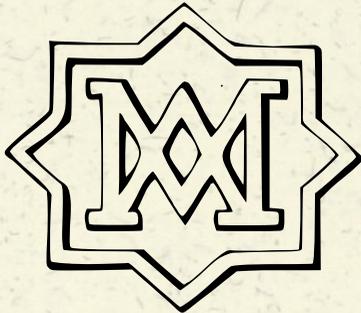


# ANNALS OF MATHEMATICS

**The cohomology ring of the real locus  
of the moduli space of stable curves  
of genus 0 with marked points**

By PAVEL ETINGOF, ANDRÉ HENRIQUES,  
JOEL KAMNITZER, and ERIC M. RAINS



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## Abstract

We compute the Poincaré polynomial and the cohomology algebra with rational coefficients of the manifold  $M_n$  of real points of the moduli space of algebraic curves of genus 0 with  $n$  labeled points. This cohomology is a quadratic algebra, and we conjecture that it is Koszul. We also compute the 2-local torsion in the cohomology of  $M_n$ . As was shown by the fourth author, the cohomology of  $M_n$  does not have odd torsion, so that the above determines the additive structure of the integral homology and cohomology. Further, we prove that the rational homology operad of  $M_n$  is the operad of 2-Gerstenhaber algebras, which is closely related to the Hanlon-Wachs operad of 2-Lie algebras (generated by a ternary bracket). Finally, using Drinfeld’s theory of quantization of coboundary Lie quasibialgebras, we show that a large series of representations of the quadratic dual Lie algebra  $L_n$  of  $H^*(M_n, \mathbb{Q})$  (associated to such quasibialgebras) factors through the natural projection of  $L_n$  to the associated graded Lie algebra of the pronipotent completion of the fundamental group of  $M_n$ . This leads us to conjecture that the said projection is an isomorphism, which would imply a formula for lower central series ranks of the fundamental group. On the other hand, we show that the spaces  $M_n$  are not formal starting from  $n = 6$ .

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## References

## 1. Introduction

Let  $\overline{M_{0,n}}$  be the Deligne-Mumford compactification of the moduli space of algebraic curves of genus 0 with  $n$  labelled points. It is a smooth projective variety over  $\mathbb{Q}$  ([DM69]) which parametrizes stable, possibly singular, curves of genus 0 with  $n$  labelled points. The geometry of this variety is very well understood; in particular, the cohomology ring of the manifold of complex points  $\overline{M_{0,n}}(\mathbb{C})$  was computed by Keel [Kee92], who also showed that it coincides with the Chow ring of  $\overline{M_{0,n}}$ .

In this paper we will be interested in the topology of the manifold  $M_n := \overline{M_{0,n}}(\mathbb{R})$  of *real* points of this variety. There are a number of prior results concerning the topology of this manifold. Kapranov and Devadoss ([Kap93], [Dev99]) found a cell decomposition of  $M_n$ , and Devadoss used it to determine its Euler characteristic. Davis-Januszkiewicz-Scott ([DJS03]) found a presentation of the fundamental group  $\Gamma_n$  of  $M_n$  and proved that  $M_n$  is a  $K(\pi, 1)$  space for this group. Nevertheless, the topology of  $M_n$  is less well understood than that of  $\overline{M_{0,n}}(\mathbb{C})$ . In particular, the Betti numbers of  $M_n$  have been unknown until present.

1.1. *The cohomology of  $M_n$ .* In this paper, we completely determine the cohomology of  $M_n$  with rational coefficients. In Definition 2.1, we give a presentation of the cohomology algebra  $H^*(M_n, \mathbb{Q})$  by generators and relations. Later in Theorem 6.4, we give a basis of this algebra and use this show that its Poincaré series equals

$$P_n(t) = \prod_{0 \leq k < (n-3)/2} (1 + (n-3-2k)^2 t).$$

In particular, we show that the rank of  $H_1(M_n)$  is  $\binom{n-1}{3}$  which answers a question of Morava, [Mor, p. 5].<sup>1</sup>

The variety  $\overline{M_{0,n}}$ , and hence the manifold  $M_n$ , has an action of the symmetric group  $S_n$  which permutes the labelled points. In a subsequent paper, E.R. [Rai09] computes the character of the action of  $S_n$  on  $H^*(M_n, \mathbb{Q})$ .

The manifolds  $M_n$  are not orientable for  $n \geq 5$ , and in particular their cohomology groups are not free and contain 2-torsion. We determine the 2-torsion in the cohomology, and show there is no 4-torsion. The cohomology of  $M_n$  does not have odd torsion (this has been recently shown by E.R. [Rai]), so our results give a description of the cohomology  $H^*(M_n, \mathbb{Z})$ . Note that since  $M_n$  is a  $K(\pi, 1)$ -space, this cohomology is also the cohomology of the group  $\Gamma_n$ .

The description of the cohomology of  $M_n$  has recently been generalized by E.R. [Rai] to a computation of the integral homology of the real points of any de Concini-Procesi model coming from any real subspace arrangement (the manifold  $M_n$  comes from the  $A_{n-2}$  hyperplane arrangement).

1.2. *The operad structure.* The collection of spaces  $M_n$  forms a topological operad, since stable curves of genus 0 can be attached to each other at marked points, as described in Section 2.4. Similarly, the homology  $H_*(M_n, \mathbb{Q})$  is an operad in the symmetric monoidal category of  $\mathbb{Z}$ -graded  $\mathbb{Q}$ -supervector spaces. This operad was first discussed by Morava [Mor], who suggested that it might be related to symplectic topology. Understanding this operad was one of the primary motivations for this work.

<sup>1</sup>Some partial results in this direction were obtained in [Bara], [Barb].

In [Theorem 2.14](#), we show that this operad is generated by a supercommutative associative product  $ab$  of degree 0 and skew-supercommutative ternary “2-bracket”  $[a, b, c]$ , such that the 2-bracket is a derivation in each variable and satisfies a quadratic Jacobi identity in the space of 5-ary operations. Motivated by the Hanlon-Wachs theory of Lie 2-algebras [\[HW95\]](#), we call this the operad of 2-Gerstenhaber algebras.

The structure of the homology operad of  $\overline{M}_{0,n}(\mathbb{C})$  was determined by Kontsevich and Manin ([\[KM94\]](#), see also [\[Get95\]](#)); in this case the operad (called the operad of hypercommutative algebras) turns out to be infinitely generated.

1.3. *The analogy with braid groups.* We see that the space  $\overline{M}_{0,n}(\mathbb{R})$  has very different topological properties from those of  $\overline{M}_{0,n}(\mathbb{C})$ . Indeed,  $\overline{M}_{0,n}(\mathbb{R})$  is  $K(\pi, 1)$ , its Poincaré polynomial has a simple factorization, its Betti numbers grow polynomially in  $n$ , and its homology is a finitely generated operad. In contrast,  $\overline{M}_{0,n}(\mathbb{C})$  is simply connected, its Poincaré polynomial does not have a simple factorization, its Betti numbers grow exponentially, and its homology operad is infinitely generated.

On the other hand, the properties of the configuration space  $C_{n-1} = \mathbb{C}^{n-1} \setminus \Delta$  of  $n - 1$ -tuples of distinct complex numbers are much more similar to those of  $\overline{M}_{0,n}(\mathbb{R})$ . It is a  $K(\pi, 1)$  space with Poincaré polynomial  $(1 + t) \cdots (1 + (n - 2)t)$ , so its Betti numbers grow polynomially. Also its homology operad is well known to be the operad of Gerstenhaber algebras, which has two binary generators. The analogy between  $M_n$  and  $C_{n-1}$  and between their fundamental groups (the pure cactus group  $\Gamma_n$  and the pure braid group  $PB_{n-1}$ ), discussed already in [\[Dev99\]](#), [\[Mor\]](#), and [\[HK06\]](#), is very useful and has been a source of inspiration for us while writing this paper.

1.4. *The Lie algebra  $L_n$ .* One application of the computation of the cohomology ring is that it allows us to understand various Lie algebras associated to the group  $\Gamma_n$ .

The cohomology algebra  $H^*(M_n, \mathbb{Q})$  is a quadratic algebra and thus we can consider its quadratic dual algebra  $U_n$ , which is the universal enveloping algebra of a quadratic Lie algebra  $L_n$ . Since we have a presentation of  $\Lambda_n$ , we get a presentation of  $L_n$  (see [Proposition 3.1](#)).

On the other hand, one can construct a Lie algebra  $\mathcal{L}_n$  directly from  $\Gamma_n$ , by taking the associated graded of the lower central series filtration and then quotienting by the 2-torsion. In [Theorem 3.9](#), we construct a surjective homomorphism of graded Lie algebras  $\psi_n : L_n \rightarrow \mathcal{L}_n$ . We expect that this homomorphism is actually an isomorphism, similarly to the braid group case.

We also expect that the algebra  $U_n$  is Koszul. On the other hand, somewhat disappointingly, we show that for  $n \geq 6$  the Malcev Lie algebra of  $\Gamma_n$  is not isomorphic to the degree completion of  $L_n \otimes \mathbb{Q}$ , and in particular the spaces  $M_n$  for

$n \geq 6$  are not formal. This fact reflects an essential difference between the pure cactus group and the pure braid group.

1.5. *Relation to coboundary Lie quasibialgebras and quasiHopf algebras.*

The motivation for the conjecture that the map  $\psi_n$  is an isomorphism comes from the theory of coboundary Lie quasibialgebras. Let  $\mathfrak{g}$  be a Lie algebra over a field of characteristic zero, with a coboundary Lie quasibialgebra structure  $\varphi \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$  ([Dri89]). Let  $X_1, \dots, X_{n-1}$  be representations of  $\mathfrak{g}$ . From the explicit form of generators and relations for  $L_n$ , we show that  $L_n$  acts on  $X_1 \otimes \dots \otimes X_{n-1}$  (see §3.8).

On the other hand, Drinfeld showed in [Dri89] that any coboundary Lie quasibialgebra can be quantized to a coboundary quasiHopf algebra. The representation category of such a quasiHopf algebra is a coboundary category. The group  $\Gamma_n$  acts on a tensor product  $Y_1 \otimes \dots \otimes Y_{n-1}$  in any coboundary monoidal category (see [HK06] and §3.4). From this, we get an action of  $\mathcal{L}_n$  on  $X_1 \otimes \dots \otimes X_{n-1}$ .

Theorem 3.12 shows that the action of  $L_n$  on  $X_1 \otimes \dots \otimes X_{n-1}$  factors through the morphism  $\psi_n$  and this action of  $\mathcal{L}_n$ .

The above statements are direct analogs of the corresponding results for pure braid groups and quasitriangular Lie quasibialgebras, as developed by Drinfeld in [Dri89], [Dri90]. Moreover, the entire action of the braid group on tensor products can be recovered as the monodromy of the Knizhnik-Zamolodchikov connection. However, because of the nonformality of  $M_n$ , at the moment we are pessimistic about the existence of an analogous result in our case.

1.6. *Organization.* The paper is organized as follows. Section 2 contains the statements of the main theorems describing  $H^*(M_n, \mathbb{Q})$ . In Section 3 we give additional results and conjectures, mostly concerning the Lie algebra  $L_n$ . Sections 4, 5, and 6 are devoted to the proof of the main theorem. First, in Section 4, we prove that a certain algebra related to  $H^*(M_n, \mathbb{Q})$  has a basis indexed by “basic triangle forests” (combinatorial objects we introduce for this purpose). In Section 5, we recall Keel’s description of  $H^*(\overline{M}_{0,n}(\mathbb{C}), \mathbb{Z})$  and use it to give an upper bound on the ranks of  $H^*(M_n, \mathbb{Q})$ . Finally in Section 6, we prove our main results concerning the cohomology ring of  $M_n$ .

2. The cohomology ring and the homology operad

2.1. *The algebra  $\Lambda_n$ .* We begin by introducing an algebra which will turn out to be equal to the cohomology ring of  $M_n$  over  $\mathbb{Q}$ .

*Definition 2.1.*  $\Lambda_n$  is the skew-commutative algebra generated over  $\mathbb{Z}$  by elements  $\omega_{ijkl}$ ,  $1 \leq i, j, k, l \leq n$ , which are antisymmetric in  $ijkl$ , with defining relations

$$(2.1) \quad \omega_{ijkl} + \omega_{jklm} + \omega_{klmi} + \omega_{lmij} + \omega_{mijk} = 0,$$

$$(2.2) \quad \omega_{ijkl} \omega_{ijkm} = 0,$$

$$(2.3) \quad \omega_{ijkl} \omega_{lmpi} + \omega_{klmp} \omega_{pijk} + \omega_{mpij} \omega_{jklm} = 0$$

for any distinct  $i, j, k, l, m, p$ ,

In particular,  $\Lambda_n$  is quadratic.

We will also consider the algebras  $\Lambda_n \otimes R$  for commutative rings  $R$ . They are defined over  $R$  by the same generators and relations.

*Remark.* One can show (by a somewhat tedious calculation, which we did using the program ‘‘Magma’’) that 2 times (2.3) is in the ideal generated by (2.1) and (2.2). So this relation becomes redundant if  $1/2 \in R$ .

The algebra  $\Lambda_n$  has a natural action of  $S_n$ .

PROPOSITION 2.2. *One has  $\Lambda_n[1] = \wedge^3 \mathfrak{h}_n$ , as  $S_n$ -modules, where  $\mathfrak{h}_n$  is the  $n - 1$ -dimensional submodule of the permutation representation, consisting of vectors with zero sum of coordinates. In particular,  $\Lambda_n[1]$  is free of rank*

$$(n - 1)(n - 2)(n - 3)/6.$$

*Proof.* An isomorphism  $\Lambda_n[1] \rightarrow \wedge^3 \mathfrak{h}_n$  is given by

$$\omega_{ijkl} \rightarrow (e_i - e_l) \wedge (e_j - e_l) \wedge (e_k - e_l). \quad \square$$

We now switch to a different presentation of  $\Lambda_n$ . In this presentation only the  $S_{n-1}$ -symmetry, rather than the full  $S_n$ -symmetry, is apparent. However the presentation only contains quadratic relations.

PROPOSITION 2.3. *The algebra  $\Lambda_n$  is isomorphic (in a natural way) to the skew-commutative algebra generated by  $v_{ijk}$ ,  $1 \leq i, j, k \leq n - 1$  (antisymmetric in  $ijk$ ) with defining relations*

$$v_{ijk} v_{ijl} = 0,$$

and

$$v_{ijk} v_{klm} + v_{jkl} v_{lmi} + v_{klm} v_{mij} + v_{lmi} v_{ijk} + v_{mij} v_{jkl} = 0.$$

*Proof.* Let  $\Lambda'_n$  be the algebra defined as in the proposition. Define a homomorphism  $f : \Lambda'_n \rightarrow \Lambda_n$  by the formula  $v_{ijk} \mapsto \omega_{ijkn}$ . By directly manipulating the relations, it is easy to see that this homomorphism is well-defined. Using the 5-term linear relation (2.1), we can find an inverse for  $f$ . Thus  $f$  is an isomorphism.  $\square$

THEOREM 2.4. *For each  $n$ ,  $\Lambda_n$  is a free  $\mathbb{Z}$ -module with Poincaré polynomial*

$$P_n(t) = \prod_{0 \leq k < (n-3)/2} (1 + (n - 3 - 2k)^2 t).$$

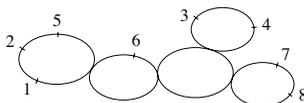
The proof of this theorem is given in Section 6.

2.2. *The real moduli space.*

2.2.1. *Stable curves.* Recall [DM69] that a stable curve of genus 0 with  $n$  labeled points is a finite union  $C$  of projective lines  $C_1, \dots, C_p$ , together with labeled distinct points  $z_1, \dots, z_n \in C$  such that the following conditions are satisfied:

- (i) each  $z_i$  belongs to a unique  $C_j$ ;
- (ii)  $C_i \cap C_j$  is either empty or consists of one point, and in the latter case the intersection is transversal;
- (iii) The graph of components (whose vertices are the lines  $C_i$  and whose edges correspond to pairs of intersecting lines) is a tree;
- (iv) The total number of special points (i.e. marked points or intersection points) that belong to a given component  $C_i$  is at least 3.

So a stable curve must have at least three labeled points.



*A stable curve with 8 marked points.*

An equivalence between two stable curves  $C = (C_1, \dots, C_p, z_1, \dots, z_n)$  and  $C' = (C'_1, \dots, C'_p, z'_1, \dots, z'_n)$  is an isomorphism of algebraic curves  $f : C \rightarrow C'$  which maps  $z_i$  to  $z'_i$  for each  $i$ . Thus  $f$  reduces to a collection of  $p$  fractional linear maps  $f_i : C_i \rightarrow C'_{\sigma(i)}$ , where  $\sigma$  is a permutation. It is easy to see that any equivalence of  $C$  to itself is the identity.

Over the real numbers, the projective lines are circles, so a stable curve is a “cactus-like” structure – a tree of circles with labeled points on them.

