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A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver

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Abstract

We extend the notion of a p-local finite group (defined in [BLO03]) to the notion of a *p-local group*. We define *morphisms* of p-local groups, obtaining thereby a category, and we introduce the notion of a *representation* of a p-local group via signalizer functors associated with groups. We construct a chain $\mathfrak{G} = (\mathcal{G}_0 \to \mathcal{G}_1 \to \cdots)$ of 2-local finite groups, via a representation of a chain $\mathfrak{G}^* = (G_0 \to G_1 \to \cdots)$ of groups, such that $\mathcal{G}_0$ is the 2-local finite group of the third Conway sporadic group Co_3, and for $n > 0$, $\mathcal{G}_n$ is one of the 2-local finite groups constructed by Levi and Oliver in [LO02]. We show that the direct limit $\mathcal{G}$ of $\mathfrak{G}$ exists in the category of 2-local groups, and that it is the 2-local group of the union of the chain $\mathfrak{G}^*$. The 2-completed classifying space of $\mathcal{G}$ is shown to be the classifying space $B\,\mathrm{DI}(4)$ of the exotic 2-compact group of Dwyer and Wilkerson [DW93].

Introduction

In [BLO03], Broto, Levi, and Oliver introduced the notion of a p-local finite group $\mathcal{G}$, consisting of a finite p-group S and a pair of categories $\mathcal{F}$ and $\mathcal{L}$ (the fusion system and the centric linking system) whose objects are subgroups of S, and which satisfy axioms which encode much of the structure that one expects from a finite group having S as a Sylow p-subgroup. If indeed G is a finite group with Sylow p-subgroup S, then there is a canonical construction which associates to G a p-local finite group $\mathcal{G} = \mathcal{G}_S(G)$, such that the p-completed nerve of $\mathcal{L}$ is homotopically equivalent to the p-completed classifying space of G. A p-local finite group $\mathcal{G}$ is said to be *exotic* if $\mathcal{G}$ is not equal to $\mathcal{G}_S(G)$ for any finite group G with Sylow group S.

From the work of various authors (cf. [BLO03, §9]), it has begun to appear that for p odd, exotic p-local finite groups are plentiful. On the other hand, exotic

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2-local finite groups are – as things stand at this date – quite exceptional. In fact, the known examples of exotic 2-local finite groups fall into a single family $\mathcal{G}_{\mathrm{Sol}}(q)$, q an odd prime power, constructed by Ran Levi and Bob Oliver [LO02]. With hindsight, the work of Ron Solomon [Sol74] in the early 1970's may be thought of as a proof that $\mathcal{G}_{\mathrm{Sol}}(q) \neq \mathcal{G}_S(G)$ for any finite simple group G with Sylow group S.

Solomon considered finite simple groups G having a Sylow 2-subgroup isomorphic to that of Co_3 (the smallest of the three sporadic groups discovered by John Conway), and he showed that any such G is isomorphic to Co_3. While proving this, he was also led to consider the situation in which G has a single conjugacy class z^G of involutions, and $C_G(z)$ has a subgroup H with the following properties:

$$H \cong \mathrm{Spin}_7(q),\ \ q = r^n,\ q \equiv 3 \text{ or } 5 \bmod 8, \quad \text{and } \ C_G(z) = O(C_G(z))H.$$

Here $\mathrm{Spin}_7(q)$ is a perfect central extension of the simple orthogonal group $\Omega_7(q)$ by a group of order 2, and for any group X, $O(X)$ denotes the largest normal subgroup of X all of whose elements are of odd order.

Solomon showed that there is no finite simple group G which satisfies the above conditions – but he was not able to do this by means of "2-local analysis" (i.e. the study of the normalizers of 2-subgroups of G). Indeed a potential counterexample possessed a rich and internally consistent 2-local structure. It was only after turning from 2-local subgroups to local subgroups for the prime r that a contradiction was reached.

One of the achievements of [LO02] is to suggest that the single "sporadic" object Co_3 in the category of groups is a member of an infinite family of exceptional objects in the category of 2-local groups. But in addition, [LO02] establishes a special relationship between the $\mathcal{G}_{\mathrm{Sol}}(q)$'s and the exotic 2-adic finite loop space $\mathrm{D\,I}(4)$ of Dwyer and Wilkerson [DW93]. Namely, in [LO02] it is shown that the classifying space $B\,\mathrm{D\,I}(4)$ is homotopy equivalent to the 2-completion of the nerve of a union of subcategories of the linking systems $\mathcal{L}_{\mathrm{Sol}}(q^n)$, with the union taken for any fixed q as n goes to infinity. (This result was prefigured in, and motivated by, work of David Benson [Ben94]. Benson showed, first, that the 2-cohomology ring $H^*(B\,\mathrm{D\,I}(4); 2)$ is finitely generated over $H^*(\mathrm{Co}_3; 2)$, and second, that the 2-cohomology of the space of fixed points in $B\,\mathrm{D\,I}(4)$ of an unstable Adams operation ψ_q, would be that of the "Solomon groups", if such groups existed.) Moreover [BLO05] introduces the notion of a "p-local compact group", and Theorem 9.8 in [BLO05] shows that each p-compact group supports the structure of a p-local compact group. As a special case, $\mathrm{D\,I}(4)$ supports such a structure. We give here an alternate, constructive proof of this fact.

Our paper is built around an alternate construction (Theorem A) of the 2-local finite groups

$$\mathcal{G}_k = \mathcal{G}_{k,r} = \mathcal{G}_{\mathrm{Sol}}(r^{2^{k+1}}) = (S_k, \mathcal{F}_k, \mathcal{L}_k)$$

where r is a prime congruent to 3 or 5 mod 8. The construction is based on a notion of the "representation" of a p-local finite group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ as the p-local group of a not necessarily finite group G, by means of a "signalizer functor" θ. This means, first of all, that S is a Sylow p-subgroup of G (in a sense which we shall make precise), and that the fusion system $\mathcal{F}$ may be identified with the fusion system $\mathcal{F}_S(G)$ consisting of all the maps between subgroups of S that are induced by conjugation by elements of G. Second, it means that whenever P is a subgroup of S which contains every p-element of its G-centralizer (i.e. whenever P is *centric* in $\mathcal{F}$), there is a direct-product factorization

$$C_G(P) = Z(P) \times \theta(P)$$

where the operator θ is inclusion-reversing and conjugation-equivariant. Then θ gives rise to a centric linking system $\mathcal{L}_\theta$ associated with $\mathcal{F}$, with the property that

$$\mathrm{Aut}_{\mathcal{L}_\theta}(P) = N_G(P)/\theta(P)$$

for any $\mathcal{F}$-centric subgroup P of S. One says that $\mathcal{G}$ is *represented in G via θ* if the p-local groups $\mathcal{G}$ and $(S, \mathcal{F}, \mathcal{L}_\theta)$ are isomorphic.

The notions of p-local finite group and of representation via a signalizer functor can be generalized to obtain a representation of the 2-local compact group of D I(4), by allowing S to be an infinite 2-group. We also introduce a notion of *morphism*, to obtain a category of p-local groups, having p-local finite groups as a full subcategory. As an application we show in Theorem B that the 2-local finite group $\mathcal{G}_0$ associated with Co_3 is a "subgroup" of each $\mathcal{G}_k$.

In the final section of this paper we introduce a notion of *direct limit* of a directed system of embeddings of p-local groups, and in this way obtain (Theorem C) a 2-local compact group $\mathcal{G}_{\mathfrak{G}}$ which is the direct limit of a directed system $\mathfrak{G} = (\mathcal{G}_k \to \mathcal{G}_{k+1})_{k \geq 0}$ of embeddings of 2-local finite groups. The identification of $\mathcal{G}_{\mathfrak{G}}$ with the 2-local compact group of D I(4) (Theorem D) is a corollary of results in [LO02], obtained by setting up a homotopy equivalence between the nerve of our direct limit and the nerve of a category $\mathcal{L}^c_{\mathrm{Sol}}(p^\infty)$ constructed in [LO02]. The 2-completion of the latter category is shown in [LO02] to be homotopy equivalent to $B\,\mathrm{D\,I}(4)$.

In view of the length of this paper, the reader may find the following outline helpful. The first three sections are concerned with general principles and supporting results. Then in Section 4, which provides information on certain spin groups, the argument actually begins to take shape.

Let p be an odd prime. For reasons which will not be immediately apparent, it will be necessary to take p to be congruent to 3 or 5 mod 8. Let $\overline{\mathbf{F}}$ be an algebraic closure of the field of p elements. There is a subfield $\mathbf{F}$ of $\overline{\mathbf{F}}$, obtained as the union of the tower of subfields of $\overline{\mathbf{F}}$ of order p^{2^n}, $n \geq 0$. Take $\bar{H}$ to be the group $\mathrm{Spin}_7(\overline{\mathbf{F}})$

– the universal covering group of the simple orthogonal group $\Omega_7(\bar{\mathbf{F}})$. Let ψ be an endomorphism of $\bar{H}$ such that $C_{\bar{H}}(\psi) \cong \mathrm{Spin}_7(p)$, and set

$$H = \bigcup_{n \geq 0} C_{\bar{H}}(\psi^{2^n}).$$

Then H is a group of $\mathbf{F}$-rational points of $\bar{H}$. One finds that all fours groups in H containing $Z(H)$ are conjugate, and that if U is such a fours group then the identity component B^0 of the group $B = N_H(U)$ is a commuting product of three copies of $\mathrm{SL}_2(\mathbf{F})$.

In Section 5 we show that there is an automorphism y of B^0 of order 3, which transitively permutes the three $\mathrm{SL}_2(\mathbf{F})$ components of B, and which when chosen carefully, interacts in a special way (to be described shortly) with the normalizer in H of a maximal torus T of B. It is at this point, in choosing an appropriate automorphism y, that we require that p be congruent to 3 or 5 mod 8. Once y has been fixed in the appropriate way, we form a group $K = \langle B, y \rangle$ which is isomorphic to a split extension of B^0 by the symmetric group of degree 3. We then form the amalgam $\mathcal{A} = (H > B < K)$, and its associated free amalgamated product

$$G = H *_B K.$$

This is the group which informs and guides our investigation.

We need the following notion of "Sylow 2-subgroup": A subgroup S of G is a Sylow 2-subgroup of G if every element of S has order a power of 2 (i.e. S is a 2-group), S is maximal with respect to inclusion among the 2-subgroups of G, and every finite 2-subgroup of G is conjugate to a subgroup of S. It turns out that the normalizer in H of a maximal torus T of B^0 contains a Sylow 2-subgroup S of G. Moreover, if T is chosen to be y-invariant then S is a Sylow 2-subgroup of each of the groups H, B, and K. The special way in which y interacts with S may be summarized as follows: for the Sylow 2-subgroup $S_\infty = S \cap T$ of T, we have

$$(*) \qquad N_G(S_\infty) = N_H(S_\infty) *_{N_B(S_\infty)} N_K(S_\infty), \quad \text{and}$$

$$(**) \qquad \mathrm{Aut}_G(S_\infty) := N_G(S_\infty)/C_G(S_\infty) \cong \mathrm{GL}(3,2) \times \mathbf{C}_2.$$

The effect of $(**)$ is that S/S_∞ may be identified with a Sylow 2-subgroup of $\mathrm{Aut}_G(S_\infty)$, and it is this property which, as is made clear in [LO02], turns out to be the key to fulfilling the axioms for "saturation" (defined in 1.5, below).

One feature of our treatment is the use of amalgams (cf. §3) to keep track of the various fusion systems which can be constructed from H, B, and K, and to distinguish the system with property $(**)$. We prove in Theorem 5.2 that the amalgam $\mathcal{A}$ with property $(**)$ is unique. Then we carry out the remainder of our analysis in the universal completion G of $\mathcal{A}$, using the "standard tree" of G

as a source of geometric intuition, and as the basis for geometric arguments. The formalization by means of the amalgam $\mathscr{A}$ and its free amalgamated product G provides, at the very least, a useful system of bookkeeping. For example, the language of amalgams provides a conceptual framework within which one can rigorously consider the question of which of the fusion systems constructed from H, B, and K is the "right" system. We mention that amalgams have also been used in recent work of G. Robinson [Rob07], and of Ian Leary and Radu Stancu [LS] as a tool for studying abstract fusion systems.

Setting $Z = Z(H)$ one has $|Z| = 2$, and $C_G(Z)$ is in fact a rather complicated subgroup of G, properly containing H. Our proof that the fusion system $\mathscr{F}_S(G)$ is saturated is modeled on the proof of saturation in [LO02] for the fusion systems $\mathscr{F}_{\mathrm{Sol}}(q)$ defined over finite 2-groups. Thus, the main step is to establish that *H controls $C_G(Z)$-fusion in S*. That is, the fusion system $\mathscr{F}_S(H)$ is equal to the *a priori* larger system $\mathscr{F}_S(C_G(Z))$.

The proof of saturation in G is accompanied by the construction of a linking system by means of a signalizer functor. These steps require information on fusion among the centric subgroups of S, obtained in Sections 6 through 8. After this, in order to prepare the way for the construction of morphisms, we determine in complete detail the *radical centric* subgroups of S and of S_σ, where σ is an automorphism of G which fixes S and which induces a Frobenius endomorphism of H. Here a subgroup P of S is defined to be radical if $\mathrm{Inn}(P) = O_2(\mathrm{Aut}_G(P))$.

One of the radical centric subgroups of S is an elementary abelian group A of order 16 which has the property, as in $(*)$, that

$$N_G(A) = N_H(A) *_{N_B(A)} N_K(A).$$

Here one can do better than to determine $\mathrm{Aut}_G(A)$ in analogy with $(**)$. Indeed, there is a surjective homomorphism

$$\phi_A \colon N_G(A) \longrightarrow L,$$

with $C_G(A) = A \times \ker(\phi_A)$, where L is a maximal subgroup of the sporadic group Co_3, isomorphic to a nonsplit extension of A by $\mathrm{Aut}(A)$. We then define a normal sub*set* $\mathbf{X}$ of G by

$$\mathbf{X} = \bigcup_{g \in G} \ker(\phi_A)^g.$$

For any centric subgroup P of S we define a subset $\theta(P)$ of $C_G(P)$ by

$$\theta(P) = C_{\mathbf{X}}(P)\, O(C_G(P)).$$

It turns out that $\theta(P)$ is a sub*group* of $C_G(P)$ and that θ is a signalizer functor (cf. Theorem 8.8 below).

We next consider the groups G_σ of fixed points of automorphisms σ of G, such that σ fixes both H and K, and such that the restriction of σ to H is a Frobenius map with $H_\sigma \cong \mathrm{Spin}_7(\mathbf{F}_q)$, $q = p^{2^n}$. The groups G_σ, for $n \geq 0$, provide representations of the 2-local finite groups of [LO02]. In particular, for each such σ (chosen so that S_σ is a Sylow 2- subgroup of G_σ), the fusion system $\mathcal{F}_{S_\sigma}(G_\sigma)$ is saturated, and the signalizer functor θ_σ given by

$$\theta_\sigma(P) = C_{\mathbf{X}_\sigma}(P)O(C_{G_\sigma}(P)),$$

for centric subgroups P of S_σ, defines a centric linking system $\mathcal{L}_{\theta_\sigma}(S_\sigma)$ associated with $\mathcal{F}_{S_\sigma}(G_\sigma)$.

This completes our outline of the proof of Theorem A. One aim of this paper is thus to suggest the possibility that many p-local finite groups may best be studied via a representation in terms of free amagalmated products and signalizer functors. For example, to study the fusion system $\mathcal{F}$ on a p-group S generated by systems $\mathcal{F}_S(G_i)$ for some family $\mathcal{G} = (G_i \mid i \in I)$ of finite groups with Sylow group S, perhaps one should study the various amalgams $\mathcal{A}$ obtained from $\mathcal{G}$, and the corresponding free amalgamated products $G = G(\mathcal{A})$. If $\mathcal{A}$ is well chosen, then S is Sylow in G and $\mathcal{F} = \mathcal{F}_S(G)$ is saturated. Then one can consider suitable overamalgams $\mathcal{B}$ of $\mathcal{A}$, and the kernels of surjections from subgroups $G(\mathcal{B})$ of G onto suitable finite groups and use these kernels to construct a signalizer functor θ and the corresponding p-local finite group from $\mathcal{F}$. If $\mathcal{A}$ is the amalgam of some family of subgroups generating a finite group $\widehat{G}$, then the kernel of the surjection $G \to \widehat{G}$ will be $\langle \theta(P)^g \mid P \in \mathcal{F}^c,\ g \in G\rangle$, whence $\mathcal{F} \cong \mathcal{F}_S(\widehat{G})$ (cf. Example 2.13, below). But in other cases one may hope for exotic p-local finite groups, such as $\mathcal{L}_{\mathrm{sol}}(q)$.

Now here are the main theorems.

THEOREM A. *Let p be a prime, $p \equiv 3$ or $5 \bmod 8$, let $\overline{\mathbf{F}}$ be an algebraic closure of the field $\mathbf{F}_p$ of p elements, and let $\mathbf{F}$ be the union of the subfields of $\overline{\mathbf{F}}$ of order $q_n = p^{2^n}$, $n \geq 0$. Then there is a group $G = G(p)$, an automorphism ψ_0 of G, a Sylow 2-subgroup S of G, and a ψ_0-invariant normal subset $\mathbf{X}$ of G such that, for any power σ of ψ_0 of the form $\psi_0^{2^n}$, we have the following.*

(1) *$G = H *_B K$ is the free amalgamated product of an amalgam*

$$\mathcal{A} = (H \longleftarrow B \longrightarrow K),$$

where H is a group of $\mathbf{F}$-rational points in $\mathrm{Spin}_7(\overline{\mathbf{F}})$, B is the normalizer in H of a fours group U of H containing $Z(H)$, and K is a group which contains B as a subgroup of index 3 where K has the property that $\mathrm{Aut}_K(U) \cong \mathrm{GL}(2,2)$.

(2) *ψ_0 leaves invariant each of the subgroups H, K, and B of G; the restriction of ψ_0 to H is the restriction of a Frobenius automorphism of $\mathrm{Spin}_7(\overline{\mathbf{F}})$, and $C_H(\psi_0) \cong \mathrm{Spin}_7(p)$.*

(3) *The group* $S_\sigma = C_S(\sigma)$ *is a finite Sylow* 2*-subgroup of the group* $G_\sigma = C_G(\sigma)$, *and there exists a unique choice of the amalgam* $\mathcal{A}$ *such that, for all* σ, *the fusion system* $\mathcal{F}_\sigma = \mathcal{F}_{S_\sigma}(G_\sigma)$ *is isomorphic to the fusion system* $\mathcal{F}_{\mathrm{Sol}}(q)$ *of* [LO02], ($q = p^{2^n}$).

(4) *For any* $\mathcal{F}_\sigma$*-centric subgroup* P *of* S_σ, *the set*

$$\theta_\sigma(P) := C_{\mathbf{X}\cap G_\sigma}(P)O(C_{G_\sigma}(P))$$

is a group, and is a complement to $Z(P)$ *in* $C_{G_\sigma}(P)$. *Moreover,* θ_σ *defines a* 2*-local finite group* $\mathcal{G}_\sigma = (S_\sigma, \mathcal{F}_\sigma, \mathcal{L}_\sigma)$ *isomorphic to the* 2*-local finite group* $\mathcal{L}_{\mathrm{sol}}(q)$ *of* [LO02].

(5) *The order of a maximal elementary abelian* 2*-subgroup* A *of* S *is* 16, *and all maximal elementary abelian* 2*-subgroups of* G *are conjugate in* G. *Moreover,* $C_G(A) = A \times C_{\mathbf{X}}(A)$, *where* $C_{\mathbf{X}}(A)$ *is a free normal subgroup of* $C_G(A)$, $N_G(A)/C_{\mathbf{X}}(A)$ *is isomorphic to a nonsplit extension of* A *by* $\mathrm{Aut}(A)$, *and* $\mathbf{X}$ *is the union of the conjugates of* $C_{\mathbf{X}}(A)$ *in* G.

THEOREM B. *Let* p, $\mathcal{A}$, G, ψ_0, *and* $\mathbf{X}$ *be as in Theorem* A. *Then there exist subgroups* H_0, K_0, *and* $B_0 = H_0 \cap K_0$ *of* H_{ψ_0}, K_{ψ_0}, *and* B_{ψ_0}, *respectively, such that the following hold.*

(1) H_0 *is isomorphic to a perfect central extension of* $\mathrm{Sp}(6,2)$ *by* $\mathbb{Z}_2$, K_0 *is a group of order* $2^{10}3^3$, *and* B_0 *is of index* 3 *in* K_0.

(2) *Setting* $G_0 = \langle H_0, K_0\rangle$, *we have*
   (a) $X \cap H_0 = X \cap K_0 = \{1\}$, *and*
   (b) $G_0/\langle X \cap G_0\rangle$ *is isomorphic to the colimit of the amalgam* $\mathcal{M}$ *of maximal subgroups of* Co_3 *containing a fixed Sylow* 2*-subgroup of* Co_3.

(3) *Let* S_0' *be a Sylow* 2*-subgroup of* Co_3 *and* S_0 *a Sylow* 2*-subgroup of* B_0. *Then there is an isomorphism of* 2*-local finite groups*

$$\mathcal{G}_{S_0'}(\mathrm{Co}_3) \cong \mathcal{G}_{S_0}(G_0).$$

THEOREM C. *For any positive integer* i, *let* $\mathcal{G}_i$ *be the* 2*-local finite group* $\mathcal{G}_{\psi_0^{2i-1}}$ *of Theorem* A, *and let* $\mathcal{G}_0$ *be the* 2*-local finite group associated with* Co_3 *as in Theorem* B. *Let* $\mathcal{G}$ *be the* 2*-local group* $(S, \mathcal{F}, \mathcal{L})$ *associated with* G *via the fusion system* $\mathcal{F} = \mathcal{F}_S(G)$ *and via the signalizer functor* θ *defined by the subset* $\mathbf{X}$ *of* G. *Then there exists a directed system*

$$\mathfrak{G} = (\beta_{i,j}\colon \mathcal{G}_i \longrightarrow \mathcal{G}_j)_{0\le i\le j}$$

of embeddings of 2*-local finite groups, possessing a limit* $\mathcal{G}_{\mathfrak{G}}$ *which is canonically isomorphic to the* 2*-local group* $\mathcal{G}$.

The 2-local group $\mathcal{G}(G)$ is a 2-*local compact group*, as defined in [BLO05].

It will be proved in [COS06] that the exotic fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$ of Levi and Oliver, defined over 2-groups S_q, are determined by the isomorphism type of S_q. This implies that for any odd prime power q, and any prime $p \equiv 3$ or $5 \mod 8$, there is a unique σ such that the Levi-Oliver fusion system $\mathcal{F}_{\mathrm{Sol}}(q)$ is isomorphic to $\mathcal{F}_{S_\sigma}(G_\sigma)$, where $G = G(p)$ is the group in characteristic p constructed here. This is needed for the proof of the following result.

THEOREM D. *Let $\mathcal{L} := \mathcal{L}_{S,\theta}(G)$ be the centric linking system over $\mathcal{F}$ as given in Theorem C. Then the* 2*-completed nerve $|\mathcal{L}|_2^\wedge$ is homotopy equivalent to* $B\,\mathrm{D\,I}(4)$. *In particular*, $\mathrm{D\,I}(4)$ *may be given the structure of the* 2*-local group* $\mathcal{G}$, *and* $\mathrm{D\,I}(4)$ *is then a* 2*-local compact group.*

We are grateful to Bob Oliver for many helpful conversations about the 2-local finite groups $\mathcal{G}_{\mathrm{Sol}}(q)$ which he and Ran Levi constructed, and for his help in understanding the space $B\,\mathrm{D\,I}(4)$ of Dwyer and Wilkerson. The proof of Theorem D was communicated to us by Levi and Oliver. We would also like to thank Ron Solomon and the other members of his seminar at Ohio State, for suggesting improvements to an earlier version of this manuscript.

Remarks and questions.

(1) One might imagine that the normal subgroup $\langle \mathbf{X} \rangle$ of G leads to an interesting factor group $G/\langle \mathbf{X} \rangle$. But the fact is that $\langle \mathbf{X} \rangle = G$. Moreover, $G_\sigma/\langle \mathbf{X}_\sigma \rangle = 1$ for any automorphism σ of G as in Theorem A, while $G_0/\langle \mathbf{X} \cap G_0 \rangle$ is in fact isomorphic to Co_3. These results will appear in [COS06].

(2) To what extent can our method of construction of the Levi-Oliver fusion and linking systems be carried out in a characteristic 0 context? For example, one might consider a subring $\mathbb{O}$ of the field of complex numbers, and ask whether there is a 7-dimensional quadratic space over $\mathbb{O}$, yielding a group $H_{\mathbb{O}} = \mathrm{Spin}_7(\mathbb{O})$, from which to build up a suitable free amalgamated product and linking system as we do here in characteristic p. One requires $1/2 \in \mathbb{O}$ in order to have an isomorphism of $\mathrm{PSL}_2(\mathbb{O})$ with a suitable 3-dimensional orthogonal group. The rings
$$\mathbb{O}_m = \mathbb{Z}[\omega/2],$$
where ω is a primitive 2^m-th root of unity, are possible candidates for this, and there may be others.

(3) The sporadic group $O'N$ (or rather, the 2-local finite group associated with $O'N$) can be shown to occur as a subgroup of some of the 2-local finite groups constructed here. Since $O'N$ and its subgroup J_1 are "pariahs", i.e. are not among the twenty sporadic simple groups which are involved in the Monster, it is of some interest to have a context in which these groups, and Co_3 (which is not a pariah) can live together in harmony. This will be the subject of another paper.

1. Fusion systems and Sylow subgroups

We shall need to consider fusion systems both over finite p-groups, and over certain infinite p-groups. In the finite case the definitions are due first of all to Lluis Puig [Pui06], and then to Broto, Levi, and Oliver [BLO03]. The latter three authors also consider a class of infinite p-groups which they call *discrete p-toral groups*, in [BLO05], and this class includes all of the p-groups that will be studied here. For reasons of exposition, however, we shall present the definitions in a somewhat more general context – but we emphasize that the main concepts, and the proofs of the basic lemmas, come from the above-cited works.

We follow the practice, peculiar to finite group theory, of using right-hand notation for conjugation within a group, and for group homomorphisms. But we use left-hand notation for functors, and for auxiliary mappings associated with some of our functors. It may also be worth mentioning that if X is a set admitting action by a group G, and g is an element of G, then the set of fixed points for g in X is denoted X_g, rather than the topologist's X^g.

If G is a group, g an element of G, and X a subset or an element of G, we write X^g for the image of X under the conjugation automorphism

$$c_g\colon G \to G, \quad (c_g\colon x \mapsto x^g := g^{-1}xg \quad \text{for all } x \in G).$$

We also write $c_g\colon P \to Q$ for the mapping of P into Q given by g-conjugation, whenever P and Q are subgroups of G with $P^g \le Q$. The *transporter* of P into Q is the set

$$N_G(P,Q) := \{g \in G \mid P^g \le Q\},$$

and we define

$$\mathrm{Hom}_G(P,Q) := \{c_g : P \to Q \mid g \in N_G(P,Q)\}.$$

Denote by $\mathrm{Inj}(P,Q)$ the set of all injective homomorphisms of P into Q. If $\alpha\colon P \to Q$ is an isomorphism, write α^* for the isomorphism from $\mathrm{Aut}(P)$ to $\mathrm{Aut}(Q)$ defined by $\alpha^*\colon \beta \mapsto \alpha^{-1}\beta\alpha$.

Definition 1.1. A *fusion system* $\mathcal{F}$ over a group S is a category whose objects are the subgroups of S, and whose morphism-sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following two conditions.

(1) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$.
(2) If $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ then the isomorphisms $\alpha\colon P \to P\alpha$ and $\alpha^{-1}\colon P\alpha \to P$ are morphisms in $\mathcal{F}$.

Example 1.2. Let G be a group and S a subgroup of G. For subgroups P and Q of S, set

$$\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_G(P,Q).$$

Then $\mathcal{F}$ is a fusion system over S, denoted $\mathcal{F}_S(G)$.

Let $\mathcal{F}$ be a fusion system over S, let P be a subgroup of S, and let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$. Set

$$N_\alpha = \{x \in N_S(P) \mid (c_x)\alpha^* \in \mathrm{Aut}_S(P)\}.$$

Thus, N_α is the largest subgroup R of $N_S(P)$ having the property that, in the group $\mathrm{Aut}_{\mathcal{F}}(P)$, the conjugation map c_α carries $\mathrm{Aut}_R(P)$ into $\mathrm{Aut}_S(P)$.

LEMMA 1.3. *Let S be a subgroup of a group G, and let P be a subgroup of S. Set $\mathcal{F} = \mathcal{F}_S(G)$, let $g \in N_G(P)$, and set $\alpha = c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.*

(a) $(N_\alpha)^g = S^g \cap (C_G(P)N_S(P))$.

(b) *If S is a p-group, and every p-subgroup of $C_G(P)N_S(P)$ is conjugate via $C_G(P)$ to a subgroup of $N_S(P)$, then there exists $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(N_\alpha, N_S(P))$ extending α.*

Proof. Set $R = N_\alpha$. By definition, R^g consists of those $x \in N_G(P)$ such that $x \in S^g$ and $c_{x|P} \in \mathrm{Aut}_S(P)$. But $c_{x|P} \in \mathrm{Aut}_S(P)$ if and only if $x \in C_G(P)N_S(P)$, and thus (a) holds.

Assume the hypothesis of (b). Then R^g is a p-subgroup of $C_G(P)N_S(P)$, by (a), so that by the hypothesis of (b) there exists $h \in C_G(P)$ such that $R^h \le N_S(P)$. Now α is the restriction to P of $\bar{\alpha} = c_{gh}\colon R \longrightarrow N_S(P)$, and we have (b). □

Definition 1.4. Let p be a prime. A group S is a *p-group* if for every $x \in S$, the order of x is a power of p. A p-subgroup S of a group G is a *Sylow p-subgroup* of G if

(1) S is maximal (with respect to inclusion) among all p-subgroups of G, and

(2) S contains a conjugate of every finite p-subgroup of G.

The set of all Sylow p-subgroups of G is denoted $\mathrm{Syl}_p(G)$. The group generated by the set of normal p-subgroups of G is itself a normal p-subgroup of G, and is denoted $O_p(G)$.

Definition 1.5. Let p be a prime, let S be a p-group, and let $\mathcal{F}$ be a fusion system over S. A subgroup P of S is *fully normalized in* $\mathcal{F}$ if, for every $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, there exists $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P\phi), N_S(P))$ such that ψ maps $P\phi$ to P. We say that $\mathcal{F}$ is *saturated* if the following two conditions hold for every subgroup P of S.

(I) There exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $P\phi$ is fully normalized in $\mathcal{F}$.

(II) If P is fully normalized in $\mathcal{F}$ then:

  (A) $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, and

  (B) each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ extends to a member of $\mathrm{Hom}_{\mathcal{F}}(N_\alpha, S)$.

The preceding definition of saturation is formulated so as to make no mention of $|N_S(P)|$ for $P \leq S$, and it is equivalent to the standard definition (cf. [BLO03]) in the case that S is finite. This follows from [BCG$^+$05, Lemma 2.3], and it is then easy to check that our definition of "fully normalized" is equivalent to the usual one, in a saturated fusion system over a finite p-group.

LEMMA 1.6. *Let p be a prime and G a group. Let $\mathcal{Y}$ be the set of subgroups Y of G such that the set $\mathcal{S}(Y)$ of maximal p-subgroups of Y is nonempty, and such that Y is transitive on $\mathcal{S}(Y)$ by conjugation. Assume for each p-subgroup P of G that*:

(1) *$N_G(P)$, $C_G(P)P$, and $C_G(P)T$ are in $\mathcal{Y}$ for $T \in \mathcal{S}(N_G(P))$.*

(2) *$\mathrm{Out}_G(P)$ is finite.*

(3) *P is Artinian (i.e., any descending chain of subgroups of P stabilizes).*

(4) *If $P \neq 1$ then $N_G(P)$ is locally finite.*

Then

(a) *G has a Sylow p-subgroup S.*

(b) *A subgroup P of S is fully normalized in $\mathcal{F}_S(G)$ if and only if $N_S(P)$ is a Sylow p-subgroup of $N_G(P)$.*

(c) *$\mathcal{F}_S(G)$ is saturated.*

Proof. We have $G \in \mathcal{Y}$, by (1) as applied to $P = 1$. Thus $\mathcal{S}(G) = \mathrm{Syl}_p(G)$, and (a) holds.

Let $P \leq S$ and set $L = N_G(P)$, $K = C_G(P)$, and $T = N_S(P)$. Applying (1) to P, we obtain $T \leq X$ for some $X \in \mathrm{Syl}_p(L)$. Let $Q \leq S$ and $g \in G$ with $Q^g = P$. Then $N_S(Q)^g$ is a p-subgroup of L, and since $L \in \mathcal{Y}$ there exists $l \in L$ with $N_S(Q)^{gl} \leq X$. We conclude that P is fully normalized if $T = X$. Further, as $G \in \mathcal{Y}$ there exists $h \in G$ with $X^h \leq S$, and so $X^h \in \mathrm{Syl}_p(L^h)$ and hence P^h is fully normalized. This verifies axiom (I) in the definition 1.5 of saturation.

Assume that P is fully normalized. Then there exists $y \in G$ with $P^{hy} = P$ and with $X^{hy} \leq T \leq X$. Then $X^{(hy)^n} \leq X$ for all $n > 0$, and it follows from (3) that $X = X^{hy}$, and then that $X = T$. This completes the proof of (b).

Set $\bar{L} = L/PK$. We have $T \in \mathrm{Syl}_p(L)$ by (b), and $\bar{L}$ is finite by (2). Let Y be the pre-image in L of a Sylow p-subgroup of $\bar{L}$ containing $\bar{T}$. By (4) there is a finite subgroup U of Y with $\bar{Y} = \bar{U}$. As $\bar{U}$ is a p-group we have $\bar{U} = \bar{V}$ for some $V \in \mathrm{Syl}_p(U)$. As $L \in \mathcal{Y}$ there exists $a \in L$ with $V^a \leq T$. Then $|\bar{Y}| = |\bar{V}| \leq |\bar{T}| \leq |\bar{Y}|$, so that $\bar{Y} = \bar{T}$, and we have verified axiom (IIA) in 1.5.

Finally, set $\mathcal{F} = \mathcal{F}_S(G)$ and let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$. As $TK \in \mathcal{Y}$, by (1), we conclude from 1.3(b) that α extends to an element of $\mathrm{Hom}_{\mathcal{F}}(N_\alpha, S)$, verifying axiom (IIB) for saturation, and completing the proof of (c). □

Notice that if G is a finite group then the hypotheses of 1.6 are satisfied by G. Thus, it is a corollary of 1.6 that for any finite group G and $S \in \mathrm{Syl}_p(G)$, the fusion system $\mathcal{F}_S(G)$ is saturated.

Let $\mathcal{F}_i$, $i = 1, 2$, be fusion systems over subgroups S_i of a group S. We say that $\mathcal{F}_1$ is a *fusion subsystem* of $\mathcal{F}_2$ (and write $\mathcal{F}_1 \leq \mathcal{F}_2$) if $S_1 \leq S_2$ and $\mathrm{Hom}_{\mathcal{F}_1}(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}_2}(P, Q)$ for all $P, Q \leq S_1$.

Given a set $\mathbf{F}$ of fusion systems over S, there is a largest fusion system

$$\mathcal{F}_{\mathbf{F}} := \bigcap_{\mathcal{F} \in \mathbf{F}} \mathcal{F}$$

which is a subsystem of each $\mathcal{F} \in \mathbf{F}$. Thus

$$\mathrm{Hom}_{\mathcal{F}_F}(P, Q) = \bigcap_{\mathcal{F} \in F} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

Given a set $\mathbf{E}$ of fusions systems, each of which is defined over a subgroup of S, define $\langle \mathbf{E} \rangle$ – the *fusion system generated by* $\mathbf{E}$ – to be the fusion system $\mathcal{F}_{\mathbf{F}}$, where $\mathbf{F}$ is the set of all fusion systems over S which contain each member of $\mathbf{E}$. The proof of the following result is straightforward:

LEMMA 1.7. *Let S be a group and let $(S_i : i \in I)$ be a collection of subgroups of S. For each $i \in I$, let $\mathcal{F}_i$ be a fusion system over S_i, and set $F = (\mathcal{F}_i \mid i \in I)$. Assume that $S_i = S$ for at least one index i, and define a fusion system $\mathcal{G}$ on S by taking* $\mathrm{Hom}_{\mathcal{G}}(P, Q)$ *to consist of the maps $\alpha_0 \dots \alpha_r$ such that, for each* $0 \leq j \leq r$, *there exists $i(j) \in I$, $P_j \leq S_{i(j)}$, and $\alpha_j \in \mathrm{Hom}_{\mathcal{F}_{i(j)}}(P_j, S_{i(j)})$, such that $P_{j+1} = P_j\alpha_j$, $P_0 = P$, and $P_{r+1} = Q$. Then $\langle F \rangle = \mathcal{G}$.* □

LEMMA 1.8. *Let G be a group,* $S \in \mathrm{Syl}_p(G)$, *and let X be a normal subgroup of G of index p in S, such that $S/X \cong N_{G/X}(S/X)$. Then $\mathcal{F}_S(G) = \langle \mathcal{F}_S(S), \mathcal{F}_X(G) \rangle$.*

Proof. Set $G^* = G/X$ and $\mathcal{E} = \langle \mathcal{F}_S(S), \mathcal{F}_X(G) \rangle$. As $\mathcal{F}_S(S)$ and $\mathcal{F}_X(G)$ are contained in $\mathcal{F} = \mathcal{F}_S(G)$, we have $\mathcal{E} \subseteq \mathcal{F}$ by definition of $\mathcal{E}$, and it remains to establish the opposite inclusion. Let $P, Q \leq S$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. If $P \not\leq X$ then as S^* is of order p and is equal to its normalizer in G^* we get $Q \not\leq X$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}_S(S)}(P, Q)$. Similarly if $P \leq X$ but $Q \not\leq X$ then $\alpha = \beta\gamma$ where $\beta \in \mathrm{Hom}_{\mathcal{F}}(P, P\beta)$, $P\beta \leq X$, and $\gamma : P\beta \to Q$ is the inclusion map. Thus it remains to show that $\mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Hom}_{\mathcal{E}}(P, Q)$ for $P, Q \leq X$, which follows since $\mathcal{F}_X(G) \subseteq \mathcal{E}$. □

The next lemma states a weak form of the Alperin-Goldschmidt fusion theorem [Gol70], in the language of fusion systems. This result will be of use in the proof of Theorem B.

LEMMA 1.9. *Let G be a finite group,* $S \in \mathrm{Syl}_p(G)$, *and denote by $\mathcal{N}$ the set of subgroups N of G having the following properties.*

(1) $N = N_G(O_p(N))$,

(2) $S \cap N \in \mathrm{Syl}_p(N)$, *and*

(3) $C_S(O_p(N)) \leq O_p(N)$.

Let $\mathcal{N}_0$ *be the set of minimal members of* $\mathcal{N}$, *with respect to inclusion. Then* $\mathcal{F}_S(G) = \langle \mathcal{F}_{S\cap N}(N) \mid N \in \mathcal{N}_0 \rangle$.

We end this section with a generalization of a well-known result concerning finite p-groups.

LEMMA 1.10. *Let* P *be a* p*-group, set* $A = \mathrm{Aut}(P)$, *and let*

$$\mathcal{C} = (P = P_0 \geq P_1 \geq \cdots \geq P_k = 1)$$

be a chain of normal subgroups of P. *Let* Λ *be a subgroup of the group*

$$C_A(\mathcal{C}) = \{\alpha \in A \mid [P_i, \alpha] \leq P_{i+1} \quad \text{for all } i, 0 \leq i < k\},$$

and assume that either Λ *is a torsion group or that* P_1 *is of bounded exponent. Then* Λ *is a* p*-group.*

Proof. Apply induction on k. The lemma is trivial when $k = 1$, so take $k > 1$ and set $P^* = P/P_{k-1}$. Then Λ centralizes the chain $\mathcal{C}^* = (P_0^* \geq \cdots \geq P_{k-1}^* = 1)$. By induction, $\mathrm{Aut}_\Lambda(\mathcal{C}^*)$ is a p-group, so it remains to show that $C_\Lambda(P^*)$ is a p-group. Thus, we may take $k = 2$. Let $\alpha \in \Lambda$, $x \in P$, and set $c = [x, \alpha]$. Then $c \in P_1 \leq C_P(\alpha)$, so that $x^{\alpha^n} = xc^n$, and hence $|\alpha_{|\langle x \rangle}| = |c|$. If α is of finite order, or P_1 is of bounded exponent, we conclude that

$$|\alpha| = \mathrm{lcm}\{|[x, \alpha]| \mid x \in P\}$$

is a power of p. □

2. Linking systems, signalizer functors, and p-local groups

Let $\mathcal{F}$ be a fusion system over a p-group S. A subgroup P of S is $\mathcal{F}$*-centric* if $C_S(P\phi) = Z(P\phi)$ for every $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, and P is $\mathcal{F}$*-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$. Write $\mathcal{F}^c$ for the set of $\mathcal{F}$-centric subgroups of S, $\mathcal{F}^r$ for the set of $\mathcal{F}$-radical subgroups of S, and $\mathcal{F}^{\mathrm{rc}}$ for $\mathcal{F}^c \cap \mathcal{F}^r$.

LEMMA 2.1. *Let* S *be a Sylow* p*-subgroup of a group* G, *set* $\mathcal{F} = \mathcal{F}_S(G)$, *and let* $P \leq S$.

(a) *If* $P \in \mathcal{F}^c$, *and* $g \in G$ *with* $P \leq S^g$, *then* $C_{S^g}(P) \leq P$.

(b) *If* P *contains every finite* p*-subgroup of* $C_G(P)$ *then* $P \in \mathcal{F}^c$.

(c) *If* P *is finite and* $P \in \mathcal{F}^c$, *then* $Z(P)$ *contains every* p*-subgroup of* $C_G(P)$.

Proof. Part (a) is immediate from the definition of $\mathcal{F}^c$. Now suppose that $Z(P)$ contains every finite p-subgroup of $C_G(P)$, and let $g \in N_G(P, S)$. Then every element of $C_S(P^g)$ is contained in $Z(P^g)$, and thus $P \in \mathcal{F}^c$, proving (b).

Finally, assume that P is finite and that $P \in \mathcal{F}^c$, and let R be a finite p-subgroup of $C_G(P)$. Then RP is a finite p-subgroup of G. Since S is a Sylow p-subgroup of G, there exists $g \in G$ with $(RP)^g \leq S$. As $P \in \mathcal{F}^c$ we have $C_S(P^g) = Z(P^g)$, and so $R \leq Z(P)$. Thus $Z(P)$ contains every finite p-subgroup of $C_G(P)$, and since every p-group is the union of its finite subgroups, we obtain (c). □

LEMMA 2.2. *Let S be a Sylow p-subgroup of a group G, and set $\mathcal{F} = \mathcal{F}_S(G)$. Let $P \in \mathcal{F}^r$ such that P contains every p-element of $C_G(P)$, and let*

$$\mathscr{C} = (P = P_0 \geq P_1 \geq \cdots \geq P_k = 1)$$

be a chain of $N_G(P)$-invariant subgroups of P. Suppose that P_1 is of bounded exponent. Then P contains every finite P-invariant p-subgroup R of $C_G(\mathscr{C})$.

Proof. First, let R_0 be a p-subgroup of $C_G(P)P$. Then $R_0P = C_{R_0P}(P)P$ by the Dedekind Lemma. As $P \trianglelefteq C_G(P)P$, R_0P is a p-group, and then $C_{R_0P}(P) \leq P$ by hypothesis. Thus

$$(*) \qquad R_0 \leq R_0P \leq C_{R_0P}(P)P \leq P.$$

Now let R be a p-subgroup of $C_G(\mathscr{C})$. Set $\Lambda = C_{\mathrm{Aut}_G(P)}(\mathscr{C})$. Then $\Lambda \trianglelefteq \mathrm{Aut}_G(P)$, and Λ is a p-group by 1.10. As P is $\mathcal{F}$-radical, it follows that $\mathrm{Aut}_R(P) \leq \mathrm{Inn}(P)$, and thus $R \leq C_G(P)P$. Then $R \leq P$ by $(*)$. □

A set $\mathcal{F}_0$ of objects in a fusion system $\mathcal{F}$ is *closed under $\mathcal{F}$-conjugation* if $P\phi \in \mathcal{F}_0$ for all $P \in \mathcal{F}_0$ and all morphisms $\phi \in \mathcal{F}$ defined on P.

LEMMA 2.3. *Let $\mathcal{F}$ be a fusion system on S. Then $\mathcal{F}^c$, $\mathcal{F}^r$, and $\mathcal{F}^{\mathrm{rc}}$ are closed under $\mathcal{F}$-conjugation.*

Proof. Let $P \leq S$ and let $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. Then $P \in \mathcal{F}^c$ if and only if $P\phi \in \mathcal{F}^c$, by definition. Now let $P \in \mathcal{F}^r$. The natural map $\phi^*\colon \mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}_{\mathcal{F}}(P\phi)$ is an isomorphism, and since $\mathrm{Inn}(P) = O_p(\mathrm{Aut}_{\mathcal{F}}(P))$, it follows that $\mathrm{Inn}(P\phi) = O_p(\mathrm{Aut}_{\mathcal{F}}(P\phi))$. Thus, $P \in \mathcal{F}^r$ if and only if $P\phi \in \mathcal{F}^r$. □

Definition 2.4. Let S be a p-group, let $\mathcal{F}$ be a fusion system over S, and let $\mathscr{E}$ be a subset of $\mathcal{F}^c$. An *$\mathscr{E}$-linking system* (or *linking system on* $\mathscr{E}$), consists of

(1) a category $\mathscr{L}$ with $\mathrm{Obj}(\mathscr{L}) = \mathscr{E}$ and composition $\cdot$ (read from left to right),
(2) a functor $\pi\colon \mathscr{L} \to \mathcal{F}$, for which the associated map of objects induces the identity map $\mathrm{Obj}(\mathscr{L}) \longrightarrow \mathscr{E}$, and
(3) a collection $\delta = \{\delta_P\colon P \to \mathrm{Aut}_{\mathscr{L}}(P) \mid P \in \mathscr{E}\}$ of injective group homomorphisms,

such that the following three conditions hold for any P and Q in $\mathscr{E}$.

(A) The left action of $Z(P)$ on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ given by

$$z\colon \phi \mapsto \delta_P(z^{-1})\cdot\phi$$

is free (i.e. $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ is a union of regular orbits for $Z(P)$), and the map $Z(P)\phi \mapsto \pi(\phi)$ is a bijection of $Z(P)\backslash\mathrm{Mor}_{\mathcal{L}}(P,Q)$ with $\mathrm{Hom}_{\mathcal{F}}(P,Q)$. In particular, π is surjective on morphism sets, and $\pi(\mathcal{L})$ is a full subcategory of $\mathcal{F}$.

(B) For all $g \in P$,

$$\pi(\delta_P(g)) = c_g \in \mathrm{Aut}_{\mathcal{F}}(P).$$

(C) For each $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{\ \psi\ } & Q \\ {\scriptstyle \delta_P(g)}\big\downarrow & & \big\downarrow{\scriptstyle \delta_Q(g\cdot\pi(\psi))} \\ P & \xrightarrow{\ \psi\ } & Q\,. \end{array}$$

A *centric linking system* on $\mathcal{F}$ is a linking system on $\mathcal{F}^c$. A *pre-local group* consists of a triple $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ where S is a p-group, $\mathcal{F}$ is a fusion system over S, and $\mathcal{L} = (\mathcal{L},\pi,\delta)$ is a linking system on a subset $\mathcal{E}$ of $\mathcal{F}^c$. If $\mathcal{F}$ is saturated, and $\mathcal{L}$ is a centric linking system, then $\mathcal{G}$ is a *p-local group*. A *p-local finite group* is a p-local group $\mathcal{G}$ in which S is finite.

We are following the notational conventions in [BLO03] in writing $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ (rather than $\mathrm{Hom}_{\mathcal{L}}(P,Q)$) for the set of morphisms in $\mathcal{L}$ from P to Q, in order to emphasize that in general, $\mathcal{L}$-morphisms are not mappings.

Definition 2.5. Let G be a group, let S be a Sylow p-subgroup of G, and set $\mathcal{F} = \mathcal{F}_S(G)$. Let $\mathcal{T}(G)$ be the set of all subgroups of G. An *$\mathcal{F}$-signalizer functor* is a mapping

$$\theta\colon \mathcal{F}^c \longrightarrow \mathcal{T}(G)$$

satisfying the following three conditions:

(1) $\theta(P)$ is a complement to $Z(P)$ in $C_G(P)$.

(2) $\theta(P^g) = \theta(P)^g$ for all $g \in N_G(P,S)$.

(3) $\theta(Q) \le \theta(P)$ for all Q with $P \le Q \le S$.

