0 \) inverts \( R_k \), we then
have \( A \cap Q_0 \leq R_k \), and \( A \) contains the unique elementary abelian subgroup \( E \) of
\( R_k \) of order 8. By 4.3, \( Q/Q_0 \) acts faithfully on \( E \), so that \( A \leq Q_0 \), and then \( A \unlhd T_2 \).
This is a contradiction, and so we conclude that \( A \cap Q_0 \) has exponent less than 4 or
has rank less than 3. Since \( Q/Q_0 \) is dihedral of order 8, it follows that \( AQ_0/Q_0 \)
is cyclic of order 4. Then \( A \cap Q_0 \) is homocyclic of rank 2 and exponent 4. Again,
\( A \cap Q_0 \leq R_k \), and now \( |A \cap E| = 4 \). The faithful action of \( Q/Q_0 \) on \( E \) implies
that \( C_{Q/Q_0}(A \cap E) \) has exponent 2. Since \( AQ_0/Q_0 \) centralizes \( A \cap E \), we again
have a contradiction, and thus \( A \leq R_k \). That is, \( \mathcal{A}(Q) = \{T_2\} \).

Let \( P \) be a Sylow 2-subgroup of \( H_\psi \) containing \( Q \). By the preceding paragraph, \( T_2 \unlhd N_P(Q) \).
It follows from 4.1 that \( N_P(Q) \) preserves the set \( \{\ell_1, \ell_2, \ell_3\} \) of hyperbolic lines, and hence \( N_P(Q) \) normalizes \( T \). Then \( N_P(Q) \unlhd TW \) by 4.3. Since \( Q \) is a Sylow 2-subgroup of \( (WT)_\psi \), we conclude that
\( N_P(Q) = Q \). Then \( P = Q \), and so \( Q \in \text{Syl}_2(H_\psi) \). We recall that \( \psi = \psi_k = \zeta^{2^k} \).
or \((\zeta w_0)^{2^k}\) for some \(k\), and that

\[ H = \bigcup_{k \geq 0} C_H((\psi_k)). \]

Then \(S\) is the union of its subgroups \(S \cap H_{\psi_k}\), and so \(\mathcal{A}(S) = \{T_2\}\).

By Zorn’s Lemma, there is a maximal 2-subgroup \(S^*\) of \(H\) containing \(S\). We have

\[ X = \bigcup_{k \geq 0} X_{\psi_k} \]

for any subgroup \(X\) of \(H\). Taking \(X = S^*\), we conclude that \(S = S^*\). Taking \(X\) to be an arbitrary subgroup of \(H\) containing \(S\), we note that every finite subgroup of \(X\) is contained in \(X_{\psi_k}\) for some \(k\), so that every finite 2-subgroup of \(X\) is \(X\)-conjugate into \(S\). Thus, \(S\) is a Sylow 2-subgroup of \(X\), and we have (a) and (b).

Notice that \(\sigma = \psi_k\) for some \(k\). Then (c) follows as \(T_2 \leq C_H(\psi_0)\), \(\mathcal{A}(S) = \{T_2\}\), and \(N_H(T_2) = N_H(T)\).

By (c) and 4.3.b, \(N_B(T_2) = TW_S = TS\). Let \(S_2\) be a subgroup of \(B\) isomorphic to \(T_2\), and \(X\) a maximal 2-subgroup of \(B\) containing \(S_2\). By (b), \(S\) is Sylow in \(B\), and so \(S^b \leq S\) for some \(b \in B\), and by (c), \(S^b_2 = T_2\). Thus we may take \(T_2 = S_2\). Similarly for each \(k\), \(X_{\psi_k}\) is contained in a conjugate of \(S\), and so by (c), \(X_{\psi_k} \leq N_B(T_2) = TS\). Hence \(X\) is a maximal 2-subgroup of \(TS\), \(X \in ST\), establishing (d).

\(\square\)

5. The amalgam \(\mathcal{A}_\lambda\), and an amalgam for \(Co_3\)

We now undertake the construction of the amalgam which provides the focus for this work. (See the beginning of Section 3 for a discussion of amalgams.)

We continue the setup and notation of the preceding section. In particular, we have \(p \equiv 3\) or \(5\) (mod 8). Let \(i\) be a square root of \(-1\) in \(F\), and let \(\tau\) be the element \(w_2[1, 1, i]\) of \(B\). Then \(B = B^0(\tau)\), by 4.4(e). By definition, \(w_2\) interchanges the two singular points of \(\ell_2\), centralizes \(\ell_1\) and \(\ell_3\), and acts as \(-1\) on \(C_V(T)\). Then \(\tau\) acts as \(-I\) on \(C_V(U) = \ell_3 + C_V(T)\), and acts as a transvection on \([V, U]\). In particular, \(\tau\) commutes with \(\phi(L_3)\), hence also with \(L_3\) (since \(L_3\) is perfect), and \(\tau\) is an involution by 4.2. Further, \(\tau\) acts as \(w_2\) on \([V, T]\), and then 4.4(c)(iii) yields

\[ \tau: [[\alpha, \beta, \gamma]] \mapsto [[\beta, \alpha, \gamma]]. \]

for all \([[\alpha, \beta, \gamma]] \in B^0\).

Define \(y_0\) to be the automorphism of \(B^0\) given by

\[
\text{(5.0)} \quad y_0: [[\alpha, \beta, \gamma]] \mapsto [[\gamma, \alpha, \beta]].
\]
Then $|y_0| = 3$, and $\langle y_0, \tau \rangle$ acts faithfully as the symmetric group $\text{Sym}(3)$ on the set $\mathcal{L} = \{L_1, L_2, L_3\}$. In the semidirect product

$$K = B^0 \langle y_0, \tau \rangle,$$

we may then identify $B$ with the subgroup $B^0 \langle \tau \rangle$ of $K$, and form the amalgam

$$\mathcal{A}_1 = (H \geq B \leq K).$$

For any $\lambda \in \text{Aut}(B)$, denote by $\lambda^*$ the composition of $\lambda$ with the inclusion map of $B$ into $K$, and form the amalgam

$$\mathcal{A}_\lambda = (H \geq B \xrightarrow{\lambda^*} K).$$

The corresponding free amalgamated product will be denoted $G_\lambda$. Subject to the usual identifications, $H$ and $K$ are subgroups of $G_\lambda$, with $H \cap K = B$. Here it is important to note that the inclusion map of $B$ into $K$, within $G$, is obtained by “twisting” via $\lambda$ the “ordinary” inclusion map occurring in $\mathcal{A}_1$.

**Lemma 5.1.** For $X \in \{H, K\}$, write $A_X$ for $\text{Aut}_{\text{Aut}(X)}(B)$. Set $\hat{\Phi} = \text{Aut}(\mathcal{F})$, and regard $\hat{\Phi}$ as the group of field automorphisms of $\text{SL}_2(\mathcal{F})$. Define a representation of $\hat{\Phi}$ on $B^0$ by

$$\lambda: [[\alpha, \beta, \gamma]] \mapsto [[\alpha, \beta, \gamma^\lambda]] \text{ for } \lambda \in \Phi.$$

Then $\Phi$ commutes with $\tau$ on $B^0$, and the representation of $\Phi$ on $B^0$ extends thereby to a representation on $B$. Moreover:

(a) $\text{Inn}(B) \leq A_H \cap A_K$.

(b) For each $X \in \{H, K\}$ we have $\text{Aut}(B) = A_X \Phi = A_H A_K \Phi$, and $A_X \cap \Phi = 1$.

(c) For $\mu, \lambda \in \text{Aut}(B)$, we have $\mathcal{A}_\lambda \cong \mathcal{A}_\mu$ if and only if $A_H A_K \lambda = A_H A_K \mu$.

**Proof.** Identify $\Phi$ with a subgroup of $\text{Aut}(B^0)$ via the prescribed representation. Evidently $[\Phi, \tau] = 1$, so we may even regard $\Phi$ as a subgroup of $\text{Aut}(B)$. As $B = H \cap K$, (a) holds. As is well known (cf. [Ste]), $\text{Aut}(\text{SL}_2(\mathcal{F})) = \text{Inn}(\text{SL}_2(\mathcal{F})) \Phi$. Then, since $B^0$ is the central product of three copies of $\text{SL}_2(\mathcal{F})$, $\text{Aut}(B^0)$ is the split extension of $\text{Aut}(\text{SL}_2(\mathcal{F}))^3$ by $\text{Sym}(3)$, where $\text{Sym}(3)$ permutes the three components of $B^0$ faithfully.

Recall that $B = B^0 \langle \tau \rangle$, where $\tau$ centralizes $L_3$ and interchanges $L_1$ and $L_2$. For $X \in \{H, K\}$ we have $A_X = \text{Inn}(B) \Phi_X$, where $\Phi_X \cong \Phi$ and $\Phi_X$ is diagonally embedded in the subgroup $\Phi^3$ of $\text{Aut}(\text{SL}_2(\mathcal{F}))^3$, and centralizes $\tau$. Similarly, $\text{Aut}(B) = \text{Inn}(B)(\Phi_X \times \Phi)$, where $\Phi$ centralizes $L_1 L_2$ and acts faithfully as $\Phi$ on $L_3$. Now (b) follows, and $\text{Aut}(B) = A_H A_K \Phi$. By [Gol80, Lemma 2.7], $\mathcal{A}_\lambda \cong \mathcal{A}_\mu$ if and only if $A_H \lambda A_K = A_H \mu A_K$, and now (c) follows from (b).

We may now state the first main result of this section.
THEOREM 5.2. Let $S_\infty$ be the Sylow 2-subgroup of $T$, and for any $\lambda \in \text{Aut}(B)$ set $N_{\lambda} = (N_H(S_\infty), N_K(S_\infty))$, where $H$ and $K$ are regarded as subgroups of $G_\lambda$ in the canonical way. Set $N_{\lambda} = \text{Aut}_{N_{\lambda}}(S_\infty)$, define $\Phi$ as in 5.1, and set

$$\Lambda = \{ \lambda \in \Phi \mid N_{\lambda} \cong \text{GL}(3, 2) \times C_2 \}.$$

Then the following hold.

(a) $|\Lambda| = 1$.

(b) For $\lambda \in \Lambda$,

$$C_{N_{\lambda}}(T_e) = \begin{cases} C_{N_{\lambda}}(S_\infty) & \text{if } e \geq 2, \text{ and} \\ C_{N_{\lambda}}(S_\infty)\langle w_0 \rangle & \text{if } e = 1. \end{cases}$$

(c) $A_H = A_K$.

(d) The map $\phi \mapsto \mathcal{A}_\phi$ is a bijection of $\Phi$ with the set of isomorphism classes of amalgams $\mathcal{A}_\mu$ with $\mu \in \text{Aut}(B)$.

Remark. It can be shown, by means of a lengthy computation based on 4.7(b), that the unique $\lambda$ in the set $\Lambda$ of Theorem 5.2 is not an algebraic endomorphism of $B$.

Let $\omega_3$ be the automorphism of $B^0$ induced by conjugation by $w_3$. By 4.7(b), $w_3 \in L_3$ and $w_3$ inverts $T \cap L_3$. Denote by $\zeta_3$ the automorphism of $B^0$ which induces the $p^{th}$ power Frobenius map on $L_3$ and which centralizes $L_1$ and $L_2$. Then let $\xi_0$ be the automorphism of $B^0$ given by

$$\xi_0 = \begin{cases} \zeta_3 \omega_3 & \text{if } p \equiv 3 \mod 8, \\ \zeta_3 & \text{if } p \equiv 5 \mod 8. \end{cases}$$

Thus,

$$\xi_0 : [\alpha, \beta, \gamma] \mapsto \begin{cases} [[\alpha, \beta, \tilde{\gamma}^w]] & \text{if } p \equiv 3 \mod 8, \\ [[\alpha, \beta, \tilde{\gamma}]] & \text{if } p \equiv 5 \mod 8, \end{cases}$$

where $\tilde{\gamma}$ is the element of $\text{SL}_2(\mathbb{F})$ whose entries are the $p^{th}$ powers of the corresponding entries of $\gamma$. For any $e \in \mathbb{N}$, set

$$\xi_e = \xi_0^{2^e}.$$

Notice that $T$ is invariant under $\xi_e$, that $[L_1 L_2, \xi_e] = 1$, and that $\xi_e = \zeta_3^{2^e}$ for $e \geq 1$. Recall from 4.8 that we have defined subgroups $T_e$ of $T_0$ by

$$T_e = \{ t \in T \mid t^{2^e} = 1 \}, \ e \geq 1,$$

and that $T_e$ is homocyclic abelian of rank 3.

LEMMA 5.3. Let $e$ be a nonnegative integer, and let $c \in \mathbb{F}$ with $c^{2^e+3} = 1$. Then
Thus (a) holds in every case. Part (b) follows from (a) and the fact that $Z = \{[1, 1, -1]\}$ and $T_{e+2} = \{[e^2, 1, 1], [1, e^2, 1], [c, c, c]\}$, where $|c| = 2^{e+3}$. 

**Proof.** Suppose first that $e = 0$ and that $p \equiv 3 \mod 8$. Then $c^8 = 1$, and
\[
[a, b, c]_{\xi_e} = [a, b, c]_{\xi_e} = [a, b, c].
\]
Since $c^5 = -c$ if $|c| = 8$, and $c^5 = c$ if $c^4 = 1$, (a) holds in this case. On the other hand, suppose that either $e > 0$ or $p \equiv 5 \mod 8$. We saw in the proof of 4.8 that for any integer $m$ with $m \equiv 5 \mod 8$,
\[
m^{2e} \equiv 1 + 2e^2 \mod 2e^3.
\]
Taking $m = p$ if $p \equiv 5 \mod 8$, and taking $m = -p$ if $p \equiv 3 \mod 8$, we then have
\[
[a, b, c]_{\xi_e} = [a, b, c^{m^{2e}}] = [a, b, c^{1+2e^2}].
\]
Thus (a) holds in every case. Part (b) follows from (a) and the fact that $Z = \{[1, 1, -1]\}$ and $T_{e+2} = \{[e^2, 1, 1], [1, e^2, 1], [c, c, c]\}$, where $|c| = 2^{e+3}$. 

**Proposition 5.4.** There is a uniquely determined sequence $(y_e | e \geq 0)$ of automorphisms of $B^0$, with $y_0$ as in 5.0, and having the following two properties.

(a) $y_e \in \{y_{e-1}, y^{\xi_{e-1}}_{e-1}\}$ for $e > 0$.
(b) The group $N_e$ of automorphisms of $T_{e+1}$ induced by the action of $(W, y_e)$ is isomorphic to $\text{GL}(3, 2) \times C_2$ for $e > 0$, and is isomorphic to $\text{GL}(3, 2)$ for $e = 0$.

The proof of 5.4 will be based on the following result.

**Lemma 5.5.** Let $N$ be the Steinberg module for $\text{GL}(3, 2)$ over the field $\mathbb{F}_2$ of two elements, and let $X$ be an extension of $N$ by $\text{GL}(3, 2)$. Then the following hold:

(a) $X$ splits over $N$.
(b) Let $D$ be a complement to $N$ in a Sylow 2-subgroup of $X$. Then $C_N(D) = \{g\}$ is of order 2.
(c) Let $D$ be as in (b), and denote by $\mathcal{P}$ the set of subgroups $P$ of $X$ such that $D \leq P \cong \text{Sym}(4)$. Then $\mathcal{P} = \{P_1, P_2, Q_1, Q_2\}$, where $Q_i = P_i^g$ for $i = 1, 2$, $(P_1, Q_1)$ is a complement to $N$ in $X$, and $(P_1, Q_2) = (P_2, Q_1) = X$.

**Proof.** Let $R \in \text{Syl}_2(X)$. Then $N$ is a free $\mathbb{F}_2 R/N$-module, so that $C_N(R) = \{g\}$ is of order 2, and for each overgroup $Y$ of $R$ in $X$ we have $H^i(Y/N, N) = 0$ for $i = 1, 2$. In particular, (a) and (b) hold.

Choose $R$ so that $R = DN$ and let $P$ and $Q$ be maximal subgroups of $X$ such that $P/N$ and $Q/N$ are the maximal parabolic subgroups of $X/N$ over $R/N$. It
follows from the preceding paragraph that for each $Y \in \{X, P, Q, R\}$, $Y$ splits over $N$ and is transitive on its complements to $N$. Thus the set $\mathcal{P}_Y$ of complements to $N$ in $Y$ containing $D$ is nonempty for $Y \in \{X, P, Q\}$, and by a standard argument (cf. 5.2.1 in [Asc86]), $N_Y(D)$ is transitive on $\mathcal{P}_Y$. Then as $N_X(D) = D \times \langle g \rangle$, $\mathcal{P}_Y = \{Y_1, Y_2\}$ with $Y_1^g = Y_2$. Since $C_N(Y) = 0$ for $Y \in \{P, Q\}$, each $P_i$ and each $Q_i$ is contained in a unique complement to $N$ in $X$. In particular $X_i \cong \text{GL}(3, 2)$ and the indexing may be chosen so that $X_i$ is generated by $P_i$ and $Q_i \in P_Q$. Then $M_i = \langle P_i, Q_{3-i} \rangle$ is not a complement to $N$ in $X$. Since $X$ is irreducible on $N$, $M_i = N$. This completes the proof. \qed

We may now prove Proposition 5.4. Since $T_1$ is elementary abelian of order 8, we have $\text{Aut}(T_1) \cong \text{GL}(3, 2)$. From 4.3(c), $W$ induces the stabilizer in $\text{GL}(T_1)$ of $Z$, and so the image of $W$ in $\text{GL}(T_1)$ is maximal. The closure of $Z$ under the action of $\langle y_0 \rangle$ is the fours group $U$, and thus $N_0 = \text{GL}(T_1)$. We may therefore assume that $e \geq 1$, and that for all indices $e'$ with $0 < e' < e$:

(*) There exists a unique $y_{e'} \in \{y_{e'-1}, y_{e'-1}^\xi\}$ such that the group $N_{e'}$ of automorphisms of $T_{e'+1}$ induced by the action of $\langle W, y_{e'} \rangle$ is isomorphic to $\text{GL}(3, 2) \times C_2$.

Set $R = T_{e+1}$, $x = y_{e-1}$, and denote by $N$ the image of $\langle W, x \rangle$ in $\text{Aut}(R)$. Applying (*) to $e' = e - 1$, we have $N/C_{N}(T_e) \cong \text{GL}(3, 2)$ if $e = 1$, and $\text{GL}(3, 2) \times C_2$ if $e > 1$. Let $d \in C_{N}(T_e)$. Writing

$$d: [a, b, c] \mapsto [\bar{a}, \bar{b}, \bar{c}],$$

for $[a, b, c] \in R$, we obtain $a^2 = \varepsilon a^2$, $b^2 = \varepsilon b^2$, and $c^2 = \varepsilon c^2$, for some $\varepsilon \in \{\pm 1\}$. Thus either $\bar{u} \in \{u, -u\}$ for all $u \in \{a, b, c\}$, or $\bar{u} \in \{iu, -iu\}$ for all $u \in \{a, b, c\}$, where $i$ is a square root of $-1$. Therefore as

$$T_1 = \langle [-1, 1, 1], [1, -1, 1], [i, i, i] \rangle,$$

d acts trivially on $R/T_1$. Then $d = 1 + \lambda_d$ for some $\lambda_d \in \text{Hom}_\mathbb{Z}(R/T_{T_e}, T_1)$, and the mapping

$$C_N(T_e) \longrightarrow \text{Hom}_\mathbb{Z}(R/T_{T_e}, T_1)$$

given by $d \mapsto \lambda_d$ is an $N$-homomorphism. Observe that $R/T_e$ and $T_1$ are isomorphic as modules for $\langle W, x \rangle$ via the map $\varphi: rT_e \mapsto r^{2e+1}$. Thus, there is an $N$-equivariant monomorphism $d \mapsto \varphi^{-1}\lambda_d$ from $C_N(T_e)$ into $M := \text{End}_\mathbb{Z}(T_1)$. Since $N$ acts as $\text{GL}(3, 2)$ on $T_1$, $M$ may be identified with the $N$-module of $3 \times 3$ matrices over the field $\mathbb{F}_2$ of two elements, and we regard $C_N(T_e)$ as an $N$-submodule of $M$.

Now $M$ is a vector space of dimension 9 over $\mathbb{F}_2$, and the subspace $M_0$ of trace-zero matrices is an 8-dimensional $N$-submodule of $M$. Indeed $N/C_N(M) \cong \text{GL}(3, 2)$, and $M_0$ is the Steinberg module for $N/C_N(M)$. The element $w_0$ of $W$ inverts $R$, and hence $\langle w_0 \rangle = C_M(N)$, and $M = M_0 \oplus C_M(N)$. Further, $N$ is an
extension of $\mathcal{N}_M = \mathcal{N} \cap M$ by $L_3(2)$, and as $C_M(\mathcal{N}) = \langle w_0 \rangle \leq \mathcal{N}_M$ and $\mathcal{N}$ is irreducible on $M_0$, $\mathcal{N}_M = M$ or $C_M(\mathcal{N})$. As $W_S/(W_S \cap T) \cong \mathbb{Z}_2 \times D_8$ (cf. 4.3), $\mathcal{N}/M_0$ or $\mathcal{N}$ is isomorphic to $\mathbb{Z}_2 \times L_3(2)$ in the respective case. Further, in the latter case, we obtain (a) and (b) of 5.4 by setting $y_{e+1} = y_e$. Thus we may assume that $\mathcal{N}_M = M$.

Set $X = [\mathcal{N}, \mathcal{N}]$. From the previous paragraph, $M_0 = X \cap M$ and $X/M_0 \cong \text{GL}(3,2)$. Denote by $D^*$ the image of $W_S$ in $\mathcal{N}$, and set $D = D^* \cap X$. Then $D^* = D \times Z(\mathcal{N})$, and $D$ is dihedral of order 8. Denote by $P^*$ and $Q^*$ the images in $\mathcal{N}$ of $W$ and $(W_S, y)$, respectively, and set $P = P^* \cap X$ and $Q = Q^* \cap X$. By 5.5(b), $C_{M_0}(D) = \langle g \rangle$ is of order 2. By definition $\mathcal{N}$ is generated by $P^*$ and $Q^*$, and so $X$ is generated by $P$ and $Q$. By 5.5(c), $\langle P, Q^8 \rangle$ is a complement to $M_0$ in $X$. By construction, $\xi_{e-1}$ centralizes $W_S$, and by 5.3(b), $\xi_{e-1}$ induces a nontrivial automorphism of $R$ centralizing a subgroup of index 2 containing $T_e$. Then, since $C_{M_0}(W_S) = \langle g \rangle$, it follows that the action of $g$ on $R$ is the same as that of $\xi_{e-1}$. Setting $y_e = x^g$, we obtain (a) and (b) of Proposition 5.4.

Let $\{\lambda_e \mid e \geq 0\}$ be the sequence of automorphisms of $B^0$ defined by $\lambda_0 = 1_{B^0}$, and for $e > 0$ by the recursive formula

$$
\lambda_e = \begin{cases} 
\lambda_{e-1} & \text{if } y_e = y_{e-1} \text{ and } \\
\lambda_{e-1} \xi_{e-1} & \text{if } y_e = y_{e-1}^\xi_{e-1}
\end{cases}
$$

where $y_e$ is as in Proposition 5.4. For $k \geq 0$, take $\psi_k$ as defined just prior to 4.3.

**Lemma 5.6.** Each $\lambda_e$ extends to an automorphism of $B$ which commutes with the element $\tau$ of $B - B^0$, and with $\psi_k$ for each $k$. Further, for each $e \geq 0$,

$$
\lambda_{e+1} \mid_{C_B(\psi_{e-1})} = \lambda_e \mid_{C_B(\psi_{e-1})}.
$$

**Proof.** Recall that $\xi_e = \xi_0^{2^e}$, where $\xi_0$ is the automorphism of $B^0$ given by

$$
\xi_0: [\langle \alpha, \beta, \gamma \rangle] \mapsto [\langle \alpha, \beta, \gamma' \rangle],
$$

for the automorphism $\gamma \mapsto \gamma'$ of $\text{SL}_2(F)$ such that

$$
\psi_0: [\langle \alpha, \beta, \gamma \rangle] \mapsto [\langle \alpha', \beta', \gamma' \rangle].
$$

It follows that $\xi_e$ commutes with $\psi_k$ for all $e$ and $k$, and also with the automorphism $\tau: [\langle \alpha, \beta, \gamma \rangle] \mapsto [\langle \beta, \alpha, \gamma \rangle]$ of $B^0$. In constructing the amalgam $\mathcal{A}_1$, we identified $B$ with the semidirect product $B^0 \langle \tau \rangle$, so $\xi_e$ may now be regarded as an automorphism of $B$. Since $\lambda_e$ is the product of some elements of $\{1, \xi_0, \ldots, \xi_e\}$, we may also regard $\lambda_e$ as an automorphism of $B$, commuting with $\psi_k$. The proof is then completed by the observation that $\xi_e$ centralizes $C_B(\psi_{e-1})$. 

$\square$
Notice that 4.8 yields
\[ B = \bigcup_{e \geq 0} C_B(\psi_e). \]
By the preceding lemma, we may then define an automorphism \( \lambda \) of \( B \) by taking \( \lambda |_{C_B(\psi_e)} = \lambda_e \) for each \( e \). Set
\[ A = A_\lambda = (H \leftarrow \iota B \xrightarrow{\lambda^*} K) \]
(where \( \iota \) denotes inclusion) and form the corresponding free amalgamated product \( G = G_\lambda \).

We may now complete the proof of Theorem 5.2. We have
\[ \Phi = \text{Aut}(F) = \lim_{\leftarrow \xi} \text{Aut}(F_{p^{2^e}}), \]
and there is an isomorphism
\[ \mathbb{Z}_{2^e} \longrightarrow \text{Aut}(F_{p^{2^e}}) \]
given by sending a residue class \([k]\) to the \( p^k\)-th power map, \( 0 \leq k < 2^e \). The sequence of inverses of these isomorphisms then defines an isomorphism of \( \text{Aut}(F) \) with the ring \( \mathbb{Z}_{(2)} \) of 2-adic integers.

Let \( \mu \in \text{Aut}(F) \), and denote by \( \mu_e \) the restriction of \( \mu \) to the subfield \( F_{p^{2^e}} \) of \( F \) of order \( p^{2^e} \). For any \( e \geq 0 \), there is then a unique integer \( k_e, 0 \leq k_e < 2^e \), such that \( \mu_e \) is given by the \( p^{k_e}\)-th power map on \( F_{2^e} \). Define a sequence \( (\varepsilon_e | e \geq 0) \) of elements of \( \{0, 1\} \) by taking \( \varepsilon_0 = k_0 \), and for \( e > 0 \) by
\[ \varepsilon_e = \begin{cases} 
0 & \text{if } k_e = k_{e-1} \\
1 & \text{if } k_e = k_{e-1} + 2^{e-1} \end{cases}. \]

We may represent the action of \( \mu \) on \( B^0 \) as in 5.1; namely \( \mu \) acts on \( \text{SL}(2, F) \) in the natural way, and on \( B^0 \) by
\[ \mu: [[a, b, c]] \longrightarrow [[a, b, c^{\mu}]]. \]
Observe that the restriction \( \mu_e \) of \( \mu \) to \( C_{B^0}(\psi_{e+1}) \) is given by
\[ \mu_e = \tilde{\xi_0}^{\varepsilon_0} \tilde{\xi_1}^{\varepsilon_1} \cdots \tilde{\xi_e}^{\varepsilon_e}, \]
where \( \tilde{\xi_0} = \xi_0 \omega_3 \) if \( p \equiv 3 \pmod{8} \), and where \( \tilde{\xi_0} = \xi_0 \) if \( p \equiv 5 \pmod{8} \).

Recall from 5.1 that we have identified \( \Phi \) with a subgroup of \( \text{Aut}(B) \), and that parts (b) and (c) of 5.1 show that for any \( \mu' \in \text{Aut}(B) \) there exists \( \mu \in \Phi \) such that \( A_\mu \cong A_{\mu'} \). In particular if \( \mu' = \alpha \mu \) with \( \alpha \in \text{Inn}(B) \) then \( A_\mu \cong A_{\mu'} \), and hence there is an induced isomorphism \( G_\mu \cong G_{\mu'} \) of universal completions.

Take \( \mu = \omega_3 \lambda \) if both \( p \equiv 3 \pmod{8} \) and \( \lambda_1 = \xi_0 \), and take \( \mu = \lambda \) otherwise. By our construction of \( \lambda \) we have \( \mu \in \Phi \), and then since \( \omega_3 \in B \) we obtain \( G_\mu \cong G_\lambda \). Adopt the notation of 5.2. In particular \( N_\lambda = \langle N_H(S_\infty), N_K(S_\infty) \rangle, N_\lambda = \langle W, y \rangle T, \)
and \( N_\lambda = \text{Aut}_{N_3} (S_\infty) \). Since \( \xi_e \) centralizes \( T_{e+1} \) by 5.3(b), we have \( y_0^\lambda T_{e+1} = y_0^\lambda e \), where \( \lambda_e \) is as defined prior to 5.6. By induction on \( e \), we then get

\[
y_0^\lambda T_{e+1} = \begin{cases} y_{e-1} & \text{if } y_e = y_{e-1} \\ y_{e-1}^\xi & \text{if } y_e \neq y_{e-1} \end{cases} \]

and so \( y_0^\lambda T_{e+1} = y_e \). Then 5.4 shows that \( \mu \) is in the set \( \Lambda \) defined in 5.2.

Now let \( \mu \) be an arbitrary element of \( \Lambda \). Set \( \nu = \omega_3 \mu \) if both \( p \equiv 3 \mod 8 \) and \( \varepsilon_0 = 1 \), and otherwise set \( \nu = \mu \). Then \( G_\mu \cong G_\nu \) and \( N_\mu \cong N_\nu \). Set \( x = \nu^{-1} y_0 \nu \), regard \( x \) as an automorphism of \( S_\infty \), and denote by \( x_e \) the restriction of \( x \) to \( T_{e+1} \).

\( e \geq 0 \). Then \( x_0 = y_0 \) and, by induction on \( e \), \( x_{e+1} = x_e^{\varepsilon_e e} \). As \( \mu \in \Lambda \), the uniqueness of the sequence in 5.4 implies that \( \varepsilon_e = 0 \) if and only if \( y_e = y_{e-1} \), and hence that \( x = y \). Since \( y : [a, b, c] \mapsto [e^{\lambda^{-1}}, b, \alpha^\lambda] \), also \( \nu = \lambda \), establishing 5.2(a). Now 5.2(b) follows from the action of \( N_\lambda \) on \( T_\infty \) in 5.4.

If \( A_H \neq A_K \) then \( A_H A_K \cap \Phi \neq 1 \), by 5.1(b), and then 5.1(c) implies that there exists \( \mu' \in \Phi - \{ \mu \} \) with \( A_\mu \cong A_{\mu'} \). This is contrary to 5.2(a), so that \( A_H = A_K \) and 5.2(c) holds. Now 5.2(d) follows from (c) and from 5.1(c), and this completes the proof of 5.2.

Regarding \( H \) and \( K \) as subgroups of \( G = G_\lambda \) in the canonical way, we have \( B = H \cap K \). From 5.6, \( \sigma = \psi_H \) commutes with \( \lambda_e \) for each \( e \), so \( \sigma \) commutes with \( \lambda \). Since \( \sigma \) commutes with \( y_0 \) and with \( \tau \) as automorphisms of \( B^0 \), and since \( y \) acts on \( B^0 \) as \( y_0^\lambda \), \( \sigma \) commutes with \( y \). Then since \( K \) is the semidirect product of \( B^0 \) with \( \langle y, \tau \rangle \), it follows that \( \sigma \) induces an automorphism \( \sigma_K \) of \( K \), commuting with \( \langle y, \tau \rangle \). The universal property of the free amalgamated product now implies that \( \sigma \) induces an automorphism of \( G \), whose restriction to \( K \) is \( \sigma_K \). We record this result for future reference.

**Lemma 5.7.** For each positive integer \( n \), \( \psi_n|_H \) extends uniquely to an automorphism \( \sigma \) of \( G \) such that \( [y, \sigma] = 1 \). \( \Box \)

We next show that the third Conway simple group Co3 is the completion of a subamalgam of \( \mathcal{A}_\lambda \), and that this subamalgam generates a fusion system which is isomorphic to that of Co3. These will be key ingredients in our proof of Theorem B.

**Theorem 5.8.** Let \( \bar{G}_0 \) be the simple group Co3, let \( S_0 \) be a Sylow 2-subgroup of \( \bar{G}_0 \), set \( Z_0 = Z(S_0) \), and let \( U_0 \) be the unique normal fours group in \( S_0 \). Set

\[
H_0 = C_{\bar{G}_0}(Z_0), \quad K_0 = N_{\bar{G}_0}(U_0), \quad B_0 = H_0 \cap K_0.
\]

and let \( \mathcal{A}_0 = (H_0 \leftarrow B_0 \rightarrow K_0) \) be the amalgam of inclusion maps among these groups, within \( \bar{G}_0 \). Set \( \sigma = \psi \) and set \( \lambda_0 = \lambda|_{B_0} \), where \( \lambda \) is an automorphism of \( B \) which satisfies the conditions of Theorem 5.2. Let \( \mathcal{A}_{\lambda_0} = (H_\sigma \leftarrow B_\sigma \rightarrow K_\sigma) \). Then the following hold:
(a) There is a morphism $\varphi: A_0 \to A_\lambda_0$ of amalgams, displaying $A_0$ as a subamalgam of $A_\lambda_0$.

(b) Let $G_0$ be the subgroup of $G$ generated by the images of $H_0$ and $K_0$ under the morphism $\varphi$ of part (a), and let $F_0$ be the fusion system $(F_{S_0}(H_0), F_{S_0}(K_0))$ contained in $F_{S_\sigma}(G_0)$. Then $F_0 = F_{S_\sigma}(G_0) = F_{S_\sigma}(\hat{G}_0)$.

