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With an Appendix by Andrew Hassell and Luc Hillairet

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Abstract

Partially rectangular domains are compact two-dimensional Riemannian manifolds X, either closed or with boundary, that contain a flat rectangle or cylinder. In this paper we are interested in partially rectangular domains with ergodic billiard flow; examples are the Bunimovich stadium, the Sinai billiard or Donnelly surfaces.

We consider a one-parameter family X_t of such domains parametrized by the aspect ratio t of their rectangular part. There is convincing theoretical and numerical evidence that the Laplacian on X_t with Dirichlet, Neumann or Robin boundary conditions is not quantum unique ergodic (QUE). We prove that this is true for all $t \in [1, 2]$ excluding, possibly, a set of Lebesgue measure zero. This yields the first examples of ergodic billiard systems proven to be non-QUE.

1. Introduction

A partially rectangular domain X is a compact Riemannian 2-manifold, either closed or with boundary, that contains a flat rectangle or cylinder, in the sense that X can be decomposed $X = R \cup W$, where R is a rectangle,[1] $R = [-\alpha, \alpha]_x \times [-\beta, \beta]_y$ (with $y = \pm\beta$ identified in the case of a cylinder) carrying the flat metric $dx^2 + dy^2$, and such that $R \cap W = R \cap \{x = \pm\alpha\}$.

The main result of this paper is that partially rectangular domains X are usually not QUE (see Theorem 1). This is primarily of interest in the case that X has ergodic billiard flow; examples include the Bunimovich stadium, the Sinai billiard, and Donnelly's surfaces [2], [21], [8]; see Figure 1. Ergodicity implies that these

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[1]For brevity we use 'rectangle' to mean 'rectangle or cylinder'.

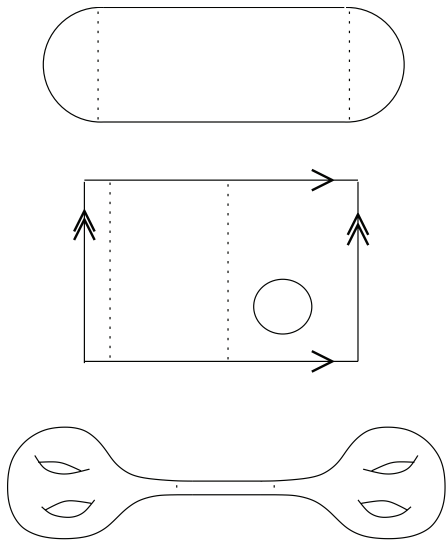

Figure 1. Examples of ergodic, partially rectangular domains: The Bunimovich stadium (top), Sinai billiard (middle), Donnelly surface (bottom).

domains are *quantum ergodic* by a theorem of Gérard-Leichtnam [12] and Zelditch-Zworski [25], generalizing work of Schnirelman [22], Zelditch [23] and Colin de Verdière [7] in the boundaryless case. Quantum ergodicity is a statement about the eigenfunctions u_j of the positive Laplacian Δ associated to the metric on X, where we assume that the Dirichlet, Neumann or Robin[2] boundary condition is specified if X has boundary. The operator Δ has a realization as a self-adjoint operator on $L^2(X)$ and has discrete spectrum $0 \le E_1 < E_2 \le E_3 \cdots \to \infty$ and corresponding orthonormal real eigenfunctions u_j, unique up to orthogonal transformations in each eigenspace.

The statement that Δ is quantum ergodic is the statement that there exists a density-one set J of natural numbers such that the subsequence $(u_j)_{j\in J}$ of eigenfunctions has the following equidistribution property: For each semiclassical pseudodifferential operator A_h, properly supported in the interior of X, we have

$$\lim_{j\in J\to\infty} \langle A_{h_j} u_j, u_j \rangle_{L^2(X)} = \frac{1}{|S^*X|} \int_{S^*X} \sigma(A). \tag{1}$$

Here $h_j = E_j^{-1/2}$ is the length scale corresponding to u_j, S^*X is the cosphere bundle of X (the bundle of unit cotangent vectors), and $|S^*X|$ denotes the measure of S^*X with respect to the natural measure induced by Liouville measure on T^*X.