2.2.2. *The moduli space  $M_n$ .* For  $n \geq 3$ , let  $M_n$  be the real locus of the Deligne-Mumford compactification of the moduli space of curves of genus zero with  $n$  marked points labeled  $1, \dots, n$ . In other words,  $M_n$  is the set of equivalence classes of stable curves of genus 0 with  $n$  labeled points defined over  $\mathbb{R}$ . Clearly,  $M_n$  carries a natural (nonfree) action of  $S_n$ .

*Example 2.5.* (i)  $M_3$  is a point.

- (ii)  $M_4$  is a circle. More precisely, the cross-ratio map defines an isomorphism  $M_4 \rightarrow \mathbb{RP}^1$ .
- (iii)  $M_5$  is a compact connected nonorientable surface of Euler characteristic  $-3$ , i.e. the connected sum of five real projective planes (see [Dev99]).

The following theorem summarizes some of the known results about  $M_n$ .

THEOREM 2.6. (i)  $M_n$  is a connected, compact, smooth manifold of dimension  $n - 3$ .

(ii) The Euler characteristic of  $M_n$  is 0 for even  $n$  and

$$(-1)^{(n+1)/2} (n-2)!! (n-4)!!$$

if  $n$  is odd.

(iii)  $M_n$  is a  $K(\pi, 1)$ -space.

Part (i) is well-known and appears in [Dev99] and [DJS03]. Part (ii) is due to Devadoss (see [Dev99, Th. 3.2.3]) and Gaiffi (see [Gai04]) and comes from understanding the natural cell structure on  $M_n$ . Part (iii) is due to Davis-Januszkiewicz-Scott [DJS03]. It is proven by showing that  $M_n$  is a  $\text{Cat}(0)$ -space.

2.3. *The cohomology of  $M_n$ .* Let us now formulate the main result of this paper. To do so, note that for any ordered  $m$ -element subset  $S = \{s_1, \dots, s_m\}$  of  $\{1, \dots, n\}$  we have a natural map  $\phi_S : M_n \rightarrow M_m$ , forgetting the points with labels outside  $S$ . More precisely, given a stable curve  $C$  with labeled points  $z_1, \dots, z_n$ ,  $\phi_S(C)$  is  $C$  with labeled points  $z_{s_1}, \dots, z_{s_m}$ , in which the components that have fewer than three special points have been collapsed in an obvious way.

Thus for any commutative ring  $R$  we have a homomorphism of algebras  $\phi_S^* : H^*(M_m, R) \rightarrow H^*(M_n, R)$ .

For  $m = 4$ ,  $M_m$  is a circle, and we denote by  $\omega_S$  the image of the standard generator of  $H^1(M_4, R)$  under  $\phi_S^*$ .

PROPOSITION 2.7. *Over any ring  $R$  in which 2 is invertible, the elements  $\omega_S$  satisfy the relations (2.1), (2.2).*

*Proof.* It is sufficient to consider the case  $R = \mathbb{Z}[1/2]$ . The skew-symmetry of  $\omega_S$  is obvious.

Next, we check the quadratic relations (2.2). By considering the maps  $\phi_S$  for  $|S| = 5$ , it suffices to check this relation on  $M_5$ . But  $H^2(M_5, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$  because  $M_5$  is nonorientable, so by the universal coefficient theorem  $H^2(M_5, R) = 0$ .

The 5-term linear relation (2.1) may also be checked on  $M_5$ . Since  $H^1(M_5, \mathbb{Z})$  is free over  $\mathbb{Z}$ , it is sufficient to check the relation after tensoring with  $\mathbb{Q}$ . As an  $S_5$ -module,  $H^1(M_5, \mathbb{Q})$  is the tensor product of the permutation and sign representations. In particular, the 5-cycle has no invariants in this representation, and hence the 5-term relation holds.  $\square$

COROLLARY 2.8. *For any ring  $R$  in which 2 is invertible, we have a homomorphism of algebras*

$$(2.4) \quad f_n^R : \Lambda_n \otimes R \rightarrow H^*(M_n, R),$$

which maps  $\omega_S$  to  $\omega_S$ .

Our main result is the following theorem.

**THEOREM 2.9.**  $f_n^{\mathbb{Q}}$  is an isomorphism.

It then follows from [Theorem 2.4](#) that the Poincaré polynomial of  $M_n$  is  $P_n(t)$ . In the process, we also prove the following result.

**THEOREM 2.10.**  $H^*(M_n, \mathbb{Z})$  does not have 4-torsion.

A description of the 2-torsion in  $H^*(M_n, \mathbb{Z})$  (which happens to be quite big) is given in Section 5.

The absence of 4-torsion in  $H^*(M_n, \mathbb{Z})$  allows us to sharpen the above statements as follows.

For any abelian group  $A$ , we denote by  $A\langle 2 \rangle$  the 2-torsion in  $A$ .

**PROPOSITION 2.11.** *Over any ring  $R$ , the elements  $\omega_S$  satisfy the relations (2.1), (2.2), and (2.3) modulo 2-torsion. Hence, we have a homomorphism of algebras*

$$f_n^R : \Lambda_n \otimes R \rightarrow H^*(M_n, R)/H^*(M_n, R)\langle 2 \rangle,$$

which maps  $\omega_S$  to  $\omega_S$ .

*Proof.* It is sufficient to consider the case  $R = \mathbb{Z}$ . Relations (2.1) and (2.2) modulo 2-torsion are proved in the same way as in the case when  $\frac{1}{2} \in R$ . It remains to prove (2.3). Although the relation (2.3) is not in the ideal generated by the other relations (over  $\mathbb{Z}$ ), if we multiply it by 2, we obtain an element of this ideal, and thus the relation holds modulo 4-torsion. Since by [Theorem 2.10](#),  $H^*(M_6, \mathbb{Z})$  has no 4-torsion<sup>2</sup>, the result follows.  $\square$

Now [Theorem 2.9](#) can be strengthened as follows. Let  $\mathbb{Z}_2$  be the ring of 2-adic integers (we could equivalently consider the ring of 2-local integers).

**THEOREM 2.12.**  $f_n^{\mathbb{Z}_2}$  is an isomorphism.

In fact, one has the following stronger result.

**THEOREM 2.13 ([Rai]).**  $H^*(M_n, \mathbb{Z})$  does not have odd torsion. In particular,  $f_n^{\mathbb{Z}}$  is an isomorphism.

*Remark.* This theorem was conjectured in the first version of the present paper, on the basis of a computation by A.H. and John Morgan, who checked, using the blowup construction of  $M_n$ , that the theorem holds for  $n \leq 8$ . This conjecture was recently proved by E.R. as a special case of the main theorem of [\[Rai\]](#).

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<sup>2</sup>This can also be verified directly, either from the chain complex of [\[Dev99\]](#) or using the Bockstein map as discussed below.

2.4. *The operad structure on the homology of  $M_n$ .* The spaces  $M_n$  form a topological operad which was first studied by Devadoss, who called it the mosaic operad [Dev99]. To define this operad, it is convenient to agree that each of the undefined moduli spaces  $M_1$  and  $M_2$  consists of one point. We will define a topological operad with a space of  $n$ -ary operations  $M(n) := M_{n+1}$ . We think of a point of  $M_{n+1}$  as an  $n$ -ary operation where the inputs sit at points  $1, \dots, n$  and the output is  $n + 1$ .

The operad structure is defined by attaching curves at marked points. More explicitly, given  $p, q \geq 0$  and  $1 \leq j \leq p$ , we have a “substitution” map  $\gamma_{j pq} : M_{p+1} \times M_{q+1} \rightarrow M_{p+q}$  given by attaching a curve  $C_1$  with  $p + 1$  marked points to a curve  $C_2$  with  $q + 1$  marked points by identifying the point  $j$  on the first curve with the point  $q + 1$  on the second curve, and then adding  $q - 1$  to the labels  $j + 1, \dots, p + 1$  on  $C_1$  and adding  $j - 1$  to the labels of the points  $1, \dots, q$  on  $C_2$ . The operad structure is obtained by iterating such maps.

Recall that a cyclic operad (see [GK95]) is an operad  $P(\bullet)$  in which the action of  $S_n$  on  $P(n)$  extends to an action of  $S_{n+1}$ , compatible with the operad structure. In our case,  $S_{n+1}$  acts on  $M(n) = M_{n+1}$  in a natural way and thus  $M(\bullet)$  is a cyclic operad.

*Remark.* For clarity, let us separately discuss special cases of the substitution map when  $p$  or  $q \leq 1$ . If  $q = 1$ , or  $p = 1$ ,  $\gamma_{j pq}$  is the identity map. If  $q = 0$ ,  $\gamma_{j pq} : M_{p+1} \rightarrow M_p$  is the map of erasing the  $i$ -th point.

Since  $M(\bullet)$  is a topological operad, the spaces  $O(n) := H_*(M_{n+1}, \mathbb{Q}) = (\Lambda_{n+1} \otimes \mathbb{Q})^*$  form an operad in the category of  $\mathbb{Z}$ -graded supervector spaces. The following result determines the structure of this operad.

**THEOREM 2.14.** *The operad  $O(n)$  is the operad of unital 2-Gerstenhaber algebras. More specifically, it is generated by  $1 \in O(0)$ ,  $\mu \in O(2)$ , and  $\tau \in O(3)$ , such that*

- (i)  $\mu$  is a commutative associative product of degree 0 with unit 1.
- (ii)  $\tau$  is a skew-symmetric ternary operation of degree  $-1$ , which is a derivation in each variable with respect to the product  $\mu$ .
- (iii)  $\tau$  satisfies the Jacobi identity:  $\text{Alt}(\tau \circ (\tau \otimes \text{Id} \otimes \text{Id})) = 0$ , where the alternator<sup>3</sup> is over  $S_5$ .

Algebras over this operad can be thought of as Lie 2-algebras with some additional structure. More precisely, we have the following result. Let  $\Sigma$  be the sign operad, which was introduced by Ginzburg-Kapranov [GK94]. Let HW denote the Hanlon-Wachs operad of Lie 2-algebras (see [HW95]). It is generated by a skew-symmetric ternary operation satisfying the Jacobi identity  $\text{Alt}(\tau \circ (\tau \otimes \text{Id} \otimes \text{Id})) = 0$ .

<sup>3</sup>As usual, the alternator is understood in the supersense.

COROLLARY 2.15. *Consider the sub-operad  $O' \subset O$ , with  $O'(2k) = 0$ ,  $O'(2k + 1) = H_k(M_{2k+2}, \mathbb{Q})$ . There is an isomorphism of operads  $O' = \text{HW} \otimes \Sigma$ .*

*Remark.* The reason for tensoring with  $\Sigma$  is that Hanlon and Wachs consider an odd supersymmetric ternary operation, while we need an odd superalternating ternary operation. As a result, the  $S_n$ -representation on  $O'(n)$  is the tensor product of the  $S_n$  representation on the Hanlon-Wachs operad  $\text{HW}(n)$  with the sign representation. However, the notions of  $O'$ -algebra and  $\text{HW}$ -algebra are essentially equivalent: one can go from one to the other by shifting by 1 the underlying  $\mathbb{Z}$ -graded super vector space. Such a twisted version is briefly mentioned in the beginning of [HW95].

2.5. *A few words on the proofs.* Theorems 2.12 and 2.14 are proved simultaneously in Section 6. The main idea is as follows, though for simplicity we give this outline over  $\mathbb{Q}$ . First, in Section 4, we introduce a twisted version of the ring  $\Lambda_n$ . We find a basis for this ring and find its Hilbert series (Corollary 4.16). Then in Section 5, we use known results about the cohomology of the complex moduli space in order to give an upper bound on the Betti numbers of  $M_n$  (Corollary 5.8). It turns out that this upper bound exactly matches the dimensions of the graded pieces of  $\Lambda_n$ . Hence the map  $f_n^{\mathbb{Q}} : \Lambda_n \otimes \mathbb{Q} \rightarrow H^*(M_n, \mathbb{Q})$  is between two graded rings with the dimensions of the left side bounding the dimensions of the right side. So it suffices to prove that  $f_n$  is injective. To do this, we use the operad structure on  $H_*(M_n, \mathbb{Q})$  to construct elements of  $H_*(M_n, \mathbb{Q})$  which pair “upper-triangularly” with the images under  $f_n$  of our basis for  $\Lambda_n$ .

### 3. Results and conjectures concerning Lie algebras

In this section we study the interplay between the cohomology ring of  $M_n$ , its fundamental group  $\Gamma_n$ , and theory of coboundary quasibalgebras. Of main interest to us are different Lie algebras which one can associate to the manifold  $M_n$  and to its fundamental group  $\Gamma_n$ . Much of what we do here is inspired by similar constructions of Kohno and Drinfeld in the configuration space/braid group setting.

#### 3.1. The quadratic dual to $\Lambda_n$ and the Lie algebra $L_n$ .

PROPOSITION 3.1. *The quadratic dual  $\Lambda_n^!$  of  $\Lambda_n$  is the algebra  $U_n$  generated over  $\mathbb{Z}$  by  $\mu_{ijk}$ ,  $1 \leq i, j, k \leq n$ , which are antisymmetric in  $ijk$ , with defining relations*

$$\sum_i \mu_{ijk} = 0, [\mu_{ijk}, \mu_{pqr}] = 0$$

for distinct  $i, j, k, p, q, r$  (with the obvious action of  $S_n$ ). It is also generated by  $\mu_{ijk}$  with  $1 \leq i, j, k \leq n - 1$  with defining relations

$$[\mu_{ijk}, \mu_{pqi} + \mu_{pqj} + \mu_{pqk}] = 0, [\mu_{ijk}, \mu_{pqr}] = 0$$

(with the obvious action of  $S_{n-1}$  which extends to an action of  $S_n$ ).

*Proof.* The equivalence of the first and second presentations of  $U_n$  follows immediately by solving the linear relations for  $\mu_{ijn}$ .

To show that the algebra  $U_n$  is dual to  $\Lambda_n$ , it is convenient to use the  $S_{n-1}$ -invariant presentation of  $\Lambda_n$ , which does not have linear relations. Recall that  $\Lambda_n[1]$  has basis  $\{\mu_{ijk}\}$ . We let  $\{v_{ijk}\}$  denote the dual basis for  $\Lambda_n^![1] = \Lambda_n[1]^*$ . By definition of quadratic dual, these  $v_{ijk}$  will generate  $\Lambda_n^![1]$ .

To find the relations for  $\Lambda_n^!$ , we must compute  $R^\perp \subset \Lambda_n^![1] \otimes \Lambda_n^![1]$ , where  $R \subset \Lambda_n[1] \otimes \Lambda_n[1]$  are the relations for  $\Lambda_n$ . A convenient way to find  $R^\perp$ , is to set  $\Omega = \sum_{i < j < k} \mu_{ijk} v_{ijk} \in \Lambda_n^! \otimes \Lambda_n$ . Then the relations of  $\Lambda_n^!$  are given by the formula  $[\Omega, \Omega] = 0$ . The above relations of  $U_n$  are obtained from this equation by a direct calculation. □

Let  $L_n$  be the Lie algebra over  $\mathbb{Z}$  generated by  $\mu_{ijk}$  with relations as above; we have  $U_n = U(L_n)$ . Thus,  $L_n \otimes \mathbb{Q}$  is the rational holonomy Lie algebra of  $M_n$  in the sense of Chen; see [PS04].

Recall that a  $\mathbb{Z}_+$ -graded algebra  $A$  over a field  $k$  with  $A[0] = k$  is called Koszul if  $\text{Ext}_A^i(k, k)$  (where  $k$  is the augmentation module) sits in degree  $i$  for all  $i \geq 1$ .

CONJECTURE 3.2. *The algebra  $\Lambda_n \otimes \mathbb{Q}$  (or, equivalently,  $U_n \otimes \mathbb{Q}$ ) is Koszul. In particular,  $U_n \otimes \mathbb{Q}$  has Hilbert series*

$$P_n^!(t) = \frac{1}{P_n(-t)} = \prod_{0 \leq k < (n-3)/2} (1 - (n - 3 - 2k)^2 t)^{-1}.$$

The second statement follows from the first by a general result about Koszul algebras.

*Remarks.* (i) This conjecture is true for  $n \leq 6$ , as in those degrees  $\Lambda_n$  has a quadratic Gröbner basis, so is Koszul.

(ii) The Hilbert series formula has been verified computationally in degree 3 for  $n \leq 9$ .

CONJECTURE 3.3.  *$U_n$  and  $L_n$  are free  $\mathbb{Z}$ -modules.*

*Remark.* Let  $A$  be a  $\mathbb{Z}_+$ -graded algebra over  $\mathbb{Z}$  such that  $A[i]$  are finitely generated free  $\mathbb{Z}$ -modules for all  $i$ . One may define  $A$  to be Koszul if for each  $j \geq 1$ ,  $\text{Ext}_A^j(\mathbb{Z}, \mathbb{Z})$  is a free  $\mathbb{Z}$ -module living in degree  $j$ . This is equivalent to

saying that the algebras  $A/pA$  are Koszul for all primes  $p$ . Thus we may strengthen Conjectures 3.2 and 3.3 by conjecturing that  $\Lambda_n$  and  $U_n$  are Koszul over  $\mathbb{Z}$ .