**Proof.** We refer to [Fin73] for the structure of the maximal subgroups of $G_0$. Thus, $H_0$ is isomorphic to the covering group of $\text{Sp}_6(2)$, which is the perfect central extension of $\text{Sp}_6(2)$ by a group of order 2. Since $\text{Sp}_6(2) \times C_2$ is a reflection group (namely, the Weyl group of type $E_7$), we have $\text{Sp}_6(2) \leq O_7(\mathbb{R})$, and then, by taking the standard $\mathbb{Z}$-form of $O_7(\mathbb{R})$ and reducing mod $p$, one obtains $\text{Sp}_6(2)$ as a subgroup of $\Omega_7(p)$ with $H_0$, we then have an inclusion of $H_0$ in $H_\sigma$. In particular, $Z_0 = Z$.

Let $S$ be $GL_2(3) \cdot S_3$ and $\hat{S} = S / Z(S)$. Set $B_1 = O^2(B_0)$ and $D = O_2(B_0)$. Sylow 2-subgroups of $H_0$ and $H_\sigma$ are of order $2^{10}$, so conjugating in $H_\sigma$, we may take $S_\sigma = S_0 \in \text{Syl}_2(H_0)$, and $U = U_0$. Next, $B_0$ is the preimage in $H_0$ of the solvable maximal parabolic subgroup of $\text{Sp}_6(2)$, so that $B_0/O_2(B_0)$ is isomorphic to $\text{Sym}(3) \times \text{Sym}(3)$, and $B_1$ is isomorphic to a subgroup of index 3 in $O_{2,3}(\hat{S})$, contained in the $\Lambda$-orbit of length 4 on such subgroups. In particular

1. $D$ is a commuting product of three quaternion groups $Q_i$, $1 \leq i \leq 3$, with the property that $\mathcal{Q} = \{Q_1, Q_2, Q_3\}$ is the set of normal subgroups of $B_1$ of order 8. Moreover $C_D(Q_i) = Q_j \times Q_k$ for any ordering $(i, j, k)$ of $(1, 2, 3)$.

The preimage $\hat{B}_1$ of $B_1$ in $S_\sigma$ is the 2-covering group of $B_1$, and so $\text{Aut}(B_1)$ acts on $\hat{B}_1$. By (1), $\text{Aut}(B_1)$ permutes $\mathcal{Q}$, and hence $\mathcal{Q} = \{\hat{Q}_i \mid 1 \leq i \leq 3\}$, where $\hat{Q}_i = \{\hat{Q}_i \mid 1 \leq i \leq 3\}$ and where $\hat{Q}_i$ is the preimage in $\Lambda$ of $Q_i$. Therefore $\text{Aut}(B_1) = \text{Aut}_\Lambda(B_1)$.

Referring once more to [Fin73], we find that $|K_0 : B_0| = 3$. Since $C_{K_0}(U) \leq B_0$ we have $B_1 \leq C_{K_0}(U) \leq K_0$, and so $B_1 \leq K_0$ and $K_0/C_{K_0}(U) \cong \text{Sym}(3)$. From the preceding paragraph, $\text{Aut}_{K_0}(B_1) \leq \Omega := \text{Aut}_\Lambda(B_1)$. Then, since the Sylow 2-subgroup $S_0$ of $B_0$ is Sylow in $K_0$ and contains the kernel $U$ of the map from $N_{\Lambda}(B_1)$ onto $\Omega$, we may regard $K_0$ as a subgroup of $\Lambda$. By (1), $\mathcal{Q}$ is $K_0$-invariant. As $K_0/C_{K_0}(U) \cong \text{Sym}(3)$, it follows that $K_0$ is transitive on $\mathcal{Q}$. Here $\Lambda / D \cong \text{Sym}(3) \cdot \text{Sym}(3)$, and $K_0/D$ is a subgroup of $\Lambda / D$ of order $2^2 \cdot 3^3$. Since $K_0/C_{K_0}(U) \cong \text{Sym}(3)$, it follows that $K_0/B_1 \cong \mathbb{Z}_2 \times \text{Sym}(3)$, and then that $K_0$ is determined up to conjugacy in $\Lambda$. The same argument shows that $K_\sigma$ is in this conjugacy class, and so $K_0 \cong K_\sigma$ and we may choose $K_0 = K_\sigma$.

Observe that $|N_{\Lambda}(B_1) : B_0| = 3$ and that $B_0$ is equal to its normalizer in $N_{\Lambda}(B_1)$. Then

2. $N_{\text{Aut}(B_1)}(B_0) = \text{Aut}_\Lambda(B_0) = \text{Inn}(B_0)$. 

By (2) we have $\text{Aut}(B_0) = \text{Inn}(B_0)C_0$, where $C_0 = C_{\text{Aut}(B_0)}(B_1)$. Then $[C_0, B_0] \leq C_{B_0}(B_1) = U$. Let $X_0 \in \text{Syl}_3(B_0)$ and let $X_0 \leq X \in \text{Syl}_2(K_0)$. Then $C_D(X_0) = U$, $C_D(X) = 1$, and as we saw above $K_0/B_1 \cong \mathbb{Z}_2 \times \text{Sym}(3)$. It follows that there is a Sylow 2-subgroup $R_0$ of $N_{B_0}(X_0)$, of the form $U \langle s \rangle \times \langle r \rangle$, where $\langle r, s \rangle$ is a Sylow 2-subgroup of $N_{K_0}(X)$, $U \langle s \rangle$ is a dihedral group of order 8, and $\langle r, s \rangle$ is a fours group.

We have $[C_0, R_0] \leq C_{B_0}(B_1) = U$, and so $R_0$ is $C_0$-invariant. Since $r \in Z(R_0)$ we then have $[C_0, r] \leq U \cap Z(R_0) = Z$. For any $\mu \in C_0$, $s^\mu$ is an involution in $sU$, so also $[C_0, s] \leq Z$. Therefore $C_0 \cong \text{Hom}(B_0/B_1, Z) \cong B_0/B_1$, and so $C_0$ is a fours group. Since $\mathbb{Z}_2 \cong \text{Aut}_U(B_0) \leq C_0$, we conclude that $|\text{Aut}(B_0) : \text{Inn}(B_0)| = |C_0 : \text{Aut}_U(B_0)| = 2$. Thus

(3) $\text{Aut}(B_0) = \text{Inn}(B_0) \cup \text{Inn}(B_0)\mu_0$, where $\mu_0 \in C_0 - \text{Aut}_U(B_0)$.

It follows from (3) and [Gol70, Lemma 2.7] that there are, up to isomorphism, at most two amalgams $(H_0 \leftarrow B_0 \xrightarrow{\alpha} K_0)$ with $B_0\alpha = B_0$, and that any such amalgam is isomorphic to $\mathcal{A}_0$ or to $\mathcal{A}_{0,\mu_0}$, where

$$\mathcal{A}_{0,\mu_0} = (H_0 \leftarrow \bar{B} \xrightarrow{\mu_0^*} K_0).$$

Thus, either $\mathcal{A}_0$ or $\mathcal{A}_{0,\mu_0}$ is a subamalgam of the amalgam $\mathcal{A}_\sigma$.

Referring again to [Fin73], there is a subgroup $M$ of $\bar{G}_0$, containing $S_0$, such that $M$ is a nonsplit extension of $E_{16}$ by $\text{GL}_4(2)$. Set $A = O_2(M)$, and denote by $M_0$ the stabilizer in $M$ of the unique $S_0$-invariant hyperplane $E_0$ of $A$. Then $[O_2(M_0), M_0]$ is homocyclic abelian of exponent 4 and rank 3, and hence $E_0 = E$ by 4.9(c). Since $[O_2(M_0), E_0] = 1$ we then have $O_2(M_0) = T_2 \langle w_0 \rangle$, by 4.3. Moreover

$$M_0 = \langle C_{M_0}(Z), N_{M_0}(U) \rangle = \langle M_0 \cap H_0, M_0 \cap K_0 \rangle.$$ 

Let $\alpha: S_0 \rightarrow N_{K_{\sigma}}(T_2)$ be the embedding of $S_0 = S_{\sigma}$ in $N_{K_0}(T_2)$. Then $S_0\alpha = S_0 \leq K_0 = K_{\sigma}$, and we have the two amalgams

$$\mathcal{A}_{M_0} = (M_0 \cap H_0 \leftarrow S_{\sigma} \xrightarrow{\alpha} M_0 \cap K_0)$$

and

$$(N_{H_{\sigma}}(T_2) \leftarrow S_0 \xrightarrow{\alpha} N_{K_{\sigma}}(T_2)).$$

On the other hand, the reader will recall from the proof of 5.4 that if $\mathcal{A}_{M_0}$ is “twisted” by $\mu_0$, to obtain an amalgam

$$\langle M_0 \cap H_0, x^{\mu_0} \rangle$$ indudes on $T_2$ the full automorphism group of $T_2$, of order $2^9|\text{GL}_3(2)|$. We therefore conclude that, of the two amalgams $\mathcal{A}_0$ and $\mathcal{A}_{0,\mu_0}$, only the first is a subamalgam of $\mathcal{A}_\sigma$. This completes the proof of (a).
Set \( \mathcal{A}_0 = \mathcal{G}_0 \) and denote by \( \mathcal{E}^* \) the set of elementary abelian subgroups \( F \) of \( S_0 \) such that \( F^\# \subseteq \mathcal{A}_0 \). Denote by \( \mathcal{N} \) the set of subgroups \( N \) of \( \mathcal{G}_0 \) such that \( N = N_{\mathcal{G}_0}(O_2(N)) \), \( S_0 \cap N \in \text{Syl}_2(N) \), and \( C_{S_0}(O_2(N)) \leq O_2(N) \). It is a property of \( \mathcal{C}_0 \) that \( x y \in \mathcal{A}_0 \) for any two distinct commuting elements \( x \) and \( y \) of \( \mathcal{A}_0 \). From (1.9) it follows that for any \( N \in \mathcal{N} \) there exists \( F \in \mathcal{E}^* \) with \( F \trianglelefteq N \). By Lemmas 5.8 and 5.9 in [Fin73], all members of \( \mathcal{E}^* \) of any given order are fused in \( \mathcal{G}_0 \), each member of \( \mathcal{E}^* \) is normal in a Sylow 2-subgroup of \( \mathcal{G}_0 \), and if \( F \in \mathcal{E}^* \) with \( |F| = 8 \) then \( N_{\mathcal{G}_0}(F) \) is contained in the normalizer of some \( F^* \in \mathcal{E}^* \) with \( |F^*| = 16 \). Then, for any \( N \in \mathcal{N} \), we have \( S_0 \leq N \), and \( N \) is contained in one of the groups \( H_0 \), \( K_0 \), or \( M \). Then 1.11 yields

\[
\mathcal{F}_{S_0}(\mathcal{C}_0) = \langle \mathcal{F}_{S_0}(H_0), \mathcal{F}_{S_0}(K_0), \mathcal{F}_{S_0}(M) \rangle.
\]

Now \( (M \cap H_0)/A \) and \( (M \cap K_0)/A \) are distinct maximal parabolic subgroups of \( M/A \cong \text{GL}_4(2) \), and so by 1.9:

\[
\mathcal{F}_{S_0}(M) = \langle \mathcal{F}_{S_0}(M \cap H_0), \mathcal{F}_{S_0}(M \cap K_0) \rangle.
\]

From (4) and (5) we have \( \mathcal{F}_{S_0}(\mathcal{G}_0) = \langle \mathcal{F}_{S_0}(H_0), \mathcal{F}_{S_0}(K_0) \rangle \), and it follows that \( \mathcal{F}_{S_0}(\mathcal{G}_0) \subseteq \mathcal{F}_{S_0}(G_0) \). Since \( \mathcal{G}_0 \) is a homomorphic image of \( G_0 \), by (a), the reverse inclusion of fusion systems is obvious, and we therefore have (b).

Some well-known properties of \( \mathcal{C}_0 \) (some of which were mentioned in the proof of 5.8(b)), which depend only on fusion, now yield corresponding properties of the subgroup \( G_0 \) of \( G \).

**COROLLARY 5.9.** Identify \( H_0 \) and \( K_0 \) with subgroups of \( G \), via the morphism \( \varphi \) of 5.8(a), and set \( G_0 = (H_0, K_0) \). Then

(a) \( G_0 \) has two classes of elements of order 2.

(b) If \( t \) and \( t' \) are distinct, commuting elements of \( z^{G_0} \), then \( tt' \in z^{G_0} \).

(c) Let \( F \) be an elementary abelian 2-subgroup of \( G_0 \). Then \( F \cap z^{G_0} \) is the set of nonidentity elements of a subgroup of \( F \).

(d) For any \( X \subseteq G_0 \), and any subgroup \( F \) of \( X \), denote by \( \mathcal{E}(X, F) \) the set of all subgroups \( P \) of \( X \) such that \( F \subseteq P \) and \( P^\# \subseteq z^{G_0} \). Write \( \mathcal{E}(X, 1) \). Then \( \{Z, U, E, E\langle w_0 \rangle\} \) is a set of representatives for the orbits of \( G_0 \) on \( \mathcal{E}(G_0) \), and for the orbits of \( H_0 \) on \( \mathcal{E}(H_0, Z) \).

**6. Discrete \( p \)-toral groups**

The notion of a discrete \( p \)-toral group, and the results in this section on such groups, come from [BLO05], particularly Sections 1 and 7 of that paper. As [BLO05] is unpublished at this time, we reproduce some of its definitions and
results here, and supply sketches of proofs in special cases, for the sake of completeness.

**Definition 6.1.** Let \( p \) be a prime and denote by \( \mathbb{Z}/p\infty \) the group of all complex roots of unity whose order is a power of \( p \). A **discrete \( p \)-toral group** is a \( p \)-group \( P \) with a normal subgroup \( P_0 \) of finite index, such that \( P_0 \) is the direct product of a finite number of copies of \( \mathbb{Z}/p\infty \). Write \( \mathcal{D}_p \) for the class of discrete \( p \)-toral groups.

We record some facts about \( \mathcal{D}_p \) from [BLO05]:

**Lemma 6.2.** Let \( P \in \mathcal{D}_p \). Then

1. \( P \) has unique subgroup \( P^0 \) which is minimal subject to the condition that \( |P : P^0| \) be finite. (Call \( P^0 \) the identity component of \( P \).)
2. \( P^0 \) is the direct product of a finite number \( r \) of copies of \( \mathbb{Z}/p\infty \). (Write \( \text{rk}(P) \) for \( r \) and call \( \text{rk}(P) \) the rank of \( P \).)
3. \( P^0 \) has no proper subgroups of finite index.
4. \( P \) is locally finite and Artinian.
5. Subgroups and homomorphic images of \( P \) are in \( \mathcal{D}_p \).
6. Torsion subgroups of \( \text{Out}(P) \) are finite.
7. Each injective homomorphism from \( P \) into \( P \) is an isomorphism.
8. If \( R \leq P \) then \( R^0 \leq P^0 \).

**Proof.** As \( P \in \mathcal{D}_p \), \( P \) has a normal subgroup \( P_0 \) of finite index which is the direct product of \( r \) copies of \( \mathbb{Z}/p\infty \) for some \( 0 \leq r \in \mathbb{Z} \). As \( \mathbb{Z}/p\infty \) has no proper subgroups of finite index, it follows that \( P^0 = P_0 \), and (1)–(3) hold. Parts (4), (5), and (6) are 1.2, 1.3, and 1.5(a) in [BLO05], respectively. Part (7) follows as \( P \) is Artinian, and (8) follows from (3). \( \square \)

**Lemma 6.3.** Let \( \mathbf{F} \) be the field of Section 4, \( V \) a finite-dimensional vector space over \( \mathbf{F} \), and \( G \leq \text{GL}(V) \). Then

1. \( G \) is locally finite.
2. All \( 2 \)-subgroups of \( G \) are in \( \mathcal{D}_2 \).
3. \( \text{Syl}_2(G) \neq \emptyset \), \( \text{Syl}_2(G) \) is the set of maximal \( 2 \)-subgroups of \( G \), and \( G \) is transitive on \( \text{Syl}_2(G) \).
4. Let \( S \in \text{Syl}_2(G) \) and \( P \leq S \). Then \( P \) is fully normalized in \( \mathcal{F}_S(G) \) if and only if \( N_S(P) \in \text{Syl}_2(N_G(P)) \).
5. \( \mathcal{F}_S(G) \) is saturated.
Proof. The proof of this lemma comes from [BLO05, §7, particularly Lemma 7.8]. The proof is a bit easier in our special case, and we supply a sketch.

As $GL(V)$ is the union of the finite groups $GL(V)_{\sigma}$, $\sigma \in \text{Aut}(F)$, (1) holds. Then (2) follows from (1) and from [Weh73, 2.6]. By [Weh73, 9.10], $G$ is transitive on its maximal 2-subgroups, and such subgroups exist, and so (3) holds.

Observe that $G$ satisfies the hypotheses of Lemma 1.6: Condition (1) of 1.6 follows from (3) applied to subgroups of $N_G(P)$. Condition (2) of 1.6 is satisfied by (1) and 6.2(6). Condition (3) holds by 6.2(4), and (4) holds by (1). Now 1.6 implies (4) and (5).

Remark 6.4. Let $H, K, B, S$ be the groups defined in Sections 4 and 5. Each of these groups has a faithful finite-dimensional representation over $F$, and so we can apply Lemma 6.3 to these groups. By 4.9(b), $S$ is a Sylow 2-subgroup of each of these groups. By 6.3(2), $S$ and each of its subgroups is a discrete 2-toral group. By 6.3(5), $\mathcal{F}_S(X)$ is saturated for each $X \in \{H, K, B\}$ and by 6.3(3), $X$ is transitive on $\text{Syl}_2(X)$ where $\text{Syl}_2(X)$ is the set of maximal 2-subgroups of $X$.

Let $G$ be the group constructed in Section 5. It will be shown, in Theorem C, that there is a 2-local group $\mathcal{G} = (S, \mathcal{F}_S(G), \mathcal{L}_S(G))$. Since $S$ is a discrete 2-toral group, $\mathcal{G}$ is then a 2-local compact group, as defined in [BLO05].

7. Local subgroups and fusion in the free amalgamated product

Let $\mathcal{A}$ be the amalgam $\mathcal{A}_\lambda$ constructed in Section 5, and let $G$ be the associated free amalgamated product, $G = H \ast_B K$. We shall view $\mathcal{A}$ as being given by the inclusion maps of $H$, $K$ and $B$ into $G$, so that

$$\mathcal{A} = (H \geq B \leq K).$$

Viewed in this way, the key point in the construction of $\mathcal{A}$ is that the element $y$ of $K$ acts on $T$ as $\lambda^{-1}y_0\lambda$. That is

$$y: [a, b, c] \mapsto [c\lambda^{-1}, a, b\lambda],$$

for all $[a, b, c] \in T$.

Recall that we have an automorphism $\sigma = \psi_n$ of $H$, with $H_\sigma \cong \text{Spin}_{\gamma}(F_q)$, $q = p^{2^n}$, and by 4.8, $S_\sigma$ is a Sylow 2-subgroup of $H_\sigma$. By 5.7, $\sigma$ induces an automorphism of $\mathcal{A}$ which induces an automorphism of $G$. Form the semidirect products $H \langle \sigma \rangle$, $B \langle \sigma \rangle$, and $K \langle \sigma \rangle$, and the amalgam

$$\hat{\mathcal{A}} = (H \langle \sigma \rangle \leftarrow B \langle \sigma \rangle \rightarrow K \langle \sigma \rangle),$$

in which the arrows are inclusion maps. Denote the free amalgamated product of $\hat{\mathcal{A}}$ by $\hat{G}$. The inclusion $\mathcal{A} \rightarrow \hat{\mathcal{A}}$ induces an isomorphism of $\hat{G}$ with the semidirect product $G \langle \sigma \rangle$, and we identify these groups via that isomorphism.

The following result is trivially verified.
Lemma 7.1. Let $\Gamma$ and $\hat{\Gamma}$ be the standard trees associated with the amalgams $\mathcal{A}$ and $\bar{\mathcal{A}}$, respectively. Then there is an isomorphism $\Gamma \longrightarrow \hat{\Gamma}$ given by
\[ Xg \mapsto X\{\sigma\}g \quad \text{for} \quad X \in \{H, B, K\} \text{ and } g \in G. \]

If $\Gamma$ and $\hat{\Gamma}$ are identified via this isomorphism, then the action of $\sigma$ on $\Gamma$ is given by
\[ (Xg)^\sigma = Xg^\sigma \quad \text{for} \quad X \in \{H, B, K\} \text{ and } g \in G. \]

For any $X \leq G\langle \sigma \rangle$, we write $\Gamma_X$ or $C_\Gamma(X)$ for the subgraph of $\Gamma$ induced on the set of fixed points of $X$ on $\Gamma$. If $\Gamma_X \neq \emptyset$ then $\Gamma_X$ is a subtree of $\Gamma$. For any graph $\Delta$ and vertex $\delta$ of $\Delta$, we write $\Delta(\delta)$ for the set of vertices $\gamma$ of $\Delta$ such that $\{\gamma, \delta\}$ is an edge of $\Delta$. If $|\Delta(\delta)| \leq 1$ then $\delta$ is a boundary vertex of $\Delta$, and otherwise $\delta$ is an interior vertex of $\Delta$.

For any subtree $\Delta$ of $\Gamma$, let $\tilde{\Delta}$ be the graph obtained by deleting the boundary vertices from $\Delta$. Thus either $\tilde{\Delta}$ is a tree or $\Delta$ has at most one edge, in which case $\tilde{\Delta}$ is empty.

Set $G_1 = H$ and $G_2 = K$, and denote by $\Gamma_i$ the set of vertices of $\Gamma$ given by the cosets of $G_i$ in $G$. For any vertex $\gamma$ of $\Gamma$, write $Z(\gamma)$ for the largest normal 2-subgroup of $G\gamma$. Define $\gamma_i$ to be the vertex of $\Gamma$ given by the coset $G_i$.

Lemma 7.2. Let $\gamma$ be a vertex of $\Gamma$.

(a) If $\gamma \in \Gamma_1$ then $Z(\gamma) = Z(G\gamma)$ is of order 2.

(b) If $\gamma \in \Gamma_2$ then $|\Gamma(\gamma)| = 3$, and $Z(\gamma)$ is a fours group, whose nonidentity cyclic subgroups are the groups $Z(\delta)$, $\delta \in \Gamma(\gamma)$.

(c) If $\gamma \in \Gamma_2$ then $C_{G\gamma}(Z(\gamma))$ is the pointwise stabilizer in $G$ of $\Gamma(\gamma)$.

Proof. The stabilizer of any vertex in $\Gamma$ is conjugate in $G$ to $G_i$, and the stabilizer of any edge is conjugate to $B$. All parts of the lemma follow trivially from these observations. \qed

Lemma 7.3. Let $X \leq G\langle \sigma \rangle$ and let $\gamma \in \Gamma_2 \cap \Gamma_X$. Then:

(a) $\gamma$ is an interior vertex of $\Gamma_X$ if and only if $X$ centralizes $Z(\gamma)$.

(b) Either of the following conditions implies that the inclusion maps from $N_H(X)$ and $N_K(X)$ into $N_G(X)$ induce an isomorphism of $N_H(X) *_{N_B(X)} N_K(X)$ with $N_G(X)$.

(i) $X \leq B^0$, and $X^H \cap B = X^K \cap B = X^B$.

(ii) $X \leq B^0\langle \sigma \rangle$ and $X^H\langle \sigma \rangle \cap B^0\langle \sigma \rangle = X^K\langle \sigma \rangle \cap B^0\langle \sigma \rangle = X^B$.

Moreover, $N_G(X)$ acts edge-transitively on $\Gamma_X$ in case (b)(i), and edge-transitively on $\tilde{\Gamma}_X$ in case (b)(ii).
Proof. Set \( \Delta = \Gamma_X \). Then \( \gamma \) is an interior vertex of \( \Delta \) if and only if \( X \) fixes at least two distinct vertices \( \alpha \) and \( \beta \) in \( \Gamma_X \). Since \( Z(\gamma) = Z(G^0_\gamma) = Z_\alpha Z_\beta \), we obtain (a).

Set \( N = \langle NG_1(X), NG_2(X) \rangle \), and assume that either (i) or (ii) holds. Take \( \Lambda = \Delta \) in case (i), and \( \Lambda = \tilde{\Delta} \) in case (ii). Then \( NG(X) \) acts on \( \Lambda \), and \( N \leq NG(X) \). By hypothesis, \( X \leq C_B(\sigma)(U) = B^0(\sigma) \), so that \( X \) fixes \( \Gamma(\gamma_2) \) pointwise, and hence \( \gamma_2 \in \Lambda \). In (i), a standard argument (cf. [Asc86, 5.21]) shows that \( NG_1(X) \) acts transitively on \( \Lambda(\gamma_i) \) for \( i = 1 \) and 2. Assume that we are in case (ii) and that \( (\gamma_i, \gamma_2, i) \) is an edge in \( \Lambda^{h^{-1}} \) for some \( h \in G_i(\sigma) \). Then \( X \leq (B(\sigma))^h \), and so \( X^{h^{-1}} \leq B(\sigma) \). Then \( X^{h^{-1}} \leq C_B(\sigma)(U) = B^0(\sigma) \) by (a). The hypotheses of (ii) then yield \( h \in B^0(\sigma)NG_1(\sigma)(X) \), so that \( NG_1(\sigma)(X) \) acts transitively on \( \Lambda(\gamma_i) \).

We now claim that \( N \) is transitive on the set of edges of \( \Lambda \). As \( \Lambda \) is connected, it suffices to show for each \( \lambda \in \Lambda \) that \( N \lambda \) is transitive on \( \Lambda(\lambda) \). Pick \( i \) and \( g \) with \( \lambda = \gamma_i g \) and set

\[
d(\lambda) = \min\{d(\lambda, \gamma_j) \mid j = 1, 2\}.
\]

Choose \( \lambda \) to be a counterexample with \( d = d(\lambda) \) minimal. By the preceding paragraph, \( d > 0 \). Thus there exists \( \alpha \in \Lambda(\lambda) \) with \( d(\alpha) < d \), and \( N \alpha \) is transitive on \( \Lambda(\alpha) \). Then there is \( \beta \in \Lambda(\alpha) \) with \( d(\beta) < d \), contrary to the choice of \( \lambda \). This completes the proof of the claim.

As the stabilizer \( NB(X) \) of an edge of \( \Lambda \) is contained in \( N \), we now obtain \( N = NG(X) \). Now [Ser80, Th. 6, p. 32] yields the conclusions concerning edge-transitivity, and the identification of \( NH(X) * NB(X) NK(X) \) with \( N \). □

Lemma 7.4. We have the following.

(a) \( C_{\tilde{\Gamma}}(\sigma) = H_\sigma \).

(b) The inclusion maps from \( H_\sigma \) and \( K_\sigma \) into \( G_\sigma \) induce an isomorphism of \( G_\sigma \) with \( H_\sigma *_{B_\sigma} K_\sigma \), and \( G_\sigma \) acts edge-transitively on the tree \( \tilde{\Gamma}_\sigma \).

(c) Define the subgroups \( H_0 \), \( K_0 \), and \( B_0 \) of \( G_\sigma \) as in 5.8. Then the inclusion maps from \( H_0 \) and \( K_0 \) into \( G_0 \) induce an isomorphism of \( G_0 \) with \( H_0 *_{B_0} K_0 \), the universal completion of the amalgam \( A_0 \) of subgroups of \( Co_3 \).

Proof. Let \( h \in H \) such that \( \sigma^h \in B^0 \). By Lang’s Theorem there exists \( b \in \tilde{B}^0 \) such that \( \sigma^h = \sigma^b \). Then \( hb^{-1} \in C_{\tilde{\Gamma}}(\sigma) = H_\sigma \) as \( \sigma = \xi_H \). Thus (a) holds and \( b \in H \). Here \( b \in B^0 \) since \( H \cap \tilde{B}^0 = B^0 \), and so \( \sigma^H \cap B^0 = \sigma^{B^0} \). Since \( K = C_K(\sigma)B^0 \), we also have \( \sigma^K \cap B^0 = \sigma^{B^0} \). Now by 7.3(b), \( G_\sigma = \{H_\sigma, K_\sigma\} \) and \( G_\sigma \) is edge-transitive on the tree \( \tilde{\Gamma}_\sigma \). Since \( H_\sigma \) and \( K_\sigma \) fix adjacent vertices in \( \tilde{\Gamma}_\sigma \), the lemma now follows from [Ser80, Th. 6, p. 32]. The same theorem implies (c). □

From now on, \( G_0 = H_0 *_{B_0} K_0 \) is the subgroup of \( G_\sigma \) defined in 7.4(c), such that \( G_0 \) is the universal completion of an amalgam of subgroups of \( Co_3 \).
LEMMA 7.5. Let \( D \in \{G, G_\sigma, G_0\} \), set \( D_i = D \cap G_i \), \( i = 1, 2 \), and set \( R = S \cap D \). Then:

(a) Hypotheses 3.4 and 3.8 hold, with \( D, D_i \) and \( D \cap B \) in the roles of \( G, G_i \) and \( B \), and with \( R \) in the role of \( S \).

(b) \( R \) is a Sylow 2-subgroup of \( D \).

(c) If \( D \) is \( G \) or \( G_\sigma \) then Hypothesis 3.12 holds.

Proof. When \( R \) is finite so is \( D_i \), and by construction \( R \) is a Sylow 2-subgroup of \( D_i \) and of \( D \cap B \). If \( R \) is infinite then \( R = S \), and by Remark 6.4, \( R \) is a Sylow 2-subgroup of \( D_i \) and of \( D \cap B \). In each case \( Z \) and \( U \) are characteristic subgroups of \( R \), so that \( N_{D_i}(R) \leq D_1 \cap D_2 \). A free amalgamated product decomposition for \( G_\sigma \) is given by 7.4(b), and for \( G \) and \( G_0 \) by the definition of these groups. Thus, Hypothesis 3.4 holds. The verification of the first three parts of Hypothesis 3.8 is immediate in each case. Part (4) of Hypothesis 3.8 holds by Sylow’s Theorem when \( D \) is finite, and by Remark 6.4 when \( D = G \). Thus (a) is established. By 3.8(4), \( R^{D_i} \cap B = R^{D_i} \cap B \). Part (b) follows from (a), 3.5(c), and this observation. Finally, when \( D = G \) or \( G_\sigma \), Hypothesis 3.12 follows from part (a) of Lemma 7.6 below.

For any subgroup \( X \) of \( G \), and any elementary abelian 2-subgroup \( F \) of \( X \), denote by \( G_n(X, F) \) the set of elementary abelian 2-subgroups of \( X \) containing \( F \), of order \( 2^n \). Write \( G_n(X) \) for \( G_n(X, 1) \). Recall from the preceding sections that

\[
Z \leq U \leq E \leq A \in G_n(G_\sigma)
\]

is a chain of elementary abelian 2-groups, where \( Z = Z(H) = \langle z \rangle \), \( U = Z(B^0) = \langle z, z_1 \rangle \), \( E = \{ e \in T \mid e^2 = 1 \} \), and \( A = E \langle w_0 \rangle \).

LEMMA 7.6. The following hold.

(a) \( G_2(H, Z) = U^H \), and \( G_2(H_\sigma, Z) = U^{H_\sigma} \).

(b) \( G_1(G) = Z^G \), and \( G_1(G_\sigma) = Z^{G_\sigma} \).

Proof. By 4.2 there is a unique class \( z_1^{\tilde{H}} \) of noncentral involutions in \( \tilde{H} \). Then since \( C_{\tilde{B}}(z_1) = B^0 \) is connected, it follows from Lang’s Theorem that \( \tilde{H}_\sigma \) has a unique class of noncentral involutions. As \( \tilde{H}_\sigma = H_\sigma \), (a) follows.

Since \( K_\sigma \) is transitive on \( U^\# \), it follows from (a) that all involutions in \( S \) (resp. \( S_\sigma \)), are fused in \( KH \) (resp \( K_\sigma H_\sigma \)). By 7.5(b), \( S \) and \( S_\sigma \) are Sylow in \( G \) and \( G_\sigma \), respectively, so (b) holds.

\]

LEMMA 7.7. We have \( E^{B^0} = G_3(B, U) \), \( E^{H} = G_3(H, Z) \), and \( E^{G} = G_3(G) \).

Proof. Let \( F \in G_3(B, U) \) and \( f \in F - U \). Then \( C_B(U) = B^0 = L_1 L_2 L_3 \), so that \( f = f_1 f_2 f_3 \) with \( f_i \in L_i \), and \( 1 = f^2 = f_1^2 f_2^2 f_3^2 \). Since \( U^\# \) is the
set of involutions in \( L_i L_j \) for \( i \neq j \), it follows that \( f_i \) is an element of order 4 in \( L_i \). Since \( L_j \) is transitive on its elements of order 4, involutions in \( B^0 - U \) are conjugate in \( B^0 \). Thus \( E^{B^0} = \varepsilon_3(B^0, U) \). The lemma now follows from 7.6. \( \square \)

**Lemma 7.8.** Set \( B(\sigma) = \langle C_{L_i}(\sigma) \mid 1 \leq i \leq 3 \rangle \). Then

(a) There exists an involution \( v \) in \( S \cap B^0\sigma - B(\sigma) \) such that \( B^0\sigma = B(\sigma)\langle v \rangle \), and \( v \) induces a diagonal automorphism on each \( C_{L_i}(\sigma) \).

(b) \( E^{B(\sigma)} = \varepsilon_3(B(\sigma), U) \) and \( \varepsilon_3(B\sigma, U) = E^{B\sigma} \cup (E')^{B(\sigma)} \), where \( E' = U \langle v \rangle \).

(c) \( \varepsilon_3(H\sigma, Z) \) is the disjoint union of \( E^{H\sigma} \) and \( (E')^{H\sigma} \).