Remark. Signalizer functors are bona fide contravariant functors. Namely, in 2.5, view $\mathcal{T} = \mathcal{T}(G)$ as a category whose morphism sets are given by

$$\mathrm{Mor}_{\mathcal{T}}(X,Y) = N_G(X,Y),$$

for any subgroups X and Y of G, and where composition is given by multiplication in G. Let $\mathcal{T}^c$ be the full subcategory of $\mathcal{T}$ whose set of objects is $\mathcal{F}^c$. Condition (2) in 2.5 implies that an $\mathcal{F}$-signalizer functor θ is a contravariant functor from $\mathcal{T}^c$ to $\mathcal{T}$, if we define

$$\theta\colon \mathrm{Mor}_{\mathcal{T}^c}(P,Q) \to \mathrm{Mor}_{\mathcal{T}}(\theta(P),\theta(Q))$$

by $\theta(g) = g^{-1}$. □

Given the setup of 2.5, define $\mathcal{L} = \mathcal{L}_\theta$ to be the category whose objects are the $\mathcal{F}$-centric subgroups of S, with morphisms

$$\mathrm{Mor}_{\mathcal{L}}(P,Q) = \theta(P)\backslash N_G(P,Q).$$

The composition of morphisms is defined by

$$\theta(P)g \cdot \theta(Q)h = \theta(P)gh,$$

for $g \in N_G(P,Q)$ and $h \in N_G(Q,R)$. In fact, this composition is no more than ordinary multiplication of subsets of G. To see this, notice that if $P^g \le Q$, then the signalizer functor axioms (2) and (3) yield $\theta(Q) \le \theta(P^g) = \theta(P)^g$. Thus $\theta(Q)^{g^{-1}} \le \theta(P)$, and so

$$(\theta(P)g)(\theta(Q)h) = \theta(P)\theta(Q)^{g^{-1}}gh = \theta(P)gh.$$

Next, define a functor

$$\pi = \pi_\theta\colon \mathcal{L}_\theta \to \mathcal{F}^c$$

by $\pi(P) = P$ and by $\pi(\theta(P)g) = c_g$ for $g \in N_G(P,Q)$. Finally, define a family $\delta = \delta_\theta = \{\delta_P \mid P \in \mathcal{F}^c\}$ of monomorphisms

$$\delta_P = \delta_{P,\theta}\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$$

by $\delta_P(g) = \theta(P)g$, for $P \in \mathcal{F}^c$.

LEMMA 2.6. *Let S be a Sylow p-subgroup of a group G, set $\mathcal{F} = \mathcal{F}_S(G)$, and let θ be an $\mathcal{F}$-signalizer functor. Then*

(a) *$(\mathcal{L}_\theta, \pi_\theta, \delta_\theta)$ is a centric linking system on $\mathcal{F}$.*

(b) *If $\mathcal{F}$ is saturated then $\mathcal{G}_{S,\theta}(G) := (S, \mathcal{F}_S(G), \mathcal{L}_\theta)$ is a p-local group.*

Proof. Let $P, Q \in \mathcal{F}^c$, let z be a nonidentity element of $Z(P)$, and let $g \in N_G(P,Q)$. Then $\theta(P)g \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, and

$$\delta_P(z)\cdot\theta(P)g = \theta(P)z\cdot\theta(P)g = \theta(P)zg.$$

Here $\theta(P)zg \ne \theta(P)g$ since $\theta(P) \cap Z(P) = 1$. Thus, $Z(P)$ acts freely on $\mathrm{Hom}_{\mathcal{L}}(P,Q)$. Similarly, since $C_G(P) = \theta(P) \times Z(P)$, the map

$$Z(P)\theta(P)g \mapsto c_g = \pi(\theta(P)g)$$

is a bijection from $\mathrm{Hom}_{\mathcal{L}}(P,Q)/Z(P)$ to $\mathrm{Hom}_{\mathcal{F}}(P,Q)$. Thus, condition (A) in Definition 2.4 is satisfied in our setup.

Now suppose that $g \in P$. By construction, we then have

$$\pi(\delta_P(g)) = \pi(\theta(P)g) = c_g,$$

and so condition (B) is satisfied. Finally, let $f = \theta(P)x \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$. We then have $\pi(f) = c_x : P \longrightarrow Q$, and $g\pi(f) = g^x$. Since also $\delta_Q(g^x) = \theta(Q)g^x$, we obtain

$$\begin{aligned} f \cdot \delta_Q((g\pi(f)) &= \theta(P)x \cdot \theta(Q)g^x = \theta(P)xg^x \\ &= \theta(P)gx = \theta(P)g \cdot \theta(P)x = \delta_P(g) \cdot f. \end{aligned}$$

Thus, condition (C) is satisfied, and (a) is proved. Part (b) follows from (a), by the definition of p-local group. □

PROPOSITION 2.7. *Let S be a Sylow p-subgroup of a group G and set $\mathcal{F} = \mathcal{F}_S(G)$. Suppose that $N_G(P)$ is finite for every $P \in \mathcal{F}^c$.*

(a) *There is a unique $\mathcal{F}$-signalizer functor θ given by $\theta(P) = O^p(C_G(P))$.*

(b) *Set $\mathcal{L}_S(G) = \mathcal{L}_\theta$, and suppose that $\mathcal{F}$ is saturated. Then*

$$\mathcal{G}_S(G) := (S, \mathcal{F}_S(G), \mathcal{L}_S(G))$$

is a p-local finite group.

Proof. Part (a) follows from 2.1(c), and then (b) follows from (a) and from 2.6(b). □

The following lemma is intended as a remark, to point out the connection between Definition 2.5 and the usual notion of "balanced signalizer functor" in finite group theory. It will not be used in the sequel.

LEMMA 2.8. *Let θ be an $\mathcal{F}$-signalizer functor, where $\mathcal{F}$ is a fusion system over a finite p-group. Then, for any $P, Q \in \mathcal{F}^c$ with $P \leq Q$,*

(a) *$Z(Q) \leq Z(P)$, and*

(b) *$\theta(Q) = C_{\theta(P)}(Q)$.*

Proof. Part (a) is immediate from 2.1(c). By definition,

$$\theta(Q) \leq C_{\theta(P)}(Q) \leq C_G(Q) = \theta(Q) \times Z(Q),$$

so that $C_{\theta(P)}(Q) = \theta(Q) \times (Z(Q) \cap \theta(P))$. Since $Z(Q) \leq Z(P)$, we obtain (b). □

Definition 2.9. Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be fusion systems over the groups S and $\widetilde{S}$, respectively. A *morphism* $\alpha: \mathcal{F} \to \widetilde{\mathcal{F}}$ *of fusion systems* consists of a functor α_1: $\mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$, and a homomorphism $\alpha_0: S \to \widetilde{S}$ of groups, satisfying the following two conditions.

(MF1) For every subgroup P of S, $\alpha_1(P) = \alpha_0(P)$, and

(MF2) for each $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, we have $\alpha_0 \circ \alpha(\phi) = \phi \circ \alpha_0$ (in right-hand notation).

We most often write α for both α_0 and α_1. In the case that α_0 is given by inclusion of S into $\widetilde{S}$, and α_1 is given by inclusion of $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ into $\mathrm{Hom}_{\widetilde{\mathcal{F}}}(P,Q)$ for all $P, Q \in \mathcal{F}$, we say that α is an *embedding*. We reserve the symbol ι to denote an embedding of fusion systems.

In general, if $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are fusion systems over finite p-groups S and $\widetilde{S}$, respectively, then a morphism $\alpha\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ of fusion systems need not send $\mathcal{F}$-centric subgroups of S to $\widetilde{\mathcal{F}}$-centric subgroups of $\widetilde{S}$. For example, let G be a finite group, take $\widetilde{G}$ to be the direct product of G with a nonidentity p-group R, let $\widetilde{S}$ be a Sylow p-subgroup of $\widetilde{G}$, and take $S = \widetilde{S} \cap G$. Then the inclusion map $\alpha\colon \mathcal{F}_S(G) \to \mathcal{F}_{\widetilde{S}}(\widetilde{G})$ carries no centric subgroup to a centric subgroup.

Recall that if $\mathcal{F}$ is a saturated fusion system over a p-group S, then $\mathcal{F}^{\mathrm{rc}}$ denotes the set of subgroups of S which are both $\mathcal{F}$-centric and $\mathcal{F}$-radical. Given a p-local group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, denote by $\mathcal{L}^{\mathrm{rc}}$ the full subcategory of $\mathcal{L}$ whose objects are the objects of $\mathcal{F}^{\mathrm{rc}}$. Thus $\mathcal{L}^{\mathrm{rc}}$ is a linking system on $\mathcal{F}^{\mathrm{rc}}$.

By [BCG+05, Th. B], the classifying spaces $|\mathcal{L}|$ and $|\mathcal{L}^{\mathrm{rc}}|$ are homotopy equivalent in the case that S is finite. This provides some justification for the following definition of morphism of p-local groups.

Definition 2.10. Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ and $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{L}})$ be pre-local groups. A *morphism of pre-local groups* from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$ is a pair (α, β), where $\alpha\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ is a morphism of fusion systems, and $\beta\colon \mathcal{L} \to \widetilde{\mathcal{L}}$ is a functor which, for each pair P, Q of objects of $\mathcal{L}$, satisfies the following conditions.

(MG1) $\alpha(P) \le \beta(P)$.

(MG2) For each $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, the restriction of $\tilde{\pi}(\beta(\psi))$ to $\alpha(P)$ maps $\alpha(P)$ into $\alpha(Q)$, and $\alpha(\pi(\psi)) = \tilde{\pi}(\beta(\psi))\,|_{\alpha(P)}$.

(MG3) $\beta \circ \delta_P = \delta_{\beta(P)} \circ \alpha_0$.

We say that the morphism (α, β) is an *embedding* if α is an embedding of fusion systems and

$$\beta\colon \mathrm{Mor}_{\mathcal{L}}(P,Q) \to \mathrm{Mor}_{\widetilde{\mathcal{L}}}(\alpha(P), \alpha(Q))$$

is an injection for all $P, Q \in \mathcal{L}$. We say that $\mathcal{G}$ is a *pre-subgroup of* $\widetilde{\mathcal{G}}$ if there is an embedding (ι, β) of $\mathcal{G}$ into $\widetilde{\mathcal{G}}$, and in this case one may say simply that $\beta\colon \mathcal{G} \longrightarrow \widetilde{\mathcal{G}}$ is an embedding. If $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are p-local groups, then a *morphism of p-local groups* from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$ is a morphism of pre-local groups

$$(\alpha, \beta)\colon (S, \mathcal{F}, \mathcal{L}^{\mathrm{rc}}) \longrightarrow (\widetilde{S}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{L}}^{\mathrm{rc}}).$$

Such a morphism is an *embedding of p-local groups* if it is an embedding of pre-local groups, and if α is given by inclusion, we say that $\mathcal{G}$ is a subgroup of $\widetilde{\mathcal{G}}$.

The next two results provide tools for carrying out the construction of morphisms, and particularly of embeddings, of p-local groups.

PROPOSITION 2.11. *Let G_1 be a subgroup of a group G_2, and assume that there are Sylow p-subgroups S_i of G_i with $S_1 = G_1 \cap S_2$. Assume that for each i, the fusion system $\mathcal{F}_i := \mathcal{F}_{S_i}(G_i)$ is saturated, and that we are given an $\mathcal{F}_i$-signalizer functor θ_i. In addition, assume given a mapping $\beta\colon \mathcal{F}_1^{\mathrm{rc}} \to \mathcal{F}_2^{\mathrm{rc}}$ such that the following conditions hold for every $P \in \mathcal{F}_1^{\mathrm{rc}}$.*

(1) $P \le \beta(P)$.

(2) *For each $g \in N_{G_1}(P, S_1)$ we have $\beta(P^g) = \beta(P)^g$.*

(3) *For each $Q \in \mathcal{F}_1^{\mathrm{rc}}$ with $P \le Q$ we have $\beta(P) \le \beta(Q)$.*

(4) $\theta_2(\beta(P)) \cap G_1 = \theta_1(P)$.

Let $\iota\colon \mathcal{F}_1 \to \mathcal{F}_2$ be the inclusion functor, and write $\mathcal{G}_i = (S_i, \mathcal{F}_i, \mathcal{L}_i)$ for the p-local group which is canonically associated with $\mathcal{F}_i$ and θ_i (cf. 2.6). Then for any $P, Q \in \mathcal{F}_1^{\mathrm{rc}}$ there is a mapping

$$(*) \qquad \beta_{P,Q}\colon \mathrm{Mor}_{\mathcal{L}_1}(P, Q) \longrightarrow \mathrm{Mor}_{\mathcal{L}_2}(P, Q)$$

given by $\theta_1(P)g \mapsto \theta_2(\beta(P))g$; and (ι, β) is an embedding of $\mathcal{G}_1$ into $\mathcal{G}_2$. That is, $\mathcal{G}_1$ is a p-local subgroup of $\mathcal{G}_2$.

Proof. Let $P, Q \in \mathcal{F}_1^{\mathrm{rc}}$ and let $g \in N_{G_1}(P, Q)$. Then $P^g \in \mathcal{F}_1^{\mathrm{rc}}$ by 2.3. Since $P^g \le Q$, (3) yields $\beta(P^g) \le \beta(Q)$. Then $\beta(P)^g \le \beta(Q)$ by (2), and so $g \in N_{G_2}(\beta(P), \beta(Q))$. We have $\theta_1(P)g = \theta_1(P)h$ if and only if $hg^{-1} \in \theta_1(P)$, while by (4), $hg^{-1} \in \theta_1(P)$ if and only if $hg^{-1} \in \theta_2(\beta(P)) \cap G$, which holds if and only if $\theta_2(\beta(P))g = \theta_2(\beta(P))h$. This shows that the mappings $\beta_{P,Q}$ in $(*)$ are well-defined injections. Visibly, β preserves composition, so that β is a functor from $\mathcal{L}_1$ to $\mathcal{L}_2$.

Axiom (MG1) in 2.10 is an immediate consequence of (1). Next,

$$\alpha(\pi_1(\theta_1(P)g)) = \alpha(c_g) = c_{g|P} = \pi_2(\theta_2(\beta(P)g)_{|P} = \pi_2(\beta(\theta_1(P)g))_{|P},$$

so that (MG2) holds. Finally, for any $g \in P$,

$$\beta(\delta_{1,P}(g)) = \beta(\theta_1(P)g) = \theta_2(\beta(P))g = \delta_{1,\beta(P)}(g) = \delta_{1,\beta(P)}(\alpha(g)),$$

and so (MG3) holds. □

PROPOSITION 2.12. *Let G_1 be a subgroup of a group G_2, assume that there are Sylow p-subgroups S_i of G_i with $S_1 = G_1 \cap S_2$, and let θ_i be a signalizer functor on the fusion system $\mathcal{F}_i := \mathcal{F}_{S_i}(G_i)$. Assume also that each $\mathcal{F}_i$ is saturated. For $P \in \mathcal{F}_1^{\mathrm{rc}}$, denote by $\mathcal{B}(P)$ the set of $N_{G_1}(P)$-invariant p-subgroups of G_2, and set $\beta(P) = \langle \mathcal{B}(P) \rangle$. Assume that the following two conditions hold for each $P \in \mathcal{F}_1^{\mathrm{rc}}$.*

(1′) $\beta(P) \in \mathcal{F}_2^{\mathrm{rc}}$. *(In particular, $\beta(P) \le S_2$.)*

(2′) $\theta_1(P) = \theta_2(\beta(P)) \cap G_1$.

Then β *satisfies conditions* (1) *through* (4) *of* 2.11, *and defines an embedding* $(\iota, \beta)\colon \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ *of p-local groups.*

Proof. By (1′), β is indeed a map from $\mathcal{F}_1^{\rm rc}$ to $\mathcal{F}_2^{\rm rc}$. Let $P \in \mathcal{F}_1^{\rm rc}$. As $P \in \mathcal{B}(P)$, condition (1) of 2.11 holds. We have $\mathcal{B}(P)^g = \mathcal{B}(P^g)$ for any $g \in N_{G_1}(P, S_1)$, so that condition (2) holds.

Let $Q \in \mathcal{F}_1^{\rm rc}$ with $P \le Q$. For any $x \in N_{G_1}(Q)$ we have $P^x \le S_1$. Then $P^x \in \mathcal{F}_1^{\rm rc}$, and so $\beta(P)^x = \beta(P^x) \le S_2$ by (2). Thus $\langle \beta(P)^{N_{G_1}(Q)} \rangle$ is a p-group, invariant under $N_{G_1}(Q)$, and hence contained in $\beta(Q)$. Thus condition (3) holds. Condition (4) is equivalent to (2′), and so the proof is complete. □

Example 2.13. Let G be a group and $K \trianglelefteq G$ such that $\bar{G} := G/K$ is finite. Let S be a p-subgroup of G such that $S \cap K = 1$, $S \in \mathrm{Syl}_p(KS)$, and $\bar{S} \in \mathrm{Syl}_p(\bar{G})$. Set $\mathcal{F} := \mathcal{F}_S(G)$, and for $P \in \mathcal{F}^c$ set $\bar{\theta}(P) = O_{p'}(C_{\bar{G}}(P))$. Let $\theta(P)$ be the preimage in $C_G(P)$ of $\bar{\theta}(P)$.

Set $\bar{\mathcal{F}} = \mathcal{F}_{\bar{S}}(\bar{G})$. Then $\bar{\theta}$ is an $\bar{\mathcal{F}}$-signalizer functor, and $\bar{\mathcal{G}} := \mathcal{G}_{\bar{S},\bar{\theta}}(\bar{G})$ is the natural p-local group $\mathcal{G}_{\bar{S}}(\bar{G})$. Let $\alpha_0\colon S \longrightarrow \bar{S}$ be the restriction to S of the quotient map $G \longrightarrow \bar{G}$. For $P \le S$ define α_1 on $\mathrm{Hom}_G(P, S)$ by $\alpha_1\colon c_g \mapsto c_{\bar{g}}$.

Observe that for $P \le S$, $\alpha_0\colon N_G(P, S) \longrightarrow N_{\bar{G}}(\bar{P}, \bar{S})$ is a surjection. Namely, if $g \in G$ with $\bar{P}^{\bar{g}} \le \bar{S}$ then, as $S \in \mathrm{Syl}_p(KS)$, there is $k \in K$ with $P^{gk} \le S$, and we have $\bar{g} = \overline{gk}$. Similarly $C_{\bar{G}}(\bar{P}) = \overline{C_G(P)}$. Also, if $g, h \in N_G(P, S)$ with $\alpha_1(c_g) = \alpha_1(c_h)$ then $P^g \le S \ge P^h$ with gh^{-1} centralizing $\bar{P}$, so $gh^{-1} \in N_G(P)$ with $[P, gh^{-1}] \le P \cap K = 1$. Thus $\alpha_1\colon \mathrm{Hom}_{\mathcal{F}}(P, S) \longrightarrow \mathrm{Hom}_{\bar{\mathcal{F}}}(\bar{P}, \bar{S})$ is a bijection. Therefore (α_0, α_1) is an isomorphism of $\mathcal{F}$ with $\bar{\mathcal{F}}$, and since $\bar{\mathcal{F}}$ is saturated, so is $\bar{\mathcal{F}}$. It is now easy to check that θ is an $\mathcal{F}$-signalizer functor, and then $\mathcal{G} := \mathcal{G}_{S,\theta}(G)$ is a p-local finite group by 2.7(b).

Define $\beta\colon \mathcal{L} \longrightarrow \bar{\mathcal{L}}$ by $\beta(P) = \bar{P}$ (on objects), and by $\beta\colon \theta(P)g \mapsto \bar{\theta}(\bar{P})\bar{g}$ (on morphisms). One may now check that $(\alpha, \beta)\colon \mathcal{G} \longrightarrow \bar{\mathcal{G}}$ is an isomorphism of p-local finite groups.

The hypothesis that $S \in \mathrm{Syl}_p(KS)$ was used only to verify that the maps $\alpha_0\colon N_G(P, S) \longrightarrow N_{\bar{G}}(\bar{P}, \bar{S})$ are surjective, and that $\bar{\theta}(\bar{P}) \le \overline{C_G(P)}$ for each $P \in \mathcal{F}^{\rm rc}$. Thus, that hypothesis may be replaced by the hypothesis that $\alpha_0\colon N_G(P, S) \longrightarrow N_{\bar{G}}(\bar{P}, \bar{S})$ is surjective and $\bar{\theta}(\bar{P}) = 1$ for each $\bar{P} \in \bar{\mathcal{F}}^{\rm rc}$.

3. **Amalgams**

In this section, an *amalgam* of groups will always mean a pair

$$\mathcal{A} = (A_1 \xleftarrow{\alpha_1} A_{1,2} \xrightarrow{\alpha_2} A_2)$$

of injective group homomorphisms. A *morphism* from the amalgam $\mathcal{A}$ to an amalgam $\mathcal{B} = (B_1 \xleftarrow{\beta_1} B_{1,2} \xrightarrow{\beta_2} B_2)$ is a triple $\gamma = (\gamma_J \mid \varnothing \ne J \subseteq \{1,2\})$ of injective group homomorphisms $\gamma_J\colon A_J \to B_J$, such that $\alpha_i\gamma_i = \gamma_{1,2}\beta_i$ for $i = 1, 2$.

For example, if G is a group with subgroups G_1 and G_2, and $G_{1,2}$ is a subgroup of $G_1 \cap G_2$, then there is a *subgroup amalgam* $\mathcal{G}$ given by the inclusion maps $\alpha_i\colon G_{1,2} \to G_i$. A *completion* of an amalgam $\mathcal{A}$ is an isomorphism $\gamma\colon \mathcal{A} \to \mathcal{G}$ of $\mathcal{A}$ with a subgroup amalgam $\mathcal{G}$ in a group G, such that $G = \langle G_1, G_2\rangle$. One often abuses the terminology and says simply that G is a *completion* of $\mathcal{A}$.

Let

$$\mathcal{A} = (G_1 \longleftarrow B \longrightarrow G_2)$$

be an amalgam and let $G = G_1 *_B G_2$ be the associated free amalgamated product. Then G is a completion of $\mathcal{A}$, and indeed the *universal completion* of $\mathcal{A}$. We identify $\mathcal{A}$ with the subgroup amalgam of G which is the image of $\mathcal{A}$ under this completion, and in particular regard G_1, G_2 and B as subgroups of G with $G_1 \cap G_2 = B$.

For any subgroup X of G, denote by $X\backslash G$ the set of right cosets of X in G. Set $\Gamma_i = G_i\backslash G$. Then $\Gamma = \Gamma(\mathcal{A})$ is the graph whose vertex set is the disjoint union $V(\Gamma) = \Gamma_1 \coprod \Gamma_2$, and whose set of edges is the set $E(\Gamma)$ of 2-subsets $\{G_1x, G_2x\}$ with $x \in G$. We call $\Gamma = \Gamma(\mathcal{A})$ the *standard tree* associated with $\mathcal{A}$ and with G, and we refer to [Se] for the fact that Γ really is a tree. Observe that G is represented as a group of automorphisms of Γ via right multiplication, and that the kernel of this representation is the largest normal subgroup of G which is contained in B. Evidently, G acts transitively on $E(\Gamma)$, while Γ_1 and Γ_2 are the (distinct) orbits for G on $V(\Gamma)$. It is also evident that G is *locally transitive* on Γ; that is the stabilizer G_δ of any vertex δ acts transitively on the set $\Gamma(\delta)$ defined by

$$\Gamma(\delta) = \{\gamma \in \Gamma \mid \{\delta, \gamma\} \in E(\Gamma)\}.$$

For any subgroup or element X of G, write Γ_X for the subgraph of Γ induced on the set of vertices which are fixed by X. For any connected graph Δ, and vertices α and β of Δ, the length of the shortest geodesic path from α to β in Δ is denoted $d(\alpha, \beta)$. Then (Δ, d) is a discrete metric space, and automorphisms of Δ are isometries. An isometry of a tree is said to be *hyperbolic* if it fixes no vertices or edges. The following two results, noticed first by J. Tits [Tit70], are elementary.

LEMMA 3.1. *Let h be an automorphism of a tree Γ, and denote by $\Lambda(h)$ the intersection of all the h-invariant subtrees of Γ. Suppose that there exists an edge $\{\gamma, \delta\}$ of Γ such that*

(1) $d(\gamma, \gamma h) = d(\delta, \delta h) \neq 0$ *and*

(2) $\{\gamma, \delta\}$ *is not fixed by h.*

Then h is hyperbolic, and the set of all edges $\{\gamma, \delta\}$ which satisfy condition (1) *is the edge set of $\Lambda(h)$.* □

LEMMA 3.2. *Let* Γ *be a tree, let* g *be a hyperbolic isometry of* Γ, *and define* $\Lambda = \Lambda(g)$ *as in* 3.1. *Set* $d = \min\{d(\delta, \delta^g) \mid \delta \in V(\Gamma)\}$. *Then*:

(a) Λ *is isomorphic to the graph* $\bar{\mathbb{Z}}$ *whose vertex set is the set* $\mathbb{Z}$ *of integers, and whose edges are the pairs* $\{n, n+1\}$ *for* $n \in \mathbb{Z}$.

(b) *There is an isomorphism* $\psi\colon \Lambda \to \bar{\mathbb{Z}}$ *such that* $\psi^{-1} g \psi\colon n \mapsto n + d$ *for all* $n \in \mathbb{Z}$.

(c) *For any vertex* γ *of* Γ, *the geodesic in* Γ *from* γ *to* γ^g *has length* $d + 2e$, *where* e *is the minimal distance from* γ *to a vertex of* Λ. □

LEMMA 3.3. *Let* Γ *be a tree, and let* $x, y \in \mathrm{Aut}(\Gamma)$, $\delta \in \Gamma_x$, $\gamma \in \Gamma_y$, *and* $(\alpha_0, \dots, \alpha_d)$ *the geodesic from* γ *to* δ. *Suppose* x *does not fix* $\alpha = \alpha_{d-1}$ *and* y *does not fix* α_1. *Then* xy *is hyperbolic.*

Proof. Observe first of all that $(\alpha_0 y, \dots, \alpha_d y)$ is the geodesic from $\gamma = \alpha_0 y$ to δy, so that the path $\rho = (\alpha_d y, \dots, \alpha_0 y, \alpha_1, \dots, \alpha_d)$ contains a geodesic g from δy to δ. Indeed as $\alpha_1 \neq \alpha_1 y$, $g = \rho$ is of length $2d$. Then $g' = (\alpha_{d-1} y, \dots, \alpha_0 y, \alpha_1, \dots, \alpha_d)$ is the geodesic from αy to δ, and since $\beta = \alpha x^{-1} \neq \alpha$, it follows that $g'\beta$ is the geodesic of length $2d$ from αy to β. But $\delta y = \delta x y$ and $\beta x y = \alpha y$, and so $d(\delta, \delta x y) = d(\beta, \beta x y)$. The lemma now follows from 3.1. □

We will work under the following hypothesis for the remainder of this section.

HYPOTHESIS 3.4. $G = G_1 *_B G_2$ *is the free amalgamated product associated with an amalgam*

$$\mathcal{A} = (G_1 \longleftarrow B \longrightarrow G_2),$$

and G_1, G_2, *and* B *are regarded as subgroups of* G *in the canonical way. Let* Γ *be the standard tree associated with* $\mathcal{A}$ *and* G, *and let* Γ_i *be the subset* $G_i \backslash G$ *of the vertex set of* Γ. *Assume*

(1) *There is a subgroup* S *of* G *such that* S *is a* (*possibly infinite*) *Sylow* p*-subgroup of each of the groups* G_1, G_2, *and* B.

(2) $N_{G_i}(S) \le B \neq G_i$, *for* $i = 1$ *and* 2.

The vertex G_i of Γ will most often be denoted γ_i.

LEMMA 3.5. *Assume Hypothesis* 3.4, *and assume also that*

$$(*) \qquad \{S^g \mid g \in G_1 \cup G_2,\ S^g \le B\} = S^B.$$

Then:

(a) $\Gamma_S = \{\gamma_1, \gamma_2\}$, *and*

(b) $S \in \mathrm{Syl}_p(G)$.

Proof. It follows from $(*)$ and from [Asc86, 5.21] that $N_{G_i}(S)$ is transitive on $\Gamma_S(\gamma_i)$. Since $N_{G_i}(S) \le B$, by 3.4, we obtain (a).

Let S^* be a p-subgroup of G containing S and let $x \in S^*$. Then $|x|$ is finite, so that 3.2 implies that $\Gamma_x \neq \varnothing$. Choose $\delta \in \Gamma_x$ and $\gamma \in \Gamma_S$ with $d := d(\delta, \gamma)$

minimal. Suppose that $d > 0$ and let $(\delta, \delta', \dots, \gamma', \gamma)$ be the geodesic from δ to γ in Γ. Then $x \notin G_{\delta'}$, and there exists $y \in S - G_{\gamma'}$. As $|xy|$ is finite, this contradicts 3.3, and so we conclude that $d = 0$. Then (a) yields $S^* \leq G_i$ for some i. Since S is a maximal p-subgroup of G_i we then have $S^* = S$, and thus S is a maximal p-subgroup of G.

Let P be a finite p-subgroup of G, and let P_0 be a subgroup of P which fixes a vertex of Γ, and which is maximal for this condition. Then $N_P(P_0)$ acts on the tree $\Gamma_0 = \Gamma_{P_0}$, and no element of $N_P(P_0) - P_0$ fixes a vertex of Γ_0. Now, $N_P(P_0) - P_0$ consists of hyperbolic isometries on Γ_0, by 3.2, and hence $N_P(P_0) = P_0 = P$. Since G is edge-transitive on Γ, it follows that P is conjugate to a subgroup of G_1 or G_2. Then P is conjugate to a subgroup of S, by 3.4(1), and S is a Sylow p-subgroup of G. □

LEMMA 3.6. *Assume Hypothesis* 3.4, *let P be a finite subgroup of S, and let $g \in N_G(P, S)$. Set $F_0 = P$. Then there exists a positive integer n, elements $g_1, \dots, g_n$ of G_1, elements $h_1, \dots, h_n$ of G_2, and subgroups E_i and F_i of S, $1 \leq i \leq n$, such that the following conditions hold.*

(a) $g_i \notin B$ *for* $1 < i \leq n$, *and* $h_i \notin B$ *for* $1 \leq i < n$.

(b) $E_i = F_{i-1}^{g_i}$ *and* $F_i = E_i^{h_i}$ *for all i with* $1 \leq i \leq n$.

(c) $g = g_1 h_1 \dots g_n h_n$.

Moreover, the minimal length of g as a word in the generating set $G_1 \cup G_2$ is $2n-2$ if $g_1, h_n \in B$, $2n-1$ if exactly one of g_1 and h_n is in B, and $2n$ if neither g_1 nor h_n is in B.

Proof. Since G_1 and G_2 generate G, we may choose elements $g_i \in G_1$ and $h_i \in G_2$, satisfying the conditions in (a) and (c). Set $\alpha_0 = \gamma_2$, $\beta_0 = \gamma_1 h_n$, and for $1 \leq i \leq n$ set $w_i = g_{n-i+1} h_{n-i+1} \dots g_n h_n$, $\alpha_i = \gamma_2 w_i$, and $\beta_i = \gamma_1 h_{n-i} w_i$. Then $q = (\alpha_0, \beta_0, \dots, \beta_{n-1})$ is a path in Γ with $\beta_i \neq \beta_{i+1}$ and $\alpha_i \neq \alpha_{i+1}$ for $i < n-1$. Thus q is a geodesic from α_0 to β_{n-1}, and if $g_1 \notin B$ then also the path $q\alpha_n$ is a geodesic. In particular if $n > 1$ or $g_1 \notin B$ then $d(a_0, a_n) > 0$, and $w_n \notin B$.

Take (b) as the definition of the groups E_i and F_i for $i > 0$. We now show that these groups are contained in B. Let $x \in F_0$, set $y_0 = x$, and for $1 \leq i \leq n$ define x_i and y_i recursively, by

$$x_i = y_{i-1}^{g_i} \quad \text{and} \quad y_i = x_i^{h_i}.$$

Suppose that for some j, either x_j or y_j is not in B, and let j be the smallest such index. Suppose that $x_j \notin B$. Then $y_{j-1} \in B$, and so $x_j = y_{j-1}^{g_j} \in G_1 - B$. Then

$$x^g = h_n^{-1} g_n^{-1} \dots h_j^{-1} x_j h_j \dots g_n h_n$$

is an alternating product of elements of G_2 and G_1, in which none of the factors lies in B except possibly for the first and the last. It follows from paragraph one that $x^g \notin B$, whereas $x^g \in S \leq B$. A similar argument shows $y_j \in B$. Therefore x_i and y_i are in B for all i, and so each of the groups E_i and F_i is contained in B. Since S is a Sylow p-subgroup of B, we may adjust our choices of the elements g_i and h_i, via right multiplication by elements of B, to ensure that E_i and F_i are in S for all i.

It remains to prove the final statement in the lemma. This follows since, by paragraph one, the length $\ell(g)$ of g as a word in the generating set $G_1 \cup G_2$ for G is equal to the shortest distance in the tree $\Gamma(\mathcal{A})$ from a vertex in $\{\gamma_1 g, \gamma_2 g\}$ to a vertex in $\{\gamma_1, \gamma_2\}$. □

We have the following immediate consequence of 3.6.

COROLLARY 3.7. *Assume Hypothesis* 3.4. *Then*

$$\mathcal{F}_S(G) = \langle \mathcal{F}_S(G_1), \mathcal{F}_S(G_2) \rangle. \qquad \square$$

For any subgroup X of G, and any elementary abelian p-subgroup A of X, denote by $\mathcal{E}_n(X, A)$ the set of elementary abelian p-subgroups of X which have order p^n and which contain A. Write $\mathcal{E}_n(X)$ for $\mathcal{E}_n(X, 1)$.

In the remainder of this section we assume the following hypothesis.

HYPOTHESIS 3.8. *Hypothesis* 3.4 *holds, and so do the following conditions.*

(1) *There is a normal subgroup Z of G_1 of order p, and Z is the unique subgroup of order p in $Z(S)$.*

(2) *There exists $U \in \mathcal{E}_2(G_2, Z)$ with $U \trianglelefteq G_2$, and G_2 acts transitively on $\mathcal{E}_1(U)$.*

(3) $B = N_{G_1}(U) = N_{G_2}(Z)$.

(4) *For each $X \in \{H, K, B\}$, X is transitive on its set of maximal p-subgroups.*

LEMMA 3.9. *Let P be a subgroup of S and let X be a subgroup of $Z(P)$ of order p.*

(a) *Let $g \in G_2$ with $P^g \leq S$. Then one of the following holds.*

   (i) *$g \in B$, and neither P nor P^g is contained in $C_G(U)$.*

   (ii) $P \leq C_G(U)$.

(b) *If $X \neq Z$ and $X \leq U$ then there exists $g \in G_2$ with $X^g = Z$ and with $P^g \leq S$.*

Proof. Let g be as in (a), and set $Q = P^g$. Since $U \trianglelefteq G_2$, we have $P \leq C_G(U)$ if and only if $Q \leq C_G(U)$. Since conclusion (ii) of (a) does not hold, neither P nor Q is contained in $C_G(U)$. Set $\overline{G}_2 = G_2/C_{G_2}(U)$. Then $\overline{P}$ and $\overline{Q}$ are nontrivial p-subgroups of $\overline{S}$. Hence by 3.8(2), $\overline{G}_2$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$ containing $\mathrm{SL}_2(p)$, and $B = N_{G_2}(Z)$ has index $p+1$ in G_2. Thus $\overline{P} = \overline{Q} = \overline{S}$, and $\bar{B} = N_{\bar{G}_2}(\bar{S})$. Since $P^g = Q$, we then have $\bar{g} \in \bar{B}$. Thus $g \in B$ and (a) is proved.

Suppose that $Z \neq X \leq U$. By 3.8(2) there exists $g \in G_2$ with $X^g = Z$. Since $X \leq Z(P)$, we have $P^g \leq C_{G_2}(Z) \leq B$. By 3.8(4) there exists $h \in B$ with $P^{gh} \leq S$. Replacing g with gh, we obtain (b). □

For $\delta \in \Gamma$ and $g \in G$ with $\delta = \gamma_i g$ for some i, Z_δ will denote Z^g if $i = 1$, and U^g if $i = 2$. This notation is well defined as a consequence of 3.8(3).

LEMMA 3.10. *Let Σ be a subtree of Γ and let $\gamma \in \Sigma$. Set $Y = \langle Z_\delta \mid \delta \in \Sigma \rangle$. Then $C_{G_\gamma}(Y)$ fixes Σ vertex-wise.*

Proof. Let $\gamma \in \Sigma$, set $X = C_{G_\gamma}(Y)$, and assume that $X \nleq G_\Sigma$. Among all pairs (δ, x) with $\delta \in \Sigma$ and $x \in X$ with $\delta x \neq \delta$, choose (δ, x) so that $d := d(\delta, \delta x)$ is minimal. Let $\alpha \in \Sigma(\delta)$ be of distance $d - 1$ from γ. Then X fixes α and centralizes Y, so X centralizes Z_δ. Thus $X \leq C_{G_\alpha}(Z_\delta) \leq G_\delta$ by 3.8(3), and contrary to our choice of δ. □

LEMMA 3.11. *Let δ and γ be distinct vertices in Γ_1 with $Z_\delta = Z_\gamma$. Then $d(\delta, \gamma) \geq 6$. Moreover, the following hold.*

(a) *Let $\alpha, \beta \in \Gamma_1$ with $d(\alpha, \beta) = 2$, and let X be a subgroup of G fixing β and centralizing Z_α and Z_β. Then X fixes α.*

(b) *Let $\alpha_0, \alpha_4 \in \Gamma_1$ with $d(\alpha_0, \alpha_4) = 4$, and such that Z_{α_4} centralizes Z_{α_0}. Then Z_{a_4} fixes a_0.*

Proof. Suppose $d := d(\delta, \gamma) < 6$. Then $d = 2$ or 4. If $d = 2$, and (δ, β, γ) is the geodesic from δ to γ, then $Z_\beta = Z_\delta Z_\gamma$ is of order p, which is not the case. Thus $d = 4$. Write $(\delta, \beta, \delta', \beta', \gamma)$ for the geodesic from δ to γ. Then

$$Z_\beta = Z_\delta Z_{\delta'} = Z_\gamma Z_{\delta'} = Z_{\beta'}.$$

By edge-transitivity, we may take $\delta' = \gamma_1$ and $\beta' = \gamma_2$. Then $U = Z_{\beta'} = Z_\beta$, and by local transitivity there exists $g \in G_1$ with $\beta^g = \beta'$. Then $U = U^g$, so $g \in B$, and $\beta = \beta'$. Then $d < 4$, and we have a contradiction.

Assume the hypothesis of (a). Without loss, $\beta = \gamma_1$ and $\{\gamma_2\} = \Gamma(\alpha) \cap \Gamma(\beta)$. As X fixes β and centralizes Z_α and Z_β, we obtain $X \leq C_H(Z_\alpha Z_\beta) = C_H(U) \leq B \leq G_\alpha$, establishing (a).

Now assume the hypothesis of (b), and take X to be Z_{α_4}. Let $(\alpha_0, \dots, \alpha_4)$ be the geodesic from α_0 to α_4. Then $X \leq Z_{a_3} = Z_{\alpha_2} Z_{a_4} \leq C_G(Z_{\alpha_2})$, and so X fixes α_2 by (a). Then, since $\mathrm{dist}(\alpha_0, \alpha_2) = 2$ and X centralizes Z_{α_i} for $i = 0, 2$, X fixes α_0 by another application of (a). □

HYPOTHESIS 3.12. *Hypothesis* 3.8 *holds, and every subgroup of G_1 of order p is conjugate in G_1 to a subgroup of U.*

LEMMA 3.13. *Assume Hypothesis* 3.12. *Let P be a subgroup of S and let X be a subgroup of $Z(P)$ of order p.*

(a) *If $X \not\leq U$ then there exists $g \in G_1$ with $X^g \leq U$ and with $P^g \leq S$.*

(b) *If $X \neq Z$ then there exists $g \in G$ with $X^g = Z$, $(XZ)^g = U$, $(PU)^g \leq S$, and $g = g_1 g_2$ where $g_i \in G_i$ and $P^{g_1} \leq S$.*

Proof. Suppose that $X \not\leq U$. By 3.12 there exists $g \in G_1$ with $X^g \leq U$. Set $Y = X^g$. Then $YZ = U$, so that $P^g \leq C_{G_1}(U) \leq B$, and by 3.8(4) there exists $h \in B$ such that $P^{gh} \leq C_S(Y)$. Replacing g with gh, we obtain (a).

Now assume that $X \neq Z$. If $X \leq U$ we appeal to 3.9(b), with PU in the role of P, in order to obtain $g_2 \in G_2$ with $X^{g_2} = Z$ and $(PU)^{g_2} \leq S$. With $g_1 = 1$, (b) holds in this case. So assume that $X \not\leq U$. If U centralizes X we apply (a) to PU, obtaining $g_1 \in G_1$ such that $X^{g_1} \leq U$ and $(PU)^{g_1} \leq S$. Then $(XZ)^{g_1} = U$, and by 3.9(b) there exists $g_2 \in G_2$ with $X^{g_1 g_2} = Z$ and $(PU)^{g_1 g_2} \leq S$. Thus (b) also holds in this case, and we are reduced to the case where U does not centralize X. Since $P \leq S$, 3.8(3) implies that $[U, X] = [U, S] = Z$. Then $XZ \trianglelefteq PU$. By 3.12 there exists $g_1 \in G_1$ with $X^{g_1} \leq U$. Then $(XZ)^{g_1} = U$, so that $(PU)^{g_1} \leq N_{G_1}(U) = B$. By 3.8(4) we may assume that g_1 was chosen so that $(PU)^{g_1} \leq S$. Then (a) applies, and completes the proof of (b). □

The next result amounts to a re-working of [LO02, Lemma 1.4] in our tree-theoretic setup. The formulation given here is different in several respects from the one in [LO02], but the main idea of the proof has not been altered. We remark that we shall only use part (c) of 3.14, and this will occur only once, in the proof of 9.2.

PROPOSITION 3.14. *Assume Hypothesis* 3.12. *Set $D = N_G(Z)$ and assume that $\mathcal{F}_S(D) \neq \mathcal{F}_S(G_1)$. Denote by Δ the set of all pairs (P, g) such that P is a finite subgroup of S, $g \in N_D(P, S)$, and $c_g \notin \mathrm{Hom}_{G_1}(P, S)$. Set*

$$\mathcal{P} = \{P \mid (P, g) \in \Delta \text{ for some } g \in D\},$$

and let $P \in \mathcal{P}$. Choose $(P, g) \in \Delta$ so that the length $\ell(g)$ of g, as a word in the set $G_1 \cup G_2$ of generators of G, is minimal. Then $[P, U] = 1$, $(PU, g) \in \Delta$, and upon replacing P with a suitable subgroup of $C_S(P)P$ containing P, we have

(a) *$\ell(g) = 5$, and $g = g_1 g_2 g_3 g_4 g_5$ where $g_i \in G_2$ for i odd, and $g_i \in G_1$ for i even.*

(b) *The elements g_1 through g_5 in* (a) *may be chosen so that $U \leq Z(P^{g_1 \cdots g_i})$, and $P^{g_1 \cdots g_i} \leq S$ for all i, $1 \leq i \leq 5$.*

(c) *There exists $E \in \mathcal{E}_3(Z(P), U)$ such that $U \leq E^g$, $C_B(E)^g \leq B$, and $C_B(E^g) \leq B^g$.*

Proof. Set $Q = P^g$. Since $g \in D$, also $g \in N_D(PZ, QZ)$, so that we may assume $Z \leq P$.

By 3.6 we have $g = g_1 \dots g_n$, where each g_i is in $G_1 \cup G_2$, and where $P^{g_1 \dots g_i} \leq S$ for all i, $1 \leq i \leq n$. Moreover, the sequence $(g_1, \dots, g_n)$ may be chosen so that $\ell(g) = n$.

Set $P_0 = P$, $Z_0 = Z$, and for $1 \leq i \leq n$ set $P_i = P_{i-1}^{g_i}$ and $Z_i = Z_{i-1}^{g_i}$. If $Z_i = Z$ for some i with $0 < i < n$, then the minimality of n implies that, for $x = g_1 \dots g_i$ and $y = g_{i+1} \dots g_n$, we have $c_x \in \mathrm{Hom}_{G_1}(P, P_i)$ and $c_y \in \mathrm{Hom}_{G_1}(P_i, Q)$. But in that case we get $c_g = c_x c_y \in \mathrm{Hom}_{G_1}(P, Q)$, contrary to hypothesis. Thus:

(1) $Z_i = Z$ if and only if $i = 0$ or n.

By 3.13(b) there exist elements v_1 through v_{n-1} of $G_1 G_2$, such that

(2) $Z_i^{v_i} = Z$, $(ZZ_i)^{v_i} = U$, and $(P_i U)^{v_i} \leq S$.

Set $v_0 = v_n = 1$, and for each i, $0 \leq i \leq n$, choose $r_i \in G_1$ and $s_i \in G_2$ with $v_i = r_i s_i$. Set $k_i = v_{i-1}^{-1} g_i v_i$ for $i \geq 1$, and set $P_i' = P_i^{v_i}$ for $i \geq 0$. Then $P_i' \leq S$ for all i, by (2), and $(P_{i-1}')^{k_i} = P_i'$ for $i \geq 1$. Notice that

$$(*) \qquad g_1 \dots g_n = k_1 \dots k_n.$$

Suppose that, for all $i \geq 1$, we have $c_{k_i} \in \mathrm{Hom}_{G_1}(P_{i-1}', P_i')$. Choose $t_i \in G_1$ so that $k_i t_i^{-1} \in C_G(P_{i-1}')$, and set $t = t_1 \dots t_n$. Then $gt^{-1} \in C_G(P)$ by $(*)$, so that $c_g \in \mathrm{Hom}_{G_1}(P, Q)$, contrary to hypothesis. Thus, $c_{k_i} \notin \mathrm{Hom}_{G_1}(P_{i-1}', P_i')$ for some $i \geq 1$. Since $k_i = (r_{i-1} s_{i-1})^{-1} g_i r_i s_i$ is of length 5, it follows from the minimality of n that $n \leq 5$.

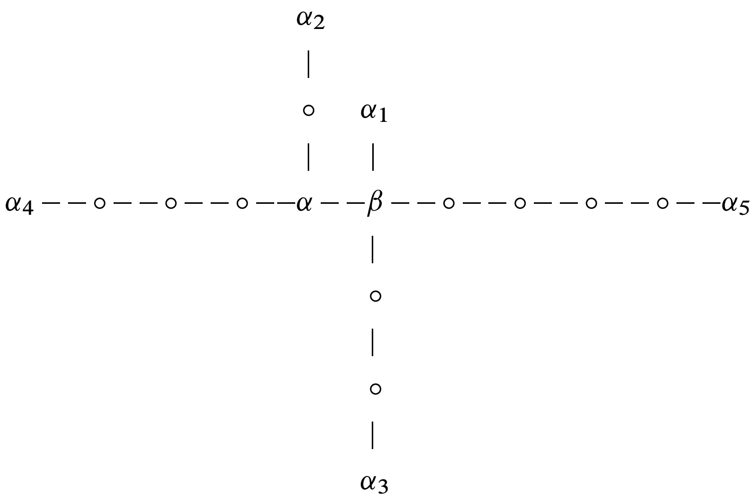
Set $\alpha = \alpha_0 = \gamma_1$, and define vertices α_1 through α_n by $\alpha_i = \alpha_{i-1} g_i$. Let Σ be the subtree of Γ generated by $\{\alpha_0, \dots, \alpha_n\}$, and let Σ_i be the subtree of Σ generated by α_0 and α_i. Thus Σ is the union of its subtrees Σ_i, and Σ_i is the geodesic from α to α_i in Γ.

Minimality of n implies that $g_1 \in G_2 - G_1$, as otherwise we may replace (P, g) with $(P^{g_1}, g_2 \dots g_n)$. Then g_i is in $G_2 - G_1$ for i odd, and in $G_1 - G_2$ for i even. Observe that

$$(**) \qquad d(\alpha, \alpha_i) = \begin{cases} i+1 & \text{if } i \text{ is odd, and} \\ i & \text{if } i \text{ is even.} \end{cases}$$

Notice that $Z_i = Z_{\alpha_i}$. Then 3.11 implies that $Z_\delta \neq Z$ for any $\delta \in V(\Sigma)$ with $d(\alpha, \delta) \leq 5$. Since $n \leq 5$, it now follows from $(**)$ that $n = 5$.

The reader may find it convenient to have a "picture" of Σ, at this point, as a visual reference for the remainder of the argument.



Since $P_i \le S \le G_{\alpha_0}$, P_i fixes every vertex of Σ_i, and then 3.10 implies that $[Z_\delta, P_i] = 1$ for every $\delta \in \Sigma_i$. Set $\beta = \gamma_2$. Then from paragraph one of the proof of 3.6, for i odd, Σ_i is the tree induced on the geodesic $(\alpha, \beta, \alpha g_i, \beta g_{i-1} g_i, \dots, \alpha g_1 \dots g_i)$. In particular, β is a vertex of Σ_i for i odd. Since $Z_\beta = U$, it follows that $[U, P_i] = 1$ for i odd. But for i odd we also have $g_i \in G_2 \le N_G(U)$, and then since $P_i = P_{i-1}^{g_i}$, we conclude that:

(3) $[U, P_i] = 1$ for all i, $0 \le i \le 5$.