3.2. *Inductive nature of  $L_n$ .* Computations using the ‘‘Magma’’ program suggest that the nonconstant coefficients of the series  $P_n^!(t)/P_{n+1}^!(t)$  are all negative and that the virtual character  $1 - P_n^!(g, t)/P_{n+1}^!(g, t)$  (where  $P_n^!(g, t) := P_n(g, -t)^{-1}$ ) is actually a character in all degrees. This suggests the following additional conjecture.

CONJECTURE 3.4. (i) *The kernel of the natural morphism  $U_{n+1} \rightarrow U_n$  (sending  $\mu_{ijn}$  to zero and  $\mu_{ijk}$  to themselves for  $i, j, k < n$ ) is a free  $U_{n+1}$ -module.*

(ii) *The kernel of the natural morphism  $L_{n+1} \rightarrow L_n$  is a free Lie algebra (with infinitely many generators).*

We note that the two statements of the conjecture become equivalent after extension of scalars from  $\mathbb{Z}$  to  $\mathbb{Q}$ . This follows from the following lemma.

LEMMA 3.5. *Let  $L$  be a  $\mathbb{N}$ -graded Lie algebra acting on another  $\mathbb{N}$ -graded Lie algebra  $F$ , both having finite-dimensional graded pieces, and let  $L \ltimes F$  be their semidirect product. Then the following conditions are equivalent:*

(i)  *$F$  is a free Lie algebra.*

(ii) *The kernel  $K$  of the natural map  $U(L \ltimes F) \rightarrow U(L)$  is a free  $U(L \ltimes F)$ -module.*

Indeed, the equivalence follows by applying the lemma for  $L = L_n$ ,  $F = \ker(L_{n+1} \rightarrow L_n)$  and  $L \ltimes F = L_{n+1}$ .

*Proof.* We have a natural isomorphism  $U(L \ltimes F) = U(L) \ltimes U(F)$ , under which the kernel  $K$  is identified with  $U(L) \otimes U(F)F$ .

If  $F$  is a free Lie algebra, then it is freely generated by a graded subspace  $G$ . Hence  $U(F)F = U(F)G$  is a free  $U(F)$ -module generated by  $G$ . It follows that  $K$  is a free  $U(L \ltimes F)$ -module generated by  $G$ . Thus (i) implies (ii).

Conversely, assume that  $K$  is a free module over  $U(L \ltimes F)$ . It is clear that  $F + (L \ltimes F)K = K$ . Let  $G \subset F$  be a graded complement to  $(L \ltimes F)K$  in  $K$ . Then  $K$  is freely generated by  $G$ , i.e.,  $K = U(L \ltimes F) \otimes G = U(L) \otimes U(F) \otimes G$ . Therefore  $U(F)F$  is generated by  $G$  over  $U(F)$ . This implies that  $F$  is generated by  $G$  as a Lie algebra, and hence (by the Hilbert series consideration) that  $F$  is freely generated by  $G$ . Thus (ii) implies (i).  $\square$

*Remark.* Note that since  $P_n^!(t)/P_{n+2}^!(t) = 1 - (n - 1)^2t$ , it is tempting to make a much more simple-looking conjecture, namely that the kernel of the homomorphism  $L_{n+2} \rightarrow L_n$  is a free Lie algebra generated by  $(n - 1)^2$  generators in degree 1. This, unfortunately, is very far from being true, since the graded  $S_n$ -character of the kernel of this homomorphism is not the same as that of a free Lie

algebra (and in particular  $1 - P_n^!(g, t)/P_{n+2}^!(g, t)$  is a virtual character, which is not a character, and is not concentrated in degree 1).

3.3. *The rational  $K(\pi, 1)$ -property.* A connected topological space  $X$  is said to be rational  $K(\pi, 1)$  if its  $\mathbb{Q}$ -completion is a  $K(\pi, 1)$ -space ([BK72]). Note that a  $K(\pi, 1)$ -space in the usual sense may not be a rational  $K(\pi, 1)$  space (e.g., the complement of the complex hyperplane arrangement of type  $D_n$ ).

It is proved in [PY99, Prop 5.2] that for a connected topological space  $X$  with finite Betti numbers, if  $H^*(X, \mathbb{Q})$  is a Koszul algebra then  $X$  is a rational  $K(\pi, 1)$  space. Thus Conjecture 3.2 implies

CONJECTURE 3.6. *The space  $M_n$  is rational  $K(\pi, 1)$  in the sense of [BK72].*

It is also shown in [PY99] that the Koszul property of  $H^*(X, \mathbb{Q})$  and the rational  $K(\pi, 1)$  property of  $X$  are equivalent if  $X$  is a formal space. However, as we show below, the spaces  $M_n$  are not formal for  $n \geq 6$ .

3.4. *The fundamental group  $\Gamma_n$  of  $M_n$  and coboundary categories.* Let  $\Gamma_n$  be the fundamental group of  $M_n$ . To understand this group, we consider another group  $J_n$  which is the orbifold fundamental group of the orbifold  $M_{n+1}/S_n$  (the group  $S_n$  leaves the point  $n + 1$  fixed). There is a short exact sequence

$$1 \rightarrow \Gamma_{n+1} \rightarrow J_n \rightarrow S_n \rightarrow 1.$$

Furthermore, it is explained in [Dev99], [DJS03], [HK06] that the group  $J_n$  has the following presentation: it is generated by elements  $s_{p,q}$ ,  $1 \leq p < q \leq n$ , with defining relations

- (i)  $s_{p,q}^2 = 1$ ;
- (ii)  $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$  if  $[p, q] \cap [m, r] = \emptyset$ ;
- (iii)  $s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q}$  if  $[m, r] \subset [p, q]$ .

The above map  $J_n \rightarrow S_n$  is defined by sending  $s_{p,q}$  to the involution that reverses the interval  $[p, q]$  and keeps the indices outside of this interval fixed. The group  $J_n$  is called the ‘‘cactus group’’ and it is analogous to the braid group.

One significance of this group  $J_n$  comes from the theory of coboundary monoidal categories. Recall ([Dri89], see also [HK06]) that a coboundary monoidal category is a monoidal category  $\mathcal{C}$  together with a commutor morphism  $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ , functorial in  $X, Y$ , such that  $c_{X,Y} c_{Y,X} = 1$ , and

$$c_{Y \otimes X, Z} c_{X, Y} = c_{X, Z \otimes Y} c_{Y, Z}$$

(for simplicity we drop the associativity isomorphisms, assuming that the category is strict, and write  $c_{X,Y}, c_{Y,Z}$  instead of  $c_{X,Y} \otimes 1_Z, 1_X \otimes c_{Y,Z}$ ).

Let  $\mathcal{C}$  be such a category, and let  $X_1, \dots, X_n$  be  $n$  objects in  $\mathcal{C}$ . Then, as shown in [HK06], every element  $g$  of the group  $J_n$  defines a morphism  $X_1 \otimes \dots \otimes X_n \rightarrow X_{g(1)} \otimes \dots \otimes X_{g(n)}$  (here  $g(i)$  is the action of the image of  $g$  in  $S_n$  on the index  $i$ ). Namely,  $s_{p,q}$  acts by

$$cX_p, X_{p+1} cX_p \otimes X_{p+1}, X_{p+2} \cdots cX_p \otimes \dots \otimes X_{q-1}, X_q.$$

The action of the cactus group on tensor products in coboundary monoidal categories is analogous to the action of the braid group on tensor products in braided monoidal categories.

For  $1 \leq p \leq q < r \leq n$ , let  $\sigma_{p,q,r} = s_{p,r} s_{p,q} s_{q+1,r}$  (we agree that  $s_{pp} = 1$ ). Clearly, such elements generate  $J_n$ . The element  $\sigma_{p,q,r}$  acts on  $X_1 \otimes \dots \otimes X_n$  by  $cX_p \otimes \dots \otimes X_q, X_{q+1} \otimes \dots \otimes X_r$ , and its inverse is given by  $\sigma_{p,q,r}^{-1} = \sigma_{p,p+r-1-q,r}$ .

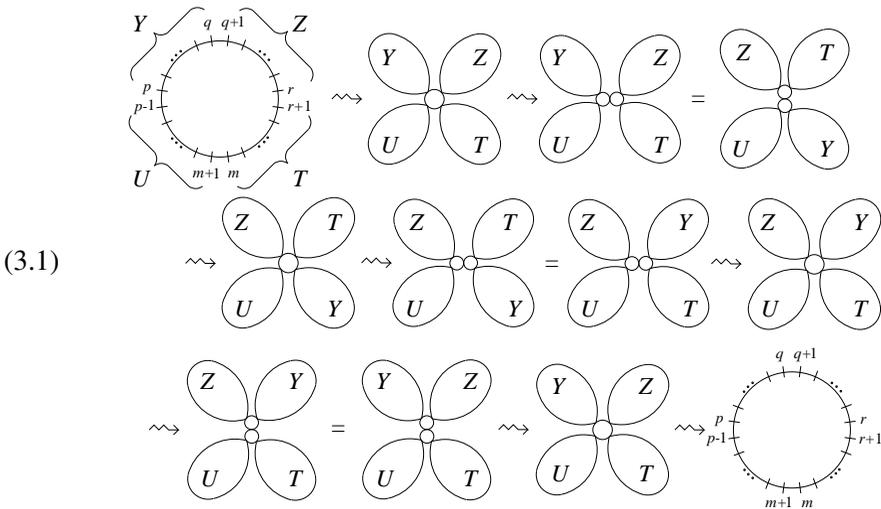
Now for  $1 \leq p \leq q < r < m \leq n$ , define

$$b_{p,q,r,m} = \sigma_{p,q,r}^{-1} \sigma_{p+r-q,r,m}^{-1} \sigma_{p,q,m}.$$

It is easy to see that  $b_{p,q,r,m} \in \Gamma_n$ . It acts on the tensor product  $X_1 \otimes \dots \otimes X_n$  by the morphism

$$cZ, Y cT, Y cY, Z \otimes T,$$

where  $Y = X_p \otimes \dots \otimes X_q$ ,  $Z = X_{q+1} \otimes \dots \otimes X_r$ ,  $T = X_{r+1} \otimes \dots \otimes X_m$ . More geometrically,  $b_{p,q,r,m}$  is represented by the following loop in  $M_n$ :



The significance of these elements is the following result.

**PROPOSITION 3.7.** *The conjugates of  $b_{p,q,r,m}$  generate  $\Gamma_n$ .*

*Proof.* Consider the quotient  $G$  of  $J_{n-1}$  by the relations  $b_{p,q,r,m} = 1$ . We claim that  $G$  is isomorphic to  $S_{n-1}$ . To show this, note that  $\sigma_{p,q,m}$  for  $m > p + 1$

can be expressed in  $G$  via elements  $\sigma_{p',q',m'}$  with  $m' - p' < m - p$ . Thus the group  $G$  is generated by  $\sigma_{p,p,p+1} = s_{p,p+1}$ , which are easily shown to satisfy the braid relations, as desired. Hence  $\Gamma_n$  is generated by the conjugates of  $b_{p,q,r,m}$ .  $\square$

### 3.5. The first homology of $M_n$ .

**THEOREM 3.8.** *One has  $H_1(M_n, \mathbb{Z}) = \wedge^3 \mathbb{Z}^{n-1} \oplus E_n$ , where  $E_n$  is a vector space over  $\mathbb{F}_2$ .*

*Proof.* For every triple  $i, j, k$  of distinct indices between 1 and  $n - 1$ , we have a submanifold of  $D_{ijk}$  of  $M_n$  (of codimension 1) which is the closure of the set of curves with two components, one containing the points  $i, j, k$  and the other containing the remaining labeled points. This submanifold is naturally isomorphic to  $M_4 \times M_{n-2} = S^1 \times M_{n-2}$ . The circle  $S^1$  has a natural orientation, which depends on the cyclic ordering of  $i, j, k$ . Let  $\mu_{ijk}$  denote the image in  $H_1(M_n, \mathbb{Z})$  of the fundamental class of this circle. It is easy to see that  $\mu_{ijk}$  are skew-symmetric in the indices and are permuted in an obvious way by  $S_{n-1}$ . Let  $\nu_{ijk} \in H^1(M_n, \mathbb{Q})$  be the images of  $\nu_{ijk}$  under the map (2.4). Recall the map  $\phi_S$  from Section 2.3. The composite

$$S^1 \hookrightarrow S^1 \times M_{n-2} = D_{ijk} \hookrightarrow M_n \xrightarrow{\phi_{i'j'k'n}} M_4 = S^1$$

is a diffeomorphism if  $\{i, j, k\} = \{i', j', k'\}$  and is constant otherwise. It follows that  $\langle \mu_S, \nu_{S'} \rangle = \pm \delta_{S,S'}$  and that the  $\mu_{ijk}$  are linearly independent over  $\mathbb{Q}$ . It remains to show that the  $\mu_{ijk}$  for  $1 \leq i, j, k < n$  span  $H_1(M_n, \mathbb{Z})/H_1(M_n, \mathbb{Z})\langle 2 \rangle$  (note that they do not span  $H_1(M_n, \mathbb{Z})$  which has a very big 2-torsion). To do so we will use the structure of the fundamental group  $\Gamma_n$ .

According to Proposition 3.7, the images of the  $b_{p,q,r,m}$  in  $\Gamma_n/[\Gamma_n, \Gamma_n]$  generate  $\Gamma_n/[\Gamma_n, \Gamma_n]$ . So it suffices to show that each  $[b_{p,q,r,m}]$  can be written as a linear combination of the  $\mu_{ijk}$ . We claim that

$$(3.2) \quad [b_{p,q,r,m}] = \sum_{p \leq i \leq q < j \leq r < k \leq m} \mu_{ijk}$$

modulo 2-torsion.

If  $n = 4$ , then  $\Gamma_n = \mathbb{Z}$  and the equality is exact (with only one term in the sum). If  $n = 5$ , then  $H_1(M_n, \mathbb{Z}) = \mathbb{Z}^4 \oplus \mathbb{F}_2$  (since  $M_5$  is the connected sum of five real projective planes) and it is therefore enough to check (3.2) over  $\mathbb{Q}$ . To do so, we compute  $\langle b_{p,q,r,m}, \nu_{ijk} \rangle$ . Namely, we compose (3.1) with  $\phi_{ijk5} : M_5 \rightarrow M_4$  and see that it produces  $1 \in H_1(M_4)$  if  $p \leq i \leq q < j \leq r < k \leq m$  and zero otherwise.

Consider now the case  $n > 5$ . It is easy to prove the following equalities:

$$\begin{aligned} b_{p,q,r,m}^{-1} &= \sigma_{p,q,r}^{-1} b_{p,p+r-q-1,r,m} \sigma_{p,q,r} \\ &= \sigma_{q+1,r,m}^{-1} b_{p,q,m+q-r-1,m} \sigma_{q+1,r,m}. \end{aligned}$$

Also, using the operad structure of  $M_n$  and the case  $n = 5$ , we get that for any  $p \leq \ell < q$ ,

$$[b_{p,q,r,m}] = [b_{\ell+1,q,r,m}] + \sigma_{p,\ell,q}^{-1}([b_{p+q-\ell,q,r,m}])$$

in  $\Gamma_n/[\Gamma_n, \Gamma_n]$  modulo 2-torsion. These two equalities imply equation (3.2) by induction on  $m - p$ . The theorem is proved.  $\square$

In particular, this gives a proof of [Theorem 2.13](#) for  $H_1(M_n, \mathbb{Z})$ .

3.6. *The lower central series of  $\Gamma_n$ .* Let  $\Gamma_n^p$  be the  $p$ -th term of the lower central series of  $\Gamma_n$ , and  $L'_n$  be the associated graded  $\mathbb{Z}$ -Lie algebra of this series, i.e.  $L'_n = \bigoplus_{p=1}^\infty \Gamma_n^p / \Gamma_n^{p+1}$ . This is a Lie algebra graded by the positive integers, and it is generated in degree 1 by definition. Also we have a natural action of  $S_{n-1}$  on  $L'_n$ , coming from the group  $J_{n-1}$ .

We have  $L'_n[1] = H_1(M_n, \mathbb{Z})$ , so  $L'_n$  has 2-torsion. Let us therefore consider the quotient Lie algebra  $\mathcal{L}_n = L'_n / L'_n \langle 2 \rangle$ . It is generated by its degree 1 part, which by [Theorem 3.8](#) is a free  $\mathbb{Z}$ -module with basis  $\mu_{ijk}, i < j < k$ .