(d) \( EE' \) is a Sylow 2-subgroup of \( C_{H(\sigma)}(E') \), and is elementary abelian of order 16.

(e) For each \( F \in \{E, E'\} \), we have \( \text{Aut}_{H(\sigma)}(F) = C_{\text{Aut}(F)}(Z) \) and \( \text{Aut}_{G(\sigma)}(F) = \text{Aut}(F) \).

**Proof.** Recall from Section 4 that we may regard \( T \) as a set of equivalence classes \([a_1, a_2, a_3]\). Let \( a \) be a 2-element in \( F \) with \( a^\sigma = -a \), and set \( f = [a, a, a] \). Then \( f \in (S_\infty \cap B^0\sigma) - B(\sigma) \), and since \( w_0 \) inverts \( S_\infty \), the element \( v := fw_0 \) is an involution in \( (S \cap B^0\sigma) - B(\sigma) \). Recall from 4.4 that \( \tilde{B}^0 \) is \( \tilde{J}/\langle i \rangle \), where \( \tilde{J} \) is the direct product of three copies of \( \text{SL}_2(\tilde{F}) \) and \( i \) is an involution diagonally embedded in \( Z(\tilde{J}) \). Thus \( \tilde{J} \) is simply connected, so that \( B(\sigma) = \tilde{J}/\langle i \rangle \), and \( B(\sigma) \) is of index \( |i| = 2 \) in \( B^0 \). Then \( B^0 = B(\sigma)\langle v \rangle \), completing the proof of (a).

Let \( X \in \{\tilde{B}^0, \tilde{H}\} \), and set

\[
\Sigma = \{(F, \tau) \mid F \in E^X, \tau \in \sigma^X, [F, \tau] = 1\}.
\]

Set

\[
\Sigma_0 = \{\tau \in \sigma^X \mid (E, \tau) \in \Sigma\} \quad \text{and} \quad \Sigma_1 = \{F \in E^X \mid (F, \sigma) \in \Sigma\}.
\]

There is a natural bijection \( \beta \) between the set of \( N_X(E) \)-orbits on \( \Sigma_0 \) and the set of \( C_X(\sigma) \)-orbits on \( \Sigma_1 \). Explicitly, if \( \{\sigma^g_i \mid i \in I\} \) is a set of representatives for the orbits of \( N_X(E) \) on \( \Sigma_0 \), then \( \{E^{\sigma^g_i^{-1}} \mid i \in I\} \) is a set of representatives for the orbits of \( C_X(\sigma) \) on \( \Sigma_1 \). By 4.3(d), \( N_{\tilde{H}}(E) = \tilde{T}W \), so that \( N_X(E) = \tilde{T}(W \cap X) \).

Let \( \tau \in \Sigma_0 \). Then

\[
\tau \in C_X(\sigma)(E) \cap \sigma^X \subseteq \tilde{T}(w_0)\sigma = \tilde{T}\sigma \cup \tilde{T}w_0\sigma.
\]

When we apply Lang’s Theorem to the connected algebraic group \( \tilde{T} \), we find that \( \tilde{T} \) is transitive on \( \tilde{T}\sigma \) and \( \tilde{T}w_0\sigma \). Since \( W \) centralizes both \( \sigma \) and \( w_0 \), we conclude that \( \tilde{T}\sigma \) and \( \tilde{T}w_0\sigma \) are the orbits for \( N_X(E) \) on \( \Sigma_0 \), with representatives \( \{\sigma, w_0\sigma\} \). Applying Lang’s Theorem to the connected group \( \tilde{B}^0 \), we obtain an element \( g \in X \) such that \( (w_0\sigma)^g = \sigma \).
When we apply the bijection \( \beta \), \( C_X(\sigma) \) has two orbits on \( \Sigma_1 \), with representatives \( E \) and \( E^g \). By 7.7, \( \Sigma_1 = \mathcal{E}_3(C_X(\langle \sigma, Z(X) \rangle)) \). Thus, \( C_X(\sigma) \) has two orbits on \( \mathcal{E}_3(X, Z(X)) \), with representatives \( E \) and \( E^g \). In the case that \( X = \tilde{B}^0 \) we have \( C_X(\sigma) = B_0^g \) by 6.4(a), and then since \( E \) is in the normal subgroup \( B(\sigma) \) of \( C_X(\sigma) \), but \( E' \) is not, it follows that \( E \) and \( E' \) are representatives for the two orbits of \( B_0^g \) on \( \mathcal{E}_3(B_0^g, U) \). Since \( [E, E'] = 1 \), and \( B_0^0 \sigma = B(\sigma)E' \) by (a), these are also the orbits for \( B(\sigma) \), establishing (b). In particular, \( E' \) is fused to \( E^g \) in \( B_\sigma \), and so \( E' \) is not fused to \( E \) in \( H_\sigma \). In the case that \( X = \tilde{H} \) we get \( C_X(\sigma) = H_\sigma \) by 7.4(a), and this yields (c).

Recall that \( \sigma = \psi_n \) for some \( n \geq 0 \). Set \( q = p^{2n} \). Let \( \delta = \pm 1 \) with \( q \equiv \delta \mod 8 \). Then \( T_\sigma \) is homocyclic abelian of rank 3 and order \( (q - \delta)^3 \), and \( C_{H_\sigma}(E) = T_\sigma \langle w_0 \rangle \) by 4.3(d). On the other hand, we have seen that \( \langle E, w_0 \sigma \rangle \in \Sigma_0 \). As \( w_0 \) inverts \( T \), \( C_T(w_0 \sigma) \) is homocyclic abelian of rank 3 and order \( (q + \delta)^3 \). In particular, \( E \) is a Sylow 2-subgroup of \( C_T(w_0 \sigma) \). Therefore a Sylow 2-subgroup of \( C_H(w_0 \sigma) \cap C_H(E) \) is of order at most 16. Since \( [E, E'] = 1 \), (d) follows.

From 4.3, \( \text{Aut}_{TW}(E) = C_{\text{Aut}(E)}(Z) \), and hence \( \text{Aut}_{T}W(E) = \text{Aut}_H(E) \). Similarly, since \( W \leq \tilde{H}_{w_0 \sigma} \), we have

\[
\text{Aut}_{\tilde{H}_{w_0 \sigma}}(E) = C_{\text{Aut}(E)}(Z).
\]

Conjugating by the element \( g \) of \( \tilde{B}^0 \) with \( (w_0 \sigma)^g = \sigma \), we obtain

\[
\text{Aut}_{\tilde{H}_\sigma}(E^g) = C_{\text{Aut}(E^g)}(Z).
\]

Since \( [E, y] = [E', y] = U \), (e) holds. \( \square \)

**Lemma 7.9.** The following hold.

(a) \( \mathcal{E}_5(G) = \emptyset \).

(b) \( \mathcal{E}_4(X) = A^X \) for \( X \in \{G, H\} \), and \( \mathcal{E}_4(B^0) = A^{B^0} = A^K \).

(c) \( \text{Aut}_H(A) = C_{\text{Aut}(A)}(Z) \), and \( A = C_H(A) \).

**Proof.** Let \( Y \) be \( H \) or \( B^0 \), and let \( A' \) be an elementary abelian subgroup of \( Y \) of maximal order. Then \( Z(Y) \leq A' \), and after conjugation in \( Y \) we may assume, by 7.7, that \( E \leq A' \). Then \( A' \leq T \langle w_0 \rangle \), by 4.3(d), so that \( |A'| = 16 \). Since \( H \) contains a Sylow 2-subgroup of \( G \), by 7.5(b), we obtain (a). Every element of \( T \) is a square, so since \( w_0 \) acts on \( T \) as inversion, all elements in \( T w_0 \) are fused by \( T \). Thus, \( A' \) is fused to \( A \) via \( T \), and this yields (b).

By 4.3(d), \( C_H(A) = C_H(E) \cap C_H(h_0) = C_{T_0}(w_0) = A \). Since \( w_0 \in Z(W) \) and \( W \leq N_H(T) \leq N_H(E) \), we have \( W \leq N_H(A) \). Set \( T_1 = \{t \in T \mid t^4 = 1\} \). Then \( T_1, w_0 = E \), and so \( T_1 \leq N_H(A) \) and \( T_1 \) induces on \( A \) the subgroup \( X(E) \) of \( \text{Aut}(A) \) consisting of all transvections with axis \( E \).
Let $F$ be a hyperplane of $A$ containing $Z$. Then $F$ is conjugate to $E$ in $H$ by 7.7. It follows that $X(F) \leq \text{Aut}_H(A)$ for all such $F$. Since

$$C_{\text{Aut}(A)}(Z) = \langle X(F) \mid F \text{ a hyperplane of } A \text{ over } Z \rangle,$$

we obtain (c). \hfill \Box

**Lemma 7.10.** Set $M = N_G(A)$, and for any subgroup $X$ of $G$ set $M_X = M \cap X$. Then the following hold.

(a) The inclusion maps from $M_H$ and $M_K$ into $M$ induce an isomorphism $M \rightarrow M_H \ast_{M_B} M_K$, and $M$ is edge-transitive on the tree $\tilde{\Gamma}_A$.

(b) $M$ is contained in the subgroup $G_0$ of $G$ defined in 5.8(b). In particular $M \leq G_\sigma$.

(c) There is a surjective homomorphism $\phi_A: M \rightarrow M_0$, where $M_0$ is a nonsplit extension of an elementary abelian group of order 16 by $\text{GL}(4, 2)$, and such that $\ker(\phi_A) \cap M_H = \ker(\phi_A) \cap M_K = 1$.

(d) For any $\phi_A$ satisfying the conditions in (c), we have $C_G(A) = \ker(\phi_A) \times A$, and $\ker(\phi_A)$ acts freely on $\Gamma$.

**Proof.** The induced isomorphism of $M$ with $M_H \ast_{M_B} M_K$ is immediate from 7.3(b) and 7.9(b). The edge-transitivity of $M$ on $\tilde{\Gamma}_A$ is given by the final statement in 7.3, so that (a) holds.

By 5.8(a), there are maximal subgroups $H_0$ and $K_0$ of the group $\text{Co}_3$ such that $H_0$ may be regarded as a subgroup of $H_\sigma$, and $K_0$ as a subgroup of $K_\sigma$, in such a way that the resulting amalgam $\mathcal{A}_0 = (H_0 \geq B_0 \leq K_0)$ of subgroups of $G_\sigma$ is isomorphic to the corresponding amalgam of subgroups of $\text{Co}_3$. Set $M_0 = N_{\text{Co}_3}(A)$. Then $M_0/A \cong \text{GL}_4(2)$, and $M_0$ does not split over $A$, as one finds from the list of maximal subgroups of $\text{Co}_3$ in [Fin73]. Moreover, as seen in the proof of 5.8, we have $M_0 = \langle M_0 \cap H_0, M_0 \cap K_0 \rangle$, as subgroups of $\text{Co}_3$. As subgroups of $G$, we have $M_0 \cap H = M_H$ and $M_0 \cap K = M_K$, so it follows from (a) that $M \leq G_{\psi_0} \leq G_\sigma$. Moreover, (a) implies that $M_0$ is a homomorphic image of $M$, via a homomorphism $\phi_A$ whose restriction to each of $M_H$ and $M_K$ is the “identity” map. In particular, the restriction of $\phi_A$ to $M_H \cup M_K$ is faithful, and this yields (b) and (c).

From (c), it is immediate that $C_G(A) = \ker(\phi_A) \times A$. Since $M$ is edge-transitive on $\Gamma_A$, and since $\ker(\phi_A)$ intersects both $H$ and $K$ trivially, it follows that $\ker(\phi_A)$ acts freely on $\Gamma_A$. That is, every nonidentity element of $\ker(\phi_A)$ induces a hyperbolic isometry of $\Gamma_A$, and hence also a hyperbolic isometry of $\Gamma$, by 3.3. Thus, (d) holds. \hfill \Box

Set $R_0 = S_{\infty, \sigma}$ and $R_1 = N_{S_{\infty}}(R_0 \langle w_0 \rangle)$. Then $R_0$ has index 8 in $R_1$. Fix a set $X$ of coset representatives for $R_0$ in $R_1$, and recall that $g_0$ denotes the exponent
of $R_0$. Then

$$X = \{ x_e \mid e \in E \}$$

where $x_{e_0} = e$. Set

$$A_e = A^{x_e}, \quad e \in E,$$

and

$$\mathcal{E} = \mathcal{E}_4(R_0(w_0)).$$

All elements of $R_0w_0$ are fused by $R_1$, so $R_1$ is transitive on $\mathcal{E}$. Since $N_{R_1}(A) \leq R_0$, $R_0$ has $|R_1 : R_0| = 8$ orbits on $\mathcal{E}$ and $\{ A_e \mid e \in E \}$ is a set of representatives for those orbits.

Every subgroup of $T$ is $\sigma$-invariant, so each $\langle x_e \rangle$ is $\sigma$-invariant. Since $\sigma$ centralizes $x_2^e$ and does not centralize $x_e$, we obtain

$$(*) \quad \sigma^{x_e} = e\sigma \quad \text{and} \quad (e\sigma)^{x_e} = \sigma \quad \text{for all } e \in E.$$

**Lemma 7.11.** For each $e \in E$:

(a) $\text{Aut}_{H_\sigma}(A_e) = C_{\text{Aut}(A_e)}(\langle z, e \rangle)$,

(b) $\text{Aut}_{K_\sigma}(A_e) = N_{\text{Aut}(A_e)}(U) \cap C_{\text{Aut}(A_e)}(e)$, and

(c) $\text{Aut}_{G_\sigma}(A_e) = C_{\text{Aut}(A_e)}(e)$.

**Proof.** For any $e \in E$, set $\sigma_e = e\sigma$, regarded as an automorphism of $G$. By 7.10, $M := N_G(A) \leq G_\sigma$ and $\text{Aut}_G(A) = \text{Aut}(A)$, so that

$$\text{Aut}_{G_\sigma}(A) = C_{\text{Aut}(A)}(e) = C_{\text{Aut}(A)}(e).$$

Then as $A_e = A^{x_e}$, conjugating this equality by $x_e$ and appealing to $(*)$, we conclude that (c) holds.

Next, $\text{Aut}_H(A) = C_{\text{Aut}(A)}(Z)$ by 7.9(c), so that

$$\text{Aut}_{H_\sigma}(A) = C_{\text{Aut}(A)}(\langle z, e \rangle).$$

Since $x_e$ centralizes $\langle z, e \rangle$, conjugation by $x_e$ yields (a).

As $\text{Aut}_{G_\sigma}(A) = \text{Aut}_G(A) = \text{Aut}(A)$, as $y \in K$, and as $N_H(U) \leq K$, we conclude that $\text{Aut}_{K_\sigma}(A) = N_{\text{Aut}(A)}(U)$. Then

$$\text{Aut}_{K_\sigma}(A) = N_{\text{Aut}(A)}(U) \cap C_{\text{Aut}(A)}(e),$$

and conjugation by $x_e$ yields (b). \hfill $\Box$

**Lemma 7.12.** Let $u \in U - Z$ and let $e \in E - U$. Then

(a) $\{ A, A_z, A_u \}$ is a set of representatives for the orbits of $H_\sigma$ on $\mathcal{E}_4(H_\sigma)$, and $H_\sigma$ fuses $A_u$ and $A_e$.

(b) $\{ A, A_z, A_e \}$ is a set of representatives for the orbits of $K_\sigma$ on $\mathcal{E}_4(K_\sigma, U)$, and $K_\sigma$ fuses $A_z$ and $A_u$.

(c) $\{ A, A_u \}$ is a set of representatives for the orbits of $G_\sigma$ on $\mathcal{E}_4(G_\sigma)$. 
Proof. Since $S_\sigma \in \text{Syl}_2(G_\sigma)$, each $A \in \mathcal{C}_4(G_\sigma)$ is fused under $G_\sigma$ into $S_\sigma$, and then by 7.6, we may take $U \leq A \leq S_\sigma$. Then by parts (b) and (d) of 7.8, we may take $E \leq A$. Since $C_{S_\sigma}(E) = R_0(w_0)$, we are reduced to the problem of fusion via $H_\sigma, K_\sigma$, and $G_\sigma$ on $\mathcal{C}$.

Notice that $W$ has the three orbits $\{1\}, \{z\}$, and $E - Z$ on $E$, and that $S_\sigma(y)$ has the three orbits $\{1\}, U^\#$ and $E - U$ on $E$. Hence there are three orbits for $S_\sigma W$ on $\mathcal{C}$, with representatives $A, A_z$, and $A_u$, and three orbits for $S_\sigma(y)$ on $\mathcal{C}$, with representatives $A, A_z$, and $A_e$. Since $(S_\sigma, W, y)$ is transitive on $\{A_f \mid f \in A^\#\}$, it is now enough to show that there is no further fusion among these groups. As $\text{Aut}_{S_\sigma}(A) = \text{Aut}(A)$, $A$ is not fused to $A_z$ or to $A_u$ in $G_\sigma$, by 7.11(c). Since $A_z$ is not fused to $A_u$ in $H_\sigma$ by 7.11(a), and $A_z$ is not fused to $A_e$ in $K_\sigma$, by 7.11(b), the lemma is proved.

Lemma 7.13. Set $N = N_G(T_2)$, and for any subgroup $X$ of $G$ set $N_X = N \cap X$. Let $\phi_A$ be defined as in 7.10, and set $D = C_{\ker(\phi_A)}(T_2)$. Then the following hold:

(a) The inclusion maps from $N_H$ and $N_K$ into $N$ induce an isomorphism of $N$ with $N_H \ast N_B \ast N_K$. In particular, $N$ is generated by $N_H$ and $N_K$.

(b) $N$ is edge-transitive on the tree $\Gamma_{S_{\infty}}$.

(c) $N = N_G(T) = N_G(T\langle w_0 \rangle)$.

(d) $C_G(T_2) = C_G(S_{\infty}) = O(T)D \times S_{\infty}$, and $O(T)D$ is $N$-invariant.

(e) $\text{Aut}_G(S_{\infty}) \cong \text{GL}(3,2) \times \mathbb{Z}_2$.

(f) $\mathcal{B}_{C_S(E)}(N_H) = \mathcal{B}_{C_S(E)}(C_N(Z))$, and $\mathcal{B}_{C_{S_\sigma}(E)}(N_{H,\sigma}) = \mathcal{B}_{C_{S_\sigma}(E)}(C_{N_\sigma}(Z))$.

Proof. By 4.9(b) $S$ is a Sylow 2-subgroup of $B$, and by 4.9(c) $T_2$ is weakly closed in $S$ with respect to $G$. Therefore

$$T_2^B = \{T_2^g \mid T_2^g \leq B, g \in H \cup K\}.$$ 

Now (a) and (b) follow from 7.3(b).

Set $T^* = T\langle w_0 \rangle$. Since $N_H(T_2) \leq N_H(E) = TW$ by 4.3(d), we have $N_H = N_H(T) = N_H(T^*)$. Then $N_B = N_B(T) = N_B(T^*)$, and $N_K = N_B(y) = N_K(T) = N_K(T^*)$. It follows now from (a) that $N \leq N_G(T)$ and $N \leq N_G(T^*)$. Since the reverse inclusions are obvious, we obtain (c).

Since $(W, y) \leq N_G(A) \cap N_G(T_2)$, $D$ is $(W, y)$-invariant, and evidently so is $O(T)$. Let $x \in C_G(S_{\infty})$. Then (c) implies that $w_0^x \in T^* - T$, and since $T$ is transitive on $T w_0$, there exists $t \in T$ with $w_0^x = w_0$. Thus $x t \in C_G(A)$, so it follows from 7.10(d) that $x t = d a$ for some $d \in \ker(\phi_A)$ and some $a \in A$. Since $T_2 \leq N_G(A)$, we have $[T_2, d] \leq \ker(\phi_A)$, and since $\ker(\phi_A)$ acts freely on $\Gamma$ we have $T \cap \ker(\phi_A) = 1$. Let $s \in T_2$. Then

$$1 = [s, x t] = [s, d a] = [s, a][s, d]^a.$$
where \([s, a] \in T_2\) and where \([s, d]^a \in \ker(\phi_A)\). It follows that \([T_2, a] = [T_2, d] = 1\), and so \(a \in E\) and \(d \in D\). Now \(x = d a t^{-1} \in DT\), and thus \(C_G(S_\infty) \leq DT\). By 5.1 and (a), \([D, S_\infty] = 1\), so that \(C_G(S_\infty) = DT\). Since \(D \cap T = 1\) we have \(S_\infty \cap O(T)D = 1\), and thus \(O(T)D\) is a complement to \(S_\infty\) in \(C_G(S_\infty)\). This completes the proof of (d). Part (e) follows from (a) and Theorem 5.2.

Let \(P \leq S\) and let \(g \in N_G(P, S) \cap C_N(Z)\). Then \(g = n d\) for some \(n \in N_H\) and \(d \in D\), by (d). But \(C_S(E) = S_\infty(w_0)\), and \(w_0 \in A \leq C_G(D)\), so \(C_S(E)\) centralizes \(D\). Thus \(d\) centralizes \(P\), so \(c_g = c_n\) on \(P\), establishing (f). ☐

8. Centric subgroups and signalizer functors

We continue the hypotheses and the notation of Sections 4, 5, and 7. Thus \(A_{\sigma}\) is the amalgam \((H_\sigma \leftarrow B_\sigma \rightarrow K_\sigma)\), and \(A_0\) is the amalgam \((H_0 \leftarrow B_0 \rightarrow K_0)\) given by 5.8. As in 7.4, we regard \(A_0\) as a subamalgam of \(A_{\sigma}\), \(H_0\) and \(K_0\) as subgroups of \(G_{\sigma}\); and we set \(G_0 = \langle H_0, K_0 \rangle\) and \(S_0 = S \cap G_0\).

There is a fair amount of notation which we now need to establish, and which will remain fixed in the remainder. First, we set

\[
\mathcal{F} = \mathcal{F}_S(G), \quad \mathcal{F}_\sigma = \mathcal{F}_{S_\sigma}(G_\sigma), \quad \mathcal{F}_0 = \mathcal{F}_{S_0}(G_0).
\]

Let \(D\) be one of the groups \(G, G_\sigma, \) or \(G_0\), and let \(\mathfrak{D}\) be the fusion system \(\mathfrak{D} = \mathcal{F}_{S \cap D}(D)\). For any subgroup \(Y\) of \(G\) such that \(S \cap D\) is a Sylow 2-subgroup of \(Y \cap D\), we write \(Y_D\) for \(Y \cap D\), and \(\mathfrak{D}_Y\) for \(\mathcal{F}_{S_D}(Y_D)\).

For any subgroup \(P\) of \(S_D\), set

\[
Z_P = \langle z^D \cap Z(P) \rangle.
\]

Thus \(Z_P = \Omega_1(Z(P))\) if \(D \neq G_0\), by 7.6(b), and in any case we have \(Z_P^\# = z^D \cap Z(P)\), by 5.9(b). Although the definition of \(Z_P\) depends on \(D\), the reader may think of \(D\) as being fixed, so there need be no cause for confusion.

Denote by \(\Gamma_0\) the smallest \(G_0\)-invariant subtree of \(\Gamma\) which contains the edge \(\{H, K\}\). Recall that \(\Gamma_1 = \gamma_1 G\) denotes the subset \(G_1 \backslash G\) of vertices of \(\Gamma\), where \(G_1 = H\) and \(G_2 = K\). Write also \(\Gamma^D\) for the standard tree for \(D\). That is, \(\Gamma^D\) is the smallest \(D\)-invariant subtree of \(\Gamma\) containing the edge \(\{H, K\}\). Thus \(\Gamma^D\) is \(\Gamma\), \(\Gamma_\sigma\), or \(\Gamma_0\), for \(D\) equal to \(G, G_\sigma, \) or \(G_0\), respectively, and \(\Gamma \supseteq \Gamma_\sigma \supseteq \Gamma_0\).

**Lemma 8.1.** Let \(Y \in \{B, K, H\}\), and let \(P\) be a 2-subgroup of \(Y\). Then \(N_Y(P, S) \neq \varnothing\).

**Proof.** This follows from Remark 6.4. ☐

**Lemma 8.2.** Let \(P\) be a subgroup of \(S_D\), and let \(Y \in \{G, H, K\}\). Then

(a) \(P \in \mathfrak{D}_Y\) if and only if \(Z(P)\) contains every finite 2-subgroup of \(C_{Y_D}(P)\).

(b) If \(P \in \mathfrak{D}_Y\) then \(P\) contains every finite \(N_{Y_D}(P)\)-invariant 2-subgroup of \(Y_D\).
Proof. We first prove (a). By 2.1 and 7.5(b), we may assume \( D = G \), \( P \) is infinite, \( P \in \mathcal{F}_Y \), and there exists a 2-element \( x \) of \( C_Y(P) \) with \( x \notin P \). Set \( P^* = \langle P, x \rangle \).

Since \( C_\Gamma(P) \) is \( x \)-invariant, and \(|x|\) is finite, it follows from 3.2 that \( x \) fixes a vertex \( \delta \) of \( C_\Gamma(P) \), and we may take \( \delta = Y \) if \( Y \in \{ H, K \} \). Now 8.1 implies that \( P^* \) is contained in a conjugate of \( S \) in \( G_\delta \), and then \( x \in P \) since \( P \in \mathcal{F}_Y \). Thus (a) is established.

Now suppose that \( P \in \mathcal{D}_Y \), set \( N = N_{Y_D}(P) \), and let \( R \) be a finite, \( N \)-invariant 2-subgroup of \( Y_D \). Set \( R_0 = N_R(P) \). Then \( R_0 \leq N \), and so \( \text{Aut}_{R_0}P(P) \leq O_2(\text{Aut}_{Y_D}(P) = \text{Inn}(P) \) as \( P \in \mathcal{D}_Y \). Then \( R_0P = CR_0P(P)P \), so that \( R_0 \leq CR_0P(P)P \leq P \) by (a). Thus \( R_0 = R \cap P \). But the \( p \)-group \( P \) induces a finite \( p \)-group \( \bar{P} \) of automorphisms on \( R \), and \( C_{N_R}(R_0)/R_0(\bar{P}) \leq N_R(P)/R_0 = R_0/R_0 \). It follows that \( R \leq P \), proving (b).

For any \( P \in \mathcal{D} \) we have \( C_{S_D}(P) \leq Z(P) \), and thus \( Z \leq Z_P \). The following lemma derives most of the remaining information that we shall need, concerning \( \mathcal{D} \)-centric subgroups of \( S_D \), including everything that is needed for the construction of signalizer functors.

**Lemma 8.3.** Let \( P \leq S_D \).

(a) Suppose that \(|Z_P| = 2\) and that either \( P \in \mathcal{D} \) or \( P \in \mathcal{F}_H \). Then \( N_{D}(P, S) \leq H \), and if \( P \in \mathcal{F}_H \) then \( N_{G}(P, S) \leq H \) and \( P \in \mathcal{F}_P \).

(b) Suppose that \( P \in \mathcal{D} \) and \(|Z_P| = 4\). Then \( C_{D}(P) \leq H \), \( Z_P = U^h \), and \( N_{D}(P) \leq K^h \) for some \( h \in H_D \). If also \( Z_P = U \) and \( P \in \mathcal{F}_K \), then \( N_{G}(P) \leq K \) and \( P \in \mathcal{F}_P \).

(c) Suppose that \( P \in \mathcal{D} \) and \(|Z_P| = 8\). Then \( Z_P \in E^{H_D} \). If also \( \mathcal{D}(P) = \emptyset \) then \( P = S_\infty \cap D \) and \( P \) is not \( \mathcal{D} \)-radical.

(d) Suppose that \( \mathcal{D}(P) \neq \emptyset \). Then \( P \in \mathcal{F}_P \) and \( O(C_G(P)) = O(C_{D}(P)) = 1 \).

**Proof.** Set \( \Sigma = C_{\Gamma}(P) \) and \( \Sigma' = \Sigma \cap \Gamma^D \). Then \( \{ \gamma_1, \gamma_2 \} \) is an edge of \( \Sigma' \), since \( P \leq S_D \leq B_D \). Let \( \mathcal{P} \) be the set of paths \( \pi = (\gamma_1, \alpha, \beta) \) in \( \Sigma \) such that \( \beta \neq \gamma_1 \), and \( \mathcal{P}' \) the paths in \( \mathcal{P} \) contained in \( \Sigma' \). If \( \pi \in \mathcal{P} \) then \( Z_\alpha = ZZ_\beta \leq C_H(P) \) by 3.11(a), so 8.2 implies that \( Z_\alpha \leq Z_P \) if either \( P \in \mathcal{F}_H \) or \( \beta \in \Sigma' \). In particular, \(|Z_P| > 2 \) and \( Z_P = Z_\alpha \) if \(|Z_P| = 4 \).

Now assume the hypothesis of (a). Then \( \mathcal{P}' = \emptyset \) by the preceding paragraph, and thus \( \gamma_1 \) is the unique vertex in \( C_{\Gamma_1}(P) \). The same is then true for \( P^\gamma \), for any \( g \in N_D(P, S) \), and thus \( N_D(P, S) \leq H \). Similarly \( N_G(P, S) \leq H \) if \( P \in \mathcal{F}_H \). Since \( C_G(P) \leq N_G(P, S) \), (a) is proved.

Suppose next that \(|Z_P| = 4 \). Since \( Z \leq Z_P \), we then have \( Z_P = U^h \) for some \( h \in H_D \), by 7.6(a) and 5.9. We conclude from paragraph one that \( \alpha = \gamma_2^h \) for each \( \pi \in \mathcal{P}' \), and hence that \( \alpha \) is the unique vertex in \( \Gamma_2 \) which is in the interior
of $\Sigma'$, so that $N_D(P) \leq K^h$. Now $C_D(P) \leq C_{K^h}(Z) = C_K(Z)^h \leq H$, and the first part of (b) is established. Now suppose that $Z_P = U$ and that $P \in \mathcal{F}_K$. Then $C_H(P) \leq N_H(U) \leq K$, so that $P \in \mathcal{F}_H$ by 8.2. Then from paragraph one, $\alpha = \gamma_2$ for each $\pi \in \mathcal{P}$, and so $\gamma_2$ is the unique vertex in $\Gamma_2$ which is in the interior of $\Sigma$. Thus $N_G(P) \leq K$, and we have (b).

Suppose next that $|Z_P| = 8$. If $Z_P$ is not conjugate to $E$ in $H_D$, then $D = G_\sigma$ by 7.7 and 5.9, and $Z_P$ is conjugate in $H_D$ to the group $E'$ defined in 7.8(a). But in that case we conclude from 7.8(d) that $P$ does not contain every 2-element of $C_D(P)$, contrary to 8.2(a). Thus, $Z_P = E^h$ for some $h \in H_D$. Set $R = C_{S_D}(E)$ and $R_0 = S_D \cap T$. Then $R = R_0 \langle w_0 \rangle$ is a Sylow 2-subgroup of $C_{H_D}(E)$, by 4.3(d), and we may choose $h$ so that $P \leq R^h$. Suppose further that $\mathcal{E}_4(P) = \varnothing$. Since $R - R_0$ consists entirely of involutions, we then have $P \leq R^h_0$. Since $P$ contains all 2-elements in $C_{H_D}(P)$ it follows that $P = R^h_0$. Then $P = R_0$ by 4.9(c). Since $w_0$ inverts $R_0$, $O_2(\text{Aut}_D(P)) \neq \text{Inn}(P)$, and therefore (c) holds.

We now remove the hypothesis that $P$ is $\mathcal{D}$-centric, and assume that $\mathcal{E}_4(P) \neq \varnothing$. Let $F \in \mathcal{E}_4(P)$. Then $F \in A^G$ by 7.9(b), so that $F$ contains every element of $C_G(F)$ of finite order by 7.10(d). The same is then true of $P$, and so $O(C_D(P)) = 1$, and $P \in \mathcal{D}$ by 2.1. That is, (d) holds.

**Corollary 8.4.** Let $P \in \mathcal{D}$, and assume that $|Z_P| \leq 4$. Then $C_D(P) = C_{H_D}(P) = Z(P) \times O(C_D(P))$.

**Proof.** By 8.3, $C_D(P) \leq H$, while $O^2(C_D(P)) = Z(P)$ by 8.2(a). Let $X$ be a finite subgroup of $C_D(P)$ containing $Z(P)$. The Schur-Zassenhaus Theorem then yields $O^2(X) = O(X)$. Since $H$ is the union of an ascending chain of finite subgroups, the result follows.  

Recall from 7.10 that there is a surjective homomorphism 

$$\phi_A : N_G(A) \to M_0,$$

where $M_0$ is a nonsplit extension of $A$ by $\text{GL}(4, 2)$, and that $M_0$ may be viewed as a subgroup of $\widetilde{G}_0 := \text{Co}_3$. From 7.4(c), $G_0$ is the free amalgamated product $H_0 \ast_{B_0} K_0$. The universal property of $G_0$ with respect to $\mathcal{A}_0$ yields a homomorphism $\lambda : G_0 \to \widetilde{G}_0$. Then $\lambda \lambda = M_0$, and we may choose $\phi_A$ to be $\lambda |_M$. For any $A' \in A^G$, choose $g \in G$ with $A' = A^g$ and let $\phi_A' : N_G(A') \to M_0$ be the homomorphism given by $c_{g^{-1}} \phi_A$. Then $\ker(\phi_A)$ does not depend on the choice of the conjugating element $g$. Set 

$$X = \bigcup_{g \in G} \ker(\phi_A)^g,$$

and for any $P \in \mathcal{F}$ define a subset $\theta(P)$ of $C_G(P)$ by 

$$\theta(P) = CX(P)O(C_G(P)).$$
Thus, $\theta(P)$ is a union of cosets of the largest normal subgroup of odd order in $C_G(P)$. For $P \in \mathcal{F}_G^c$, set $X_\sigma = X \cap G_\sigma$, and

$$\theta_\sigma(P) = C_{X_\sigma}(P)O(C_G(P))_\sigma.$$ 

For $P \in \mathcal{F}_G^c$ set $X_0 = X \cap G_0$ and

$$\theta_0(P) = C_{X_0}(P).$$

Write $\theta_D$ for $\theta_0, \theta_\sigma$, when $D = G_0, G_\sigma$, or $G$, respectively.