[2]We only consider Robin boundary conditions of the form $d_n u = bu$ where $b \in \mathbb{R}$ is constant; see Remark 8.

In particular, this holds when A_h is multiplication by a smooth function η supported in the interior X° of X. In that case, (1) reads

$$\lim_{j\in J\to\infty}\int_X \eta\, u_j^2\, dg = \frac{1}{|X|}\int_X \eta\, dg$$

which implies that the probability measures u_j^2 tend weakly to uniform measure on X (for $j \in J$); the condition (1) is a finer version of this statement that can be interpreted as equidistribution of the u_j, $j \in J$ in phase space. *Quantum Unique Ergodicity* (QUE) is the stronger property that (1) holds for the full sequence of eigenfunctions, i.e. that J can be taken to be $\mathbb{N}$.

These properties can also be expressed in terms of quantum limits, or semiclassical measures. These are measures on T^*X obtained as weak limits of subsequences of the measures μ_j which act on compactly supported functions on T^*X° according to

$$C_c^\infty(T^*X^\circ) \ni a(x,\xi) \mapsto \langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle.$$

Here Op_h is a semiclassical quantization of the symbol a. Thus quantum ergodicity is the statement that for a density-one sequence J of integers, the sequence μ_j converges weakly to uniform measure on S^*X, and QUE has the same property with $J = \mathbb{N}$.

There are few results, either positive or negative, on quantum unique ergodicity. Rudnick-Sarnak [18] conjectured that closed hyperbolic manifolds are always QUE. This has been verified by Lindenstrauss, Silbermann-Venkatesh and Holowinsky-Soundararajan in some arithmetic cases [16], [19], [20], [14], provided one restricts to Hecke eigenfunctions which removes any eigenvalue degeneracy which might be present in the spectrum. In the negative direction, Faure-Nonnenmacher and De Bièvre-Faure-Nonnenmacher [9], [10] showed that certain quantized cat maps on the torus are non-QUE. In related work, Anantharaman [1] has shown that quantum limits on a closed, negatively curved manifold have positive entropy, which rules out quantum limits supported on a finite number of periodic geodesics. Until now, no billiard systems have been rigorously proved to be either QUE or non-QUE.

The results of the present paper are based crucially on the fact that partially rectangular domains X may be considered part of a one-parameter family X_t where we fix the height β of the rectangle and vary the length α. Here we arbitrarily set $\beta = \pi/2$ and let $\alpha = t\beta$ with $t \in [1,2]$. Our main result is

THEOREM 1. *The Laplacian on X_t, with either Dirichlet, Neumann or Robin boundary conditions, is non-*QUE *for almost every value of $t \in [1,2]$.*

This is proved first in the simple setting of the stadium billiard with the Dirichlet boundary condition. In the appendix, which is joint work with Luc Hillairet, we show how to obtain the result for any partially rectangular domain and the other boundary conditions.

The proof is based on the original argument of Heller and O'Connor [17] as refined by Zelditch [24], using 'bouncing ball' quasimodes. Their argument shows that QUE fails provided that one can find a subsequence of intervals of the form $[n^2 - a, n^2 + a]$, for arbitrary fixed $a > 0$, such that the number of eigenvalues in this interval is bounded uniformly as $n \to \infty$ along this subsequence. Note that in two dimensions, the expected number of eigenvalues in the interval $[E - a, E + a]$ is independent of E, so this is a very plausible condition.