From the description of Σ_5 above, βg is the vertex of Σ_5 at distance 5 from α, and $\alpha g \ne \alpha_1 g \in \Gamma(\beta g)$. Set $\beta' = \beta g$, let α' be the vertex in $\Gamma_{\beta'}$ at distance 4 from α, and let α'' be the vertex of Σ_5 at distance 2 from α. By (3), U centralizes $Z_{\alpha'}$, so as $Z_{\beta'} = Z Z_{\alpha'} \ge Z_1^g$, we get $Z_1^g \le C_G(U) \le C_G(Z_{\alpha''})$. Then Z_1^g fixes α'', by 3.11(b). Also $[Z_1^g, Z] = 1$, so that 3.11(a) implies Z_1^g fixes α. Since $U^g = (Z Z_1)^g = Z Z_1^g$, we conclude that $U^g \le G_\alpha \cap G_\beta$. That is, $U^g \le B$, and so QU^g is a p-subgroup of B. By 3.8(4) there exists $h \in B$ with $(QU^g)^h \le S$, and we may replace (P, g) with (PU, gh), without increasing the length n. Thus, we may assume henceforth that $U \le P$. By symmetry between (P, g) and (Q, g^{-1})), we may assume also that $U \le Q$. Then since $g_1 \in N_G(U)$), also $U \le P_1$. Since $g_5 \in N_G(U)$ we then obtain $U \le P_4$.

As $g_2, g_4 \in G_1$ and $U \le P_1 \cap P_4$, we have $Z \le P_2 \cap P_3$. Since $g_3 \in N_G(U)$, we have $U \le P_2$ if and only if $U \le P_3$. Suppose that $U \not\le P_2$. Then

$$Z^{g_3} = (U \cap P_2)^{g_3} = U \cap P_3 = Z,$$

contrary to $g_3 \in G_2 - G_1$. Therefore $U \le P_i$ for all i. Then by (3):

(4) $U \le Z(P_i)$ for all i, $0 \le i \le 5$.

Set $U_0 = U = U_{-1}$, and for $1 \le i \le 5$, let $U_i = U^{g_1 \cdots g_i}$. For $0 \le i \le 5$ set $E_i = U_{i-1}U_i$. As $g_1 \in G_2$, we have $U = U_1 \le E_2$, and so

$$Z = Z^{g_4} \le U^{g_4} = U^{g_3 g_4} \le E_2^{g_3 g_4} = E_4.$$

Also $g_5 \in G_2 - G_1$, so that $Z \ne Z^{g_5^{-1}} \le E_5^{-1} = E_4$. Then $U = ZZ^{g_5^{-1}} \le E_4$. Since $g_5 \in G_2$, also $U \le E_5$.

Set $F = C_B(E_0)$. Then $F^g = C_{B^g}(E_5)$, and B^g is the stabilizer in G of the edge $\{\alpha_5, \beta g\}$ of Σ_5. Next $U^g \le E_0^g = E_5$, and $U^g = Z_{\beta g}$, so that F^g centralizes $Z_{\beta g}$. Then F^g fixes the vertex α' of Σ_5, by 3.11(a). Denote by β'' the vertex of Σ_5 at distance 3 from α, and hence adjacent to α'. From an earlier remark, we have $\beta'' = \beta g_4 g_5$, and so

$$Z_{\beta''} = U^{g_4 g_5} = U^{g_3 g_4 g_5} \le E_2^{g_3 g_4 g_5} = E_5.$$

Thus F^g centralizes $Z_{\beta''}$, and so F^g fixes every vertex in $\Gamma(\beta'')$ by 3.11(a). In particular, F^g fixes α''. Since $[U, F^g] = 1$, F^g fixes every vertex in $\Gamma(\beta)$ by 3.11(a). Thus, F^g fixes α and β, and so $F^g \le B$. This yields the first part of (c), with E_0 in the role of E. Since $Z_{\beta''}Z_{\beta'} \le E_5$, 3.11(a) yields $C_B(E_5) \le B^g$, and thus all parts of 3.14 have been established. □

4. $\mathrm{Spin}_7(\mathbf{F})$

Let p be an odd prime, let $\overline{\mathbf{F}}$ be an algebraic closure of the field of p elements, let $\widetilde{V}$ be a vector space over $\overline{\mathbf{F}}$ (of finite dimension d), and let f be a symmetric, nondegenerate bilinear form on $\widetilde{V}$. The form f is essentially unique, as $\widetilde{V}$ has an orthonormal basis with respect to f. The isometry group $O(\widetilde{V}, f)$ will be denoted also $O(\widetilde{V})$ (and $O_d(\overline{\mathbf{F}})$). The identity component of $O(\widetilde{V})$ is denoted $\Omega(\widetilde{V})$, and has index 2 in $O(\widetilde{V})$. Indeed, we have $O(\widetilde{V}) = \Omega(\widetilde{V})\langle\tau\rangle$, where τ is a reflection on $\widetilde{V}$. In the case that d is odd, we have $O(\widetilde{V}) = \Omega(\widetilde{V}) \times \{\pm I\}$. The universal covering group of $\Omega(\widetilde{V})$ is denoted $\mathrm{Spin}(\widetilde{V})$ (or $\mathrm{Spin}_d(\overline{\mathbf{F}})$).

There is a rational representation $\phi\colon \mathrm{Spin}(\widetilde{V}) \longrightarrow \Omega(\widetilde{V})$, with kernel contained in $Z(\mathrm{Spin}(\widetilde{V}))$. From [C], one has $|\ker(\phi)| = 2$, and $\ker(\phi) = Z(\mathrm{Spin}(\widetilde{V}))$ if d is odd.

For any subset or element D of $\mathrm{Spin}(\widetilde{V})$, we write $C_{\widetilde{V}}(D)$ and $[\widetilde{V}, D]$ for $C_{\widetilde{V}}(D\phi)$ and $[\widetilde{V}, D\phi]$, respectively.

Let $\widetilde{T}$ be a maximal torus of $\mathrm{Spin}(\widetilde{V})$. By a *weight* of $\widetilde{T}$ on $\widetilde{V}$ we mean a homomorphism $\lambda\colon \widetilde{T} \longrightarrow \overline{\mathbf{F}}^{\times}$ such that the space $\widetilde{V}_\lambda = \{v \in \widetilde{V} \mid va = \lambda(a)v$ for all $a \in \widetilde{T}\}$ is nonzero. The set of such weights is denoted $\Lambda(\widetilde{T})$.

A *hyperbolic line* in $\widetilde{V}$ is a nondegenerate subspace ℓ of $\widetilde{V}$ of dimension 2. Such a subspace has exactly two 1-dimensional singular subspaces (or *points*), and from this one may easily deduce that $\Omega(\ell) \cong \bar{\mathbf{F}}^{\times}$ and that $O(\ell) = \Omega(\ell)\langle t \rangle$, where t is an involution which interchanges the singular points of ℓ.

The following result is well known, and its proof is elementary.

LEMMA 4.1. *Let $(\tilde{V}, f)$ be a nondegenerate orthogonal space over $\bar{\mathbf{F}}$ of dimension d, and let $\tilde{T}$ be a maximal torus of $\Omega(\tilde{V})$. Then there exists a set $\ell(\tilde{T}) = \{\ell_1, \dots, \ell_k\}$ of $\tilde{T}$-invariant, pairwise orthogonal hyperbolic lines in $\tilde{V}$, such that the following hold.*

(a) *$d = 2k$ or $d = 2k + 1$.*

(b) *$[\tilde{V}, \tilde{T}] = \ell_1 + \cdots + \ell_k$, and either $C_{\tilde{V}}(T) = 0$ or $C_{\tilde{V}}(\tilde{T})$ is a nonsingular 1-space, orthogonal to $[\tilde{V}, \tilde{T}]$.*

(c) *Each ℓ_i is a sum of two weight spaces $\tilde{V}_\lambda$ and $\tilde{V}_{\lambda^{-1}}$, where $\lambda \neq \lambda^{-1}$. These weight spaces are the singular points of ℓ_i.*

The set $\ell(\tilde{T})$ is uniquely determined by the conditions (a) *and* (c). *Conversely, for any maximal set $\mathcal{U}$ of pairwise orthogonal, hyperbolic lines in $\tilde{V}$, there is a unique maximal torus $\tilde{T}$ in $\Omega(\tilde{V})$ with $\ell(\tilde{T}) = \mathcal{U}$.*

From now on, let $\tilde{V}$ be a vector space of dimension 7 over $\bar{\mathbf{F}}$, and set $\tilde{H} = \mathrm{Spin}(\tilde{V})$. Write Z for the kernel of ϕ. Then $Z = \langle z \rangle$ where z is of order 2.

It is well known that an involution t in an orthogonal group (over a field of characteristic different from 2) lifts to an involution in the corresponding spin group if and only if the dimension of the commutator space of t is divisible by 4. This implies the following result.

LEMMA 4.2. *Let $x \in \tilde{H}$ with $|\phi(x)| = 2$. Then $|x| = 2$ if and only if $\dim([\tilde{V}, x]) = 4$.*

Let $\tilde{T}$ be a maximal torus of $\tilde{H}$. By 4.1, the commutator space $[\tilde{V}, \tilde{T}]$ is the orthogonal direct sum of three hyperbolic lines ℓ_1, ℓ_2, and ℓ_3, where each ℓ_i is a sum of two weight spaces for $\tilde{T}$, with weights λ_i and λ_i^{-1}. For $i = 1, 2, 3$, fix a basis $\{x_{2i-1}, x_{2i}\}$ for ℓ_i of singular vectors, with $f(x_{2i-1}, x_{2i}) = 1$. Then 4.1 yields

(4.2.1) $$[\tilde{V}, \tilde{T}] = \ell_1 + \ell_2 + \ell_3.$$

Let $x_7 \in C_{\tilde{V}}(\tilde{T})$, with $f(x_7, x_7) = 1$. Then

$$C_{\tilde{V}}(\tilde{T}) = [\tilde{V}, \tilde{T}]^{\perp} = \bar{\mathbf{F}}x_7.$$

Identify $\tilde{V}$ with $\bar{\mathbf{F}}^{(7)}$, via the ordered basis $(x_1, \dots, x_7)$.

The semidirect product $\tilde{V}\tilde{H}$ is an algebraic group in which $\tilde{T}$ is a maximal torus, and in which $\tilde{V}$ is the unipotent radical. Let ζ be a Frobenius endomorphism of $\tilde{V}\tilde{H}$ which induces the p^{th}-power map on $\tilde{T}$. Then ζ fixes $\ell(\tilde{T})$ pointwise, and fixes the vectors x_1 through x_7.

Denote also by ζ the p^{th}-power automorphism of $\overline{\mathbf{F}}$, and set

$$\mathbf{F} = \bigcup_{k \geq 0} C_{\overline{\mathbf{F}}}(\zeta^{2^k}).$$

Evidently, $\mathbf{F}$ is a subfield of $\overline{\mathbf{F}}$. Denote by V the $\mathbf{F}$-span of $\{x_1, \dots, x_7\}$ in $\widetilde{V}$, and by H the group of $\mathbf{F}$-rational points in $\widetilde{H}$, with respect to the matrix representation given by the chosen basis for $\widetilde{V}$. The restriction of f to $V \times V$ defines an orthogonal form on V, and $\phi(H)$ is contained in $O(V)$. Set $T = \widetilde{T} \cap H$, and set $E = \{x \in T \mid x^2 = 1\}$.

Since $O(\widetilde{V}) = \Omega(\widetilde{V})\langle t \rangle$ for any reflection $t \in O(\widetilde{V})$, it follows that $\Omega(\widetilde{V})$ is the group $SO(\widetilde{V})$ of determinant 1 isometries. Any element x of $O([\widetilde{V}, \widetilde{T}])$ extends to an element of $\Omega(\widetilde{V})$, since we are free to adjust the action of x on $C_{\widetilde{V}}(\widetilde{T})$ by ± 1. In particular, there exists an element w_0 of $\widetilde{H}$ such that w_0 acts on $[\widetilde{V}, \widetilde{T}]$ by the permutation $(x_1\ x_2)(x_3\ x_4)(x_5\ x_6)$ of the basis vectors; and then $x_7 w_0 = -x_7$. Evidently w_0 commutes with ζ, so $w_0 \in H$. By 4.1, $w_0 \in N_H(T)$, and one can check that w_0 acts on $\phi(T)$ by inversion. Since every element of $\mathbf{F}$ is a square, it follows that w_0 acts on T by inversion.

Similarly we choose elements w_1, w_2, w_3, w, and ρ of $N_H(T)$ so that:

$$\begin{aligned} \phi(w) &= (x_1\ x_3\ x_5)(x_2\ x_4\ x_6), \\ \phi(\rho) &= (x_1\ x_3)(x_2\ x_4), \quad \text{and} \\ \phi(w_i) &= (x_{2i-1}\ x_{2i}), \quad 1 \leq i \leq 3. \end{aligned}$$

Thus, we may take $w_0 = w_1 w_2 w_3$.

(4.2.2) Fix $n \geq 0$ and set $q = p^{2^n}$. Denote by ω both the inner automorphism of $\widetilde{H}$ induced by w_0, and the identity map on $\overline{\mathbf{F}}$. For any $k \geq 0$, let ψ_k be the automorphism of $\widetilde{H}$ defined by

$$\psi_k = \begin{cases} (\zeta\omega)^{2^k} & \text{if } p \equiv 3 \mod 4, \quad \text{and} \\ \zeta^{2^k} & \text{if } p \equiv 1 \mod 4. \end{cases}$$

We now fix n, and set

$$\sigma = \psi_n.$$

Notice that since $\omega^2 = 1$, we in fact have $\sigma = \zeta^{2^n}$ unless $n = 0$ and $p \equiv 3 \mod 4$.

The restriction of σ to H will again be denoted σ. For any subgroup D of H, write D_σ for $C_D(\sigma)$, and write $\mathbf{F}_\sigma$ for $C_{\mathbf{F}}(\sigma)$.

LEMMA 4.3. *Set $W = \langle z, w_1, w_2, w_3, w, \rho \rangle$. Then $W \leq H_\sigma$, and the following hold.*

(a) *$T = C_H(T)$, and w_0 acts on T as inversion.*

(b) $N_H(T) = WT$.

(c) $\mathrm{Aut}_H(T) \cong \mathrm{Sym}(4) \times C_2$.

(d) *Set $E = \{x \in T \mid x^2 = 1\}$. Then E is an elementary abelian subgroup of T_σ of rank 3, $C_H(E) = T\langle w_0\rangle$, $N_{\widetilde{H}}(E) = \widetilde{T}W$, and $N_H(E) = N_H(T)$.*

Proof. Set $\mathscr{L} = \{\ell_1, \ell_2, \ell_3\}$, and denote by $\widetilde{T}^*$ the pointwise stabilizer of $\mathscr{L}$ in $\widetilde{H}$. Thus $T \le \widetilde{T}^*$, and $\phi(\widetilde{T}^*)$ is contained in the direct product of the orthogonal groups $O(\ell_i)$, so that $\widetilde{T}^* = \widetilde{T}\langle w_1, w_2, w_3\rangle$.

By 4.1, $C_{\widetilde{H}}(\widetilde{T}) \le \widetilde{T}^*$ and $N_{\widetilde{H}}(\widetilde{T})$ permutes $\mathscr{L}$. Since $\mathrm{Aut}_{\widetilde{T}}(\ell_i)$ contains its centralizer in $\mathrm{GL}(\ell_i)$, we have $\widetilde{T} = C_{\widetilde{H}}(\widetilde{T})$. Similarly $C_H(T) = T$.

Evidently, each of the elements w_i, w, and ρ commutes with both σ and w_0, and so $W \le N_{H_\sigma}(T)$. From the definitions of these elements, we obtain $\phi(w_0) \in Z(\phi(W))$, $\phi(\langle w_1, w_2, w_3\rangle)$ is elementary abelian of order 8, $\phi(\langle w, \rho\rangle) \cong \mathrm{Sym}(3)$, and $\langle w, \rho\rangle$ acts naturally on $\phi(\{w_1, w_2, w_3\})$ and on $\mathscr{L}$. We conclude that WT contains $H \cap T^*$, $WT = N_H(T)$, that $WT/T \cong \mathrm{Sym}(4) \times C_2$, and that $\langle w_0\rangle T/T = Z(WT/T)$. As w_0 inverts $\widetilde{T}$, it inverts T. Thus, parts (a) through (c) hold.

Notice that $C_H([V, T]) = Z$ since $\phi(H)$ contains no reflections. As ω inverts T and ζ induces a power map on $\widetilde{T}$, T_σ contains the group $E = \{t \in T \mid t^2 = 1\}$. From 4.2, $\phi(E)$ is a fours group and E is elementary abelian of order 8. The lines ℓ_i are the fixed point spaces for the three involutions in $\phi(E)$ on $[\widetilde{V}, E]$, so that $N_{\widetilde{H}}(E) = N_{\widetilde{H}}(\widetilde{T})$, and hence $N_H(E) = N_H(T)$. Since w_0 inverts T, $w_0 \in C_H(E)$.

Since $C_H(E) \le \widetilde{T}^*$, we have $C_H(E) = T\langle w_1, w_2, w_3\rangle$. By 4.2, for $\{i, j, k\} = \{1, 2, 3\}$ and $x \in E$ such that $[\widetilde{V}, x] = \ell_j + \ell_k$, each of w_i, $w_i x$, $w_j w_k$, and $w_j w_k x$ is of order 4. Then x does not centralize w_i or $w_j w_k$, and $C_H(E) = T\langle w_0\rangle$, completing the proof of (d). □

We may choose $z_1 \in \widetilde{T}$ so that z_1 acts as the scalar -1 on $\ell_1 + \ell_2$, and as 1 on ℓ_3. Then z_1 centralizes x_7. Set $U = \langle z, z_1\rangle$, $\widetilde{B} = N_{\widetilde{H}}(U)$, $B = \widetilde{B} \cap H$, and denote the identity component of $\widetilde{B}$ by $\widetilde{B}^0$. Set $B^0 = \widetilde{B}^0 \cap H$.

LEMMA 4.4. *The following hold.*

(a) *$\widetilde{V}$ is the orthogonal direct sum of $[\widetilde{V}, U]$ and $C_{\widetilde{V}}(U)$, of dimensions 4 and 3, respectively, over $\bar{\bar{F}}$.*

(b) *$\widetilde{B}$ is the stabilizer in $\widetilde{H}$ of $[\widetilde{V}, U]$ and of $C_{\widetilde{V}}(U)$.*

(c) *$\widetilde{B}^0 = C_{\widetilde{H}}(U) = \widetilde{L}_1\widetilde{L}_2\widetilde{L}_3$, where $\widetilde{L}_i \cong \mathrm{SL}_2(\overline{\mathbf{F}})$ and where $\widetilde{L}_i\widetilde{L}_j \cong \mathrm{SL}_2(\overline{\mathbf{F}}) \times \mathrm{SL}_2(\overline{\mathbf{F}})$ for all distinct i and j. Moreover, the indexing may be chosen so that*

  (i) *$\widetilde{L}_3$ centralizes $[\widetilde{V}, U]$ and $\widetilde{L}_3\phi = \Omega(C_{\widetilde{V}}(U))$.*

  (ii) *$\widetilde{L}_1\widetilde{L}_2$ centralizes $C_{\widetilde{V}}(U)$ and $\widetilde{L}_1\widetilde{L}_2\phi = \Omega([\widetilde{V}, U])$.*

(iii) *The maximal singular subspaces of* $[\widetilde{V}, U]$ *spanned by* $\{x_1, x_4\}$ *and* $\{x_2, x_3\}$ *over* $\overline{\mathbf{F}}$ *are natural* $\mathrm{SL}_2(\overline{\mathbf{F}})$*-modules for* $\widetilde{L}_1$*, and the maximal singular subspaces spanned by* $\{x_1, x_3\}$ *and* $\{x_2, x_4\}$ *are natural* $\mathrm{SL}_2(\overline{\mathbf{F}})$*-modules for* $\widetilde{L}_2$.

(d) $U = Z(\widetilde{B}^0)$, *and notation may be chosen so that* $z_1 \in \widetilde{L}_1$. *When* z_i *is the involution in* $\widetilde{L}_i$, *then*

$$z = z_1 z_2 = z_3.$$

(e) $\widetilde{B} = \widetilde{B}^0\langle w_1\rangle = \widetilde{B}^0\langle w_2\rangle$, *where both* w_1 *and* w_2 *interchange* $\widetilde{L}_1$ *and* $\widetilde{L}_2$ *by conjugation.*

Proof. As $[\widetilde{V}, U] = [\widetilde{V}, z_1]$ and $C_{\widetilde{V}}(U) = C_{\widetilde{V}}(z_1)$, part (a) is immediate from our choice of z_1. The stabilizer in $\widetilde{H}$ of $[\widetilde{V}, U]$ normalizes the unique subgroup U of $\widetilde{H}$ containing Z which acts as $-I$ on $[\widetilde{V}, U]$ and as I on $[\widetilde{V}, U]^{\perp}$. Similarly, the stabilizer in $\widetilde{H}$ of $C_{\widetilde{V}}(U)$ normalizes U, establishing (b).

Set $\widetilde{K} = C_{\widetilde{H}}(C_{\widetilde{V}}(U))^0$ and $\widetilde{L}_3 = C_{\widetilde{H}}([\widetilde{V}, U])^0$. From the Steinberg relations, $\widetilde{K} = \widetilde{L}_1 \times \widetilde{L}_2$, and $\widetilde{L}_i \cong \mathrm{SL}_2(\mathbf{F})$ for $i = 1, 2, 3$. Thus $\widetilde{B}^0$ is a commuting product of these three copies of $\mathrm{SL}_2(\overline{\mathbf{F}})$, and $U \leq Z(\widetilde{B}^0)$. Here $[\widetilde{V}, U]$ is a natural $\Omega_4(\overline{\mathbf{F}})$-module for $\widetilde{L}_1\widetilde{L}_2$, and is therefore a direct sum of two natural $\mathrm{SL}_2(\overline{\mathbf{F}})$-modules for each of $\widetilde{L}_1$ and $\widetilde{L}_2$. Observe that $\widetilde{T} \leq \widetilde{B}^0$ since $\widetilde{T}$ is connected. Then $\widetilde{T}$ is a maximal torus of $\widetilde{B}^0$, and hence $\widetilde{T} = \widetilde{T}_1\widetilde{T}_2\widetilde{T}_3$, where $\widetilde{T}_i := \widetilde{T} \cap \widetilde{L}_i$. Since $[\widetilde{L}_1, \widetilde{L}_2\widetilde{L}_3] = 1$, the irreducible $\widetilde{L}_1\widetilde{T}$-submodules of $[\widetilde{V}, U]$ are weight spaces for $\widetilde{T}_2\widetilde{T}_3$. Since these irreducible $\widetilde{L}_1\widetilde{T}$-submodules are also maximal singular subspaces of $[\widetilde{V}, U]$, the only possibilities are that the two irreducible $\widetilde{L}_1\widetilde{T}$-submodules of $[\widetilde{V}, U]$ are

$$\{\langle x_1, x_3\rangle, \langle x_2, x_4\rangle\} \quad \text{or} \quad \{\langle x_1, x_4\rangle, \langle x_2, x_3\rangle\}.$$

We may therefore choose the indexing so that (c)(iii) holds.

To complete the proof of (c), it remains to show that $\widetilde{B}^0 = C_{\widetilde{H}}(U)$. This will follow from (e), and since (d) is immediate from the parts of (c) which have already been established, we now need only prove (e).

The group $O([\widetilde{V}, U])$ is generated by $\Omega([\widetilde{V}, U])$ together with a reflection interchanging $\phi(\widetilde{L}_1)$ and $\phi(\widetilde{L}_2)$. Similarly, $O(C_{\widetilde{V}}(U))$ is generated by $\Omega(C_{\widetilde{V}}(U))$ together with a reflection on $C_{\widetilde{V}}(U)$. Since $\phi(\widetilde{H})$ contains no reflections on $\widetilde{V}$, we have $|\widetilde{B}\colon \widetilde{B}^0| = 2$, and then (e) follows from the definitions of w_1 and w_2. □

Notice that each of the groups $\widetilde{L}_i$ is ζ-invariant. Set $L_i = \widetilde{L}_i \cap H$, $1 \leq i \leq 3$. The following result should then be evident:

LEMMA 4.5. *All parts of* 4.4 *hold, with* B, B^0, $\mathbf{F}$, *and* L_i *in place of* $\widetilde{B}$, $\widetilde{B}^0$, $\overline{\mathbf{F}}$, *and* $\widetilde{L}_i$. □

Given a basis $\mathbf{v}_1, \dots, \mathbf{v}_m$ of an $\mathbf{F}$-space $\mathbf{V}$, and elements d_i of $\mathbf{F}$, $1 \le i \le m$, we write $\partial(d_1, d_2, \dots, d_m)$ for the diagonal map $\mathbf{v}_i \mapsto d_i \mathbf{v}_i$ for each i.

LEMMA 4.6. *Let* $\mathbf{V}$ *be a* 2*-dimensional* $\mathbf{F}$*-space with basis* $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$, *set* $\mathbf{L} = \mathrm{SL}(\mathbf{V})$, *and let* $\mathbf{T}$ *be the maximal torus* $\{\partial(a, a^{-1}) \mid a \in \mathbf{F}\}$ *of* $\mathbf{L}$ *determined by* $\mathcal{B}$. *Set* $X = \mathbf{L} \times \mathbf{L} \times \mathbf{L}$, *set* $[[X]] = X/\langle(-I, -I, -I)\rangle$, *and write* $[[a, b, c]]$ *for the image of* $(a, b, c) \in X$ *under the canonical surjection* $X \to [[X]]$. *Finally, let* $\gamma_1, \delta_1, \gamma_2$, *and* δ_2 *be maps from* $\mathcal{B}$ *into* V *which send the ordered basis* $(\mathbf{v}_1, \mathbf{v}_2)$ *to the pairs* (x_1, x_4), (x_3, x_2), (x_1, x_3), *and* (x_4, x_2), *respectively.*

(a) *There are isomorphisms* $\alpha_i : \mathbf{L} \to L_i$, $i = 1, 2, 3$, *such that*

  (i) α_1, γ_1 *and* α_1, δ_1 *are quasi-equivalences of the representation of* $\mathbf{L}$ *on* $\mathbf{V}$ *with the representations of* L_1 *on* $\langle x_1, x_4\rangle$ *and* $\langle x_3, x_2\rangle$, *respectively.*

  (ii) α_2, γ_2 *and* α_2, δ_2 *are quasi-equivalences of the representation of* $\mathbf{L}$ *on* $\mathbf{V}$ *with the representations of* L_2 *on* $\langle x_1, x_3\rangle$ *and* $\langle x_4, x_2\rangle$, *respectively.*

  (iii) *The map* α_3 *is the* 3*-dimensional orthogonal representation of* $\mathbf{L}$ *in which* $\partial(c, c^{-1})$ *acts as* $\partial(c^2, 1, c^{-2})$ *with respect to the ordered basis* (x_5, x_7, x_6) *of* $C_V(U)$.

(b) *The map* $\alpha_1 \times \alpha_2 \times \alpha_3 : X \to B^0$ *given by*

$$(a, b, c) \mapsto (a\alpha_1)(b\alpha_2)(c\alpha_3)$$

induces an isomorphism of $[[X]]$ *with* B^0.

(c) $(T \cap L_i)\alpha_i^{-1}$ *is the set of diagonal maps in* $\mathbf{L}$. *For each i, let* $\beta_i : F \to T \cap L_i$ *be the composition of* ∂ *with* α_i. *Set* $Y = \mathbf{F} \times \mathbf{F} \times \mathbf{F}$, *and* $[Y] = Y/\langle(-1, -1, -1)\rangle$, *with* $[a, b, c]$ *the image of* $(a, b, c) \in Y$ *in* $[Y]$. *Then the map* $\beta_1 \times \beta_2 \times \beta_2$: $Y \to T$ *induces an isomorphism* $[a, b, c] \mapsto (a\beta_1)(b\beta_2)(c\beta_3)$ *of* $[Y]$ *with* T.

Proof. This is straightforward, given 4.4. □

From now on we use 4.6 to identify B^0 with the set of equivalence classes

$$[[a, b, c]] = [[-a, -b, -c]], \quad a, b, c \in \mathrm{SL}_2(\mathbf{F}),$$

and identify T with the set of equivalence classes

$$[a, b, c] = [-a, -b, -c] \ \ a, b, c \in \mathbf{F}^\times.$$

LEMMA 4.7. (a) *The element* $[a, b, c]$ *of* T *acts diagonally as*

$$\partial(ab, a^{-1}b^{-1}, ab^{-1}, a^{-1}b, c^2, c^{-2}, 1)$$

with respect to the ordered basis $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ *of* V.

(b) *The action of W on T is given as follows.*

$$\begin{aligned} w_1&:[a,b,c] \mapsto [b^{-1},a^{-1},c],\\ w_2&:[a,b,c] \mapsto [b,a,c],\\ w_3&:[a,b,c] \mapsto [a,b,c^{-1}],\\ w&:[a^2,b^2,c^2] \mapsto [abc^2,a^{-1}b^{-1}c^2,ab^{-1}],\\ \rho&:[a,b,c] \mapsto [a,b^{-1},c]. \end{aligned}$$

Proof. Again, this is straightforward, given 4.6. □

Denote by S_∞ the set of elements of T whose order is a power of 2. Set $W_S = \langle w_1, w_2, w_3, \rho\rangle$, and set $S = S_\infty W_S$. Also, for any $k \geq 1$ set

$$T_k = \{t \in T \mid t^{2^k} = 1\}.$$

Thus T_k is a subgroup of S_∞, and T_1 is the group E appearing in 4.2.

We shall henceforth take p to be congruent to 3 or 5 mod 8. One reason for this choice is that it allows us to keep track of the structure of Sylow 2-subgroups of the groups H and H_σ, as in the following two lemmas.

LEMMA 4.8. *Let k be a nonnegative integer, and set $\psi = \psi_k$. Then $C_{S_\infty}(\psi) = T_{k+2}$ is homocyclic abelian of rank* 3 *and exponent* 2^{k+2}.

Proof. For any integer m, and any $k \geq 1$,

$$m^{2^k} - 1 = (m^{2^{k-1}} + 1)(m^{2^{k-1}} - 1).$$

A straightforward induction argument then yields the following fact.

(∗) For any integer m with $m \equiv 5 \mod 8$, and for any nonnegative integer k, we have

$$m^{2^k} \equiv 1 + 2^{k+2} \mod 2^{k+3}.$$

Set $D = \{d \in \mathbf{F} \mid d^{2^{k+2}} = 1\}$, set $D' = \{f \in \mathbf{F} \mid f^{2^{k+3}} = 1\}$, and fix $f \in D' - D$. Then $D' = D \cup Df$. Set $Q_k = \{[a,b,c] \in T \mid a,b,c \in D\}$ and $R_k = T_{k+2}$. That is $R_k = \{x \in T \mid x^{2^{k+2}} = 1\}$. As $[a,b,c]^{2^k} = 1$ in T if and only if $a^{2^k} = b^{2^k} = c^{2^k} = \pm 1$, it follows that $R_k = Q_k \bigcup Q_k[f,f,f]$. Thus Q_k has index 2 in R_k. Let A be a homocyclic abelian group of rank 3 and exponent 2^{k+2}. Since $[a,b,c] = [-a,-b,-c]$, there is an exact sequence

$$1 \longrightarrow C_2 \longrightarrow A \longrightarrow Q_k \longrightarrow 1,$$

and thus $|A| = 2|Q_k| = |R_k|$. Since R_k is abelian of rank 3, exponent 2^{k+2}, and order $|A|$, it follows from the fundamental theorem of finite abelian groups that $R_k \cong A$.

Suppose that $p \equiv 3 \mod 8$ and that $k = 0$. For $[a, b, c] \in T$,

$$[a, b, c]\psi = [a^{-p}, b^{-p}, c^{-p}],$$

so that $[a, b, c] \in T_\psi$ if and only if $a^{p+1} = b^{p+1} = c^{p+1} = \pm 1$. As $p \equiv 3 \mod 8$, 4 is the largest power of 2 dividing $p + 1$, and it follows from the preceding paragraph that $C_{S_\infty}(\psi) = R_k$ in this case. On the other hand, suppose that $p \equiv 5 \mod 8$ or that $k > 0$. Then

$$[a, b, c]\psi = [a^{p^{2^k}}, b^{p^{2^k}}, c^{p^{2^k}}].$$

Notice that if $p \equiv 3 \mod 8$ then $-p \equiv 5 \mod 8$, while for $k > 0$ we have $p^{2^k} = (-p)^{2^k}$. Now $(*)$ shows that, in any case, we have $C_{S_\infty}(\psi) = R_k$. This yields the lemma. □

LEMMA 4.9. *The following hold.*

(a) $S_\sigma = C_{S_\infty}(\sigma)W_S$, *and* S_σ *is a Sylow* 2*-subgroup of* H_σ.

(b) S *is a Sylow* 2*-subgroup of every subgroup* X *of* H *which contains* S.

(c) T_2 *is the unique homocyclic abelian subgroup of* S *of rank* 3 *and exponent* 4. *Moreover, we have* $T = C_H(T_2)$, *and* $T_2 \le T_\sigma = C_{H_\sigma}(T_2)$.

(d) S^B *is the set of maximal* 2*-subgroups of* B *containing a subgroup isomorphic to* T_2.

Proof. For any subgroup P of S, denote by $\mathcal{A}(P)$ the set of homocyclic abelian subgroups of P of rank 3 and exponent 4. Let ψ and R_k be defined as in the preceding lemma, set $Q = W_S R_k$, and set $Q_0 = \langle w_0 \rangle R_k$.

Suppose first that that there exists $A \in \mathcal{A}(Q)$ with $A \neq T_2$. Then $A \not\le T$. Suppose that $A \cap Q_0$ has rank 3 and exponent 4. Since w_0 inverts R_k, we then have $A \cap Q_0 \le R_k$, and A contains the unique elementary abelian subgroup E of R_k of order 8. By 4.3, Q/Q_0 acts faithfully on E, so that $A \le Q_0$, and then $A \le T$. This is a contradiction, and so we conclude that $A \cap Q_0$ has exponent less than 4 or has rank less than 3. Since Q/Q_0 is dihedral of order 8, it follows that AQ_0/Q_0 is cyclic of order 4. Then $A \cap Q_0$ is homocyclic of rank 2 and exponent 4. Again, $A \cap Q_0 \le R_k$, and now $|A \cap E| = 4$. The faithful action of Q/Q_0 on E implies that $C_{Q/Q_0}(A \cap E)$ has exponent 2. Since AQ_0/Q_0 centralizes $A \cap E$, we again have a contradiction, and thus $A \le R_k$. That is, $\mathcal{A}(Q) = \{T_2\}$.

Let P be a Sylow 2-subgroup of (the finite group) H_ψ containing Q. By the preceding paragraph, $T_2 \trianglelefteq N_P(Q)$. It follows from 4.1 that $N_P(Q)$ preserves the set $\{\ell_1, \ell_2, \ell_2\}$ of hyperbolic lines, and hence $N_P(Q)$ normalizes T. Then $N_P(Q) \le TW$ by 4.3. Since Q is a Sylow 2-subgroup of $(WT)_\psi$, we conclude that $N_P(Q) = Q$. Then $P = Q$, and so $Q \in \mathrm{Syl}_2(H_\psi)$. We recall that $\psi = \psi_k = \zeta^{2^k}$

or $(\zeta w_0)^{2^k}$ for some k, and that

$$H = \bigcup_{k \geq 0} C_H((\psi_k).$$

Then S is the union of its subgroups $S \cap H_{\psi_k}$, and so $\mathscr{A}(S) = \{T_2\}$.

By Zorn's Lemma, there is a maximal 2-subgroup S^* of H containing S. We have

$$X = \bigcup_{k \geq 0} X_{\psi_k}$$

for any subgroup X of H. Taking $X = S^*$, we conclude that $S = S^*$. Taking X to be an arbitrary subgroup of H containing S, we note that every finite subgroup of X is contained in X_{ψ_k} for some k, so that every finite 2-subgroup of X is X-conjugate into S. Thus, S is a Sylow 2-subgroup of X, and we have (a) and (b).

Notice that $\sigma = \psi_k$ for some k. Then (c) follows as $T_2 \leq C_H(\psi_0)$, $\mathscr{A}(S) = \{T_2\}$, and $N_H(T_2) = N_H(T)$.

By (c) and 4.3.b, $N_B(T_2) = TW_S = TS$. Let S_2 be a subgroup of B isomorphic to T_2, and X a maximal 2-subgroup of B containing S_2. By (b), S is Sylow in B, and so $S_2^b \leq S$ for some $b \in B$, and by (c), $S_2^b = T_2$. Thus we may take $T_2 = S_2$. Similarly for each k, X_{ψ_k} is contained in a conjugate of S, and so by (c), $X_{\psi_k} \leq N_B(T_2) = TS$. Hence X is a maximal 2-subgroup of TS, $X \in S^T$, establishing (d). □

5. The amalgam $\mathscr{A}_\lambda$, and an amalgam for Co_3

We now undertake the construction of the amalgam which provides the focus for this work. (See the beginning of Section 3 for a discussion of amalgams.)

We continue the setup and notation of the preceding section. In particular, we have $p \equiv 3$ or 5 (mod 8). Let i be a square root of -1 in $\mathbf{F}$, and let τ be the element $w_2[1, 1, i]$ of B. Then $B = B^0\langle\tau\rangle$, by 4.4(e). By definition, w_2 interchanges the two singular points of ℓ_2, centralizes ℓ_1 and ℓ_3, and acts as -1 on $C_V(T)$. Then τ acts as $-I$ on $C_V(U) = \ell_3 + C_V(T)$, and acts as a transvection on $[V, U]$. In particular, τ commutes with $\phi(L_3)$, hence also with L_3 (since L_3 is perfect), and τ is an involution by 4.2. Further, τ acts as w_2 on $[V, T]$, and then 4.4(c)(iii) yields

$$\tau\colon [[\alpha, \beta, \gamma]] \mapsto [[\beta, \alpha, \gamma]],$$

for all $[[\alpha, \beta, \gamma]] \in B^0$.

Define y_0 to be the automorphism of B^0 given by

$$y_0\colon [[\alpha, \beta, \gamma]] \mapsto [[\gamma, \alpha, \beta]]. \tag{5.0}$$

Then $|y_0| = 3$, and $\langle y_0, \tau\rangle$ acts faithfully as the symmetric group Sym(3) on the set $\mathscr{L} = \{L_1, L_2, L_3\}$. In the semidirect product

$$K = B^0\langle y_0, \tau\rangle,$$

we may then identify B with the subgroup $B^0\langle\tau\rangle$ of K, and form the amalgam

$$\mathscr{A}_1 = (H \geq B \leq K).$$

For any $\lambda \in \mathrm{Aut}(B)$, denote by λ^* the composition of λ with the inclusion map of B into K, and form the amalgam

$$\mathscr{A}_\lambda = (H \geq B \xrightarrow{\lambda^*} K).$$

The corresponding free amalgamated product will be denoted G_λ. Subject to the usual identifications, H and K are subgroups of G_λ, with $H \cap K = B$. Here it is important to note that the inclusion map of B into K, within G_λ, is obtained by "twisting" via λ the "ordinary" inclusion map occurring in $\mathscr{A}_1$.

LEMMA 5.1. *For $X \in \{H, K\}$, write A_X for* $\mathrm{Aut}_{\mathrm{Aut}(X)}(B)$. *Set* $\Phi = \mathrm{Aut}(\mathbf{F})$, *and regard Φ as the group of field automorphisms of* $\mathrm{SL}_2(\mathbf{F})$. *Define a representation of Φ on B^0 by*

$$\lambda\colon [[\alpha, \beta, \gamma]] \mapsto [[\alpha, \beta, \gamma^\lambda]] \text{ for } \lambda \in \Phi.$$

Then Φ commutes with τ on B^0, and the representation of Φ on B^0 extends thereby to a representation on B. Moreover:

(a) $\mathrm{Inn}(B) \leq A_H \cap A_K$.

(b) *For each $X \in \{H, K\}$ we have* $\mathrm{Aut}(B) = A_X\Phi = A_H A_K \Phi$, *and* $A_X \cap \Phi = 1$.

(c) *For $\mu, \lambda \in \mathrm{Aut}(B)$, we have $\mathscr{A}_\lambda \cong \mathscr{A}_\mu$ if and only if $A_H A_K \lambda = A_H A_K \mu$.*

Proof. Identify Φ with a subgroup of $\mathrm{Aut}(B^0)$ via the prescribed representation. Evidently $[\Phi, \tau] = 1$, so we may even regard Φ as a subgroup of $\mathrm{Aut}(B)$. As $B = H \cap K$, (a) holds. As is well known (cf. [Ste]), $\mathrm{Aut}(\mathrm{SL}_2(\mathbf{F})) = \mathrm{Inn}(\mathrm{SL}_2(\mathbf{F}))\Phi$. Then, since B^0 is the central product of three copies of $\mathrm{SL}_2(\mathbf{F})$, $\mathrm{Aut}(B^0)$ is the split extension of $\mathrm{Aut}(\mathrm{SL}_2(\mathbf{F}))^3$ by Sym(3), where Sym(3) permutes the three components of B^0 faithfully.

Recall that $B = B^0\langle\tau\rangle$, where τ centralizes L_3 and interchanges L_1 and L_2. For $X \in \{H, K\}$ we have $A_X = \mathrm{Inn}(B)\Phi_X$, where $\Phi_X \cong \Phi$ and Φ_X is diagonally embedded in the subgroup Φ^3 of $\mathrm{Aut}(\mathrm{SL}_2(\mathbf{F}))^3$, and centralizes τ. Similarly, $\mathrm{Aut}(B) = \mathrm{Inn}(B)(\Phi_X \times \Phi)$, where Φ centralizes $L_1 L_2$ and acts faithfully as Φ on L_3. Now (b) follows, and $\mathrm{Aut}(B) = A_H A_K \Phi$. By [Gol80, Lemma 2.7], $\mathscr{A}_\lambda \cong \mathscr{A}_\mu$ if and only if $A_H \lambda A_K = A_H \mu A_K$, and now (c) follows from (b). □

We may now state the first main result of this section.

THEOREM 5.2. *Let S_∞ be the Sylow* 2*-subgroup of T, and for any $\lambda \in \mathrm{Aut}(B)$ set $N_\lambda = \langle N_H(S_\infty), N_K(S_\infty)\rangle$, where H and K are regarded as subgroups of G_λ in the canonical way. Set $\mathcal{N}_\lambda = \mathrm{Aut}_{N_\lambda}(S_\infty)$, define Φ as in* 5.1, *and set*

$$\Lambda = \{\lambda \in \Phi \mid \mathcal{N}_\lambda \cong \mathrm{GL}(3,2) \times C_2\}.$$

Then the following hold.

(a) $|\Lambda| = 1$.

(b) *For $\lambda \in \Lambda$,*

$$C_{N_\lambda}(T_e) = \begin{cases} C_{N_\lambda}(S_\infty) & \text{if } e \geq 2, \text{ and} \\ C_{N_\lambda}(S_\infty)\langle w_0\rangle & \text{if } e = 1. \end{cases}$$

(c) $A_H = A_K$.

(d) *The map $\phi \mapsto \mathcal{A}_\phi$ is a bijection of Φ with the set of isomorphism classes of amalgams $\mathcal{A}_\mu$ with $\mu \in \mathrm{Aut}(B)$.*

Remark. It can be shown, by means of a lengthy computation based on 4.7(b), that the unique λ in the set Λ of Theorem 5.2 is not an algebraic endomorphism of B.

Let ω_3 be the automorphism of B^0 induced by conjugation by w_3. By 4.7(b), $w_3 \in L_3$ and w_3 inverts $T \cap L_3$. Denote by ζ_3 the automorphism of B^0 which induces the p^{th} power Frobenius map on L_3 and which centralizes L_1 and L_2. Then let ξ_0 be the automorphism of B^0 given by

$$\xi_0 = \begin{cases} \zeta_3\omega_3 & \text{if } p \equiv 3 \mod 8, \\ \zeta_3 & \text{if } p \equiv 5 \mod 8. \end{cases}$$

Thus,

$$\xi_0\colon [[\alpha,\beta,\gamma]] \mapsto \begin{cases} [[\alpha,\beta,\bar{\gamma}^{w_3}]] & \text{if } p \equiv 3 \mod 8, \\ [[\alpha,\beta,\bar{\gamma}]] & \text{if } p \equiv 5 \mod 8, \end{cases}$$

where $\bar{\gamma}$ is the element of $\mathrm{SL}_2(\mathbb{F})$ whose entries are the p^{th} powers of the corresponding entries of γ. For any $e \in \mathbb{N}$, set

$$\xi_e = \xi_0^{2^e}.$$

Notice that T is invariant under ξ_e, that $[L_1L_2, \xi_e] = 1$, and that $\xi_e = \zeta_3^{2^e}$ for $e \geq 1$. Recall from 4.8 that we have defined subgroups T_e of T_0 by

$$T_e = \{t \in T \mid t^{2^e} = 1\},\ e \geq 1,$$

and that T_e is homocyclic abelian of rank 3.

LEMMA 5.3. *Let e be a nonnegative integer, and let $c \in \mathbf{F}$ with $c^{2^{e+3}} = 1$. Then*

(a) $[a,b,c]\xi_e = [a,b,c]$ *if* $c^{2^{e+2}} = 1$, *and otherwise* $[a,b,c]\xi_e = [a,b,-c]$.

(b) ξ_e *centralizes a subgroup of index* 2 *in* T_{e+2} *containing* T_{e+1}, *and* $[T_{e+2},\xi_e] = Z$.

Proof. Suppose first that $e = 0$ and that $p \equiv 3 \mod 8$. Then $c^8 = 1$, and

$$[a,b,c]\xi_e = [a,b,c^{-p}] = [a,b,c^5].$$

Since $c^5 = -c$ if $|c| = 8$, and $c^5 = c$ if $c^4 = 1$, (a) holds in this case. On the other hand, suppose that either $e > 0$ or $p \equiv 5 \mod 8$. We saw in the proof of 4.8 that for any integer m with $m \equiv 5 \mod 8$,

$$m^{2^e} \equiv 1 + 2^{e+2} \mod 2^{e+3}.$$

Taking $m = p$ if $p \equiv 5 \mod 8$, and taking $m = -p$ if $p \equiv 3 \mod 8$, we then have

$$[a,b,c]\xi_e = [a,b,c^{m^{2^e}}] = [a,b,c^{1+2^{e+2}}].$$

Thus (a) holds in every case. Part (b) follows from (a) and the fact that $Z = \langle [1,1,-1] \rangle$ and $T_{e+2} = \langle [c^2,1,1],[1,c^2,1],[c,c,c] \rangle$, where $|c| = 2^{e+3}$. □

PROPOSITION 5.4. *There is a uniquely determined sequence* $(y_e \mid e \geq 0)$ *of automorphisms of* B^0, *with* y_0 *as in* 5.0, *and having the following two properties.*

(a) $y_e \in \{y_{e-1}, y_{e-1}^{\xi_{e-1}}\}$ *for* $e > 0$.

(b) *The group* $\mathcal{N}_e$ *of automorphisms of* T_{e+1} *induced by the action of* $\langle W, y_e \rangle$ *is isomorphic to* GL(3, 2) $\times C_2$ *for* $e > 0$, *and is isomorphic to* GL(3, 2) *for* $e = 0$.

The proof of 5.4 will be based on the following result.

LEMMA 5.5. *Let* N *be the Steinberg module for* GL(3, 2) *over the field* $\mathbb{F}_2$ *of two elements*, *and let* X *be an extension of* N *by* GL(3, 2). *Then the following hold*:

(a) X *splits over* N.

(b) *Let* D *be a complement to* N *in a Sylow* 2*-subgroup of* X. *Then* $C_N(D) = \langle g \rangle$ *is of order* 2.

(c) *Let* D *be as in* (b), *and denote by* $\mathcal{P}$ *the set of subgroups* P *of* X *such that* $D \leq P \cong$ Sym(4). *Then* $\mathcal{P} = \{P_1, P_2, Q_1, Q_2\}$, *where* $Q_i = P_i^g$ *for* $i = 1, 2$, $\langle P_i, Q_i \rangle$ *is a complement to* N *in* X, *and* $\langle P_1, Q_2 \rangle = \langle P_2, Q_1 \rangle = X$.

Proof. Let $R \in \mathrm{Syl}_2(X)$. Then N is a free $\mathbb{F}_2 R/N$-module, so that $C_N(R) = \langle g \rangle$ is of order 2, and for each overgroup Y of R in X we have $H^i(Y/N, N) = 0$ for $i = 1, 2$. In particular, (a) and (b) hold.

Choose R so that $R = DN$ and let P and Q be maximal subgroups of X such that P/N and Q/N are the maximal parabolic subgroups of X/N over R/N. It follows from the preceding paragraph that for each $Y \in \{X, P, Q, R\}$, Y splits over N and is transitive on its complements to N. Thus the set $\mathcal{P}_Y$ of complements to N in Y containing D is nonempty for $Y \in \{X, P, Q\}$, and by a standard argument (cf. 5.2.1 in [Asc86]), $N_Y(D)$ is transitive on $\mathcal{P}_Y$. Then as $N_X(D) = D \times \langle g \rangle$, $\mathcal{P}_Y = \{Y_1, Y_2\}$ with $Y_1^g = Y_2$. Since $C_N(Y) = 0$ for $Y \in \{P, Q\}$, each P_i and each Q_i is contained in a unique complement to N in X. In particular $X_i \cong \mathrm{GL}(3,2)$ and the indexing may be chosen so that X_i is generated by P_i and $Q_i \in P_Q$. Then $M_i = \langle P_i, Q_{3-i} \rangle$ is not a complement to N in X. Since X is irreducible on N, $M_i = N$. This completes the proof. □

We may now prove Proposition 5.4. Since T_1 is elementary abelian of order 8, we have $\mathrm{Aut}(T_1) \cong \mathrm{GL}(3,2)$. From 4.3(c), W induces the stabilizer in $\mathrm{GL}(T_1)$ of Z, and so the image of W in $\mathrm{GL}(T_1)$ is maximal. The closure of Z under the action of $\langle y_0 \rangle$ is the fours group U, and thus $\mathcal{N}_0 = \mathrm{GL}(T_1)$. We may therefore assume that $e \geq 1$, and that for all indices e' with $0 < e' < e$:

(∗) There exists a unique $y_{e'} \in \{y_{e'-1}, y_{e'-1}^{\xi_{e'-1}}\}$ such that the group $\mathcal{N}_{e'}$ of automorphisms of $T_{e'+1}$ induced by the action of $\langle W, y_{e'} \rangle$ is isomorphic to $\mathrm{GL}(3,2) \times \mathbf{C}_2$.