Recall from 7.2 that, for any vertex $\gamma$ of $\Gamma$, the largest normal 2-subgroup of $G_\gamma$ is denoted $Z(\gamma)$.

**Lemma 8.5.** Let $x \in X$, and let $A' \in A^G$ with $x \in C_G(A')$. Denote by $\Lambda(x)$ the intersection of all the $x$-invariant subtrees of $\Gamma$, set $E(x) = (Z(\gamma) \mid \gamma \in \Lambda(x))$, and denote by $G_{\Lambda(x)}$ the vertex-wise stabilizer of $\Lambda(x)$ in $G$. Then the following hold.

(a) $\Gamma_x = \emptyset$, and $x$ induces a hyperbolic isometry on $\Gamma$.
(b) $E(x) \leq A'$, and $|A' : E(x)| \leq 2$.
(c) Let $\{\gamma, \delta\}$ be an edge of $\Lambda(x)$. Then $G_{\Lambda(x)} = C_{G_\gamma \cap G_\delta}(E(x))$.
(d) If $E(x) \neq A'$, and $\{H, K\}$ is an edge of $\Lambda(x)$, then $G_{\Lambda(x)}$ is a $B$-conjugate of $T(w_0)$.

**Proof.** By the definition of $X$ we have $x \in \ker(\phi_{A^*})$ for some $A^* \in A^G$. By 7.10(d), $x$ fixes no vertices and inverts no edges of $\Gamma$. That is, $x$ induces a hyperbolic isometry of $\Gamma$, in the sense of Section 3, and we have (a). Then 3.2 shows that $\Lambda(x)$ is a linear subtree of $\Gamma$, on which $x$ acts as a translation. Since $\Lambda(x)$ is contained in every $x$-invariant subtree of $\Gamma$, and since $x$ centralizes $A'$, we have $\Lambda(x) \subseteq \Gamma_{A'}$. Then 7.2 implies that $A'$ centralizes $E(x)$. Since $A'$ contains every 2-element in $C_G(A')$, by 7.10(d), we then have $E(x) \leq A'$.

Let $(\delta_0, \delta_1, \delta_2)$ be a geodesic in $\Lambda(x)$ with $\delta_1 \in \Gamma_1$. Then $\delta_2 = \delta_0 h$ for some $h \in G_{\delta_1} - G_{\delta_0}$, and so also $Z(\delta_0)^h = Z(\delta_2)$. Since $B = N_H(U)$, we have $Z(\delta_0) \neq Z(\delta_2)$, and since $Z(\delta_0)Z(\delta_2) \leq E(x)$ we conclude that $|E(x)| \geq 8$. This yields (b). As $G_{\Lambda(x)}$ centralizes $Z(\alpha)$ for each $\alpha \in \Gamma_1 \cap \Lambda(x)$, we get $G_{\Lambda(x)} \leq J := C_{G_{\gamma, \delta}}(E(x))$. Conversely $J \leq G_{\Lambda(x)}$ by 3.10, and this proves (c).

Suppose that $E(x) \neq A'$ and that $\{H, K\}$ is an edge of $\Lambda(x)$. Since $\mathcal{E}_4(B, U) = A^B$ by 7.9(b), we may assume that $A' = A$. By 7.9(c), all hyperplanes of $A$ containing $U$ are fused in $N_B(A)$, so we may assume also that $E(x) = E$. Then (c) yields $G_{\Lambda(x)} = C_B(E)$, and now (d) follows from 4.3(d).

Our aim is to show that $\theta_D$ is a signalizer functor on $\mathcal{D}$, as defined in 2.5. The key to this is the next result.

**Lemma 8.6.** The following hold.
(a) $C_X(A) = \ker(\phi_A)$.
(b) $C_X(S_\infty) = C_X(T_2) \subseteq C_{\ker(\phi_A)}(T_2)O(T)$.

**Proof.** Let $x \in C_X(A)$, and suppose that $x \notin \ker(\phi_A)$. Then $x \in \ker(\phi_{A'})$ for some $A' \in \mathcal{A}(G) - \{A\}$. Let $\Lambda(x)$ and $E(x)$ be as in 8.5. Then 8.5 yields $\Lambda(x) \subseteq \Gamma_A \cap \Gamma_{A'}$, $E(x) \leq A \cap A'$, and $|E(x)| = 8$. Since $N_G(A)$ is edge-transitive on $\tilde{\Gamma}_A$, we may assume that $\Lambda(x)$ contains the edge $\{H, K\}$. By 8.5(d) and as $T$ is transitive on $A^G \cap C_B(E)$, we may then assume also that $G_{\Lambda(x)} = T\langle w_0 \rangle$. Thus $T\langle w_0 \rangle$ is $x$-invariant, and then so is $T$. We then have

(1) $x \in N_G(T) \cap C_G(A) = N_G(T) \cap \ker(\phi_A)A = N_{\ker(\phi_A)}(T)A$.

Since $A = E\langle w_0 \rangle$, we have $N_T(A) = T_2$. Then

(2) $N_{\ker(\phi_A)}(T) \leq N_{\ker(\phi_A)}(T_2) = C_{\ker(\phi_A)}(T_2)$,

since $\ker(\phi_A)$ is invariant under $T_2$ and intersects $T_2$ trivially.

Since all involutions in $T\langle w_0 \rangle - E$ are fused by $T$, there exists $t \in T$ such that $(A')^t = A$. Then $x^t \in \ker(\phi_A)$. By (1) and (2), $x = ga$ for some $g \in C_{\ker(\phi_A)}(T_2)$ and $a \in A$. By 7.13(d) we have $g^t \in C_{\ker(\phi_A)}(T_2)O(T)$, so that $x^t a^t = g^t = k y$ for some $k \in \ker(\phi_A)$ and $y \in O(T)$. Thus $ya^t = k^{-1} x^t \in \ker(\phi_A)$, and then since $ya^t$ is of finite order and $\ker(\phi_A)$ is torsion-free, we have $ya^t = 1$. Therefore $y = a = 1$, and $x = g \in \ker(\phi_A)$, contrary to our choice of $x$. This contradiction proves (a).

Let $x \in C_X(S_\infty)$. Then there exists $A' \in A^G$ with $x \in C_{\ker(\phi_{A'})}(S_\infty)$. As in the proof of (a), $x$ induces a hyperbolic isometry of $\Gamma$, and $\Lambda(x)$ is contained in $\Gamma_{S_\infty} \cap \Gamma_{A'}$. Setting $F = \langle Z_{S_\infty} \mid \delta \in \Lambda(x) \rangle$, we find that $F \leq C_H(S_\infty)$, and since $N_G(S_\infty)$ is edge-transitive on $\Gamma_{S_\infty}$, $F^g \leq C_H(S_\infty)$ for some $g \in N_G(S_\infty)$. Since all involutions in $C_H(S_\infty)$ are contained in $E$, we conclude that $F = F^g = E$.

Since $\Lambda(x) \subseteq \Gamma_{A'}$ we get $[E, A'] = 1$, and then $E \leq A'$ since $A' = O^t(C_G(A'))$. Again by the edge-transitivity of $N_G(S_\infty)$ on $\Gamma_{S_\infty}$, we have $A'^g \leq C_H(E) = T^*$, for some $g \in N_G(S_\infty)$. Then $A'^gt = A$ for some $t \in T$, and so

$$C_{\ker(\phi_{A'})}(S_\infty) = C_{\ker(\phi_A)}(S_\infty)^{(gt)^{-1}} \leq C_{\ker(\phi_A)}(S_\infty)O(T),$$

by 7.13(d). This completes the proof of (b). \hfill \Box

**PROPOSITION 8.7.** $X_0 = \bigcup \ker(\phi_A)^{G_0}$.

**Proof.** Let $x \in X_0$. By definition, there exists $A' \in \mathcal{A}^G$ with $x \in \ker(\phi_{A'})$. Since $x \in G_0$, 8.5(a) implies that the axis $\Lambda(x)$ is contained in $G_{A'} \cap \Gamma_0$. Set $E' = Z(\Lambda(x))$. Then 8.5(b) yields $E' \leq A'$. We have $Z = Z(S_0)$, and $\langle Z^{K_0} \rangle = U$, so that $U \leq G_0$. Since $G_0 = H_0 \ast_{B_0} K_0$, $G_0$ is edge-transitive on $\Gamma_0$, and so $Z(\gamma) \leq G_0$ for any $\gamma \in \Gamma_0$. Thus, $E' \leq G_0$. 


Thus, we may assume:

Denote by $\mathcal{E}^*$ the set of elementary abelian 2-subgroups $F$ of $G_0$ such that $F = \{Z^{G_0} \cap F\}$, and by $\mathcal{E}_n^*$ the set of all $F \in \mathcal{E}^*$ with $|F| = 2^n$. By construction, $S_0 := S \cap B_0$ is a Sylow 2-subgroup of $G_0$, and then 5.9 implies that for any $n$, all members of $\mathcal{E}_n^*$ are fused in $G_0$. We have $A \in \mathcal{E}_0^*$ by 7.10, and evidently $E' \in \mathcal{E}^*$. If $E' = A'$ we conclude that $A' \in A^{G_0}$, and there is nothing more to show. Thus, we may assume henceforth that $E'$ is a proper subgroup of $A'$, and then 8.5(b) yields $|E'| = 8$. Moreover, $E'$ is conjugate to $E$ in $G_0$, since $G_0$ is transitive on $\mathcal{E}_3^*$.

Since $G_0$ is edge-transitive on $\Gamma_0$ we may assume that $\{H, K\}$ is an edge of $\Lambda(x)$. Then 8.5(d) implies that $G_{\Lambda(x)}$ is conjugate in $B$ to $T(w_0)$. Let $T'$ be the abelian subgroup of index 2 in $G_{\Lambda(x)}$. By 8.5(c), $[T', E'] = 1$, so that $E' \leq T'$ and $G_{\Lambda(x)} = C_B(E') = T'A'$. Let $R$ be the Sylow 2-subgroup of $T'$, and set $N = N_G(R)$. Then 7.13 yields

$$C_N(E') = (O(T')C_{\ker(\phi_{A'})}(R) \times R)A',$$

and $C_N(A') = C_{\ker(\phi_{A'})}(R)A'$. Since $x \in C_N(A')$, 8.6(a) now yields $x \in C_{\ker(\phi_{A'})}(R)$. Thus, $x$ centralizes the Sylow 2-subgroup $RA'$ of $T'A'$. Since $O(T')C_{\ker(\phi_{A'})}(R)$ contains no nontrivial 2-elements, it follows that:

1. $x$ centralizes every 2-subgroup of $C_N(E')$ that $x$ normalizes.

By 5.8(a), there is a surjective homomorphism $G_0 \to C_{\mathfrak{Co}_3}$ whose kernel intersects $B_0$ trivially. Then $C_{B_0}(E')$ is isomorphic to a subgroup of $C_{\mathfrak{Co}_3}(E')$. Since $E'$ is conjugate to $E$ in $G_0$, and since $C_{\mathfrak{Co}_3}(E)$ is of order $2^n$, we conclude that $C_{B_0}(E')$ is a 2-group. Since $C_{B_0}(E') = G_0 \cap G_{\Lambda(x)}$, (1) now yields:

2. $x$ centralizes $C_{B_0}(E')$.

If there exists $F \in \mathcal{E}_4^*$ with $F \leq C_{B_0}(E')$ then $x \in \ker(\phi_{F})$ by (2) and 8.6(a). Thus, we may assume:

3. $C_{B_0}(E')$ contains no member of $\mathcal{E}_4^*$.

Set $\Sigma = \Gamma_0 \cap \Gamma_{E'}$. Then $\overline{\Sigma}$ is a subtree of $\Gamma_0$ containing $\Lambda(x)$. For any $d > 0$ denote by $\Lambda^{(d)}$ the subtree of $\overline{\Sigma}$ induced on the set of vertices of $\overline{\Sigma}$ at distance at most $d$ from $\Lambda(x)$, and set $Z^{(d)} = Z(\Lambda^{(d)})$. Thus $E' = Z^0 \leq G_{\Lambda(x)}$, and we claim that $Z^{(d)} \leq G_{\Lambda(x)}$ for all $d \geq 0$. Suppose false, and let $d$ be minimal subject to the condition that, for some vertex $\gamma$ of $\overline{\Sigma}$ at distance $d$ from $\Lambda(x)$, we have $Z(\gamma) \not\leq G_{\Lambda(x)}$. Then $Z^{(d-1)} \leq G_{\Lambda(x)}$, and thus $Z^{(d-1)} \leq G_0 \cap B = B_0$. Now (3) yields $Z^{(d-1)} = E'$. Notice that $E'$ centralizes $Z(\gamma)$ since $E'$ fixes every edge in $\Gamma_0$ at every vertex of $\overline{\Sigma}$. Thus $Z(\gamma)$ centralizes $Z^{(d-1)}$. Arguing as in the proof of 8.5(c), it follows that $Z(\gamma)$ fixes every vertex of $\Lambda^{(d-1)}$, and thus $Z^{(d)} \leq G_{\Lambda(x)}$ as claimed.

It now follows that $Z(\overline{\Sigma}) \leq C_{B_0}(E')$, and then (3) yields $Z(\overline{\Sigma}) = E'$. On the other hand, since $E'$ is fused to $E$ in $G_0$ there exists $F \in \mathcal{E}_4^*$ with $E' \leq F$. 


Then $F \in A^{G_0}$. Set $\Theta = \tilde{\Gamma}_F \cap \Gamma_0$. Since $\Theta \subseteq \tilde{\Sigma}$ and $Z(\tilde{\Sigma}) = E'$, we conclude that $Z(\Theta) = E'$, and hence $E' \subseteq N_G(F)$. This is contrary to 7.10(c), and the proposition is thereby proved. 

**Theorem 8.8.** $\theta_D$ is a signalizer functor on $\mathcal{D}$.

**Proof.** Let $P \in \mathcal{D}^c$ and set $Y = X \cap D$. We first verify that $\theta_D(P)$ is a complementary subgroup to $Z(P)$ in $C_D(P)$. This is the case if $|Z_P| \leq 4$, by 8.4, so assume that $|Z_P| \geq 8$. Suppose that $\mathcal{E}_4(P) \neq \emptyset$ and choose $F \in \mathcal{E}_4(P)$. Then every subgroup of $P$ which contains $F$ is in $\mathcal{D}^c$, by 7.3(d). We have $F \in A^G$ by 7.9, so that 7.10(d) yields $C_G(F) = F \times \ker(\phi_F)$. Since $\ker(\phi_F) = C_X(F)$ by 8.6(a), $C_Y(F)$ is a subgroup of $C_D(F)$, and we get

$$C_D(F) = F \times C_Y(F).$$

In particular $O(C_D(F)) = 1$, so $C_Y(F) = \theta_D(F)$ by definition. Thus $\theta_D(F)$ is a complement to $F$ in $C_D(F)$.

Let $P_1$ be maximal in $P$ subject to the conditions: $F \leq P_1$ and $C_D(P_1) = Z(P_1) \times \theta_D(P_1)$. If $P_1 \neq P$ then $P_1 < P_2 := N_P(P_1)$ and $C_D(P_2) \leq C_D(P_1)$. Both $Z(P_1)$ and $Y$ are $P_2$-invariant, so that $P_2$ also acts on $C_Y(P_1) = \theta_D(P_1)$. Thus

$$C_D(P_2) = C_Z(P_1)(P_2) \times C_{\theta_D(P_1)}(P_2) = Z(P_2) \times \theta_D(P_2),$$

contrary to the maximality of $P_1$. Thus $P_1 = P$ and $\theta_D(P)$ is a complement to $Z(P)$ in $C_D(P)$.

On the other hand, suppose that $\mathcal{E}_4(P) = \emptyset$. Then $P = D \cap S_\infty$ by 8.3(c). Set $I = C_{\ker(\phi_A)}(T_2)$. Then $C_G(P) = T \times I$ by 7.13(d). Since $\ker(\phi_A) \leq G_0 \leq G_\sigma$, we have $C_D(P) = (D \cap T) \times I$. If $D = G_\sigma$ then $D \cap T = T_\sigma = P \times O(C_D(P))$, while if $D = G_0$ then $D \cap T = T_2$. Thus $\theta_D(P) = O(C_D(P))I$ is a complement to $P$ in $C_D(P)$ for any $P \in \mathcal{D}^c$.

Evidently $\theta_D(P^g) = \theta_D(P)^g$ for any $g \in N_D(P, S)$, so that by 2.6 it remains to show, for any $Q \in \mathcal{D}^c$ with $P \leq Q$, that $\Delta(Q) \leq \Delta(P)$. Since $C_Y(Q) \subseteq C_Y(P)$, this amounts to showing that $O(C_D(Q)) \leq O(C_D(P))$. But $O(C_D(P)) = O^2(C_D(P))$, by 7.4 if $|Z_P| \leq 4$, and by 7.6 if $P = D \cap S_\infty$, while in all other cases we have just seen that $O^2(C_D(P)) = 1$. Thus $O(C_D(Q)) \leq O(C_D(P))$ as required. 

### 9. Saturation and Theorem A

We continue the notation that was introduced at the start of Section 8.

**Proposition 9.1.** Let $D \in \{G_0, G_\sigma, G\}$. Then the fusion systems $\mathcal{D}_H$ and $\mathcal{D}_K$ are saturated.
Proof. If \( D = G_0 \) or \( G_\sigma \) then \( H_D \) and \( K_D \) are finite, and the result is then immediate from 1.6(c). In the case that \( D = G \) we appeal to Remark 6.4. □

Until further notice, we take \( D \in \{ G, G_\sigma \} \).

**Proposition 9.2.** We have \( \mathcal{U}_H = \mathcal{F}_{SD}(C_D(Z)) \) and \( \mathcal{U}_K = \mathcal{F}_{SD}(N_G(U)) \).

**Proof.** Supposing first that \( \mathcal{U}_H = \mathcal{F}_{SD}(C_D(Z)) \), we show that
\[
\mathcal{U}_K = \mathcal{F}_{SD}(N_G(U)),
\]
as follows. Let \( g \in N_G(U) \) and let \( P \leq S \) with \( Q := P^g \leq S \). Then \((PU)^g = P^gU \leq S \), and we may therefore take \( U \leq P \). Since \( \text{Aut}_K(U) = \text{Aut}(U) \), we may write \( g = g'k \) with \( g' \in C_G(U) \) and \( k \in K \). Set \( Q' = P^{g'} \). Then \( Q' \leq C_K(Z) = B \), so by 8.1 we may choose \( g' \) so that \( Q' \leq S \). By assumption, we then have \( c_{g'} = c_h \) for some \( h \in N_H(P, Q') \) with \( U^h = U \). Then \( h \in N_H(U) \leq K \), and \( c_{g'} = c_hc_k = c_{hk} \) where \( hk \in K \). We are therefore reduced to proving that \( \mathcal{U}_H = \mathcal{F}_{SD}(C_D(Z)) \).

Suppose that \( \mathcal{U}_H \neq \mathcal{F}_{SD}(C_D(Z)) \). By 7.5 and 3.14(c) there then exist \( P \leq S_D \), \( F \in \mathcal{E}_3(Z(P), U) \), and \( g \in N_{C_D(Z)}(P, S) \), such that

1. \( c_g \notin \text{Hom}_{H_D}(P, S) \),
2. \( U \leq F \cap F^g \), and
3. \( C_{B_D}(F)^g \leq B_D \) and \( C_{B_D}(F^g) \leq (B_D)^g \).

Suppose that both \( F \) and \( F^g \) are conjugate to \( E \) in \( B_D \), and choose elements \( b, b' \) in \( B_D \) with \( F^b = E = F^{gb'} \). From the first statement in (3), \( C_{B_D}(E)^g \leq C_{B_D}(E) \), and thus \( g' \) normalizes \( S_D \cap T \) by 4.9(c). We may then adjust \( b' \) in \( T \cap D \) so that \( g' \) normalizes \( (S_D \cap T)(w_0) = C_{SD}(E) \). Set \( N = N_D(C_{SD}(E)) \). It follows from 7.13(f) that \( N \cap H \) controls strong fusion in \( C_N(Z) \), and so there exists \( t \in N \cap H \) with \( c_t = c_{g'} \) on \( P^b \). Then \( c_g = c_{bt(b')^{-1}} \) on \( P \), contrary to (1).

We may therefore assume that either \( F \) or \( F^g \) is not conjugate to \( E \) in \( B_D \). Then 7.8 yields \( D = G_\sigma \), and every member of \( \mathcal{E}_3(B_D, U) \) is fused in \( B_D \) to \( E \) or to \( E' \), where \( E' \) is as defined in 7.8(b). Suppose that \( F \) is fused to \( E \) and that \( F^g \) is fused to \( E' \) in \( B_D \). A Sylow 2-subgroup of \( C_{B_\sigma}(F^g) \) is elementary abelian, by 7.8(d), whereas a Sylow 2-subgroup of \( C_{B_\sigma}(F) \) has exponent 4, contrary to (3). Similarly if \( F \) is conjugate to \( E' \) then \( F^g \) is not conjugate to \( E \). Thus we are reduced to the case where both \( F \) and \( F^g \) are fused to \( E' \) in \( B_\sigma \), and hence in \( B_\sigma^0 \), by 7.8(b).

Let \( b, b' \in B_\sigma^0 \) with \( F^b = E' = F^{gb'} \). Then \( b^{-1}gb' \in N_D(E') \). By 7.8(e) there exists \( h \in H_\sigma \) such that \( c_h = c_{b^{-1}gb'} \) as elements of \( \text{Aut}_D(E') \). Then \( c_g = c_{bh(b')^{-1}} \) in \( N_D(F, F^g) \), and so \( P \neq F \) by (1). Since \( P \leq C_S(F) \) and \( EE' \) is a Sylow 2-subgroup of \( C_{H_\sigma}(E') \), we conclude that \( P^b, P^{gb'} \), and \( A' = EE' \) are Sylow in \( C_{H_\sigma}(E') \). Thus there exist \( a, a' \in H_\sigma \) with \( Pa = A' = P^{ga'} \). Then
\[ a^{-1}g_{a'} \in N_{D}(A'). \] Observe next that \( A' \) is in the set \( \mathcal{E} \) defined just prior to 7.11, so that 7.11 implies there exists \( h \in H_{\sigma} \) with \( c_{h} = c_{a^{-1}g_{a'}} \) as elements of \( \text{Aut}_{D}(A') \). Then \( c_{g} = c_{ah(a')^{-1}} \) on \( P \), contrary to (1). \( \square \)

**Lemma 9.3.** Let \( P \leq S_{D} \), and assume that there exists no \( N_{D}(P) \)-invariant subgroup \( X \) of \( Z_{P} \) with \( |X| = 2 \) or 4. Then one of the following holds.

(a) \( P \in E^{D} \cup A^{D} \).

(b) \( D = G_{\sigma} \) and \( P \) is \( D \)-conjugate to the group \( E' \) defined in 7.8(b).

(c) \( P \leq C_{S}(E) \), and either \( S_{\infty} \leq P \) or \( P \cap T \) is homocyclic of rank 3 and exponent at least 4.

**Proof.** By hypothesis, \(|Z_{P}| \geq 8\). Suppose first that \(|Z_{P}| > 8\). Then \( P \in A^{G} \) by 7.9, and if \( P \notin A^{D} \) then 7.11(c) and 7.12 show that there is an \( N_{D}(P) \)-invariant subgroup of \( P \) of order 2. Thus \( P \in A^{D} \), and (a) holds in this case. Also, if \( P \in E^{G} - E^{D} \) then 7.8 yields (b). We may therefore assume that \( P \) is not elementary abelian. Then \(|Z_{P}| = 8\), and \( Z \leq Z_{P} \).

By 7.7 we have \((Z_{P})^{h} = E\) for some \( h \in H \), and by 8.1 we may choose \( h \) so that \( P^{h} \leq S \). Let \( P_{0} \) be the group generated by the noninvolutions in \( P \). Since \( C_{S}(E) - S_{\infty} \) consists entirely of involutions, \( P_{0} \) is a characteristic abelian subgroup of \( P \), of index at most 2, containing \( Z_{P} \). Since \( P \) has no characteristic subgroups of order 2 or 4, it follows that either \( P = S_{\infty} \) or that \( P_{0} \) is homocyclic. Since \( P \notin \mathcal{E}(S) \), \( P \) contains a conjugate of \( T_{2} \), and then 4.9(c) implies that \( P_{0} = T_{n} \) for some \( n \geq 2 \) or \( P_{0} = S_{\infty} \). Thus, (c) holds. \( \square \)

**Lemma 9.4.** Let \( P \leq S_{D} \) and let \( X \) be a nonidentity \( N_{D}(P) \)-invariant subgroup of \( Z_{P} \) of minimal order. Then \( N_{D}(P) \) acts transitively on \( X^{\#} \), and the following hold.

(a) There exists an element \( f = hkh' \) of \( D \), with \( h, h' \in H_{D} \) and \( k \in K_{D} \), such that \( N_{S_{P}}(P)^{f} \leq S_{D} \) and such that either \( X^{f} \in \{Z, U, E, A\} \), or \( D = G_{\sigma} \) and \( X^{f} = E' \), where \( E' \) is as defined in 7.8(b).

(b) If \( P \) is fully normalized in \( \mathcal{Q} \) then so is \( P^{f} \), for any \( f \) as in (a).

**Proof.** We first show that \( N_{D}(P) \) acts transitively on \( X^{\#} \). This is trivial if \(|X| = 2\), and is immediate from the minimality of \( X \) if \(|X| = 4\). Suppose that \(|X| = 8\). If \( P \in \mathcal{E}_{4}(S_{D}) \) then 7.12 and 7.11(c) show that \( \text{Aut}_{D}(P) \) leaves no maximal subgroup of \( P \) invariant, while if \( X = P \in \mathcal{E}_{3}(D) \) then 7.7 and 7.8(e) show that \( N_{D}(P) \) acts transitively on \( P^{\#} \). Thus, we may assume that \( P \) is not elementary abelian, and then 9.3 yields \( P \leq C_{S}(E) \), and either \( P = S_{\infty} \) or \( P \cap T \) is homocyclic of rank 3 and exponent at least 4. Then \( X = E \), and 7.13 implies that \( N_{D}(P) \) induces the full automorphism group of \( X \).
We next prove (a). Let \( x \neq y \in C_X(N_{SD}(P)) \) and suppose that \( x \neq z \). By 3.13(b) there exist elements \( h \in H \) and \( k \in K \) such that \( x^{hk} = z \) and such that \( N_{SD}(P)^{hk} \leq SD \). Thus, we may assume that \( z \in X \), after replacing \( P \) by a suitable conjugate. Then, by 7.6 through 7.9, there exists \( h' \in H_D \) with \( X^{h'} \in \{Z, U, E, A\} \), or else \( D = G_0 \) and there exists \( h' \in H_D \) with \( X^{h'} \in \{E', A\} \). If \( X^{h'} = A_z \) then 7.11(c) contradicts the minimality of \( X \), so that this case does not arise in our context.

Set \( Y = X^{h'} \). In order to complete the proof of (a) it suffices to show that, in each case, every 2-subgroup of \( N_{HD}(Y) \) is fused into \( N_{SD}(Y) \) in \( N_{HD}(Y) \). But in each case we have \( N_{SD}(Y) \in \text{Syl}_2(N_{HD}(Y)) \), and so the required fusion follows from Sylow’s Theorem, or from 6.3 and 6.4 when \( D = G \) and \( N_S(Y) \) is infinite. Thus, (a) holds.

Now suppose that \( P \) is fully normalized in \( \mathcal{D} \), let \( f \) be given as in (a), and set \( Q = P^f \). Let \( R = Q^g \) be a \( \mathcal{D} \)-conjugate of \( Q \) contained in \( S \). Then \( R = P^{fg} \), and since \( P \) is fully normalized there exists \( d \in G \) such that \( R^d = P \) and \( N_S(R)^d \leq N_S(P) \). Also, as \( P \) is fully normalized there exists \( d' \in G \) with \( Q^{d'} = P \) and with \( N_S(Q)^{d'} \leq N_S(P) \). Since \( N_S(P)^f \leq N_S(Q) \) we conclude that \( N_S(Q)^{d'} = N_S(P) \), and \( N_S(R)^{dd'd'^{-1}} \leq N_S(Q) \). Since \( R^{dd'd'^{-1}} = P \), we have thus shown that \( Q \) is fully normalized, proving (b).

**Lemma 9.5.** Let \( P \leq SD \) be fully normalized in \( \mathcal{D}_H \). Assume that there exists a minimal, nonidentity, \( N_D(P) \)-invariant subgroup \( X \) of \( Z_P \) with \( Z \leq X \). Then \( P \) is fully normalized in \( \mathcal{D} \).

**Proof.** Let \( g \in N_D(P, SD) \) and set \( Q = P^g \) and \( Y = X^g \). There then exists \( y \in Y^g \) such that \([y, N_{SD}(Q)] = 1\), and since \( N_D(Q) \) acts transitively on \( Y^g \), by 9.4, we may assume that \( y = z^g \). By 3.13(b) there exist elements \( h \in H_D \) and \( k \in K_D \) such that \( y^{hk} = z \) and such that \( N_{SD}(P)^{hk} \leq SD \). Set \( d = hk \) and \( R = Q^d \). As \( gd \in C_D(Z) \) it follows from 9.1 and 9.2 that there exists \( h' \in H_D \) with \( R^{h'} = P \) and with \( N_S(R)^{h'} \leq N_S(P) \). Then \( Q^{dh'} = P \) and \( N_S(Q)^{dh'} \leq N_S(P) \). This shows that \( P \) is fully normalized in \( \mathcal{D} \).

**Lemma 9.6.** Let \( P \) and \( Q \) be subgroups of \( S \), and let \( x, y \in G \) such that \( P^x \leq Q \) and \( Q^y \leq P \). Then \( P^x = Q \) and \( Q^y = P \).

**Proof.** The map \( c_{xy} : P \to P \) is injective, so by 6.4 and 6.2(7), \( c_{xy} \) is an isomorphism. Thus \( P = P^{xy} = Q^y \), and similarly \( P^x = Q \).

**Theorem 9.7.** \( \mathcal{D} \) is saturated.

**Proof.** Let \( 1 \neq P \leq SD \) and let \( X \) be a minimal, nonidentity, \( N_D(P) \)-invariant subgroup of \( Z_P \). By 9.4 there is \( g \in D \) with \( N_{SD}(P)^g \leq SD \) and with \( Z \leq X^g \). Adjusting in \( H \) and appealing to 9.1, we may assume that \( Q := P^g \) is fully normalized in \( \mathcal{D}_H \). Then \( Q \) is fully normalized in \( \mathcal{D} \), by 9.5, and thus \( \mathcal{D} \) satisfies the
saturation condition (I) in 1.5. It remains to verify the two parts of condition (II). Thus we may take \( P \) to be fully normalized in \( \mathcal{D} \).

By 9.4 there exists a fully normalized conjugate \( P' = P^y \) of \( P \) such that \( X' = X^y \) is in \( \{Z, U, E, E', A\} \). Then \( y \) can be chosen so that \( N_S(P)^y \leq N_S(P') \), and similarly (since \( P \) is fully normalized) there exists \( y' \in N_D(P', P) \) such that \( N_S(P')^{y'} \leq N_S(P) \). By 9.6, \( N_S(P)^y = N_S(P') \), so that if \( P' \) satisfies saturation condition (II) then so does \( P \). Thus, we may assume that \( X \in \{Z, U, E, E', A\} \) and, in particular, that \( Z \leq X \). Also, since \( \mathcal{D}_H \) is saturated there exists a \( \mathcal{D}_H \)-conjugate \( P'' \) of \( P \) which is fully normalized in \( \mathcal{D}_H \), and hence in \( \mathcal{D} \) by 9.4(b); the preceding argument then shows that \( N_S(P) \) and \( N_S(P'') \) are \( \mathcal{D} \)-isomorphic. We may therefore assume that \( P \) is fully normalized in \( \mathcal{D}_H \). Then \( P \) is fully normalized in \( \mathcal{F}_S(D(C_D(Z))) \), by 9.2.

If \( X \in \{Z, U\} \) then fusion in \( N_D(P) \) is controlled by fusion in \( N_{HD}(P) \) or in \( N_{KD}(P) \), by 9.2, and since \( \mathcal{D}_H \) and \( \mathcal{D}_K \) are saturated, there is nothing more to prove in this case. Thus, we may assume that \( |X| \geq 8 \). By 9.3 we then have \( P = X \in \{E, E', A\} \), or else \( P \leq C_{SD}(E) \) with \( P \cap T = S_\infty \) or with \( P \cap T \) homocyclic and of exponent at least 4. In the first case, where \( P \) is elementary abelian, we have \( \text{Aut}_{SD}(P) \in \text{Syl}_2(\text{Aut}_{D}(P)) \) by 7.8 and 7.10. In the second case we obtain \( \text{Aut}_{SD}(P) \in \text{Syl}_2(\text{Aut}_{D}(P)) \) from 7.13. This establishes the saturation condition (IIA).