**THEOREM 3.9.** *There is a surjective  $S_{n-1}$ -equivariant homomorphism of graded Lie algebras  $\psi_n : L_n \rightarrow \mathcal{L}_n$ , which maps  $\mu_{ijk}$  to  $\mu_{ijk}$ .*

*Proof.* The only thing we need to prove is that  $\mu_{ijk} \in \mathcal{L}_n[1]$  satisfy the quadratic relations of  $L_n$ . Because of the  $S_{n-1}$ -symmetry, it is sufficient to show that

$$(3.3) \quad [\mu_{123}, \mu_{145} + \mu_{245} + \mu_{345}] = 0, \quad [\mu_{123}, \mu_{456}] = 0.$$

To do so, note that, as was explained in the proof of [Theorem 3.8](#),  $\mu_{123}, \mu_{456}$  are the classes in  $\mathcal{L}_n[1]$  of the elements  $b_{1,1,2,3}, b_{4,4,5,6}$ , while  $\mu_{145} + \mu_{245} + \mu_{345}$  is the class of the element  $b_{1,3,4,5}$ . Now the identities (3.3) follow from the relations  $b_{1,1,2,3}b_{1,3,4,5} = b_{1,3,4,5}b_{1,1,2,3}$  and  $b_{1,1,2,3}b_{4,4,5,6} = b_{4,4,5,6}b_{1,1,2,3}$  in  $\Gamma_n$ , which easily follow from the defining relations of  $J_{n-1}$ .  $\square$

**CONJECTURE 3.10.** (i)  $\psi_n$  is an isomorphism. In particular, (assuming [Conjecture 3.3](#)),  $\mathcal{L}_n$  is a free  $\mathbb{Z}$ -module, and thus the only torsion in the lower central series of  $\Gamma_n$  is 2-torsion.

(ii)  $\bigcap_{k \geq 1} \Gamma_n^k = \{1\}$ . In other words, the group  $\Gamma_n$  is residually nilpotent.

3.7. *The pronilpotent completion.* Let  $\widehat{\Gamma}_n$  be the pronilpotent (=Malcev) completion of  $\Gamma_n$  (over  $\mathbb{Q}$ ). This is a pronilpotent proalgebraic group. We can also define the proalgebraic group  $\widehat{J_{n-1}} = J_{n-1} \times_{\Gamma_n} \widehat{\Gamma}_n$ .

Let  $\text{Lie}\widehat{\Gamma}_n$  be the Lie algebra of the group  $\widehat{\Gamma}_n$ , which is called the Malcev Lie algebra of  $\Gamma_n$  (see [PS04] for a more general discussion about such Lie algebras). Let  $\text{grLie}\widehat{\Gamma}_n$  be the associated graded of this Lie algebra with respect to the lower central series filtration. It is easy to see that  $\text{grLie}\widehat{\Gamma}_n = \mathcal{L}_n \otimes \mathbb{Q}$ . Thus we have a surjective homomorphism  $\psi_n^{\mathbb{Q}} : L_n \otimes \mathbb{Q} \rightarrow \text{grLie}\widehat{\Gamma}_n$ , and we can make the  $\mathbb{Q}$ -version of [Conjecture 3.10](#):

CONJECTURE 3.11. (i)  $\psi_n^{\mathbb{Q}}$  is an isomorphism.

(ii)  $\Gamma_n$  injects into its pronipotent completion.

*Remark.* Note that since  $\Gamma_n$  is the fundamental group of an aspherical manifold, it is torsion free. This is a necessary condition for part (ii) of the conjecture.

3.8. *Coboundary Lie quasibialgebras, quasiHopf algebras and representations of  $L_n$  and  $\Gamma_n$ .* Recall [Dri89] that a coboundary Lie quasibialgebra is a Lie algebra  $\mathfrak{g}$  together with an element<sup>4</sup>  $\varphi \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ . Given such a  $(\mathfrak{g}, \varphi)$  (over a field of characteristic zero), one can define a family of homomorphisms  $\beta_{n,\mathfrak{g},\varphi} : L_n \rightarrow U(\mathfrak{g})^{\otimes n-1}$ , defined by the formula  $\beta_{n,\mathfrak{g},\varphi}(\mu_{ijk}) = \varphi_{ijk}$ . (Here  $\varphi_{ijk}$  denotes the image of  $\varphi$  under the embedding  $\mathfrak{g}^{\otimes 3} \rightarrow U(\mathfrak{g})^{\otimes n-1}$  which puts 1s in all factors other than  $i, j, k$ .) The invariance of  $\varphi$  implies that the quadratic relations of  $L_n$  are preserved under this assignment.

THEOREM 3.12. *The representations  $\beta_{n,\mathfrak{g},\varphi}$  factor through  $\mathcal{L}_n$ .*

*Proof.* The proof is based on the fact, due to Drinfeld [Dri89], that the representations  $\beta_{n,\mathfrak{g},\varphi}$  can be quantized, by quantizing the Lie quasibialgebra  $(\mathfrak{g}, \varphi)$ . Namely, Drinfeld showed that there exists an associator  $\Phi = \Phi(\hbar^2) = 1 + \hbar^2 \varphi/3 + O(\hbar^4)$  in  $U(\mathfrak{g})^{\otimes 3}[[\hbar]]$  (given by some universal formula in terms of  $\hbar^2 \varphi$ ), such that  $(U(\mathfrak{g})^{\otimes 3}[[\hbar]], \Phi)$  is a coboundary quasiHopf algebra. (See [Dri89, Prop. 3.10].)

Let  $\mathcal{C}$  be the associated category (the objects are representations of  $\mathfrak{g}$  and the morphisms are power series in  $\hbar$  whose coefficients are morphisms of representations). This is a coboundary category and hence as explained in [Section 3.4](#), there is an action of  $\Gamma_n$  on tensor products of objects. In fact, from Drinfeld's construction, this action comes from a map  $B_{n,\mathfrak{g},\varphi} : \Gamma_n \rightarrow 1 + \hbar^2 U(\mathfrak{g})^{\otimes n-1}[[\hbar]]$ . This homomorphism factors through the pronipotent completion  $\widehat{\Gamma}_n$  of  $\Gamma_n$ , and thus defines a Lie algebra homomorphism  $B_{n,\mathfrak{g},\varphi} : \text{Lie}\widehat{\Gamma}_n \rightarrow \hbar^2 U(\mathfrak{g})^{\otimes n-1}[[\hbar]]$ .

Under this homomorphism, the image of the  $p$ -th term  $\text{Lie}\widehat{\Gamma}_n^p$  of the lower central series of  $\text{Lie}\widehat{\Gamma}_n$  is contained in  $\hbar^{2p} U(\mathfrak{g})^{\otimes n-1}[[\hbar]]$ . Therefore, we have a natural homomorphism  $B_{n,\mathfrak{g},\varphi}^0 : \text{grLie}\widehat{\Gamma}_n \rightarrow U(\mathfrak{g})^{\otimes n-1}$ , defined by the formula  $B_{n,\mathfrak{g},\varphi}^0(z) = \hbar^{-2p} B_{n,\mathfrak{g},\varphi}(\hat{z}) \text{ mod } \hbar$ , where  $z$  is of degree  $p$  and  $\hat{z}$  is a lift of  $z$ .

<sup>4</sup>We consider Lie quasibialgebras up to twisting.

Now we claim that  $\beta_{n,g,\varphi} = B_{n,g,\varphi}^0 \circ \psi_n$ . Both sides are invariant under the  $S_{n-1}$ -action, and thus it suffices to check the equality on the element  $\mu_{123}$ . This is straightforward to do by computing the  $\hbar^2$ -part of  $B_{n,g,\varphi}(b_{1,1,2,3})$  using that  $\Phi = 1 + \hbar^2\varphi/3 + \dots$ .  $\square$

*Remark.* This theorem gives supporting evidence for the above conjecture that  $\psi_n$  is injective.

3.9. *The nonformality of  $M_n$ .* Let  $\widehat{L_n \otimes \mathbb{Q}}$  be the degree completion of  $L_n \otimes \mathbb{Q}$ .

**PROPOSITION 3.13.** *For  $n \geq 6$ , there does not exist an isomorphism  $\xi : \text{Lie} \widehat{\Gamma_n} \rightarrow \widehat{L_n \otimes \mathbb{Q}}$  whose associated graded is the identity in degree 1.*

*Proof.* It suffices to prove the statement for  $n = 6$ , since we have homomorphisms  $\Gamma_{n-1} \rightarrow \Gamma_n$  and  $\Gamma_n \rightarrow \Gamma_{n-1}$  whose composition is the identity map  $\Gamma_{n-1} \rightarrow \Gamma_{n-1}$ , and similar homomorphisms for  $L_n$ .

For  $n = 6$ , the statement was checked computationally using the ‘‘Magma’’ program.

More precisely, consider the exact sequence

$$1 \rightarrow \widehat{\Gamma_6} \rightarrow \widehat{J_5} \rightarrow S_5 \rightarrow 1.$$

Since  $\widehat{\Gamma_6}$  is prounipotent over  $\mathbb{Q}$  and  $S_5$  is finite, this exact sequence is split. Thus we get  $\widehat{J_5} = S_5 \ltimes \widehat{\Gamma_6}$ , for some action of  $S_5$  on  $\widehat{\Gamma_6}$ .

Now assume that an isomorphism  $\xi$  exists. Then all such isomorphisms form a torsor over a prounipotent group of unipotent automorphisms of the target, so by averaging we can choose  $\xi$  to be  $S_5$ -invariant. Then  $\xi$  can be lifted to a homomorphism  $\xi' : \widehat{J_5} \rightarrow S_5 \ltimes \exp(\widehat{L_6 \otimes \mathbb{Q}})$ . This can be restricted to a homomorphism  $\xi' : J_5 \rightarrow S_5 \ltimes \exp(\widehat{L_6 \otimes \mathbb{Q}})$ , which maps  $b_{1,1,2,3}$  into  $\exp(\mu_{123} + \dots)$ , where  $\dots$  are higher order terms. Since  $J_5$  is given by simple generators and relations, one is able to look for such homomorphisms using ‘‘Magma’’ (modulo  $m$ -th commutators for some  $m$ , to make the computations finite). The computation showed that already modulo the fourth commutators there is no such homomorphism under which the element  $b_{1,1,2,3}$  goes to  $\exp(\mu_{123} + \text{higher terms})$ . This implies the required statement.  $\square$

In particular, the proposition implies that for  $n \geq 6$  the Malcev Lie algebra  $\text{Lie} \widehat{\Gamma_n}$  of  $\Gamma_n$  is not isomorphic to the degree completion of the rational holonomy Lie algebra  $L_n \otimes \mathbb{Q}$  of  $M_n$ , which means that the group  $\Gamma_n$  is not 1-formal in the sense of [PS04]. Therefore, by the results of [Mor78], [Sul77],  $M_n$  is not a formal space in the sense of Sullivan (see e.g. [PS04] for more explanations).

*Remark.* For  $n \leq 5$ , the spaces  $M_n$  are formal and the Malcev Lie algebra of  $\Gamma_n$  is isomorphic to its associated graded.

3.10. *The Drinfeld-Kohno Lie algebra.* The Drinfeld-Kohno Lie algebra  $\mathbf{L}_n$ , defined for  $n > 2$ , is generated over  $\mathbb{Z}$  by generators  $t_{ij} = t_{ji}$  for distinct indices  $1 \leq i, j \leq n-1$ , and relations

$$[t_{ij}, t_{ik} + t_{jk}] = 0, [t_{ij}, t_{kl}] = 0$$

for distinct  $i, j, k, l$  (see [Koh85], [Dri89], [Dri90]).  $\mathbf{L}_n$  is a free  $\mathbb{Z}$ -module (since it is an iterated semidirect product of free Lie algebras).

Kohno proved in [Koh85] (using the Knizhnik-Zamolodchikov equations) that  $\mathbf{L}_{n+1} \otimes \mathbb{Q}$  is the Lie algebra of the pronipotent completion of the pure braid group  $PB_n$ . Let  $\mathbf{U}_n$  be the universal enveloping algebra of  $\mathbf{L}_n$ . The algebra  $\mathbf{U}_n \otimes \mathbb{Q}$  is Koszul, and is the quadratic dual of the Orlik-Solomon algebra  $OS_{n-1}$ , which is the cohomology algebra of the configuration space  $C_{n-1} = \{(z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1} : z_i \neq z_j\}$ . The Hilbert series of  $\mathbf{U}_n$  is

$$\mathbf{P}_n(t) = \prod_{m=1}^{n-2} (1 - mt)^{-1}.$$

(See e.g. [Yuz01].)

If  $\mathfrak{g}$  is a Lie algebra and  $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$  is an invariant element (i.e.,  $(\mathfrak{g}, \Omega)$  is a quasitriangular Lie quasibialgebra [Dri89]), then we have a homomorphism  $\gamma_{n,\mathfrak{g},\Omega} : \mathbf{L}_n \rightarrow U(\mathfrak{g})^{\otimes n-1}$  given by  $\gamma_{n,\mathfrak{g},\Omega}(t_{ij}) = \Omega_{ij}$  (this is analogous to  $\beta_{n,\mathfrak{g},\varphi}$  above). In the braid group setting, as  $\mathbf{L}_n$  is isomorphic to the Lie algebra of the pronipotent completion of  $PB_n$ , the analogous statement to Theorem 3.12 is that the two different representations of  $\mathbf{L}_n$  agree (one comes from  $\gamma$ , the other from quantizing  $\Omega$ ).

3.11. *Relation between  $L_n$  and  $\mathbf{L}_n$ .* We have a homomorphism of Lie algebras  $\xi_n : L_n \rightarrow \mathbf{L}_n$  given by the formula  $\xi_n(\mu_{ijk}) = [t_{ij}, t_{jk}]$ . Note that under this homomorphism,  $L_n[1]$  is identified with  $\mathbf{L}_n[2]$ . It is clear that  $\gamma_{n,\mathfrak{g},\omega} \circ \xi_n = \beta_{n,\mathfrak{g},\varphi}$ , where  $\varphi = [\Omega^{12}, \Omega^{23}]$ .

**THEOREM 3.14.** *The map  $\xi_n^{\mathbb{Q}} : L_n \otimes \mathbb{Q} \rightarrow \mathbf{L}_n \otimes \mathbb{Q}$  factors through  $\mathcal{L}_n \otimes \mathbb{Q}$ .*

*Proof.* Using Drinfeld's "unitarization" trick, one can define a homomorphism of pronipotent completions  $\Xi_n : \widehat{\Gamma}_n \rightarrow \widehat{PB_{n-1}}$  (see [HK06]). Namely, let  $\widehat{B_{n-1}} = B_{n-1} \times_{PB_{n-1}} \widehat{PB_{n-1}}$ . Then we can define a homomorphism  $\Xi_n : \widehat{J_{n-1}} \rightarrow \widehat{B_{n-1}}$  by setting

$$\Xi_n(\sigma_{p,q,r}) = \beta_{[p,q],[q+1,r]} (\beta_{[p,p+r-q-1],[p+r-q,r]} \beta_{[p,q],[q+1,r]})^{-1/2},$$

where  $\beta_{[p,q],[q+1,r]}$  is the braid which interchanges the intervals  $[p,q]$  and  $[q+1,r]$ .

At the Lie algebra level,  $\Xi_n$  defines a homomorphism of filtered Lie algebras  $\text{Lie}\widehat{\Gamma}_n \rightarrow \text{Lie}\widehat{PB_{n-1}}$ . Taking the associated graded of this map, and using the fact that  $\text{grLie}\widehat{\Gamma}_n = \mathcal{L}_n \otimes \mathbb{Q}$ , we obtain the required statement.  $\square$

CONJECTURE 3.15.  $\xi_n$  is injective.

Remarks. (i) Conjecture 3.15 implies Conjecture 3.11 (i) and Conjecture 3.3.

(ii) Conjecture 3.15 also implies Conjecture 3.4, as we have natural morphisms  $\mathbf{L}_{n+1} \rightarrow \mathbf{L}_n$ , whose kernels are known to be free Lie algebras. On the other hand, by the Shirshov-Witt theorem, a Lie subalgebra of a free Lie algebra is free.

#### 4. Poset homology and a twisted version of $\Lambda_n$

We now leave this world of Lie algebras and turn to the task of proving our main theorem describing the cohomology ring. As a first step, we will examine our ring  $\Lambda_n$  more closely. More precisely, we will consider a twisted version of  $\Lambda_n$  which has close connections to the homology of the poset of odd set partitions (which in turn has close connections with the operad of Lie 2-algebras [HW95]).

4.1. *Homology of the poset of odd set partitions.* Given a poset  $P$  with bottom and top elements  $\hat{0}$  and  $\hat{1}$ , there is an associated chain complex  $\tilde{C}_*$  in which  $\tilde{C}_{r+1}$  is the free  $\mathbb{Z}$ -module spanned by chains

$$(\hat{0} < x_1 < \dots < x_r < \hat{1})$$

with differential

$$\begin{aligned} \partial_r(\hat{0} < x_1 < \dots < x_r < \hat{1}) \\ = \sum_{1 \leq i \leq r} (-1)^{i-1} (\hat{0} < x_1 < \dots < x_{i-1} < x_{i+1} < \dots < x_r < \hat{1}). \end{aligned}$$

By our convention, if  $\hat{0} = \hat{1}$  then  $\tilde{C}_0(P) = \mathbb{Z}$  and  $\tilde{C}_r(P) = 0$  for  $r \neq 0$ . The (shifted reduced) homology  $\tilde{H}_*(P)$  is defined to be the homology of this chain complex. If  $P$  does not have a top element, then we define  $\tilde{H}_n(P) := \tilde{H}_{n+1}(P^+)$ , where  $P^+ = P \sqcup \{\hat{1}\}$ . Let  $\tilde{h}_n(P)$  be the rank of  $\tilde{H}_n(P)$ . Note that we have shifted the degrees of the chains from the usual convention, in order for the following to hold.