Set $R = T_{e+1}$, $x = y_{e-1}$, and denote by $\mathcal{N}$ the image of $\langle W, x \rangle$ in $\mathrm{Aut}(R)$. Applying (∗) to $e' = e-1$, we have $\mathcal{N}/C_{\mathcal{N}}(T_e) \cong \mathrm{GL}(3,2)$ if $e = 1$, and $\mathrm{GL}(3,2) \times C_2$ if $e > 1$. Let $d \in C_{\mathcal{N}}(T_e)$. Writing

$$d\colon [a, b, c] \longrightarrow [\widetilde{a}, \widetilde{b}, \widetilde{c}],$$

for $[a, b, c] \in R$, we obtain $a^2 = \varepsilon \widetilde{a}^2$, $b^2 = \varepsilon \widetilde{b}^2$, and $c^2 = \varepsilon \widetilde{c}^2$, for some $\varepsilon \in \{\pm 1\}$. Thus either $\widetilde{u} \in \{u, -u\}$ for all $u \in \{a, b, c\}$, or $\widetilde{u} \in \{iu, -iu\}$ for all $u \in \{a, b, c\}$, where i is a square root of -1. Therefore as

$$T_1 = \langle [-1, 1, 1], [1, -1, 1], [i, i, i] \rangle,$$

d acts trivially on R/T_1. Then $d = 1 + \lambda_d$ for some $\lambda_d \in \mathrm{Hom}_{\mathbb{Z}}(R/T_e, T_1)$, and the mapping

$$C_{\mathcal{N}}(T_e) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R/T_e, T_1)$$

given by $d \mapsto \lambda_d$ is an $\mathcal{N}$-homomorphism. Observe that R/T_e and T_1 are isomorphic as modules for $\langle W, x \rangle$ via the map $\varphi\colon rT_e \mapsto r^{2^{e+1}}$. Thus, there is an $\mathcal{N}$-equivariant monomorphism $d \mapsto \varphi^{-1}\lambda_d$ from $C_{\mathcal{N}}(T_e)$ into $M := \mathrm{End}_{\mathbb{Z}}(T_1)$. Since $\mathcal{N}$ acts as $\mathrm{GL}(3,2)$ on T_1, M may be identified with the $\mathcal{N}$-module of 3×3 matrices over the field $\mathbf{F}_2$ of two elements, and we regard $C_{\mathcal{N}}(T_e)$ as an $\mathcal{N}$-submodule of M.

Now M is a vector space of dimension 9 over $\mathbf{F}_2$, and the subspace M_0 of trace-zero matrices is an 8-dimensional $\mathcal{N}$-submodule of M. Indeed $\mathcal{N}/C_{\mathcal{N}}(M) \cong \mathrm{GL}(3,2)$, and M_0 is the Steinberg module for $\mathcal{N}/C_{\mathcal{N}}(M)$. The element w_0 of W inverts R, and hence $\langle w_0 \rangle = C_M(\mathcal{N})$, and $M = M_0 \oplus C_M(\mathcal{N})$. Further, $\mathcal{N}$ is an extension of $\mathcal{N}_M = \mathcal{N} \cap M$ by $L_3(2)$, and as $C_M(\mathcal{N}) = \langle w_0 \rangle \leq \mathcal{N}_M$ and $\mathcal{N}$ is irreducible on M_0, $\mathcal{N}_M = M$ or $C_M(\mathcal{N})$. As $W_S/(W_S \cap T) \cong \mathbf{Z}_2 \times D_8$ (cf. 4.3), $\mathcal{N}/M_0$ or $\mathcal{N}$ is isomorphic to $\mathbf{Z}_2 \times L_3(2)$ in the respective case. Further, in the latter case, we obtain (a) and (b) of 5.4 by setting $y_{e+1} = y_e$. Thus we may assume that $\mathcal{N}_M = M$.

Set $X = [\mathcal{N}, \mathcal{N}]$. From the previous paragraph, $M_0 = X \cap M$ and $X/M_0 \cong \mathrm{GL}(3,2)$. Denote by D^* the image of W_S in $\mathcal{N}$, and set $D = D^* \cap X$. Then $D^* = D \times Z(\mathcal{N})$, and D is dihedral of order 8. Denote by P^* and Q^* the images in $\mathcal{N}$ of W and $\langle W_S, y \rangle$, respectively, and set $P = P^* \cap X$ and $Q = Q^* \cap X$. By 5.5(b), $C_{M_0}(D) = \langle g \rangle$ is of order 2. By definition $\mathcal{N}$ is generated by P^* and Q^*, and so X is generated by P and Q. By 5.5(c), $\langle P, Q^g \rangle$ is a complement to M_0 in X. By construction, ξ_{e-1} centralizes W_S, and by 5.3(b), ξ_{e-1} induces a nontrivial automorphism of R centralizing a subgroup of index 2 containing T_e. Then, since $C_{M_0}(W_S) = \langle g \rangle$, it follows that the action of g on R is the same as that of ξ_{e-1}. Setting $y_e = x^g$, we obtain (a) and (b) of Proposition 5.4. □

Let $\{\lambda_e \mid e \geq 0\}$ be the sequence of automorphisms of B^0 defined by $\lambda_0 = 1_{B^0}$, and for $e > 0$ by the recursive formula

$$\lambda_e = \begin{cases} \lambda_{e-1} & \text{if } y_e = y_{e-1} \text{ and} \\ \lambda_{e-1}\xi_{e-1} & \text{if } y_e = y_{e-1}^{\xi_{e-1}} \end{cases}$$

where y_e is as in Proposition 5.4. For $k \geq 0$, take ψ_k as defined just prior to 4.3.

LEMMA 5.6. *Each λ_e extends to an automorphism of B which commutes with the element τ of $B - B^0$, and with ψ_k for each k. Further, for each $e \geq 0$,*

$$\lambda_{e+1}|_{C_B(\psi_{e-1})} = \lambda_e|_{C_B(\psi_{e-1})}\,.$$

Proof. Recall that $\xi_e = \xi_0^{2^e}$, where ξ_0 is the automorphism of B^0 given by

$$\xi_0 \colon [[\alpha, \beta, \gamma]] \mapsto [[\alpha, \beta, \gamma']],$$

for the automorphism $\gamma \mapsto \gamma'$ of $\mathrm{SL}_2(\mathbf{F})$ such that

$$\psi_0 \colon [[\alpha, \beta, \gamma]] \mapsto [[\alpha', \beta', \gamma']].$$

It follows that ξ_e commutes with ψ_k for all e and k, and also with the automorphism $\tau\colon [[\alpha, \beta, \gamma]] \mapsto [[\beta, \alpha, \gamma]]$ of B^0. In constructing the amalgam $\mathcal{A}_1$, we identified B with the semidirect product $B^0\langle \tau \rangle$, so ξ_e may now be regarded as an automorphism of B. Since λ_e is the product of some elements of $\{1, \xi_0, \ldots, \xi_e\}$, we may also regard λ_e as an automorphism of B, commuting with ψ_k. The proof is then completed by the observation that ξ_e centralizes $C_B(\psi_{e-1})$. □

Notice that 4.8 yields

$$B = \bigcup_{e \geq 0} C_B(\psi_e).$$

By the preceding lemma, we may then define an automorphism λ of B by taking $\lambda\,|_{C_B(\psi_e)} = \lambda_e$ for each e. Set

$$\mathcal{A} = \mathcal{A}_\lambda = (H \xleftarrow{\ \iota\ } B \xrightarrow{\ \lambda^*\ } K)$$

(where ι denotes inclusion) and form the corresponding free amalgamated product $G = G_\lambda$.

We may now complete the proof of Theorem 5.2. We have

$$\Phi = \mathrm{Aut}(\mathbf{F}) = \varprojlim \mathrm{Aut}(\mathbf{F}_{p^{2^e}}),$$

and there is an isomorphism

$$\mathbb{Z}_{2^e} \longrightarrow \mathrm{Aut}(\mathbf{F}_{p^{2^e}})$$

given by sending a residue class $[k]$ to the p^k-th power map, $0 \leq k < 2^e$. The sequence of inverses of these isomorphisms then defines an isomorphism of Aut($\mathbf{F}$) with the ring $\mathbb{Z}_{(2)}$ of 2-adic integers.

Let $\mu \in \mathrm{Aut}(\mathbf{F})$, and denote by μ_e the restriction of μ to the subfield $\mathbf{F}_{p^e}$ of $\mathbf{F}$ of order p^{2^e}. For any $e \geq 0$, there is then a unique integer k_e, $0 \leq k_e < 2^e$, such that μ_e is given by the p^{k_e}-th power map on $\mathbf{F}_{2^e}$. Define a sequence $(\varepsilon_e \mid e \geq 0)$ of elements of $\{0, 1\}$ by taking $\varepsilon_0 = k_0$, and for $e > 0$ by

$$\varepsilon_e = \begin{Bmatrix} 0 & \text{if } k_e = k_{e-1} \\ 1 & \text{if } k_e = k_{e-1} + 2^{e-1} \end{Bmatrix}.$$

We may represent the action of μ on B^0 as in 5.1; namely μ acts on $\mathrm{SL}(2, \mathbf{F})$ in the natural way, and on B^0 by

$$\mu\colon [[a, b, c]] \longrightarrow [[a, b, c^\mu]].$$

Observe that the restriction μ_e of μ to $C_{B^0}(\psi_{e+1})$ is given by

$$\mu_e = \tilde{\xi}_0^{\varepsilon_0} \xi_1^{\varepsilon_1} \dots \xi_e^{\varepsilon_e},$$

where $\tilde{\xi}_0 = \xi_0 \omega_3$ if $p \equiv 3 \bmod 8$, and where $\tilde{\xi}_0 = \xi_0$ if $p \equiv 5 \bmod 8$.

Recall from 5.1 that we have identified Φ with a subgroup of Aut(B), and that parts (b) and (c) of 5.1 show that for any $\mu' \in \mathrm{Aut}(B)$ there exists $\mu \in \Phi$ such that $\mathcal{A}_\mu \cong \mathcal{A}_{\mu'}$. In particular if $\mu' = \alpha\mu$ with $\alpha \in \mathrm{Inn}(B)$ then $\mathcal{A}_\mu \cong \mathcal{A}_{\mu'}$, and hence there is an induced isomorphism $G_\mu \cong G_{\mu'}$ of universal completions.

Take $\mu = \omega_3\lambda$ if both $p \equiv 3 \bmod 8$ and $\lambda_1 = \xi_0$, and take $\mu = \lambda$ otherwise. By our construction of λ we have $\mu \in \Phi$, and then since $\omega_3 \in B$ we obtain $G_\mu \cong G_\lambda$. Adopt the notation of 5.2. In particular $N_\lambda = \langle N_H(S_\infty), N_K(S_\infty)\rangle$, $N_\lambda = \langle W, y\rangle T$, and $\mathcal{N}_\lambda = \mathrm{Aut}_{N_\lambda}(S_\infty)$. Since ξ_e centralizes T_{e+1} by 5.3(b), we have $y^\lambda_{0|T_{e+1}} = y_0^{\lambda_e}$, where λ_e is as defined prior to 5.6. By induction on e, we then get

$$y^\lambda_{0|T_{e+1}} = \begin{Bmatrix} y_{e-1} & \text{if } y_e = y_{e-1} \\ y_{e-1}^{\xi_{e-1}} & \text{if } y_e \neq y_{e-1} \end{Bmatrix},$$

and so $y^\lambda_{0|T_{e+1}} = y_e$. Then 5.4 shows that μ is in the set Λ defined in 5.2.

Now let μ be an arbitrary element of Λ. Set $\nu = \omega_3\mu$ if both $p \equiv 3 \bmod 8$ and $\varepsilon_0 = 1$, and otherwise set $\nu = \mu$. Then $G_\mu \cong G_\nu$ and $N_\mu \cong N_\nu$. Set $x = \nu^{-1} y_0 \nu$, regard x as an automorphism of S_∞, and denote by x_e the restriction of x to T_{e+1}, $e \geq 0$. Then $x_0 = y_0$ and, by induction on e, $x_{e+1} = x_e^{\xi_e^{\varepsilon_e}}$. As $\mu \in \Lambda$, the uniqueness of the sequence in 5.4 implies that $\varepsilon_e = 0$ if and only if $y_e = y_{e-1}$, and hence that $x = y$. Since $y\colon [a,b,c] \mapsto [c^{\lambda^{-1}}, b, a^\lambda]$, also $\nu = \lambda$, establishing 5.2(a). Now 5.2(b) follows from the action of $\mathcal{N}_\lambda$ on T_∞ in 5.4.

If $A_H \neq A_K$ then $A_H A_K \cap \Phi \neq 1$, by 5.1(b), and then 5.1(c) implies that there exists $\mu' \in \Phi - \{\mu\}$ with $\mathcal{A}_\mu \cong \mathcal{A}_{\mu'}$. This is contrary to 5.2(a), so that $A_H = A_K$ and 5.2(c) holds. Now 5.2(d) follows from (c) and from 5.1(c), and this completes the proof of 5.2. □

Regarding H and K as subgroups of $G = G_\lambda$ in the canonical way, we have $B = H \cap K$. From 5.6, $\sigma = \psi_n$ commutes with λ_e for each e, so σ commutes with λ. Since σ commutes with y_0 and with τ as automorphisms of B^0, and since y acts on B^0 as y_0^λ, σ commutes with y. Then since K is the semidirect product of B^0 with $\langle y, \tau\rangle$, it follows that σ induces an automorphism σ_K of K, commuting with $\langle y, \tau\rangle$. The universal property of the free amalgamated product now implies that σ induces an automorphism of G, whose restriction to K is σ_K. We record this result for future reference.

LEMMA 5.7. *For each positive integer n, $\psi_{n|H}$ extends uniquely to an automorphism σ of G such that $[y,\sigma] = 1$.* □

We next show that the third Conway simple group Co_3 is the completion of a subamalgam of $\mathcal{A}_\lambda$, and that this subamalgam generates a fusion system which is isomorphic to that of Co_3. These will be key ingredients in our proof of Theorem B.

THEOREM 5.8. *Let $\bar{G}_0$ be the simple group Co_3, let S_0 be a Sylow 2-subgroup of $\bar{G}_0$, set $Z_0 = Z(S_0)$, and let U_0 be the unique normal fours group in S_0. Set*

$$H_0 = C_{\bar{G}_0}(Z_0), \quad K_0 = N_{\bar{G}_0}(U_0), \quad B_0 = H_0 \cap K_0,$$

and let $\mathcal{A}_0 = (H_0 \longleftarrow B_0 \longrightarrow K_0)$ be the amalgam of inclusion maps among these groups, within $\bar{G}_0$. Set $\sigma = \psi$ and set $\lambda_0 = \lambda\,|_{B_\sigma}$, where λ is an automorphism of B which satisfies the conditions of Theorem 5.2. Let $\mathcal{A}_{\lambda_0} = (H_\sigma \overset{\iota}{\longleftarrow} B_\sigma \overset{\lambda_0^*}{\longrightarrow} K_\sigma)$. Then the following hold:

(a) *There is a morphism* $\varphi\colon \mathcal{A}_0 \to \mathcal{A}_{\lambda_0}$ *of amalgams, displaying* $\mathcal{A}_0$ *as a subamalgam of* $\mathcal{A}_{\lambda_0}$.

(b) *Let* G_0 *be the subgroup of* G *generated by the images of* H_0 *and* K_0 *under the morphism* φ *of part* (a), *and let* $\mathcal{F}_0$ *be the fusion system* $\langle \mathcal{F}_{S_0}(H_0), \mathcal{F}_{S_0}(K_0)\rangle$ *contained in* $\mathcal{F}_{S_\sigma}(G_0)$. *Then* $\mathcal{F}_0 = \mathcal{F}_{S_\sigma}(G_0) = \mathcal{F}_{S_0}(\overline{G}_0)$.

Proof. We refer to [Fin73] for the structure of the maximal subgroups of G_0. Thus, H_0 is isomorphic to the covering group of $\mathrm{Sp}_6(2)$, which is the perfect central extension of $\mathrm{Sp}_6(2)$ by a group of order 2. Since $\mathrm{Sp}_6(2) \times C_2$ is a reflection group (namely, the Weyl group of type E_7), we have $\mathrm{Sp}_6(2) \leq O_7(\mathbb{R})$, and then, by taking the standard $\mathbb{Z}$-form of $O_7(\mathbb{R})$ and reducing mod p, one obtains $\mathrm{Sp}_6(2)$ as a subgroup of $\Omega_7(p)$. Identifying $\Omega_7(p)$ with H_σ, we then have an inclusion of H_0 in H_σ. In particular, $Z_0 = Z$.

Let Λ be $\mathrm{GL}_2(3) \wr S_3$ and $\bar{\Lambda} = \Lambda/Z(\Lambda)$. Set $B_1 = O^2(B_0)$ and $D = O_2(B_0)$. Sylow 2-subgroups of H_0 and H_σ are of order 2^{10}, so conjugating in H_σ, we may take $S_\sigma = S_0 \in \mathrm{Syl}_2(H_0)$, and $U = U_0$. Next, B_0 is the preimage in H_0 of the solvable maximal parabolic subgroup of $\mathrm{Sp}_6(2)$, so that $B_0/O_2(B_0)$ is isomorphic to $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$, and B_1 is isomorphic to a subgroup of index 3 in $O_{2,3}(\bar{\Lambda})$, contained in the $\bar{\Lambda}$-orbit of length 4 on such subgroups. In particular

(1) D is a commuting product of three quaternion groups Q_i, $1 \leq i \leq 3$, with the property that $\mathcal{Q} = \{Q_1, Q_2, Q_3\}$ is the set of normal subgroups of B_1 of order 8. Moreover $C_D(Q_i) = Q_j \times Q_k$ for any ordering (i, j, k) of $(1, 2, 3)$.

The preimage $\widehat{B}_1$ of B_1 in Λ is the 2-covering group of B_1, and so $\mathrm{Aut}(B_1)$ acts on $\widehat{B}_1$. By (1), $\mathrm{Aut}(B_1)$ permutes $\mathcal{Q}$, and hence $\widehat{\mathcal{Q}} = \{\widehat{Q}_i \mid 1 \leq i \leq 3\}$, where $\widehat{Q}_i = [\widetilde{Q}_1, \widehat{B}_1]$ and where $\widetilde{Q}_i$ is the preimage in Λ of Q_i. Therefore $\mathrm{Aut}(B_1) = \mathrm{Aut}_{\bar{\Lambda}}(B_1)$.

Referring once more to [Fin73], we find that $|K_0 : B_0| = 3$. Since $C_{K_0}(U) \leq B_0$ we have $B_1 \leq C_{B_0}(U) \trianglelefteq K_0$, and so $B_1 \trianglelefteq K_0$ and $K_0/C_{B_0}(U) \cong \mathrm{Sym}(3)$. From the preceding paragraph, $\mathrm{Aut}_{K_0}(B_1) \leq \Omega := \mathrm{Aut}_{\bar{\Lambda}}(B_1)$. Then, since the Sylow 2-subgroup S_0 of B_0 is Sylow in K_0 and contains the kernel U of the map from $N_{\bar{\Lambda}}(B_1)$ onto Ω, we may regard K_0 as a subgroup of $\bar{\Lambda}$. By (1), $\mathcal{Q}$ is K_0-invariant. As $K_0/C_{K_0}(U) \cong \mathrm{Sym}(3)$, it follows that K_0 is transitive on $\mathcal{Q}$. Here $\bar{\Lambda}/D \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(3)$, and K_0/D is a subgroup of $\bar{\Lambda}/D$ of order $2^2 \cdot 3^3$. Since $K_0/C_{K_0}(U) \cong \mathrm{Sym}(3)$, it follows that $K_0/B_1 \cong \mathbf{Z}_2 \times \mathrm{Sym}(3)$, and then that K_0 is determined up to conjugacy in $\bar{\Lambda}$. The same argument shows that K_σ is in this conjugacy class, and so $K_0 \cong K_\sigma$ and we may choose $K_0 = K_\sigma$.

Observe that $|N_{\bar{\Lambda}}(B_1) : B_0| = 3$ and that B_0 is equal to its normalizer in $N_{\bar{\Lambda}}(B_1)$. Then

(2) $N_{\mathrm{Aut}(B_1)}(B_0) = \mathrm{Aut}_{\bar{\Lambda}}(B_0) = \mathrm{Inn}(B_0)$.

By (2) we have $\mathrm{Aut}(B_0) = \mathrm{Inn}(B_0)C_0$, where $C_0 = C_{\mathrm{Aut}(B_0)}(B_1)$. Then $[C_0, B_0] \leq C_{B_0}(B_1) = U$. Let $X_0 \in \mathrm{Syl}_3(B_0)$ and let $X_0 \leq X \in \mathrm{Syl}_2(K_0)$. Then $C_D(X_0) = U$, $C_D(X) = 1$, and as we saw above $K_0/B_1 \cong \mathbf{Z}_2 \times \mathrm{Sym}(3)$. It follows that there is a Sylow 2-subgroup R_0 of $N_{B_0}(X_0)$, of the form $U\langle s\rangle \times \langle r\rangle$, where $\langle r, s\rangle$ is a Sylow 2-subgroup of $N_{K_0}(X)$, $U\langle s\rangle$ is a dihedral group of order 8, and $\langle r, s\rangle$ is a fours group.

We have $[C_0, R_0] \leq C_{B_0}(B_1) = U$, and so R_0 is C_0-invariant. Since $r \in Z(R_0)$ we then have $[C_0, r] \leq U \cap Z(R_0) = Z$. For any $\mu \in C_0$, s^μ is an involution in sU, so also $[C_0, s] \leq Z$. Therefore $C_0 \cong \mathrm{Hom}(B_0/B_1, Z) \cong B_0/B_1$, and so C_0 is a fours group. Since $\mathbf{Z}_2 \cong \mathrm{Aut}_U(B_0) \leq C_0$, we conclude that $|\mathrm{Aut}(B_0) : \mathrm{Inn}(B_0)| = |C_0 : \mathrm{Aut}_U(B_0)| = 2$. Thus

(3) $\mathrm{Aut}(B_0) = \mathrm{Inn}(B_0) \cup \mathrm{Inn}(B_0)\mu_0$, where $\mu_0 \in C_0 - \mathrm{Aut}_U(B_0)$.

It follows from (3) and [Gol70, Lemma 2.7] that there are, up to isomorphism, at most two amalgams $(H_0 \overset{\iota}{\longleftarrow} B_0 \overset{\alpha}{\longrightarrow} K_0)$ with $B_0\alpha = B_0$, and that any such amalgam is isomorphic to $\mathcal{A}_0$ or to $\mathcal{A}_{0,\mu_0}$, where

$$\mathcal{A}_{0,\mu_0} = (H_0 \overset{\iota}{\longleftarrow} B \overset{\mu_0^*}{\longrightarrow} K_0).$$

Thus, either $\mathcal{A}_0$ or $\mathcal{A}_{0,\mu_0}$ is a subamalgam of the amalgam $\mathcal{A}_\sigma$.

Referring again to [Fin73], there is a subgroup M of $\overline{G}_0$, containing S_0, such that M is a nonsplit extension of E_{16} by $\mathrm{GL}_4(2)$. Set $A = O_2(M)$, and denote by M_0 the stabilizer in M of the unique S_0-invariant hyperplane E_0 of A. Then $[O_2(M_0), M_0]$ is homocyclic abelian of exponent 4 and rank 3, and hence $E_0 = E$ by 4.9(c). Since $[O_2(M_0), E_0] = 1$ we then have $O_2(M_0) = T_2\langle w_0\rangle$, by 4.3. Moreover

$$M_0 = \langle C_{M_0}(Z), N_{M_0}(U)\rangle = \langle M_0 \cap H_0, M_0 \cap K_0\rangle.$$

Let $\alpha\colon S_0 \to N_{K_\sigma}(T_2)$ be the embedding of $S_0 = S_\sigma$ in $N_{K_0}(T_2)$. Then $S_0\alpha = S_0 \leq K_0 = K_\sigma$, and we have the two amalgams

$$\mathcal{A}_{M_0} = (M_0 \cap H_0 \longleftarrow S_\sigma \longrightarrow M_0 \cap K_0)$$

and

$$(N_{H_\sigma}(T_2) \overset{\iota}{\longleftarrow} S_0 \overset{\alpha}{\longrightarrow} N_{K_\sigma}(T_2)).$$

On the other hand, the reader will recall from the proof of 5.4 that if $\mathcal{A}_{M_0}$ is "twisted" by μ_0, to obtain an amalgam

$$(M_0 \cap H_0 \overset{\iota}{\longleftarrow} S_\sigma \overset{\mu_0^*}{\longrightarrow} M_0 \cap K_0),$$

then $\langle M_0 \cap H_0, x^{\mu_0}\rangle$ induces on T_2 the full automorphism group of T_2, of order $2^9|\mathrm{GL}_3(2)|$. We therefore conclude that, of the two amalgams $\mathcal{A}_0$ and $\mathcal{A}_{0,\mu_0}$, only the first is a subamalgam of $\mathcal{A}_\sigma$. This completes the proof of (a).

Set $\mathscr{X}_0 = z^{G_0}$ and denote by $\mathscr{E}^*$ the set of elementary abelian subgroups F of S_0 such that $F^\# \subseteq \mathscr{X}_0$. Denote by $\mathscr{N}$ the set of subgroups N of $\overline{G}_0$ such that $N = N_{\overline{G}_0}(O_2(N))$, $S_0 \cap N \in \mathrm{Syl}_2(N)$, and $C_{S_0}(O_2(N)) \le O_2(N)$. It is a property of Co_3 that $xy \in \mathscr{X}_0$ for any two distinct commuting elements x and y of $\mathscr{X}_0$ (cf. [Fin73]), from which it follows that for any $N \in \mathscr{N}$ there exists $F \in \mathscr{E}^*$ with $F \trianglelefteq N$. By Lemmas 5.8 and 5.9 in [Fin73], all members of $\mathscr{E}^*$ of any given order are fused in $\overline{G}_0$, each member of $\mathscr{E}^*$ is normal in a Sylow 2-subgroup of $\overline{G}_0$, and if $F \in \mathscr{E}^*$ with $|F| = 8$ then $N_{\bar{G}_0}(F)$ is contained in the normalizer of some $F^* \in \mathscr{E}^*$ with $|F^*| = 16$. Then, for any $N \in \mathscr{N}$, we have $S_0 \le N$, and N is contained in one of the groups H_0, K_0, or M. Then 1.11 yields

$$\mathscr{F}_{S_0}(\mathrm{Co}_3) = \langle \mathscr{F}_{S_0}(H_0), \mathscr{F}_{S_0}(K_0), \mathscr{F}_{S_0}(M) \rangle. \tag{4}$$

Now $(M \cap H_0)/A$ and $(M \cap K_0)/A$ are distinct maximal parabolic subgroups of $M/A \cong \mathrm{GL}_4(2)$, and so by 1.9:

$$\mathscr{F}_{S_0}(M) = \langle \mathscr{F}_{S_0}(M \cap H_0), \mathscr{F}_{S_0}(M \cap K_0) \rangle. \tag{5}$$

From (4) and (5) we have $\mathscr{F}_{S_0}(\overline{G}_0) = \langle \mathscr{F}_{S_0}(H_0), \mathscr{F}_{S_0}(K_0) \rangle$, and it follows that $\mathscr{F}_{S_0}(\overline{G}_0) \subseteq \mathscr{F}_{S_0}(G_0)$. Since $\overline{G}_0$ is a homomorphic image of G_0, by (a), the reverse inclusion of fusion systems is obvious, and we therefore have (b). □

Some well-known properties of Co_3 (some of which were mentioned in the proof of 5.8(b)), which depend only on fusion, now yield corresponding properties of the subgroup G_0 of G.

COROLLARY 5.9. *Identify H_0 and K_0 with subgroups of G, via the morphism φ of* 5.8(a), *and set $G_0 = \langle H_0, K_0 \rangle$. Then*

(a) *G_0 has two classes of elements of order* 2.

(b) *If t and t' are distinct, commuting elements of z^{G_0}, then $tt' \in z^{G_0}$.*

(c) *Let F be an elementary abelian* 2*-subgroup of G_0. Then $F \cap z^{G_0}$ is the set of nonidentity elements of a subgroup of F.*

(d) *For any $X \le G_0$, and any subgroup F of X, denote by $\widetilde{\mathscr{E}}(X, F)$ the set of all subgroups P of X such that $F \le P$ and $P^\# \subseteq z^{G_0}$. Write $\widetilde{\mathscr{E}}(X)$ for $\widetilde{\mathscr{E}}(X, 1)$. Then $\{Z, U, E, E\langle w_0 \rangle\}$ is a set of representatives for the orbits of G_0 on $\widetilde{\mathscr{E}}(G_0)$, and for the orbits of H_0 on $\widetilde{\mathscr{E}}(H_0, Z)$.*

6. Discrete p-toral groups

The notion of a discrete p-toral group, and the results in this section on such groups, come from [BLO05], particularly Sections 1 and 7 of that paper. As [BLO05] is unpublished at this time, we reproduce some of its definitions and results here, and supply sketches of proofs in special cases, for the sake of completeness.

Definition 6.1. Let p be a prime and denote by $\mathbf{Z}/p^\infty$ the group of all complex roots of unity whose order is a power of p. A *discrete p-toral group* is a p-group P with a normal subgroup P_0 of finite index, such that P_0 is the direct product of a finite number of copies of $\mathbf{Z}/p^\infty$. Write $\mathcal{D}_p$ for the class of discrete p-toral groups.

We record some facts about $\mathcal{D}_p$ from [BLO05]:

LEMMA 6.2. *Let $P \in \mathcal{D}_p$. Then*

(1) *P has unique subgroup P^0 which is minimal subject to the condition that $|P : P^0|$ be finite.* (*Call P^0 the identity component of P.*)

(2) *P^0 is the direct product of a finite number r of copies of* $\mathbf{Z}/p^\infty$. (*Write* $\mathrm{rk}(P)$ *for r and call* $\mathrm{rk}(P)$ *the rank of P.*)

(3) *P^0 has no proper subgroups of finite index.*

(4) *P is locally finite and Artinian.*

(5) *Subgroups and homomorphic images of P are in $\mathcal{D}_p$.*

(6) *Torsion subgroups of $Out(P)$ are finite.*

(7) *Each injective homomorphism from P into P is an isomorphism.*

(8) *If $R \le P$ then $R^0 \le P^0$.*

Proof. As $P \in \mathcal{D}_p$, P has a normal subgroup P_0 of finite index which is the direct product of r copies of $\mathbf{Z}/p^\infty$ for some $0 \le r \in \mathbf{Z}$. As $\mathbf{Z}/p^\infty$ has no proper subgroups of finite index, it follows that $P^0 = P_0$, and (1)–(3) hold. Parts (4), (5), and (6) are 1.2, 1.3, and 1.5(a) in [BLO05], respectively. Part (7) follows as P is Artinian, and (8) follows from (3). □

LEMMA 6.3. *Let* $\mathbf{F}$ *be the field of Section* 4, *V a finite-dimensional vector space over* $\mathbf{F}$, *and* $G \le \mathrm{GL}(V)$. *Then*

(1) *G is locally finite.*

(2) *All* 2*-subgroups of G are in $\mathcal{D}_2$.*

(3) $\mathrm{Syl}_2(G) \neq \varnothing$, $\mathrm{Syl}_2(G)$ *is the set of maximal* 2*-subgroups of G, and G is transitive on* $\mathrm{Syl}_2(G)$.

(4) *Let* $S \in \mathrm{Syl}_2(G)$ *and $P \le S$. Then P is fully normalized in $\mathcal{F}_S(G)$ if and only if* $N_S(P) \in \mathrm{Syl}_2(N_G(P))$.

(5) *$\mathcal{F}_S(G)$ is saturated.*

Proof. The proof of this lemma comes from [BLO05, §7, particularly Lemma 7.8]. The proof is a bit easier in our special case, and we supply a sketch.

As $GL(V)$ is the union of the finite groups $GL(V)_\sigma$, $\sigma \in Aut(\mathbf{F})$, (1) holds. Then (2) follows from (1) and from [Weh73, 2.6]. By [Weh73, 9.10], G is transitive on its maximal 2-subgroups, and such subgroups exist, and so (3) holds.

Observe that G satisfies the hypotheses of Lemma 1.6: Condition (1) of 1.6 follows from (3) applied to subgroups of $N_G(P)$. Condition (2) of 1.6 is satisfied by (1) and 6.2(6). Condition (3) holds by 6.2(4), and (4) holds by (1). Now 1.6 implies (4) and (5). □

Remark 6.4. Let H, K, B, S be the groups defined in Sections 4 and 5. Each of these groups has a faithful finite-dimensional representation over $\mathbf{F}$, and so we can apply Lemma 6.3 to these groups. By 4.9(b), S is a Sylow 2-subgroup of each of these groups. By 6.3(2), S and each of its subgroups is a discrete 2-toral group. By 6.3(5), $\mathcal{F}_S(X)$ is saturated for each $X \in \{H, K, B\}$ and by 6.3(3), X is transitive on $\mathrm{Syl}_2(X)$ where $\mathrm{Syl}_2(X)$ is the set of maximal 2-subgroups of X.

Let G be the group constructed in Section 5. It will be shown, in Theorem C, that there is a 2-local group $\mathcal{G} = (S, \mathcal{F}_S(G), \mathcal{L}_S(G))$. Since S is a discrete 2-toral group, $\mathcal{G}$ is then a 2-*local compact group*, as defined in [BLO05].

7. Local subgroups and fusion in the free amalgamated product

Let $\mathcal{A}$ be the amalgam $\mathcal{A}_\lambda$ constructed in Section 5, and let G be the associated free amalgamated product, $G = H *_B K$. We shall view $\mathcal{A}$ as being given by the inclusion maps of H, K and B into G, so that

$$\mathcal{A} = (H \geq B \leq K).$$

Viewed in this way, the key point in the construction of $\mathcal{A}$ is that the element y of K acts on T as $\lambda^{-1} y_0 \lambda$. That is

$$y\colon [a, b, c] \mapsto [c\lambda^{-1}, a, b\lambda],$$

for all $[a, b, c] \in T$.

Recall that we have an automorphism $\sigma = \psi_n$ of H, with $H_\sigma \cong \mathrm{Spin}_7(\mathbf{F}_q)$, $q = p^{2^n}$, and by 4.8, S_σ is a Sylow 2-subgroup of H_σ. By 5.7, σ induces an automorphism of $\mathcal{A}$ which induces an automorphism of G. Form the semidirect products $H\langle\sigma\rangle$, $B\langle\sigma\rangle$, and $K\langle\sigma\rangle$, and the amalgam

$$\widehat{\mathcal{A}} = (H\langle\sigma\rangle \longleftarrow B\langle\sigma\rangle \longrightarrow K\langle\sigma\rangle),$$

in which the arrows are inclusion maps. Denote the free amalgamated product of $\widehat{\mathcal{A}}$ by $\widehat{G}$. The inclusion $\mathcal{A} \to \widehat{\mathcal{A}}$ induces an isomorphism of $\widehat{G}$ with the semidirect product $G\langle\sigma\rangle$, and we identify these groups via that isomorphism.

The following result is trivially verified.

LEMMA 7.1. *Let Γ and $\widehat{\Gamma}$ be the standard trees associated with the amalgams $\mathcal{A}$ and $\widehat{\mathcal{A}}$, respectively. Then there is an isomorphism $\Gamma \longrightarrow \widehat{\Gamma}$ given by*

$$Xg \mapsto X\langle\sigma\rangle g \quad \text{for } X \in \{H, B, K\} \text{ and } g \in G.$$

If Γ and $\widehat{\Gamma}$ are identified via this isomorphism, then the action of σ on Γ is given by

$$(Xg)^\sigma = Xg^\sigma \quad \text{for } X \in \{H, B, K\} \text{ and } g \in G. \qquad \square$$

For any $X \le G\langle\sigma\rangle$, we write Γ_X or $C_\Gamma(X)$ for the subgraph of Γ induced on the set of fixed points of X on Γ. If $\Gamma_X \ne \varnothing$ then Γ_X is a subtree of Γ. For any graph Δ and vertex δ of Δ, we write $\Delta(\delta)$ for the set of vertices γ of Δ such that $\{\gamma, \delta\}$ is an edge of Δ. If $|\Delta(\delta)| \le 1$ then δ is a *boundary vertex* of Δ, and otherwise δ is an *interior vertex* of Δ.

For any subtree Δ of Γ, let $\widetilde{\Delta}$ be the graph obtained by deleting the boundary vertices from Δ. Thus either $\widetilde{\Delta}$ is a tree or Δ has at most one edge, in which case $\widetilde{\Delta}$ is empty.

Set $G_1 = H$ and $G_2 = K$, and denote by Γ_i the set of vertices of Γ given by the cosets of G_i in G. For any vertex γ of Γ, write $Z(\gamma)$ for the largest normal 2-subgroup of G_γ. Define γ_i to be the vertex of Γ given by the coset G_i.

LEMMA 7.2. *Let γ be a vertex of Γ.*

(a) *If $\gamma \in \Gamma_1$ then $Z(\gamma) = Z(G_\gamma)$ is of order 2.*

(b) *If $\gamma \in \Gamma_2$ then $|\Gamma(\gamma)| = 3$, and $Z(\gamma)$ is a fours group, whose nonidentity cyclic subgroups are the groups $Z(\delta)$, $\delta \in \Gamma(\gamma)$.*

(c) *If $\gamma \in \Gamma_2$ then $C_{G_\gamma}(Z(\gamma))$ is the pointwise stabilizer in G of $\Gamma(\gamma)$.*

Proof. The stabilizer of any vertex in Γ_i is conjugate in G to G_i, and the stabilizer of any edge is conjugate to B. All parts of the lemma follow trivially from these observations. $\square$

LEMMA 7.3. *Let $X \le G\langle\sigma\rangle$ and let $\gamma \in \Gamma_2 \cap \Gamma_X$. Then:*

(a) *γ is an interior vertex of Γ_X if and only if X centralizes $Z(\gamma)$.*

(b) *Either of the following conditions implies that the inclusion maps from $N_H(X)$ and $N_K(X)$ into $N_G(X)$ induce an isomorphism of $N_H(X) *_{N_B(X)} N_K(X)$ with $N_G(X)$.*

  (i) *$X \le B^0$, and $X^H \cap B = X^K \cap B = X^B$.*
  (ii) *$X \le B^0\langle\sigma\rangle$ and $X^{H\langle\sigma\rangle} \cap B^0\langle\sigma\rangle = X^{K\langle\sigma\rangle} \cap B^0\langle\sigma\rangle = X^{B^0}$.*

Moreover, $N_G(X)$ acts edge-transitively on Γ_X in case (b)(i), *and edge-transitively on $\widetilde{\Gamma}_X$ in case* (b)(ii).

Proof. Set $\Delta = \Gamma_X$. Then γ is an interior vertex of Δ if and only if X fixes at least two distinct vertices α and β in Γ_X. Since $Z(\gamma) = Z(G_\gamma^0) = Z_\alpha Z_\beta$, we obtain (a).

Set $N = \langle N_{G_1}(X), N_{G_2}(X)\rangle$, and assume that either (i) or (ii) holds. Take $\Lambda = \Delta$ in case (i), and $\Lambda = \widehat{\Delta}$ in case (ii). Then $N_G(X)$ acts on Λ, and $N \le N_G(X)$. By hypothesis, $X \le C_{B\langle\sigma\rangle}(U) = B^0\langle\sigma\rangle$, so that X fixes $\Gamma(\gamma_2)$ pointwise, and hence $\gamma_2 \in \Lambda$. In (i), a standard argument (cf. [Asc86, 5.21]) shows that $N_{G_i}(X)$ acts transitively on $\Lambda(\gamma_i)$ for $i = 1$ and 2. Assume that we are in case (ii) and that (γ_i, γ_{3-i}) is an edge in $\Lambda^{h^{-1}}$ for some $h \in G_i\langle\sigma\rangle$. Then $X \le (B\langle\sigma\rangle)^h$, and so $X^{h^{-1}} \le B\langle\sigma\rangle$. Then $X^{h^{-1}} \le C_{B\langle\sigma\rangle}(U) = B^0\langle\sigma\rangle$ by (a). The hypotheses of (ii) then yield $h \in B^0\langle\sigma\rangle N_{G_i\langle\sigma\rangle}(X)$, so that $N_{G_i\langle\sigma\rangle}(X)$ acts transitively on $\Lambda(\gamma_i)$.

We now claim that N is transitive on the set of edges of Λ. As Λ is connected, it suffices to show for each $\lambda \in \Lambda$ that N_λ is transitive on $\Lambda(\lambda)$. Pick i and g with $\lambda = \gamma_i g$ and set

$$d(\lambda) = \min\{d(\lambda, \gamma_j) \mid j = 1, 2\}.$$

Choose λ to be a counterexample with $d = d(\lambda)$ minimal. By the preceding paragraph, $d > 0$. Thus there exists $\alpha \in \Lambda(\lambda)$ with $d(\alpha) < d$, and N_α is transitive on $\Lambda(\alpha)$. Then there is $\beta \in \Lambda(\alpha)$ with $d(\beta) < d$, contrary to the choice of λ. This completes the proof of the claim.

As the stabilizer $N_B(X)$ of an edge of Λ is contained in N, we now obtain $N = N_G(X)$. Now [Ser80, Th. 6, p. 32] yields the conclusions concerning edge-transitivity, and the identification of $N_H(X) *_{N_B(X)} N_K(X)$ with N. □

LEMMA 7.4. *We have the following.*

(a) $C_{\widetilde{H}}(\sigma) = H_\sigma$.

(b) *The inclusion maps from* H_σ *and* K_σ *into* G_σ *induce an isomorphism of* G_σ *with* $H_\sigma *_{B_\sigma} K_\sigma$, *and* G_σ *acts edge-transitively on the tree* $\widetilde{\Gamma}_\sigma$.

(c) *Define the subgroups* G_0, H_0, K_0, *and* B_0 *of* G_σ *as in* 5.8. *Then the inclusion maps from* H_0 *and* K_0 *into* G_0 *induce an isomorphism of* G_0 *with* $H_0 *_{B_0} K_0$, *the universal completion of the amalgam* $\mathscr{A}_0$ *of subgroups of* Co_3.

Proof. Let $h \in H$ such that $\sigma^h \in B^0\sigma$. By Lang's Theorem there exists $b \in \widetilde{B}^0$ such that $\sigma^h = \sigma^b$. Then $hb^{-1} \in C_{\widetilde{H}}(\sigma) = H_\sigma$ as $\sigma = \xi_n$. Thus (a) holds and $b \in H$. Here $b \in B^0$ since $H \cap \widetilde{B}^0 = B^0$, and so $\sigma^H \cap B^0 = \sigma^{B^0}$. Since $K = C_K(\sigma)B^0$, we also have $\sigma^K \cap B^0 = \sigma^{B^0}$. Now by 7.3(b), $G_\sigma = \langle H_\sigma, K_\sigma\rangle$ and G_σ is edge-transitive on the tree $\widetilde{\Gamma}_\sigma$. Since H_σ and K_σ fix adjacent vertices in $\widetilde{\Gamma}_\sigma$, the lemma now follows from [Ser80, Th. 6, p. 32]. The same theorem implies (c). □

From now on, $G_0 = H_0 *_{B_0} K_0$ is the subgroup of G_σ defined in 7.4(c), such that G_0 is the universal completion of an amalgam of subgroups of Co_3.

LEMMA 7.5. *Let* $D \in \{G, G_\sigma, G_0\}$, *set* $D_i = D \cap G_i$, $i = 1, 2$, *and set* $R = S \cap D$. *Then*:

(a) *Hypotheses* 3.4 *and* 3.8 *hold, with* D, D_i *and* $D \cap B$ *in the roles of* G, G_i *and* B, *and with* R *in the role of* S.

(b) R *is a Sylow* 2*-subgroup of* D.

(c) *If* D *is* G *or* G_σ *then Hypothesis* 3.12 *holds.*

Proof. When R is finite so is D_i, and by construction R is a Sylow 2-subgroup of D_i and of $D \cap B$. If R is infinite then $R = S$, and by Remark 6.4, R is a Sylow 2-subgroup of D_i and of $D \cap B$. In each case Z and U are characteristic subgroups of R, so that $N_{D_i}(R) \le D_1 \cap D_2$. A free amalgamated product decomposition for G_σ is given by 7.4(b), and for G and G_0 by the definition of these groups. Thus, Hypothesis 3.4 holds. The verification of the first three parts of Hypothesis 3.8 is immediate in each case. Part (4) of Hypothesis 3.8 holds by Sylow's Theorem when D is finite, and by Remark 6.4 when $D = G$. Thus (a) is established. By 3.8(4), $R^{D_i} \cap B = R^{D_i \cap B}$. Part (b) follows from (a), 3.5(c), and this observation. Finally, when $D = G$ or G_σ, Hypothesis 3.12 follows from part (a) of Lemma 7.6 below. □

For any subgroup X of G, and any elementary abelian 2-subgroup F of X, denote by $\mathscr{E}_n(X, F)$ the set of elementary abelian 2-subgroups of X containing F, of order 2^n. Write $\mathscr{E}_n(X)$ for $\mathscr{E}_n(X, 1)$. Recall from the preceding sections that

$$Z \le U \le E \le A \in \mathscr{E}_4(G_\sigma)$$

is a chain of elementary abelian 2-groups, where $Z = Z(H) = \langle z \rangle$, $U = Z(B^0) = \langle z, z_1 \rangle$, $E = \{e \in T \mid e^2 = 1\}$, and $A = E\langle w_0 \rangle$.

LEMMA 7.6. *The following hold.*

(a) $\mathscr{E}_2(H, Z) = U^H$, *and* $\mathscr{E}_2(H_\sigma, Z) = U^{H_\sigma}$.

(b) $\mathscr{E}_1(G) = Z^G$, *and* $\mathscr{E}_1(G_\sigma) = Z^{G_\sigma}$.

Proof. By 4.2 there is a unique class $z_1^{\widetilde{H}}$ of noncentral involutions in $\widetilde{H}$. Then since $C_{\widetilde{H}}(z_1) = \widetilde{B}^0$ is connected, it follows from Lang's Theorem that $\widetilde{H}_\sigma$ has a unique class of noncentral involutions. As $\widetilde{H}_\sigma = H_\sigma$, (a) follows.

Since K_σ is transitive on $U^\#$, it follows from (a) that all involutions in S (resp. S_σ), are fused in KH (resp $K_\sigma H_\sigma$). By 7.5(b), S and S_σ are Sylow in G and G_σ, respectively, so (b) holds. □

LEMMA 7.7. *We have* $E^{B^0} = \mathscr{E}_3(B, U)$, $E^H = \mathscr{E}_3(H, Z)$, *and* $E^G = \mathscr{E}_3(G)$.

Proof. Let $F \in \mathscr{E}_3(B, U)$ and $f \in F - U$. Then $C_B(U) = B^0 = L_1L_2L_3$, so that $f = f_1f_2f_3$ with $f_i \in L_i$, and $1 = f^2 = f_1^2f_2^2f_3^2$. Since $U^\#$ is the set of involutions in $L_i L_j$ for $i \neq j$, it follows that f_i is an element of order 4 in L_i. Since L_i is transitive on its elements of order 4, involutions in $B^0 - U$ are conjugate in B^0. Thus $E^{B^0} = \mathscr{E}_3(B^0, U)$. The lemma now follows from 7.6. $\square$

LEMMA 7.8. *Set* $B(\sigma) = \langle C_{L_i}(\sigma) \mid 1 \leq i \leq 3 \rangle$. *Then*

(a) *There exists an involution* v *in* $S \cap B^0_\sigma - B(\sigma)$ *such that* $B^0_\sigma = B(\sigma)\langle v \rangle$, *and* v *induces a diagonal automorphism on each* $C_{L_i}(\sigma)$.

(b) $E^{B(\sigma)} = \mathscr{E}_3(B(\sigma), U)$ *and* $\mathscr{E}_3(B_\sigma, U) = E^{B_\sigma} \cup (E')^{B(\sigma)}$, *where* $E' = U\langle v \rangle$.

(c) $\mathscr{E}_3(H_\sigma, Z)$ *is the disjoint union of* E^{H_σ} *and* $(E')^{H_\sigma}$.

(d) EE' *is a Sylow* 2*-subgroup of* $C_{H_\sigma}(E')$, *and is elementary abelian of order* 16.

(e) *For each* $F \in \{E, E'\}$, *we have* $\mathrm{Aut}_{H_\sigma}(F) = C_{\mathrm{Aut}(F)}(Z)$ *and* $\mathrm{Aut}_{G_\sigma}(F) = \mathrm{Aut}(F)$.