Now let \( \alpha = c_g \in \text{Aut}_G(P) \), where \( g \in N_D(P) \). Set \( Z' = Zg^{-1} \). By definition, we have \( N_{\alpha}^g \leq C_D(P)S_D \). Thus \( N_{\alpha} \leq C_{SD}(Z') \). By 3.13(b) there exists \( a \in N_D(N_{\alpha}, S_D) \) with \( (Z')^a = Z \). Set \( Q = P^a \) and \( b = a^{-1}g \). Then \( b \in C_D(Z) \) and \( Q^b = P \). As \( P \) is fully normalized in \( \mathcal{F}_S(D(C_D(Z))) \), it follows from the “standard” axioms for saturation in [BLO03] – equivalent to those in 1.5 – that \( \beta := c_b \) extends to a \( \mathcal{D} \)-morphism \( \bar{c}_b \) of \( N_\beta \) into \( S_D \). That is, there exists \( d \in N_D(N_\beta, S_D) \) such that \( d^{-1} \) centralizes \( Q \). Set \( \bar{g} = ad \). Then

\[
g^{-1}\bar{g} = g^{-1}ad = g^{-1}gb^{-1}d = b^{-1}d,
\]

and so \( g^{-1}\bar{g} \) centralizes \( P \). But also \( (N_{\alpha})^a \leq N_\beta \) since

\[
(N_{\alpha})^{ab} = (N_{\alpha})^{a^{-1}ag} = (N_{\alpha})^g,
\]

and \( \text{Aut}_{N_\alpha}(P)^{c_g} \leq \text{Aut}_{S_D}(P) \) by definition of \( N_\alpha \). Now

\[
(N_{\alpha})^{\bar{g}} = (N_{\alpha})^{ad} \leq (N_\beta)^g \leq S_D,
\]

and thus \( c_\bar{g} \) is an extension of \( c_g \) from \( P \) to a \( \mathcal{D} \)-morphism of \( N_\alpha \) into \( S_D \). This shows that \( \mathcal{D} \) satisfies the saturation condition (IIB), and the proof is thereby complete.

In 2.6 it was shown that if \( \mathcal{D} \) is saturated then a \( \mathcal{D} \)-signalizer functor \( \theta_D \) determines an associated centric linking system and an associated \( p \)-local group.
Theorem A.10 of [BLO03]. A modification of this proof when \( \text{Aut} \) (a) and (d) of 9.7, the amalgam \( \text{Theorem } A \) have been proved. Part (5) is given by 7.9, 7.10, and the definition of \( G \), respectively. Theorems 8.8 and 9.7 therefore have the following immediate corollary:

**Corollary 9.8.** \( \mathcal{G} \) is a 2-local finite group, and \( \mathcal{G} \) is a 2-local group. \( \Box \)

**Theorem 9.9.** For any fixed \( p \), \( p \equiv 3 \) or \( 5 \) mod 8, and for any integer of the form \( q = p^{2n} \), there is a unique \( \sigma = \psi_n \) such that the fusion system \( \mathcal{F}_{\text{Sol}}(q) \) constructed in [L002] is isomorphic to \( \mathcal{F}_{\sigma} \). Moreover the associated 2-local finite group constructed in [L002] is isomorphic to \( \mathcal{G}_{\sigma} \).

**Proof.** The fusion system \( \mathcal{E} = \mathcal{F}_{\text{Sol}}(q) \) is constructed as

\[
\langle \mathcal{F}_{\sigma}(H_q), \mathcal{F}_{\sigma}(U_q)(K_q) \rangle,
\]

where \( H_q = H_{\sigma} \) is Spin\( \gamma(q) \), \( S_q = S_{\sigma} \) is a Sylow 2-group of \( H_q \), \( S_{U,q} = C_{S_{\sigma}}(U) \), and \( K_q = K_{\sigma} \), subject to a choice of embedding \( \alpha \) of \( B_q = N_{H_q}(U) \) in \( K_q \) such that, for \( I = \langle N_{H_q}(S_{U,q}), N_{K_q}(S_{U,q}) \rangle \), \( \text{Aut}(S_{U,q}) \cong GL(3,2) \times C_2 \). By parts (a) and (d) of 9.7, the amalgam \( \mathcal{A}_{\alpha} = (H \xleftarrow{\iota} B \xrightarrow{\alpha^*} K) \) is determined up to isomorphism by these properties, so \( A_{\alpha} = A_{\lambda,\sigma} = (H_{\sigma} > B_{\sigma} < K_{\sigma}) \). Thus \( \mathcal{E} = \langle \mathcal{F}_{\sigma}(H_{\sigma}), \mathcal{F}_{C_{S_{\sigma}}(U)}(K_{\sigma}) \rangle \). By 1.10, \( \mathcal{F}_{\sigma}(K_{\sigma}) = \langle \mathcal{F}_{\sigma}(S_{\sigma}), \mathcal{F}_{C_{S_{\sigma}}(U)}(K_{\sigma}) \rangle \), so \( \mathcal{E} = \langle \mathcal{F}_{\sigma}(S_{\sigma}), \mathcal{F}_{\sigma}(K_{\sigma}) \rangle \), and then \( \mathcal{E} = \mathcal{F}_{\sigma}(G_{\sigma}) = \mathcal{F}_{\sigma} \) by 3.7, completing the proof of the first part of the theorem. The remainder of the theorem follows from the uniqueness of the 2-local finite group associated to \( \mathcal{E} \), proved in [L002]. \( \Box \)

To sum up: parts (1) and (2) of Theorem A are given by the construction of \( G \) in Section 5, while parts (3) and (4) of Theorem A are given by the preceding theorem. Part (5) is given by 7.9, 7.10, and the definition of \( X \). Thus all parts of Theorem A have been proved.

**Lemma 9.10.** \( \mathcal{F} \) is frc-generated. That is,

\[
\mathcal{F} = \langle A_{\mathcal{F}}(P) \mid P \in \mathcal{F}_{\text{frc}} \rangle,
\]

where \( \mathcal{F}_{\text{frc}} \) is the set of fully normalized radical centric subgroups in \( \mathcal{F} \), and \( A_{\mathcal{F}}(P) \) is the fusion system on \( P \) whose morphisms are the restrictions of members of \( \text{Aut}_{\mathcal{F}}(P) \) to subgroups of \( P \).

**Proof.** This generalization of Alperin’s Fusion Theorem is well known for saturated fusion systems on finite \( p \)-groups \( S \); a short proof appears in Theorem A.10 of [BLO03]. A modification of this proof when \( S \) is a discrete \( p \)-toral group appears in Theorem 3.6 in [BLO05]. We sketch that proof in our special case, where things are much easier.

Pick \( P, P' \leq S \) and an \( \mathcal{F} \)-isomorphism \( \varphi: P \to P' \) such that \( \varphi \) is not in \( \mathcal{A} = \langle A_{\mathcal{F}}(P) \mid P \in \mathcal{F}_{\text{frc}} \rangle \). Then \( \varphi = c_g \) for some \( g \in G \). As in the proof of Theorem A.10 in [BLO03], using the fact that \( \mathcal{F} \) is saturated, we may assume
$P'$ is fully normalized in $\mathcal{F}$. Pick $P$ so that $|T_2 \cap P|$ is maximal. If $T_2 \nsubseteq P$ then $T_2 \cap P < N_{T_2 \cap P}(P)$, and as in the proof of A.10, after we replace $P$ by $N_{T_2}(P)P$, the maximality of $|T_2 \cap P|$ supplies a contradiction.

As $T_2 \leq P$, 4.9(c) says that $g \in N_G(T_2)$. As $S_{\infty} \leq N_G(T_2)$, replacing $P, P'$ by $PS_{\infty}, P'S_{\infty}$, we may assume $S_{\infty} \leq P$. In particular $P$ is centric. Finally choosing $|P : S_{\infty}|$ maximal, and arguing as in the proof of Theorem A.10 in [BLO03], we first reduce to the case where $g \in N_G(P)$, and then show $P$ is radical, contradicting the assumption that $c_g \notin \mathcal{A}$.

10. Radical centric subgroups

In this section, we determine the members of $\mathcal{D}^{rc}$, and make the necessary preparations for obtaining embeddings among the 2-local groups constructed in the preceding section.

**Lemma 10.1.** Let $Y \in \{H, K\}$, and let $P \leq S$ with $N_D(P) \leq Y$. Then the following are equivalent.

(a) $P \in \mathcal{D}^{rc}$.

(b) $P \in \mathcal{D}^c_Y$.

(c) $P$ contains every 2-element in $C_Y(P)$, and $O_2(\text{Out}_Y(P)) = 1$.

**Proof.** As $C_D(P) \leq Y$, 8.2(a) says that $P \in \mathcal{D}^c$ if and only if $P$ contains every 2-element in $C_{Y_D}(P)$, and that this holds if and only if $P \in \mathcal{D}^c_Y$. Thus we may assume $P \in \mathcal{D}^c$. As $N_D(P) \leq Y$, we have

$$O_2(\text{Aut}_D(P)) = O_2(\text{Aut}_{Y_D}(P)).$$

Thus $P \in \mathcal{D}^{rc}$ if and only if $\text{Inn}_D(P) = O_2(\text{Aut}_D(P))$ if and only if $\text{Inn}_D(P) = O_2(\text{Aut}_{Y_D}(P))$ if and only if $P \in \mathcal{D}^{rc}_Y$ if and only if $O_2(\text{Out}_{Y_D}(P)) = 1$, completing the proof. \hfill \Box

Recall from 4.5 that $B^0$ is the commuting product of “components” $L_1, L_2,$ and $L_3$, where each $L_i$ is isomorphic to $\text{SL}(2, F)$.

**Lemma 10.2.** Let $P \in \mathcal{D}^{rc}_K$, set $N = N_{K_D}(P)$, $R = S_D$, $J_i = L_i \cap D$, and $J = J_1J_2J_3$. For any subgroup $X$ of $K_D$, and any $i$ with $1 \leq i \leq 3$, denote by $X_i$ the projection of $X \cap J$ in $J_i$. Denote by $\mathcal{Q}_i$ the set of subgroups of $R_i$ that are isomorphic to the quaternion group $Q_8$. Then the following hold:

(a) $C_K(P) \leq P$.

(b) $Z_P \in \{Z, U\}$.

(c) $P \cap J = P_1P_2P_3$, and for each $i$ with $1 \leq i \leq 3$, either $P_i = R_i$ or $P_i \in \mathcal{Q}_i$.

(d) Either

(i) $P \in \{C_R(U), R\}$,
(ii) $P = P \cap J$ and $P_i \in \mathcal{G}_i$ for at least two indices $i$, or

(iii) $P = (P \cap J)(s)$ for some $s \in P - \text{C}_P(U)$, and either $P_3 \in \mathcal{G}_3$ or $P_i \in \mathcal{G}_i$ for both $i = 1$ and $2$. Moreover, if $P_3 \not\subseteq R_3$ then $O^2(N \cap J_3) \neq 1$.

(e) $P \in \mathcal{F}_K^c$, and every $N$-invariant 2-subgroup of $K$ is contained in $S$.

(f) If $U'$ is a fours group in $K$ such that $[P, U'] \leq Z$, then $U' = U$.

Conversely, any subgroup $P$ of $S$ which satisfies (c) and (d) is in $\mathcal{G}_K^c$.

**Proof.** Set $\mathcal{G} = \{J_i \mid 1 \leq i \leq 3\}$ and set $P_J = \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3$. Since $\mathcal{G}$ is $K$$_D$-invariant, $P_J$ is $N$-invariant, and then $P_J \leq P$ by 8.2(b). Thus $P_1 \leq P$ for all $i$, and $P_J = P \cap J$.

Suppose that $P_i \leq U$ for some $i$. Then $P^* := \langle P_i^P \rangle \leq U$, and $R^* := \langle R_i^P \rangle$ is a $P$-invariant 2-group which properly contains $P^*$. Pick $r \in N_{R^*}(P) - P$ with $r^2 \in U$. Then $r$ centralizes the chain $P \geq U \geq 1$, and since $U \leq N$ we conclude from 2.2 that $r \in P$. This contradiction shows that no $P_i$ is contained in $U$.

Set $N_i^* = N_{J_i}(P_i)$. Then $N^* = N_1^* N_2^* N_3^*$ is $N$-invariant, so by 8.2(b), $O_2(N_i^*) = P_i$. Let $\mathcal{G}$ be the class of 2-groups each of whose finite subgroups is cyclic. Suppose that $P_i \notin \mathcal{G}_i$. There is then a unique maximal subgroup $X$ of $P_i$ in $\mathcal{G}$ of order at least 8. Since $|X| \geq 8$ there is also a unique maximal 2-subgroup $V_i$ of $J_i$ in $\mathcal{G}$ containing $X$. Then $V_i$ is $N_i^*$-invariant, and 8.2(b) yields $V_i \leq P_i$. Then also $\text{Aut}_{R_i}(P) \leq O_2(\text{Aut}_{K_D}(P))$, and $P_i = R_i$. Thus, (c) is proved. It follows from (c) that $C_{L_i}(P_i) \leq P_i$, so $C_{B^0}(P_J) \leq P_J$. Since any element of $K - B^0$ acts nontrivially on $U$, and since $U \leq P_J$, we obtain (a) and (b).

Suppose next that no $P_i$ is a quaternion group. Then for all $i$ we have $P_i = R_i \notin \mathcal{G}_i$. If $D = G$ then $P_J = C_S(U)$ is of index 2 in $P$, and (d)(i) holds. On the other hand, suppose that $D \neq G$. Since $R_i \notin \mathcal{G}_i$ if $D = G_0$, we conclude that $D = G_\sigma$. By 7.8(a) there exists an element $x$ of $S_D$ such that $x$ induces a diagonal outer automorphism on each $J_i$. Here $N \cap J = P J$ and $P J \langle x \rangle \leq S_D$, so as $P$ is $\mathcal{G}_K$-radical it follows that $x \in P$. Then $P J \langle x \rangle = C_R(U) \leq P$, and again (d)(i) holds.

If $C_P(U) \neq P_J$ then there exists $x \in C_P(U) - J$. As in the preceding paragraph, $D \neq G$, and for all $i$ either $J_i \cong \text{SL}(2, \mathbb{F}_\sigma)$, or $D = G_0$ and $J_i \in \mathcal{G}_i$. As $[N_i, x] \leq P$ for all $i$, $O^2(N_i)$ is not isomorphic to $\text{SL}(2, 3)$, and $P_i = R_i$ for all $i$. Thus $P \geq (R \cap J) \langle x \rangle = C_R(U)$. Once again, we obtain (d)(i).

Now assume that (d)(i) does not hold. We conclude from the discussion above that $P_i \in \mathcal{G}_i$ for some $i$ and that $P_J = C_P(U)$. Suppose next that $P = P_J$, and that neither $P_1$ nor $P_2$ is a quaternion group. Recall that $B = B^0(s)$ where $s \in S_0$, $s$ centralizes $L_3$, and $s$ interchanges $L_1$ and $L_2$. One may now check that $N = (N \cap J)^{(s)}$ and that $O^2(N \cap J) \leq J_3$, whence $P \langle s \rangle \leq N$. As $P$ is $\mathcal{G}_K$-radical, $s \in P$, contrary to the assumption that $P = P_J$. Thus, either $P_1$ or $P_2$ is a quaternion group. Since $K/B^0 \cong \text{Sym}(3)$, a similar argument shows that for any
two distinct indices $j$ and $k$, at least one of $P_j$ and $P_k$ is a quaternion group. Thus (d)(ii) holds in this case.

Now suppose that $P \neq P_j$. As $P_j = C_P(U)$ we then have $P = P_j \langle s \rangle$ for some $s \in R - C_R(U)$. Then $s$ interchanges $P_1$ and $P_2$, and since some $P_i$ is a quaternion group we get either $P_3 \in \mathcal{Q}_3$ or $P_1 \in \mathcal{Q}_1$ for both $i = 1$ and $2$. If $P_3 = R_3$ then (d)(iii) holds. Thus we may assume $P_3 \neq R_3$, so that $P_3 \in \mathcal{Q}_3$. Also $R_3 \notin \mathcal{Q}_3$, so that in particular $D \neq G_0$. Let $\bar{N} = N/P_j$. If $P_1 = R_1$ then $O^2(N) = O^2(C_N(\bar{s})) \leq O^2(N_3^*)$, while if $P_1 \in \mathcal{Q}_1$ then $O^2(N) = P_j \langle d \rangle$ or $P_j \langle d \rangle O^2(N_3^*)$ for a suitable element $d$ of order 3 in $N_3^*N_2^*$. In particular if $O^3(N_3^*)$ does not centralize $\bar{s}$, then $N_{R_3}(P) \leq O_2(N)$, again contrary to $R_3 \neq P_3$ and $P \in \mathcal{R}_P$. Thus (d)(iii) holds, completing the proof of (d).

Suppose next that $P$ is a 2-element in $K - U$, such that $[P, x] \leq Z$. If $[P, x] = Z$ assume that $x$ is an involution. Since $C_{L_i}(P_i) = Z(P_i) \leq U$, and all involutions in $L_3$ are in $Z$, it follows that $x \notin B^0$. But then $x$ interchanges $P_i$ and $P_j$ for some pair of distinct indices $i$ and $j$, and so $[P, x] \notin Z$. Thus no such $x$ exists. This proves (f), and shows also that $P \in \mathcal{R}_K$.

Let $F$ be an $N$-invariant 2-subgroup of $K$ with $F \neq S$. Then $FP$ is a 2-group. For any $i$, set $S_i = S \cap L_i$, and recall $F_i$, $N_i$ are the projections of $F \cap B^0$, $N \cap B^0$ on $L_i$. Then $F_i$ is $N_i$-invariant. It is a property of the group $\bar{L} = \text{PSL}(2, F)$ that the intersection of any pair of distinct Sylow 2-subgroups is abelian (either cyclic or a fours group), and therefore $S_i$ is the unique Sylow 2-subgroup of $L_i$ containing $P_i$. Thus $F_i \leq S_i$, and we conclude that $F \cap B^0 \leq S$.

Now $F = (F \cap B^0) \langle t \rangle$, where $t$ acts nontrivially on $\mathcal{L} := \{L_1, L_2, L_3\}$. Thus, there is an ordering $(1', 2', 3')$ of $\{1, 2, 3\}$ such that $t$ interchanges $L_1'$ and $L_2'$, and fixes $L_3'$. Suppose that $P \neq B^0$. Then there exists $s \in P$ such that $s$ interchanges $L_1$ and $L_2$, and since $\langle s, t \rangle$ is a 2-group it follows that $i = i'$ for all $i$. Without loss, we may assume that $P \leq F$, so that $st \in F \cap B^0$. But then $t \in S$ and $F \leq S$. Thus $P \leq B^0$, and it follows from (c) and (d) that $N_i \leq N$ for all $i$.

Suppose that $P \neq J$. Then (d) implies that $C_R(U) \leq P$, and since $P \leq B^0$ we get $P = C_R(U)$. The Frattini Argument then implies that there exists an element $x$ of $N$ which permutes the components $L_i$ transitively, and then $\langle x, t \rangle$ is not a 2-group. This shows that $P \leq J$, and a similar argument shows that there exists $j$ with $P_j \neq R_j$.

Since $P_i^J \leq F \cap B^0 \leq S$, we have $P_i^J \leq S \cap S_i^J$, so that $S_i^J \leq S$ for all $i$. Since $\langle t^{N_j} \rangle$ is a 2-group, we obtain $N_{1'} = R_{1'}$ and $N_{2'} = R_{2'}$. Thus $P_{3'} \neq R_{3'}$, and so $N_{3'} \cong GL_2(3)^+$. Then $N_{K}(N_{3'}) = C_K(O^2(N))N_{3'}$. Since $N_{3'} \leq N$ we have $N = C_N(O^2(N))N_{3'}$, and since $C_R(O^2(N))$ is a 2-group we get $N = N_{3'}P$. Thus, $N = N_1N_2N_3$. On the other hand, there exists an element $t'$ of $C_{K^D}(J_{3'})$ such that $t'$ interchanges $R_{1'}$ and $R_{2'}$. Then $t' \in P$, by 7.1(b), contradicting $P \leq J$, and completing the proof of (e).
It remains to establish the final statement in the lemma. Thus let $P \leq S$ such that $P$ satisfies (c) and (d). Then there exists $Q \leq P$ such that $Q \geq Q_i \cong Q_8$ for all $i$, and one checks that $C_K(Q) = U \leq Q$. Then $Q \in \mathcal{D}_K^c$, and hence also $P \in \mathcal{D}_K^c$.

Set $N = N_{K_D}(P)$. As $C_K(Q) = U$ we have $O(N) = 1$, and it remains to show that $P = O_2(N)$ if $d(i)$ holds, then $P = R$ or $P = C_R(U)$, so that $N = R$ or $N/R \cong \text{Sym}(3)$, and in particular $P = O_2(N)$. Suppose that $d(ii)$ holds. If $P = Q$ then it is easy to check that $P = O_2(N)$; so we may assume that $P \neq Q$. Then $P_j = R_j \notin \mathfrak{Q}_i$ for exactly one index $j$, $N/P \cong \text{Sym}(3) \cap C_2$, and we are done in this case.

Finally, suppose that $d(iii)$ holds, set $P_J = P \cap J$, and let $s \in P - J$. Since $P_J = C_P(U)$ we have $N \leq M := N_{K_D}(P_J)$. Set $\bar{M} = M/P_J$; it follows that $N$ is the preimage of $M$ of $C_{\bar{M}}(\bar{s})$. Thus it suffices to show that

\[(\bar{s}) = O_2(C_{\bar{M}}(\bar{s})).\]

If $R_i \in \mathfrak{Q}_i$ then $D \in \{G_0, G_{\psi_0}\}$, $P_J = Q$, $R/Q$ is a 4-group, and every involution in $R/Q$ has nontrivial fixed points on $O_3(\bar{M})$. Since $N_{K_D}(R) = R$ we conclude that (*) holds in this case. We may therefore assume that $R_i \notin \mathfrak{Q}_i$. Suppose that $P_J = Q$. Then $\bar{M} \cong \text{Sym}(3) \cap \text{Sym}(3)$, and $P_3 \neq R_3$. Then (d)(iii) requires $O^2(N_3) \neq 1$, and hence $N$ contains an element $g$ of $J_3$ of order 3. Then $|O_3(\bar{N})| = 9$, $\langle \bar{s} \rangle$ is a Sylow 2-subgroup of $C_{\bar{M}}(O_3(\bar{N}))$, and we have (*).

If $P_3 = R_3 \notin \mathfrak{Q}_3$ then $P_1$ and $P_2$ are quaternion groups, $\bar{M} \cong \text{Sym}(3) \cap C_2$, and $\bar{N} \cong D_{12}$. This yields (*), so we are reduced to the case where $P_i = R_i \notin \mathfrak{Q}_i$ for both $i = 1$ and 2. Then $P_3 \neq R_3$, so $O^2(N \cap J_3) \neq 1$, and $\bar{M} = \bar{N}$ with $\langle \bar{s} \rangle = O_2(\bar{M})$. Again (*) holds, so the proof is complete.

**Lemma 10.3.** Let $g \in G$ and $R \leq S$ such that $R = R^1 R^2 R^3$, with $R^i = R \cap L_i^g \cong Q_8$. Then $U = U^g = Z(R)$, and $R$ is special of order $2^8$.

**Proof.** For each $i \neq j$ we have $R^i R^j = R^i \times R^j$, so that $R$ is special with center $U^g$. Thus it remains to show $U = U^g$. Set $\bar{S} = S/S_\infty$. Then $\bar{S}$ has no $Q_8$-subgroups, so that $R^i \cap S_\infty \neq 1$, and hence $U^g \leq S_\infty$.

Set $Y = C_R(E)$ and $Y_0 = R \cap S_\infty$, and suppose first that $|R/Y| \leq 2$. Then $|Y_0| \geq 64$, so as $R$ has no elements of order 8 it follows that $Y_0 = T_2$. Since $\Phi(R) = Z(R)$, this is a contradiction and so we conclude that $|R/Y| \geq 4$. Since $U^g \leq Y$, $R/Y$ is elementary abelian, and is then a maximal elementary abelian subgroup of $\text{Aut}_S(E)$. Since the fixed point groups in $E$ for the two maximal elementary abelian subgroups of $\text{Aut}_S(E)$ are $Z$ and $U$, we conclude $U = U^g$. □

**Lemma 10.4.** Let $P \in \mathcal{D}_K^c$ with $|Z_P| \leq 4$. Then:

(a) $P$ contains every $P$-invariant subgroup of $D$ of order 4.

(b) $Z_P \in \{Z, U\}$. 


Proof. Let $F$ be a $P$-invariant subgroup of $D$ of order 4. Then $|[F, P]| \leq 2$, and so 8.2(a) implies that $[F, P] \leq Z_P$. If $[Z_P, F] = 1$ then $F$ centralizes the chain $P \geq Z_P \geq 1$, whence $F \leq P$ by 8.2(a) and 2.2. On the other hand, suppose that $[Z_P, F] \neq 1$. Then $|Z_P| = 4$ and $Z_P \not\leq \Phi(P)$. By 8.3(b) we have $Z_P = U^h$ for some $h \in H_D$. Then $E^h \leq C_D(Z_P)$, and so $P \neq Z_P$. Thus $\Phi(P) \neq 1$, so that $Z_P \cap \Phi(P) = Z$, and $F$ centralizes the chain $P \geq Z_P \geq Z \geq 1$ of characteristic subgroups of $P$. Thus $F \leq P$ by 2.2, and (a) holds.

Now suppose that $Z_P \neq Z$. Then $Z_P^g = U$ for some $g \in H_D$, and $P^g \leq C_{H_D}(U) \leq K_D$ by 8.3(b). By 6.4 there exists $k \in K_D$ with $P^{g^k} \leq S_D$, and replacing $g$ by $gk$, we may assume $P^g \leq S_D$. Then $P^g \in \mathfrak{D}_K^c$ by 8.3(b) and 10.1, and then 10.2 implies that $P$ contains a subgroup $R$ satisfying the hypothesis of 9.3, with $g^{-1}$ in the role of $g$. Then $U = U^{g^{-1}}$ by 10.3, proving (b). \[\Box\]

**Lemma 10.5.** $P \in \mathfrak{D}_c$ with $|Z_P| > 4$ if and only if $P \in A^D$ or $P = C_{S_D}(E)$.

**Proof.** Let $P \leq S_D$. Suppose that $|Z_P| > 8$. Then $P = C_D(P) = Z_P \in \mathcal{C}_4(S_D)$, by 7.9. If $P \in A^D$ then 7.10 shows that $O_2(\text{Aut}_D(P)) = 1$, whence $P \in \mathfrak{D}_c$. On the other hand suppose that $P \notin A^D$. Then $D = G_\sigma$ by 5.9 and 7.9, and then 7.12 shows that $P \in A^D_u$ for some $u \in U - Z$. Now $\text{Aut}_D(A_u) = C_{\text{Aut}(A_u)}(u)$ by 7.11(c). On the other hand, the definition of $A_u$ preceding 7.11 shows that $\text{Aut}_D(A_u)$ is an elementary abelian subgroup of $\text{Aut}_D(A_u)$ of order 8. It follows that $O_2(\text{Aut}_D(A_u)) \neq 1$. Thus $P \notin \mathfrak{D}_c$. Hence the lemma holds when $|Z_P| > 8$, so we are reduced to the case where $|Z_P| = 8$.

If $\mathcal{C}_4(P) = \emptyset$ then $P \notin \mathfrak{D}_c$ by 8.3(c). Thus we may assume $\mathcal{C}_4(P) \neq \emptyset$.

Suppose $Z_P \notin E^D$. Then $D \neq G$ by 7.7, and $D \neq G_0$ by 5.9, so that $D = G_\sigma$. Then $C_{S_D}(Z_P) \in \mathcal{C}_4(S_D)$ by 7.8. But in that case $P = Z_P$, and $P$ is not centric. Thus we may assume $Z_P \in E^D$. Set $R = S_D$.

By 5.9, 7.7, and 7.8, $H_D$ is transitive on $E^G \cap H_D$. Since $E$ is normal in the Sylow 2-group $R$ of $H_D$, there is $h \in H_D$ with $E^h = Z_P$, $P \leq R^h$, and $Z_P \leq R^h$. In particular $P \leq C_{R^h}(Z_P)$, so that $P_0 = P \cap S^h$ is of index 2 in $P$.

Now $Z_P$ is generated by the involutions in $P_0$, and $P - P_0$ consists entirely of involutions, and so $P_0$ is a characteristic subgroup of $P$. Let $R_0$ be the unique conjugate of $T_2$ in $R^h$. Then $R_0$ centralizes the chain $P \geq R_0 \cap P \geq 1$, and so $R_0 \leq P$ by 2.2. Thus $R_0 = T_2$ is weakly closed in $P$ by 4.9(c), and so $\langle h, N_D(P) \rangle \leq N_D(T_2)$. Then $Z_P = E^h = E$ and $\text{Aut}_{C_R(E)}(P) \leq O_2(\text{Aut}_D(P))$ by 7.13. Thus if $P \in \mathfrak{D}_c$ then $P = C_{R}(E)$. On the other hand if $P = C_{R}(E)$ then 7.13 says $P \in \mathfrak{D}_c$. This completes the proof. \[\Box\]

Recall from Section 4 that $H$ acts on an orthogonal space $V$ of dimension 7 over $F$, and that there is a distinguished basis $\{x_1, \ldots, x_7\}$ of $V$ such that $T$ acts on $F x_i$ for each $i$. Define $F_D$ to be $F$ if $D = G$, $F_\sigma$ if $D = G_\sigma$, and $F_{\psi_0}$ if $D = G_0$. 

Let $V_D$ be the $F_D$-span of $\{x_1, \ldots, x_7\}$. Then the quadratic form $f$ associated with $V$ restricts to a quadratic form $f_D$ on $V_D$, preserved by $H_D$.

Denote by $\Lambda(V_D)$ the collection of all sets $\Lambda$ of pairwise orthogonal subspaces of $V_D$ whose sum is $V_D$. For any subspace $X$ of $V_D$, denote by $X^F$ the $F$-span of $X$ in $V$. For any $\Lambda \in \Lambda(V_D)$ define $\Lambda^F \in \Lambda(V)$ by

$$\Lambda^F = \{X^F \mid X \in \Lambda\},$$

and define the type of $\Lambda$ to be the nondecreasing sequence $\tau = \tau(\Lambda)$ of integers given by the dimensions of the members of $\Lambda$. We will abbreviate such sequences, using exponential notation. For example, $\tau(\Lambda) = 1^7$ means that each member of $\Lambda$ is a 1-space, while $\tau(\Lambda) = 1^52$ means that $\Lambda$ consists of five 1-spaces and one 2-space. Write $\Lambda(V_D, \tau)$ for the set of $\Lambda \in \Lambda(V_D)$ with $\tau(\Lambda) = \tau$. For $\Lambda \in \Lambda(V)$ and $X$ a subgroup (or subset) of $H\langle \sigma \rangle$, write $C_X(\Lambda)$, $N_X(\Lambda)$ for the set of all $x \in X$ which acts on each member of $\Lambda$, and permutes the members of $\Lambda$, respectively.

**Lemma 10.6.** Let $\Lambda \in \Lambda(V)$ with $\sigma \in C_{H\sigma}(\Lambda)$, and set $\tau = \tau(\Lambda)$. Then $H$ acts transitively on $\Lambda(V, \tau)$.

**Proof.** Since every member of $F$ is a square, all nondegenerate subspaces of $V$ of a given dimension are isometric, and so the result follows from Witt’s Lemma. $\square$

**Lemma 10.7.** Let $D \in \{G_\sigma, G\}$, let $P \in B^\infty_H$ with $N_{H_D}(P) \not\leq K$, and set $N = N_{H_D}(P)$. Denote by $B(P)$ the set of $N$-invariant 2-subgroups of $H$, and set $\beta(P) = (B(P))$. Then:

(a) One of the following holds.

1. There exists $\Lambda \in \Lambda(V_D, 1^7)$ such that $N = N_{H_D}(\Lambda)$ and $P = C_{H_D}(\Lambda)$. Moreover either
   (i) $P \cong D_8^3$, and $N/P \cong \text{Alt}(7)$ if $D = G_{\psi_0}$, while $N/P \cong \text{Sym}(7)$ if $D \neq G_{\psi_0}$, or
   (ii) $D = G_\sigma$, $P \cong \mathbb{Z}_4 \ast Q_8^2$, and $N/P \cong \text{Sym}(6)$.
2. $D \neq \psi_0$, and there exists $\Lambda \in \Lambda(V_D, 1^52)$ with $N \leq N_{H_D}(\Lambda)$. More precisely $P = O_2(N_{H_D}(\Lambda))(t)$, where $t$ acts as $-1$ on every point in $\Lambda$ and as a reflection on the line $l$ in $\Lambda$. Further, $l^F$ is one of the lines $l_i$, $1 \leq i \leq 3$, from Section 4, and $N/P \cong \text{Sym}(5)$.
3. $P = O_2(N_{H_D}(E))$.

(b) $C_G(P) \leq P$.

(c) $\beta(P) \leq S$. More precisely: $\beta(P) = O_2(N_H(\Lambda^F))$ in case (a)(1), $O_2(N_H(\Lambda^F))(t)$ in case (a)(2), and $C_S(E)$ in case (a)(3).