PROPOSITION 4.1. *If  $P$  and  $Q$  are posets with bottom and top and at least one of  $\tilde{H}_*(P)$  and  $\tilde{H}_*(Q)$  is free, then*

$$\tilde{H}_*(P \times Q) = \tilde{H}_*(P) \otimes \tilde{H}_*(Q)$$

as graded modules. In particular, the induced isomorphism

$$\tilde{H}_*(P \times Q) \rightarrow \tilde{H}_*(Q \times P)$$

is the standard exchange map for the tensor product of supermodules.

Here and below, as usual in homological algebra, the tensor product of supermodules is always taken in the symmetric monoidal category of supermodules, which means that the commutativity isomorphism is given by the formula  $v \otimes w \rightarrow (-1)^{\deg(v)\deg(w)} w \otimes v$  for homogeneous vectors  $v, w$ .

*Proof.* In terms of the order complex  $|\Delta(P)|$  of  $P$ , we have

$$\tilde{H}_*(P) = H_*(|\Delta(P)|, |\Delta(P)| \setminus \{x\}),$$

where  $x$  is the midpoint of the edge between  $\hat{0}$  and  $\hat{1}$  (the local homology at that point). The result then follows from the Künneth formula for relative homology. □

Let  $\Pi_n^{\text{odd}}$  denote the poset of partitions of  $\{1, 2, \dots, n\}$  in which each part has odd order.

**THEOREM 4.2** ([Bjö80], [CHR86]). *The poset  $\Pi_n^{\text{odd}}$  is totally semimodular, and thus Cohen-Macaulay; that is, the homology of any closed interval in the poset is a free  $\mathbb{Z}$ -module concentrated in the top degree.*

In particular,  $\tilde{H}_*(\Pi_{2n+1}^{\text{odd}})$  is concentrated in degree  $n$ , as is  $\tilde{H}_*(\Pi_{2n+2}^{\text{odd}})$ ; note that as the latter does not naturally have a top element, it is thus implicitly added (and the degree shifted by 1). The further structure of these modules has been determined in [CHR86].

**THEOREM 4.3** ([CHR86]). *We have the exponential generating functions*

$$\sum_{n \geq 0} \tilde{h}_n(\Pi_{2n+1}^{\text{odd}}) \frac{u^{2n+1}}{(2n+1)!} = \arcsin(u)$$

and

$$\sum_{n \geq 0} \tilde{h}_n(\Pi_{2n+2}^{\text{odd}}) \frac{u^{2n+2}}{(2n+2)!} = 1 - \sqrt{1-u^2}.$$

In [Koz01], Kozlov considers the following spectral sequence. Let  $P$  be a Cohen-Macaulay poset. The rank of an element  $x \in P$  is then the unique  $r$  such that  $\tilde{H}_r([\hat{0}, x]) \neq 0$  (we take by convention that  $\text{rank}(\hat{0}) = 0$ ). Consider the filtration of the associated chain complex given by

$$F^r C_{k+1} = \langle \hat{0} < x_1 < \dots < x_k < \hat{1} : \text{rank}(x_k) \leq r \rangle.$$

Namely, for our purposes, we consider chains of set partitions such that the coarsest (= biggest) partition in the chain has at least  $n - 2r$  parts. Then Kozlov observes that the associated spectral sequence looks like

$$E_{k,l}^1 = \bigoplus_{\text{rank}(x)=l} \tilde{H}_k([\hat{0}, x]) \Rightarrow \tilde{H}_{k+1}(P),$$

where the indices  $k, l$  run from 0 to  $\text{rank}(\hat{1}) - 1$ . But this is 0 unless  $k = l$ , and thus the spectral sequence collapses on the  $E^2$  page. Since the spectral sequence converges to  $\tilde{H}_*(P)$  and this is concentrated in the top degree, we find that  $E^2_{k,l} = 0$  unless  $k = l = \text{rank}(\hat{1}) - 1$ , in which case it equals the homology of  $P$ .

In other words, we obtain the following result.

**THEOREM 4.4.** *If  $P$  is a Cohen-Macaulay poset, then the Whitney homology groups*

$$W_k(P) = \bigoplus_{x \in P} \tilde{H}_k([\hat{0}, x]) = \bigoplus_{\text{rank}(x)=k} \tilde{H}_*([\hat{0}, x])$$

form a canonical exact sequence

$$0 \rightarrow W_{\text{rank}(\hat{1})} \rightarrow \dots \rightarrow W_1 \rightarrow W_0 \rightarrow 0.$$

*Remark.* Note that  $W_{\text{rank}(\hat{1})} = \tilde{H}_*(P)$  and  $W_0 = \mathbb{Z}$ . If  $P$  has a bottom but not a top, we by convention do not include the  $\tilde{H}_*(P)$  term in  $W_*(P)$ .

By construction, this exact sequence respects products when the reduced homology modules are free. The differential corresponding to this chain complex acts on saturated chains (in which each step is a covering relation) by

$$\partial_W(\hat{0} < x_1 < \dots < x_r) = (-1)^{r-1}(\hat{0} < x_1 < \dots < x_{r-1}).$$

Note that since  $P$  is Cohen-Macaulay, only saturated chains appear in the Whitney homology.

4.2. *A twisted version of  $\Lambda$ .* For a finite set  $S$ , let  $\tilde{\Lambda}(S)$  be the supercommutative  $\mathbb{Z}$ -algebra with symmetric degree 1 generators  $\tau_{ijk}$  for  $i, j, k$  distinct elements of  $S$ , subject to the quadratic relations

$$\begin{aligned} \tau_{ijk}\tau_{jkl} &= 0, \\ \tau_{ijk}\tau_{klm} + \tau_{jkl}\tau_{lmi} + \tau_{klm}\tau_{mij} + \tau_{lmi}\tau_{ijk} + \tau_{mij}\tau_{jkl} &= 0. \end{aligned}$$

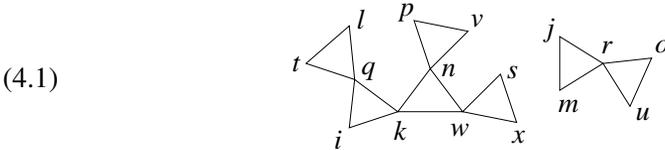
This is of course quite closely related to the  $S_n$ -invariant presentation of  $\Lambda_{n+1}$  considered above, except for the change in symmetry of the generators. In fact, experimentation for small  $n$  immediately suggests that the two algebras have the same Hilbert series, a fact which we will confirm below.

A triangle graph is a pair  $(V, E)$  where  $V$  is a finite set and  $E$  a subset of the set of three element subsets of  $V$ .  $V$  is called the set of vertices and  $E$  the set of edges. More generally, there is the notion of hypergraph, where subsets of any size are allowed.

Many notions from graph theory may be extended to triangle graphs. A path of length  $n$  in a triangle graph, from vertex  $x$  to a vertex  $y$ , is a sequence of edges  $e_1, \dots, e_n$  and vertices  $x = v_0, \dots, v_n = y$  with  $v_i \in e_i \cap e_{i+1}$  for  $0 < i < n$ ,

$v_0 \in e_1$  and  $v_n \in e_n$ . A cycle is a path of length greater than 1 from a vertex to itself which consists of distinct edges and vertices. A triangle forest is a triangle graph without any cycles and a connected triangle forest is called a triangle tree.

Given a monomial in the triples  $\tau_{ijk}$ , there is a natural triangle graph with vertex set  $S$  and with an edge for each variable. We can consider the associated partition of the set  $S$  into connected components. This gives a grading of the exterior algebra with respect to the lattice of partitions. That is, the product of the homogeneous components associated to two partitions  $\pi_1, \pi_2$  lies in the homogeneous component associated to their join. Since the ideal of relations of  $\tilde{\Lambda}(S)$  is homogeneous with respect to this grading,  $\tilde{\Lambda}(S)$  inherits this grading. Thus for each partition  $\pi$  of  $S$ , we associate a corresponding space  $\tilde{\Lambda}[\pi]$  and we have  $\tilde{\Lambda}(S) = \bigoplus_{\pi} \tilde{\Lambda}[\pi]$ .



The triangle graph for the monomial  $\tau_{tlq}\tau_{pvn}\tau_{qik}\tau_{knw}\tau_{wsx}\tau_{mjr}\tau_{rou}$ .

LEMMA 4.5. If  $\pi_1$  and  $\pi_2$  are partitions of disjoint sets  $S_1$  and  $S_2$ , then

$$\tilde{\Lambda}[\pi_1 \cup \pi_2] = \tilde{\Lambda}[\pi_1] \otimes \tilde{\Lambda}[\pi_2]$$

under the natural embedding of  $\tilde{\Lambda}(S_1) \otimes \tilde{\Lambda}(S_2)$  in  $\tilde{\Lambda}(S_1 \cup S_2)$ .

In particular, the structure of these homogeneous components is determined by the structure for partitions with a single part.

4.3. *Spanning the algebra  $\tilde{\Lambda}(S)$ .* Direct calculation gives some small cases of  $\tilde{\Lambda}[\pi]$ :

$$\begin{aligned} \tilde{\Lambda}[\{1\}] &= \langle 1 \rangle \\ \tilde{\Lambda}[\{1, 2\}] &= 0 \\ \tilde{\Lambda}[\{1, 2, 3\}] &= \langle \tau_{123} \rangle \\ \tilde{\Lambda}[\{1, 2, 3, 4\}] &= 0. \end{aligned}$$

For  $n = 5$ , we find the following.

LEMMA 4.6. The space  $\tilde{\Lambda}[\{1, 2, 3, 4, 5\}]$  is spanned by the nine products  $\tau_{12i}\tau_{jkl}$  for which  $|\{1, 2, i, j, k, l\}| = 5$ .

LEMMA 4.7. For any pair of distinct elements  $j, j' \in S$ ,  $\tilde{\Lambda}[S]$  is spanned by monomials in which at least one triple contains  $\{j, j'\}$ .

*Proof.* Any monomial in  $\tilde{\Lambda}[S]$  corresponds to a connected triangle graph, and thus that triangle graph will contain a shortest path from  $j$  to  $j'$ . We proceed by induction on the length of that path, noting that if the path has length 1, we are done.

Assuming therefore that the length is greater than 1, let the shortest path begin by  $j, k, l$ . Then the monomial contains triples

$$\tau_{ijk}\tau_{klm}$$

for some  $i, m \in S$ . If we apply the quadratic relation, the resulting monomials all contain a triple joining  $j$  and  $l$ , and thus the length of the shortest path from  $j$  to  $j'$  in the new monomials has been decreased by 1.  $\square$

**COROLLARY 4.8.** *If the triangle graph of a monomial in  $\tilde{\Lambda}[S]$  contains a cycle, then that monomial is 0.*

*Proof.* The same argument shows that we can reduce the cycle to length 2. But a cycle of length 2 is of the form  $\tau_{ijk}\tau_{ijl} = 0$ .  $\square$

In other words, the triangle graph of any nonzero monomial in  $\tilde{\Lambda}(S)$  is a forest. This immediately implies that the grading by partitions refines the grading by degree.

**THEOREM 4.9.** *Let  $\pi$  be a partition of a set  $S$  of cardinality  $n$ . Then any element of  $\tilde{\Lambda}[\pi]$  has degree  $(n - |\pi|)/2$ . In particular, if  $n - |\pi|$  is odd, then  $\tilde{\Lambda}[\pi] = 0$ .*

*Proof.* Consider a nonvanishing monomial of degree  $d$  in  $\tilde{\Lambda}[\pi]$ . The incidence graph of its associated triangle graph has  $n + d$  vertices and  $3d$  edges; since it is a forest, it must therefore have  $n - 2d$  connected components.  $\square$

**COROLLARY 4.10.** *If  $\pi \notin \Pi_n^{\text{odd}}$ , then  $\tilde{\Lambda}[\pi] = 0$ .*

*Proof.* Indeed, the components of a triangle forest all have odd order.  $\square$

Standard results on exponential generating functions then imply the following.

**COROLLARY 4.11.** *Let  $f(u, t)$  be the generating function*

$$\sum_{0 \leq k} \dim(\tilde{\Lambda}[\{1, 2, \dots, 2k + 1\}])t^k \frac{u^{2k+1}}{(2k + 1)!},$$

and let  $P(u, t)$  be the generating function

$$\sum_{0 \leq n} P_n(t) \frac{u^n}{n!},$$

where  $P_n(t)$  is the Poincaré polynomial of  $\tilde{\Lambda}_n$ . Then  $P(u, t) = \exp(f(u, t))$ .

Using [Lemma 4.7](#) to its fullest, we can obtain the following spanning set of  $\tilde{\Lambda}(S)$ . Choose a total ordering on  $S$ .

*Definition 4.12.* We define *basic triangle trees* and *forests* as follows, by induction on the number of triangles.

1. A single point is a basic triangle tree.
2. A nontrivial triangle tree is basic if and only if the two smallest vertices are on a common triangle, and each of the three components after removing that triangle is basic.
3. A triangle forest is basic if and only if each component is basic.

For example, [\(4.1\)](#) is a basic triangle forest with respect to the usual order on the letter  $i, j, \dots, x$ .

**PROPOSITION 4.13.** *The algebra  $\tilde{\Lambda}(S)$  is spanned by the monomials associated to basic triangle forests.*

*Proof.* This is a simple induction; by [Lemma 4.7](#), any monomial associated to a tree can be expanded in terms of monomial associated to trees in which the two smallest vertices are on a common triangle, and similarly for the monomials associated to the subtrees.  $\square$

If  $h(u)$  is the exponential generating function for basic trees, then we claim that  $h''(u) = uh'(u)^3$ . Indeed, the left-hand side is the exponential generating function for basic trees on  $n + 2$  vertices, while the right-hand side corresponds to the following:

- a. A choice of element  $k \in \{3, \dots, n + 2\}$ ,
- b. An ordered set partition of  $\{3, \dots, n + 2\} \setminus \{k\}$  into three parts  $P_1, P_2, P_3$ ,
- c. Basic trees on  $P_1 \cup \{1\}, P_2 \cup \{2\}, P_3 \cup \{k\}$ .

Adjoining the triangle  $\{1, 2, k\}$  to the three basic trees gives a basic tree on  $n + 2$  vertices, establishing the differential equation. The unique solution with  $h(0) = 0$ ,  $h'(0) = 1$  is  $h(u) = \arcsin(u)$ .

We will now show that basic triangle trees form a basis of  $\tilde{\Lambda}[S]$  (and similarly for basic triangle forests and  $\tilde{\Lambda}(S)$ ). As a consequence,  $\arcsin(u)$  is also the exponential generating function for  $\dim \tilde{\Lambda}[S]$ .

#### 4.4. A basis for $\tilde{\Lambda}(S)$ .

**THEOREM 4.14.** *For a partition  $\pi \in \Pi_n^{\text{odd}}$ , let  $S_\pi \subset S_n$  be the corresponding product of symmetric groups. Then there is a canonical isomorphism of graded  $\mathbb{Z}[S_\pi]$ -modules between  $\tilde{\Lambda}[\pi]$  and  $\tilde{H}_*(\hat{0}, \pi)$ .*

*Moreover, these abelian groups have bases indexed by basic triangle forests with component partition  $\pi$ .*

*Proof.* Since both  $\tilde{\Lambda}[\pi]$  and  $\tilde{H}_*(\hat{0}, \pi)$  are multiplicative under disjoint union of partitions, it is enough to treat the case  $|\pi| = 1$ , namely  $[\hat{0}, \pi] = \Pi_n^{\text{odd}}$ .

Given a triangle tree, any ordering of the triangles gives rise to a chain of set partitions; the  $k$ -th partition in the chain is the set of components of the subgraph spanned by the first  $k$  triangles. We thus obtain a map  $\tilde{\Lambda}[\hat{1}] \rightarrow \tilde{C}_*(\Pi_n^{\text{odd}})$ , taking a tree to the alternating sum over the chains resulting from all orderings of the triangles. The five-term relation (and any tree multiple thereof) is annihilated by this map, so it is indeed well-defined.

We claim that the image of a tree is a cycle, and we thus obtain a well-defined map to  $\tilde{H}_*(\Pi_n^{\text{odd}})$ . Indeed, the operation of removing the  $l$ -th partition in a chain is nearly invariant under swapping the  $l$ -th and  $l + 1$ -st triangles, except that swapping the triangles gives rise to an overall sign.

Since the number of basic trees is equal to the rank of the free  $\mathbb{Z}$ -module  $\tilde{H}_*(\Pi_n^{\text{odd}})$  (by the remark following [Proposition 4.13](#) and [Theorem 4.3](#)), the result will follow if we can exhibit a set of cochains such that the induced pairing with basic trees is triangular with unit diagonal. Note from the description of the map from trees to chains that the pairing of an elementary cochain (corresponding to a maximal chain in  $\Pi_n^{\text{odd}}$ ) with a triangle tree is  $\pm 1$  or 0. Moreover, the pairing is nonzero precisely if there exists an ordering of the triangles in the tree corresponding to the same chain of odd set partitions.

First, some additional terminology. The “root” of a basic tree is the triangle containing the two minimal elements of its support. If we remove the root, we obtain three components; the one *not* containing one of the original minimal elements will be called the “stepchild”.