Proof. Recall from Section 4 that we may regard T as a set of equivalence classes $[a_1, a_2, a_3]$. Let a be a 2-element in $\mathbf{F}$ with $a^\sigma = -a$, and set $f = [a, a, a]$. Then $f \in (S_\infty \cap B^0_\sigma) - B(\sigma)$, and since w_0 inverts S_∞, the element $v := f w_0$ is an involution in $(S \cap B^0_\sigma) - B(\sigma)$. Recall from 4.4 that $\widetilde{B}^0$ is $\widetilde{J}/\langle i \rangle$, where $\widetilde{J}$ is the direct product of three copies of $\mathrm{SL}_2(\overline{\mathbf{F}})$ and i is an involution diagonally embedded in $Z(\widetilde{J})$. Thus $\widetilde{J}$ is simply connected, so that $B(\sigma) = \widetilde{J}_\sigma/\langle i \rangle$, and $B(\sigma)$ is of index $|i| = 2$ in B^0_σ. Then $B^0_\sigma = B(\sigma)\langle v \rangle$, completing the proof of (a).

Let $X \in \{\widetilde{B}^0, \widetilde{H}\}$, and set

$$\Sigma = \{(F, \tau) \mid F \in E^X,\ \tau \in \sigma^X,\ [F, \tau] = 1\}.$$

Set

$$\Sigma_0 = \{\tau \in \sigma^X \mid (E, \tau) \in \Sigma\} \quad \text{and} \quad \Sigma_1 = \{F \in E^X \mid (F, \sigma) \in \Sigma\}.$$

There is a natural bijection β between the set of $N_X(E)$-orbits on Σ_0 and the set of $C_X(\sigma)$-orbits on Σ_1. Explicitly, if $\{\sigma^{g_i} \mid i \in I\}$ is a set of representatives for the orbits of $N_X(E)$ on Σ_0, then $\{E^{g_i^{-1}} \mid i \in I\}$ is a set of representatives for the orbits of $C_X(\sigma)$ on Σ_1. By 4.3(d), $N_{\widetilde{H}}(E) = \widetilde{T}W$, so that $N_X(E) = \widetilde{T}(W \cap X)$.

Let $\tau \in \Sigma_0$. Then

$$\tau \in C_{X\langle\sigma\rangle}(E) \cap \sigma^X \subseteq \widetilde{T}\langle w_0 \rangle \sigma = \widetilde{T}\sigma \cup \widetilde{T} w_0 \sigma.$$

When we apply Lang's Theorem to the connected algebraic group $\widetilde{T}$, we find that $\widetilde{T}$ is transitive on $\widetilde{T}\sigma$ and $\widetilde{T} w_0 \sigma$. Since W centralizes both σ and w_0, we conclude that $\widetilde{T}\sigma$ and $\widetilde{T} w_0 \sigma$ are the orbits for $N_X(E)$ on Σ_0, with representatives $\{\sigma, w_0\sigma\}$. Applying Lang's Theorem to the connected group $\widetilde{B}^0$, we obtain an element $g \in X$ such that $(w_0\sigma)^g = \sigma$.

When we apply the bijection β, $C_X(\sigma)$ has two orbits on Σ_1, with representatives E and E^g. By 7.7, $\Sigma_1 = \mathscr{E}_3(C_X(\langle\sigma, Z(X)\rangle))$. Thus, $C_X(\sigma)$ has two orbits on $\mathscr{E}_3(X, Z(X))$, with representatives E and E^g. In the case that $X = \widetilde{B}^0$ we have $C_X(\sigma) = B^0_\sigma$ by 6.4(a), and then since E is in the normal subgroup $B(\sigma)$ of $C_X(\sigma)$, but E' is not, it follows that E and E' are representatives for the two orbits of B^0_σ on $\mathscr{E}_3(B^0_\sigma, U)$. Since $[E, E'] = 1$, and $B^0\sigma = B(\sigma)E'$ by (a), these are also the orbits for $B(\sigma)$, establishing (b). In particular, E' is fused to E^g in B_σ, and so E' is not fused to E in H_σ. In the case that $X = \widetilde{H}$ we get $C_X(\sigma) = H_\sigma$ by 7.4(a), and this yields (c).

Recall that $\sigma = \psi_n$ for some $n \geq 0$. Set $q = p^{2^n}$. Let $\delta = \pm 1$ with $q \equiv \delta$ mod 8. Then T_σ is homocyclic abelian of rank 3 and order $(q-\delta)^3$, and $C_{H_\sigma}(E) = T_\sigma\langle w_0\rangle$ by 4.3(d). On the other hand, we have seen that $(E, w_0\sigma) \in \Sigma_0$. As w_0 inverts $\widetilde{T}$, $C_T(w_0\sigma)$ is homocyclic abelian of rank 3 and order $(q+\delta)^3$. In particular, E is a Sylow 2-subgroup of $C_T(w_0\sigma)$. Therefore a Sylow 2-subgroup of $C_H(w_0\sigma) \cap C_H(E)$ is of order at most 16. Since $[E, E'] = 1$, (d) follows.

From 4.3, $\mathrm{Aut}_{TW}(E) = C_{\mathrm{Aut}(E)}(Z)$, and hence $\mathrm{Aut}_{TW}(E) = \mathrm{Aut}_H(E)$. Similarly, since $W \leq \widetilde{H}_{w_0\sigma}$, we have

$$\mathrm{Aut}_{\widetilde{H}_{w_0\sigma}}(E) = C_{\mathrm{Aut}(E)}(Z).$$

Conjugating by the element g of $\widetilde{B}^0$ with $(w_0\sigma)^g = \sigma$, we obtain

$$\mathrm{Aut}_{\widetilde{H}_\sigma}(E^g) = C_{\mathrm{Aut}(E^g)}(Z).$$

Since $[E, y] = [E', y] = U$, (e) holds. □

LEMMA 7.9. *The following hold.*

(a) $\mathscr{E}_5(G) = \varnothing$.

(b) $\mathscr{E}_4(X) = A^X$ *for* $X \in \{G, H\}$, *and* $\mathscr{E}_4(B^0) = A^{B^0} = A^K$.

(c) $\mathrm{Aut}_H(A) = C_{\mathrm{Aut}(A)}(Z)$, *and* $A = C_H(A)$.

Proof. Let Y be H or B^0, and let A' be an elementary abelian subgroup of Y of maximal order. Then $Z(Y) \leq A'$, and after conjugation in Y we may assume, by 7.7, that $E \leq A'$. Then $A' \leq T\langle w_0\rangle$, by 4.3(d), so that $|A'| = 16$. Since H contains a Sylow 2-subgroup of G, by 7.5(b), we obtain (a). Every element of T is a square, so since w_0 acts on T as inversion, all elements in Tw_0 are fused by T. Thus, A' is fused to A via T, and this yields (b).

By 4.3(d), $C_H(A) = C_H(E) \cap C_H(w_0) = C_{T\langle w_0\rangle}(w_0) = A$. Since $w_0 \in Z(W)$ and $W \leq N_H(T) \leq N_H(E)$, we have $W \leq N_H(A)$. Set $T_1 = \{t \in T \mid t^4 = 1\}$. Then $[T_1, w_0] = E$, and so $T_1 \leq N_H(A)$ and T_1 induces on A the subgroup $X(E)$ of $\mathrm{Aut}(A)$ consisting of all transvections with axis E.

Let F be a hyperplane of A containing Z. Then F is conjugate to E in H by 7.7. It follows that $X(F) \leq \mathrm{Aut}_H(A)$ for all such F. Since

$$C_{\mathrm{Aut}(A)}(Z) = \langle X(F) \mid F \text{ a hyperplane of } A \text{ over } Z \rangle,$$

we obtain (c). □

LEMMA 7.10. *Set $M = N_G(A)$, and for any subgroup X of G set $M_X = M \cap X$. Then the following hold.*

(a) *The inclusion maps from M_H and M_K into M induce an isomorphism $M \longrightarrow M_H *_{M_B} M_K$, and M is edge-transitive on the tree $\widetilde{\Gamma}_A$.*

(b) *M is contained in the subgroup G_0 of G defined in* 5.8(b). *In particular $M \leq G_\sigma$.*

(c) *There is a surjective homomorphism $\phi_A\colon M \longrightarrow M_0$, where M_0 is a nonsplit extension of an elementary abelian group of order* 16 *by* GL(4, 2), *and such that* $\ker(\phi_A) \cap M_H = \ker(\phi_A) \cap M_K = 1$.

(d) *For any ϕ_A satisfying the conditions in* (c), *we have* $C_G(A) = \ker(\phi_A) \times A$, *and* $\ker(\phi_A)$ *acts freely on* Γ.

Proof. The induced isomorphism of M with $M_H *_{M_B} M_K$ is immediate from 7.3(b) and 7.9(b). The edge-transitivity of M on $\widetilde{\Gamma}_A$ is given by the final statement in 7.3, so that (a) holds.

By 5.8(a), there are maximal subgroups H_0 and K_0 of the group Co_3 such that H_0 may be regarded as a subgroup of H_σ, and K_0 as a subgroup of K_σ, in such a way that the resulting amalgam $\mathscr{A}_0 = (H_0 \geq B_0 \leq K_0)$ of subgroups of G_σ is isomorphic to the corresponding amalgam of subgroups of Co_3. Set $M_0 = N_{\mathrm{Co}_3}(A)$. Then $M_0/A \cong \mathrm{GL}_4(2)$, and M_0 does not split over A, as one finds from the list of maximal subgroups of Co_3 in [Fin73]. Moreover, as seen in the proof of 5.8, we have $M_0 = \langle M_0 \cap H_0, M_0 \cap K_0 \rangle$, as subgroups of Co_3. As subgroups of G, we have $M_0 \cap H = M_H$ and $M_0 \cap K = M_K$, so it follows from (a) that $M \leq G_{\psi_0} \leq G_\sigma$. Moreover, (a) implies that M_0 is a homomorphic image of M, via a homomorphism ϕ_A whose restriction to each of M_H and M_K is the "identity" map. In particular, the restriction of ϕ_A to $M_H \cup M_K$ is faithful, and this yields (b) and (c).

From (c), it is immediate that $C_G(A) = \ker(\phi_A) \times A$. Since M is edge-transitive on Γ_A, and since $\ker(\phi_A)$ intersects both H and K trivially, it follows that $\ker(\phi_A)$ acts freely on Γ_A. That is, every nonidentity element of $\ker(\phi_A)$ induces a hyperbolic isometry of Γ_A, and hence also a hyperbolic isometry of Γ, by 3.3. Thus, (d) holds. □

Set $R_0 = S_{\infty,\sigma}$ and $R_1 = N_{S_\infty}(R_0\langle w_0 \rangle)$. Then R_0 has index 8 in R_1. Fix a set $\mathbf{X}$ of coset representatives for R_0 in R_1, and recall that q_0 denotes the exponent of R_0. Then

$$\mathbf{X} = \{x_e \mid e \in E\}$$

where $x_e^{q_0} = e$. Set

$$A_e = A^{x_e}, \quad e \in E,$$

and

$$\mathscr{E} = \mathscr{E}_4(R_0\langle w_0\rangle).$$

All elements of $R_0 w_0$ are fused by R_1, so R_1 is transitive on $\mathscr{E}$. Since $N_{R_1}(A) \leq R_0$, R_0 has $|R_1 : R_0| = 8$ orbits on $\mathscr{E}$ and $\{A_e \mid e \in E\}$ is a set of representatives for those orbits.

Every subgroup of T is σ-invariant, so each $\langle x_e\rangle$ is σ-invariant. Since σ centralizes x_e^2 and does not centralize x_e, we obtain

$$(*) \qquad \sigma^{x_e} = e\sigma \quad \text{and} \quad (e\sigma)^{x_e} = \sigma \quad \text{for all } e \in E.$$

LEMMA 7.11. *For each $e \in E$:*

(a) $\mathrm{Aut}_{H_\sigma}(A_e) = C_{\mathrm{Aut}(A_e)}(\langle z, e\rangle)$,

(b) $\mathrm{Aut}_{K_\sigma}(A_e) = N_{\mathrm{Aut}(A_e)}(U) \cap C_{\mathrm{Aut}(A_e)}(e)$, *and*

(c) $\mathrm{Aut}_{G_\sigma}(A_e) = C_{\mathrm{Aut}(A_e)}(e)$.

Proof. For any $e \in E$, set $\sigma_e = e\sigma$, regarded as an automorphism of G. By 7.10, $M := N_G(A) \leq G_\sigma$ and $\mathrm{Aut}_G(A) = \mathrm{Aut}(A)$, so that

$$\mathrm{Aut}_{G_{\sigma_e}}(A) = C_{\mathrm{Aut}_G(A)}(e) = C_{\mathrm{Aut}(A)}(e).$$

Then as $A_e = A^{x_e}$, conjugating this equality by x_e and appealing to $(*)$, we conclude that (c) holds.

Next, $\mathrm{Aut}_H(A) = C_{\mathrm{Aut}(A)}(Z)$ by 7.9(c), so that

$$\mathrm{Aut}_{H_{\sigma_e}}(A) = C_{\mathrm{Aut}(A)}(\langle z, e\rangle).$$

Since x_e centralizes $\langle z, e\rangle$, conjugation by x_e yields (a).

As $\mathrm{Aut}_{G_\sigma}(A) = \mathrm{Aut}_G(A) = \mathrm{Aut}(A)$, as $y \in K$, and as $N_H(U) \leq K$, we conclude that $\mathrm{Aut}_{K_\sigma}(A) = N_{\mathrm{Aut}(A)}(U)$. Then

$$\mathrm{Aut}_{K_{\sigma_e}}(A) = N_{\mathrm{Aut}(A)}(U) \cap C_{\mathrm{Aut}(A)}(e),$$

and conjugation by x_e yields (b). □

LEMMA 7.12. *Let $u \in U - Z$ and let $e \in E - U$. Then*

(a) *$\{A, A_z, A_u\}$ is a set of representatives for the orbits of H_σ on $\mathscr{E}_4(H_\sigma)$, and H_σ fuses A_u and A_e.*

(b) *$\{A, A_z, A_e\}$ is a set of representatives for the orbits of K_σ on $\mathscr{E}_4(K_\sigma, U)$, and K_σ fuses A_z and A_u.*

(c) *$\{A, A_u\}$ is a set of representatives for the orbits of G_σ on $\mathscr{E}_4(G_\sigma)$.*

Proof. Since $S_\sigma \in \mathrm{Syl}_2(G_\sigma)$, each $A \in \mathscr{E}_4(G_\sigma)$ is fused under G_σ into S_σ, and then by 7.6, we may take $U \le A \le S_\sigma$. Then by parts (b) and (d) of 7.8, we may take $E \le A$. Since $C_{S_\sigma}(E) = R_0\langle w_0\rangle$, we are reduced to the problem of fusion via H_σ, K_σ, and G_σ on $\mathscr{E}$.

Notice that W has the three orbits $\{1\}$, $\{z\}$, and $E - Z$ on E, and that $S_\sigma\langle y\rangle$ has the three orbits $\{1\}$, $U^\#$ and $E - U$ on E. Hence there are three orbits for $S_\sigma W$ on $\mathscr{E}$, with representatives A, A_z, and A_u, and three orbits for $S_\sigma\langle y\rangle$ on $\mathscr{E}$, with representatives A, A_z, and A_e. Since $\langle S_\sigma, W, y\rangle$ is transitive on $\{A_f \mid f \in A^\#\}$, it is now enough to show that there is no further fusion among these groups. As $\mathrm{Aut}_{G_\sigma}(A) = \mathrm{Aut}(A)$, A is not fused to A_z or to A_u in G_σ, by 7.11(c). Since A_z is not fused to A_u in H_σ by 7.11(a), and A_z is not fused to A_e in K_σ, by 7.11(b), the lemma is proved. □

LEMMA 7.13. *Set $N = N_G(T_2)$, and for any subgroup X of G set $N_X = N \cap X$. Let ϕ_A be defined as in* 7.10, *and set $D = C_{\ker(\phi_A)}(T_2)$. Then the following hold.*

(a) *The inclusion maps from N_H and N_K into N induce an isomorphism of N with $N_H *_{N_B} N_K$. In particular, N is generated by N_H and N_K.*

(b) *N is edge-transitive on the tree Γ_{S_∞}.*

(c) $N = N_G(T) = N_G(T\langle w_0\rangle)$.

(d) $C_G(T_2) = C_G(S_\infty) = O(T)D \times S_\infty$, *and $O(T)D$ is N-invariant.*

(e) $\mathrm{Aut}_G(S_\infty) \cong \mathrm{GL}(3,2) \times \mathbf{Z}_2$.

(f) $\mathscr{F}_{C_S(E)}(N_H) = \mathscr{F}_{C_S(E)}(C_N(Z))$, *and* $\mathscr{F}_{C_{S_\sigma}(E)}(N_{H,\alpha}) = \mathscr{F}_{C_{S_\sigma}(E)}(C_{N_\sigma}(Z))$.

Proof. By 4.9(b) S is a Sylow 2-subgroup of B, and by 4.9(c) T_2 is weakly closed in S with respect to G. Therefore

$$T_2^B = \{T_2^g \mid T_2^g \le B,\ g \in H \cup K\}.$$

Now (a) and (b) follow from 7.3(b).

Set $T^* = T\langle w_0\rangle$. Since $N_H(T_2) \le N_H(E) = TW$ by 4.3(d), we have $N_H = N_H(T) = N_H(T^*)$. Then $N_B = N_B(T) = N_B(T^*)$, and $N_K = N_B\langle y\rangle = N_K(T) = N_K(T^*)$. It follows now from (a) that $N \le N_G(T)$ and $N \le N_G(T^*)$. Since the reverse inclusions are obvious, we obtain (c).

Since $\langle W, y\rangle \le N_G(A) \cap N_G(T_2)$, D is $\langle W, y\rangle$-invariant, and evidently so is $O(T)$. Let $x \in C_G(S_\infty)$. Then (c) implies that $w_0^x \in T^* - T$, and since T is transitive on Tw_0, there exists $t \in T$ with $w_0^{xt} = w_0$. Thus $xt \in C_G(A)$, so it follows from 7.10(d) that $xt = da$ for some $d \in \ker(\phi_A)$ and some $a \in A$. Since $T_2 \le N_G(A)$, we have $[T_2, d] \le \ker(\phi_A)$, and since $\ker(\phi_A)$ acts freely on Γ we have $T \cap \ker(\phi_A) = 1$. Let $s \in T_2$. Then

$$1 = [s, xt] = [s, da] = [s, a][s, d]^a,$$

where $[s,a] \in T_2$ and where $[s,d]^a \in \ker(\phi_A)$. It follows that $[T_2,a]=[T_2,d]=1$, and so $a \in E$ and $d \in D$. Now $x = dat^{-1} \in DT$, and thus $C_G(S_\infty) \leq DT$. By 5.1 and (a), $[D, S_\infty] = 1$, so that $C_G(S_\infty) = DT$. Since $D \cap T = 1$ we have $S_\infty \cap O(T)D = 1$, and thus $O(T)D$ is a complement to S_∞ in $C_G(S_\infty)$. This completes the proof of (d). Part (e) follows from (a) and Theorem 5.2.

Let $P \leq S$ and let $g \in N_G(P,S) \cap C_N(Z)$. Then $g = nd$ for some $n \in N_H$ and $d \in D$, by (d). But $C_S(E) = S_\infty\langle w_0\rangle$, and $w_0 \in A \leq C_G(D)$, so $C_S(E)$ centralizes D. Thus d centralizes P, so $c_g = c_n$ on P, establishing (f). □

8. **Centric subgroups and signalizer functors**

We continue the hypotheses and the notation of Sections 4, 5, and 7. Thus $\mathcal{A}_\sigma$ is the amalgam $(H_\sigma \longleftarrow B_\sigma \longrightarrow K_\sigma)$, and $\mathcal{A}_0$ is the amalgam $(H_0 \longleftarrow B_0 \longrightarrow K_0)$ given by 5.8. As in 7.4, we regard $\mathcal{A}_0$ as a subamalgam of $\mathcal{A}_\sigma$, H_0 and K_0 as subgroups of G_σ; and we set $G_0 = \langle H_0, K_0\rangle$ and $S_0 = S \cap G_0$.

There is a fair amount of notation which we now need to establish, and which will remain fixed in the remainder. First, we set

$$\mathcal{F} = \mathcal{F}_S(G), \quad \mathcal{F}_\sigma = \mathcal{F}_{S_\sigma}(G_\sigma), \quad \mathcal{F}_0 = \mathcal{F}_{S_0}(G_0).$$

Let D be one of the groups G, G_σ, or G_0, and let $\mathcal{D}$ be the fusion system $\mathcal{D} = \mathcal{F}_{S\cap D}(D)$. For any subgroup Y of G such that $S \cap D$ is a Sylow 2-subgroup of $Y \cap D$, we write Y_D for $Y \cap D$, and $\mathcal{D}_Y$ for $\mathcal{F}_{S_D}(Y_D)$.

For any subgroup P of S_D, set

$$Z_P = \langle z^D \cap Z(P)\rangle.$$

Thus $Z_P = \Omega_1(Z(P))$ if $D \neq G_0$, by 7.6(b), and in any case we have $Z_P^\# = z^D \cap Z(P)$, by 5.9(b). Although the definition of Z_P depends on D, the reader may think of D as being fixed, so there need be no cause for confusion.

Denote by Γ_0 the smallest G_0-invariant subtree of Γ which contains the edge $\{H, K\}$. Recall that $\Gamma_i = \gamma_i G$ denotes the subset $G_i\backslash G$ of vertices of Γ, where $G_1 = H$ and $G_2 = K$. Write also Γ^D for the standard tree for D. That is, Γ^D is the smallest D-invariant subtree of Γ containing the edge $\{H, K\}$. Thus Γ^D is Γ, Γ_σ, or Γ_0, for D equal to G, G_σ, or G_0, respectively, and $\Gamma \supseteq \Gamma_\sigma \supseteq \Gamma_0$.

LEMMA 8.1. *Let $Y \in \{B, K, H\}$, and let P be a 2-subgroup of Y. Then $N_Y(P,S) \neq \varnothing$.*

Proof. This follows from Remark 6.4. □

LEMMA 8.2. *Let P be a subgroup of S_D, and let $Y \in \{G, H, K\}$. Then*

(a) *$P \in \mathcal{D}_Y^c$ if and only if $Z(P)$ contains every finite 2-subgroup of $C_{Y_D}(P)$.*

(b) *If $P \in \mathcal{D}_Y^{rc}$ then P contains every finite $N_{Y_D}(P)$-invariant 2-subgroup of Y_D.*

Proof. We first prove (a). By 2.1 and 7.5(b), we may assume $D = G$, P is infinite, $P \in \mathcal{F}^c_Y$, and there exists a 2-element x of $C_Y(P)$ with $x \notin P$. Set $P^* = \langle P, x \rangle$.

Since $C_\Gamma(P)$ is x-invariant, and $|x|$ is finite, it follows from 3.2 that x fixes a vertex δ of $C_\Gamma(P)$, and we may take $\delta = Y$ if $Y \in \{H, K\}$. Now 8.1 implies that P^* is contained in a conjugate of S in G_δ, and then $x \in P$ since $P \in \mathcal{F}^c_Y$. Thus (a) is established.

Now suppose that $P \in \mathcal{D}^{\rm rc}_Y$, set $N = N_{Y_D}(P)$, and let R be a finite, N-invariant 2-subgroup of Y_D. Set $R_0 = N_R(P)$. Then $R_0 \trianglelefteq N$, and so $\mathrm{Aut}_{R_0P}(P) \leq O_2(\mathrm{Aut}_{Y_D}(P) = \mathrm{Inn}(P)$ as $P \in \mathcal{D}^{\rm rc}_Y$. Then $R_0P = C_{R_0P}(P)P$, so that $R_0 \leq C_{R_0P}(P)P \leq P$ by (a). Thus $R_0 = R \cap P$. But the p-group P induces a finite p-group $\widetilde{P}$ of automorphisms on R, and $C_{N_R(R_0)/R_0}(\widetilde{P}) \leq N_R(P)/R_0 = R_0/R_0$. It follows that $R \leq P$, proving (b). □

For any $P \in \mathcal{D}^c$ we have $C_{S_D}(P) \leq Z(P)$, and thus $Z \leq Z_P$. The following lemma derives most of the remaining information that we shall need, concerning $\mathcal{D}$-centric subgroups of S_D, including everything that is needed for the construction of signalizer functors.

LEMMA 8.3. *Let* $P \leq S_D$.

(a) *Suppose that* $|Z_P| = 2$ *and that either* $P \in \mathcal{D}^c$ *or* $P \in \mathcal{F}^c_H$. *Then* $N_D(P, S) \subseteq H$, *and if* $P \in \mathcal{F}^c_H$ *then* $N_G(P, S) \subseteq H$ *and* $P \in \mathcal{F}^c$.

(b) *Suppose that* $P \in \mathcal{D}^c$ *and* $|Z_P| = 4$. *Then* $C_D(P) \leq H$, $Z_P = U^h$, *and* $N_D(P) \leq K^h$ *for some* $h \in H_D$. *If also* $Z_P = U$ *and* $P \in \mathcal{F}^c_K$, *then* $N_G(P) \leq K$ *and* $P \in \mathcal{F}^c$.

(c) *Suppose that* $P \in \mathcal{D}^c$ *and* $|Z_P| = 8$. *Then* $Z_P \in E^{H_D}$. *If also* $\mathcal{E}_4(P) = \varnothing$ *then* $P = S_\infty \cap D$ *and* P *is not* $\mathcal{D}$*-radical.*

(d) *Suppose that* $\mathcal{E}_4(P) \neq \varnothing$. *Then* $P \in \mathcal{F}^c$ *and* $O(C_G(P)) = O(C_D(P)) = 1$.

Proof. Set $\Sigma = C_\Gamma(P)$ and $\Sigma' = \Sigma \cap \Gamma^D$. Then $\{\gamma_1, \gamma_2\}$ is an edge of Σ', since $P \leq S_D \leq B_D$. Let $\mathcal{P}$ be the set of paths $\pi = (\gamma_1, \alpha, \beta)$ in Σ such that $\beta \neq \gamma_1$, and $\mathcal{P}'$ the paths in $\mathcal{P}$ contained in Σ'. If $\pi \in \mathcal{P}$ then $Z_\alpha = ZZ_\beta \leq C_H(P)$ by 3.11(a), so 8.2 implies that $Z_\alpha \leq Z_P$ if either $P \in \mathcal{F}^c_H$ or $\beta \in \Sigma'$. In particular, $|Z_P| > 2$ and $Z_P = Z_\alpha$ if $|Z_P| = 4$.

Now assume the hypothesis of (a). Then $\mathcal{P}' = \varnothing$ by the preceding paragraph, and thus γ_1 is the unique vertex in $C_{\Gamma_1}(P)$. The same is then true for P^g, for any $g \in N_D(P, S)$, and thus $N_D(P, S) \subseteq H$. Similarly $N_G(P, S) \subseteq H$ if $P \in \mathcal{F}^c_H$. Since $C_G(P) \subseteq N_G(P, S)$, (a) is proved.

Suppose next that $|Z_P| = 4$. Since $Z \leq Z_P$, we then have $Z_P = U^h$ for some $h \in H_D$, by 7.6(a) and 5.9. We conclude from paragraph one that $\alpha = \gamma_2^h$ for each $\pi \in \mathcal{P}'$, and hence that α is the unique vertex in Γ_2 which is in the interior of Σ', so that $N_D(P) \leq K^h$. Now $C_D(P) \leq C_{K^h}(Z) = C_K(Z)^h \leq H$, and the first part of (b) is established. Now suppose that $Z_P = U$ and that $P \in \mathcal{F}^c_K$. Then $C_H(P) \leq N_H(U) \leq K$, so that $P \in \mathcal{F}^c_H$ by 8.2. Then from paragraph one, $\alpha = \gamma_2$ for each $\pi \in \mathcal{P}$, and so γ_2 is the unique vertex in Γ_2 which is in the interior of Σ. Thus $N_G(P) \leq K$, and we have (b).

Suppose next that $|Z_P| = 8$. If Z_P is not conjugate to E in H_D, then $D = G_\sigma$ by 7.7 and 5.9, and Z_P is conjugate in H_D to the group E' defined in 7.8(a). But in that case we conclude from 7.8(d) that P does not contain every 2-element of $C_D(P)$, contrary to 8.2(a). Thus, $Z_P = E^h$ for some $h \in H_D$. Set $R = C_{S_D}(E)$ and $R_0 = S_D \cap T$. Then $R = R_0\langle w_0\rangle$ is a Sylow 2-subgroup of $C_{H_D}(E)$, by 4.3(d), and we may choose h so that $P \leq R^h$. Suppose further that $\mathcal{E}_4(P) = \varnothing$. Since $R - R_0$ consists entirely of involutions, we then have $P \leq R_0^h$. Since P contains all 2-elements in $C_{H_D}(P)$ it follows that $P = R_0^h$. Then $P = R_0$ by 4.9(c). Since w_0 inverts R_0, $O_2(\mathrm{Aut}_D(P)) \neq \mathrm{Inn}(P)$, and therefore (c) holds.

We now remove the hypothesis that P is $\mathcal{D}$-centric, and assume that $\mathcal{E}_4(P) \neq \varnothing$. Let $F \in \mathcal{E}_4(P)$. Then $F \in A^G$ by 7.9(b), so that F contains every element of $C_G(F)$ of finite order by 7.10(d). The same is then true of P, and so $O(C_D(P))=1$, and $P \in \mathcal{D}^c$ by 2.1. That is, (d) holds. □

COROLLARY 8.4. *Let $P \in \mathcal{D}^c$, and assume that $|Z_P| \leq 4$. Then $C_D(P) = C_{H_D}(P) = Z(P) \times O(C_D(P))$.*

Proof. By 8.3, $C_D(P) \leq H$, while $O^{2'}(C_D(P)) = Z(P)$ by 8.2(a). Let X be a finite subgroup of $C_D(P)$ containing $Z(P)$. The Schur-Zassenhaus Theorem then yields $O^2(X) = O(X)$. Since H is the union of an ascending chain of finite subgroups, the result follows. □

Recall from 7.10 that there is a surjective homomorphism

$$\phi_A \colon N_G(A) \to M_0,$$

where M_0 is a nonsplit extension of A by $\mathrm{GL}(4,2)$, and that M_0 may be viewed as a subgroup of $\overline{G}_0 := \mathrm{Co}_3$. From 7.4(c), G_0 is the free amalgamated product $H_0 *_{B_0} K_0$. The universal property of G_0 with respect to $\mathcal{A}_0$ yields a homomorphism $\lambda \colon G_0 \to \overline{G}_0$. Then $M\lambda = M_0$, and we may choose ϕ_A to be $\lambda|_M$. For any $A' \in A^G$, choose $g \in G$ with $A' = A^g$ and let $\phi_{A'} \colon N_G(A') \to M_0$ be the homomorphism given by $c_{g^{-1}}\phi_A$. Then $\ker(\phi_{A'})$ does not depend on the choice of the conjugating element g. Set

$$\mathbf{X} = \bigcup_{g \in G} \ker(\phi_A)^g,$$

and for any $P \in \mathcal{F}^c$ define a subset $\theta(P)$ of $C_G(P)$ by

$$\theta(P) = C_{\mathbf{X}}(P)O(C_G(P)).$$

Thus, $\theta(P)$ is a union of cosets of the largest normal subgroup of odd order in $C_G(P)$. For $P \in \mathscr{F}^c_\sigma$, set $\mathbf{X}_\sigma = X \cap G_\sigma$, and

$$\theta_\sigma(P) = C_{\mathbf{X}_\sigma}(P) O(C_G(P))_\sigma.$$

For $P \in \mathscr{F}^c_0$ set $\mathbf{X}_0 = \mathbf{X} \cap G_0$ and

$$\theta_0(P) = C_{\mathbf{X}_0}(P).$$

Write θ_D for θ_0, θ_σ, when $D = G_0$, G_σ, or G, respectively.

Recall from 7.2 that, for any vertex γ of Γ, the largest normal 2-subgroup of G_γ is denoted $Z(\gamma)$.

LEMMA 8.5. *Let $x \in \mathbf{X}$, and let $A' \in A^G$ with $x \in C_G(A')$. Denote by $\Lambda(x)$ the intersection of all the x-invariant subtrees of Γ, set $E(x) = \langle Z(\gamma) \mid \gamma \in \Lambda(x)\rangle$, and denote by $G_{\Lambda(x)}$ the vertex-wise stabilizer of $\Lambda(x)$ in G. Then the following hold.*

(a) *$\Gamma_x = \varnothing$, and x induces a hyperbolic isometry on Γ.*

(b) *$E(x) \leq A'$, and $|A' : E(x)| \leq 2$.*

(c) *Let $\{\gamma, \delta\}$ be an edge of $\Lambda(x)$. Then $G_{\Lambda(x)} = C_{G_\gamma \cap G_\delta}(E(x))$.*

(d) *If $E(x) \neq A'$, and $\{H, K\}$ is an edge of $\Lambda(x)$, then $G_{\Lambda(x)}$ is a B-conjugate of $T\langle w_0\rangle$.*

Proof. By the definition of $\mathbf{X}$ we have $x \in \ker(\phi_{A^*})$ for some $A^* \in A^G$. By 7.10(d), x fixes no vertices (and inverts no edges) of Γ. That is, x induces a hyperbolic isometry of Γ, in the sense of Section 3, and we have (a). Then 3.2 shows that $\Lambda(x)$ is a linear subtree of Γ, on which x acts as a translation. Since $\Lambda(x)$ is contained in every x-invariant subtree of Γ, and since x centralizes A', we have $\Lambda(x) \subseteq \Gamma_{A'}$. Then 7.2 implies that A' centralizes $E(x)$. Since A' contains every 2-element in $C_G(A')$, by 7.10(d), we then have $E(x) \leq A'$.

Let $(\delta_0, \delta_1, \delta_2)$ be a geodesic in $\Lambda(x)$ with $\delta_1 \in \Gamma_1$. Then $\delta_2 = \delta_0 h$ for some h in $G_{\delta_1} - G_{\delta_0}$, and so also $Z(\delta_0)^h = Z(\delta_2)$. Since $B = N_H(U)$, we have $Z(\delta_0) \neq Z(\delta_2)$, and since $Z(\delta_0)Z(\delta_2) \leq E(x)$ we conclude that $|E(x)| \geq 8$. This yields (b). As $G_{\Lambda(x)}$ centralizes $Z(\alpha)$ for each $\alpha \in \Gamma_1 \cap \Lambda(x)$, we get $G_{\Lambda(x)} \leq J := C_{G_{\gamma,\delta}}(E(x))$. Conversely $J \leq G_{\Lambda(x)}$ by 3.10, and this proves (c).

Suppose that $E(x) \neq A'$ and that $\{H, K\}$ is an edge of $\Lambda(x)$. Since $\mathscr{E}_4(B, U) = A^B$ by 7.9(b), we may assume that $A' = A$. By 7.9(c), all hyperplanes of A containing U are fused in $N_B(A)$, so we may assume also that $E(x) = E$. Then (c) yields $G_{\Lambda(x)} = C_B(E)$, and now (d) follows from 4.3(d). □

Our aim is to show that θ_D is a signalizer functor on $\mathscr{D}$, as defined in 2.5. The key to this is the next result.

LEMMA 8.6. *The following hold.*

(a) $C_{\mathbf{X}}(A) = \ker(\phi_A)$.

(b) $C_{\mathbf{X}}(S_\infty) = C_{\mathbf{X}}(T_2) \subseteq C_{\ker(\phi_A)}(T_2)O(T)$.

Proof. Let $x \in C_{\mathbf{X}}(A)$, and suppose that $x \notin \ker(\phi_A)$. Then $x \in \ker(\phi_{A'})$ for some $A' \in \mathscr{E}_4(G) - \{A\}$. Let $\Lambda(x)$ and $E(x)$ be as in 8.5. Then 8.5 yields $\Lambda(x) \subseteq \Gamma_A \cap \Gamma_{A'}$, $E(x) \leq A \cap A'$, and $|E(x)| = 8$. Since $N_G(A)$ is edge-transitive on $\widetilde{\Gamma}_A$, we may assume that $\Lambda(x)$ contains the edge $\{H, K\}$. By 8.5(d) and as T is transitive on $A^G \cap C_B(E)$, we may then assume also that $G_{\Lambda(x)} = T\langle w_0\rangle$. Thus $T\langle w_0\rangle$ is x-invariant, and then so is T. We then have

$$(1) \qquad x \in N_G(T) \cap C_G(A) = N_G(T) \cap \ker(\phi_A)A = N_{\ker(\phi_A)}(T)A.$$

Since $A = E\langle w_0\rangle$, we have $N_T(A) = T_2$. Then

$$(2) \qquad N_{\ker(\phi_A)}(T) \leq N_{\ker(\phi_A)}(T_2) = C_{\ker(\phi_A)}(T_2),$$

since $\ker(\phi_A)$ is invariant under T_2 and intersects T_2 trivially.

Since all involutions in $T\langle w_0\rangle - E$ are fused by T, there exists $t \in T$ such that $(A')^t = A$. Then $x^t \in \ker(\phi_A)$. By (1) and (2), $x = ga$ for some $g \in C_{\ker(\phi_A)}(T_2)$ and $a \in A$. By 7.13(d) we have $g^t \in C_{\ker(\phi_A)}(T_2)O(T)$, so that $x^t a^t = g^t = ky$ for some $k \in \ker(\phi_A)$ and $y \in O(T)$. Thus $ya^t = k^{-1}x^t \in \ker(\phi_A)$, and then since ya^t is of finite order and $\ker(\phi_A)$ is torsion-free, we have $ya^t = 1$. Therefore $y = a = 1$, and $x = g \in \ker(\phi_A)$, contrary to our choice of x. This contradiction proves (a).

Let $x \in C_{\mathbf{X}}(S_\infty)$. Then there exists $A' \in A^G$ with $x \in C_{\ker(\phi_{A'})}(S_\infty)$. As in the proof of (a), x induces a hyperbolic isometry of Γ, and $\Lambda(x)$ is contained in $\Gamma_{S_\infty} \cap \Gamma_{A'}$. Setting $F = \langle Z_\delta \mid \delta \in \Lambda(x)\rangle$, we find that $F \leq C_G(S_\infty)$, and since $N_G(S_\infty)$ is edge-transitive on Γ_{S_∞}, $F^g \leq C_H(S_\infty)$ for some $g \in N_G(S_\infty)$. Since all involutions in $C_H(S_\infty)$ are contained in E, we conclude that $F = F^g = E$. Since $\Lambda(x) \subseteq \Gamma_{A'}$ we get $[E, A'] = 1$, and then $E \leq A'$ since $A' = O^{2'}(C_G(A'))$. Again by the edge-transitivity of $N_G(S_\infty)$ on Γ_{S_∞}, we have $A'^g \leq C_H(E) = T^*$, for some $g \in N_G(S_\infty)$. Then $A'^{gt} = A$ for some $t \in T$, and so

$$C_{\ker(\phi_{A'})}(S_\infty) = C_{\ker(\phi_A)}(S_\infty)^{(gt)^{-1}} \leq C_{\ker(\phi_A)}(S_\infty)O(T),$$

by 7.13(d). This completes the proof of (b). □

PROPOSITION 8.7. $\mathbf{X}_0 = \bigcup \ker(\phi_A)^{G_0}$.

Proof. Let $x \in \mathbf{X}_0$. By definition, there exists $A' \in \mathscr{A}^G$ with $x \in \ker(\phi_{A'})$. Since $x \in G_0$, 8.5(a) implies that the axis $\Lambda(x)$ is contained in $G_{A'} \cap \Gamma_0$. Set $E' = Z(\Lambda(x))$. Then 8.5(b) yields $E' \leq A'$. We have $Z = Z(S_0)$, and $\langle Z^{K_0}\rangle = U$, so that $U \leq G_0$. Since $G_0 = H_0 *_{B_0} K_0$, G_0 is edge-transitive on Γ_0, and so $Z(\gamma) \leq G_0$ for any $\gamma \in \Gamma_0$. Thus, $E' \leq G_0$.

Denote by $\mathscr{E}^*$ the set of elementary abelian 2-subgroups F of G_0 such that $F = \langle Z^{G_0} \cap F \rangle$, and by $\mathscr{E}_n^*$ the set of all $F \in \mathscr{E}^*$ with $|F| = 2^n$. By construction, $S_0 := S \cap B_0$ is a Sylow 2-subgroup of G_0, and then 5.9 implies that for any n, all members of $\mathscr{E}_n^*$ are fused in G_0. We have $A \in \mathscr{E}_4^*$ by 7.10, and evidently $E' \in \mathscr{E}^*$. If $E' = A'$ we conclude that $A' \in A^{G_0}$, and there is nothing more to show. Thus, we may assume henceforth that E' is a proper subgroup of A', and then 8.5(b) yields $|E'| = 8$. Moreover, E' is conjugate to E in G_0, since G_0 is transitive on $\mathscr{E}_3^*$.

Since G_0 is edge-transitive on Γ_0 we may assume that $\{H, K\}$ is an edge of $\Lambda(x)$. Then 8.5(d) implies that $G_{\Lambda(x)}$ is conjugate in B to $T\langle w_0 \rangle$. Let T' be the abelian subgroup of index 2 in $G_{\Lambda(x)}$. By 8.5(c), $[T', E'] = 1$, so that $E' \leq T'$ and $G_{\Lambda(x)} = C_B(E') = T'A'$. Let R be the Sylow 2-subgroup of T', and set $N = N_G(R)$. Then 7.13 yields

$$C_N(E') = (O(T')C_{\ker(\phi_{A'})}(R) \times R)A',$$

and $C_N(A') = C_{\ker(\phi_{A'})}(R)A'$. Since $x \in C_N(A')$, 8.6(a) now yields $x \in C_{\ker(\phi_{A'})}(R)$. Thus, x centralizes the Sylow 2-subgroup RA' of $T'A'$. Since $O(T')C_{\ker(\phi_{A'})}(R)$ contains no nontrivial 2-elements, it follows that:

(1) x centralizes every 2-subgroup of $C_N(E')$ that x normalizes.

By 5.8(a), there is a surjective homomorphism $G_0 \to \mathrm{Co}_3$ whose kernel intersects B_0 trivially. Then $C_{B_0}(E')$ is isomorphic to a subgroup of $C_{\mathrm{Co}_3}(E')$. Since E' is conjugate to E in G_0, and since $C_{\mathrm{Co}_3}(E)$ is of order 2^7, we conclude that $C_{B_0}(E')$ is a 2-group. Since $C_{B_0}(E') = G_0 \cap G_{\Lambda(x)}$, (1) now yields:

(2) x centralizes $C_{B_0}(E')$.

If there exists $F \in \mathscr{E}_4^*$ with $F \leq C_{B_0}(E')$ then $x \in \ker(\phi_F)$ by (2) and 8.6(a). Thus, we may assume:

(3) $C_{B_0}(E')$ contains no member of $\mathscr{E}_4^*$.

Set $\Sigma = \Gamma_0 \cap \Gamma_{E'}$. Then $\widetilde{\Sigma}$ is a subtree of Γ_0 containing $\Lambda(x)$. For any $d > 0$ denote by $\Lambda^{(d)}$ the subtree of $\widetilde{\Sigma}$ induced on the set of vertices of $\widetilde{\Sigma}$ at distance at most d from $\Lambda(x)$, and set $Z^{(d)} = Z(\Lambda^{(d)})$. Thus $E' = Z^0 \leq G_{\Lambda(x)}$, and we claim that $Z^{(d)} \leq G_{\Lambda(x)}$ for all $d \geq 0$. Suppose false, and let d be minimal subject to the condition that, for some vertex γ of $\widetilde{\Sigma}$ at distance d from $\Lambda(x)$, we have $Z(\gamma) \not\leq G_{\Lambda(x)}$. Then $Z^{(d-1)} \leq G_{\Lambda(x)}$, and thus $Z^{(d-1)} \leq G_0 \cap B = B_0$. Now (3) yields $Z^{(d-1)} = E'$. Notice that E' centralizes $Z(\gamma)$ since E' fixes every edge in Γ_0 at every vertex of $\widetilde{\Sigma}$. Thus $Z(\gamma)$ centralizes $Z^{(d-1)}$. Arguing as in the proof of 8.5(c), it follows that $Z(\gamma)$ fixes every vertex of $\Lambda^{(d-1)}$, and thus $Z^{(d)} \leq G_{\Lambda(x)}$ as claimed.

It now follows that $Z(\widetilde{\Sigma}) \leq C_{B_0}(E')$, and then (3) yields $Z(\widetilde{\Sigma}) = E'$. On the other hand, since E' is fused to E in G_0 there exists $F \in \mathscr{E}_4^*$ with $E' \leq F$.

Then $F \in A^{G_0}$. Set $\Theta = \widetilde{\Gamma}_F \cap \Gamma_0$. Since $\Theta \subseteq \widetilde{\Sigma}$ and $Z(\widetilde{\Sigma}) = E'$, we conclude that $Z(\Theta) = E'$, and hence $E' \trianglelefteq N_G(F)$. This is contrary to 7.10(c), and the proposition is thereby proved. □

THEOREM 8.8. *θ_D is a signalizer functor on $\mathfrak{D}$.*

Proof. Let $P \in \mathfrak{D}^c$ and set $Y = \mathbf{X} \cap D$. We first verify that $\theta_D(P)$ is a complementary subgroup to $Z(P)$ in $C_D(P)$. This is the case if $|Z_P| \leq 4$, by 8.4, so assume that $|Z_P| \geq 8$. Suppose that $\mathscr{E}_4(P) \neq \varnothing$ and choose $F \in \mathscr{E}_4(P)$. Then every subgroup of P which contains F is in $\mathfrak{D}^c$, by 7.3(d). We have $F \in A^G$ by 7.9, so that 7.10(d) yields $C_G(F) = F \times \ker(\phi_F)$. Since $\ker(\phi_F) = C_X(F)$ by 8.6(a), $C_Y(F)$ is a subgroup of $C_D(F)$, and we get

$$C_D(F) = F \times C_Y(F).$$

In particular $O(C_D(F)) = 1$, so $C_Y(F) = \theta_D(F)$ by definition. Thus $\theta_D(F)$ is a complement to F in $C_D(F)$.

Let P_1 be maximal in P subject to the conditions: $F \leq P_1$ and $C_D(P_1) = Z(P_1) \times \theta_D(P_1)$. If $P_1 \neq P$ then $P_1 < P_2 := N_P(P_1)$ and $C_D(P_2) \leq C_D(P_1)$. Both $Z(P_1)$ and Y are P_2-invariant, so that P_2 also acts on $C_Y(P_1) = \theta_D(P_1)$. Thus

$$C_D(P_2) = C_{Z(P_1)}(P_2) \times C_{\theta_D(P_1)}(P_2) = Z(P_2) \times \theta_D(P_2),$$

contrary to the maximality of P_1. Thus $P_1 = P$ and $\theta_D(P)$ is a complement to $Z(P)$ in $C_D(P)$.

On the other hand, suppose that $\mathscr{E}_4(P) = \varnothing$. Then $P = D \cap S_\infty$ by 8.3(c). Set $I = C_{\ker(\phi_A)}(T_2)$. Then $C_G(P) = T \times I$ by 7.13(d). Since $\ker(\phi_A) \leq G_0 \leq G_\sigma$, we have $C_D(P) = (D \cap T) \times I$. If $D = G_\sigma$ then $D \cap T = T_\sigma = P \times O(C_D(P))$, while if $D = G_0$ then $D \cap T = T_2$. Thus $\theta_D(P) = O(C_D(P))I$ is a complement to P in $C_D(P)$ for any $P \in \mathfrak{D}^c$.

Evidently $\theta_D(P^g) = \theta_D(P)^g$ for any $g \in N_D(P, S)$, so that by 2.6 it remains to show, for any $Q \in \mathfrak{D}^c$ with $P \leq Q$, that $\Delta(Q) \leq \Delta(P)$. Since $C_Y(Q) \subseteq C_Y(P)$, this amounts to showing that $O(C_D(Q)) \leq O(C_D(P))$. But $O(C_D(P)) = O^2(C_D(P))$, by 7.4 if $|Z_P| \leq 4$, and by 7.6 if $P = D \cap S_\infty$, while in all other cases we have just seen that $O^2(C_D(P)) = 1$. Thus $O(C_D(Q)) \leq O(C_D(P))$ as required. □

9. **Saturation and Theorem A**

We continue the notation that was introduced at the start of Section 8.

PROPOSITION 9.1. *Let $D \in \{G_0, G_\sigma, G\}$. Then the fusion systems $\mathfrak{D}_H$ and $\mathfrak{D}_K$ are saturated.*

Proof. If $D = G_0$ or G_σ then H_D and K_D are finite, and the result is then immediate from 1.6(c). In the case that $D = G$ we appeal to Remark 6.4. $\square$

Until further notice, we take $D \in \{G, G_\sigma\}$.

PROPOSITION 9.2. *We have* $\mathfrak{D}_H = \mathcal{F}_{S_D}(C_D(Z))$ *and* $\mathfrak{D}_K = \mathcal{F}_{S_D}(N_G(U))$.