**Proof.** Observe that $U \leq P$ by 10.4(a). Set $H^* = H/Z$. We first prove (a).
Let $Q^*$ be an $N_H(P)$-invariant elementary abelian 2-subgroup of $Z(P^*)$ containing $U^*$; for example as $U^* \leq Z(P^*)$, $Q^*_{U'} = \langle U^*N_H(P) \rangle$ is such a subgroup. Let $\Lambda = \Lambda(Q)$ be the set of weight spaces of $Q^*$ on $V_D$; then $\Lambda \in (V_D)$ and $N_{HD}(P) \leq N_{HD}(\Lambda) = N_{HD}(\Lambda^F)$. Let $R = O_2(N_H(\Lambda^F))$. As $N_{HD}(P) \leq N_H(\Lambda^F) \leq N_H(R)$,

$$R \cap D \leq O_2(N_{HD}(\Lambda^F)) \leq O_2(N_{HD}(P)) \leq P$$

by 8.2(b). On the other hand $P^*$ centralizes $Q^*$ and hence stabilizes each member of $\Lambda^F$. Further if $\text{Aut}_P(Y) \leq \langle -1 \rangle$ for each $Y \in \Lambda$, then $P \leq R \cap D$, so that

$$P = R \cap D = O_2(N_{HD}(\Lambda^F)) = O_2(N_{HD}(P)).$$

Suppose $\Lambda$ is of type $1^7$. Then each $Y \in \Lambda$ is a 1-dimensional $P$-invariant orthogonal space, so that $\text{Aut}_P(Y) \leq \langle -1 \rangle$, and hence $P = R \cap D = O_2(N_{HD}(P))$ by the previous paragraph. Further, $\Lambda = \Lambda(P)$, so that $N_{HD}(P) = N_{HD}(\Lambda)$, and as $P = O_2(N_{HD}(\Lambda^F)), P = C_{HD}(\Lambda)$. If $D = G_\sigma$ it follows that (a)(1.i) holds, so we may take

$$D = G_\sigma. \text{ As } [\sigma, P] = 1, \sigma \in C_{H(\sigma)}(\Lambda^F), \text{ so by 10.6 there exists } h \in H, \Theta \in \Lambda(H), \text{ and } \sigma' \in \sigma^H \text{ such that } \sigma \text{ centralizes } N_H(\Theta) = N \text{ modulo } Z, \sigma' \text{ centralizes } \Theta, \text{ and } (\sigma, \Lambda^F)^h = (\sigma', \Theta). \text{ In particular } N_{HD}(P) \cong N_{H_\sigma}(P^h). \text{ As } \sigma' \text{ centralizes } \Theta, \sigma' = r\sigma \text{ for some } r \in R_0 = O_2(N_H(\Theta)). \text{ We may regard } R_0^* \text{ as the core of the permutation module for } S_7 \cong N_H(\Theta)/R_0. \text{ Thus } r^* \text{ is of weight 0, 2, 4, or 6 in } R_0^*. \text{ If } \sigma = \psi_0 \text{ then } [\sigma, g] = z \text{ for } g \in N - O^2(N), \text{ so that } O^2(N) = C_N(\sigma), \text{ and hence if in addition } r^* \text{ is of weight 0 then } N_{HD}(P) \cong O^2(N). \text{ In this case (a) holds, so we may assume one of the remaining cases holds. Then } N \text{ centralizes } \sigma' \text{ if } r \in Z \text{ and } rz \in rR_0 \text{ if } r \notin Z, \text{ so that } C_{N^*}(r^*) = R_0^*C_N(r\sigma)^* \text{ by a Frattini argument. Further, } \sigma \text{ centralizes } R_0, \text{ so that}

$$P \cong C_{R_0}(r\sigma) \cong C_{R_0}(r) \cong D_3^3 \times Q_{8}^2, Z_4 \times Q_{8}^2, Z_2 \times Q_{8}^2, \text{ or } Z_4 \times Q_{8}^2,$$

for the respective choices of $r^*$, and $N_{HD}(P)^*/P^*$ is isomorphic to the stabilizer in $S_7$ of $r^*$, and hence is $S_7, S_5 \times Z_2, S_4 \times S_3, S_6,$ respectively. So as $R \cap D = O_2(N_{HD}(P)), r^* \text{ is of weight 0 or 6, and hence (1) holds.}

Assume next that $\Lambda$ is of type $1^52$, and let $l$ be the line in $\Lambda$. Then

$$O_2(N_{HD}(P)) \leq P$$

and $P$ acts on each member of $\Lambda$. Let $\Theta \in \Lambda(V)$ be of type $1^52, l_N$ the line in $\Theta$, and $N_H(\Theta) = N$. Then $N = N_1N_2\langle l_N \rangle$, where $N_1 = C_G(l_N), N_2 = C_N(l_N^1), t_N$ inverts $l_N^1$ and induces a reflection on $l_N$, $O_2(N_1) \cong Q_8D_8$, and $N_1/O_2(N_1) \cong S_5$. If $D = G_\sigma$, then as above we can pick $\Theta$ so that $(\sigma, \Lambda^F)$ is conjugate in $H$ to $(r\sigma, \Theta)$ such that $\sigma$ centralizes $O^2(N_1)$, and $r$ fixes each member of $\Theta$. Thus $r = r_1r_2$ with $r_1 \in O_2(N_1), r_2 \in N_2$, and $r_1^*$ is of weight 0, 2, or 4 in the permutation module $O_2(N_1)^*$ for $N_1/O_2(N_1) \cong S_5$. But again if $r_1^* \neq 1$ then $O_2(C_N(r\sigma))$ does not act on each point in $\Theta$, contrary to an earlier remark.
Thus we may take \( \Theta = \Lambda^F \) and \( H_1 = O^2(C_{H_D}(l)) \cong O_2(N_1) \cong A_5 \times Q_8 \) with \( P_1 = O_2(H_1) \leq P \) by 8.2(b). Let \( H_2 = C_{H_D}(l^+) \). Then \( H_2 \) is cyclic and \( P_2 = O_2(H_2) \leq P \) by 8.2(b). Also there is \( t \in N_{H_D}(\Lambda) \) inverting \( l^+ \) and inducing a reflection on \( l \). Then \( t \) induces an automorphism in \( O_2(\text{Aut}_{H_D}(P)) \) on \( P \), so that \( t \in P \) as \( P \in \mathcal{F}_H \). Indeed \( P = P_1 P_2(t) \).

If \( |P_2^*| \leq 2 \) then \( P^* \) is elementary abelian with weight spaces of dimension 1, and we obtain a contradiction from our treatment of the case \( \Lambda \) of type \( 1^7 \). Thus \( |P_2^*| > 2 \), and if \( D = G_{\sigma} \) then \( |P_2^*| = (q-\varepsilon)2/2 \) where \( q = |F_D| = \varepsilon = \pm 1 \) mod 4. It follows that \( \sigma \neq \psi_0 \). Next \( U \leq Q \leq P_1 P_3 \) where \( P_3 \) is the subgroup of \( P_2 \) of order 4, so \( P_2 C_{P_1}(U) \leq C_S(U) \). As \( \text{Aut}_{C_S(U)}(S_{\infty}) \cong E_8 \), \( P_4 = \Phi(P_2) \leq S_{\infty} \). As \( |P_4| > 2 \), \( l^1 = C_{V_D}(P_4) \), and then as \( P_4 \notin S_{\infty} \cap D \), \( l^F \) is one of the three lines \( l_i \) from Section 4. Thus (2) holds in this case.

Suppose \( N_{H_D}(P) \leq B_D^h \) for some \( h \in H_D \). Arguing as in the first few paragraphs of the proof of 9.2, there is \( R_+ \leq P \) such that \( R_+ = R_1^+ R_2^+ R_3^+ \) with \( R_+ = R_0 \cap L^h \cong Q_8 \). Then by 10.3, \( U = U^h \), contrary to our hypothesis that \( N_{H_D}(P) \notin K \). Thus we may assume \( N_{H_D}(P) \) is contained in no \( H_D \)-conjugate of \( B_D \).

Suppose \( Z \neq P_0 \leq P \) is normal in \( N_{H_D}(P) \) with \( \Phi(P_0) = 1 \). By the previous paragraph, \( m_2(P_0) > 2 \), and so \( m_2(P_0) = 3 \) or 4. If \( m_2(P_0) = 4 \) then by 7.11, 7.12, and as \( N_{H_D}(P_0) \) acts on no 4-subgroup of \( P_0 \), \( \text{Aut}_{H_D}(P_0) = C_{GL}(P_0)(z) \). Thus \( P = O_2(N_{H_D}(P_0)) \) by 8.2(b), and \( P^* \cong E_{64} \), so that \( \Lambda(P) \in (D) \) is of type \( 1^7 \), and we obtain a contradiction from our treatment of this case. Therefore \( m_2(P_0) = 3 \).

Hence by 7.8, \( P_0 \) is \( B_D \)-conjugate to \( E \) or \( E' \), and in the latter case \( EE' \cong E_{16} \) is Sylow in \( C_{H_D}(E') \). In the latter case a \( P \)-invariant Sylow 2-subgroup \( P_1 \) of \( C_{H_D}(P_0) \) satisfies \( [P, P_1] \leq P_0 \leq C_P(P_1) \); so \( P_1 \leq P \) by 2.2, and we obtain a contradiction from our treatment of the case \( m_2(P_0) = 4 \). Thus \( P_0 = E_8^g \) for some \( g \in B_D \). Let \( S_1 = S_{\infty} \cap D \). Then \( |S_1 \cap P| \leq S_{1}^g \cap P \leq N_{H_D}(P) \); so \( S_1 \notin P \) by 2.2. Hence \( S_1 \cong S_{1}^g \) by 4.9(c), and so \( P_0 = E \). Similarly \( C_{S_D}(E) \leq P \), and then (3) holds by 8.2(b).

We have reduced to the case where \( Z \) is the largest elementary abelian 2-subgroup of \( P \) normal in \( N_{H_D}(P) \). It follows that \( Q \) is of symplectic type and hence (cf. [Asc86, 23.9]) \( Q = Q_0 \star Z(Q) \) with \( Q_0 \) extraspecial and \( Z(Q) \) cyclic of order 2 or 4. Further we may choose \( Q_0 \) with \( U \leq Q_0 \). As \( \text{Inn}(Q_0) = C_{\text{Aut}(Q_0)}(Q_0^*) \), \( P = Q_0 C_P(Q_0) \).

As \( [E, P] \leq [E, Q] \leq U \leq P \), \( E \) acts on \( P \), and similarly \( E \) acts on \( Q_0 \). Thus \( [E, C_P(Q_0)] \leq C_U(Q_0) = Z \) and if \( E \) does not centralize \( P^* \) then \( E \) does not centralize \( Q^* \). Then as \( E \) acts on \( Q_0 \), \( E \) induces a transvection on the orthogonal space \( Q_0^* \) with center \( U^* \). This is impossible as \( U^* \) is a singular point in \( Q_0^* \). Thus \( E \) centralizes \( P^* \), so that \( E \leq P \) by 2.2, and hence \( E^* \leq Z(P^*) \); so replacing \( Q \) by \( Q \langle E^{N_{H_D}(P)} \rangle \), we may assume \( E \leq Q \).
Finally let $\Theta \in \Lambda(V)$ be of type $1^7$ such that each member of $\Lambda F$ is a sum of points in $\Theta$, and let $R_0 = O_2(N_H(\Theta))$. Thus $Q$ centralizes $\Theta$, $Q \leq R_0$, and we view $R_0^*$ as the core of the permutation module for $N_H(\Theta)/R_0 \cong S_7$ on $\Theta$. Thus we identify $f^* \in Q^*$ with the points of $\Theta$ inverted by $f$. Observe $\Lambda F(E) = \{l_0, l_1, l_2, l_3\}$, where $l_0 = C_V(E)$ is a point. As $Z(Q)$ is cyclic, for each $e \in E - Z$ there is $f \in Q$ with $[e, f] = z$, and hence $|f^* \cap e^*|$ is odd. It follows that for at least two $i \in \{1, 2, 3\}$, the eigenspaces of $f$ on $l_i$ are 1-dimensional. Hence $\Lambda$ is of type $1^52$ or $1^7$, cases we have already handled. This completes the proof of (a).

Set $X = C_H(P)$. Since $Z = Z_P$ by (a), we conclude from 8.3(a) that if $X \leq P$ then $X = C_H(P)$, so that (b) holds. Thus to prove (b), it suffices to show $X \leq P$. If $P = C_{SD}(E)$ then $A \leq P$, and then $X \leq P$ since $C_H(A) = A$. On the other hand, if $P$ satisfies (a)(1) then $X \leq C_H(\Lambda F)$. If $\Lambda$ is of type $1^7$ then $P$ has index at most 2 in $O_2(N_H(\Lambda F))$ and $X = Z$. If $\Lambda$ is of type $1^52$ then $P = P_0(t)$ where $P_0 = O_2(N_{HD}(\Lambda F))$ and where $t$ inverts every element of $C_H(P_0)$. Thus $X \leq P$ in all cases, establishing (b).

Let $Q \in \mathcal{B}(P)$, $R_1$ our candidate for $\beta(P)$ in (c), $R_2 = N_{R_1}(Q)$, and $Q_2 = N_Q(R_1)$. Then $Q_2$ acts on $R_2$. Observe that $P \in \mathcal{B}^c_H$ and $Z_P = Z$ by (a) and (b), so $P \in \mathcal{B}^c$ by 8.3(a). Thus the same holds for any overgroup $Q' \in S$; so $Q' \in \mathcal{B}^c$ and $N_G(Q') \leq H$ by 8.3(a). In particular $Q_2$ and $N_Q(R_2)$ are contained in $H$.

We claim that $N_Q(R_2)$ acts on $R_1$, so that $Q_2 = N_Q(R_2)$. In case (a)(1), from the treatment of that case above, each of $R_1$ and $P$, and hence also $R_2$ has $\Lambda$ as its set of weight spaces, so $Q_2$ acts on $O_2(N_H(\Lambda)) = R_1$. Similarly in case (a)(2), each of $R_1$ and $P$, and hence also $R_2$ has the same set of 1-dimensional weight spaces, so again $Q_2$ acts on $R_3 = O_2(N_H(\Lambda))$, and hence also on $R_1 = R_3 < t$ as $t \in P$ acts on $Q_2$. Finally in case (3), $Q_2$ acts on $T_2$ by 4.9(c), so that $Q_2$ acts on $S_{\infty}R_2 = R_1$. This completes the proof of the claim.

Next from the structure of $N_H(R_1)$ and $N_{HD}(P)$, $N_{HD}(P)$ acts on no nontrivial 2-subgroup of $N_H(R_1)/R_1$; so $Q_2 \leq R_1$, and hence also $Q_2 \leq R_2$. Therefore as $Q$ is finite, $Q = Q_2 \leq R_1$, completing the proof of the lemma.

**Lemma 10.8.** Let $D = G_0$ and $H_0^* = H_0/Z$. Write $\mathcal{R}(H_0)$ for the set of $P_0 \leq S$ such that $Z \leq P$ and $P^*$ is the radical of some proper parabolic of $H_0^* \cong \text{Sp}_6(2)$ containing $S_0^*$.

(1) If $P \in \mathcal{B}^c$ and $N_D(P) \leq H$ then $P \in \mathcal{R}(H_0)$.

(2) $\mathcal{R}(H_0) \subseteq \mathcal{B}^c$.

(3) $B_0^* = C_{H_0^*}(a^*)$, where $a^*$ is the involution of type $a_2$ in $Z(S_0^*)$. For each $P \in \mathcal{R}(H_0)$ with $O_2(B_0) \leq P$, $N_D(P) \leq K_0$.

(4) $N_D(P) \leq H_0$ and $P \in \mathcal{B}^c_{S_0^*}(H_0^*)$ for each $P \in \mathcal{R}(H_0)$ with $O_2(B_0) \not\leq P$. 

As
Conversely
by the Borel-Tits Theorem. Thus (1) holds.

Thus in any event, \( N \subset Z \) in the former case. Thus in any event,

\( \text{Parabolic} \) of
in \( \mathbb{H}^{104} \) of (4). Then (2) follows from (3) and (4).

By 10.3, \( \text{Aut} \) is the unique normal 4-subgroup of \( \mathbb{H}^{104} \).

Next, by 7.10, \( \text{Aut} \) of \( \mathbb{H}^{104} \) by 10.1. Assume that
\( \text{Parabolic} \) of

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Proof. Assume the hypothesis of (1). Then by 10.1, \( P \) contains each 2-element in \( C \mathbb{H}_0(P) \), and \( O_2(N \mathbb{H}_0(P)P/C \mathbb{H}_0(P)) \).

Hence as \( H^*_0 \cong \text{Sp}_6(2) \), \( P \in \mathbb{R}(\mathbb{H}_0) \) by the Borel-Tits Theorem. Thus (1) holds.

Next \( U \) is the unique normal 4-subgroup of \( S_0 \), so that \( U^* \) is generated by the unique involution in \( Z(S^+_0) \) lifting to an involution of \( \mathbb{H}_0 \). As \( \mathbb{H}_0 \) is the covering group of \( \text{Sp}_6(2) \) and \( B_0 = N \mathbb{H}_0(U) \), it follows that the first statement in (3) holds.

By 10.3, \( O_2(B_0) \) is weakly closed in \( S_0 \) with respect to \( D \) and so the remaining statement in (3) follows.

Next, by 7.10, \( \text{Aut} \cong L_4(2) \) and so \( (\mathbb{H} \cdot \mathbb{A})/A \cong L_3(2) \). This implies (5).

Also \( N \mathbb{H}_0(A)^* \) contains two minimal parabolics: \( N \mathbb{B}_0(D)^* \) and \( Y_0^* \), where \( Y_0 = N \mathbb{H}_0(A) \cap N \mathbb{D}(E) = N \mathbb{H}_0(E) \).

Thus (6) holds.

Let \( P \in \mathbb{R}(H_0) \) with \( O_2(B_0) \not= P \). Then \( N \mathbb{H}_0(P)^* \) contains the minimal parabolic \( Y_0^* \); so \( N \mathbb{H}_0(P) \) is \( Y_0, N \mathbb{H}_0(A) \), or the preimage of the third maximal parabolic of \( H_0^* \), isomorphic to \( S_6/\mathbb{Z}_2 \).

In the first two cases \( Z = Z_P \) from the action of \( N \mathbb{D}(A) \) on \( A \), and in the third case \( P \cong \mathbb{Z}_4 * \mathbb{Z}_2 \), and \( Z = Z_P \). Thus \( N \mathbb{D}(P) \subseteq N \mathbb{D}(Z) \).

As \( S_0 \leq N \mathbb{D}(P) \) and \( S_0 \in \text{Syl}_2(D) \), \( P \) contains each element in \( C \mathbb{D}(P) \) by 5.8 and so \( P \in \mathbb{F}_C \) by 8.2(a). Then \( N \mathbb{D}(P) \leq \mathbb{H}_0 \) by 8.3(a). Also \( S_0 = S_{\psi_0} \) acts on \( P \) and as \( C \mathbb{S}_{\psi_0}(P) \leq P \), \( P \in \mathbb{F}_C \mathbb{S}_{\psi_0}(\mathbb{H}_{\psi_0}) \).

Similarly \( N \mathbb{D}(P) \leq \mathbb{H}_{\psi_0} \) and \( P = O_2(N \mathbb{D}(P)) \), so that \( P \in \mathbb{F}_C \mathbb{S}_{\psi_0}(\mathbb{H}_{\psi_0}) \) which completes the proof of (4). Then (2) follows from (3) and (4).

Proof. Assume that \( N \mathbb{D}(P) \leq \mathbb{H} \) by 8.3(a), so that \( P \in \mathbb{F}_H \).

Next \( \text{Parabolic} \) of
In the latter case, \( Z_P = U \) and \( N \mathbb{D}(P) \leq \mathbb{K} \) by 10.4 and 8.3(b); certainly \( N \mathbb{D}(P) \leq \mathbb{K} \) in the former case. Thus in any event, \( N \mathbb{D}(P) \leq \mathbb{K} \), so that \( P \in \mathbb{F}_K \) by 10.1.

Proposition 10.9. Let \( P \in \mathbb{F}_C \).

(a) There is no nontrivial \( \mathbb{N}_D(P) \)-invariant subgroup of \( \mathbb{C}_G(P) \) of odd order.

(b) \( P \in \mathbb{F}_C \).

(c) One of the following holds.

(1) \( P \in \mathbb{Z}_P \).

(2) \( P = \mathbb{C}_D \).

(3) \( P \in \mathbb{F}_D \text{ for some } Y \in \{H, K\}, and \mathbb{N}_G(P) \leq Y \).

Conversely, every subgroup \( P \) of \( S_D \) which satisfies one of the conditions in (c) is in \( \mathbb{F}_C \).

Proof. Assume first that \( |Z_P| = 2 \). Then \( N \mathbb{D}(P) \leq H \) by 8.3(a), so that \( P \in \mathbb{F}_H \) by 10.1. Assume that \( N \mathbb{D}(P) \not\leq B \).

Then (10.7(b) and 10.8(4) say that \( \mathbb{C}_G(P) \not\leq P \), so that (a) and (b) hold and \( P \in \mathbb{F}_C \).

Then (3) follows from 8.3(a).

Suppose next that either \( |Z_P| = 2 \) and \( N \mathbb{D}(P) \leq B \), or \( |Z_P| = 4 \). In the latter case, \( Z_P = U \) and \( N \mathbb{D}(P) \leq \mathbb{K} \) by 10.4 and 8.3(b); certainly \( N \mathbb{D}(P) \leq \mathbb{K} \) in the former case. Thus in any event, \( N \mathbb{D}(P) \leq \mathbb{K} \), so that \( P \in \mathbb{F}_K \) by 10.1.
\( C_K(P) \leq P \) by 10.2(a), and \( P \in \mathcal{F}_K^c \) by 10.2(e). Then as \( U \leq P \), \( C_H(P) \leq C_H(U) \leq K \); so \( C_H(P) \leq C_K(P) \leq P \), and hence \( P \in \mathcal{F}_H^c \). Now it follows from parts (a) and (b) of 8.3 that \( C_G(P) \leq P \) and (a)-(c) hold.

If \( |Z_P| = 8 \) then \( P = C_{SD}(E) \) by 9.5. Then \( A \leq P \), and so \( P \in \mathcal{F}_N^c \) by 8.3(d). Also, it follows from 7.13 that (a) holds, and we have (a) through (c).

If \( |Z_P| > 8 \) then \( P \in A^D \) by 9.5. Then 8.3(d) yields (b), and 7.10 yields (a). Thus, we are reduced to establishing the final statement in the theorem.

If \( P \in A^D \) or \( P = C_{SD}(E) \) then \( P \in \mathcal{D}_Y^c \), and \( N_D(P) \nsubseteq H \cup K \), by 7.10 and 7.13. Finally assume \( P \in \mathcal{D}_Y^c \) for some \( Y \in \{H, K\} \) with \( N_G(P) \leq Y \). Then 10.2, 10.7, and 10.8 yield \( Z_P \in \{Z, U\} \) and \( P \in \mathcal{F}_S^c \), so that \( P \in \mathcal{F}_N^c \) by 8.2(a), and \( P \in \mathcal{D}_Y^c \) by 10.1.

**Proposition 10.10.** Let \( P \in \mathcal{D}_Y^c \), let \( \mathcal{B}(P) \) be the set of finite \( N_D(P) \)-invariant 2-subgroups of \( G \), and set \( \beta(P) = (\mathcal{B}(P)) \). Then \( \beta(P) \leq S \), and one of the following holds.

1. \( P \in A^D \) and \( \beta(P) = P \).
2. \( P = C_{SD}(E) \) and \( \beta(P) = C_S(E) \).
3. There exists \( Y \in \{H, K\} \) such that \( P \in \mathcal{D}_Y^c \), \( N_D(P) \leq Y \), \( \beta(P) \in \mathcal{F}_S(Y)^c \), and \( N_G(\beta(P)) \leq Y \).

**Proof.** Set \( N = N_D(P) \), let \( R \) be a finite \( N \)-invariant 2-subgroup of \( G \) containing \( P \), and set \( R_0 = N_R(P) \).

If \( P \in A^D \) then \( N = N_G(P) \) by 7.10; so \( R_0 \leq O_2(N) = P \), and (1) holds. Suppose that \( P = C_{SD}(E) \). Define \( R_i \) by \( R_i = N_R(R_{i-1}) \) for \( i \geq 1 \), and set \( M_i = N_G(R_i) \). Set \( M = N_G(C_S(E)) \). By 7.13, \( C_M(E) = X \times O_2(M) \), where \( X \) is a free normal subgroup of \( M \), \( M/C_M(E) \cong GL(3, 2) \), and \( M = (X \times C_S(E)) \). As \( R_0 \) is \( N \)-invariant, we get \( R_0 \leq C_S(E) \), and \( R_0 = T_k(w_0) \) for some \( k \geq 2 \). By 4.9(a) and 7.13, \( N_G(R_i) \leq M \). Then a straightforward induction argument yields \( R_i \leq C_S(E) \) for all \( i \), and thus \( \beta(P) \leq C_S(E) \). Since \( C_S(E) \) is the union of finite \( N \)-invariant subgroups, we conclude that \( \beta(P) = C_S(E) \), and (2) holds.

By 10.9(c) we are reduced to the case where there exists \( Y \in \{H, K\} \) with \( P \in \mathcal{D}_Y^c \) and \( N_G(P) \leq Y \). Suppose further that \( \beta(P) \leq S \), and set \( M = N_G(\beta(P)) \). As \( P \leq \beta(P) \), and since \( P \in \mathcal{F}_c \) by 10.9(b), we conclude \( \beta(P) \in \mathcal{F}_c \) and \( Z_{\beta(P)} \leq Z_{P} \). If \( N \leq H \) and \( N \nsubseteq K \) then 10.7 yields \( Z_P = Z \), and so \( Z_{\beta(P)} = Z \), and then \( M \leq H \) by 8.3(a). On the other hand, suppose that \( N \subseteq K \). Then \( U \) is the unique 4-group in \( K \) which centralizes \( P/Z \), by 10.2(f). Since we are assuming that \( \beta(P) \leq S \), it follows that also \( U \) is the unique 4-group in \( \beta(P) \) which centralizes \( \beta(P)/Z \), and hence \( U \) is the unique 4-group in \( \beta(P) \) which is normal in \( M \). Then 8.3 says \( M \) is contained in \( H \) or \( K \), and since \( N_H(U) \leq K \) we get \( M \leq K \). Thus, \( M \leq Y \).
We now show that $\beta(P) \leq S$. If $N \leq H$ and $N \nleq K$, this follows from 10.7(c) and 10.8(4). Suppose $N \leq K$, and set $R_1 = R \cap K$. Then $R_0 \leq R_1 \leq S$ by 10.2(e). Also, 10.2(e) says that $P \in \mathcal{F}_K^c$, and hence $R_1 \in \mathcal{F}_K^c$. As $U$ is the unique normal 4-subgroup of $R_1$, we have $N_R(R_1) \leq N_R(U)$. Since $Z_{R_1} \leq Z_P \leq U$, it follows from 8.3 that $N_R(R_1)$ is contained in $H$ or $K$, and since $N_H(U) \leq K$ we get $N_R(R_1) \leq K$. Then $R = R_1$, and so $R \leq S$ for each $R \in \mathcal{B}(P)$. That is, $\beta(P) \leq S$.

In order to complete the proof of (3), it remains to show that $\beta(P) \in \mathcal{F}_Y^r$. Let $Q$ be the preimage in $M$ of $O_2(\text{Aut}_M(\beta(P)))$. As $M \leq Y$ we have $\theta(\beta(P)) = O(C_G(\beta(P))) = 1$, by 8.4 and 10.9(a). Thus $Q$ is a 2-group, $Q \in \mathcal{B}(P)$, and $Q = \beta(P) \in \mathcal{F}_Y^r$ as required. \qed

11. Theorem B and embeddings

We begin the section with a refinement of Theorem 5.8.

THEOREM 11.1. Let $G_0$ be the group $C_0$, identify $S_0$ with a Sylow 2-subgroup of $G_0$ as in Theorem 5.8, and set $\bar{G}_0 = \mathcal{F}_S_0(G_0)$. Let $\lambda: G_0 \to \bar{G}_0$ be the canonical homomorphism $\lambda: g \mapsto \bar{g}$ induced by the inclusion maps of $H_0$ and $K_0$ into $G_0$, and let $\mathcal{G}_0$ be the 2-local finite group $(S_0, \bar{G}_0, \mathcal{F}_0^c)$ associated with $G_0$ as in Proposition 2.7. Then there is an isomorphism (in the sense of 2.10)

$$\begin{align*}
(\alpha, \beta): &\mathcal{G}_0 \to \bar{G}_0,
\end{align*}$$

in which $\alpha: \mathcal{G}_0 \to \bar{G}_0$ is the identity map on objects and, on morphisms, $\alpha: c_g \mapsto c_{\bar{g}}$; and where $\beta: \mathcal{L}_0^c \to \bar{G}_0$ is the identity map on objects, and

$$\beta_{P, Q}: \text{Mor}_{\mathcal{G}_0}(P, Q) \to \text{Mor}_{\bar{G}_0}(P, Q)$$

is given by $\theta_0(\beta(P))g \mapsto \bar{g}$ for $P$ and $Q$ in $\mathcal{L}_0$ and $g \in N_{G_0}(P, Q)$.

Proof. Recall from the discussion following 8.4 that there is a surjection $\lambda: G_0 \to \bar{G}_0$, where $\lambda$ may be regarded as the “identity map” on $H_0 \cup K_0$, and $\ker(\phi_A) = \ker(\lambda)|_M$. As $\ker(\phi_A) = \ker(\lambda)|_M \leq \ker(\lambda)$, 8.7 yields $X_0 \subseteq \ker(\lambda)$. Thus $\theta_0(\beta(P)) \leq \ker(\lambda)$ for any $P \in \mathcal{F}_0^c$. Then since $O(C_{G_0}(P)) = 1$, the lemma follows from 5.8 and the last paragraph of 2.13. \qed

We may now establish Theorem B. Part (1) of Theorem B follows from the construction of $G_0$ in 5.8, and part (3) follows from 10.1. Part (2a) holds since $H_0$ and $K_0$ are finite while the nontrivial elements of $X$ are torsion-free. Thus it only remains to verify part (2b) of Theorem B.

As we just saw during the proof of 11.1, there is a surjective homomorphism $\lambda: G_0 \to \bar{G}_0 = C_0$ induced by the inclusion maps of $H_0$ and $K_0$ into $\bar{G}_0$. Set $M_0 := N_{\bar{G}_0}(A)$, and denote by $\bar{C}$ the colimit of the subgroup amalgam defined by the inclusion maps among the intersections of the members of $\mathcal{M} := \{H_0, K_0, M_0\}$. 


Then there is a surjective homomorphism $\beta: \bar{C} \to \bar{G}_0$ and a surjection $\delta: G_0 \to \bar{C}$ with $\delta \beta = \lambda$. Set $M = N_G(A)$. As seen in the proof of 7.5, $M \leq G_0$ and $\phi_A: M \to M_0$ is the restriction of $\lambda$ to $M$. Thus $\ker(\delta) = \langle \ker(\phi_A)G_0 \rangle$, and then part (2b) of Theorem B follows from 8.7.

For any $i > 0$, set $m_i = 2^{i-1}$, $\sigma_i = \psi_0^{m_i}$, $G_i = G_{\sigma_i}$, $S_i = S \cap G_i$, and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$. Write $\Lambda$ for the poset $\mathbb{N}$, under the usual total ordering. There is then a directed system of embeddings of fusion systems

$$\mathcal{F} = (\iota_{i,j}: \mathcal{F}_i \to \mathcal{F}_j)_{i \leq j \in \Lambda},$$

where $\iota_{i,j}$ is an inclusion map, and $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$ is the fusion system of $\mathrm{Co}_3$.

Let $\iota_i: \mathcal{F}_i \to \mathcal{F}$ be the inclusion, and observe that $\iota_j \circ \iota_{i,j} = \iota_i$ for $i \leq j$. For $i, j \in \Lambda$ with $i \leq j$ and $P \in \mathcal{F}_i^c$, write $\mathcal{B}_i(P)$ for the set of finite $N_{G_i}(P)$-invariant $2$-subgroups of $G$, and set $\beta_i(P) = \langle \mathcal{B}_i(P) \rangle$ and $\beta_{i,j}(P) = \beta_i(P) \cap G_j$.

**Lemma 11.2.** Let $i, j \in \Lambda$ with $i \leq j$, and let $P \in \mathcal{F}_i^c$. Then

(a) $\beta_i(P) \in \mathcal{F}_i^c$,

(b) $\beta_{i,j}(P) \in \mathcal{F}_j^c$ and

(c) $\beta_{i,j}(P) = P$.

**Proof.** Set $D = G_j$, $\mathcal{D} = \mathcal{F}_j$, $N = N_D(P)$, and $\beta = \beta_i$. Also set $Q = \beta(P)$, $\bar{D} = G_j$, $\bar{\mathcal{D}} = \mathcal{F}_j$, $\bar{P} = \beta_{i,j}(P) = Q \cap \bar{D}$, and $\bar{N} = N_{\bar{D}}(\bar{P})$. Since $P \in \mathcal{F}_i^c$ by 10.9(b), it follows from 8.2(a) that $\bar{P} \in \bar{\mathcal{D}}^c$ and that $Q \in \mathcal{F}_c^c$. If $P \in A_D$ or $P = C_{S_D}(P)$ then $Q = P$ or $C_S(E)$ by 10.10. Then 10.5 says $Q \in \mathcal{F}_c^c$, $\bar{P} = P$ or $C_{S_j}(E)$, and $\bar{P} \in \bar{\mathcal{D}}^c$. That is, the lemma holds in these two cases.

By 10.9 we may assume that $N_D(P) \leq Y$ for some $Y \in \{H, K\}$, and that $P \in \mathcal{D}_Y^c$. Then (a) is given by 10.9 and 10.10. In order to complete the proof of (b), it remains to show that $\bar{P} \in \bar{\mathcal{D}}_j^c$. Let $R$ be the pre-image in $Y$ of $O_2(\mathrm{Aut}_R(\bar{P}))$.

As in the final lines of the proof of 10.10, we find that $\theta_j(\bar{P}) = 1$; hence $R$ is a 2-group, and since $R$ is $N$-invariant we get $R \leq Q$. Then $R = \bar{P}$ and $\bar{P} \in \bar{\mathcal{D}}_j^c$ as required.

Finally suppose $i = j$. Here $P \leq \bar{P}$ and $N_{G_i}(P) \leq N_{G_i}(\bar{P})$, and since $P \in \mathcal{F}_i^c$ we get $\mathrm{Aut}_{\mathcal{F}_i}(P) \leq O_P(\mathrm{Aut}_{G_i}(P)) = \mathrm{Inn}(P)$. Then $\bar{P} = P C_{\bar{P}}(P) = P$ as $P \in \mathcal{F}_i^c$, and (c) holds.