Given a basic tree  $T$ ,  $\text{size}(T)$  represents the size of its support, and we define

$$\text{rank}(T) := \frac{\text{size}(T) - \text{size}(\text{stepchild}(T))}{2}.$$

Then if  $\text{size}(T) = 2n + 1$ , we can associate a composition  $\mu(T)$  of  $n$  with first part  $\text{rank}(T)$ , second part the rank of its stepchild, and so forth. This then induces a partial order on basic trees by lexicographically ordering the associated compositions.

Finally, we define the “keystone” of a basic tree as follows. If the stepchild of  $T$  is a single vertex, then its keystone is simply the root. Otherwise, the keystone of  $T$  is the keystone of its stepchild.

**LEMMA 4.15.** *Let  $T$  be a basic triangle tree and  $F$  be the forest obtained by removing its keystone. Then any other basic tree  $T'$  containing  $F$  has  $\mu(T') > \mu(T)$ .*

*Proof.* If the root of  $T$  was the keystone, so  $T$  had maximal composition  $n$ , then  $F$  consists of the component of 1, the component of 2, and a single isolated

vertex  $k$ , and thus the only basic tree containing  $F$  is obtained by adjoining the triangle  $12k$ , so the result holds in this case.

Otherwise, the root of  $T$  is still in  $F$ . If we remove it, we obtain a five tree forest, consisting of the component of 1, the component of 2, and three other components all contained in the stepchild of  $T$ . Now,  $T'$  is obtained from  $T$  by removing the keystone and adding another triangle instead. If that triangle meets the component of 1 or 2, then the composition clearly increases. Otherwise, the result follows by induction applied to the stepchild of  $T$ .  $\square$

Now, for each basic forest  $F$ , we choose a triangle  $t_F$  which is the keystone of some component. This then inductively gives rise to an ordering  $(t_i)_{i=1}^n$  on the triangles of any tree  $T$  by letting  $t_n := t_T$  and  $t_{i-1} := t_{T \setminus \{t_i, \dots, t_n\}}$ . Let  $\alpha_T$  be the sequence of set partitions obtained by successively adding  $t_1, t_2$ , etc, which we identify with the corresponding elementary cochain.

By Lemma 4.15, we can reconstruct the sequence of triangles from  $\alpha_T$  by taking at each step the triangle that minimizes the composition associated to the resulting tree. In other words,  $T$  is reconstructed from  $\alpha_T$  as the minimal tree having a nonzero (and thus unit) pairing with  $\alpha_T$ . This immediately gives us the desired triangularity.  $\square$

COROLLARY 4.16. *Basic forests form a basis of  $\tilde{\Lambda}(S)$ . The corresponding exponential generating function (see Corollary 4.11) is given by*

$$P(u, t) = \exp(\arcsin(u\sqrt{t})/\sqrt{t}),$$

and the Hilbert series of  $\tilde{\Lambda}(\{1, 2, \dots, n\})$  is

$$P_{n+1}(t) = \prod_{0 \leq k < (n-2)/2} (1 + (n-2-2k)^2 t).$$

*Proof.* The only thing remaining to show is that the exponential generating function implies the Hilbert series, or in other words that

$$\exp(\arcsin(u\sqrt{t})/\sqrt{t}) = \sum_{n \geq 0} \prod_{0 \leq k < (n-2)/2} (1 + (n-2-2k)^2 t) \frac{u^n}{n!}.$$

To this end, we observe that if  $t = -1/k^2$  for a positive integer  $k$ , then

$$(4.2) \quad \exp(\arcsin(u\sqrt{t})/\sqrt{t}) = (u/k + \sqrt{1 + u^2/k^2})^k.$$

But (4.2) is the sum of a polynomial of degree  $k$  and of a function satisfying  $f(u) = (-1)^{k+1} f(-u)$ . Thus, if  $n - k$  is an even positive integer, we have  $\frac{d^n}{du^n} P(u, -1/k^2)|_{u=0} = 0$ . The coefficient of  $u^n/n!$  in  $P(u, t)$  is therefore divisible by  $(1 + (n - 2i)^2 t)$  for  $1 \leq i < n/2$ . Since we know the degree of  $P_{n+1}(t)$ ,

this determines the above expansion up to a constant which can be set via the limit  $P(u, 0) = \exp(u)$ . □

COROLLARY 4.17. *There is a canonical isomorphism*

$$(4.3) \quad \tilde{\Lambda}(\{1, 2, \dots, n\}) \cong W_*(\Pi_n^{\text{odd}}).$$

It will be of use to know how the canonical differential on  $W_*$  is expressed on  $\tilde{\Lambda}(\{1, 2, \dots, n\})$ .

LEMMA 4.18. *Under the isomorphism (4.3), the canonical differential on  $W_*$  takes a forest monomial*

$$\tau_{a_1 b_1 c_1} \tau_{a_2 b_2 c_2} \cdots \tau_{a_l b_l c_l}$$

to

$$\partial(\tau_{a_1 b_1 c_1} \tau_{a_2 b_2 c_2} \cdots \tau_{a_l b_l c_l}) = \sum_{1 \leq i \leq l} (-1)^{i-1} \prod_{j \neq i} \tau_{a_j b_j c_j}.$$

*Proof.* The action of the canonical differential is to remove the last element in the chain of partitions. This is equivalent to removing the last triangle in the ordering. Summing over the possible last triangles gives the stated result. □

This is the differential for the reduced homology of a simplicial complex, with a vertex for each triangle and a simplex for each basic forest.

COROLLARY 4.19. *The simplicial complex of basic forests on  $\{1, 2, \dots, 2n+1\}$  is contractible. The homology of the simplicial complex of disconnected basic forests on  $\{1, 2, \dots, n\}$  is isomorphic to the homology of  $\Pi_n^{\text{odd}}$ .*

### 5. The cohomology of the Bockstein

As a key step in determining the cohomology ring of  $M_n$ , we will compute the cohomology of  $M_n$  with coefficients in  $\mathbb{Z}/4\mathbb{Z}$ , modulo 2-torsion. To do so we will start with the cohomology ring of the complex moduli space  $M_n^{\mathbb{C}} := \overline{M_{0,n}}(\mathbb{C})$ . In [Theorem 5.5](#), we will use Keel’s work to find an explicit basis for  $H^*(M_n^{\mathbb{C}}, \mathbb{Z})$  which is convenient for our purposes. We use this cohomology ring to determine the cohomology of the real moduli space  $M_n$  with coefficients in  $\mathbb{F}_2$  ([Theorem 5.6](#)). This mod 2 cohomology  $H^*(M_n, \mathbb{F}_2)$  has a differential, the Bockstein, whose cohomology is the cohomology of  $M_n$  with coefficients in  $\mathbb{Z}/4\mathbb{Z}$ , modulo 2-torsion. In [Theorem 5.7](#), we compute the cohomology of this Bockstein differential and show that it is isomorphic to  $\tilde{\Lambda}(\{1, \dots, n-1\}) \otimes \mathbb{F}_2$ . Thus the resulting Betti numbers give us the desired upper bound on the Betti numbers of  $M_n$  ([Corollary 5.8](#)).

5.1. *Cohomology of  $M_n^{\mathbb{C}}$ .* We begin by recalling the following presentation, due to Keel, of  $H^*(M_n^{\mathbb{C}}, \mathbb{Z})$ . For our purposes, it will be useful to single out one of the marked points, and thus consider  $M_{n+1}^{\mathbb{C}}$ , with points marked  $0, 1, \dots, n$ .

THEOREM 5.1 ([Kee92]). *The commutative ring  $H^*(M_{n+1}^{\mathbb{C}}, \mathbb{Z})$  is generated by elements (of degree 2)  $D_S$ , one for each subset  $S \subset \{0, 1, 2, \dots, n\}$  with  $2 \leq |S| \leq n-1$ , subject to the following relations:*

1.  $D_S = D_{\{0,1,\dots,n\} \setminus S}$ .
2. For distinct elements  $i, j, k, l \in \{0, 1, \dots, n\}$ ,

$$\sum_{\substack{i,j \in S \\ k,l \notin S}} D_S = \sum_{\substack{i,k \in S \\ j,l \notin S}} D_S.$$

3. If  $S \cap T \notin \{\emptyset, S, T\}$  and  $S \cup T \neq \{0, 1, \dots, n\}$ , then  $D_S D_T = 0$ .

The exponential generating function

$$A(u, t) := \sum_{n \geq 2} \sum_{k=0}^{n-2} \dim(H^{2k}(M_{n+1}^{\mathbb{C}}, \mathbb{Z})) t^k \frac{u^n}{n!}$$

satisfies the differential equation

$$(5.1) \quad \frac{\partial}{\partial u} A(u, t) = u + (1+t)A(u, t) + tA(u, t) \frac{\partial A(u, t)}{\partial u}.$$

*Remark.* The class  $D_S$  has a geometrical interpretation as the class of the divisor of  $M_{n+1}^{\mathbb{C}}$  consisting of singular genus 0 curves in which the removal of a singular point separates the points in  $S$  from the points not in  $S$ .

For our purposes, we will need an alternate presentation involving only  $S \subset \{1, \dots, n\}$ . First, define an element

$$D_{\{1,2,\dots,n\}} = - \sum_{\{1,2\} \subset S \subsetneq \{1,2,\dots,n\}} D_S.$$

LEMMA 5.2. *The element  $D_{\{1,2,\dots,n\}}$  is invariant under the action of  $S_n$ .*

*Proof.* We need simply to show that

$$\sum_{\{1,2\} \subset S \subsetneq \{1,2,\dots,n\}} D_S = \sum_{\{1,3\} \subset S \subsetneq \{1,2,\dots,n\}} D_S.$$

But if we eliminate the common terms from both sides, this becomes

$$\sum_{\substack{1,2 \in S \\ 0,3 \notin S}} D_S = \sum_{\substack{1,3 \in S \\ 0,2 \notin S}} D_S,$$

which holds by the linear relation.  $\square$

*Remark.* It follows from Proposition 1.6.3 of [KM96] that

$$D_{\{1,2,\dots,n\}} = -c_1(\ell_0),$$

where  $\ell_0$  is the line bundle obtained by taking the tangent line at the 0-th marked point, and  $c_1$  is the first Chern class.

Now, for  $2 \leq |S| \leq n$ , define

$$\Pi_S = - \sum_{S \subset T \subset \{1,2,\dots,n\}} D_T.$$

Since

$$D_S = - \sum_{S \subset T \subset \{1,2,\dots,n\}} (-1)^{|T|-|S|} \Pi_T,$$

these elements span  $H^*(M_{n+1}^C, \mathbb{Z})$ . However, by the definition of  $D_{\{1,2,\dots,n\}}$ , we find that  $\Pi_S = 0$  whenever  $|S| = 2$ , and thus  $H^1(M_{n+1}^C, \mathbb{Z})$  is spanned by the elements  $\Pi_S$  for  $|S| \geq 3$ . There are  $2^n - \binom{n}{2} - n - 1$  such elements, which equals the rank of  $H^1(M_{n+1}^C, \mathbb{Z})$ , and thus the elements  $\Pi_S$  for  $|S| \geq 3$  form a basis of  $H^1(M_{n+1}^C, \mathbb{Z})$ , and generate the cohomology ring.

PROPOSITION 5.3. *The elements  $\Pi_S$  satisfy the following quadratic relation. For any subsets  $S, T \subset \{1, 2, \dots, n\}$  such that  $S \cap T \notin \{\emptyset, S, T\}$ ,*

$$(5.2) \quad (\Pi_S - \Pi_{S \cup T})(\Pi_T - \Pi_{S \cup T}) = 0.$$

*Proof.* Note that the conditions on  $S$  and  $T$  imply  $2 \leq |S|, |T| \leq n - 1$ . We have

$$\Pi_S - \Pi_{S \cup T} = \sum_{\substack{S \subset U \\ T \not\subset U}} D_U \quad \text{and} \quad \Pi_T - \Pi_{S \cup T} = \sum_{\substack{T \subset V \\ S \not\subset V}} D_V.$$

In particular, for any terms  $D_U$  and  $D_V$  in the respective sums, we have  $S \cap T \subset U \cap V$ , so  $U \cap V \neq \emptyset$ , and  $S, T \not\subset U \cap V$ , so  $U \cap V \notin \{U, V\}$ . In other words,  $D_U D_V = 0$  for any such pair, and the product of the two sums vanishes termwise.  $\square$

*Remark.* When  $|T| = 2$ , we have  $\Pi_T = 0$ . Since  $S \cup T = S \cup \{i\}$  for some  $i \notin S$ ,

$$(5.3) \quad \Pi_{S \cup \{i\}}(\Pi_S - \Pi_{S \cup \{i\}}) = 0$$

is a special case of (5.2). More generally, it follows from an easy induction that for any disjoint sets  $S, T$ ,

$$(5.4) \quad \Pi_{S \cup T}^{|S|+1} = \Pi_{S \cup T}^{|S|} \Pi_T.$$

We claim that (5.2) are the only relations satisfied by the  $\Pi_S$ . To prove this, it will in fact be simplest to give a basis for the ring thus presented, and show that it is a free  $\mathbb{Z}$ -module with the correct Hilbert series.

In fact, we can give a Gröbner basis for the ring, which will moreover be invariant under the  $S_n$  action. We need one more set of relations, which can be deduced from (5.2).

LEMMA 5.4. *The elements  $\Pi_S$  satisfy the following relation for all  $k \geq 0$ . Let  $S_0, S_1, \dots, S_k$  be disjoint sets, with union  $S$ ; suppose moreover that  $|S_i| \geq 3$  for  $1 \leq i \leq k$ . Then*

$$(5.5) \quad \Pi_S^{|S_0|+k-1} \prod_{1 \leq i \leq k} (\Pi_{S_i} - \Pi_S) = 0.$$

*Proof.* For  $k = 0$ , this becomes the statement

$$(5.6) \quad \Pi_S^{|S|-1} = 0,$$

which follows from (5.3) and the fact that  $\Pi_T = 0$  if  $|T| = 2$ .

For  $k = 1$  the claim is equivalent to (5.4), so let us assume that  $k \geq 2$ . By applying (5.4) multiple times, we get

$$\Pi_S^{|S_0|+k-1} \prod_{1 \leq i \leq k} (\Pi_{S_i} - \Pi_S) = \Pi_S^{|S_0|} \Pi_{S \setminus S_0}^{k-1} \prod_{1 \leq i \leq k} (\Pi_{S_i} - \Pi_{S \setminus S_0})$$

and may thus assume  $S_0 = \emptyset$ .

For  $k = 2$ , choose  $i \in S_2$  and consider the known relation

$$(5.7) \quad (\Pi_{S_1} - \Pi_{S_1 \cup \{i\}}) \Pi_{S_2} \Pi_{S_1 \cup \{i\}} = 0.$$

By (5.2), we have

$$\Pi_{S_2} \Pi_{S_1 \cup \{i\}} = \Pi_{S_2} \Pi_{S_1 \cup S_2} + \Pi_{S_1 \cup \{i\}} \Pi_{S_1 \cup S_2} - \Pi_{S_1 \cup S_2}^2,$$

so we can simplify (5.7) to

$$\begin{aligned} 0 &= (\Pi_{S_1} - \Pi_{S_1 \cup \{i\}})(\Pi_{S_2} - \Pi_{S_1 \cup S_2}) \Pi_{S_1 \cup S_2} \\ &= (\Pi_{S_1} - \Pi_{S_1 \cup S_2})(\Pi_{S_2} - \Pi_{S_1 \cup S_2}) \Pi_{S_1 \cup S_2} \end{aligned}$$

as required.

For  $k > 2$ , let  $T = S_1 \cup S_2$ . We need to show that

$$(5.8) \quad \Pi_S^{k-1} \prod_{1 \leq i \leq k} (\Pi_{S_i} - \Pi_S) = 0.$$

If we subtract from (5.8) the relation

$$(\Pi_{S_1} - \Pi_T)(\Pi_{S_2} - \Pi_T) \Pi_T \Pi_S^{k-2} \prod_{i>2} (\Pi_{S_i} - \Pi_S),$$

we find by setting  $\Pi_T = \Pi_S$  that the result is divisible by  $\Pi_T - \Pi_S$ . It is therefore divisible by

$$\Pi_S^{k-2}(\Pi_T - \Pi_S) \prod_{i>2} (\Pi_{S_i} - \Pi_S),$$

and so is 0 by induction. □

**THEOREM 5.5.** *The algebra  $H^*(M_{n+1}^{\mathbb{C}}, \mathbb{Z})$  is freely spanned by monomials of the form  $\prod_{|S| \geq 3} \Pi_S^{d_S}$  satisfying the following conditions:*

1. *If  $d_S > 0, d_T > 0$ , then  $S \cap T \in \{\emptyset, S, T\}$ .*
2. *For each  $S$  such that  $d_S > 0$ , let  $S_1, \dots, S_k$  be the maximal proper subsets of  $S$  such that  $d_{S_i} > 0$ , disjoint by condition 1. Then*

$$d_S < k - 1 + |S| - \sum_i |S_i|.$$

*Equivalently, the relations (5.2) and (5.5) form a Gröbner basis of (the ideal of relations of) the cohomology ring  $H^*(M_{n+1}^{\mathbb{C}}, \mathbb{Z})$  with respect to the grevlex order, relative to any ordering on the variables  $\Pi_S$  extending inclusion.*

*Proof.* Any monomial not satisfying the above conditions can be expanded in smaller monomials using the given relations; as a result, the “good” monomials span. The theorem will thus follow if we can show that they form a basis.