Proof. Supposing first that $\mathfrak{D}_H = \mathcal{F}_{S_D}(C_D(Z))$, we show that

$$\mathfrak{D}_K = \mathcal{F}_{S_D}(N_G(U)),$$

as follows. Let $g \in N_G(U)$ and let $P \le S$ with $Q := P^g \le S$. Then $(PU)^g = P^g U \le S$, and we may therefore take $U \le P$. Since $\mathrm{Aut}_K(U) = \mathrm{Aut}(U)$, we may write $g = g'k$ with $g' \in C_G(U)$ and $k \in K$. Set $Q' = P^{g'}$. Then $Q' \le C_K(Z) = B$, so by 8.1 we may choose g' so that $Q' \le S$. By assumption, we then have $c_{g'} = c_h$ for some $h \in N_H(P, Q')$ with $U^h = U$. Then $h \in N_H(U) \le K$, and $c_g = c_h c_k = c_{hk}$ where $hk \in K$. We are therefore reduced to proving that $\mathfrak{D}_H = \mathcal{F}_{S_D}(C_D(Z))$.

Suppose that $\mathfrak{D}_H \ne \mathcal{F}_{S_D}(C_D(Z))$. By 7.5 and 3.14(c) there then exist $P \le S_D$, $F \in \mathcal{E}_3(Z(P), U)$, and $g \in N_{C_D(Z)}(P, S)$, such that

(1) $c_g \notin \mathrm{Hom}_{H_D}(P, S)$,

(2) $U \le F \cap F^g$, and

(3) $C_{B_D}(F)^g \le B_D$ and $C_{B_D}(F^g) \le (B_D)^g$.

Suppose that both F and F^g are conjugate to E in B_D, and choose elements b, b' in B_D with $F^b = E = F^{gb'}$. Set $g' := b^{-1}gb'$. From the first statement in (3), $C_{B_D}(E)^{g'} \le C_{B_D}(E)$, and thus g' normalizes $S_D \cap T$ by 4.9(c). We may then adjust b' in $T \cap D$ so that g' normalizes $(S_D \cap T)\langle w_0 \rangle = C_{S_D}(E)$. Set $N = N_D(C_{S_D}(E))$. It follows from 7.13(f) that $N \cap H$ controls strong fusion in $C_N(Z)$, and so there exists $t \in N \cap H$ with $c_t = c_{g'}$ on P^b. Then $c_g = c_{bt(b')^{-1}}$ on P, contrary to (1).

We may therefore assume that either F or F^g is not conjugate to E in B_D. Then 7.8 yields $D = G_\sigma$, and every member of $\mathcal{E}_3(B_D, U)$ is fused in B_D to E or to E', where E' is as defined in 7.8(b). Suppose that F is fused to E and that F^g is fused to E' in B_D. A Sylow 2-subgroup of $C_{B_\sigma}(F^g)$ is elementary abelian, by 7.8(d), whereas a Sylow 2-subgroup of $C_{B_\sigma}(F)$ has exponent 4, contrary to (3). Similarly if F is conjugate to E' then F^g is not conjugate to E. Thus we are reduced to the case where both F and F^g are fused to E' in B_σ, and hence in B_σ^0, by 7.8(b).

Let $b, b' \in B_\sigma^0$ with $F^b = E' = F^{gb'}$. Then $b^{-1}gb' \in N_D(E')$. By 7.8(e) there exists $h \in H_\sigma$ such that $c_h = c_{b^{-1}gb'}$ as elements of $\mathrm{Aut}_D(E')$. Then $c_g = c_{bh(b')^{-1}}$ in $N_D(F, F^g)$, and so $P \ne F$ by (1). Since $P \le C_S(F)$ and EE' is a Sylow 2-subgroup of $C_{H_\sigma}(E')$, we conclude that P^b, $P^{gb'}$, and $A' = EE'$ are Sylow in $C_{H_\sigma}(E')$. Thus there exist $a, a' \in H_\sigma$ with $P^a = A' = P^{ga'}$. Then

$a^{-1}ga' \in N_D(A')$. Observe next that A' is in the set $\mathscr{E}$ defined just prior to 7.11, so that 7.11 implies there exists $h \in H_\sigma$ with $c_h = c_{a^{-1}ga'}$ as elements of $\mathrm{Aut}_D(A')$. Then $c_g = c_{ah(a')^{-1}}$ on P, contrary to (1). $\square$

LEMMA 9.3. *Let $P \leq S_D$, and assume that there exists no $N_D(P)$-invariant subgroup X of Z_P with $|X| = 2$ or 4. Then one of the following holds.*

(a) $P \in E^D \cup A^D$.

(b) *$D = G_\sigma$ and P is D-conjugate to the group E' defined in* 7.8(b).

(c) *$P \leq C_S(E)$, and either $S_\infty \leq P$ or $P \cap T$ is homocyclic of rank* 3 *and exponent at least* 4.

Proof. By hypothesis, $|Z_P| \geq 8$. Suppose first that $|Z_P| > 8$. Then $P \in A^G$ by 7.9, and if $P \notin A^D$ then 7.11(c) and 7.12 show that there is an $N_D(P)$-invariant subgroup of P of order 2. Thus $P \in A^D$, and (a) holds in this case. Also, if $P \in E^G - E^D$ then 7.8 yields (b). We may therefore assume that P is not elementary abelian. Then $|Z_P| = 8$, and $Z \leq Z_P$.

By 7.7 we have $(Z_P)^h = E$ for some $h \in H$, and by 8.1 we may choose h so that $P^h \leq S$. Let P_0 be the group generated by the noninvolutions in P. Since $C_S(E) - S_\infty$ consists entirely of involutions, P_0 is a characteristic abelian subgroup of P, of index at most 2, containing Z_P. Since P has no characteristic subgroups of order 2 or 4, it follows that either $P = S_\infty$ or that P_0 is homocyclic. Since $P \notin \mathscr{E}(S)$, P contains a conjugate of T_2, and then 4.9(c) implies that $P_0 = T_n$ for some $n \geq 2$ or $P_0 = S_\infty$. Thus, (c) holds. $\square$

LEMMA 9.4. *Let $P \leq S_D$ and let X be a nonidentity $N_D(P)$-invariant subgroup of Z_P of minimal order. Then $N_D(P)$ acts transitively on $X^\#$, and the following hold.*

(a) *There exists an element $f = hkh'$ of D, with $h, h' \in H_D$ and $k \in K_D$, such that $N_{S_D}(P)^f \leq S_D$ and such that either $X^f \in \{Z, U, E, A\}$, or $D = G_\sigma$ and $X^f = E'$, where E' is as defined in* 7.8(b).

(b) *If P is fully normalized in $\mathfrak{D}$ then so is P^f, for any f as in* (a).

Proof. We first show that $N_D(P)$ acts transitively on $X^\#$. This is trivial if $|X| = 2$, and is immediate from the minimality of X if $|X| = 4$. Suppose that $|X| = 8$. If $P \in \mathscr{E}_4(S_D)$ then 7.12 and 7.11(c) show that $\mathrm{Aut}_D(P)$ leaves no maximal subgroup of P invariant, while if $X = P \in \mathscr{E}_3(D)$ then 7.7 and 7.8(e) show that $N_D(P)$ acts transitively on $P^\#$. Thus, we may assume that P is not elementary abelian, and then 9.3 yields $P \leq C_S(E)$, and either $P = S_\infty$ or $P \cap T$ is homocyclic of rank 3 and exponent at least 4. Then $X = E$, and 7.13 implies that $N_D(P)$ induces the full automorphism group of X.

We next prove (a). Let $1 \neq x \in C_X(N_{S_D}(P))$ and suppose that $x \neq z$. By 3.13(b) there exist elements $h \in H$ and $k \in K$ such that $x^{hk} = z$ and such that $N_{S_D}(P)^{hk} \leq S_D$. Thus, we may assume that $z \in X$, after replacing P by a suitable conjugate. Then, by 7.6 through 7.9, there exists $h' \in H_D$ with $X^{h'} \in \{Z, U, E, A\}$, or else $D = G_\sigma$ and there exists $h' \in H_D$ with $X^{h'} \in \{E', A_z\}$. If $X^{h'} = A_z$ then 7.11(c) contradicts the minimality of X, so that this case does not arise in our context.

Set $Y = X^{h'}$. In order to complete the proof of (a) it suffices to show that, in each case, every 2-subgroup of $N_{H_D}(Y)$ is fused into $N_{S_D}(Y)$ in $N_{H_D}(Y)$. But in each case we have $N_{S_D}(Y) \in \mathrm{Syl}_2(N_{H_D}(Y)$, and so the required fusion follows from Sylow's Theorem, or from 6.3 and 6.4 when $D = G$ and $N_S(Y)$ is infinite. Thus, (a) holds.

Now suppose that P is fully normalized in $\mathfrak{D}$, let f be given as in (a), and set $Q = P^f$. Let $R = Q^g$ be a $\mathfrak{D}$-conjugate of Q contained in S. Then $R = P^{fg}$, and since P is fully normalized there exists $d \in G$ such that $R^d = P$ and $N_S(R)^d \leq N_S(P)$. Also, as P is fully normalized there exists $d' \in G$ with $Q^{d'} = P$ and with $N_S(Q)^{d'} \leq N_S(P)$. Since $N_S(P)^f \leq N_S(Q)$ we conclude that $N_S(Q)^{d'} = N_S(P)$, and $N_S(R)^{dd'^{-1}} \leq N_S(Q)$. Since $R^{dd'^{-1}} = P$, we have thus shown that Q is fully normalized, proving (b). □

LEMMA 9.5. *Let $P \leq S_D$ be fully normalized in $\mathfrak{D}_H$. Assume that there exists a minimal, nonidentity, $N_D(P)$-invariant subgroup X of Z_P with $Z \leq X$. Then P is fully normalized in $\mathfrak{D}$.*

Proof. Let $g \in N_D(P, S_D)$ and set $Q = P^g$ and $Y = X^g$. There then exists $y \in Y^\#$ such that $[y, N_{S_D}(Q)] = 1$, and since $N_D(Q)$ acts transitively on $Y^\#$, by 9.4, we may assume that $y = z^g$. By 3.13(b) there exist elements $h \in H_D$ and $k \in K_D$ such that $y^{hk} = z$ and such that $N_{S_D}(P)^{hk} \leq S_D$. Set $d = hk$ and $R = Q^d$. As $gd \in C_D(Z)$ it follows from 9.1 and 9.2 that there exists $h' \in H_D$ with $R^{h'} = P$ and with $N_S(R)^{h'} \leq N_S(P)$. Then $Q^{dh'} = P$ and $N_S(Q)^{dh'} \leq N_S(P)$. This shows that P is fully normalized in $\mathfrak{D}$. □

LEMMA 9.6. *Let P and Q be subgroups of S, and let $x, y \in G$ such that $P^x \leq Q$ and $Q^y \leq P$. Then $P^x = Q$ and $Q^y = P$.*

Proof. The map $c_{xy}\colon P \longrightarrow P$ is injective, so by 6.4 and 6.2(7), c_{xy} is an isomorphism. Thus $P = P^{xy} = Q^y$, and similarly $P^x = Q$. □

THEOREM 9.7. *$\mathfrak{D}$ is saturated.*

Proof. Let $1 \neq P \leq S_D$ and let X be a minimal, nonidentity, $N_D(P)$-invariant subgroup of Z_P. By 9.4 there is $g \in D$ with $N_{S_D}(P)^g \leq S_D$ and with $Z \leq X^g$. Adjusting in H and appealing to 9.1, we may assume that $Q := P^g$ is fully normalized in $\mathfrak{D}_H$. Then Q is fully normalized in $\mathfrak{D}$, by 9.5, and thus $\mathfrak{D}$ satisfies the saturation condition (I) in 1.5. It remains to verify the two parts of condition (II). Thus we may take P to be fully normalized in $\mathfrak{D}$.

By 9.4 there exists a fully normalized conjugate $P' = P^y$ of P such that $X' := X^y$ is in $\{Z, U, E, E', A\}$. Then y can be chosen so that $N_S(P)^y \leq N_S(P')$, and similarly (since P is fully normalized) there exists $y' \in N_D(P', P)$ such that $N_S(P')^{y'} \leq N_S(P)$. By 9.6, $N_S(P)^y = N_S(P')$, so that if P' satisfies saturation condition (II) then so does P. Thus, we may assume that $X \in \{Z, U, E, E', A\}$ and, in particular, that $Z \leq X$. Also, since $\mathfrak{D}_H$ is saturated there exists a $\mathfrak{D}_H$-conjugate P'' of P which is fully normalized in $\mathfrak{D}_H$, and hence in $\mathfrak{D}$ by 9.4(b); the preceding argument then shows that $N_S(P)$ and $N_S(P'')$ are $\mathfrak{D}$-isomorphic. We may therefore assume that P is fully normalized in $\mathfrak{D}_H$. Then P is fully normalized in $\mathcal{F}_{S_D}(C_D(Z))$, by 9.2.

If $X \in \{Z, U\}$ then fusion in $N_D(P)$ is controlled by fusion in $N_{H_D}(P)$ or in $N_{K_D}(P)$, by 9.2, and since $\mathfrak{D}_H$ and $\mathfrak{D}_K$ are saturated, there is nothing more to prove in these cases. Thus, we may assume that $|X| \geq 8$. By 9.3 we then have $P = X \in \{E, E', A\}$, or else $P \leq C_{S_D}(E)$ with $P \cap T = S_\infty$ or with $P \cap T$ homocyclic and of exponent at least 4. In the first case, where P is elementary abelian, we have $\mathrm{Aut}_{S_D}(P) \in \mathrm{Syl}_2(\mathrm{Aut}_D(P))$ by 7.8 and 7.10. In the second case we obtain $\mathrm{Aut}_{S_D}(P) \in \mathrm{Syl}_2(\mathrm{Aut}_D(P))$ from 7.13. This establishes the saturation condition (IIA).

Now let $\alpha = c_g \in \mathrm{Aut}_{\mathfrak{D}}(P)$, where $g \in N_D(P)$. Set $Z' = Z^{g^{-1}}$. By definition, we have $N_\alpha^g \leq C_D(P)S_D$. Thus $N_\alpha \leq C_{S_D}(Z')$. By 3.13(b) there exists $a \in N_D(N_\alpha, S_D)$ with $(Z')^a = Z$. Set $Q = P^a$ and $b = a^{-1}g$. Then $b \in C_D(Z)$ and $Q^b = P$. As P is fully normalized in $\mathcal{F}_{S_D}(C_D(Z))$, it follows from the "standard" axioms for saturation in [BLO03] – equivalent to those in 1.5 – that $\beta := c_b$ extends to a $\mathfrak{D}$-morphism $\bar{c}_b$ of N_β into S_D. That is, there exists $d \in N_D(N_\beta, S_D)$ such that d^{-1} centralizes Q. Set $\bar{g} = ad$. Then

$$g^{-1}\bar{g} = g^{-1}ad = g^{-1}gb^{-1}d = b^{-1}d,$$

and so $g^{-1}\bar{g}$ centralizes P. But also $(N_\alpha)^a \leq N_\beta$ since

$$(N_\alpha)^{ab} = (N_\alpha)^{a^{-1}ag} = (N_\alpha)^g,$$

and $\mathrm{Aut}_{N_\alpha}(P)^{c_g} \leq \mathrm{Aut}_{S_D}(P)$ by definition of N_α. Now

$$(N_\alpha)^{\bar{g}} = (N_\alpha)^{ad} \leq (N_\beta)^g \leq S_D,$$

and thus $c_{\bar{g}}$ is an extension of c_g from P to a $\mathfrak{D}$-morphism of N_α into S_D. This shows that $\mathfrak{D}$ satisfies the saturation condition (IIB), and the proof is thereby complete. □

In 2.6 it was shown that if $\mathfrak{D}$ is saturated then a $\mathfrak{D}$-signalizer functor θ_D determines an associated centric linking system and an associated p-local group

$\mathcal{G}_{S_D,\theta_D}(D) = (S_D, \mathcal{F}_{S_D}(D), \mathcal{L}_{\theta_D})$. Write $\mathcal{G}_0 = (S_0, \mathcal{F}_0, \mathcal{L}_0)$, $\mathcal{G}_\sigma = (S_\sigma, \mathcal{F}_\sigma, \mathcal{L}_\sigma)$, $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, for $\mathcal{G}_{S_D,\theta_D}(D)$, when D is G_0, G_σ, G, respectively. Theorems 8.8 and 9.7 therefore have the following immediate corollary:

COROLLARY 9.8. *$\mathcal{G}_\sigma$ is a 2-local finite group, and $\mathcal{G}$ is a 2-local group.* □

THEOREM 9.9. *For any fixed p, $p \equiv 3$ or 5 mod 8, and for any integer of the form $q = p^{2^n}$, there is a unique $\sigma = \psi_n$ such that the fusion system $\mathcal{F}_{\mathrm{Sol}}(q)$ constructed in* [LO02] *is isomorphic to $\mathcal{F}_\sigma$. Moreover the associated 2-local finite group constructed in* [LO02] *is isomorphic to $\mathcal{G}_\sigma$.*

Proof. The fusion system $\mathcal{E} = \mathcal{F}_{\mathrm{Sol}}(q)$ is constructed as

$$\langle \mathcal{F}_{S_q}(H_q), \mathcal{F}_{S_{U,q}}(K_q)\rangle,$$

where $H_q = H_\sigma$ is $\mathrm{Spin}_7(q)$, $S_q = S_\sigma$ is a Sylow 2-group of H_q, $S_{U,q} = C_{S_\sigma}(U)$, and $K_q = K_\sigma$, subject to a choice of embedding α of $B_q = N_{H_q}(U)$ in K_q such that, for $I = \langle N_{H_q}(S_{U,q}), N_{K_q}(S_{U,q})\rangle$, $\mathrm{Aut}_I(S_{U,q}) \cong \mathrm{GL}(3,2) \times C_2$. By parts (a) and (d) of 9.7, the amalgam $\mathcal{A}_\alpha = (H \xleftarrow{\iota} B \xrightarrow{\alpha^*} K)$ is determined up to isomorphism by these properties, so $\mathcal{A}_\alpha = \mathcal{A}_{\lambda,\sigma} = (H_\sigma > B_\sigma < K_\sigma)$. Thus $\mathcal{E} = \langle \mathcal{F}_{S_\sigma}(H_\sigma), \mathcal{F}_{C_{S_\sigma}(U)}(K_\sigma)\rangle$. By 1.10, $\mathcal{F}_{S_\sigma}(K_\sigma) = \langle \mathcal{F}_{S_\sigma}(S_\sigma), \mathcal{F}_{C_{S_\sigma}(U)}(K_\sigma)\rangle$, so $\mathcal{E} = \langle \mathcal{F}_{S_\sigma}(H_\sigma), \mathcal{F}_{S_\sigma}(K_\sigma)\rangle$, and then $\mathcal{E} = \mathcal{F}_S(G_\sigma) = \mathcal{F}_\sigma$ by 3.7, completing the proof of the first part of the theorem. The remainder of the theorem follows from the uniqueness of the 2-local finite group associated to $\mathcal{E}$, proved in [LO02]. □

To sum up: parts (1) and (2) of Theorem A are given by the construction of G in Section 5, while parts (3) and (4) of Theorem A are given by the preceding theorem. Part (5) is given by 7.9, 7.10, and the definition of **X**. Thus all parts of Theorem A have been proved.

LEMMA 9.10. *$\mathcal{F}$ is* frc*-generated. That is,*

$$\mathcal{F} = \langle A_{\mathcal{F}}(P) \mid P \in \mathcal{F}^{\mathrm{frc}}\rangle,$$

where $\mathcal{F}^{\mathrm{frc}}$ is the set of fully normalized radical centric subgroups in $\mathcal{F}$, and $A_{\mathcal{F}}(P)$ is the fusion system on P whose morphisms are the restrictions of members of $\mathrm{Aut}_{\mathcal{F}}(P)$ to subgroups of P.

Proof. This generalization of Alperin's Fusion Theorem is well known for saturated fusion systems on finite p-groups S; a short proof appears in Theorem A.10 of [BLO03]. A modification of this proof when S is a discrete p-toral group appears in Theorem 3.6 in [BLO05]. We sketch that proof in our special case, where things are much easier.

Pick $P, P' \le S$ and an $\mathcal{F}$-isomorphism $\varphi\colon P \to P'$ such that φ is not in $\mathcal{A} = \langle A_{\mathcal{F}}(P) \mid P \in \mathcal{F}^{\mathrm{frc}}\rangle$. Then $\varphi = c_g$ for some $g \in G$. As in the proof of Theorem A.10 in [BLO03], using the fact that $\mathcal{F}$ is saturated, we may assume

P' is fully normalized in $\mathcal{F}$. Pick P so that $|T_2 \cap P|$ is maximal. If $T_2 \not\leq P$ then $T_2 \cap P < N_{T_2 \cap P}(P)$, and as in the proof of A.10, after we replace P by $N_{T_2}(P)P$, the maximality of $|T_2 \cap P|$ supplies a contradiction.

As $T_2 \leq P$, 4.9(c) says that $g \in N_G(T_2)$. As $S_\infty \trianglelefteq N_G(T_2)$, replacing P, P' by $PS_\infty, P'S_\infty$, we may assume $S_\infty \leq P$. In particular P is centric. Finally choosing $|P : S_\infty|$ maximal, and arguing as in the proof of Theorem A.10 in [BLO03], we first reduce to the case where $g \in N_G(P)$, and then show P is radical, contradicting the assumption that $c_g \notin \mathcal{A}$. □

10. Radical centric subgroups

In this section, we determine the members of $\mathfrak{D}^{\mathrm{rc}}$, and make the necessary preparations for obtaining embeddings among the 2-local groups constructed in the preceding section.

LEMMA 10.1. *Let $Y \in \{H, K\}$, and let $P \leq S$ with $N_D(P) \leq Y$. Then the following are equivalent.*

(a) $P \in \mathfrak{D}^{\mathrm{rc}}$.

(b) $P \in \mathfrak{D}^{\mathrm{rc}}_Y$.

(c) *P contains every 2-element in $C_Y(P)$, and $O_2(\mathrm{Out}_{Y_D}(P)) = 1$.*

Proof. As $C_D(P) \leq Y$, 8.2(a) says that $P \in \mathfrak{D}^c$ if and only if P contains every 2-element in $C_{Y_D}(P)$, and that this holds if and only if $P \in \mathfrak{D}^c_Y$. Thus we may assume $P \in \mathfrak{D}^c$. As $N_D(P) \leq Y$, we have

$$O_2(\mathrm{Aut}_D(P)) = O_2(\mathrm{Aut}_{Y_D}(P)).$$

Thus $P \in \mathfrak{D}^{\mathrm{rc}}$ if and only if $\mathrm{Inn}_D(P) = O_2(\mathrm{Aut}_D(P))$ if and only if $\mathrm{Inn}_D(P) = O_2(\mathrm{Aut}_{Y_D}(P))$ if and only if $P \in \mathfrak{D}^{\mathrm{rc}}_Y$ if and only if $O_2(\mathrm{Out}_{Y_D}(P)) = 1$, completing the proof. □

Recall from 4.5 that B^0 is the commuting product of “components” L_1, L_2, and L_3, where each L_i is isomorphic to $\mathrm{SL}(2, \mathbf{F})$.

LEMMA 10.2. *Let $P \in \mathfrak{D}^{\mathrm{rc}}_K$, set $N = N_{K_D}(P)$, $R = S_D$, $J_i = L_i \cap D$, and $J = J_1J_2J_3$. For any subgroup X of K_D, and any i with $1 \leq i \leq 3$, denote by X_i the projection of $X \cap J$ in J_i. Denote by $\mathfrak{Q}_i$ the set of subgroups of R_i that are isomorphic to the quaternion group $\mathbf{Q}_8$. Then the following hold*:

(a) $C_K(P) \leq P$.

(b) $Z_P \in \{Z, U\}$.

(c) *$P \cap J = P_1P_2P_3$, and for each i with $1 \leq i \leq 3$, either $P_i = R_i$ or $P_i \in \mathfrak{Q}_i$.*

(d) *Either*

(i) $P \in \{C_R(U), R\}$,

(ii) *$P = P \cap J$ and $P_i \in \mathfrak{Q}_i$ for at least two indices i, or*

(iii) *$P = (P \cap J)\langle s\rangle$ for some $s \in P - C_P(U)$, and either $P_3 \in \mathfrak{Q}_3$ or $P_i \in \mathfrak{Q}_i$ for both $i = 1$ and 2. Moreover, if $P_3 \neq R_3$ then $O^2(N \cap J_3) \neq 1$.*

(e) *$P \in \mathcal{F}^c_K$, and every N-invariant 2-subgroup of K is contained in S.*

(f) *If U' is a fours group in K such that $[P, U'] \leq Z$, then $U' = U$.*

Conversely, any subgroup P of S which satisfies (c) *and* (d) *is in $\mathfrak{D}^{\mathrm{rc}}_K$.*

Proof. Set $\mathcal{J} = \{J_i \mid 1 \leq i \leq 3\}$ and set $P_J = P_1P_2P_3$. Since $\mathcal{J}$ is K_D-invariant, P_J is N-invariant, and then $P_J \leq P$ by 8.2(b). Thus $P_i \leq P$ for all i, and $P_J = P \cap J$.

Suppose that $P_i \leq U$ for some i. Then $P^* := \langle P_i^P\rangle \leq U$, and $R^* := \langle R_i^P\rangle$ is a P-invariant 2-group which properly contains P^*. Pick $r \in N_{R^*}(P) - P$ with $r^2 \in U$. Then r centralizes the chain $P \geq U \geq 1$, and since $U \trianglelefteq N$ we conclude from 2.2 that $r \in P$. This contradiction shows that no P_i is contained in U.

Set $N_i^* = N_{J_i}(P_i)$. Then $N^* = N_1^*N_2^*N_3^*$ is N-invariant, so by 8.2(b), $O_2(N_i^*) = P_i$. Let $\mathcal{S}$ be the class of 2-groups each of whose finite subgroups is cyclic. Suppose that $P_i \notin \mathfrak{Q}_i$. There is then a unique maximal subgroup X of P_i in $\mathcal{S}$ of order at least 8. Since $|X| \geq 8$ there is also a unique maximal 2-subgroup V_i of J_i in $\mathcal{S}$ containing X. Then V_i is N_i^*-invariant, and 8.2(b) yields $V_i \leq P_i$. Then also $\mathrm{Aut}_{R_i}(P) \leq O_2(\mathrm{Aut}_{K_D}(P))$, and $P_i = R_i$. Thus, (c) is proved. It follows from (c) that $C_{L_i}(P_i) \leq P_i$, so $C_{B^0}(P_J) \leq P_J$. Since any element of $K - B^0$ acts nontrivially on U, and since $U \leq P_J$, we obtain (a) and (b).

Suppose next that no P_i is a quaternion group. Then for all i we have $P_i = R_i \notin \mathfrak{Q}_i$. If $D = G$ then $P_J = C_S(U)$ is of index 2 in P, and (d)(i) holds. On the other hand, suppose that $D \neq G$. Since $R_i \in \mathfrak{Q}_i$ if $D = G_0$, we conclude that $D = G_\sigma$. By 7.8(a) there exists an element x of S_D such that x induces a diagonal outer automorphism on each J_i. Here $N \cap J = P_J$ and $P_J\langle x\rangle \trianglelefteq S_D$, so as P is $\mathfrak{D}_K$-radical it follows that $x \in P$. Then $P_J\langle x\rangle = C_R(U) \leq P$, and again (d)(i) holds.

If $C_P(U) \neq P_J$ then there exists $x \in C_P(U) - J$. As in the preceding paragraph, $D \neq G$, and for all i either $J_i \cong \mathrm{SL}(2, \mathbf{F}_\sigma)$, or $D = G_0$ and $J_i \in \mathfrak{Q}_i$. As $[N_i, x] \leq P$ for all i, $O^2(N_i)$ is not isomorphic to $\mathrm{SL}(2, 3)$, and $P_i = R_i$ for all i. Thus $P \geq (R \cap J)\langle x\rangle = C_R(U)$. Once again, we obtain (d)(i).

Now assume that (d)(i) does not hold. We conclude from the discussion above that $P_i \in \mathfrak{Q}_i$ for some i and that $P_J = C_P(U)$. Suppose next that $P = P_J$, and that neither P_1 nor P_2 is a quaternion group. Recall that $B = B^0\langle s\rangle$ where $s \in S_0$, s centralizes L_3, and s interchanges L_1 and L_2. One may now check that $N = (N \cap J)\langle s\rangle$ and that $O^2(N \cap J) \leq J_3$, whence $P\langle s\rangle \trianglelefteq N$. As P is $\mathfrak{D}_K$-radical, $s \in P$, contrary to the assumption that $P = P_J$. Thus, either P_1 or P_2 is a quaternion group. Since $K/B^0 \cong \mathrm{Sym}(3)$, a similar argument shows that for any two distinct indices j and k, at least one of P_j and P_k is a quaternion group. Thus (d)(ii) holds in this case.

Now suppose that $P \neq P_J$. As $P_J = C_P(U)$ we then have $P = P_J\langle s\rangle$ for some $s \in R - C_R(U)$. Then s interchanges P_1 and P_2, and since some P_i is a quaternion group we get either $P_3 \in \mathcal{Q}_3$ or $P_i \in \mathcal{Q}_i$ for both $i = 1$ and 2. If $P_3 = R_3$ then (d)(iii) holds. Thus we may assume $P_3 \neq R_3$, so that $P_3 \in \mathcal{Q}_3$. Also $R_3 \notin \mathcal{Q}_3$, so that in particular $D \neq G_0$. Let $\bar{N} = N/P_J$. If $P_1 = R_1$ then $O^2(N) = O^2(C_N(\bar{s})) \leq O^2(N_3^*)$, while if $P_1 \in \mathcal{Q}_1$ then $O^2(N) = P_J\langle d\rangle$ or $P_J\langle d\rangle O^2(N_3^*)$ for a suitable element d of order 3 in $N_1^*N_2^*$. In particular if $O^3(N_3^*)$ does not centralize $\bar{s}$, then $N_{R_3}(P) \leq O_2(N)$, again contrary to $R_3 \neq P_3$ and $P \in \mathcal{H}_D^r$. Thus (d)(iii) holds, completing the proof of (d).

Suppose next that x is a 2-element in $K - U$, such that $[P, x] \leq Z$. If $[P, x] = Z$ assume that x is an involution. Since $C_{L_i}(P_i) = Z(P_i) \leq U$, and all involutions in L_3 are in Z, it follows that $x \notin B^0$. But then x interchanges P_i and P_j for some pair of distinct indices i and j, and so $[P, x] \not\leq Z$. Thus no such x exists. This proves (f), and shows also that $P \in \mathcal{F}_K^c$.

Let F be an N-invariant 2-subgroup of K with $F \not\leq S$. Then FP is a 2-group. For any i, set $S_i = S \cap L_i$, and recall F_i, N_i are the projections of $F \cap B^0$, $N \cap B^0$ on L_i. Then F_i is N_i-invariant. It is a property of the group $\bar{L} = \mathrm{PSL}(2, \mathbf{F})$ that the intersection of any pair of distinct Sylow 2-subgroups is abelian (either cyclic or a fours group), and therefore S_i is the unique Sylow 2-subgroup of L_i containing P_i. Thus $F_i \leq S_i$, and we conclude that $F \cap B^0 \leq S$.

Now $F = (F \cap B^0)\langle t\rangle$, where t acts nontrivially on $\mathcal{L} := \{L_1, L_2, L_3\}$. Thus, there is an ordering $(1', 2', 3')$ of $\{1, 2, 3\}$ such that t interchanges $L_{1'}$ and $L_{2'}$, and fixes $L_{3'}$. Suppose that $P \not\leq B^0$. Then there exists $s \in P$ such that s interchanges L_1 and L_2, and since $\langle s, t\rangle$ is a 2-group it follows that $i = i'$ for all i. Without loss, we may assume that $P \leq F$, so that $st \in F \cap B^0$. But then $t \in S$ and $F \leq S$. Thus $P \leq B^0$, and it follows from (c) and (d) that $N_i \leq N$ for all i.

Suppose that $P \not\leq J$. Then (d) implies that $C_R(U) \leq P$, and since $P \leq B^0$ we get $P = C_R(U)$. The Frattini Argument then implies that there exists an element x of N which permutes the components L_i transitively, and then $\langle x, t\rangle$ is not a 2-group. This shows that $P \leq J$, and a similar argument shows that there exists j with $P_j \neq R_j$.

Since $P_i^t \leq F \cap B^0 \leq S$, we have $P_i^t \leq S \cap S_i^t$, so that $S_i^t \leq S$ for all i. Since $\langle t^{N_i}\rangle$ is a 2-group, we obtain $N_{1'} = R_{1'}$ and $N_{2'} = R_{2'}$. Then $P_{3'} \neq R_{3'}$, and so $N_{3'} \cong \mathrm{GL}_2(3)^+$. Then $N_K(N_{3'}) = C_K(O^2(N))N_{3'}$. Since $N_{3'} \trianglelefteq N$ we have $N = C_N(O^2(N))N_{3'}$, and since $C_N(O^2(N))$ is a 2-group we get $N = N_{3'}P$. Thus, $N = N_1N_2N_3$. On the other hand, there exists an element t' of $C_{K_D}(J_{3'})$ such that t' interchanges $R_{1'}$ and $R_{2'}$. Then $t' \in P$, by 7.1(b), contradicting $P \leq J$, and completing the proof of (e).

It remains to establish the final statement in the lemma. Thus let $P \leq S$ such that P satisfies (c) and (d). Then there exists $Q \leq P$ such that $Q \geq Q_i \cong \mathbf{Q}_8$ for all i, and one checks that $C_K(Q) = U \leq Q$. Then $Q \in \mathfrak{D}^c_K$, and hence also $P \in \mathfrak{D}^c_K$.

Set $N = N_{K_D}(P)$. As $C_K(Q) = U$ we have $O(N) = 1$, and it remains to show that $P = O_2(N)$. If (d)(i) holds, then $P = R$ or $P = C_R(U)$, so that $N = R$ or $N/R \cong \mathrm{Sym}(3)$, and in particular $P = O_2(N)$. Suppose that (d)(ii) holds. If $P = Q$ then it is easy to check that $P = O_2(N)$; so we may assume that $P \neq Q$. Then $P_j = R_j \notin \mathfrak{Q}_j$ for exactly one index j, $N/P \cong \mathrm{Sym}(3) \wr \mathbf{C}_2$, and we are done in this case.

Finally, suppose that (d)(iii) holds, set $P_J = P \cap J$, and let $s \in P - J$. Since $P_J = C_P(U)$ we have $N \leq M := N_{K_D}(P_J)$. Set $\bar{M} = M/P_J$; it follows that N is the preimage in M of $C_{\bar{M}}(\bar{s})$. Thus it suffices to show that

$$(*) \qquad \langle \bar{s} \rangle = O_2(C_{\bar{M}}(\bar{s})).$$

If $R_i \in \mathfrak{Q}_i$ then $D \in \{G_0, G_{\psi_0}\}$, $P_J = Q$, R/Q is a 4-group, and every involution in R/Q has nontrivial fixed points on $O_3(\bar{M})$. Since $N_{K_D}(R) = R$ we conclude that $(*)$ holds in this case. We may therefore assume that $R_i \notin \mathfrak{Q}_i$. Suppose that $P_J = Q$. Then $\bar{M} \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(3)$, and $P_3 \neq R_3$. Then (d)(iii) requires $O^2(N_3) \neq 1$, and hence N contains an element g of J_3 of order 3. Then $|O_3(\bar{N})| = 9$, $\langle \bar{s} \rangle$ is a Sylow 2-subgroup of $C_{\bar{M}}(O_3(\bar{N}))$, and we have $(*)$.

If $P_3 = R_3 \notin \mathfrak{Q}_3$ then P_1 and P_2 are quaternion groups, $\bar{M} \cong \mathrm{Sym}(3) \wr \mathbf{C}_2$, and $\bar{N} \cong \mathbf{D}_{12}$. This yields $(*)$, so we are reduced to the case where $P_i = R_i \notin \mathfrak{Q}_i$ for both $i = 1$ and 2. Then $P_3 \neq R_3$, so $O^2(N \cap J_3) \neq 1$, and $\bar{M} = \bar{N}$ with $\langle \bar{s} \rangle = O_2(\bar{M})$. Again $(*)$ holds, so the proof is complete. □

LEMMA 10.3. *Let $g \in G$ and $R \leq S$ such that $R = R^1R^2R^3$, with $R^i = R \cap L_i^g \cong \mathbf{Q}_8$. Then $U = U^g = Z(R)$, and R is special of order 2^8.*

Proof. For each $i \neq j$ we have $R^iR^j = R^i \times R^j$, so that R is special with center U^g. Thus it remains to show $U = U^g$. Set $\widetilde{S} = S/S_\infty$. Then $\widetilde{S}$ has no $\mathbf{Q}_8$-subgroups, so that $R^i \cap S_\infty \neq 1$, and hence $U^g \leq S_\infty$.

Set $Y = C_R(E)$ and $Y_0 = R \cap S_\infty$, and suppose first that $|R/Y| \leq 2$. Then $|Y_0| \geq 64$, so as R has no elements of order 8 it follows that $Y_0 = T_2$. Since $\Phi(R) = Z(R)$, this is a contradiction and so we conclude that $|R/Y| \geq 4$. Since $U^g \leq Y$, R/Y is elementary abelian, and is then a maximal elementary abelian subgroup of $\mathrm{Aut}_S(E)$. Since the fixed point groups in E for the two maximal elementary abelian subgroups of $\mathrm{Aut}_S(E)$ are Z and U, we conclude $U = U^g$. □

LEMMA 10.4. *Let $P \in \mathfrak{D}^{\mathrm{rc}}$ with $|Z_P| \leq 4$. Then*:

(a) *P contains every P-invariant subgroup of D of order* 4.

(b) $Z_P \in \{Z, U\}$.

Proof. Let F be a P-invariant subgroup of D of order 4. Then $|[F, P]| \le 2$, and so 8.2(a) implies that $[F, P] \le Z_P$. If $[Z_P, F] = 1$ then F centralizes the chain $P \ge Z_P \ge 1$, whence $F \le P$ by 8.2(a) and 2.2. On the other hand, suppose that $[Z_P, F] \ne 1$. Then $|Z_P| = 4$ and $Z_P \not\le \Phi(P)$. By 8.3(b) we have $Z_P = U^h$ for some $h \in H_D$. Then $E^h \le C_D(Z_P)$, and so $P \ne Z_P$. Thus $\Phi(P) \ne 1$, so that $Z_P \cap \Phi(P) = Z$, and F centralizes the chain $P \ge Z_P \ge Z \ge 1$ of characteristic subgroups of P. Thus $F \le P$ by 2.2, and (a) holds.

Now suppose that $Z_P \ne Z$. Then $Z_P^g = U$ for some $g \in H_D$, and $P^g \le C_{H_D}(U) \le K_D$ by 8.3(b). By 6.4 there exists $k \in K_D$ with $P^{gk} \le S_D$, and replacing g by gk, we may assume $P^g \le S_D$. Then $P^g \in \mathcal{D}_K^{\mathrm{rc}}$ by 8.3(b) and 10.1, and then 10.2 implies that P contains a subgroup R satisfying the hypothesis of 9.3, with g^{-1} in the role of g. Then $U = U^{g^{-1}}$ by 10.3, proving (b). □

LEMMA 10.5. *$P \in \mathcal{D}^{\mathrm{rc}}$ with $|Z_P| > 4$ if and only if $P \in A^D$ or $P = C_{S_D}(E)$.*

Proof. Let $P \le S_D$. Suppose that $|Z_P| > 8$. Then $P = C_D(P) = Z_P \in \mathscr{E}_4(S_D)$, by 7.9. If $P \in A^D$ then 7.10 shows that $O_2(\mathrm{Aut}_D(P)) = 1$, whence $P \in \mathcal{D}^{\mathrm{rc}}$. On the other hand suppose that $P \notin A^D$. Then $D = G_\sigma$ by 5.9 and 7.9, and then 7.12 shows that $P \in A_u^D$ for some $u \in U - Z$. Now $\mathrm{Aut}_D(A_u) = C_{\mathrm{Aut}(A_u)}(u)$ by 7.11(c). On the other hand, the definition of A_u preceding 7.11 shows that $\mathrm{Aut}_{T_2}(A_u)$ is an elementary abelian subgroup of $\mathrm{Aut}_D(A_u)$ of order 8. It follows that $O_2(\mathrm{Aut}_D(A_u)) \ne 1$. Thus $P \notin \mathcal{D}^{\mathrm{rc}}$. Hence the lemma holds when $|Z_P| > 8$, so we are reduced to the case where $|Z_P| = 8$.

If $\mathscr{E}_4(P) = \varnothing$ then $P \notin \mathcal{D}^{\mathrm{rc}}$ by 8.3(c). Thus we may assume $\mathscr{E}_4(P) \ne \varnothing$.

Suppose $Z_P \notin E^D$. Then $D \ne G$ by 7.7, and $D \ne G_0$ by 5.9, so that $D = G_\sigma$. Then $C_{S_D}(Z_P) \in \mathscr{E}_4(S_D)$ by 7.8. But in that case $P = Z_P$, and P is not centric. Thus we may assume $Z_P \in E^D$. Set $R = S_D$.

By 5.9, 7.7, and 7.8, H_D is transitive on $E^G \cap H_D$. Since E is normal in the Sylow 2-group R of H_D, there is $h \in H_D$ with $E^h = Z_P$, $P \le R^h$, and $Z_P \trianglelefteq R^h$. In particular $P \le C_{R^h}(Z_P)$, so that $P_0 = P \cap S_\infty^h$ is of index 2 in P.

Now Z_P is generated by the involutions in P_0, and $P - P_0$ consists entirely of involutions, and so P_0 is a characteristic subgroup of P. Let R_0 be the unique conjugate of T_2 in R^h. Then R_0 centralizes the chain $P \ge R_0 \cap P \ge 1$, and so $R_0 \le P$ by 2.2. Thus $R_0 = T_2$ is weakly closed in P by 4.9(c), and so $\langle h, N_D(P)\rangle \le N_D(T_2)$. Then $Z_P = E^h = E$ and $\mathrm{Aut}_{C_R(E)}(P) \le O_2(\mathrm{Aut}_D(P))$ by 7.13. Thus if $P \in \mathcal{D}^{\mathrm{rc}}$ then $P = C_R(E)$. On the other hand if $P = C_R(E)$ then 7.13 says $P \in \mathcal{D}^{\mathrm{rc}}$. This completes the proof. □

Recall from Section 4 that H acts on an orthogonal space V of dimension 7 over $\mathbf{F}$, and that there is a distinguished basis $\{x_1, \dots, x_7\}$ of V such that T acts on $\mathbf{F}x_i$ for each i. Define $\mathbf{F}_D$ to be $\mathbf{F}$ if $D = G$, $\mathbf{F}_\sigma$ if $D = G_\sigma$, and $\mathbf{F}_{\psi_0}$ if $D = G_0$.

Let V_D be the $\mathbf{F}_D$-span of $\{x_1, \dots, x_7\}$. Then the quadratic form f associated with V restricts to a quadratic form f_D on V_D, preserved by H_D.

Denote by $\mathbf{\Lambda}(V_D)$ the collection of all sets Λ of pairwise orthogonal subspaces of V_D whose sum is V_D. For any subspace X of V_D, denote by $X^{\mathbf{F}}$ the $\mathbf{F}$-span of X in V. For any $\Lambda \in \mathbf{\Lambda}(V_D)$ define $\Lambda^{\mathbf{F}} \in \mathbf{\Lambda}(V)$ by

$$\Lambda^{\mathbf{F}} = \{X^{\mathbf{F}} \mid X \in \Lambda\},$$

and define the *type* of Λ to be the nondecreasing sequence $\tau = \tau(\Lambda)$ of integers given by the dimensions of the members of Λ. We will abbreviate such sequences, using exponential notation. For example, $\tau(\Lambda) = 1^7$ means that each member of Λ is a 1-space, while $\tau(\Lambda) = 1^5 2$ means that Λ consists of five 1-spaces and one 2-space. Write $\mathbf{\Lambda}(V_D, \tau)$ for the set of $\Lambda \in \mathbf{\Lambda}(V_D)$ with $\tau(\Lambda) = \tau$. For $\Lambda \in \mathbf{\Lambda}(V)$ and X a subgroup (or subset) of $H\langle\sigma\rangle$, write $C_X(\Lambda)$, $N_X(\Lambda)$ for the set of all $x \in X$ which acts on each member of Λ, and permutes the members of Λ, respectively.

LEMMA 10.6. *Let $\Lambda \in \mathbf{\Lambda}(V)$ with $\sigma \in C_{H\sigma}(\Lambda)$, and set $\tau = \tau(\Lambda)$. Then H acts transitively on $\mathbf{\Lambda}(V, \tau)$.*

Proof. Since every member of $\mathbf{F}$ is a square, all nondegenerate subspaces of V of a given dimension are isometric, and so the result follows from Witt's Lemma. □

LEMMA 10.7. *Let $D \in \{G_\sigma, G\}$, let $P \in \mathfrak{D}_H^{\mathrm{rc}}$ with $N_{H_D}(P) \not\leq K$, and set $N = N_{H_D}(P)$. Denote by $\mathcal{B}(P)$ the set of N-invariant 2-subgroups of H, and set $\beta(P) = \langle \mathcal{B}(P) \rangle$. Then*:

(a) *One of the following holds.*

 (1) *There exists $\Lambda \in \mathbf{\Lambda}(V_D, 1^7)$ such that $N = N_{H_D}(\Lambda)$ and $P = C_{H_D}(\Lambda)$. Moreover either*
 (i) *$P \cong D_8^3$, and $N/P \cong$ Alt(7) if $D = G_{\psi_0}$, while $N/P \cong$ Sym(7) if $D \neq G_{\psi_0}$, or*
 (ii) *$D = G_\sigma$, $P \cong \mathbf{Z}_4 * Q_8^2$, and $N/P \cong$ Sym(6).*

 (2) *$D \neq \psi_0$, and there exists $\Lambda \in \mathbf{\Lambda}(V_D, 1^5 2)$ with $N \leq N_{H_D}(\Lambda)$. More precisely $P = O_2(N_{H_D}(\Lambda))\langle t \rangle$, where t acts as -1 on every point in Λ and as a reflection on the line ℓ in Λ. Further, $\ell^{\mathbf{F}}$ is one of the lines l_i, $1 \leq i \leq 3$, from Section 4, and $N/P \cong$ Sym(5).*

 (3) *$P = O_2(N_{H_D}(E))$.*

(b) *$C_G(P) \leq P$.*

(c) *$\beta(P) \leq S$. More precisely*: *$\beta(P) = O_2(N_H(\Lambda^{\mathbf{F}}))$ in case* (a)(1), *$O_2(N_H(\Lambda^{\mathbf{F}}))\langle t \rangle$ in case* (a)(2), *and $C_S(E)$ in case* (a)(3).

Proof. Observe that $U \leq P$ by 10.4(a). Set $H^* = H/Z$. We first prove (a).

Let Q^* be an $N_H(P)$-invariant elementary abelian 2-subgroup of $Z(P^*)$ containing U^*; for example as $U^* \leq Z(P^*)$, $Q^*_U = \langle U^{*N_H(P)} \rangle$ is such a subgroup. Let $\Lambda = \Lambda(Q)$ be the set of weight spaces of Q^* on V_D; then $\Lambda \in (V_D)$ and $N_{H_D}(P) \leq N_{H_D}(\Lambda) = N_{H_D}(\Lambda^{\mathbf{F}})$. Let $R = O_2(N_H(\Lambda^{\mathbf{F}}))$. As $N_{H_D}(P) \leq N_H(\Lambda^{\mathbf{F}}) \leq N_H(R)$,

$$R \cap D \leq O_2(N_{H_D}(\Lambda^{\mathbf{F}})) \leq O_2(N_{H_D}(P)) \leq P$$

by 8.2(b). On the other hand P^* centralizes Q^* and hence stabilizes each member of $\Lambda^{\mathbf{F}}$. Further if $\mathrm{Aut}_P(Y) \leq \langle -1_Y \rangle$ for each $Y \in \Lambda$, then $P \leq R \cap D$, so that $P = R \cap D = O_2(N_{H_D}(\Lambda^{\mathbf{F}})) = O_2(N_{H_D}(P))$.

Suppose Λ is of type 1^7. Then each $Y \in \Lambda$ is a 1-dimensional P-invariant orthogonal space, so that $\mathrm{Aut}_P(Y) \leq \langle -1_Y \rangle$, and hence $P = R \cap D = O_2(N_{H_D}(P))$ by the previous paragraph. Further, $\Lambda = \Lambda(P)$, so that $N_{H_D}(P) = N_{H_D}(\Lambda)$, and as $P = O_2(N_{H_D}(\Lambda^{\mathbf{F}}))$, $P = C_{H_D}(\Lambda)$. If $D = G$ it follows that (a)(1.i) holds, so we may take $D = G_\sigma$. As $[\sigma, P] = 1$, $\sigma \in C_{H\langle\sigma\rangle}(\Lambda^{\mathbf{F}})$, so by 10.6 there exists $h \in H$, $\Theta \in \Lambda(H)$, and $\sigma' \in \sigma^H$ such that σ centralizes $N_H(\Theta) = N$ modulo Z, σ' centralizes Θ, and $(\sigma, \Lambda^{\mathbf{F}})^h = (\sigma', \Theta)$. In particular $N_{H_D}(P) \cong N_{H_{\sigma'}}(P^h)$. As σ' centralizes Θ, $\sigma' = r\sigma$ for some $r \in R_0 = O_2(N_H(\Theta))$. We may regard R_0^* as the core of the permutation module for $S_7 \cong N_H(\Theta)/R_0$. Thus r^* is of weight 0, 2, 4, or 6 in R_0^*. If $\sigma = \psi_0$ then $[\sigma, g] = z$ for $g \in N - O^2(N)$, so that $O^2(N) = C_N(\sigma)$, and hence if in addition r^* is of weight 0 then $N_{H_D}(P) \cong O^2(N)$. In this case (a) holds, so we may assume one of the remaining cases holds. Then N centralizes σ' if $r \in Z$ and $rz \in r^{R_0}$ if $r \notin Z$, so that $C_{N^*}(r^*) = R_0^* C_N(r\sigma)^*$ by a Frattini argument. Further, σ centralizes R_0, so that

$$P \cong C_{R_0}(r\sigma) \cong C_{R_0}(r) \cong D_8^3,\ \mathbf{Z}_4 * Q_8^2,\ \mathbf{Z}_2 \times Q_8^2,\ \text{or } \mathbf{Z}_4 * Q_8^2,$$

for the respective choices of r^*, and $N_{H_D}(P)^*/P^*$ is isomorphic to the stabilizer in S_7 of r^*, and hence is S_7, $S_5 \times \mathbf{Z}_2$, $S_4 \times S_3$, S_6, respectively. So as $R \cap D = O_2(N_{H_D}(P))$, r^* is of weight 0 or 6, and hence (1) holds.