**Lemma 11.3.** Let $\theta$, $\theta_0$, and $\theta_i = \theta_{\sigma_i}$ be the signalizer functors on $\mathcal{F}$, $\mathcal{F}_0$, and $\mathcal{F}_i$, respectively, given by 8.8. Let $i, j \in \Lambda$ with $i \leq j$, and let $P \in \mathcal{F}_i^c$. Then

$$\theta_i(P) = \theta(\beta_i(P)) \cap G_i = \theta_j(\beta_{i,j}(P)) \cap G_i.$$

**Proof.** Recall that by definition,

$$\theta(P) = O(C_G(P))C_X(P) \quad \text{and} \quad \theta_i(P) = O(C_{G_i}(P))C_{X \cap G_i}(P).$$
But \(O(C_G(P)) = O(C_{G_i}(P)) = 1\) by 10.9(a), so that \(\theta_i(P) = \theta(P) \cap G_i\). If \(P \in A^{G_i}\) then \(\beta_i(P) = \beta_{i,j}(P) = P\), and the lemma follows from the preceding observation. If \(P = C_{S_j}(E)\) then \(\beta(P) = C_{S}(E)\) and \(\beta_{i,j}(P) = C_{S_j}(P)\) by 10.10, and the lemma then follows from 7.13(d). If \(P \in F_{S_j}(Y \cap G_i)^{rc}\) for some \(Y \in \{H, K\}\) then also \(\beta(P) \in F_S(Y)^{rc}\) by 10.10, and \(\theta_i(P) = \theta(\beta(P)) = 1\). The lemma holds trivially in this case, and there are no more cases to consider, by 10.9.

\[\square\]

**Lemma 11.4.** \(F^{rc} = \{\beta_i(P) \mid P \in F_i^{rc}, i \geq 0\}\).

**Proof.** Let \(B = \{\beta_i(P) : P \in F_i^{rc}, i \geq 0\}\). Then \(B \subseteq F^{rc}\) by 11.2(a). Let \(\tilde{P} \in F^{rc}\). If \(\tilde{P} \in A^G\) then \(\tilde{P} \leq G_i\) for some \(i\), and \(\tilde{P} = \beta_i(\tilde{P})\). If \(\tilde{P} = C_S(E)\) then \(\tilde{P} = \beta_0(C_S(E))\) by 10.10. Suppose that \(\tilde{P} \in F_S(H)^{rc}\), such that \(N_G(\tilde{P}) \leq H\) and \(N_G(\tilde{P}) \not\leq K\). The possibilities for \(N_G(\tilde{P})\) are listed in 10.7(a), and we shall deal with them case by case.

In case (1) and (2) of 10.7(a), \(N_G(\tilde{P}) = N_H(\Lambda)\) for some \(\Lambda \in \Lambda(V, \tau)\), with \(\tau = 1^7\) or \(1^5, 2\). Let \(\tilde{Q} = \Omega_2(O_2(N_H(\Lambda)))\). Then \(\tilde{Q}\) is finite; so \(\tilde{Q} \leq G_i\) for some \(i > 0\). Further \(\Lambda = \Lambda(\tilde{Q})\) is the set of weight spaces of \(\tilde{Q}\) on \(V\), and as \(\tilde{Q} \leq G_i\), \(\Lambda = \Theta^F\), where \(\Theta\) is the set of weight spaces for \(\tilde{Q}\) on \(V_i = V_{F_i}\). If \(\tau = 1^7\), let \(P = \tilde{Q} = \tilde{P}\), while if \(\tau = 1^5, 2\), let \(P = O_2(N_H(\Theta))(t)\), where \(t\) is as in 10.7(a). Then \(P \in F_i^{rc}\) and \(\tilde{P} = \beta_i(\tilde{P})\) by 10.7(c).

It remains to consider case (3), where \(\tilde{P} = O_2(N_H(E))\). We take \(P = O_2(N_H \cap G_i(E))\), obtaining \(\beta_1(P) = \tilde{P}\); again from 10.7(c).

By 10.9(c) we may now assume that \(N_G(\tilde{P}) \leq K\), so that \(\tilde{P} \in F_S(K)^{rc}\). By 10.2 we have \(\tilde{P} \cap B^0 = \tilde{P}_1 \tilde{P}_2 \tilde{P}_3\), where \(\tilde{P}_k = \tilde{P} \cap L_k\) is either a quaternion group or a Sylow 2-subgroup of \(L_k\). Since \(K\) is locally finite, and since 10.2 shows that \(N_G(\tilde{P})/\tilde{P}\) is finite, we may choose \(i\) sufficiently large so that \(P := \tilde{P} \cap G_i\) has the following properties:

1. For all \(k\) for which \(\tilde{P}_k \in Syl_2(L_k)\) we have \(|P \cap L_k| \geq 16\), and for all other \(k\) we have \(P \cap L_k = \tilde{P}_k\).
2. \(N_G(\tilde{P}) = N_K(\tilde{P}) = N_G(P)\tilde{P}\).

Set \(N = N_K \cap G_i(P)\) and \(\tilde{N} = N_K(\tilde{P})\). It follows from (1) and (2), and from the final statement in 10.2, that \(P \in F_{S_j}(K \cap G_i)^{rc}\) and that \(N \leq \tilde{N}\). As \(N \leq \tilde{N}\), \(\tilde{P} \leq \beta_i(P)\), so that it remains to show \(\beta_i(P) \leq \tilde{P}\).

Let \(P \leq R \in B(P)\) and set \(R_0 = R \cap K\). Then \(R \leq S\) by 10.2(e). As \(\tilde{N} = N_G(P)^{\tilde{P}}, (R_0, \tilde{P})\) is an \(\tilde{N}\)-invariant 2-group, so since \(\tilde{P} \in F^{rc}\) we conclude that \(R_0 \leq \tilde{P}\). By 10.9(b), \(P \in F^{rc}\), so \(R_0 \in F^{rc}\). By 10.2(f), \(U\) is the unique normal fours group in \(R_0\), and since \(Z_{R_0} \leq Z_P\) it follows from 8.3 that \(N_G(R_0) \leq K\). Then \(\tilde{R}_0 = R\), and the proof is complete. \(\square\)
12. Limits, and Theorem C

Our aim in this section is to introduce limits of directed systems of $p$-local groups, and to obtain Theorem C as an application. See for example [Jac80, §2.5] for a discussion of directed systems and their limits. Theorem D will then be obtained as a corollary to [LO02, Th. 4.5].

Let $(\Lambda, \leq)$ be a directed set. For $\lambda \in \Lambda$, write $\Lambda(\lambda)$ for $\{\mu \in \Lambda \mid \lambda \leq \mu\}$. A subset $\Omega$ of $\Lambda$ is closed if $\Lambda(\lambda) \subseteq \Omega$ for all $\lambda \in \Omega$. In particular, each of the sets $\Lambda(\lambda)$ is closed.

Recall the notion of “pre-local group” from 2.4. Fix a prime $p$, and assume that for each $\lambda \in \Lambda$ we are given a pre-local group $G_\lambda = (S_\lambda, \mathcal{F}_\lambda, \mathcal{L}_\lambda)$, where each $S_\lambda$ is a $p$-group. We write $\mathcal{E}_\lambda$ for $\text{Obj}(\mathcal{F}_\lambda)$, and given subgroups $P$ and $Q$ of $S_\lambda$ we write $\text{Hom}_\lambda(P, Q)$ for $\text{Hom}_{\mathcal{F}_\lambda}(P, Q)$, and $\text{Mor}_\lambda(P, Q)$ for $\text{Mor}_{\mathcal{E}_\lambda}(P, Q)$ if $P$ and $Q$ are in $\mathcal{E}_\lambda$. Assume that for all pairs $(\lambda, \mu)$ with $\lambda \leq \mu$ in $\Lambda$, we are given an embedding

$$(i_{\lambda, \mu}, \beta_{\lambda, \mu}) : G_\lambda \longrightarrow G_\mu,$$

of pre-local groups (cf. 2.10). We may write simply $\beta_{\lambda, \mu}$ for the pair $(i_{\lambda, \mu}, \beta_{\lambda, \mu})$. We assume further that

$$\mathcal{G} = (\beta_{\lambda, \mu} : G_\lambda \longrightarrow G_\mu)_{\lambda \leq \mu \in \Lambda}$$

is a directed system of pre-local groups. That is, we have $\beta_{\mu, \nu} \circ \beta_{\lambda, \mu} = \beta_{\lambda, \nu}$ whenever $\lambda \leq \mu \leq \nu$ in $\Lambda$, and each $\beta_{\lambda, \lambda}$ is the “identity morphism” on $G_\lambda$, consisting of a pair of identity functors.

Let $S := S_\infty$ be the limit of the $\Lambda$-directed system of $p$-groups

$$(i_{\lambda, \mu} : S_\lambda \longrightarrow S_\mu)_{\lambda \leq \mu}.$$ 

By [Jac80, 2.8], the limit exists and is a group, and there are monomorphisms $i_\lambda : S_\lambda \rightarrow S$, compatible with the monomorphisms $i_{\lambda, \mu}$. We may then view all of these monomorphisms as ordinary inclusion maps, in order to obtain the following result:

**Lemma 12.1.** $S = \bigcup_{\lambda \in \Lambda} S_\lambda$, and $S$ is a $p$-group. \qed

Let $P$ and $Q$ be subgroups of $S$. For any $\phi \in \text{Inj}(P, Q)$ and any $\lambda \in \Lambda$, define $\phi_\lambda$ to be the restriction of $\phi$ to $P \cap S_\lambda$, and set

$$\Lambda_\phi = \{\lambda \in \Lambda \mid \phi_\lambda \in \text{Hom}_\lambda(P \cap S_\lambda, Q \cap S_\lambda)\}.$$ 

Define $\text{Hom}_\infty(P, Q)$ to be the set of all $\phi \in \text{Inj}(P, Q)$ such that $\Lambda_\phi$ contains a nonempty closed subset of $\Lambda$. As $\Lambda$ is a directed set, each pair of elements of $\Lambda$ has an upper bound, and it follows that the composition of $\phi \in \text{Hom}_\infty(P, Q)$ with $\psi \in \text{Hom}_\infty(Q, R)$ is in $\text{Hom}_\infty(P, R)$. Thus we may form the category $\mathcal{F}_\infty$, whose objects are the subgroups of $S$, and whose morphisms are given by the sets
Hom_\infty(P, Q). Moreover, if P and Q are subgroups of S_\lambda and \phi \in \text{Hom}_\lambda(P, Q), then \phi \in \text{Hom}_\infty(P, Q). This natural inclusion of sets may be denoted

\iota_\lambda: \text{Hom}_\lambda(P, Q) \to \text{Hom}_\infty(P, Q).

Allowing P and Q to vary over the set of all subgroups of S_\lambda, \iota_\lambda is then an embedding of fusion systems (cf. 2.9). We record this in the following lemma, whose proof is straightforward and left to the reader.

**Lemma 12.2.** \( \mathcal{F}_\infty \) is a fusion system on S and \( \iota_\lambda: \mathcal{F}_\lambda \to \mathcal{F}_\infty \) is an embedding of fusion systems.

Recall that \( \mathcal{C}_\lambda \) is the set of objects of the linking system \( \mathcal{L}_\lambda \). For \( P \in \mathcal{C}_\lambda \) there is then a subgroup \( \beta_\lambda(P) \) of S defined by

\[ \beta_\lambda(P) = \bigcup_{\mu \in \Lambda(\lambda)} \beta_{\lambda, \mu}(P). \]

Note that for any \( \mu \in \Lambda(\lambda) \), we have \( \beta_\lambda(P) = \beta_\mu(\beta_{\lambda, \mu}(P)) \). Set

\[ \mathcal{C}_\infty = \{ \beta_\lambda(P) \mid \lambda \in \Lambda, P \in \mathcal{C}_\lambda \}. \]

**Definition 12.3.** Let \( \mathcal{C} \) be a category and \( \Delta = (C_\lambda, c_{\lambda, \mu}; \lambda \leq \mu) \) a directed system in \( \mathcal{C} \). A family \( \Sigma = (\gamma_\lambda: C_\lambda \to C: \lambda \in \Lambda) \) of morphisms in \( \mathcal{C} \) is said to be compatible with \( \Delta \) if for all \( \lambda \leq \mu \) in \( \Lambda \), \( \gamma_\lambda = \gamma_\mu \circ c_{\lambda, \mu} \).

Now specialize to the case where \( \mathcal{C} \) is the category of sets. A compatible family \( \Sigma \) is said to be nearly injective on \( \Delta \) if

\[ \mathcal{C} = \bigcup_{\lambda \in \Lambda} \gamma_\lambda(C_\lambda), \]

and for each \( \lambda \in \Lambda \), whenever \( a, b \in C_\lambda \) with \( \gamma_\lambda(a) = \gamma_\lambda(b) \), then there exists \( \mu \in \Lambda(\lambda) \) with \( \beta_{\lambda, \mu}(a) = \beta_{\lambda, \mu}(b) \). For example if \( \gamma_\lambda: C_\lambda \to C \) is injective for each \( \lambda \in \Lambda \), then \( \Sigma \) is nearly injective.

Now take \( \Delta = \Delta(\mathcal{G}) \) to be \( (\mathcal{C}_\lambda, \beta_{\lambda, \mu}; \lambda \leq \mu) \), and observe \( \Delta \) is a directed system in the category of sets, if we regard \( \beta_{\lambda, \mu} \) as the function from \( \mathcal{C}_\lambda \) to \( \mathcal{C}_\mu \) defined by \( \beta_{\lambda, \mu}: P \to \beta_{\lambda, \mu}(P) \). We say that \( \Delta \) is nearly injective if \( \Sigma(\Delta) \) is nearly injective on \( \Delta(\mathcal{G}) \), where \( \Sigma(\Delta) = (\beta_\lambda: \mathcal{C}_\lambda \to \text{Ob}(\mathcal{F}_\infty); \lambda \in \Lambda) \). Similarly define \( \mathcal{G} \) to be nearly injective if \( \Delta(\mathcal{G}) \) is nearly injective.

**Lemma 12.4.** Assume \( \Sigma = (\gamma_\lambda: \mathcal{C}_\lambda \to C: \lambda \in \Lambda) \) is nearly injective on \( \Delta(\mathcal{G}) \) and \( \Gamma = (\delta_\lambda: \mathcal{C}_\lambda \to D: \lambda \in \Lambda) \) is a family of functions compatible with \( \Delta(\mathcal{G}) \).

1. Suppose \( \lambda, \nu \in \Lambda \), \( P \in \mathcal{C}_\lambda \), and \( Q \in \mathcal{C}_\nu \) such that \( \beta_\lambda(P) = c = \beta_\nu(Q) \). Then there exists \( \mu \in \Lambda(\lambda) \cap \Lambda(\nu) \) such that \( R = \beta_{\lambda, \mu}(P) = \beta_{\nu, \mu}(Q) \). Moreover \( \beta_\mu(R) = c \).
(2) $\delta: C \rightarrow D$ is a well-defined function, where $\delta(c) = \delta_\lambda(P)$ for $\lambda \in \Lambda$ and $P \in \mathcal{E}_\lambda$ such that $\gamma_\lambda(P) = c$.

(3) If $\Gamma$ is also nearly injective on $\Delta(\mathfrak{S})$ then $\delta$ is a bijection.

**Proof.** As $\Lambda$ is directed there is $\eta \in \Lambda(\lambda) \cap \Lambda(\nu)$. Set $P' = \beta_{\lambda, \eta}(P)$ and $Q' = \beta_{\nu, \eta}(Q)$. Then

$$
\beta_{\eta}(P') = \beta_\lambda(P) = \hat{P} = \beta_\nu(Q) = \beta_{\eta}(Q').
$$

As $\Sigma$ is nearly injective, there exists $\mu \geq \eta$ with $\beta_{\eta, \mu}(P') = R = \beta_{\eta, \mu}(Q')$. Then $\beta_{\lambda, \mu}(P) = \beta_{\eta, \mu}(\beta_{\lambda, \eta}(P)) = R$, and similarly $\beta_{\nu, \mu}(Q) = R$. Since $\beta_{\mu}(R) = \beta_{\mu}(\beta_{\lambda, \mu}(P)) = \beta_\lambda(P) = c$, (1) holds.

To see that $\delta$ is well defined, suppose that $\delta_\lambda(P) = \delta_\lambda(Q)$ for some $P, Q \in \mathcal{E}_\nu$. Choose $\mu$ as in (1). Then $\delta_\lambda(P) = \delta_\mu(\beta_{\lambda, \mu}(P)) = \delta_\mu(\beta_{\nu, \mu}(Q)) = \delta_\mu(Q)$ and so $\gamma$ is well defined, establishing (2).

Assume the hypothesis of (3). Then by (2) applied to $\Gamma$, the map $\alpha: \gamma_\lambda(P) \mapsto \delta_\lambda(P)$ is a well-defined function from $D$ to $C$, and visibly $\alpha$ is an inverse for $\delta$; so (3) holds. □

We now assume that $\mathfrak{S}$ is nearly injective, and define a category $\mathcal{L}_\infty$ whose set of objects is $\mathcal{E}_\infty$, and which will be shown to be the direct limit of the directed system $(\beta_{\lambda, \mu}: \mathcal{L}_\lambda \rightarrow \mathcal{L}_\mu)_{\lambda \leq \mu}$ of categories.

Let $\hat{P}, \hat{Q} \in \mathcal{E}_\infty$. Then there exist $\lambda \in \Lambda$, and $P, Q \in \mathcal{E}_\lambda$, such that $\hat{P} = \beta_\lambda(P)$ and $\hat{Q} = \beta_\lambda(Q)$. In the following discussion, leading up to 12.5, we take $\lambda, P,$ and $Q$ to be fixed. Define $\text{Mor}_\infty(\hat{P}, \hat{Q})$ to be the set of equivalence classes $\left[f\right]$ of mappings

$$
f: \Omega_f \rightarrow \bigcup_{\mu \in \Omega_f} \text{Mor}_\mu(\beta_{\lambda, \mu}(P), \beta_{\lambda, \mu}(Q))
$$

where

(i) $\Omega_f$ is a nonempty closed subset of $\Lambda(\lambda)$,

(ii) $f(\mu) \in \text{Mor}_\mu(\beta_{\lambda, \mu}(P), \beta_{\lambda, \mu}(Q))$ for all $\mu \in \Omega_f$,

(iii) $\beta_{\mu, \nu}(f(\mu)) = f(\nu)$ whenever $\mu \leq \nu$,

and where two such mappings $f$ and $f'$ are defined to be equivalent if they agree on a nonempty closed set. There is then a well-defined composition

$$
\text{Mor}_\infty(\hat{Q}, \hat{R}) \times \text{Mor}_\infty(\hat{P}, \hat{Q}) \rightarrow \text{Mor}_\infty(\hat{P}, \hat{R})$

for any $\hat{R} \in \mathcal{E}_\infty$. Namely, one may assume $\lambda$ chosen so that also $\hat{R} = \beta_\lambda(R)$ for some $\hat{R} \in \mathcal{E}_\lambda$. Then, for any $[g] \in \text{Mor}_\infty(\beta_\lambda(Q), \beta_\lambda(R))$, define $[g] \cdot [f]$ to be the equivalence class of the mapping $g \cdot f$, where $(g \cdot f)(\mu) = g(\mu)f(\mu)$. This defines the category $\mathcal{L}_\infty$. Notice, using 12.4 and increasing $\lambda$ if necessary, that
these definitions are independent of the choice of $P$, $Q$, and $R$. Thus $\mathcal{L}_\infty$ is well defined.

For any $\psi \in \text{Mor}_\lambda(P,Q)$, there is an element $[f_\psi]$ of $\text{Mor}_\infty(\hat{P},\hat{Q})$, defined by $f_\psi(\mu) = \beta_{\lambda,\mu}(\psi)$ for any $\mu \in \Lambda(\lambda)$. The map $\beta_{\lambda}: \mathcal{E}_\lambda \rightarrow \mathcal{E}_\infty$ extends to a functor $\beta_{\lambda}: \mathcal{L}_\lambda \rightarrow \mathcal{L}_\infty$, where $\beta_{\lambda}$ is defined on morphisms by $\beta_{\lambda}(\psi) = [f_\psi]$.

**Lemma 12.5.** $(\beta_{\lambda}: \mathcal{L}_\lambda \rightarrow \mathcal{L}_\infty)_{\lambda \in \Lambda}$ is the direct limit of the nearly injective directed system $L = (\beta_{\lambda,\mu}: \mathcal{L}_\lambda \rightarrow \mathcal{L}_\mu)_{\lambda \leq \mu}$ of categories.

**Proof.** Let $(\gamma_{\lambda}: \mathcal{L}_\lambda \rightarrow \mathcal{E}_\lambda)_{\lambda \in \Lambda}$ be a family of functors compatible with the directed system $L$ of categories. By 12.4.2, we can define a function $\gamma: \mathcal{E}_\infty \rightarrow \text{Obj}(\mathcal{E})$ by $\gamma(P) = \gamma_{\lambda}(P)$ for $P \in \mathcal{E}_\infty$, where $P \in \mathcal{E}_\lambda$ and $\hat{P} = \beta_{\lambda}(P)$.

A similar argument allows us to define $\gamma$ on morphisms: Let $[f] \in \text{Mor}_\infty(\hat{P},\hat{Q})$ and pick a representative $f$ of $[f]$. We may choose $\lambda$ so that $\hat{P} = \beta_{\lambda}(P)$ and $\hat{Q} = \beta_{\lambda}(Q)$ for some $P,Q \in \mathcal{E}_\lambda$, and so that $\lambda \in \Omega_f$. Setting $\psi = f(\lambda)$, we have $\psi \in \text{Mor}_\lambda(P,Q)$ and $\beta_{\lambda}(\psi) = [f]$. We now “define” $\gamma([f])$ to be $\gamma_{\lambda}(\psi)$. As in the preceding paragraph, if $\gamma_{\lambda}(\psi) = \gamma_{\nu}(\phi)$ for some $\nu$ and some $\mathcal{L}_\nu$-morphism $\phi$, we may replace $\psi$ and $\phi$ by their images under the maps $\beta_{\lambda,\mu}$ and $\beta_{\nu,\mu}$, and reduce to the case where $\lambda = \nu$. Then $\beta_{\lambda}(\psi) = \beta_{\lambda}(\phi) = [f]$, whence $f(\lambda) = \psi = \phi$, and $\gamma$ is well defined on morphisms.

It is now straightforward to check that $\gamma$ is a functor, and $\gamma$ is then visibly the unique functor such that $\gamma_{\lambda} = \gamma \circ \beta_{\lambda}$ for all $\lambda \in \Lambda$. \qed

There is a functor $\pi_\infty: \mathcal{L}_\infty \rightarrow \mathcal{F}_\infty$, defined as follows. As a map from $\text{Obj}(\mathcal{L}_\infty)$ to $\text{Obj}(\mathcal{F}_\infty)$ we take $\pi_\infty$ to induce the identity map on $\mathcal{E}_\infty$. As a map of morphisms, define $\pi_\infty([f]): \hat{P} \rightarrow \hat{Q}$ by

$$\pi_\infty([f]): x \mapsto \pi_{\mu}(f(\mu))(x)$$

for any $\mu \in \Omega_f$ such that $x \in \beta_{\lambda,\mu}(P)$. The definition is independent of $\mu$, and the verification that $\pi_\infty([f])$ is in $\text{Hom}_\infty(\hat{P},\hat{Q})$ is straightforward, as is the verification of functoriality.

Next, define a family of monomorphisms of groups

$$\delta = \delta_\infty = (\delta_{\hat{P}}: \hat{P} \rightarrow \text{Aut}_{\mathcal{L}_\infty}(\hat{P}))_{\hat{P} \in \mathcal{E}_\infty},$$

as follows. Let $P \in \mathcal{E}_\lambda$ with $\hat{P} = \beta_{\lambda}(P)$, and let $x \in \hat{P}$. Then $x \in \beta_{\lambda,\mu}(P) = P_\mu$ for some $\mu \in \Lambda(\lambda)$. Define $\delta_{\hat{P}}(x)$ to be $[g_x]$, where $g_x(\nu) = \beta_{\mu,\nu}(\delta_{\mu}(x))$ for $\nu \in \Lambda(\mu)$. The verification that each $\delta_{\hat{P}}(x)$ is in $\text{Aut}_{\mathcal{L}_\infty}(\hat{P})$ and that $\delta_{\hat{P}}$ is a monomorphism reduces to the corresponding facts concerning the family $\delta_{\mu}$ of monomorphisms associated with $\mathcal{E}_\mu$.

Henceforth $\mathcal{L}_\infty$ will denote the triple consisting of the category $\mathcal{L}_\infty$, the functor $\pi_\infty$, and the collection $\delta_\infty$ of monomorphisms.
LEMMA 12.6. Let \( \hat{P}, \hat{Q} \in \mathcal{E}_\infty \). 

(a) \( \hat{P} \) acts semiregularly on \( \text{Mor}_\infty(\hat{P}, \hat{Q}) \) via \( x : [f] \mapsto \delta_{\hat{P}}(x^{-1}) \cdot [f] \).

(b) The orbits of \( Z(\hat{P}) \) on \( \text{Mor}_\infty(\hat{P}, \hat{Q}) \) are the fibers of \( \pi_\infty \).

Proof. As \( \delta_{\hat{P}} : \hat{P} \rightarrow \text{Aut}_{\mathcal{E}_\infty}(\hat{P}) \) is a monomorphism, the action in (1) defines an injective representation of \( \hat{P} \) on \( \text{Mor}_\infty(\hat{P}, \hat{Q}) \). Let \([f] \in \text{Mor}_\infty(\hat{P}, \hat{Q}), \lambda \in \Lambda, \)

and \( P, Q \in \mathcal{E}_\lambda \) with \( \hat{P} = \beta_\lambda(P) \) and \( \hat{Q} = \beta_\lambda(Q) \). Let \( x \in \hat{P} \), and suppose that \([f] \) is a fixed point for \( \delta_{\hat{P}}(x) \). Without loss of generality, we may assume that \( x \in P_\lambda \)

and that \( \lambda \in \Omega_f \). Set \( \psi = f(\lambda) \) and \( \phi = \pi_\lambda(\psi) \). Then \( \delta_{\lambda, P}(x) \cdot \psi = \psi \), and hence by conditions (B) and (C) in 2.4, \( \phi = \pi(\delta_{\lambda, P}(x) \cdot \psi) = c_x \phi \). Thus as \( \phi \) is injective, it follows that \( x \in Z(P) \), and then 2.4(A) yields \( x = 1 \). Thus (a) holds.

We next prove (b). Let \([f], [h] \in \text{Mor}_\infty(\hat{P}, \hat{Q}) \). We may assume that \( \Lambda(\lambda) = \Omega_f = \Omega_h \). Then \( \pi_\infty[f] = \pi_\infty[h] \) if and only if for each \( \mu \in \Lambda(\lambda), \pi_\mu(f(\mu)) = \pi_\mu(h(\mu)) \). As \( \delta_{\mu} \) is a pre-local group, this holds if and only if there exists \( z_\mu \in Z(P_\mu) \) with \( h(\mu) = \delta_{\mu, P_\mu}(z_\mu) \cdot f(\mu) \). (cf. 2.4(A)).

But \[
\delta_{\mu, P_\mu}(z_\mu) \cdot f(\mu) = h(\mu) = \beta_{\lambda, \mu}(h(\lambda)) = \beta_{\lambda, \mu}(\delta_{\lambda, P}(z_\lambda) \cdot f(\lambda)) = \beta_{\lambda, \mu}(\delta_{\lambda, P}(z_\lambda)) \cdot \beta_{\lambda, \mu}(f(\lambda)) = \delta_{\mu, P_\mu}(z_\lambda) \cdot f(\mu).
\]

By (1) applied in \( \mathcal{G}_\mu \) this holds if and only if \( z_\mu = z_\lambda \). Therefore we have shown that \( \pi_\infty[f] = \pi_\infty[h] \) if and only if for all \( \mu \in \Lambda(\lambda), z_\lambda = z_\mu \in Z(P_\mu) \) and \( h(\mu) = \delta_{\mu, P_\mu} \cdot f(\mu) \). Since \( \hat{P} \) is the union of the groups \( P_\mu \) for \( \mu \in \Lambda(\lambda) \), we conclude that \( \pi_\infty[f] = \pi_\infty[h] \) if and only if \( z_\lambda = z \in Z(\hat{P}) \). Thus (b) holds.

The fusion system \( \mathcal{F}_\infty \) is in general “too large”, in various ways. In particular, \( \pi_\infty \) need not map morphism sets in \( \mathcal{L}_\infty \) onto homomorphism sets in \( \mathcal{F}_\infty \), and as a result these homomorphism sets are in general too large for \( \mathcal{F}_\infty \) to serve as the fusion system in the limit of the direct system \( \mathcal{G} \).

The smallest fusion system on \( S_\infty \) containing \( \pi_\infty(\text{Mor}_\infty(\hat{P}, \hat{Q})) \) for all \( \hat{P} \) and \( \hat{Q} \in \mathcal{E}_\infty \) will be denoted \( \text{Im}(\pi_\infty) \). If each \( \mathcal{G}_\lambda \) is a \( p \)-local finite group then \( \text{Im}(\pi_\infty) \) will contain \( \iota_\lambda(\mathcal{F}_\lambda) \) for all \( \lambda \), as a consequence of Alperin’s Fusion Theorem. More generally, set
\[
\mathcal{F}_\Theta = \{ \text{Im}(\pi_\infty), \iota_\lambda(\mathcal{F}_\lambda) \mid \lambda \in \Lambda \}.
\]
Set
\[
\mathcal{G}_\infty = (S_\infty, \mathcal{F}_\infty, \mathcal{L}_\infty) \quad \text{and} \quad \mathcal{G}_\Theta = (S_\infty, \mathcal{F}_\Theta, \mathcal{L}_\infty).
\]
Let \( \phi \in \text{Hom}_\lambda(P, Q) \), with \( P \) and \( Q \) in \( \mathcal{E}_\lambda \). By 2.4(A) there is \( \psi \in \text{Mor}_\lambda(P, Q) \) such that \( \pi_\lambda(\psi) = \phi \). Let \( \lambda \leq \mu \leq \nu \) in \( \Lambda \). Condition (MG2) in 2.10 yields:
\[
(12.7) \quad \pi_\nu(\beta_{\lambda, \nu}(\psi)) \mid_{P_\mu} = \pi_\nu(\beta_{\mu, \nu}(\beta_{\lambda, \mu}(\psi))) \mid_{P_\mu} = \pi_\mu(\beta_{\lambda, \mu}(\psi)).
\]
Definition 12.8. We say that the element $\mu$ of $\Lambda(\lambda)$ is $\phi$-good provided that, for all $\nu \in \Lambda(\mu)$, $\pi_\nu(\beta_{\lambda,\mu}(\psi))$ is the unique member of $\text{Hom}_\nu(P_\nu, Q_\nu)$ which restricts to $\pi_\mu(\beta_{\lambda,\mu}(\psi))$ on $P_\mu$. Denote by $E_\phi$ the set of all $\phi$-good elements of $\Lambda_{\phi}$.

Lemma 12.9. (a) Either of the following conditions implies that $\mathcal{G}_\infty$ is a pre-local group.

1. $\pi_\infty : \text{Mor}_\infty(\hat{P}, \hat{Q}) \to \text{Hom}_\infty(\hat{P}, \hat{Q})$ is a surjection, for all $\hat{P}, \hat{Q} \in \mathcal{E}_\infty$.

2. For every $\lambda$, and for every $\mathcal{F}_\lambda$-morphism $\phi$, $E_\phi$ is nonempty.

(b) If $\mathcal{G}_\infty$ is a pre-local group, then so is $\mathcal{G}_\phi$.

Proof. In order to show that $\mathcal{G}_\infty$ is a pre-local group, we must verify conditions (A) through (C) in 2.4, and that $\mathcal{E}_\infty \subseteq \mathcal{F}_\infty$. Under the hypothesis of (1), Condition (A) is an immediate consequence of 12.6; we leave the remaining verifications in (1) to the reader.

Assume next that $\mathcal{G}_\infty$ is a pre-local group. Then

$$\pi_\infty(\text{Mor}_\infty(\hat{P}, \hat{Q})) = \text{Hom}_\infty(\hat{P}, \hat{Q}),$$

and so $\text{Hom}_{\mathcal{F}_\phi}(\hat{P}, \hat{Q}) = \pi_\infty(\text{Mor}_\infty(\hat{P}, \hat{Q}))$. The argument of the preceding paragraph then yields (b).

Finally assume the hypothesis of (2). Let $\phi \in \text{Hom}_\infty(\hat{P}, \hat{Q})$. We may assume $\lambda \in \Lambda_{\phi}$, and thus the restriction $\phi_{\mu}$ of $\phi$ to $P_\mu$ is in $\text{Hom}_\mu(P_\mu, Q_\mu)$ for all $\mu \in \Lambda(\lambda)$. Further, we may assume that $\lambda \in \mathcal{E}_{\phi, \lambda}$. Choose $\psi \in \pi_{\lambda}^{-1}(\phi_{\lambda})$, and consider the map $f = f_{\lambda}$ on $\Lambda(\lambda)$ defined before 12.5, such that $[f] \in \text{Hom}_\lambda(\hat{P}, \hat{Q})$ and $\beta_\lambda(\psi) = [f]$. Thus $f(\mu) = \beta_{\lambda,\mu}(\psi)$, and so $\mu(\mu f(\mu)) = \pi_\mu(\beta_{\lambda,\mu}(\psi))$ on $P_\lambda$. Thus as $\lambda \in E_{\phi, \lambda}$, $\pi_\lambda(f(\mu)) = \phi_\mu$, so that $\pi_\infty([f]) = \phi$. Thus we have verified the hypothesis of (1). Therefore by (1), $\mathcal{G}_\infty$ is a pre-local group, and the proof of (a) is complete. \qed

Lemma 12.10. Assume that $\mathcal{G}$ is nearly injective.