Let us recall this argument in more detail. For simplicity, suppose the Dirichlet boundary condition is imposed on the horizontal sides of the rectangle R. Consider the function $v_n \in \operatorname{dom}(\Delta)$ given by $\chi(x) \sin ny$ for even n and $\chi(x) \cos ny$ for odd n, where $\chi(x)$ is supported in $x \in [-\pi/4, \pi/4]$. (For other boundary conditions, we replace $\sin ny$ and $\cos ny$ by the corresponding one-dimensional eigenfunctions; in the cylindrical case, we use $e^{\pm iny}$ and take n even.) For convenience, we choose χ so that $\|v_n\|_{L^2} = 1$ for all n. The v_n are so-called 'bouncing ball' quasimodes; they concentrate semiclassically as $n \to \infty$ onto a subset of the bouncing ball trajectories, which are the periodic trajectories that bounce vertically (i.e. with x fixed) between the horizontal sides of the rectangle R. They satisfy $\|(\Delta - n^2)v_n\|_{L^2} \le K$, uniformly in n. It follows from basic spectral theory that

$$\|P_{[n^2-2K,n^2+2K]}v_n\|^2 \ge \frac{3}{4} \tag{2}$$

where P_I is the spectral projection of the operator Δ corresponding to the set $I \subset \mathbb{R}$. Suppose there exists a subsequence n_j of integers with the property that there exists M, independent of j, such that

(3) there are at most M eigenvalues of Δ in the interval $[n_j^2 - 2K, n_j^2 + 2K]$.

Then for each n_j there is a normalized eigenfunction u_{k_j} such that $\langle u_{k_j}, v_{n_j}\rangle \ge \sqrt{3/4M}$. (Choose the normalized eigenfunction with eigenvalue in the interval $[n_j^2 - 2K, n_j^2 + 2K]$ with the largest component in the direction of v_n. There is at least one eigenfunction with eigenvalue in this range thanks to (2).) Then the sequence (u_{k_j}) of eigenfunctions has positive mass along bouncing ball trajectories, and in particular is not equidistributed. To see this, given any $\varepsilon > 0$, let A be a self-adjoint semiclassical pseudodifferential operator, properly supported in the rectangle in both variables, so that $\sigma(A) \le 1$ and so that $\|(\mathrm{Id} - A)v_n\| \le \varepsilon$ for sufficiently large n. Then, we can compute

$$\begin{aligned}\langle A^2 u_{k_j}, u_{k_j}\rangle = \|Au_{k_j}\|^2 \ge \Big|\langle Au_{k_j}, v_{n_j}\rangle\Big|^2 \\ = \Big|\langle u_{k_j}, Av_{n_j}\rangle\Big|^2 \ge \Big(|\langle u_{k_j}, v_{n_j}\rangle| - \varepsilon\Big)^2 \ge \Big(\sqrt{3/4M} - \varepsilon\Big)^2.\end{aligned}$$

This is bounded away from zero for small ε. By choosing a sequence of operators A such that $\|(\mathrm{Id}-A)v_n\| \to 0$ and such that the support of the symbol of A shrinks to the set of bouncing ball covectors (i.e. multiples of dy supported in the rectangle), we see that the mass of any quantum limit obtained by subsequences of the u_{k_j} must have mass at least $3/4M$ on the bouncing ball trajectories.

The missing step in this argument, supplied by the present paper (at least for a large measure set in the parameter t), is to show that there are indeed sequences $n_j \to \infty$ so that (3) holds.

Remark 2. Burq and Zworski [5] have shown that $o(1)$ quasimodes, unlike $O(1)$ quasimodes, cannot concentrate asymptotically *strictly* inside the rectangle $R = [-\alpha, \alpha] \times [-\beta, \beta]$, in the sense that they cannot concentrate in subrectangles $\omega \times [-\beta, \beta]$ with ω a strict closed subinterval of $[-\alpha, \alpha]$.