Assume next that Λ is of type $1^5 2$, and let l be the line in Λ. Then

$$O_2(N_{H_D}(P)) \leq P$$

and P acts on each member of Λ. Let $\Theta \in \Lambda(V)$ be of type $1^5 2$, l_N the line in Θ, and $N_H(\Theta) = N$. Then $N = N_1 N_2 \langle t_N \rangle$, where $N_1 = C_G(l_N)$, $N_2 = C_N(l_N^{\perp})$, t_N inverts $l_N^{\perp}$ and induces a reflection on l_N, $O_2(N_1) \cong Q_8 D_8$, and $N_1/O_2(N_1) \cong S_5$. If $D = G_\sigma$, then as above we can pick Θ so that $(\sigma, \Lambda^{\mathbf{F}})$ is conjugate in H to $(r\sigma, \Theta)$ such that σ centralizes $O^2(N_1)$, and r fixes each member of Θ. Thus $r = r_1 r_2$ with $r_1 \in O_2(N_1)$, $r_2 \in N_2$, and r_1^* is of weight 0, 2, or 4 in the permutation module $O_2(N_1)^*$ for $N_1/O_2(N_1) \cong S_5$. But again if $r_1^* \neq 1$ then $O_2(C_N(r\sigma))$ does not act on each point in Θ, contrary to an earlier remark.

Thus we may take $\Theta = \Lambda^{\mathbf{F}}$ and $H_1 = O^2(C_{H_D}(l)) \cong O_2(N_1) \cong A_5/Q_8D_8$ with $P_1 = O_2(H_1) \le P$ by 8.2(b). Let $H_2 = C_{H_D}(l^{\perp})$. Then H_2 is cyclic and $P_2 = O_2(H_2) \le P$ by 8.2(b). Also there is $t \in N_{H_D}(\Lambda)$ inverting $l^{\perp}$ and inducing a reflection on l. Then t induces an automorphism in $O_2(\mathrm{Aut}_{H_D}(P))$ on P, so that $t \in P$ as $P \in \mathfrak{D}_H^{\mathrm{rc}}$. Indeed $P = P_1P_2\langle t\rangle$.

If $|P_2^*| \le 2$ then P^* is elementary abelian with weight spaces of dimension 1, and we obtain a contradiction from our treatment of the case Λ of type 1^7. Thus $|P_2^*| > 2$, and if $D = G_\sigma$ then $|P_2^*| = (q-\varepsilon)_2/2$ where $q = |\mathbf{F}_D| \equiv \varepsilon = \pm 1 \mod 4$. It follows that $\sigma \neq \psi_0$. Next $U \le Q \le P_1P_3$ where P_3 is the subgroup of P_2 of order 4, so $P_2C_{P_1}(U) \le C_S(U)$. As $\mathrm{Aut}_{C_S(U)}(S_\infty) \cong E_8$, $P_4 = \Phi(P_2) \le S_\infty$. As $|P_4| > 2$, $l^{\perp} = C_{V_D}(P_4)$, and then as $P_4 \le S_\infty \cap D$, $l^{\mathbf{F}}$ is one of the three lines l_i from Section 4. Thus (2) holds in this case.

Suppose $N_{H_D}(P) \le B_D^h$ for some $h \in H_D$. Arguing as in the first few paragraphs of the proof of 9.2, there is $R_+ \le P$ such that $R_+ = R_+^1R_+^2R_+^3$ with $R_+^i = R_+ \cap L_i^h \cong Q_8$. Then by 10.3, $U = U^h$, contrary to our hypothesis that $N_{H_D}(P) \not\le K$. Thus we may assume $N_{H_D}(P)$ is contained in no H_D-conjugate of B_D.

Suppose $Z \neq P_0 \le P$ is normal in $N_{H_D}(P)$ with $\Phi(P_0) = 1$. By the previous paragraph, $m_2(P_0) > 2$, and so $m_2(P_0) = 3$ or 4. If $m_2(P_0) = 4$ then by 7.11, 7.12, and as $N_{H_D}(P_0)$ acts on no 4-subgroup of P_0, $\mathrm{Aut}_{H_D}(P_0) = C_{\mathrm{GL}(P_0)}(z)$. Thus $P = O_2(N_{H_D}(P_0))$ by 8.2(b), and $P^* \cong E_{64}$, so that $\Lambda(P) \in (D)$ is of type 1^7, and we obtain a contradiction from our treatment of this case. Therefore $m_2(P_0) = 3$. Hence by 7.8, P_0 is B_D-conjugate to E or E', and in the latter case $EE' \cong E_{16}$ is Sylow in $C_{H_D}(E')$. In the latter case a P-invariant Sylow 2-subgroup P_1 of $C_{H_D}(P_0)$ satisfies $[P, P_1] \le P_0 \le C_P(P_1)$; so $P_1 \le P$ by 2.2, and we obtain a contradiction from our treatment of the case $m_2(P_0) = 4$. Thus $P_0 = E^g$ for some $g \in B_D$. Let $S_1 = S_\infty \cap D$. Then $[S_1^g, P] \le S_1^g \cap P \trianglelefteq N_{H_D}(P)$; so $S_1^g \le P$ by 2.2. Hence $S_1 = S_1^g$ by 4.9(c), and so $P_0 = E$. Similarly $C_{S_D}(E) \le P$, and then (3) holds by 8.2(b).

We have reduced to the case where Z is the largest elementary abelian 2-subgroup of P normal in $N_{H_D}(P)$. It follows that Q is of symplectic type and hence (cf. [Asc86, 23.9]) $Q = Q_0 * Z(Q)$ with Q_0 extraspecial and $Z(Q)$ cyclic of order 2 or 4. Further we may choose Q_0 with $U \le Q_0$. As $\mathrm{Inn}(Q_0) = C_{\mathrm{Aut}(Q_0)}(Q_0^*)$, $P = Q_0C_P(Q_0)$.

As $[E, P] \le [E, S] \le U \le P$, E acts on P, and similarly E acts on Q_0. Thus $[E, C_P(Q_0)] \le C_U(Q_0) = Z$ and if E does not centralize P^* then E does not centralize Q^*. Then as E acts on Q_0, E induces a transvection on the orthogonal space Q_0^* with center U^*. This is impossible as U^* is a singular point in Q_0^*. Thus E centralizes P^*, so that $E \le P$ by 2.2, and hence $E^* \le Z(P^*)$; so replacing Q by $Q\langle E^{N_{H_D}(P)}\rangle$, we may assume $E \le Q$.

Finally let $\Theta \in \Lambda(V)$ be of type 1^7 such that each member of $\Lambda^{\mathbf{F}}$ is a sum of points in Θ, and let $R_0 = O_2(N_H(\Theta))$. Thus Q centralizes Θ, $Q \le R_0$, and we view R_0^* as the core of the permutation module for $N_H(\Theta)/R_0 \cong S_7$ on Θ. Thus we identify $f^* \in Q^*$ with the points of Θ inverted by f. Observe $\Lambda^{\mathbf{F}}(E) = \{l_0, l_1, l_2, l_3\}$, where $l_0 = C_V(E)$ is a point. As $Z(Q)$ is cyclic, for each $e \in E - Z$ there is $f \in Q$ with $[e, f] = z$, and hence $|f^* \cap e^*|$ is odd. It follows that for at least two $i \in \{1, 2, 3\}$, the eigenspaces of f on l_i are 1-dimensional. Hence Λ is of type $1^5 2$ or 1^7, cases we have already handled. This completes the proof of (a).

Set $X = C_H(P)$. Since $Z = Z_P$ by (a), we conclude from 8.3(a) that if $X \le P$ then $X = C_H(P)$, so that (b) holds. Thus to prove (b), it suffices to show $X \le P$. If $P = C_{S_D}(E)$ then $A \le P$, and then $X \le P$ since $C_H(A) = A$. On the other hand, if P satisfies (a)(1) then $X \le C_H(\Lambda^{\mathbf{F}})$. If Λ is of type 1^7 then P has index at most 2 in $O_2(N_H(\Lambda^{\mathbf{F}}))$ and $X = Z$. If Λ is of type $1^5 2$ then $P = P_0\langle t\rangle$ where $P_0 = O_2(N_{H_D}(\Lambda^{\mathbf{F}})$ and where t inverts every element of $C_H(P_0)$. Thus $X \le P$ in all cases, establishing (b).

Let $Q \in \mathcal{B}(P)$, R_1 our candidate for $\beta(P)$ in (c), $R_2 = N_{R_1}(Q)$, and $Q_2 = N_Q(R_1)$. Then Q_2 acts on R_2. Observe that $P \in \mathcal{F}_H^c$ and $Z_P = Z$ by (a) and (b), so $P \in \mathcal{F}^c$ by 8.3(a). Thus the same holds for any overgroup Q' of P in S; so $Q' \in \mathcal{F}^c$ and $N_G(Q') \le H$ by 8.3(a). In particular Q_2 and $N_Q(R_2)$ are contained in H.

We claim that $N_Q(R_2)$ acts on R_1, so that $Q_2 = N_Q(R_2)$. In case (a)(1), from the treatment of that case above, each of R_1 and P, and hence also R_2 has Λ as its set of weight spaces, so Q_2 acts on $O_2(N_H(\Lambda)) = R_1$. Similarly in case (a)(2), each of R_1 and P, and hence also R_2 has the same set of 1-dimensional weight spaces, so again Q_2 acts on $R_3 = O_2(N_H(\Lambda))$, and hence also on $R_1 = R_3\langle t\rangle$ as $t \in P$ acts on Q_2. Finally in case (3), Q_2 acts on T_2 by 4.9(c), so that Q_2 acts on $S_\infty R_2 = R_1$. This completes the proof of the claim.

Next from the structure of $N_H(R_1)$ and $N_{H_D}(P)$, $N_{H_D}(P)$ acts on no nontrivial 2-subgroup of $N_H(R_1)/R_1$; so $Q_2 \le R_1$, and hence also $Q_2 \le R_2$. Therefore as Q is finite, $Q = Q_2 \le R_1$, completing the proof of the lemma. □

LEMMA 10.8. *Let $D = G_0$ and $H_0^* = H_0/Z$. Write $\mathcal{R}(H_0)$ for the set of $P_0 \le S$ such that $Z \le P$ and P^* is the radical of some proper parabolic of $H_0^* \cong \mathrm{Sp}_6(2)$ containing S_0^*.*

(1) *If $P \in \mathcal{D}^{\mathrm{rc}}$ and $N_D(P) \le H$ then $P \in \mathcal{R}(H_0)$.*

(2) *$\mathcal{R}(H_0) \subseteq \mathcal{D}^{\mathrm{rc}}$.*

(3) *$B_0^* = C_{H_0^*}(a^*)$, where a^* is the involution of type a_2 in $Z(S_0^*)$. For each $P \in \mathcal{R}(H_0)$ with $O_2(B_0) \le P$, $N_D(P) \le K_0$.*

(4) *$N_D(P) \le H_0$ and $P \in \mathcal{F}^{\mathrm{rc}}_{S_{\psi_0}}(H_{\psi_0})$ for each $P \in \mathcal{R}(H_0)$ with $O_2(B_0) \not\le P$.*

(5) $N_{H_0}(A)^*$ *is the maximal parabolic isomorphic to* $L_3(2)/E_{64}$.

(6) $N_{H_0}(E)^*$ *is the minimal parabolic not contained in* B_0^*.

Proof. Assume the hypothesis of (1). Then by 10.1, P contains each 2-element in $C_{H_0}(P)$, and $O_2(N_{H_0}(P)/PC_{H_0}(P)) = 1$. Hence as $H_0^* \cong \mathrm{Sp}_6(2)$, $P \in \mathcal{R}(H_0)$ by the Borel-Tits Theorem. Thus (1) holds.

Next U is the unique normal 4-subgroup of S_0, so that U^* is generated by the unique involution in $Z(S_0^*)$ lifting to an involution of H_0. As H_0 is the covering group of $\mathrm{Sp}_6(2)$ and $B_0 = N_{H_0}(U)$, it follows that the first statement in (3) holds. By 10.3, $O_2(B_0)$ is weakly closed in S_0 with respect to D and so the remaining statement in (3) follows.

Next, by 7.10, $\mathrm{Aut}_D(A) \cong L_4(2)$ and so $N_{H_0}(A)/A \cong L_3(2)$. This implies (5). Also $N_{H_0}(A)^*$ contains two minimal parabolics: $N_{B_0}(A)^*$ and Y_0^*, where $Y_0 = N_{H_0}(A) \cap N_D(E) = N_{H_0}(E)$. Thus (6) holds.

Let $P \in \mathcal{R}(H_0)$ with $O_2(B_0) \not\leq P$. Then $N_{H_0}(P)^*$ contains the minimal parabolic Y_0^*; so $N_{H_0}(P)$ is Y_0, $N_{H_0}(A)$, or the preimage of the third maximal parabolic of H_0^*, isomorphic to S_6/E_{32}. In the first two cases $Z = Z_P$ from the action of $N_D(A)$ on A, and in the third case $P \cong \mathbf{Z}_4 * Q_8^2$ and again $Z = Z_P$. Thus $N_D(P) \leq N_D(Z)$. As $S_0 \leq N_D(P)$ and $S_0 \in \mathrm{Syl}_2(D)$, P contains each element in $C_D(P)$ by 5.8 and so $P \in \mathfrak{D}^c$ by 8.2(a). Then $N_D(P) \leq H_0$ by 8.3(a). Also $S_0 = S_{\psi_0}$ acts on P and as $C_{S_0}(P) \leq P$, $P \in \mathcal{F}^c_{S_{\psi_0}}(H_{\psi_0})$. Similarly $N_D(P) \leq H_{\psi_0}$ and $P = O_2(N_D(P))$, so that $P \in \mathcal{F}^r_{S_{\psi_0}}(H_{\psi_0})$ which completes the proof of (4). Then (2) follows from (3) and (4). □

PROPOSITION 10.9. *Let* $P \in \mathfrak{D}^{\mathrm{rc}}$. *Then*

(a) *There is no nontrivial* $N_D(P)$*-invariant subgroup of* $C_G(P)$ *of odd order.*

(b) $P \in \mathcal{F}^c$.

(c) *One of the following holds.*

(1) $P \in A^D$,

(2) $P = C_{S_D}(E)$, *or*

(3) $P \in \mathfrak{D}^{\mathrm{rc}}_Y$ *for some* $Y \in \{H, K\}$, *and* $N_G(P) \leq Y$.

Conversely, every subgroup P *of* S_D *which satisfies one of the conditions in* (c) *is in* $\mathfrak{D}^{\mathrm{rc}}$.

Proof. Suppose first that $|Z_P| = 2$. Then $N_D(P) \leq H$ by 8.3(a), so that $P \in \mathfrak{D}^{\mathrm{rc}}_H$ by 10.1. Assume that $N_D(P) \not\leq B$. Then 10.7(b) and 10.8(4) say that $C_G(P) \leq P$, so that (a) and (b) hold and $P \in \mathcal{F}^c_H$. Then (3)(c) follows from 8.3(a).

Suppose next that either $|Z_P| = 2$ and $N_D(P) \leq B$, or $|Z_P| = 4$. In the latter case, $Z_P = U$ and $N_D(P) \leq K$ by 10.4 and 8.3(b); certainly $N_D(P) \leq K$ in the former case. Thus in any event, $N_D(P) \leq K$, so that $P \in \mathfrak{D}^{\mathrm{rc}}_K$ by 10.1. Then

$C_K(P) \leq P$ by 10.2(a), and $P \in \mathcal{F}_K^c$ by 10.2(e). Then as $U \leq P$, $C_H(P) \leq C_H(U) \leq K$; so $C_H(P) \leq C_K(P) \leq P$, and hence $P \in \mathcal{F}_H^c$. Now it follows from parts (a) and (b) of 8.3 that $C_G(P) \leq P$ and (a)-(c) hold.

If $|Z_P| = 8$ then $P = C_{S_D}(E)$ by 9.5. Then $A \leq P$, and so $P \in \mathcal{F}^c$ by 8.3(d). Also, it follows from 7.13 that (a) holds, and we have (a) through (c).

If $|Z_P| > 8$ then $P \in A^D$ by 9.5. Then 8.3(d) yields (b), and 7.10 yields (a). Thus, we are reduced to establishing the final statement in the theorem.

If $P \in A^D$ or $P = C_{S_D}(E)$ then $P \in \mathfrak{D}^{\mathrm{rc}}$, and $N_D(P) \not\leq H \cup K$, by 7.10 and 7.13. Finally assume $P \in \mathfrak{D}_Y^{\mathrm{rc}}$ for some $Y \in \{H, K\}$ with $N_G(P) \leq Y$. Then 10.2, 10.7, and 10.8 yield $Z_P \in \{Z, U\}$ and $P \in \mathcal{F}_Y^c$, so that $P \in \mathcal{F}^c$ by 8.2(a), and $P \in \mathfrak{D}^{\mathrm{rc}}$ by 10.1. □

PROPOSITION 10.10. *Let $P \in \mathfrak{D}^{\mathrm{rc}}$, let $\mathfrak{B}(P)$ be the set of finite $N_D(P)$-invariant 2-subgroups of G, and set $\beta(P) = \langle \mathfrak{B}(P) \rangle$. Then $\beta(P) \leq S$, and one of the following holds.*

(1) *$P \in A^D$ and $\beta(P) = P$.*

(2) *$P = C_{S_D}(E)$ and $\beta(P) = C_S(E)$.*

(3) *There exists $Y \in \{H, K\}$ such that $P \in \mathfrak{D}_Y^{\mathrm{rc}}$, $N_D(P) \leq Y$, $\beta(P) \in \mathcal{F}_S(Y)^{\mathrm{rc}}$, and $N_G(\beta(P)) \leq Y$.*

Proof. Set $N = N_D(P)$, let R be a finite N-invariant 2-subgroup of G containing P, and set $R_0 = N_R(P)$.

If $P \in A^D$ then $N = N_G(P)$ by 7.10; so $R_0 \leq O_2(N) = P$, and (1) holds. Suppose that $P = C_{S_D}(E)$. Define R_i by $R_i = N_R(R_{i-1})$ for $i \geq 1$, and set $M_i = N_G(R_i)$. Set $M = N_G(C_S(E))$. By 7.13, $C_M(E) = X \times O_2(M)$, where X is a free normal subgroup of M, $M/C_M(E) \cong \mathrm{GL}(3, 2)$, and $M = (X \times C_S(E))N$. As R_0 is N-invariant, we get $R_0 \leq C_S(E)$, and $R_0 = T_k\langle w_0 \rangle$ for some $k \geq 2$. By 4.9(a) and 7.13, $N_G(R_i) \leq M$. Then a straightforward induction argument yields $R_i \leq C_S(E)$ for all i, and thus $\beta(P) \leq C_S(E)$. Since $C_S(E)$ is the union of finite N-invariant subgroups, we conclude that $\beta(P) = C_S(E)$, and (2) holds.

By 10.9(c) we are reduced to the case where there exists $Y \in \{H, K\}$ with $P \in \mathfrak{D}_Y^{\mathrm{rc}}$ and $N_G(P) \leq Y$. Suppose further that $\beta(P) \leq S$, and set $M = N_G(\beta(P))$. As $P \leq \beta(P)$, and since $P \in \mathcal{F}^c$ by 10.9(b), we conclude $\beta(P) \in \mathcal{F}^c$ and $Z_{\beta(P)} \leq Z_P$. If $N \leq H$ and $N \not\leq K$ then 10.7 yields $Z_P = Z$, and so $Z_{\beta(P)} = Z$, and then $M \leq H$ by 8.3(a). On the other hand, suppose that $N \leq K$. Then U is the unique 4-group in K which centralizes P/Z, by 10.2(f). Since we are assuming that $\beta(P) \leq S$, it follows that also U is the unique 4-group in $\beta(P)$ which centralizes $\beta(P)/Z$, and hence U is the unique 4-group in $\beta(P)$ which is normal in M. Then 8.3 says M is contained in H or K, and since $N_H(U) \leq K$ we get $M \leq K$. Thus, $M \leq Y$.

We now show that $\beta(P) \le S$. If $N \le H$ and $N \not\le K$, this follows from 10.7(c) and 10.8(4). Suppose $N \le K$, and set $R_1 = R \cap K$. Then $R_0 \le R_1 \le S$ by 10.2(e). Also, 10.2(e) says that $P \in \mathcal{F}^c_K$, and hence $R_1 \in \mathcal{F}^c_K$. As U is the unique normal 4-subgroup of R_1, we have $N_R(R_1) \le N_R(U)$. Since $Z_{R_1} \le Z_P \le U$, it follows from 8.3 that $N_R(R_1)$ is contained in H or K, and since $N_H(U) \le K$ we get $N_R(R_1) \le K$. Then $R = R_1$, and so $R \le S$ for each $R \in \mathcal{B}(P)$. That is, $\beta(P) \le S$.

In order to complete the proof of (3), it remains to show that $\beta(P) \in \mathcal{F}^r_Y$. Let Q be the preimage in M of $O_2(\mathrm{Aut}_M(\beta(P)))$. As $M \le Y$ we have $\theta(\beta(P)) = O(C_G(\beta(P))) = 1$, by 8.4 and 10.9(a). Thus Q is a 2-group, $Q \in \mathcal{B}(P)$, and $Q = \beta(P) \in \mathcal{F}^r$ as required. □

11. **Theorem B and embeddings**

We begin the section with a refinement of Theorem 5.8.

THEOREM 11.1. *Let $\overline{G}_0$ be the group* Co_3, *identify S_0 with a Sylow* 2-*subgroup of $\overline{G}_0$ as in Theorem* 5.8, *and set $\overline{\mathcal{F}}_0 = \mathcal{F}_{S_0}(\overline{G}_0)$. Let $\lambda\colon G_0 \to \overline{G}_0$ be the canonical homomorphism $\lambda\colon g \mapsto \bar{g}$ induced by the inclusion maps of H_0 and K_0 into $\overline{G}_0$, and let $\bar{\mathcal{G}}_0$ be the* 2-*local finite group $(S_0, \overline{\mathcal{F}}_0, \bar{\mathcal{L}}^c_0)$ associated with $\overline{G}_0$ as in Proposition* 2.7. *Then there is an isomorphism* (*in the sense of* 2.10)

$$(\alpha, \beta)\colon \mathcal{G}_0 \to \bar{\mathcal{G}}_0,$$

in which $\alpha\colon \mathcal{F}_0 \to \bar{\mathcal{F}}_0$ is the identity map on objects and, *on morphisms*, $\alpha\colon c_g \mapsto c_{\bar{g}}$; *and where $\beta\colon \mathcal{L}^{\mathrm{rc}}_0 \to \bar{\mathcal{L}}^{\mathrm{rc}}_0$ is the identity map on objects*, *and*

$$\beta_{P,Q}\colon \mathrm{Mor}_{\mathcal{L}_0}(P,Q) \to \mathrm{Mor}_{\bar{\mathcal{L}}_0}(P,Q)$$

is given by $\theta_0(P)g \mapsto \bar{g}$ for P and Q in $\mathcal{L}_0$ and $g \in N_{G_0}(P,Q)$.

Proof. Recall from the discussion following 8.4 that there is a surjection $\lambda\colon G_0 \to \overline{G}_0$, where λ may be regarded as the "identity map" on $H_0 \cup K_0$, and $\ker(\phi_A) = \ker(\lambda)_{|M}$. As $\ker(\phi_A) = \ker(\lambda)_{|M} \le \ker(\lambda)$, 8.7 yields $X_0 \subseteq \ker(\lambda)$. Thus $\theta_0(P) \le \ker(\lambda)$ for any $P \in \mathcal{F}^c_0$. Then since $O(C_{\overline{G}_0}(P)) = 1$, the lemma follows from 5.8 and the last paragraph of 2.13. □

We may now establish Theorem B. Part (1) of Theorem B follows from the construction of G_0 in 5.8, and part (3) follows from 10.1. Part (2a) holds since H_0 and K_0 are finite while the nontrivial elements of X are torsion-free. Thus it only remains to verify part (2b) of Theorem B.

As we just saw during the proof of 11.1, there is a surjective homomorphism $\lambda\colon G_0 \to \overline{G}_0 = \mathrm{Co}_3$ induced by the inclusion maps of H_0 and K_0 into $\overline{G}_0$. Set $M_0 := N_{\overline{G}_0}(A)$, and denote by $\widetilde{C}$ the colimit of the subgroup amalgam defined by the inclusion maps among the intersections of the members of $\mathcal{M} := \{H_0, K_0, M_0\}$.

Then there is a surjective homomorphism $\beta: \widetilde{C} \to \overline{G}_0$ and a surjection $\delta: G_0 \to \widetilde{C}$ with $\delta\beta = \lambda$. Set $M = N_G(A)$. As seen in the proof of 7.5, $M \leq G_0$ and $\phi_A: M \to M_0$ is the restriction of λ to M. Thus $\ker(\delta) = \langle \ker(\phi_A)^{G_0} \rangle$, and then part (2b) of Theorem B follows from 8.7. □

For any $i > 0$, set $m_i = 2^{i-1}$, $\sigma_i = \psi_0^{m_i}$, $G_i = G_{\sigma_i}$, $S_i = S \cap G_i$, and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$. Write Λ for the poset $\mathbb{N}$, under the usual total ordering. There is then a directed system of embeddings of fusion systems

$$\mathfrak{F} = (\iota_{i,j}: \mathcal{F}_i \to \mathcal{F}_j)_{i \leq j \in \Lambda},$$

where $\iota_{i,j}$ is an inclusion map, and $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$ is the fusion system of Co_3.

Let $\iota_i: \mathcal{F}_i \to \mathcal{F}$ be the inclusion, and observe that $\iota_j \circ \iota_{i,j} = \iota_i$ for $i \leq j$. For $i, j \in \Lambda$ with $i \leq j$ and $P \in \mathcal{F}_i^{\mathrm{rc}}$, write $\mathcal{B}_i(P)$ for the set of finite $N_{G_i}(P)$-invariant 2-subgroups of G, and set $\beta_i(P) = \langle \mathcal{B}_i(P) \rangle$ and $\beta_{i,j}(P) = \beta_i(P) \cap G_j$.

LEMMA 11.2. *Let $i, j \in \Lambda$ with $i \leq j$, and let $P \in \mathcal{F}_i^{\mathrm{rc}}$. Then*

(a) $\beta_i(P) \in \mathcal{F}^{\mathrm{rc}}$,

(b) $\beta_{i,j}(P) \in \mathcal{F}_j^{\mathrm{rc}}$ *and*

(c) $\beta_{i,i}(P) = P$.

Proof. Set $D = G_i$, $\mathcal{D} = \mathcal{F}_i$, $N = N_D(P)$, and $\beta = \beta_i$. Also set $Q = \beta(P)$, $\widetilde{D} = G_j$, $\widetilde{\mathcal{D}} = \mathcal{F}_j$, $\widetilde{P} = \beta_{i,j}(P) = Q \cap \widetilde{D}$, and $\widetilde{N} = N_{\widetilde{D}}(\widetilde{P})$. Since $P \in \mathcal{F}^c$ by 10.9(b), it follows from 8.2(a) that $\widetilde{P} \in \widetilde{\mathcal{D}}^c$ and that $Q \in \mathcal{F}^c$. If $P \in A^D$ or $P = C_{S_D}(P)$ then $Q = P$ or $C_S(E)$ by 10.10. Then 10.5 says $Q \in \mathcal{F}^{\mathrm{rc}}$, $\widetilde{P} = P$ or $C_{S_j}(E)$, and $\widetilde{P} \in \widetilde{\mathcal{D}}^{\mathrm{rc}}$. That is, the lemma holds in these two cases.

By 10.9 we may assume that $N_D(P) \leq Y$ for some $Y \in \{H, K\}$, and that $P \in \mathcal{D}_Y^{\mathrm{rc}}$. Then (a) is given by 10.9 and 10.10. In order to complete the proof of (b), it remains to show that $\widetilde{P} \in \mathcal{F}_j^r$. Let R be the pre-image in Y of $O_2(\mathrm{Aut}_{\widetilde{N}}(\widetilde{P}))$. As in the final lines of the proof of 10.10, we find that $\theta_j(\widetilde{P}) = 1$; hence R is a 2-group, and since R is N-invariant we get $R \leq Q$. Then $R = \widetilde{P}$ and $\widetilde{P} \in \mathcal{F}_j^r$ as required.

Finally suppose $i = j$. Here $P \leq \widetilde{P}$ and $N_{G_i}(P) \leq N_{G_i}(\widetilde{P})$, and since $P \in \mathcal{F}_i^r$ we get $\mathrm{Aut}_{\widetilde{P}}(P) \leq O_p(\mathrm{Aut}_{G_i}(P)) = \mathrm{Inn}(P)$. Then $\widetilde{P} = PC_{\widetilde{P}}(P) = P$ as $P \in \mathcal{F}_i^c$, and (c) holds. □

LEMMA 11.3. *Let θ, θ_0, and $\theta_i = \theta_{\sigma_i}$ be the signalizer functors on $\mathcal{F}$, $\mathcal{F}_0$, and $\mathcal{F}_i$, respectively, given by* 8.8. *Let $i, j \in \Lambda$ with $i \leq j$, and let $P \in \mathcal{F}_i^{\mathrm{rc}}$. Then*

$$\theta_i(P) = \theta(\beta_i(P)) \cap G_i = \theta_j(\beta_{i,j}(P)) \cap G_i.$$

Proof. Recall that by definition,

$$\theta(P) = O(C_G(P))C_{\mathbf{X}}(P) \quad \text{and} \quad \theta_i(P) = O(C_{G_i}(P))C_{\mathbf{X} \cap G_i}(P).$$

But $O(C_G(P)) = O(C_{G_i}(P)) = 1$ by 10.9(a), so that $\theta_i(P) = \theta(P) \cap G_i$. If $P \in A^{G_i}$ then $\beta_i(P) = \beta_{i,j}(P) = P$, and the lemma follows from the preceding observation. If $P = C_{S_i}(E)$ then $\beta(P) = C_S(E)$ and $\beta_{i,j}(P) = C_{S_j}(P)$ by 10.10, and the lemma then follows from 7.13(d). If $P \in \mathcal{F}_{S_i}(Y \cap G_i)^{\mathrm{rc}}$ for some $Y \in \{H, K\}$ then also $\beta(P) \in \mathcal{F}_S(Y)^{\mathrm{rc}}$ by 10.10, and $\theta_i(P) = \theta(\beta(P)) = 1$. The lemma holds trivially in this case, and there are no more cases to consider, by 10.9. □

LEMMA 11.4. $\mathcal{F}^{\mathrm{rc}} = \{\beta_i(P) \mid P \in \mathcal{F}_i^{\mathrm{rc}},\ i \geq 0\}$.

Proof. Let $\mathcal{B} = \{\beta_i(P) \colon P \in \mathcal{F}_i^{\mathrm{rc}}, i \geq 0\}$. Then $\mathcal{B} \subseteq \mathcal{F}^{\mathrm{rc}}$ by 11.2(a). Let $\widetilde{P} \in \mathcal{F}^{\mathrm{rc}}$. If $\widetilde{P} \in A^G$ then $\widetilde{P} \leq G_i$ for some i, and $\widetilde{P} = \beta_i(\widetilde{P})$. If $\widetilde{P} = C_S(E)$ then $\widetilde{P} = \beta_0(C_{S_0}(E))$ by 10.10. Suppose that $\widetilde{P} \in \mathcal{F}_S(H)^{\mathrm{rc}}$, such that $N_G(\widetilde{P}) \leq H$ and $N_G(\widetilde{P}) \not\leq K$. The possibilities for $N_G(\widetilde{P})$ are listed in 10.7(a), and we shall deal with them case by case.

In case (1) and (2) of 10.7(a), $N_G(\widetilde{P}) = N_H(\Lambda)$ for some $\Lambda \in \mathbf{\Lambda}(V, \tau)$, with $\tau = 1^7$ or $1^5, 2$. Let $\widetilde{Q} = \Omega_2(O_2(N_H(\Lambda)))$. Then $\widetilde{Q}$ is finite; so $\widetilde{Q} \leq G_i$ for some $i > 0$. Further $\Lambda = \Lambda(\widetilde{Q})$ is the set of weight spaces of $\widetilde{Q}$ on V, and as $\widetilde{Q} \leq G_i$, $\Lambda = \Theta^{\mathbf{F}}$, where Θ is the set of weight spaces for $\widetilde{Q}$ on $V_i = V_{\mathbf{F}_\sigma}$. If $\tau = 1^7$, let $P = \widetilde{Q} = \widetilde{P}$, while if $\tau = 1^5, 2$, let $P = O_2(N_{H_i}(\Theta))\langle t \rangle$, where t is as in 10.7(a). Then $P \in \mathcal{F}_i^{\mathrm{rc}}$ and $\widetilde{P} = \beta_i(\widetilde{P})$ by 10.7(c).

It remains to consider case (3), where $\widetilde{P} = O_2(N_H(E))$. We take $P = O_2(N_{H \cap G_1}(E))$, obtaining $\beta_1(P) = \widetilde{P}$; again from 10.7(c).

By 10.9(c) we may now assume that $N_G(P) \leq K$, so that $\widetilde{P} \in \mathcal{F}_S(K)^{\mathrm{rc}}$. By 10.2 we have $\widetilde{P} \cap B^0 = \widetilde{P}_1 \widetilde{P}_2 \widetilde{P}_3$, where $\widetilde{P}_k = \widetilde{P} \cap L_k$ is either a quaternion group or a Sylow 2-subgroup of L_k. Since K is locally finite, and since 10.2 shows that $N_G(\widetilde{P})/\widetilde{P}$ is finite, we may choose i sufficiently large so that $P := \widetilde{P} \cap G_i$ has the following properties:

(1) For all k for which $\widetilde{P}_k \in \mathrm{Syl}_2(L_k)$ we have $|P \cap L_k| \geq 16$, and for all other k we have $P \cap L_k = \widetilde{P}_k$.

(2) $N_G(\widetilde{P}) = N_K(\widetilde{P}) = N_{G_i}(P)\widetilde{P}$.

Set $N = N_{K \cap G_i}(P)$ and $\widetilde{N} = N_K(\widetilde{P})$. It follows from (1) and (2), and from the final statement in 10.2, that $P \in \mathcal{F}_{S_i}(K \cap G_i)^{\mathrm{rc}}$ and that $N \leq \widetilde{N}$. As $N \leq \widetilde{N}$, $\widetilde{P} \leq \beta_i(P)$, so that it remains to show $\beta_i(P) \leq \widetilde{P}$.

Let $P \leq R \in \mathcal{B}(P)$ and set $R_0 = R \cap K$. Then $R \leq S$ by 10.2(e). As $\widetilde{N} = N_{G_i}(P)\widetilde{P}$, $\langle R_0, \widetilde{P} \rangle$ is an $\widetilde{N}$-invariant 2-group, so since $\widetilde{P} \in \mathcal{F}^r$ we conclude that $R_0 \leq \widetilde{P}$. By 10.9(b), $P \in \mathcal{F}^c$, so $R_0 \in \mathcal{F}^c$. By 10.2(f), U is the unique normal fours group in R_0, and since $Z_{R_0} \leq Z_P$ it follows from 8.3 that $N_G(R_0) \leq K$. Then $R_0 = R$, and the proof is complete. □

12. **Limits, and Theorem C**

Our aim in this section is to introduce limits of directed systems of p-local groups, and to obtain Theorem C as an application. See for example [Jac80, §2.5] for a discussion of directed systems and their limits. Theorem D will then be obtained as a corollary to [LO02, Th. 4.5].

Let $(\Lambda, \leq)$ be a directed set. For $\lambda \in \Lambda$, write $\Lambda(\lambda)$ for $\{\mu \in \Lambda \mid \lambda \leq \mu\}$. A subset Ω of Λ is *closed* if $\Lambda(\lambda) \subseteq \Omega$ for all $\lambda \in \Omega$. In particular, each of the sets $\Lambda(\lambda)$ is closed.

Recall the notion of "pre-local group" from 2.4. Fix a prime p, and assume that for each $\lambda \in \Lambda$ we are given a pre-local group $\mathcal{G}_\lambda = (S_\lambda, \mathcal{F}_\lambda, \mathcal{L}_\lambda)$, where each S_λ is a p-group. We write $\mathcal{E}_\lambda$ for $\mathrm{Obj}(\mathcal{L}_\lambda)$, and given subgroups P and Q of S_λ we write $\mathrm{Hom}_\lambda(P, Q)$ for $\mathrm{Hom}_{\mathcal{F}_\lambda}(P, Q)$, and $\mathrm{Mor}_\lambda(P, Q)$ for $\mathrm{Mor}_{\mathcal{L}_\lambda}(P, Q)$ if P and Q are in $\mathcal{E}_\lambda$. Assume that for all pairs (λ, μ) with $\lambda \leq \mu$ in Λ, we are given an embedding

$$(\iota_{\lambda,\mu}, \beta_{\lambda,\mu})\colon \mathcal{G}_\lambda \longrightarrow \mathcal{G}_\mu,$$

of pre-local groups (cf. 2.10). We may write simply $\beta_{\lambda,\mu}$ for the pair $(\iota_{\lambda,\mu}, \beta_{\lambda,\mu})$. We assume further that

$$\mathfrak{G} = (\beta_{\lambda,\mu}\colon \mathcal{G}_\lambda \longrightarrow \mathcal{G}_\mu)_{\lambda \leq \mu \in \Lambda}$$

is a directed system of pre-local groups. That is, we have $\beta_{\mu,\nu} \circ \beta_{\lambda,\mu} = \beta_{\lambda,\nu}$ whenever $\lambda \leq \mu \leq \nu$ in Λ, and each $\beta_{\lambda,\lambda}$ is the "identity morphism" on $\mathcal{G}_\lambda$, consisting of a pair of identity functors.

Let $S := S_\infty$ be the limit of the Λ-directed system of p-groups

$$(\iota_{\lambda,\mu}\colon S_\lambda \longrightarrow S_\mu)_{\lambda \leq \mu}.$$

By [Jac80, 2.8], the limit exists and is a group, and there are monomorphisms $\iota_\lambda\colon S_\lambda \to S$, compatible with the monomorphisms $\iota_{\lambda,\mu}$. We may then view all of these monomorphisms as ordinary inclusion maps, in order to obtain the following result:

LEMMA 12.1. *$S = \bigcup_{\lambda \in \Lambda} S_\lambda$, and S is a p-group.* □

Let P and Q be subgroups of S. For any $\phi \in \mathrm{Inj}(P, Q)$ and any $\lambda \in \Lambda$, define ϕ_λ to be the restriction of ϕ to $P \cap S_\lambda$, and set

$$\Lambda_\phi = \{\lambda \in \Lambda \mid \phi_\lambda \in \mathrm{Hom}_\lambda(P \cap S_\lambda, Q \cap S_\lambda)\}.$$

Define $\mathrm{Hom}_\infty(P, Q)$ to be the set of all $\phi \in \mathrm{Inj}(P, Q)$ such that Λ_ϕ contains a nonempty closed subset of Λ. As Λ is a directed set, each pair of elements of Λ has an upper bound, and it follows that the composition of $\phi \in \mathrm{Hom}_\infty(P, Q)$ with $\psi \in \mathrm{Hom}_\infty(Q, R)$ is in $\mathrm{Hom}_\infty(P, R)$. Thus we may form the category $\mathcal{F}_\infty$, whose objects are the subgroups of S, and whose morphisms are given by the sets

$\mathrm{Hom}_\infty(P, Q)$. Moreover, if P and Q are subgroups of S_λ and $\phi \in \mathrm{Hom}_\lambda(P, Q)$, then $\phi \in \mathrm{Hom}_\infty(P, Q)$. This natural inclusion of sets may be denoted

$$\iota_\lambda\colon \mathrm{Hom}_\lambda(P, Q) \to \mathrm{Hom}_\infty(P, Q).$$

Allowing P and Q to vary over the set of all subgroups of S_λ, ι_λ is then an embedding of fusion systems (cf. 2.9). We record this in the following lemma, whose proof is straightforward and left to the reader.

LEMMA 12.2. *$\mathcal{F}_\infty$ is a fusion system on S and $\iota_\lambda\colon \mathcal{F}_\lambda \to \mathcal{F}_\infty$ is an embedding of fusion systems.*

Recall that $\mathscr{E}_\lambda$ is the set of objects of the linking system $\mathscr{L}_\lambda$. For $P \in \mathscr{E}_\lambda$ there is then a subgroup $\beta_\lambda(P)$ of S defined by

$$\beta_\lambda(P) = \bigcup_{\mu \in \Lambda(\lambda)} \beta_{\lambda,\mu}(P).$$

Note that for any $\mu \in \Lambda(\lambda)$, we have $\beta_\lambda(P) = \beta_\mu(\beta_{\lambda,\mu}(P))$. Set

$$\mathscr{E}_\infty = \{\beta_\lambda(P) \mid \lambda \in \Lambda, P \in \mathscr{E}_\lambda\}.$$

Definition 12.3. Let $\mathscr{C}$ be a category and $\Delta = (C_\lambda, c_{\lambda,\mu}\colon \lambda \le \mu)$ a directed system in $\mathscr{C}$. A family $\Sigma = (\gamma_\lambda\colon C_\lambda \to C\colon \lambda \in \Lambda)$ of morphisms in $\mathscr{C}$ is said to be *compatible* with Δ if for all $\lambda \le \mu$ in Λ, $\gamma_\lambda = \gamma_\mu \circ c_{\lambda,\mu}$.

Now specialize to the case where $\mathscr{C}$ is the category of sets. A compatible family Σ is said to be *nearly injective* on Δ if

$$C = \bigcup_{\lambda \in \Lambda} \gamma_\lambda(C_\lambda),$$

and for each $\lambda \in \Lambda$, whenever $a, b \in C_\lambda$ with $\gamma_\lambda(a) = \gamma_\lambda(b)$, then there exists $\mu \in \Lambda(\lambda)$ with $\beta_{\lambda,\mu}(a) = \beta_{\lambda,\mu}(b)$. For example if $\gamma_\lambda\colon C_\lambda \to C$ is injective for each $\lambda \in \Lambda$, then Σ is nearly injective.

Now take $\Delta = \Delta(\mathfrak{G})$ to be $(\mathscr{E}_\lambda, \beta_{\lambda,\mu}\colon \lambda \le \mu)$, and observe Δ *is* a directed system in the category of sets, if we regard $\beta_{\lambda,\mu}$ as the function from $\mathscr{E}_\lambda$ to $\mathscr{E}_\mu$ defined by $\beta_{\lambda,\mu}\colon P \to \beta_{\lambda,\mu}(P)$. We say that Δ is *nearly injective* if $\Sigma(\Delta)$ is nearly injective on $\Delta(\mathfrak{G})$, where $\Sigma(\Delta) = (\beta_\lambda\colon \mathscr{E}_\lambda \to Ob(\mathscr{L}_\infty)\colon \lambda \in \Lambda)$. Similarly define $\mathfrak{G}$ to be *nearly injective* if $\Delta(\mathfrak{G})$ is nearly injective.

LEMMA 12.4. *Assume $\Sigma = (\gamma_\lambda\colon \mathscr{E}_\lambda \to C\colon \lambda \in \Lambda)$ is nearly injective on $\Delta(\mathfrak{G})$ and $\Gamma = (\delta_\lambda\colon \mathscr{E}_\lambda \to D\colon \lambda \in \Lambda)$ is a family of functions compatible with $\Delta(\mathfrak{G})$.*

(1) *Suppose $\lambda, \nu \in \Lambda$, $P \in \mathscr{E}_\lambda$, and $Q \in \mathscr{E}_\nu$ such that $\beta_\lambda(P) = c = \beta_\nu(Q)$. Then there exists $\mu \in \Lambda(\lambda) \cap \Lambda(\nu)$ such that $R = \beta_{\lambda,\mu}(P) = \beta_{\lambda,\mu}(Q)$. Moreover $\beta_\mu(R) = c$.*

(2) *$\delta: C \to D$ is a well-defined function, where $\delta(c) = \delta_\lambda(P)$ for $\lambda \in \Lambda$ and $P \in \mathscr{E}_\lambda$ such that $\gamma_\lambda(P) = c$.*

(3) *If Γ is also nearly injective on $\Delta(\mathfrak{G})$ then δ is a bijection.*

Proof. As Λ is directed there is $\eta \in \Lambda(\lambda) \cap \Lambda(\nu)$. Set $P' = \beta_{\lambda,\eta}(P)$ and $Q' = \beta_{\nu,\eta}(Q)$. Then

$$\beta_\eta(P') = \beta_\lambda(P) = \hat{P} = \beta_\nu(Q) = \beta_\eta(Q').$$

As Σ is nearly injective, there exists $\mu \geq \eta$ with $\beta_{\eta,\mu}(P') = R = \beta_{\eta,\mu}(Q')$. Then $\beta_{\lambda,\mu}(P) = \beta_{\eta,\mu}(\beta_{\lambda,\eta}(P)) = R$, and similarly $\beta_{\nu,\mu}(Q) = R$. Since $\beta_\mu(R) = \beta_\mu(\beta_{\lambda,\mu}(P)) = \beta_\lambda(P) = c$, (1) holds.

To see that δ is well defined, suppose that $\delta_\lambda(P) = \delta_\nu(Q)$ for some ν and some $Q \in \mathscr{E}_\nu$. Choose μ as in (1). Then $\delta_\lambda(P) = \delta_\mu(\beta_{\lambda,\mu}(P)) = \delta_\mu(\beta_{\nu,\mu}(Q)) = \delta_\mu(Q)$ and so γ is well defined, establishing (2).

Assume the hypothesis of (3). Then by (2) applied to Γ, the map $\alpha: \gamma_\lambda(P) \mapsto \delta_\lambda(P)$ is a well-defined function from D to C, and visibly α is an inverse for δ; so (3) holds. □

We now assume that $\mathfrak{G}$ is nearly injective, and define a category $\mathscr{L}_\infty$ whose set of objects is $\mathscr{E}_\infty$, and which will be shown to be the direct limit of the directed system $(\beta_{\lambda,\mu}: \mathscr{L}_\lambda \longrightarrow \mathscr{L}_\mu)_{\lambda \leq \mu}$ of categories.

Let $\widehat{P}, \widehat{Q} \in \mathscr{E}_\infty$. Then there exist $\lambda \in \Lambda$, and $P, Q \in \mathscr{E}_\lambda$, such that $\widehat{P} = \beta_\lambda(P)$ and $\widehat{Q} = \beta_\lambda(Q)$. In the following discussion, leading up to 12.5, we take λ, P, and Q to be fixed. Define $\mathrm{Mor}_\infty(\widehat{P}, \widehat{Q})$ to be the set of equivalence classes $[f]$ of mappings

$$f: \Omega_f \longrightarrow \bigcup_{\mu \in \Omega_f} \mathrm{Mor}_\mu(\beta_{\lambda,\mu}(P), \beta_{\lambda,\mu}(Q))$$

where

(i) Ω_f is a nonempty closed subset of $\Lambda(\lambda)$,

(ii) $f(\mu) \in \mathrm{Mor}_\mu(\beta_{\lambda,\mu}(P), \beta_{\lambda,\mu}(Q))$ for all $\mu \in \Omega_f$,

(iii) $\beta_{\mu,\nu}(f(\mu)) = f(\nu)$ whenever $\mu \leq \nu$,

and where two such mappings f and f' are defined to be equivalent if they agree on a nonempty closed set. There is then a well-defined composition

$$\mathrm{Mor}_\infty(\widehat{Q}, \widehat{R}) \times \mathrm{Mor}_\infty(\widehat{P}, \widehat{Q}) \longrightarrow \mathrm{Mor}_\infty(\widehat{P}, \widehat{R})$$

for any $\widehat{R} \in \mathscr{E}_\infty$. Namely, one may assume λ chosen so that also $\widehat{R} = \beta_\lambda(R)$ for some $R \in \mathscr{E}_\lambda$. Then, for any $[g] \in \mathrm{Mor}_\infty(\beta_\lambda(Q), \beta_\lambda(R))$, define $[g] \cdot [f]$ to be the equivalence class of the mapping $g \cdot f$, where $(g \cdot f)(\mu) = g(\mu) f(\mu)$. This defines the category $\mathscr{L}_\infty$. Notice, using 12.4 and increasing λ if necessary, that these definitions are independent of the choice of P, Q, and R. Thus $\mathscr{L}_\infty$ is well defined.