For this, we need simply count the monomials and show that we obtain the correct exponential generating function. Now, consider the exponential generating function  $C(u, t)$  of monomials with  $d_{\{1,2,\dots,n\}} > 0$ , extended to include the constant 1 for  $n = 1$ . By standard manipulations of exponential generating functions, this satisfies

$$C(u, t) = u + \sum_{m \geq 3} \sum_{l=1}^{m-2} t^l C(u, t)^m / m!,$$

where  $m = k + |S| - \sum_i |S_i|$  and  $l = d_S$ . If we multiply by  $1 - t$  and simplify, this becomes the functional equation

$$\exp(tC(u, t)) = 1 + tu + t^2(\exp(C(u, t)) - 1 - u).$$

Now, the exponential generating function of all monomials (including 1 for  $n \leq 2$ ) is  $B(u, t) := \exp(C(u, t))$ , and thus satisfies

$$B(u, t)^t = 1 + tu + t^2(B(u, t) - 1 - u).$$

Differentiating with respect to  $u$ , we get

$$\frac{\partial B(u, t)}{\partial u} = \frac{B(u, t)}{1 - t(B(u, t) - 1 - u)}.$$

Therefore  $B(u, t) - 1 - u$  satisfies the differential equation (5.1), and we indeed have the correct number of monomials.  $\square$

*Remark.* A similar  $\mathbb{Z}$ -basis of  $H^*(M_{n+1}^{\mathbb{C}}, \mathbb{Z})$  was given by Yuzvinsky [Yuz97], based on the generators  $D_S$  for  $S \subset \{1, 2, \dots, n\}$ ,  $|S| > 2$ , but otherwise the same. It differs in two important respects: first, while it respects a filtration by set partitions, it does not respect the grading by set partitions; and second, the associated Gröbner basis is more complicated. On the other hand, Yuzvinsky's basis is more amenable to computation of the Poincaré pairing.

One important consequence of this presentation is that there is a natural grading of  $H^*(M_{n+1}^{\mathbb{C}}, \mathbb{Z})$  indexed by the lattice of set partitions of  $\{1, 2, \dots, n\}$ . Indeed, the hypergraphs associated to each monomial of each relation all have the same set of connected components; that set of connected components is the associated set partition. By inspection of the canonical monomials, we also find that the partitions that appear never have sets of size 2.

5.2. *Cohomology with coefficients in  $\mathbb{F}_2$ .* The presentation for  $H^*(M_{n+1}^{\mathbb{C}}, \mathbb{Z})$ , stated above, immediately gives rise to a presentation for  $H^*(M_{n+1}^{\mathbb{R}}, \mathbb{F}_2)$ , by the following result.

**THEOREM 5.6.** *There is a canonical isomorphism between  $H^{2k}(M_{n+1}^{\mathbb{C}}, \mathbb{F}_2)$  and  $H^k(M_{n+1}^{\mathbb{R}}, \mathbb{F}_2)$  for each  $k$ , compatible with the ring structures.*

*Proof.* Indeed, Keel [Kee92] showed that  $H^*(M_{n+1}^{\mathbb{C}}, \mathbb{Z})$  is generated by algebraic divisors, all of which are in fact rational over  $\mathbb{R}$ . It follows that  $\overline{M_{0, n+1}}$  is an “algebraically maximal” variety in the sense of [Kra03] and thus the isomorphism follows from the main result of that paper.  $\square$

*Remark.* Januszkiewicz (personal communication) has suggested another proof. One can construct  $M_{n+1}$  as an iterated blowup of projective  $n$ -space along the standard  $A_{n-1}$  hyperplane arrangement. The desired agreement of mod 2 cohomology holds for  $\mathbb{R}\mathbb{P}^n$  and  $\mathbb{C}\mathbb{P}^n$  and is preserved under blowing up of a linear subspace, and the result follows by induction.

See also [HHP05] for a more general class of spaces satisfying this same relation to their real locus.

We denote the piece of  $H^k(M_{n+1}^{\mathbb{R}}, \mathbb{F}_2)$  indexed by the set partition  $\pi$  by  $\mathbb{E}^k[\pi]$ ; if  $\pi$  is a partition of a subset of  $\{1, 2, \dots, n\}$ , we adjoin singletons as necessary to make it a partition. We will also use  $\mathbb{E}_n^*$  to denote the entire cohomology ring  $H^*(M_{n+1}^{\mathbb{R}}, \mathbb{F}_2)$ .

5.3. *The Bockstein map and its cohomology.* For any space  $X$ , consider the Bockstein map  $\beta : H^*(X, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(X, \mathbb{Z}/2\mathbb{Z})$ , which is the connecting map

for the exact sequence

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

The Bockstein is a differential. From the long exact sequence in cohomology, we see that its cohomology is isomorphic to the quotient of the mod 4 cohomology by the 2-torsion,

$$H_\beta^*(X) \cong H^*(X, \mathbb{Z}/4\mathbb{Z})/H^*(X, \mathbb{Z}/4\mathbb{Z})\langle 2 \rangle.$$

One goal is to compute the cohomology of the Bockstein map acting on

$$H^*(M_{n+1}, \mathbb{Z}/2\mathbb{Z}) = \mathfrak{E}_n^*$$

in order to determine

$$H^*(M_{n+1}, \mathbb{Z}/4\mathbb{Z})/H^*(M_{n+1}, \mathbb{Z}/4\mathbb{Z})\langle 2 \rangle.$$

We denote this cohomology  $H_\beta^*(\mathfrak{E}_n^*)$ . The action of  $\beta$  can be determined from the following two properties (corresponding to the fact [Hat02, §4.L] that  $\beta$  is the first Steenrod square):

1.  $\beta$  is a derivation of degree 1.
2. If  $x \in \mathfrak{E}_n^1$ , then  $\beta(x) = x^2$ .

Since  $\mathfrak{E}_n^*$  is generated in degree 1, this uniquely determines the action of  $\beta$  on the entire algebra:

$$\beta(v) = \sum_S \Pi_S^2 \frac{\partial}{\partial \Pi_S} v.$$

(This is well-defined since  $\beta(I) \subset I$  where  $I$  is the defining ideal; in fact, if  $r$  is one of the defining relations (5.2), then  $\beta(r)$  is a multiple of  $r$ .) In particular,  $\beta$  is homogeneous with respect to the grading by set partitions, and thus

$$H_\beta^*(\mathfrak{E}_n^*) = \bigoplus_{\pi} H_\beta^*(\mathfrak{E}^*[\pi]).$$

Moreover,

$$\mathfrak{E}^*[S_1, S_2, \dots, S_k] = \bigotimes_i \mathfrak{E}^*[S_i],$$

and thus

$$H_\beta^*(\mathfrak{E}^*[S_1, S_2, \dots, S_k]) = \bigotimes_i H_\beta^*(\mathfrak{E}^*[S_i]).$$

We therefore need only to determine  $H_\beta^*(\mathfrak{E}^*[\{1, 2, \dots, n\}])$ .

The main result of this section is the following.

**THEOREM 5.7.** *There is a natural graded module isomorphism*

$$H_\beta^*(\mathfrak{E}^*[\{1, 2, \dots, 2n + 1\}]) \cong \tilde{\Lambda}[\{1, 2, \dots, 2n + 1\}] \otimes \mathbb{F}_2,$$

and thus the former is concentrated in degree  $n$ . For even lengths,

$$H_\beta^*(\Xi^*\{1, 2, \dots, 2n\}) = 0.$$

*Proof.* We first observe that there is a map from  $\widetilde{\Lambda}(S) \otimes \mathbb{F}_2$  to  $H_\beta^*(\Xi^*(S))$ , taking  $\tau_{ijk}$  to  $\Pi_{\{i,j,k\}}$ . To see that this is well-defined, we check by (5.6) that  $\Pi_{\{i,j,k\}}$  is  $\beta$ -closed, and verify that the relations hold modulo the image of  $\beta$ . Indeed, we find

$$\begin{aligned} \Pi_{\{i,j,k\}}^2 &= 0 \\ \Pi_{\{i,j,k\}}\Pi_{\{j,k,l\}} &= \Pi_{\{i,j,k\}}\Pi_{\{i,j,k,l\}} + \Pi_{\{j,k,l\}}\Pi_{\{i,j,k,l\}} + \Pi_{\{i,j,k,l\}}^2 \\ &= \beta(\Pi_{\{i,j,k,l\}}) \end{aligned}$$

by (5.2) and (5.3). For the 5-term relation, we take the identity

$$\Pi_{\{i,j,k\}}\Pi_{\{k,l,m\}} = \Pi_{\{i,j,k\}}\Pi_{\{i,j,k,l,m\}} + \Pi_{\{k,l,m\}}\Pi_{\{i,j,k,l,m\}} + \beta(\Pi_{\{i,j,k,l,m\}})$$

and sum over cyclic shifts to obtain

$$\begin{aligned} \Pi_{\{i,j,k\}}\Pi_{\{k,l,m\}} + \Pi_{\{j,k,l\}}\Pi_{\{l,m,i\}} + \Pi_{\{k,l,m\}}\Pi_{\{m,i,j\}} \\ + \Pi_{\{l,m,i\}}\Pi_{\{i,j,k\}} + \Pi_{\{m,i,j\}}\Pi_{\{j,k,l\}} = \beta(\Pi_{\{i,j,k,l,m\}}), \end{aligned}$$

since each term  $\Pi_{\{i,j,k\}}\Pi_{\{i,j,k,l,m\}}$  appears exactly twice in the sum.

We will prove the theorem by induction, using a spectral sequence. Consider the filtration of  $\Xi^*\{1, \dots, n\}$  given by

$$V_i = \Pi_{\{1,2,\dots,n\}}^i \Xi_n^*,$$

for  $1 \leq i \leq n - 2$ ; each of these is a homogeneous ideal of  $\Xi_n^*$ . The quotient  $V_i/V_{i+1}$  is spanned by monomials  $\prod \Pi_S^{d_S}$  having  $d_{\{1,2,\dots,n\}} = i$ . By Theorem 5.5, we therefore have a natural identification (including the action of  $\beta$ )

$$V_i/V_{i+1} = \Pi_{\{1,2,\dots,n\}}^i \bigoplus_{\pi:|\pi|>i+1} \Xi^*[\pi] \cong \bigoplus_{\pi:|\pi|>i+1} \Xi^*[\pi].$$

The filtration  $V_i$  gives rise to a spectral sequence converging to  $H_\beta^*(\Xi_n^*)$  with first page

$$E_1^{p,q} = H_\beta^{p+q}(V_p/V_{p+1}) \cong \bigoplus_{\pi:|\pi|>p+1} H_\beta^q(\Xi^*[\pi]).$$

Since each set in each partition  $\pi$  has size  $< n$ , we find by the inductive hypothesis that

$$H_\beta^*(\Xi^*[\pi]) = 0$$

unless all parts of  $\pi$  have odd order, in which case it is the homology of the interval  $[\widehat{0}, \pi]$ , where  $\widehat{0}$  is the partition into singletons. More precisely, the interval splits naturally as a product of intervals  $[\widehat{0}, \{1, 2, \dots, k\}]$ , and by Proposition 4.1,



We may as well assume that  $x \in \prod_{\{1, \dots, n\}}^{2p+1} \Xi^*[\pi]$  for some partition  $\pi$  with  $2p + 3$  parts, i.e. that

$$x = \left( \prod_{\{1, \dots, n\}}^{2p+1} \prod_i \Pi_{\pi_i} \right) y$$

for some  $y$ . Multiplying by  $\prod_{\{1, \dots, n\}}$ , we use (5.5) to compute

$$d_2(x + V_{2p+2}) = \left( \prod_{\{1, \dots, n\}}^{2p+2} \prod_i \Pi_{\pi_i} \right) y \equiv \left( \sum_i \prod_{\{1, \dots, n\}}^{2p+3} \prod_{j \neq i} \Pi_{\pi_j} \right) y$$

modulo  $V_{2p+4}$ . In particular, the  $d_2$  differential is induced by the natural maps  $\delta :=$ “divide by  $\Pi_{\pi_i}$ ” on each piece of  $\pi$

$$\delta : H_{\beta}^*(V_1/V_2) \rightarrow H_{\beta}^{*-1}(V_0/V_1).$$

So it suffices to prove that  $\delta$  agrees with the canonical differential in its action on  $\tilde{\Lambda}$ . For this we proceed by induction, observing that it holds for  $\tilde{\Lambda}[\{1, 2, 3\}]$  (where both maps take  $\tau_{123}$  to 1), and that it acts as a derivation on trees. Indeed, using (5.2), we check that

$$\begin{aligned} \delta((\Pi_S y)(\Pi_T z)) &= \delta((\Pi_{S \cup T} \Pi_S + \Pi_{S \cup T} \Pi_T - \Pi_{S \cup T}^2) yz) \\ &\equiv \delta(\Pi_{S \cup T} (\Pi_S + \Pi_T) yz) = (\Pi_S + \Pi_T) yz = \delta(\Pi_S y) \Pi_T z + \delta(\Pi_T z) \Pi_S y. \end{aligned}$$

The theorem follows.  $\square$

Let  $h^k(X, \mathbb{Q})$  denote the  $k$ -th Betti number of a space  $X$ .

COROLLARY 5.8. *We have the coefficient-wise upper bound*

$$\sum_{0 \leq n, k} h^k(M_{n+1}^{\mathbb{R}}, \mathbb{Q}) t^k \frac{u^n}{n!} \leq \exp(\arcsin(u\sqrt{t})/\sqrt{t}).$$

*Proof.* Indeed, we have

$$h^k(M_{n+1}^{\mathbb{R}}, \mathbb{Q}) \leq h_{\beta}^k(H^*(M_{n+1}^{\mathbb{R}}, \mathbb{F}_2)) = h_{\beta}^k(\Xi_n^*) = \dim \tilde{\Lambda}(\{1, \dots, n\}),$$

and the latter has the stated exponential generating function by Corollary 4.16.  $\square$

A twisted version may also be of interest.

COROLLARY 5.9. *Let  $\beta'$  be the twisted differential defined by*

$$\beta'(v) = \beta(v) + \prod_{\{1, 2, \dots, n\}} v.$$

*Then there is a canonical isomorphism*

$$H_{\beta'}^*(\Xi_{2n}^*) \cong \tilde{H}_*(\Pi_{2n}^{\text{odd}}, \mathbb{Z}) \otimes \mathbb{F}_2$$

and

$$H_{\beta'}^*(\Xi_{2n+1}^*) = 0.$$

### 6. The operad structure and the proof of the main results

The main remaining step of the proof of the main theorem is to show that  $f_n^{\mathbb{Z}}$  is split-injective. To do this we will use the operad structure to construct elements of the homology  $H_*(M_n)$  which pair upper-triangularly with the images under  $f_n$  of the basis vectors in  $\Lambda_n$ . Along the way we will also determine the structure of the homology operad.

6.1. *The cyclic (co)operad  $\Lambda$ .* The collection of algebras  $\Lambda_n$  forms a cyclic cooperad  $\Lambda$ . To describe this cyclic cooperad structure, it will be convenient to replace the index set  $\{1, 2, \dots, n\}$  by an arbitrary nonempty finite set. Thus let  $\Lambda\langle S \rangle$  be the  $\mathbb{Z}$ -algebra with antisymmetric generators  $\omega_{ijkl}$  for  $i, j, k, l \in S$  satisfying the relations (2.1), (2.2), (2.3). Similarly, let  $\Lambda(S)$  denote the  $\mathbb{Z}$ -algebra with antisymmetric generators  $\nu_{ijk}$  for  $i, j, k \in S$  satisfying the relations of  $\Lambda_{|S|+1}$ .

Given nonempty finite sets  $S, T$  and a function  $f : S \rightarrow T$ , define a function  $h_t : S \rightarrow f^{-1}(t) \sqcup \{t\}$  for each  $t \in T$  by

$$h_t(s) = \begin{cases} s, & \text{if } f(s) = t, \\ t, & \text{otherwise.} \end{cases}$$

Then we define a homomorphism (which will be the structure map of the cooperad)

$$\Delta_f : \Lambda\langle S \rangle \rightarrow \Lambda\langle T \rangle \otimes \bigotimes_{t \in T} \Lambda\langle f^{-1}(t) \sqcup \{t\} \rangle$$

in degree one as follows:

$$\Delta_f(\omega_{ijkl}) = \omega_{f(i)f(j)f(k)f(l)}^0 + \sum_{t \in T} \omega_{h_t(i)h_t(j)h_t(k)h_t(l)}^t,$$

where each  $\omega_{ijkl} = 0$  if two indices agree. (The superscripts are used merely to distinguish the generators of  $\Lambda\langle T \rangle$ , denoted  $\omega_{ijkl}^0$ , from the generators of each  $\Lambda\langle f^{-1}(t) \sqcup \{t\} \rangle$ .) Note that this implies that at most one term appears on the right. (N.b., since we are defining the map on generators, the order of the tensor product is irrelevant. Also, if  $|f^{-1}(t)| < 3$ , we may freely omit the corresponding factor on the right, since  $\Lambda\langle S \rangle = \mathbb{Z}$  if  $1 \leq |S| \leq 3$ .) This should be viewed as corresponding to the geometric operad map

$$M_T \times \prod_{t \in T} M_{f^{-1}(t) \sqcup \{t\}} \rightarrow M_S$$

obtained by gluing together each pair of points labelled  $t$ .