2. **Hadamard variational formula**

Let S_t denote the stadium billiard with aspect ratio t, given explicitly as the union of the rectangle $[-t\pi/2, t\pi/2]_x \times [-\pi/2, \pi/2]_y$ and the circles centred at $(\pm t\pi/2, 0)$ with radius $\pi/2$. Let Δ_t denote the Laplacian on S_t with Dirichlet boundary conditions. Define $E_j(t)$ to be the jth eigenvalue (counted with multiplicity) of Δ_t. The key to the proof of Theorem 1 for S_t will be a consideration of how $E_j(t)$ varies with t. Let $u_j(t)$ denote an eigenfunction of Δ_t with eigenvalue $E_j(t)$ (chosen orthonormally for each t), and let $\psi_j(t)$ denote $E_j^{-1/2}$ times the outward-pointing normal derivative $d_n u_j(t)$ of $u_j(t)$ at the boundary of S_t. Let $\rho_t(s)$ denote the function on ∂S_t given by $\rho_t(s) = (\operatorname{sgn} x)\partial_x \cdot n/2$, where n is the outward-pointing unit normal vector at ∂S_t. The function ρ_t is the 'normal variation' of the boundary ∂S_t with respect to t. Notice that $\rho_t \geq 0$ everywhere.

We first observe that the eigenvalue branches $E(t)$ can be chosen holomorphic in t. To see this, we fix a reference domain S_1 and consider the family of metrics

$$g_t = (1 + (t-1)\phi(x))^2 dx^2 + dy^2,$$

where $\phi(x)$ is nonnegative, positive at $x = 0$ and supported close to $x = 0$. If $\int \phi = 1$, then S_1 with this metric is isometric to S_t for $1 \leq t \leq 2$. Note that g_t is a real analytic family of metrics. Then Δ_t is (unitarily equivalent to) the Laplacian with respect to the metric g_t on S_1.

The analytic family of metrics g_t gives rise to a holomorphic family of elliptic operators $\tilde{L}_t$ for t in a complex neighbourhood of $[1, 2]$ (with complex coefficients for t nonreal), equal to Δ_t for real t. This operator acts on $L^2(S_1; dg_t)$ with domain $H^2(S_1) \cap H^1_0(S_1)$, where dg_t is the measure $(1 + (t-1)\phi(x))dxdy$. Define the operator V_t by $V_t(f) = (1 + (t-1)\phi(x))^{1/2} f$, which for t real is a unitary operator from $L^2(S_1; dg_t) \to L^2(S_t; dg_1)$. Then $\tilde{L}_t$ is similar to the holomorphic family of operators $L_t = V_t \tilde{L}_t V_t^{-1}$ acting on $L^2(S_1; dg_1)$ with domain $H^2(S_1) \cap H_0^1(S_1)$, and is unitarily equivalent to L_t for real t. The family L_t is a holomorphic family of type A in the sense of Kato's book [15]. Accordingly, the eigenvalues $E(t)$ and eigenprojections can be chosen holomorphic in t. Let $u(t)$ be a holomorphic family of (Dirichlet) eigenfunctions, normalized for real t, corresponding to $E(t)$.

LEMMA 3 (Hadamard variational formula). *We have*

$$\frac{d}{dt} E(t) = -\int_{\partial S_t} \rho_t(s) (d_n u(t)(s))^2 \, ds. \tag{4}$$

This is a standard formula (see e.g. [11]). It can also be derived from the proof of Proposition 7 using the formula $\dot{L}_t = [L_t, \partial_x^t \Phi + \Phi \partial_x^t]$ where $\Phi = \int \phi$.

Now we return to ordering the eigenfunctions $u(t)$ by eigenvalue for each fixed t. It follows from holomorphy of the eigenprojections that either $E_j(t) = E_k(t)$ for all $t \in [1, 2]$, or $E_j(t) = E_k(t)$ for at most finitely many $t \in [1, 2]$. Thus $E_j(t)$ is piecewise smooth and, except for finitely many values of t, according to (4) its derivative satisfies

$$E_j^{-1} \frac{d}{dt} E_j(t) = -\int_{\partial S_t} \rho_t(s) \psi_j(s)^2 \, ds. \tag{5}$$

This formula is the basic tool we shall use to prove Theorem 1.