For any $\psi \in \mathrm{Mor}_\lambda(P, Q)$, there is an element $[f_\psi]$ of $\mathrm{Mor}_\infty(\widehat{P}, \widehat{Q})$, defined by $f_\psi(\mu) = \beta_{\lambda,\mu}(\psi)$ for any $\mu \in \Lambda(\lambda)$. The map $\beta_\lambda: \mathscr{E}_\lambda \to \mathscr{E}_\infty$ extends to a functor $\beta_\lambda: \mathscr{L}_\lambda \longrightarrow \mathscr{L}_\infty$, where β_λ is defined on morphisms by $\beta_\lambda(\psi) = [f_\psi]$.

LEMMA 12.5. *$(\beta_\lambda: \mathscr{L}_\lambda \longrightarrow \mathscr{L}_\infty)_{\lambda\in\Lambda}$ is the direct limit of the nearly injective directed system $L = (\beta_{\lambda,\mu}: \mathscr{L}_\lambda \longrightarrow \mathscr{L}_\mu)_{\lambda\leq\mu}$ of categories.*

Proof. Let $(\gamma_\lambda: \mathscr{L}_\lambda \to \mathscr{C})_{\lambda\in\Lambda}$ be a family of functors compatible with the directed system L of categories. By 12.4.2, we can define a function $\gamma: \mathscr{E}_\infty \to \mathrm{Obj}(\mathscr{C})$ by $\gamma(\widehat{P}) = \gamma_\lambda(P)$ for $\widehat{P} \in \mathscr{E}_\infty$, where $P \in \mathscr{E}_\lambda$ and $\widehat{P} = \beta_\lambda(P)$.

A similar argument allows us to define γ on morphisms: Let $[f]\in\mathrm{Mor}_\infty(\widehat{P},\widehat{Q})$ and pick a representative f of $[f]$. We may choose λ so that $\widehat{P} = \beta_\lambda(P)$ and $\widehat{Q} = \beta_\lambda(Q)$ for some $P, Q \in \mathscr{E}_\lambda$, and so that $\lambda \in \Omega_f$. Setting $\psi = f(\lambda)$, we have $\psi \in \mathrm{Mor}_\lambda(P, Q)$ and $\beta_\lambda(\psi) = [f]$. We now "define" $\gamma([f])$ to be $\gamma_\lambda(\psi)$. As in the preceding paragraph, if $\gamma_\lambda(\psi) = \gamma_\nu(\phi)$ for some ν and some $\mathscr{L}_\nu$-morphism ϕ, we may replace ψ and ϕ by their images under the maps $\beta_{\lambda,\mu}$ and $\beta_{\nu,\mu}$, and reduce to the case where $\lambda = \nu$. Then $\beta_\lambda(\psi) = \beta_\lambda(\phi) = [f]$, whence $f(\lambda) = \psi = \phi$, and γ is well defined on morphisms.

It is now straightforward to check that γ is a functor, and γ is then visibly the unique functor such that $\gamma_\lambda = \gamma \circ \beta_\lambda$ for all $\lambda \in \Lambda$. □

There is a functor $\pi_\infty: \mathscr{L}_\infty \to \mathscr{F}_\infty$, defined as follows. As a map from $\mathrm{Obj}(\mathscr{L}_\infty)$ to $\mathrm{Obj}(\mathscr{F}_\infty)$ we take π_∞ to induce the identity map on $\mathscr{E}_\infty$. As a map of morphisms, define $\pi_\infty([f]): \widehat{P} \to \widehat{Q}$ by

$$\pi_\infty([f]): x \mapsto \pi_\mu(f(\mu))(x)$$

for any $\mu \in \Omega_f$ such that $x \in \beta_{\lambda,\mu}(P)$. The definition is independent of μ, and the verification that $\pi_\infty([f])$ is in $\mathrm{Hom}_\infty(\widehat{P}, \widehat{Q})$ is straightforward, as is the verification of functoriality.

Next, define a family of monomorphisms of groups

$$\delta = \delta_\infty = (\delta_{\widehat{P}}: \widehat{P} \to \mathrm{Aut}_{\mathscr{L}_\infty}(\widehat{P}))_{\widehat{P}\in\mathscr{E}_\infty},$$

as follows. Let $P \in \mathscr{E}_\lambda$ with $\widehat{P} = \beta_\lambda(P)$, and let $x \in \widehat{P}$. Then $x \in \beta_{\lambda,\mu}(P) = P_\mu$ for some $\mu \in \Lambda(\lambda)$. Define $\delta_{\widehat{P}}(x)$ to be $[g_x]$, where $g_x(\nu) = \beta_{\mu,\nu}(\delta_\mu(x))$ for $\nu \in \Lambda(\mu)$. The verification that each $\delta_{\widehat{P}}(x)$ is in $\mathrm{Aut}_{\mathscr{L}_\infty}(\widehat{P}))$ and that $\delta_{\widehat{P}}$ is a monomorphism reduces to the corresponding facts concerning the family δ_μ of monomorphisms associated with $\mathscr{G}_\mu$.

Henceforth $\mathscr{L}_\infty$ will denote the triple consisting of the category $\mathscr{L}_\infty$, the functor π_∞, and the collection δ_∞ of monomorphisms.

LEMMA 12.6. *Let* $\widehat{P}, \widehat{Q} \in \mathscr{E}_\infty$.

(a) $\widehat{P}$ *acts semiregularly on* $\mathrm{Mor}_\infty(\widehat{P}, \widehat{Q})$ *via* $x\colon [f] \mapsto \delta_{\widehat{P}}(x^{-1}) \cdot [f]$.

(b) *The orbits of* $Z(\widehat{P})$ *on* $\mathrm{Mor}_\infty(\widehat{P}, \widehat{Q})$ *are the fibers of* π_∞.

Proof. As $\delta_{\hat{P}}\colon \hat{P} \to \mathrm{Aut}_{\mathscr{L}_\infty}(\hat{P})$ is a monomorphism, the action in (1) defines an injective representation of $\hat{P}$ on $\mathrm{Mor}_\infty(\hat{P}, \hat{Q})$. Let $[f] \in \mathrm{Mor}_\infty(\widehat{P}, \widehat{Q})$, $\lambda \in \Lambda$, and $P, Q \in \mathscr{E}_\lambda$ with $\widehat{P} = \beta_\lambda(P)$ and $\widehat{Q} = \beta_\lambda(Q)$. Let $x \in \widehat{P}$, and suppose that $[f]$ is a fixed point for $\delta_{\widehat{P}}(x)$. Without loss of generality, we may assume that $x \in P_\lambda$ and that $\lambda \in \Omega_f$. Set $\psi = f(\lambda)$ and $\phi = \pi_\lambda(\psi)$. Then $\delta_{\lambda,P}(x) \cdot \psi = \psi$, and hence by conditions (B) and (C) in 2.4, $\phi = \pi(\delta_{\lambda,P}(x) \cdot \psi) = c_x\phi$. Thus as ϕ is injective, it follows that $x \in Z(P)$, and then 2.4(A) yields $x = 1$. Thus (a) holds.

We next prove (b). Let $[f], [h] \in \mathrm{Mor}_\infty(\widehat{P}, \widehat{Q})$. We may assume that $\Lambda(\lambda) = \Omega_f = \Omega_h$. Then $\pi_\infty[f] = \pi_\infty[h]$ if and only if for each $\mu \in \Lambda(\lambda)$, $\pi_\mu(f(\mu)) = \pi_\mu(h(\mu))$. As $\mathscr{G}_\mu$ is a pre-local group, this holds if and only if there exists $z_\mu \in Z(P_\mu)$ with $h(\mu) = \delta_{\mu,P_\mu}(z_\mu) \cdot f(\mu)$. (cf. 2.4(A)). But

$$\begin{aligned}\delta_{\mu,P_\mu}(z_\mu) \cdot f(\mu) = h(\mu) &= \beta_{\lambda,\mu}(h(\lambda)) = \beta_{\lambda,\mu}(\delta_{\lambda,P_\lambda}(z_\lambda) \cdot f(\lambda)) \\ &= \beta_{\lambda,\mu}(\delta_{\lambda,P_\lambda}(z_\lambda)) \cdot \beta_{\lambda,\mu}(f(\lambda)) = \delta_{\mu,P_\mu}(z_\lambda) \cdot f(\mu).\end{aligned}$$

By (1) applied in $\mathscr{G}_\mu$ this holds if and only if $z_\mu = z_\lambda$. Therefore we have shown that $\pi_\infty[f] = \pi_\infty[h]$ if and only if for all $\mu \in \Lambda(\lambda)$, $z_\lambda = z_\mu \in Z(P_\mu)$ and $h(\mu) = \delta_{\mu,P_\mu} \cdot f(\mu)$. Since $\widehat{P}$ is the union of the groups P_μ for $\mu \in \Lambda(\lambda)$, we conclude that $\pi_\infty[f] = \pi_\infty[h]$ if and only if $z_\lambda = z \in Z(\hat{P})$ and $[h] = \delta_{\hat{P}}(z) \cdot [f] = [f]z$. Thus (b) holds. □

The fusion system $\mathscr{F}_\infty$ is in general "too large", in various ways. In particular, π_∞ need not map morphism sets in $\mathscr{L}_\infty$ onto homomorphism sets in $\mathscr{F}_\infty$, and as a result these homomorphism sets are in general too large for $\mathscr{F}_\infty$ to serve as the fusion system in the limit of the direct system $\mathfrak{G}$.

The smallest fusion system on S_∞ containing $\pi_\infty(\mathrm{Mor}_\infty(\widehat{P}, \widehat{Q}))$ for all $\widehat{P}$ and $\widehat{Q}$ in $\mathscr{E}_\infty$ will be denoted $\mathrm{Im}(\pi_\infty)$. If each $\mathscr{G}_\lambda$ is a p-local finite group then $\mathrm{Im}(\pi_\infty)$ will contain $\iota_\lambda(\mathscr{F}_\lambda)$ for all λ, as a consequence of Alperin's Fusion Theorem. More generally, set

$$\mathscr{F}_{\mathfrak{G}} = \langle \mathrm{Im}(\pi_\infty), \iota_\lambda(\mathscr{F}_\lambda) \mid \lambda \in \Lambda \rangle.$$

Set

$$\mathscr{G}_\infty = (S_\infty, \mathscr{F}_\infty, \mathscr{L}_\infty) \quad \text{and} \quad \mathscr{G}_{\mathfrak{G}} = (S_\infty, \mathscr{F}_{\mathfrak{G}}, \mathscr{L}_\infty).$$

Let $\phi \in \mathrm{Hom}_\lambda(P, Q)$, with P and Q in $\mathscr{E}_\lambda$. By 2.4(A) there is $\psi \in \mathrm{Mor}_\lambda(P, Q)$ such that $\pi_\lambda(\psi) = \phi$. Let $\lambda \le \mu \le \nu$ in Λ. Condition (MG2) in 2.10 yields:

$$\pi_\nu(\beta_{\lambda,\nu}(\psi)) \,|_{P_\mu} = \pi_\nu(\beta_{\mu,\nu}(\beta_{\lambda,\mu}(\psi))) \,|_{P_\mu} = \pi_\mu(\beta_{\lambda,\mu}(\psi)). \tag{12.7}$$

Definition 12.8. We say that the element μ of $\Lambda(\lambda)$ is *ϕ-good* provided that, for all $\nu \in \Lambda(\mu)$, $\pi_\nu(\beta_{\lambda,\nu}(\psi))$ is the *unique* member of $\mathrm{Hom}_\nu(P_\nu, Q_\nu)$ which restricts to $\pi_\mu(\beta_{\lambda,\mu}(\psi))$ on P_μ. Denote by E_ϕ the set of all ϕ-good elements of Λ_ϕ.

LEMMA 12.9. (a) *Either of the following conditions implies that $\mathcal{G}_\infty$ is a pre-local group.*

(1) $\pi_\infty\colon \mathrm{Mor}_\infty(\hat{P}, \hat{Q}) \to \mathrm{Hom}_\infty(\hat{P}, \hat{Q})$ *is a surjection, for all* $\hat{P}, \hat{Q} \in \mathcal{E}_\infty$.

(2) *For every λ, and for every $\mathcal{F}_\lambda$-morphism ϕ, E_ϕ is nonempty.*

(b) *If $\mathcal{G}_\infty$ is a pre-local group, then so is $\mathcal{G}_{\mathfrak{G}}$.*

Proof. In order to show that $\mathcal{G}_\infty$ is a pre-local group, we must verify conditions (A) through (C) in 2.4, and that $\mathcal{E}_\infty \subseteq \mathcal{F}^c_\infty$. Under the hypothesis of (1), Condition (A) is an immediate consequence of 12.6; we leave the remaining verifications in (1) to the reader.

Assume next that $\mathcal{G}_\infty$ is a pre-local group. Then

$$\pi_\infty(\mathrm{Mor}_\infty(\hat{P}, \hat{Q})) = \mathrm{Hom}_\infty(\hat{P}, \hat{Q}),$$

and so $\mathrm{Hom}_{\mathcal{F}_{\mathfrak{G}}}(\hat{P}, \hat{Q}) = \pi_\infty(\mathrm{Mor}_\infty(\hat{P}, \hat{Q}))$. The argument of the preceding paragraph then yields (b).

Finally assume the hypothesis of (2). Let $\phi \in \mathrm{Hom}_\infty(\hat{P}, \hat{Q})$. We may assume $\lambda \in \Lambda_\phi$, and thus the restriction ϕ_μ of ϕ to P_μ is in $\mathrm{Hom}_\mu(P_\mu, Q_\mu)$ for all $\mu \in \Lambda(\lambda)$. Further, we may assume that $\lambda \in \mathcal{E}_{\phi_\lambda}$. Choose $\psi \in \pi_\lambda^{-1}(\phi_\lambda)$, and consider the map $f = f_\lambda$ on $\Lambda(\lambda)$ defined before 12.5, such that $[f] \in \mathrm{Hom}_\infty(\hat{P}, \hat{Q})$ and $\beta_\lambda(\psi) = [f]$. Thus $f(\mu) = \beta_{\lambda,\mu}(\psi)$, and so $\mu_\mu(f(\mu)) = \pi_\mu(\beta_{\lambda,\mu}(\psi))$ on P_λ. Thus as $\lambda \in E_{\phi_\lambda}$, $\pi_\mu(f(\mu)) = \phi_\mu$, so that $\pi_\infty([f]) = \phi$. Thus we have verified the hypothesis of (1). Therefore by (1), $\mathcal{G}_\infty$ is a pre-local group, and the proof of (a) is complete. □

LEMMA 12.10. *Assume that $\mathfrak{G}$ is nearly injective.*

(a) *If $\mathcal{G}_{\mathfrak{G}}$ is a pre-local group then $\beta_\lambda = (\iota_\lambda, \beta_\lambda)\colon \mathcal{G}_\lambda \to \mathcal{G}_{\mathfrak{G}}$ is an embedding of pre-local groups, and $(\beta_\lambda\colon \mathcal{G}_\lambda \to \mathcal{G}_{\mathfrak{G}})_{\lambda\in\Lambda}$ is the direct limit of $\mathfrak{G}$ in the category of pre-local groups and embeddings.*

(b) *Assume that there exists a pre-local group $\widetilde{\mathcal{G}} = (\widetilde{S}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{L}})$, and a family Σ of embeddings*

$$\gamma_\lambda = (\alpha_\lambda, \gamma_\lambda)\colon \mathcal{G}_\lambda \to \widetilde{\mathcal{G}},\ \lambda \in \Lambda,$$

of pre-local groups compatible with $\mathfrak{G}$. Assume that the following conditions hold for all λ.

(i) $\widetilde{S} = \bigcup_{\lambda\in\Lambda} \alpha_\lambda(S_\lambda)$.

(ii) *For each $\mathcal{F}_\lambda$-morphism ϕ, $E_\phi \neq \varnothing$.*

(iii) *For all* $P \in \mathscr{E}_\lambda$,

$$\gamma_\lambda(P) = \bigcup_{\lambda \le \mu} \alpha_\mu(\beta_{\lambda,\mu}(P)).$$

(iv) *For each* $\widetilde{P}, \widetilde{Q} \in \widetilde{\mathscr{E}}$ *and* $\widetilde{\phi} \in \mathrm{Hom}_{\widetilde{\mathscr{F}}}(\widetilde{P}, \widetilde{Q})$, *there exist* $\lambda \in \Lambda$ *and* $P, Q \in \mathscr{E}_\lambda$ *such that* $\gamma_\lambda(P) = \widetilde{P}$, $\gamma_\lambda(Q) = \widetilde{Q}$, *and such that for each* $\mu \in \Lambda(\lambda)$,

$$\widetilde{\phi}_{|\alpha_\mu(\beta_{\lambda,\mu}P)} \in \alpha_\mu(\mathrm{Hom}_\mu(\beta_{\lambda,\mu}(P), \beta_{\lambda,\mu}(Q))).$$

(v) $\widetilde{\mathscr{F}}$ *has the Alperin generation property with respect to* $\widetilde{\mathscr{E}}$ (*cf.* 2.14).

(vi) Σ *is nearly injective on* $\Delta(\mathfrak{G})$.

Then $\mathscr{G}_{\mathfrak{G}}$ *is a pre-local group*, $\mathscr{G}_{\mathfrak{G}}$ *is the limit of* $\mathfrak{G}$, *and* $\widetilde{\mathscr{G}} \cong \mathscr{G}_{\mathfrak{G}}$ *as pre-local groups.*

Proof. Assume that $\mathscr{G}_{\mathfrak{G}}$ is a pre-local group. We check that β_λ satisfies the conditions (MG1) through (MG3) in 2.10. We have (MG1) since, for any $P \in \mathscr{E}_\lambda$,

$$P = \beta_{\lambda,\lambda}(P) \le \bigcup_{\mu \in \Lambda(\lambda)} \beta_{\lambda,\mu}(P) = \beta_\lambda(P).$$

Let $\psi \in \mathrm{Mor}_\lambda(P, Q)$. Then $\pi_\infty(\beta_\lambda(\psi))_{|P} = \pi_\lambda(\psi)$ by definition of π_∞, and thus (MG2) holds. Condition (MG3) is the assertion that $\beta_\lambda \circ \delta_{\lambda,P} = \delta_{\beta_\lambda(P)}$ on P, which holds by definition of δ_P. Thus β_λ is a morphism of pre-local groups. Recall that $\beta_\lambda(\psi) = [f_\psi]$, where $f_\psi(\mu) = \beta_{\lambda,\mu}(\psi)$ for all $\mu \in \Lambda(\lambda)$. In particular by (ii), β_λ is injective as a mapping from $\mathrm{Mor}_\lambda(P, Q)$ into $\mathscr{L}_\infty$-morphisms, and β_λ is therefore an embedding.

Let $\widetilde{\mathscr{G}}$ satisfy the initial hypothesis in (b), but for the moment do not assume the conditions (i) through (vi) in (b). Define the functor $\gamma: \mathscr{L}_\infty \to \widetilde{\mathscr{L}}$ as in the proof of 12.5. That is, $\gamma(\widehat{P}) = \gamma_\lambda(P)$ and $\gamma([f]) = \gamma_\lambda(f(\lambda))$, where $\beta_\lambda(P) = \widehat{P}$ and $\beta_\lambda(f(\lambda)) = [f]$. Then γ is the unique functor satisfying $\gamma_\lambda = \gamma \circ \beta_\lambda$ for all $\lambda \in \Lambda$.

Let $\lambda \in \Lambda$. Define $\alpha_0: S_\infty \to \widetilde{S}$ by $\alpha_{0|S_\lambda} = \alpha_\lambda$. As each α_λ is injective, α_0 is an injective homomorphism. For any pair of subgroups P, Q of S, and any $\phi \in \mathrm{Hom}_{\mathscr{F}_{\mathfrak{G}}}(P, Q)$, define $\alpha(\phi)$ to be the homomorphism

$$\alpha_0^{-1} \circ \phi \circ \alpha_0: P\alpha_0 \longrightarrow Q\alpha_0.$$

Then α is a morphism of fusion systems if and only if the following condition holds for all $P, Q \le S_\infty$:

(∗) $\alpha(\phi) \in \mathrm{Hom}_{\widetilde{\mathscr{F}}}(\alpha_0(P), \alpha_0(Q))$.

If $P, Q \le S_\lambda$ and $\phi \in \mathrm{Hom}_\lambda(P, Q)$, then (∗) holds as α_λ is a morphism of fusion systems. If $P, Q \in \mathscr{E}_\infty$, then $\phi = \pi_\infty(\beta_\lambda(f))$ for some $f \in \mathrm{Mor}_\lambda(P_\lambda, Q_\lambda)$. Write ϕ_λ for the restriction of ϕ to P_λ. Then on P_λ:

$$\widetilde{\pi}(\gamma(\beta_\lambda(f))) = \widetilde{\pi}(\gamma_\lambda(f)) = \alpha_\lambda(\pi_\lambda(f)) = \alpha_\lambda(\phi_\lambda)) = \alpha_\lambda \circ \phi_\lambda \circ \alpha_\lambda^{-1},$$

and as this holds for all λ in a closed subset of Λ, we conclude that $\tilde{\pi}(\gamma(\beta_\lambda(f))) = \alpha \circ \phi \circ \alpha^{-1} = \alpha(\phi)$, and we again obtain $(*)$. Since, by definition, $\mathcal{F}_{\mathfrak{G}}$ is generated by such morphisms, $(*)$ holds in general, and α is a morphism of fusion systems. We leave it to the reader to check that (α, γ) satisfies the axioms in Definition 2.10, and hence is a morphism of pre-local groups, yielding (a).

Now assume all of the hypotheses of (b). Then by (b)(ii) and 12.9, $\mathcal{G}_{\mathfrak{G}}$ is pre-local group. Then (a) says that $\mathcal{G}_{\mathfrak{G}}$ is the limit of $\mathfrak{G}$, and supplies the morphism $\gamma = (\alpha, \gamma)\colon \mathcal{G}_\infty \to \tilde{\mathcal{G}}$ described above. By (b)(i) and the definition of α, $\alpha_0\colon S_\infty \to \tilde{S}$ is an isomorphism of groups.

The key step in the proof of (b) is to show that each of the sets $\mathrm{Hom}_{\tilde{\mathcal{F}}}(\tilde{P}, \tilde{Q})$, with $\tilde{P}$ and $\tilde{Q}$ in $\tilde{\mathcal{E}}$, lies in the image of α. So let $\tilde{P}, \tilde{Q} \in \tilde{E}$ and $\tilde{\phi} \in \mathrm{Hom}_{\tilde{\mathcal{F}}}(\tilde{P}, \tilde{Q})$. Let λ, P, and Q be as in (iv). By (iii),

$$(**) \qquad \alpha(\beta_\lambda(P)) = \bigcup_{\lambda \le \mu} \alpha_\mu(\beta_{\lambda,\mu}(P)) = \gamma_\lambda(P) = \tilde{P},$$

so that, in particular, $\tilde{\mathcal{E}} \subseteq \alpha(\mathcal{E}_\infty)$.

For $\mu \in \Lambda(\lambda)$ write P_μ for $\beta_{\lambda,\mu}(P)$ and Q_μ for $\beta_{\lambda,\mu}(Q)$, and set $\tilde{\phi}_\mu = \tilde{\phi}_{|\alpha_\mu(P_\mu)}$. Then (iv) says that for all such μ there exists $\phi_\mu \in \mathrm{Hom}_\mu(P_\mu, Q_\mu)$ with $\alpha_\mu(\phi_\mu) = \tilde{\phi}_\mu$. By (ii), we may assume λ is chosen so that $\lambda \in E_{\phi_\lambda}$. Let $\psi_\lambda \in \pi_\lambda^{-1}(\phi_\lambda)$ and set $\tilde{\psi} = \gamma_\lambda(\psi_\lambda)$, $\tilde{\eta} = \tilde{\pi}(\tilde{\psi})$, and $\psi_\mu = \beta_{\lambda,\mu}(\psi_\lambda)$. Then arguing as in the proof of (a), we get $\tilde{\eta} = \alpha_\mu(\pi_\mu(\psi_\mu))$ on $\alpha_\mu(P_\mu)$, so that, in particular, $\tilde{\eta} = \tilde{\phi}_\lambda$ on $\alpha_\lambda(P)$. But also $\tilde{\eta}_\lambda = \pi_\lambda(\psi_\lambda) = \pi_\mu(\psi_\mu)$ on P. As $\lambda \in E_{\phi_\lambda}$ we conclude that $\pi_\mu(\psi_\mu)$ is the unique extension of ϕ_λ to P_μ. Then

$$\tilde{\phi}_\mu = \alpha_\mu(\phi_\mu) = \alpha_\mu(\pi_\mu(\psi_\mu)) = \tilde{\eta}$$

on $\alpha_\mu(P_\mu)$. That is $\tilde{\eta} = \tilde{\phi}$. Thus $\tilde{\phi} = \tilde{\pi}(\gamma(\beta_\lambda(\psi))) = \alpha(\pi_\infty(\beta_\lambda(\psi))$ on $\alpha(\beta_\lambda(P_\lambda) = \tilde{P}$, so indeed $\tilde{\phi}$ is in $\alpha(\mathrm{Hom}_{\mathfrak{G}}(P_\lambda, Q_\lambda))$.

Next $\alpha(\mathrm{Hom}_{\mathfrak{G}}(P, Q)) \subseteq \mathrm{Hom}_{\tilde{\mathcal{F}}}(\alpha(P), \alpha(Q))$ as α is a morphism of fusion systems; further, this map is injective by definition. In particular $\alpha(\mathrm{Im}(\pi_\infty)) \subseteq \tilde{\mathcal{F}}$ and of course $\alpha(\iota_\lambda(\mathcal{F}_\lambda)) \subseteq \tilde{\mathcal{F}}$, so that by definition of $\mathcal{F}_{\mathfrak{G}}$,

$$\alpha(\mathcal{F}_{\mathfrak{G}}) = \alpha(\langle \mathrm{Im}(\pi_\infty), \iota_\lambda(\mathcal{F}_\lambda) \mid \lambda \in \Lambda \rangle) = \langle \alpha(\mathrm{Im}(\pi_\infty)), \alpha(\iota_\lambda(\mathcal{F}_\lambda)) \mid \lambda \in \Lambda \rangle \subseteq \tilde{\mathcal{F}}.$$

By the preceding paragraph, $\mathrm{Hom}_{\tilde{\mathcal{F}}}(\alpha(P), \alpha(Q)) \subseteq \alpha(\mathrm{Hom}_{\mathfrak{G}}(P, Q))$, and so $A_{\tilde{\mathcal{F}}}(\tilde{P}) \subseteq \alpha(\mathcal{F}_{\mathfrak{G}})$ (cf. 2.14). Thus by (v), $\alpha(\mathcal{F}_{\mathfrak{G}}) = \tilde{\mathcal{F}}$. Therefore α induces a bijection $\mathrm{Hom}_{\mathfrak{G}}(P, Q) \to \mathrm{Hom}_{\tilde{\mathcal{F}}}(\tilde{P}, \tilde{Q})$ for all $\tilde{P}$ and $\tilde{Q}$ in $\tilde{\mathcal{F}}$, and hence $\alpha\colon \mathcal{F}_{\mathfrak{G}} \to \tilde{\mathcal{F}}$ is an isomorphism of fusion systems.

By (a), $\gamma : \mathcal{G}_{\mathfrak{G}} \to \widetilde{\mathcal{G}}$ is an embedding, and $\gamma_\lambda = \gamma_\mu \circ \beta_{\lambda,\mu}$ for $\lambda \le \mu$. In order to show that γ is an isomorphism, it remains to show that γ defines a bijection $\mathscr{E}_\infty \to \widetilde{\mathscr{E}}$, and defines bijections on morphism sets. The first condition is a consequence of (vi) and 12.4(c), so it remains to verify the second condition.

Let $\widetilde{\psi} \in \mathrm{Mor}_{\widetilde{\mathscr{L}}}(\widetilde{P}, \widetilde{Q})$ be an $\widetilde{\mathscr{L}}$-morphism. Set $\widetilde{\phi} = \widetilde{\pi}(\widetilde{\psi})$. As α is an isomorphism of fusion systems we may choose $\widehat{P}$, $\widehat{Q}$, and $\phi \in \mathrm{Hom}_{\mathfrak{G}}(\widehat{P}, \widehat{Q})$ so that $\alpha(\phi) = \widetilde{\phi}$. Choose $\psi \in \pi_\infty^{-1}(\phi)$. Then $\gamma(\psi)$ lies in the $\widetilde{\pi}$-fiber over $\widetilde{\phi}$, as γ satisfies (MG2) and $\alpha(\widehat{P}) = \widetilde{P}$. As $\widetilde{\mathcal{G}}$ is a pre-local group there then exists $y \in Z(\widetilde{P})$ such that $\gamma(\psi) = \widetilde{\delta}_{\widetilde{P}}(y) \cdot \widetilde{\phi}$. Let x be the element of $Z(\widehat{P})$ which is mapped to y^{-1} by α_0. As γ satisfies (MG3) we obtain $\gamma(\delta_{\widehat{P}}(x) \circ \psi) = \widetilde{\psi}$, and thus γ is surjective on morphism sets.

Finally, let $[f], [h] \in \mathrm{Mor}_\infty(\widehat{P}, \widehat{Q})$ with $\gamma([f]) = \gamma([h])$. As α is injective on homomorphisms, it follows from (MG2) that $[f]$ and $[h]$ lie in the same π_∞-fiber, and thus $[h] = \delta_{\widehat{P}}(z) \cdot [f]$ for some $z \in Z(\widehat{P})$. We may choose λ so that $z \in \widehat{P} \cap S_\lambda := P$, and also so that $\lambda \in \Omega_f \cap \Omega_h$ for suitable representatives f and h of the given morphisms. Then $z \in Z(P)$ and

$$\begin{aligned}\gamma([f]) = \gamma([h]) = \gamma_\lambda(h(\lambda)) = \gamma_\lambda(\delta_{\lambda,P}(z) \cdot f(\lambda)) = \gamma_\lambda(\delta_{\lambda,P}(z)) \cdot \gamma_\lambda(f(\lambda)) \\ = \widetilde{\delta}_{\widetilde{P}}(\alpha_0(z)) \cdot \gamma([f]).\end{aligned}$$

As $\widetilde{\delta}_{\widetilde{P}}$ defines a free action of $Z(\widetilde{P})$ on $\mathrm{Mor}_{\widetilde{\mathscr{L}}}(\widetilde{P}, \widetilde{Q})$, we conclude that $z = 1$, and that γ is bijective on morphism sets. Thus γ is an isomorphism of categories, and (b) is proved. □

Theorem C is the following result:

THEOREM 12.11. *Take Λ to be the set of nonnegative integers, and for each $i \in \Lambda$ let $\mathcal{G}_i = (S_i, \mathcal{F}_i, \mathscr{L}_i)$ and $\mathcal{G} = (S, \mathcal{F}, \mathscr{L})$ be the 2-local groups defined prior to* 11.2. *Let θ and θ_i be the signalizer functors in* 11.3, *and let $\beta_{i,j} : \mathcal{F}_i^{\mathrm{rc}} \to \mathcal{F}_j^{\mathrm{rc}}$ and $\beta_i : \mathcal{F}_i^{\mathrm{rc}} \to \mathcal{F}^{\mathrm{rc}}$ be the mappings defined prior to* 11.2. *Then the following hold*:

(a) *For each $i, j \in \Lambda$, with $i \le j$, the mappings $\beta_{i,j}$ and β_i extend to embeddings*

$$\beta_{i,j} = (\iota_{i,j}, \beta_{i,j}) : \mathcal{G}_i \to \mathcal{G}_j \quad \textit{and} \quad \beta_i = (\iota_i, \beta_i) : \mathcal{G}_i \to \mathcal{G}$$

of 2*-local groups, and $\beta_j \circ \beta_{i,j} = \beta_i$.*

(b) *$\mathfrak{G} := (\beta_{i,j} : \mathcal{G}_i \longrightarrow \mathcal{G}_j)_{i \le j \in \Lambda}$ is a directed system of embeddings of* 2*-local finite groups.*

(c) *The direct limit $\mathcal{G}_{\mathfrak{G}}$ of $\mathfrak{G}$, in the category of pre-local groups and embeddings, admits the structure of a* 2*-local group isomorphic to $\mathcal{G}$.*

Proof. Write $\mathscr{L}_i^{\mathrm{rc}}$ and $\mathscr{L}^{\mathrm{rc}}$ for the restriction of the centric linking systems $\mathscr{L}_i$ and $\mathscr{L}$ to radical centric linking systems on $\mathcal{F}_i^{\mathrm{rc}}$ and $\mathcal{F}^{\mathrm{rc}}$, respectively.

First, 11.2 and 11.3 show that the mappings β_i and $\beta_{i,j}$ are well defined and satisfy conditions (1′) and (2′) in Proposition 2.12, relative to the appropriate signalizer functors. Then 2.11 and 2.12 yield embeddings of 2-local groups, as in (a), in which β_i and $\beta_{i,j}$ act on $\mathscr{L}_i^{\mathrm{rc}}$-morphisms via

$$\beta_i : \theta_i(P)g \mapsto \theta(\beta(P))g \quad \text{and} \quad \beta_{i,j} : \theta_i(P)g \mapsto \theta(\beta_j(P))g$$

for $P \in \mathscr{F}_i^{\mathrm{rc}}$ and $g \in N_{G_i}(P, S)$. In order to check that $\beta_j \circ \beta_{i,j} = \beta_i$ in the category of 2-local groups, it suffices to check the equality on objects. This follows from 10.10 in cases (1) and (2) of 10.9(c), and from 10.7 in case (3) when $Y = H$. Suppose $Y = K$. By construction, $N_{G_i}(P) \leq N_{G_j}(\beta_{i,j}(P))$, so that $(\beta_j \circ \beta_{i,j})(P) \leq \beta_i(P)$. We check from 10.2 that no proper $N_{G_i}(P)$-invariant subgroup of $\beta_i(P)$ is in $\mathscr{F}^{\mathrm{rc}}$, completing the proof of (a).

The equality $\beta_j \circ \beta_{i,j} = \beta_i$ implies that

$$(\beta_{j,k} \circ \beta_{i,j})(P) = \beta_{j,k}(\beta_{i,j}(P)) = \beta_j(\beta_{i,j}(P)) \cap G_k = \beta_i(P) \cap G_k = \beta_{i,k}(P),$$

so that $\beta_{j,k} \circ \beta_{i,j} = \beta_{i,k}$. Also $\beta_{i,i} = 1$ by 11.12(c). Hence (b) holds.

We next check that the hypotheses of 12.10(b) are satisfied with $\mathscr{G}$, $\mathscr{L}_i^{\mathrm{rc}}$, β_i, $i \in \Lambda$, in the roles of $\widetilde{\mathscr{G}}$, $\widetilde{\mathscr{E}}$, γ_λ, $\lambda \in \Lambda$, respectively.

First, S is the union of the groups S_i for $i \in \Lambda$, so that condition (i) of 12.10(b) holds. Second, for any $\phi = c_g \in \mathrm{Hom}_{\mathscr{F}}(P, Q)$ we have $g \in G_i$ for some i, so that $\phi_{|S_j} \in \mathrm{Hom}_{\mathscr{F}_j}(P \cap S_j, Q \cap S_j)$ for all $j \geq i$, and hence condition (iv) holds.

Let $P \in \mathscr{F}_i^{\mathrm{rc}}$, and for $j \geq i$ set $P_j = \beta_{i,j}(P_i)$. We have $P_j = \beta_i(P_i) \cap G_j$ for such j, and S is the union of its subgroups S_j, and so condition (iii) of 12.10(b) holds.

We claim that there exists $j \geq i$ such that $C_G(P_j)$ centralizes P_k for all $k \geq j$. Suppose first that $|Z_{P_i}| \leq 4$. Then from the proof of 10.9, $C_G(P_i) \leq P_i$. But $Z(\beta_i(P_i))$ is finite, and $P_k = \beta_i(P) \cap G_k$ for $k \geq i$ and so, from (ii), we may choose $j \geq i$ with $Z(P_j) = Z(\beta_i(P_i))$. Thus $C_G(P_j) = Z(P_k)$ for $k \geq j$. On the other hand if $|Z_{P_i}| > 4$ then $P_i \in A^G$ or $P_i = C_{S_i}(E)$ by 10.5. In the first of these two cases, $P_i = P_k$ for all $k \geq i$, while in the second $C_G(P_i) = EC_{\theta(A)}(T_2) = C_G(P_k)$ for any $k \geq i$, by 7.10 and 7.13. Thus the claim is established.

We now verify condition (ii) of 12.10(b). Let $\phi \in \mathrm{Hom}_{\mathscr{F}_j}(P_j, Q_j)$ and suppose that for some $k \geq j$, we have $\phi_1, \phi_2 \in \mathrm{Hom}_{\mathscr{F}_k}(P_k, Q_k)$ extending ϕ. Then ϕ_r is the restriction of c_{g_r} to P_k for some $g_r \in G$, $r = 1, 2$. Then $g_1 g_2^{-1} \in C_G(P_j) \leq C_G(P_k)$ by the claim, so $\phi_1 = \phi_2$. This yields (ii).

Condition (v) of 12.10(b) follows from 9.10. Finally each β_i is injective on objects, since $\beta_i(P) \cap S_i = P$ by 11.12(c). Thus $\mathfrak{G}$ is nearly injective, and condition (vi) of 12.10(b) holds. Therefore we conclude from 12.10(b) that $(S, \mathscr{F}, \mathscr{L}^{\mathrm{rc}}) \cong \mathscr{G}_{\mathfrak{G}}$ as pre-local groups, via the isomorphism $\gamma = (\alpha, \gamma)$ constructed in the proof of 12.5 and 12.10. In particular $\alpha(\mathscr{E}) = \mathscr{F}^{\mathrm{rc}}$, so as $\alpha : \mathscr{F}_{\mathfrak{G}} \to \mathscr{F}$ is an isomorphism of fusion systems, $\mathscr{E} = \mathscr{F}^{\mathrm{rc}}_{\mathfrak{G}}$, and $\mathscr{F}_{\mathfrak{G}}$ is saturated as $\mathscr{F}$ is saturated. Transferring the centric linking system $\mathscr{L}$ on $\mathscr{F}$ to $\mathscr{F}_{\mathfrak{G}}$ via α, we may regard $\mathscr{G}_{\mathfrak{G}}$ as a 2-local group, and the isomorphism γ of pre-local groups is then also an isomorphism of 2-local groups. This completes the proof of (c). □

Theorem D is essentially the following result. We thank Ran Levi and Bob Oliver for guiding us through a proof.

THEOREM 12.12. *The 2-completed nerve* $|\mathscr{L}_{\mathfrak{G}}|^{\wedge}_2$ *is homotopy equivalent to* $B\,\mathrm{DI}(4)$.

Proof. Let $\mathscr{L}^{cc}_m$ be the full subcategory of $\mathscr{L}^{c}_m$ whose objects are centric in $\mathscr{F}^{c}_n$ for all $n \geq m$. Then $\mathscr{F}^{\mathrm{rc}}_m \subseteq \mathrm{Obj}(\mathscr{L}^{cc}_m)$ for all m, by 10.9(b). Set $\gamma_m = \beta_{m,m+1}$, and consider the diagram of categories and functors

$$(*) \qquad \begin{array}{ccc} \mathscr{L}^{\mathrm{rc}}_m & \xrightarrow{\gamma_m} & \mathscr{L}^{\mathrm{rc}}_{m+1} \\ {\scriptstyle\iota_m}\downarrow & & \downarrow{\scriptstyle\iota_{m+1}} \\ \mathscr{L}^{cc}_m & \xrightarrow{\iota^{cc}_m} & \mathscr{L}^{cc}_{m+1}, \end{array}$$

where ι is in every instance an inclusion functor. We claim that this diagram commutes up to a natural homomorphism $\mu\colon \iota^{cc} \circ \iota_m \to \iota_{m+1} \circ \gamma_m$. Since $\iota^{cc} \circ \iota_m$ is the identity map on objects, what this means is that for all $P, Q \in \mathscr{L}^{\mathrm{rc}}_m$, there are $\mathscr{L}_{m+1}$-morphisms μ_P and μ_Q such that, for each $\psi \in \mathrm{Mor}_m(P, Q)$, the following diagram commutes.

$$(**) \qquad \begin{array}{ccc} P & \xrightarrow{\iota^{cc}(\iota_m(\psi))} & Q \\ {\scriptstyle\mu_P}\downarrow & & \downarrow{\scriptstyle\mu_Q} \\ \gamma_m(P) & \xrightarrow{\iota_{m+1}(\gamma(\psi))} & \gamma_m(Q). \end{array}$$

Indeed, for any $R \in \mathscr{L}^{\mathrm{rc}}_m$ define μ_R to be $\theta_{m+1}(R)$. Recall that $\psi = \theta_m(P)g$ for some $g \in N_{G_m}(P, Q)$, by the definition of $\mathscr{L}_m$. The functor ι^{cc} sends ψ to ψ regarded as an element of $\mathrm{Mor}_{m+1}(P, Q)$. That is, we have $\iota^{cc}(\psi) = \theta_{m+1}(P)g$, and hence (in our mix of left- and right-hand notation, as set forth in §1)

$$\iota^{cc}(\iota_m(\psi)) \cdot \mu_Q = \theta_{m+1}(P)g\theta_{m+1}(Q) = \theta_{m+1}(P)g,$$

while also

$$\mu_P \cdot \iota_{m+1}(\gamma_m(\psi)) = \theta_{m+1}(P)\theta_{m+1}(\gamma_m(P))g = \theta_{m+1}(P)g.$$

Thus μ is a natural transformation as desired, and the claim is proved.

Recall that the nerve of a small category $\mathscr{C}$ is a simplicial set (or equivalently, the topological realization of a simplicial set) whose k-simplices are chains

$(\alpha_0, \dots, \alpha_k)$ of composable morphisms in $\mathscr{C}$. If $f\colon \mathscr{C} \to \mathscr{D}$ is a functor of small categories, then there is a continuous map $|f|\colon |\mathscr{C}| \to |\mathscr{D}|$ of spaces, given by $(\alpha_0, \dots, \alpha_k) \mapsto (f(\alpha_0), \dots, f(\alpha_k))$.

Set $X_m = |\mathscr{L}_m^{\mathrm{rc}}|$, $Y_m = |\mathscr{L}_m^{cc}|$, and consider the following diagram of spaces and continuous maps.

$$(***) \qquad \begin{array}{ccccccc} X_1 & \xrightarrow{|\gamma_1|} & X_2 & \xrightarrow{|\gamma_2|} & X_3 & \xrightarrow{|\gamma_3|} & \dots \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ Y_1 & \xrightarrow{f_1^{cc}} & Y_2 & \xrightarrow{f_2^{cc}} & Y_3 & \xrightarrow{f_3^{cc}} & \dots . \end{array}$$

Here, we are taking $f_m := |\iota_m|$ and $f_m^{cc} := |\iota_m^{cc}|$. Each f_m^{cc} may be viewed as inclusion, since ι_m^{cc} induces an inclusion of $\mathrm{Mor}_m(P, Q)$ into $\mathrm{Mor}_{m+1}(P, Q)$ for any $P, Q \in \mathscr{L}_m^{cc}$. Similarly, since each γ_m is an embedding, γ_m induces an injective mapping $\mathrm{Mor}_m(P, Q) \to \mathrm{Mor}_{m+1}(P, Q)$ for $P, Q \in \mathscr{L}_m^{\mathrm{rc}}$, and hence $|\gamma_m|$ is injective. There is then no harm in viewing each $|\gamma_m|$ as an ordinary inclusion of topological spaces (and in adjusting the vertical arrows by suitable homeomorphisms, to compensate for this). The direct limit X of the top row in $(***)$ is then the union of the spaces X_n. No such adjustment is necessary for the bottom row, whose union we denote by Y.

It is the content of [LO02, Prop. 4.3] that the 2-completion $(Y)_2^{\wedge}$ of Y is $B\,\mathrm{DI}(4)$, up to homotopy equivalence. That this is so requires some explanation, since the union taken in [LO02] is that of a somewhat different collection of spaces than $\{Y_m\}_{m>0}$. Namely, Levi and Oliver choose a sequence $(n_i)_{i>0}$ of positive integers, so that each n_i divides n_{i+1} and so that every positive integer divides some n_i. They then show that $B\,\mathrm{DI}(4)$ is the 2-completion of the union of the spaces $|\mathscr{L}_{\mathrm{Sol}}(q^{n_i})^{cc}|$, for any odd prime power q. We may take $q = p$, and may take the sequence $(n_i)_{i>0}$ so that 2^i is the highest power of 2 dividing n_i. Then $\mathscr{F}_{\mathrm{Sol}}(p^{n_i})$ is a fusion system over the Sylow 2-subgroup S_i of $\mathrm{Spin}_7(p^{n_i})$. As the 2-shares of $\mathrm{Spin}_7(p^{n_i})$ and $\mathrm{Spin}_7(p^{2^i})$ are the same, $\mathrm{Spin}_7(p^{n_i})$ has the same Sylow 2-subgroup as $\mathrm{Spin}_7(p^{2^i})$. By a result in [COS06] the fusion systems $\mathscr{F}_{\mathrm{Sol}}(p^{n_i})$ and $\mathscr{F}_{\mathrm{Sol}}(p^{2^i})$ are isomorphic, and their corresponding linking systems are then isomorphic [LO02, Lemma 3.2]. Thus, the union of the nerves $|\mathscr{L}_{\mathrm{Sol}}(p^{n_i})^{cc}|$ is homeomorphic to Y, and $(Y)_2^{\wedge}$ may be identified with $B\,\mathrm{DI}(4)$.

Since it is obvious from the definitions that the nerve of an increasing union of categories is the union of the nerves, it follows from 12.5 that the space X is homeomorphic to $|\mathscr{L}_{\mathfrak{G}}|$. Thus, it remains only to show that X is homotopy equivalent to Y.

The existence of a natural transformation μ as in $(**)$ implies that each of the squares in $(***)$ commutes up to homotopy (cf. [Dwy01, Prop. 5.2]. For any m, the inclusion of the simplicial set X_m in X_{m+1} is a CW-pair, and so the homotopy extension property for CW-pairs [Hat02, Prop. 0.16] implies that f_2 may be replaced by a map f_2' which is homotopic to f_2 and which extends f_1. We continue up the chain, replacing f_{m+1} by a map f_{m+1}' which is homotopic to f_{m+1} and extends f_m'. Now define $f\colon X \to Y$ to be the union of the maps f_m'.

Observe that every finite subcomplex of the CW-complex X (or Y) is contained in some X_i (or Y_i). Since every compact subset of a CW-complex is contained in a finite subcomplex, every compact subset of X or Y is contained in some X_i or Y_i. From this, it follows directly from the definition of homotopy groups that for each n, $\pi_n(X)$ is the direct limit of the $\pi_n(X_i)$, and similarly for $\pi_n(Y)$. That each f_m (and hence also each f_m') is a homotopy equivalence is given by [BCG+05, Th. B], and thus $\pi_n(f)$ is an isomorphism for all n. Then f is a homotopy equivalence by Whitehead's Theorem, and the proof is complete. □

We close with an example.

Example 12.13. Let p be a prime and let $G = G(F)$ be a Chevalley group over the algebraic closure F of $\mathbf{F}_p$ of Lie rank l. Let Σ be the set of positive integers, partially ordered by $n \le m$ if n divides m. Set $I = \{1, \dots, l\}$ and let $(P_J \mid \varnothing \ne J \subseteq I)$ be the set of proper parabolic subgroups of G over a fixed Borel subgroup $B = P_I$. For $J \subseteq I$ let S_J be the unipotent radical of P_J, and set $S = S_I$. Let $\psi_1\colon a \mapsto a^p$ be the Frobenius map on F, and regard ψ_1 also as a field automorphism of G. For $k \ge 1$ set $\psi_k = \psi_1^k$, and let $G_k = G_{\psi_k}$ be the group of fixed points of ψ_k on G. Set $S_k = S \cap G_k$, $S_{J,k} = S_J \cap G_k$, and let

$$\mathcal{G}_k = \mathcal{G}_{S_k}(G_k) = (S_k, \mathcal{F}_k, \mathcal{L}_k)$$

be the p-local finite group associated with G_k.

By Borel-Tits, $\mathcal{F}_k^{\mathrm{rc}} = (S_{J,k} : J \subseteq I)$, and $\theta_k(P) := O^p(C_{G_k}(P)) = 1$ for all $P \in \mathcal{F}_k^{\mathrm{rc}}$. When k divides j, we have the inclusion map $\beta_{k,j}\colon \mathcal{G}_k \to \mathcal{G}_j$ with $\beta_{k,j}(S_{J,k}) = S_{J,j}$. It follows from 2.11 that $\mathfrak{G} := (\mathcal{G}_k, \beta_{k,j} : k \le j)$ is a directed system of p-local finite groups. Further one can check that the hypotheses of 12.10(b) are satisfied by $\mathfrak{G}$, so, as in the proof of Theorem 12.11, the limit $\mathcal{G}_{\mathfrak{G}}$ of $\mathfrak{G}$ is isomorphic to $\mathcal{G}(G) = (\mathcal{F}_S(G), \mathcal{F}_S^{\mathrm{rc}}(G), \mathcal{L})$, where $\mathcal{F}_S^{\mathrm{rc}}(G)$ has object set $(S_J : J)$, $\mathrm{Mor}(S_J, S_K) = \mathrm{Hom}(S_J, S_K) = N_G(S_J, S_K)$, and π and δ are identity maps.

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