(a) If $\mathcal{G}_\phi$ is a pre-local group then $\beta_\lambda = (\iota_\lambda, \beta_\lambda) : \mathcal{G}_\lambda \to \mathcal{G}_\phi$ is an embedding of pre-local groups, and $(\beta_\lambda : \mathcal{G}_\lambda \to \mathcal{G}_\phi)_{\lambda \in \Lambda}$ is the direct limit of $\mathcal{G}$ in the category of pre-local groups and embeddings.

(b) Assume that there exists a pre-local group $\mathcal{G} = (\mathcal{S}, \mathcal{F}, \mathcal{D})$, and a family $\Sigma$ of embeddings

$$\gamma_\lambda = (\alpha_\lambda, \gamma_\lambda) : \mathcal{G}_\lambda \to \mathcal{G}, \lambda \in \Lambda,$$

of pre-local groups compatible with $\mathcal{G}$. Assume that the following conditions hold for all $\lambda$.

(i) $\mathcal{S} = \bigcup_{\lambda \in \Lambda} \alpha_\lambda(S_\lambda)$.

(ii) For each $\mathcal{F}_\lambda$-morphism $\phi$, $E_\phi \neq \emptyset$. 

(iii) For all $P \in \mathcal{E}_\lambda$,
\[
\gamma_\lambda(P) = \bigcup_{\lambda \leq \mu} \alpha_\mu(\beta_{\lambda,\mu}(P)).
\]

(iv) For each $\bar{P}, \bar{Q} \in \mathcal{E}$ and $\phi \in \text{Hom}_{\mathcal{F}}(\bar{P}, \bar{Q})$, there exist $\lambda \in \Lambda$ and $P, Q \in \mathcal{E}_\lambda$ such that $\gamma_\lambda(P) = \bar{P}$, $\gamma_\lambda(Q) = \bar{Q}$, and such that for each $\mu \in \Lambda(\lambda)$,
\[
\bar{\phi}(\alpha_\mu(\beta_{\lambda,\mu}P)) \in \alpha_\mu(\text{Hom}_\mu(\beta_{\lambda,\mu}(P), \beta_{\lambda,\mu}(Q))).
\]

(v) $\tilde{\mathcal{F}}$ has the Alperin generation property with respect to $\mathcal{E}$ (cf. 2.14).

(vi) $\Sigma$ is nearly injective on $\Delta(\mathcal{G})$.

Then $\mathcal{G}_{\mathcal{G}}$ is a pre-local group, $\mathcal{G}_{\mathcal{G}}$ is the limit of $\mathcal{G}$, and $\mathcal{G} \cong \mathcal{G}_{\mathcal{G}}$ as pre-local groups.

Proof. Assume that $\mathcal{G}_{\mathcal{G}}$ is a pre-local group. We check that $\beta_\lambda$ satisfies the conditions (MG1) through (MG3) in 2.10. We have (MG1) since, for any $P \in \mathcal{E}_\lambda$,
\[
P = \beta_{\lambda,\lambda}(P) \leq \bigcup_{\mu \in \Lambda(\lambda)} \beta_{\lambda,\mu}(P) = \beta_\lambda(P).
\]

Let $\psi \in \text{Mor}_\lambda(P, Q)$. Then $\pi_\infty(\beta_\lambda(\psi))|_P = \pi_\lambda(\psi)$ by definition of $\pi_\infty$, and thus (MG2) holds. Condition (MG3) is the assertion that $\beta_\lambda \circ \delta_{\lambda,\mu} P = \delta_{\beta_\lambda(P)}$ on $P$, which holds by definition of $\delta_P$. Thus $\beta_\lambda$ is a morphism of pre-local groups.

Recall that $\beta_\lambda(\psi) = [f_\psi]$, where $f_\psi(\mu) = \beta_{\lambda,\mu}(\psi)$ for all $\mu \in \Lambda(\lambda)$. In particular by (ii), $\beta_\lambda$ is injective as a mapping from $\text{Mor}_\lambda(P, Q)$ into $\mathcal{L}_\infty$-morphisms, and $\beta_\lambda$ is therefore an embedding.

Let $\tilde{\mathcal{G}}$ satisfy the initial hypothesis in (b), but for the moment do not assume the conditions (i) through (vi) in (b). Define the functor $\gamma: \mathcal{L}_\infty \rightarrow \tilde{\mathcal{F}}$ as in the proof of 12.5. That is, $\gamma(\bar{P}) = \gamma_\lambda(P)$ and $\gamma([f]) = \gamma_\lambda(f(\lambda))$, where $\beta_\lambda(P) = \bar{P}$ and $\beta_\lambda(f(\lambda)) = [f]$. Then $\gamma$ is the unique functor satisfying $\gamma_\lambda = \gamma \circ \beta_\lambda$ for all $\lambda \in \Lambda$.

Let $\lambda \in \Lambda$. Define $\alpha_0: S_\infty \rightarrow \tilde{\mathcal{S}}$ by $\alpha_0|_{S_\lambda} = \alpha_\lambda$. As each $\alpha_\lambda$ is injective, $\alpha_0$ is an injective homomorphism. For any pair of subgroups $P, Q$ of $S$, and any $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$, define $\alpha(\phi)$ to be the homomorphism $\alpha_0^{-1} \circ \phi \circ \alpha_0: P \alpha_0 \rightarrow Q \alpha_0$.

Then $\alpha$ is a morphism of fusion systems if and only if the following condition holds for all $P, Q \leq S_\infty$:

(*) $\alpha(\phi) \in \text{Hom}_{\mathcal{F}}(\alpha_0(P), \alpha_0(Q))$.

If $P, Q \leq S_\lambda$ and $\phi \in \text{Hom}_\lambda(P, Q)$, then (*) holds as $\alpha_\lambda$ is a morphism of fusion systems. If $P, Q \in \mathcal{E}_\infty$, then $\phi = \pi_\infty(\beta_\lambda(f))$ for some $f \in \text{Mor}_\lambda(P, Q_\lambda)$. Write $\phi_\lambda$ for the restriction of $\phi$ to $P_\lambda$. Then on $P_\lambda$:
\[
\tilde{\pi}(\gamma(\beta_\lambda(f))) = \tilde{\pi}(\gamma_\lambda(f)) = \alpha_\lambda(\pi_\lambda(f)) = \alpha_\lambda(\phi_\lambda) = \alpha_\lambda \circ \phi_\lambda \circ \alpha_\lambda^{-1},
\]
and as this holds for all $\lambda$ in a closed subset of $\Lambda$, we conclude that $\widetilde{\tau}(\gamma(\beta_{\lambda}(f))) = \alpha \circ \phi \circ \alpha^{-1} = \alpha(\phi)$, and we again obtain (*). Since, by definition, $\mathcal{F}_\Phi$ is generated by such morphisms, (*) holds in general, and $\alpha$ is a morphism of fusion systems. We leave it to the reader to check that $(\alpha, \gamma)$ satisfies the axioms in Definition 2.10, and hence is a morphism of pre-local groups, yielding (a).

Now assume all of the hypotheses of (b). Then by (b)(ii) and 12.9, $\mathcal{G}_\Phi$ is pre-local group. Then (a) says that $\mathcal{G}_\Phi$ is the limit of $\mathcal{G}$, and supplies the morphism $\gamma = (\alpha, \gamma): \mathcal{G}_\infty \rightarrow \mathcal{G}$ described above. By (b)(i) and the definition of $\alpha$, $\alpha_0: S_\infty \rightarrow \mathcal{S}$ is an isomorphism of groups.

The key step in the proof of (b) is to show that each of the sets $\text{Hom}_{\mathcal{G}}(\widetilde{P}, \widetilde{Q})$, with $\widetilde{P}$ and $\widetilde{Q}$ in $\mathcal{E}$, lies in the image of $\alpha$. So let $\widetilde{P}, \widetilde{Q} \in \mathcal{E}$ and $\tilde{\phi} \in \text{Hom}_{\mathcal{G}}(\widetilde{P}, \widetilde{Q})$. Let $\lambda$, $P$, and $Q$ be as in (iv). By (iii),

$$\alpha(\beta_{\lambda}(P)) = \bigcup_{\lambda \leq \mu} \alpha(\beta_{\lambda,\mu}(P)) = \gamma_{\lambda}(P) = \widetilde{P},$$

so that, in particular, $\mathcal{E} \subseteq \alpha(\mathcal{E}_\infty)$.

For $\mu \in \Lambda(\lambda)$ write $P_\mu$ for $\beta_{\lambda,\mu}(P)$ and $Q_\mu$ for $\beta_{\lambda,\mu}(Q)$, and set $\tilde{\phi}_\mu = \tilde{\phi}|_{\alpha(\mu)(P_\mu)}$. Then (iv) says that for all such $\mu$ there exists $\phi_\mu \in \text{Hom}_{\mathcal{G}}(P_\mu, Q_\mu)$ with $\alpha(\mu)(\phi_\mu) = \tilde{\phi}_\mu$. By (ii), we may assume $\lambda$ is chosen so that $\lambda \in E_{\Phi, \lambda}$. Let $\psi_\lambda \in \pi_{\lambda}^{-1}(\phi_\lambda)$ and set $\overline{\psi} = \gamma_\lambda(\psi_\lambda)$, $\overline{\eta} = \overline{\tau}(\overline{\psi})$, and $\eta_\lambda = \beta_{\lambda,\mu}(\psi_\mu)$. Then arguing as in the proof of (a), we get $\overline{\eta} = \alpha_\mu(\pi_\mu(\psi_\mu))$ on $\alpha_\mu(P_\mu)$, so that, in particular, $\overline{\eta} = \tilde{\phi}_\mu$ on $\alpha_\mu(P)$. But also $\eta_{\lambda} = \pi_\lambda(\psi_\lambda) = \pi_\mu(\psi_\mu)$ on $P$. As $\lambda \in E_{\Phi, \lambda}$ we conclude that $\pi_\mu(\psi_\mu)$ is the unique extension of $\phi_\mu$ to $P_\mu$. Then

$$\tilde{\phi}_\mu = \alpha_\mu(\phi_\mu) = \alpha_\mu(\pi_\mu(\psi_\mu)) = \overline{\eta}$$
on $\alpha_\mu(P_\mu)$. That is $\overline{\eta} = \tilde{\phi}$. Thus $\tilde{\phi} = \overline{\tau}(\gamma(\beta_{\lambda}(\psi))) = \alpha(\pi_{\infty}(\beta_{\lambda}(\psi)))$ on $\alpha(\beta_{\lambda}(P_\lambda)) = \widetilde{P}$, so indeed $\tilde{\phi}$ is in $\alpha(\text{Hom}_{\mathcal{G}}(P_\lambda, Q_\lambda))$.

Next $\alpha(\text{Hom}_{\mathcal{G}}(P, Q)) \subseteq \text{Hom}_{\mathcal{G}}(\alpha(P), \alpha(Q))$ as $\alpha$ is a morphism of fusion systems; further, this map is injective by definition. In particular $\alpha(\text{Im}(\pi_{\infty})) \subseteq \mathcal{F}$ and of course $\alpha(\iota_{\lambda}(\mathcal{F}_\lambda)) \subseteq \mathcal{F}$, so that by definition of $\mathcal{F}_\Phi$,

$$\alpha(\mathcal{F}_\Phi) = \alpha(\langle \text{Im}(\pi_{\infty}), \iota_{\lambda}(\mathcal{F}_\lambda) \mid \lambda \in \Lambda \rangle) = \langle \alpha(\text{Im}(\pi_{\infty})), \alpha(\iota_{\lambda}(\mathcal{F}_\lambda)) \mid \lambda \in \Lambda \rangle \subseteq \mathcal{F}.$$
By (a), \( \gamma : \mathcal{O}_\varnothing \to \widetilde{\mathcal{O}} \) is an embedding, and \( \gamma_\lambda = \gamma_\mu \circ \beta_\lambda,\mu \) for \( \lambda \leq \mu \). In order to show that \( \gamma \) is an isomorphism, it remains to show that \( \gamma \) defines a bijection \( \mathcal{E}_\infty \to \widetilde{\mathcal{E}} \), and defines bijections on morphism sets. The first condition is a consequence of (vi) and 12.4(c), so it remains to verify the second condition.

Let \( \psi \in \text{Mor}_{\tilde{T}}(\tilde{P}, \tilde{Q}) \) be an \( \tilde{T} \)-morphism. Set \( \tilde{\phi} = \tilde{\pi}(\psi) \). As \( \alpha \) is an isomorphism of fusion systems we may choose \( \tilde{\phi}, \tilde{\pi} \in \text{Hom}_{\tilde{T}}(\tilde{P}, \tilde{Q}) \) so that \( \alpha(\phi) = \tilde{\phi} \). Choose \( \psi \in \pi^{-1}(\phi) \). Then \( \gamma(\psi) \) lies in the \( \pi \)-fiber over \( \tilde{\phi} \), as \( \gamma \) satisfies (MG2) and \( \alpha(\tilde{P}) = \tilde{P} \). As \( \tilde{\mathcal{O}} \) is a pre-local group there then exists \( y \in Z(\tilde{P}) \) such that \( \gamma(\psi) = \delta_{\tilde{P}}(y) \cdot \tilde{\phi} \). Let \( x \) be the element of \( Z(\tilde{P}) \) which is mapped to \( y^{-1} \) by \( \alpha_0 \). As \( \gamma \) satisfies (MG3) we obtain \( \gamma(\delta_{\tilde{P}}(x) \circ \psi) = \tilde{\psi} \), and thus \( \gamma \) is surjective on morphism sets.

Finally, let \([f],[h] \in \text{Mor}_\infty(\tilde{P}, \tilde{Q}) \) with \( \gamma([f]) = \gamma([h]) \). As \( \alpha \) is injective on homomorphisms, it follows from (MG2) that \( [f] \) and \( [h] \) lie in the same \( \pi_\infty \) fiber, and thus \( [h] = \delta_{\tilde{P}}(z) \cdot [f] \) for some \( z \in Z(\tilde{P}) \). We may choose \( \lambda \) so that \( z \in \tilde{P} \cap S_\lambda := P \), and also so that \( \lambda \in \Omega_f \cap \Omega_h \) for suitable representatives \( f \) and \( h \) of the given morphisms. Then \( z \in Z(P \rangle \) and

\[
\gamma([f]) = \gamma([h]) = \gamma_\lambda(h(\lambda)) = \gamma_\lambda(\delta_\lambda,p(z) \cdot f(\lambda)) = \gamma_\lambda(\delta_\lambda,p(z) \cdot f(\lambda)) = \gamma_\lambda(\delta_p, p(z)) \cdot \gamma_\lambda(f(\lambda)) = \delta_{\tilde{P}}(\alpha_0(z)) \cdot \gamma([f]).
\]

As \( \delta_{\tilde{P}} \) defines a free action of \( Z(\tilde{P}) \) on \( \text{Mor}_\infty(\tilde{P}, \tilde{Q}) \), we conclude that \( z = 1 \), and that \( \gamma \) is bijective on morphism sets. Thus \( \gamma \) is an isomorphism of categories, and (b) is proved. \( \square \)

Theorem C is the following result:

**Theorem 12.11.** Take \( \Lambda \) to be the set of nonnegative integers, and for each \( i \in \Lambda \) let \( \mathcal{G}_i = (S_i, \mathcal{F}_i, \mathcal{L}_i) \) and \( \mathcal{G} = (S, \mathcal{F}, \mathcal{L}) \) be the 2-local groups defined prior to 11.2. Let \( \theta \) and \( \theta_i \) be the signalizer functors in 11.3, and let \( \beta_{i,j} : \mathcal{F}^\text{rc}_i \to \mathcal{F}^\text{rc}_j \) and \( \beta_i : \mathcal{F}^\text{rc}_i \to \mathcal{F}^\text{rc}_i \) be the mappings defined prior to 11.2. Then the following hold:

(a) For each \( i, j \in \Lambda \), with \( i \leq j \), the mappings \( \beta_{i,j} \) and \( \beta_i \) extend to embeddings

\[
\beta_{i,j} = (\iota_{i,j}, \beta_{i,j}) : \mathcal{G}_i \to \mathcal{G}_j \quad \text{and} \quad \beta_i = (\iota_i, \beta_i) : \mathcal{G}_i \to \mathcal{G}
\]

of 2-local groups, and \( \beta_j \circ \beta_{i,j} = \beta_i \).

(b) \( \mathcal{G} := (\mathcal{G}_i : \mathcal{G}_i \to \mathcal{G}_j)_{i \leq j \in \Lambda} \) is a directed system of embeddings of 2-local finite groups.

(c) The direct limit \( \mathcal{G}_\infty \) of \( \mathcal{G} \), in the category of pre-local groups and embeddings, admits the structure of a 2-local group isomorphic to \( \mathcal{G} \).

**Proof.** Write \( \mathcal{L}^\text{rc}_i \) and \( \mathcal{L}^\text{rc} \) for the restriction of the centric linking systems \( \mathcal{L}_i \) and \( \mathcal{L} \) to radical centric linking systems on \( \mathcal{F}^\text{rc}_i \) and \( \mathcal{F}^\text{rc} \), respectively.
First, 11.2 and 11.3 show that the mappings $\beta_i$ and $\beta_{i,j}$ are well defined and satisfy conditions (1') and (2') in Proposition 2.12, relative to the appropriate signalizer functors. Then 2.11 and 2.12 yield embeddings of 2-local groups, as in (a), in which $\beta_i$ and $\beta_{i,j}$ act on $\F_i$-morphisms via

$$\beta_i: \theta_i(P)g \mapsto \theta(\beta_i(P))g \quad \text{and} \quad \beta_{i,j}: \theta_i(P)g \mapsto \theta(\beta_{j}(P))g$$

for $P \in \F_i$ and $g \in N_{G_i}(P, S)$. In order to check that $\beta_j \circ \beta_{i,j} = \beta_i$ in the category of 2-local groups, it suffices to check the equality on objects. This follows from 10.10 in cases (1) and (2) of 10.9(c), and from 10.7 in case (3) when $Y = H$. Suppose $Y = K$. By construction, $N_{G_i}(P) \leq N_{G_j}(\beta_{i,j}(P))$, so that $(\beta_j \circ \beta_{i,j})(P) \leq \beta_i(P)$. We check from 10.2 that no proper $N_{G_i}(P)$-invariant subgroup of $\beta_i(P)$ is in $\F_i$, completing the proof of (a).

The equality $\beta_j \circ \beta_{i,j} = \beta_i$ implies that

$$(\beta_{j,k} \circ \beta_{i,j})(P) = \beta_{j,k}(\beta_{i,j}(P)) = \beta_j(\beta_{i,j}(P)) \cap G_k = \beta_i(P) \cap G_k = \beta_{i,k}(P),$$

so that $\beta_{j,k} \circ \beta_{i,j} = \beta_{i,k}$. Also $\beta_{i,i} = 1$ by 11.12(c). Hence (b) holds.

We next check that the hypotheses of 12.10(b) are satisfied with $\mathcal{G}, \mathcal{F}_i, \beta_i, i \in \Lambda$, in the roles of $\mathcal{G}, \mathcal{F}_i, \gamma_\lambda, \lambda \in \Lambda$, respectively.

First, $S$ is the union of the groups $S_i$ for $i \in \Lambda$, so that condition (i) of 12.10(b) holds. Second, for any $\phi = c_g \in \text{Hom}_{\mathcal{F}}(P, Q)$ we have $g \in G_i$ for some $i$, so that $\phi|_{S_i} \in \text{Hom}_{\mathcal{F}}(P \cap S_i, Q \cap S_i)$ for all $j \geq i$, and hence condition (iv) holds.

Let $P \in \F_i$, and for $j \geq i$ set $P_j = \beta_{i,j}(P_i)$. We have $P_j = \beta_i(P_i) \cap G_j$ for such $j$, and $S$ is the union of its subgroups $S_j$, and so condition (ii) of 12.10(b) holds.

We claim that there exists $j \geq i$ such that $C_G(P_j)$ centralizes $P_k$ for all $k \geq j$. Suppose first that $|Z_{P_j}| \leq 4$. Then from the proof of 10.9, $C_G(P_i) \leq P_i$. But $Z(\beta_i(P_i))$ is finite, and $P_k = \beta_i(P) \cap G_k$ for $k \geq i$ and so, from (ii), we may choose $j \geq i$ with $Z(P_j) = Z(\beta_i(P_i))$. Thus $C_G(P_j) = Z(P_k)$ for $k \geq j$. On the other hand if $|Z_{P_j}| > 4$ then $P_i \in A^G$ or $P_i = C_{S_i}(E)$ by 10.5. In the first of the these two cases, $P_i = P_k$ for all $k \geq i$, while in the second $C_G(P_i) = EC_{\theta(A)}(T_2) = C_G(P_k)$ for any $k \geq i$, by 7.10 and 7.13. Thus the claim is established.

We now verify condition (ii) of 12.10(b). Let $\phi \in \text{Hom}_{\mathcal{F}}(P_j, Q_j)$ and suppose that for some $k \geq j$, we have $\phi_1, \phi_2 \in \text{Hom}_{\mathcal{F}}(P_k, Q_k)$ extending $\phi$. Then $\phi_i$ is the restriction of $c_{g_i}$ to $P_k$ for some $g_r \in G, r = 1, 2$. Then $g_1 g_2^{-1} \in C_G(P_j) \leq C_G(P_k)$ by the claim, so $\phi_1 = \phi_2$. This yields (ii).

Condition (v) of 12.10(b) follows from 9.10. Finally each $\beta_i$ is injective on objects, since $\beta_i(P) \cap S_i = P$ by 11.12(c). Thus $\mathcal{G}$ is nearly injective, and condition (vi) of 12.10(b) holds. Therefore we conclude from 12.10(b) that $(S, \mathcal{F}, \mathcal{L}_r) \cong \mathcal{G}_{\mathcal{F}_i}$ as pre-local groups, via the isomorphism $\gamma = (\alpha, \gamma)$ constructed in the proof of 12.5 and 12.10. In particular $\alpha(\mathcal{F}) = \mathcal{F}_i$, so as $\alpha: \mathcal{F}_{\mathcal{G}_i} \to \mathcal{F}$ is an isomorphism of
fusion systems, \( \mathcal{C} = \mathcal{F}_\mathcal{c} \), and \( \mathcal{F}_\mathcal{c} \) is saturated as \( \mathcal{F} \) is saturated. Transferring the centric linking system \( \mathcal{L} \) on \( \mathcal{F} \) to \( \mathcal{F}_\mathcal{c} \) via \( \alpha \), we may regard \( \mathcal{G}_\mathcal{c} \) as a 2-local group, and the isomorphism \( \gamma \) of pre-local groups is then also an isomorphism of 2-local groups. This completes the proof of (c).

Theorem D is essentially the following result. We thank Ran Levi and Bob Oliver for guiding us through a proof.

**Theorem 12.12.** The 2-completed nerve \( |\mathcal{L}_\mathcal{c}|_2^\wedge \) is homotopy equivalent to \( B\mathcal{D}1(4) \).

**Proof.** Let \( \mathcal{L}^\mathcal{c}_m \) be the full subcategory of \( \mathcal{L}_m^\mathcal{c} \) whose objects are centric in \( \mathcal{F}_n^\mathcal{c} \) for all \( n \geq m \). Then \( \mathcal{F}_m^\mathcal{c} \subseteq \text{Obj}(\mathcal{L}^\mathcal{c}_m) \) for all \( m \), by 10.9(b). Set \( \gamma_m = \beta_{m,m+1} \), and consider the diagram of categories and functors

\[
\begin{array}{ccc}
\mathcal{L}^\mathcal{c}_m & \xrightarrow{\gamma_m} & \mathcal{L}^\mathcal{c}_{m+1} \\
\iota_m \downarrow & & \downarrow \iota_{m+1} \\
\mathcal{F}^\mathcal{c}_m & \xrightarrow{\iota_m^\mathcal{c}} & \mathcal{F}^\mathcal{c}_{m+1},
\end{array}
\]

where \( \iota \) is in every instance an inclusion functor. We claim that this diagram commutes up to a natural homomorphism \( \mu: \iota^\mathcal{c} \circ \iota_m \rightarrow \iota_{m+1} \circ \gamma_m \). Since \( \iota^\mathcal{c} \circ \iota_m \) is the identity map on objects, what this means is that for all \( P, Q \in \mathcal{F}^\mathcal{c}_m \), there are \( \mathcal{L}^\mathcal{c}_{m+1} \)-morphisms \( \mu_P \) and \( \mu_Q \) such that, for each \( \psi \in \text{Mor}_m(P, Q) \), the following diagram commutes.

\[
\begin{array}{ccc}
P & \xrightarrow{\iota_m^\mathcal{c}(\psi)} & Q \\
\mu_P \downarrow & & \downarrow \mu_Q \\
\gamma_m(P) & \xrightarrow{\iota_{m+1}(\gamma_m(\psi))} & \gamma_m(Q).
\end{array}
\]

Indeed, for any \( R \in \mathcal{L}^\mathcal{c}_m \) define \( \mu_R \) to be \( \theta_{m+1}(R) \). Recall that \( \psi = \theta_m(P)g \) for some \( g \in N_{G_m}(P, Q) \), by the definition of \( \mathcal{L}_m \). The functor \( \iota^\mathcal{c} \) sends \( \psi \) to \( \psi \) regarded as an element of \( \text{Mor}_{m+1}(P, Q) \). That is, we have \( \iota^\mathcal{c}(\psi) = \theta_{m+1}(P)g \), and hence (in our mix of left- and right-hand notation, as set forth in §1)

\[
\iota^\mathcal{c}(\iota_m(\psi)) \cdot \mu_Q = \theta_{m+1}(P)g \theta_{m+1}(Q) = \theta_{m+1}(P)g,
\]

while also

\[
\mu_P \cdot \iota_{m+1}(\gamma_m(\psi)) = \theta_{m+1}(P) \theta_{m+1}(\gamma_m(P))g = \theta_{m+1}(P)g.
\]

Thus \( \mu \) is a natural transformation as desired, and the claim is proved.

Recall that the nerve of a small category \( \mathcal{C} \) is a simplicial set (or equivalently, the topological realization of a simplicial set) whose \( k \)-simplices are chains.
(\alpha_0, \ldots, \alpha_k) \) of composable morphisms in \( \mathcal{C} \). If \( f : \mathcal{C} \to \mathcal{D} \) is a functor of small categories, then there is a continuous map \( |f| : |\mathcal{C}| \to |\mathcal{D}| \) of spaces, given by \( (\alpha_0, \ldots, \alpha_k) \mapsto (f(\alpha_0), \ldots, f(\alpha_k)) \).

Set \( X_m = |\mathcal{F}_m^{cc}|, Y_m = |\mathcal{F}_m^{cc}| \), and consider the following diagram of spaces and continuous maps.

\[
\begin{array}{cccccc}
X_1 & \xrightarrow{|\gamma_1|} & X_2 & \xrightarrow{|\gamma_2|} & X_3 & \xrightarrow{|\gamma_3|} & \ldots \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
Y_1 & \xrightarrow{f_1^{cc}} & Y_2 & \xrightarrow{f_2^{cc}} & Y_3 & \xrightarrow{f_3^{cc}} & \ldots
\end{array}
\]

Here, we are taking \( f_m := |t_m| \) and \( f_m^{cc} := |t_m^{cc}| \). Each \( f_m^{cc} \) may be viewed as inclusion, since \( t_m^{cc} \) induces an inclusion of \( \text{Mor}_m(P, Q) \) into \( \text{Mor}_{m+1}(P, Q) \) for any \( P, Q \in \mathcal{F}_m^{cc} \). Similarly, since each \( \gamma_m \) is an embedding, \( \gamma_m \) induces an injective mapping \( \text{Mor}_m(P, Q) \to \text{Mor}_{m+1}(P, Q) \) for \( P, Q \in \mathcal{F}_m^{cc} \), and hence \( |\gamma_m| \) is injective. There is then no harm in viewing each \( |\gamma_m| \) as an ordinary inclusion of topological spaces (and in adjusting the vertical arrows by suitable homeomorphisms, to compensate for this). The direct limit \( X \) of the top row in \((***)\) is then the union of the spaces \( X_n \). No such adjustment is necessary for the bottom row, whose union we denote by \( Y \).

It is the content of [LO02, Prop. 4.3] that the 2-completion \( (Y)^\wedge_2 \) of \( Y \) is \( B \text{DI}(4) \), up to homotopy equivalence. That this is so requires some explanation, since the union taken in [LO02] is that of a somewhat different collection of spaces than \( \{Y_m\}_{m>0} \). Namely, Levi and Oliver choose a sequence \( (n_i)_{i>0} \) of positive integers, so that each \( n_i \) divides \( n_{i+1} \) and so that every positive integer divides some \( n_i \). They then show that \( B \text{DI}(4) \) is the 2-completion of the union of the spaces \( |\mathcal{F}_{sol}(q^{ni})^{cc}| \), for any odd prime power \( q \). We may take \( q = p \), and may take the sequence \( (n_i)_{i>0} \) so that \( 2^i \) is the highest power of 2 dividing \( n_i \). Then \( \mathcal{F}_{sol}(p^{ni}) \) is a fusion system over the Sylow 2-subgroup \( S_i \) of \( \text{Spin}_7(p^{ni}) \). As the 2-shares of \( \text{Spin}_7(p^{ni}) \) and \( \text{Spin}_2(p^{2i}) \) are the same, \( \text{Spin}_7(p^{ni}) \) has the same Sylow 2-subgroup as \( \text{Spin}_2(p^{2i}) \). By a result in [COS06] the fusion systems \( \mathcal{F}_{sol}(p^{ni}) \) and \( \mathcal{F}_{sol}(p^{2i}) \) are isomorphic, and their corresponding linking systems are then isomorphic [LO02, Lemma 3.2]. Thus, the union of the nerves \( |\mathcal{F}_{sol}(p^{ni})^{cc}| \) is homeomorphic to \( Y \), and \( (Y)^\wedge_2 \) may be identified with \( B \text{DI}(4) \).

Since it is obvious from the definitions that the nerve of an increasing union of categories is the union of the nerves, it follows from 12.5 that the space \( X \) is homeomorphic to \( |\mathcal{F}_\emptyset| \). Thus, it remains only to show that \( X \) is homotopy equivalent to \( Y \).

The existence of a natural transformation \( \mu \) as in \((***)\) implies that each of the squares in \((***)\) commutes up to homotopy (cf. [Dwy01, Prop. 5.2]. For
any $m$, the inclusion of the simplicial set $X_m$ in $X_{m+1}$ is a CW-pair, and so the homotopy extension property for CW-pairs [Hat02, Prop. 0.16] implies that $f_2$ may be replaced by a map $f'_2$ which is homotopic to $f_2$ and which extends $f_1$. We continue up the chain, replacing $f_{m+1}$ by a map $f'_{m+1}$ which is homotopic to $f_{m+1}$ and extends $f'_m$. Now define $f : X \to Y$ to be the union of the maps $f'_m$.

Observe that every finite subcomplex of the CW-complex $X$ (or $Y$) is contained in some $X_i$ (or $Y_i$). Since every compact subset of a CW-complex is contained in a finite subcomplex, every compact subset of $X$ or $Y$ is contained in some $X_i$ or $Y_i$. From this, it follows directly from the definition of homotopy groups that for each $n$, $\pi_n(X)$ is the direct limit of the $\pi_n(X_i)$, and similarly for $\pi_n(Y)$. That each $f_m$ (and hence also each $f'_m$) is a homotopy equivalence is given by [BCG+05, Th. B], and thus $\pi_n(f)$ is an isomorphism for all $n$. Then $f$ is a homotopy equivalence by Whitehead’s Theorem, and the proof is complete. 

We close with an example.

Example 12.13. Let $p$ be a prime and let $G = G(F)$ be a Chevalley group over the algebraic closure $F$ of $\mathbb{F}_p$ of Lie rank $l$. Let $\Sigma$ be the set of positive integers, partially ordered by $n \leq m$ if $n$ divides $m$. Set $I = \{1, \ldots, l\}$ and let $(P_J \mid \emptyset \neq J \subseteq I)$ be the set of proper parabolic subgroups of $G$ over a fixed Borel subgroup $B = P_I$. For $J \subseteq I$ let $S_J$ be the unipotent radical of $P_J$, and set $S = S_I$. Let $\psi_1 : a \mapsto a^p$ be the Frobenius map on $F$, and regard $\psi_1$ also as a field automorphism of $G$. For $k \geq 1$ set $\psi_k = \psi_1^k$, and let $G_k = G_{\psi_k}$ be the group of fixed points of $\psi_k$ on $G$. Set $S_k = S \cap G_k$, $S_{J,k} = S_J \cap G_k$, and let

$\mathcal{G}_k = \mathcal{G}_{S_k}(G_k) = (S_k, \mathcal{F}_k, \mathcal{L}_k)$

be the $p$-local finite group associated with $G_k$.

By Borel-Tits, $\mathcal{F}_{S,k}^{rc} = (S_{J,k} : J \subseteq I)$, and $\theta_k(P) := O^p(C_{G_k}(P)) = 1$ for all $P \in \mathcal{F}_{S,k}^{rc}$. When $k$ divides $j$, we have the inclusion map $\beta_{k,j} : \mathcal{G}_k \to \mathcal{G}_j$ with $\beta_{k,j}(S_{J,k}) = S_{J,j}$. It follows from 2.11 that $\mathcal{G} := (\mathcal{G}_k, \beta_{k,j} : k \leq j)$ is a directed system of $p$-local finite groups. Further one can check that the hypotheses of 12.10(b) are satisfied by $\mathcal{G}$, so, as in the proof of Theorem 12.11, the limit $\mathcal{G}_\mathcal{G}$ of $\mathcal{G}$ is isomorphic to $\mathcal{G}(G) = (\mathcal{F}_S(G), \mathcal{F}_{S,k}^{rc}(G), \mathcal{L})$, where $\mathcal{F}_{S,k}^{rc}(G)$ has object set $(S_J : J)$, $\text{Mor}(S_J, S_K) = \text{Hom}(S_J, S_K) = N_G(S_J, S_K)$, and $\pi$ and $\delta$ are identity maps.

References


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