The map  $\Delta_f$  is easily seen to respect relation (2.1). For relation (2.2), we find that either one of the two generators maps to 0 or both map to elements of the same algebra; in either case, the relation automatically holds. To check relation (2.3), there are a few more cases. Either all the generators map to the same algebra, or at

least one generator in each monomial maps to 0, or we have  $f(i) = f(j) = f(k) \neq f(l) = f(m) = f(p)$ , or we have  $f(i) = f(j) = f(k)$ ,  $|\{f(i), f(l), f(m), f(p)\}| = 4$  (up to cyclic permutation of the indices). In either of the latter cases, the relation is easily verified.

Given another map  $g : T \rightarrow U$ , we define for each  $u \in U$  a map

$$f_u : (g \circ f)^{-1}(u) \sqcup \{u\} \rightarrow g^{-1}(u) \sqcup \{u\}$$

by  $f_u(s) = f(s)$  for  $s \in (g \circ f)^{-1}(u)$ , and  $f_u(u) = u$ .

**THEOREM 6.1.** *Given nonempty finite sets  $S, T, U$  and functions  $f : S \rightarrow T, g : T \rightarrow U$ , we have the identity*

$$(\Delta_g \otimes 1) \circ \Delta_f = (1 \otimes \bigotimes_{u \in U} \Delta_{f_u}) \circ \Delta_{g \circ f}$$

(up to the symmetry of the tensor product).

In other words, the maps  $\Delta_f$  furnish the algebras  $\Lambda\langle S \rangle$  with the structure of a cyclic cooperad in the category of superalgebras.

*Proof.* It suffices to check the relation on the generators of  $\Lambda\langle S \rangle$ . And indeed, both sides map  $\omega_{ijkl}$  to

$$\begin{aligned} \omega_{g(f(i))g(f(j))g(f(k))g(f(l))}^0 &+ \sum_{u \in U} \omega_{h_u(f(i))h_u(f(j))h_u(f(k))h_u(f(l))}^u \\ &+ \sum_{t \in T} \omega_{h_t(i)h_t(j)h_t(k)h_t(l)}^t. \end{aligned} \quad \square$$

The following proposition follows easily from definitions.

**PROPOSITION 6.2.** *The above cyclic cooperad structure is compatible with the natural map  $f_n^{\mathbb{Z}_2} : \Lambda_n \rightarrow H^*(M_n, \mathbb{Z}_2)/H^*(M_n, \mathbb{Z}_2)\langle 2 \rangle$  and the cyclic cooperad structure induced on cohomology by the cyclic operad structure of  $M_n$ .*

We now turn to the  $S_{n-1}$ -symmetric algebras  $\Lambda(S)$ . If we add to  $S$  a new label  $\infty$ , we know by [Proposition 2.3](#) that  $v_{ijk} \mapsto \omega_{ijk\infty}$  induces an isomorphism  $\Lambda(S) \cong \Lambda(S \sqcup \{\infty\})$ .

Given a function  $f : S \rightarrow T$ , we can extend it to a function  $f : S \sqcup \{\infty\} \rightarrow T \sqcup \{\infty\}$  by taking  $f(\infty) = \infty$ . This then gives a map which we abusively denote  $\Delta_f$ :

$$\Delta_f : \Lambda(S) \rightarrow \Lambda(T) \otimes \bigotimes_{t \in T} \Lambda(f^{-1}(t)).$$

This map then satisfies the axioms of a (noncyclic) operad. On the generators  $v$ , we have

$$\Delta_f(v_{abc}) = \begin{cases} v_{f(i)f(j)f(k)}^0, & \text{if } |f(\{i, j, k\})| = 3, \\ 0, & \text{if } |f(\{i, j, k\})| = 2, \\ v_{ijk}^t, & \text{if } f(i) = f(j) = f(k) =: t. \end{cases}$$

Now, if we quotient by the augmentation ideal of  $\Lambda(T)$ , we obtain a map  $\eta_f : \Lambda(S) \rightarrow \bigotimes_{t \in T} \Lambda(f^{-1}(t))$ .

**PROPOSITION 6.3.** *The map  $\eta_f$  is a split surjection.*

*Proof.* Indeed, for each  $t \in T$ , we have a map  $\Lambda(f^{-1}(t)) \rightarrow \Lambda(S)$  induced from the inclusion map; taking the product of these maps gives the desired splitting.  $\square$

Since  $\Lambda$  is a cooperad, its dual  $\Lambda^*$  is an operad. The simplest operation of  $\Lambda^*$  not in the suboperad of commutative superalgebras (i.e. the degree 0 part) is the ternary operation  $\tau$ , of degree -1, corresponding to the linear functional on  $\Lambda(\{1, 2, 3\})$  that takes 1 to 0 and  $v_{123}$  to 1. Together with the supercommutative product  $\cdot$ , the operation  $\tau$  satisfies the following relations:

$$(6.1) \quad \tau(x, y, z) = (-1)^{1+|x||y|} \tau(y, x, z) = (-1)^{1+|y||z|} \tau(x, z, y)$$

$$(6.2) \quad \tau(w, x, y \cdot z) = \tau(w, x, y) \cdot z + (-1)^{|y||z|} \tau(w, x, z) \cdot y,$$

and a 10-term relation stating that the various permutations (with appropriate signs compatible with superantisymmetry) of  $\tau(\tau(v, w, x), y, z)$  sum to 0. (That some 10-term relation holds follows from the fact that  $\Lambda_{5+1}[2]$  is free of rank 9, while the space of compositions of  $\tau$  is free of rank 10; that it has the stated form follows from the fact that  $\text{Hom}(\Lambda_{5+1}[2], \mathbb{Z})$  does not contain a copy of the sign representation of  $S_5$ .)

Let  $\Lambda^\#$  be the operad generated by  $\cdot$  and  $\tau$ , with relations given by (6.1), (6.2) and the above 10-term relation. We have a map  $\Lambda^\# \rightarrow \Lambda^*$  which we will soon show is an isomorphism.

If we ignore the product, the operad  $\Lambda^\#(\bullet)$  generated by  $\tau$  with superantisymmetry and the 10-term relation is a twisted version of the Lie 2-algebra operad  $\text{HW}(\bullet)$  discussed in [HW95]. One of their main results is that  $\text{HW}(n)$  is isomorphic to  $H^*(\Pi_n^{\text{odd}})$ . It follows from Theorem 4.3 that the exponential generating function for  $\dim(\text{HW}(n))$  is  $\arcsin(u\sqrt{t})/\sqrt{t}$ . Since the twisting just amounts to tensoring with the sign operad, the same generating function applies to our twisted operad.

Using (6.2), any operation in  $\Lambda^\#$  can be expressed as a sum of operations where  $\cdot$  is only used after  $\tau$ . It follows that the exponential generating function for the number of additive generators of  $\Lambda^\#$  is bounded above by  $\exp(\arcsin(u\sqrt{t})/\sqrt{t})$ .

Recall that by [Corollary 4.16](#) this was also the exponential generating function for the ranks of  $\tilde{\Lambda}_n$ . A similar result holds for  $\Lambda_n$ .

**THEOREM 6.4.** *The basic forests form a basis of  $\Lambda(S)$ , which is thus a free  $\mathbb{Z}$ -module. The corresponding exponential generating function is*

$$P(u, t) = \exp\left(\arcsin(u\sqrt{t})/\sqrt{t}\right),$$

and the Hilbert series of  $\Lambda_n$  is

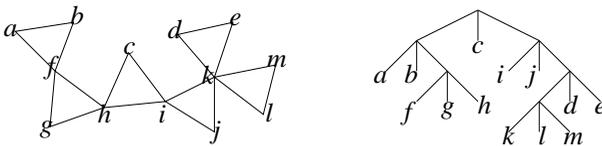
$$P_n(t) = \prod_{0 \leq k < (n-3)/2} (1 + (n-3-2k)^2 t).$$

*Proof.* The argument in [Proposition 4.13](#) was purely combinatorial, and thus changing signs in the relations will have no effect. As a result,  $\Lambda(S)$  is also spanned by basic forests.

Unfortunately, by twisting the  $S_n$  action, we have destroyed the canonical isomorphism with partition homology, making the argument in [Theorem 4.14](#) no longer valid. Recall that the main tool of [Theorem 4.14](#) was a pairing between saturated chains of odd partitions and triangle trees. We can achieve the same effect using ternary forests and the operad structure.

To each saturated chain of odd partitions, we associate a ternary forest, with a node for each set that appears as the part of some partition in the chain. To each ternary tree  $U$ , let  $\tau_U \in \Lambda^*(S)$  denote the result of composition of the operation  $\tau$  according to the tree  $U$ ; we then extend this to forests using the product operation  $\cdot$ . Recall that to each triangle forest  $F$ , we have a monomial  $v^F \in \Lambda(S)$ . Now given  $F$  a triangle forest and  $G$  a ternary forest, we can consider the pairing  $\langle \tau_G, v^F \rangle$ .

The action of composition with  $\tau$  (corresponding to a partition into three components) is such that a triangle forest has nonzero image if and only if there is a triangle in the forest that hits each component, while all other triangles are contained in a component; the action of composition with  $\cdot$  is such that a triangle forest has nonzero image if and only if each component of the forest is contained in a component of the composition. In particular, the pairing between ternary forests and triangle forests takes on only the values  $\pm 1$  and  $0$ , nonzero if and only if there exists an ordering on the triangles inducing a partition chain with the given associated ternary forest.



*A triangle tree and a ternary tree that pair nontrivially.*

By mimicking the proof of [Theorem 4.14](#), with ternary forests instead of chains of odd partitions we obtain the result; note that the action of  $\cdot$  is such that the pairing between forests with the same component partition is just (up to sign) the product of the pairings between the individual trees.  $\square$

The proof of the theorem shows that the pairing between  $\Lambda^\#$  and  $\Lambda$  induces a surjective map  $\Lambda^\# \rightarrow \text{Hom}(\Lambda, \mathbb{Z})$ . Moreover, the number of additive generators of  $\Lambda^\#$  is bounded above by the rank of  $\Lambda$ . It follows that  $\Lambda^\#$  is torsion free and that the natural map  $\Lambda^\# \rightarrow \Lambda^*$  is an isomorphism. We conclude the following.

**COROLLARY 6.5.** *The operad  $\Lambda^*$  is presented over the operad of commutative superalgebras by a ternary operation  $\tau$  satisfying [\(6.1\)](#), [\(6.2\)](#) and the 10-term relation  $\text{Alt}(\tau \circ (\tau \otimes \text{Id} \otimes \text{Id})) = 0$ .*

**6.2. Determination of the cohomology ring.** We are now in a position to prove our main theorems.

**THEOREM 6.6.** *The map  $f_n : \Lambda_n \rightarrow H^*(M_n, \mathbb{Z})/H^*(M_n, \mathbb{Z})\langle 2 \rangle$  is injective and splits as a map of  $\mathbb{Z}$ -modules.*

*Proof.* Recall that  $\Lambda_n$  is a free  $\mathbb{Z}$  module with basis of monomials  $v^F$  indexed by basic triangle forests  $F$ . It suffices to find a collection of elements in  $H_*(M_n, \mathbb{Z})$ , also indexed by basic triangle forests which pair upper triangularly with the images  $f_n(v^F)$ .

To do this we follow the proof of [Theorem 4.14](#), or more precisely its modification in [Theorem 6.4](#). In the proof of [Theorem 4.14](#), for each basic triangle forest (more precisely, tree, but by the product structure, this extends to forests), we associated a saturated chain of set partitions, which paired with the desired property. In the proof of [Theorem 6.4](#), we used this saturated chain of set partitions to build a ternary forest.

For each ternary tree  $U$ , we can consider a map  $g_U : M_4^d \rightarrow M_n$  which is given by gluing curves according to the tree  $U$ . Let  $\rho^U$  denote the image under the map  $g_U$  of the fundamental class in  $H_*(M_n)$ . These elements  $\rho^U$  are exactly what would be obtained by using  $U$  to specify a particular composition of the homology operad generator  $\tau \in H_1(M_4, \mathbb{Q})$  (after tensoring with  $\mathbb{Q}$ ). More generally, for a ternary forest  $G$ , we consider the associated map  $g_G : M_4^{d'} \times M_c \rightarrow M_n$ , where  $c$  is the number of components of  $G$ , first gluing together curves according to the component trees of the forest, then gluing their roots to  $M_c$ . The class  $\rho^G$  is then the image of the product of the fundamental class of  $M_4^d$  with the point class in  $M_c$ , and again corresponds to the appropriate element of the homology operad.

The pairing between  $\rho^G$  and  $f_n(v^F)$  is again described combinatorially as in the proof of [Theorem 6.4](#), and thus we again have the desired upper triangularity property.

In particular, we have constructed a collection of linear functionals on

$$H^*(M_n, \mathbb{Z})/H^*(M_n, \mathbb{Z})\langle 2 \rangle$$

that, when evaluated on our basis of  $\Lambda_n$ , produce a triangular matrix with unit diagonal. Composing these functionals with the inverse of that upper triangular matrix produces a left inverse of  $f_n$ . It follows that  $\Lambda_n$  is a direct summand of  $H^*(M_n, \mathbb{Z})/H^*(M_n, \mathbb{Z})\langle 2 \rangle$ .  $\square$

So we can write  $H^*(M_n, \mathbb{Z})/H^*(M_n, \mathbb{Z})\langle 2 \rangle = \Lambda_n \oplus B$ . At this point we do not know much about the complement  $B$ . Here we need the information from [Section 5](#).

**THEOREM 6.7.**  *$B$  is a product of odd torsion modules and  $H^*(M_n, \mathbb{Z})$  has no 4-torsion.*

*Proof.* Let us split  $H^*(M_n, \mathbb{Z})$  (noncanonically) as  $H^*(M_n, \mathbb{Z}) = F \oplus G \oplus H \oplus I$ , where  $F$  is free,  $G$  is a product of odd torsion modules,  $H$  is a product of  $\mathbb{Z}/2\mathbb{Z}$  and  $I$  is a product of  $\mathbb{Z}/2^k\mathbb{Z}$  for  $k \geq 2$ . Then,  $H^*(M_n, \mathbb{Z})/H^*(M_n, \mathbb{Z})\langle 2 \rangle$  is isomorphic to  $F \oplus G \oplus I'$  where  $I' = I/I\langle 2 \rangle$ . By [Theorem 6.6](#), we see that the  $\text{rk}(F) \geq \text{rk}(\Lambda_n)$ .

On the other hand,  $H^*(M_n, \mathbb{Z}/4\mathbb{Z})/H^*(M_n, \mathbb{Z}/4\mathbb{Z})\langle 2 \rangle$  is an  $\mathbb{F}_2$  vector space of dimension  $\text{rk}(F)$  plus twice the number of factors in  $I$ .

We know from [Theorem 5.7](#) that there are isomorphisms

$$\tilde{\Lambda}_n \otimes \mathbb{F}_2 \cong \Lambda_n \otimes \mathbb{F}_2 \rightarrow H^*(M_n, \mathbb{Z}/4\mathbb{Z})/H^*(M_n, \mathbb{Z}/4\mathbb{Z})\langle 2 \rangle.$$

Since  $\Lambda_n$  is free, the dimension of  $\Lambda_n \otimes \mathbb{F}_2$  is  $\text{rk}(\Lambda_n)$ . Combining together these observations we see that

$$\begin{aligned} \text{rk}(\Lambda_n) &= \dim(H^*(M_n, \mathbb{Z}/4\mathbb{Z})/H^*(M_n, \mathbb{Z}/4\mathbb{Z})\langle 2 \rangle) \\ &= \text{rk}(F) + 2(\# \text{ of factors in } I) \geq \text{rk}(F) \geq \text{rk}(\Lambda_n). \end{aligned}$$

Hence we conclude that  $\text{rk}(F) = \text{rk}(\Lambda_n)$  and that  $I = 0$ . Since  $I = 0$ , there is no 4-torsion. Also we see that the complement  $B$  is a product of odd torsion modules.  $\square$

**COROLLARY 6.8.** *The map  $f^{\mathbb{Z}_2} : \Lambda_n \otimes \mathbb{Z}_2 \rightarrow H^*(M_n, \mathbb{Z}_2)/H^*(M_n, \mathbb{Z}_2)\langle 2 \rangle$  is an isomorphism.*

As a corollary of our proof we also see the following.

**COROLLARY 6.9.**  *$H_*(M_n, \mathbb{Q})$  has a basis given by fundamental classes tensor point classes coming from maps  $M_4^d \times M_c \rightarrow M_n$  corresponding to basic triangle forests.*

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