In Section 4, we will prove the following stronger version of Theorem 1, which gives more information about non-Liouville quantum limits on S_t.

THEOREM 4. *For every $\varepsilon > 0$ there exists a subset $B_\varepsilon \subset [1, 2]$ of measure at least $1 - 4\varepsilon$, and a strictly positive constant $m(\varepsilon)$ with the following property. For every $t \in B_\varepsilon$, there exists a quantum limit formed from Dirichlet eigenfunctions on the stadium S_t that has mass at least $m(\varepsilon)$ on the bouncing ball trajectories.*

3. The main idea

Before we give the proof of Theorem 4, we sketch the main idea. For simplicity, in this section we only attempt to argue that there is at least one $t \in [1, 2]$ such that Δ_t is non-QUE. To do so, let us assume that Δ_t is QUE for all for $t \in [1, 2]$, and seek a contradiction.

We begin with some heuristics. Let $A(t)$ denote the area of S_t. By Weyl's law, we have $E_j(t) \approx c A(t)^{-1} j$. Therefore, since the area of S_t grows linearly with t, we have $\dot{E}_j \approx -\text{const}\, A(t)^{-1} E_j$, on the average. *The* QUE *assumption implies that this is true*, *asymptotically*, *at the level of each individual eigenvalue.*

Indeed, let

$$f_j(t) = \int_{\partial S_t} \rho_t(s)|\psi_j(t;s)|^2\, ds. \tag{6}$$

Then (5) says that $\dot{E}_j = -E_j f_j$, while the QUE assumption implies that the boundary values $|\psi_j(t)|^2$ tend weakly to $A(t)^{-1}$ on the boundary ∂S_t [12], [13], and [3]. In particular, this shows that

$$f_j(t) \to kA(t)^{-1} > 0, \tag{7}$$

where $k = \int_{\partial S_t} \rho_t(s)\, ds > 0$ is independent of t. So, this gives

$$E_j^{-1}\dot{E}_j = -kA(t)^{-1}(1 + o(1)), \quad j \to \infty. \tag{8}$$

In particular, the magnitude of $E_j(t)^{-1}\dot{E}_j(t)$ is bounded below for large j. This prevents the eigenvalues conspiring to concentrate in intervals $[n^2 - a, n^2 + a]$. Indeed, such concentration, for every $t \in [1, 2]$, would require that at least some eigenvalues 'loiter' near $E = n^2$ for significant intervals of time t, which is ruled out by (8). The Heller-O'Connor-Zelditch argument from the introduction then gives a contradiction to the QUE assumption.

Rather than employing such a contradiction argument, however, we use a slightly more elaborate direct approach, which yields more information.

4. Proof of Theorem 4

We begin by dividing the interval $[1, 2]$ into two sets $Z_1 \cup Z_2$, where Z_1 is the set of t such that

$$\liminf_{j\to\infty} f_j(t) = 0, \tag{9}$$

where $f_j(t)$ is defined in (6), and Z_2 is the complement (i.e. where the lim inf above is positive).

First, consider any $t \in Z_1$. Consider the semiclassical measures ν on the unit ball bundle of ∂S_t studied in [12]. The relation (9) implies that there exists a ν which vanishes on the curved sides of the stadium. Such a ν cannot have mass on the boundary of the unit ball bundle, since the straight part of the boundary is non-strictly gliding [4]. The relation between quantum limits μ and boundary measures ν in Theorem 2.3 of [12] then shows that there exists a quantum limit μ supported on (interior) rays that do not meet the curved sides of the stadium. The only such trajectories are the bouncing ball trajectories. Therefore, every $t \in Z_1$ satisfies the conditions of the theorem.

Next consider $t \in Z_2$. Given $\varepsilon > 0$, there is a subset H_ε of Z_2, whose measure is at least $|Z_2| - \varepsilon$, such that

$$t \in H_\varepsilon \implies \liminf_{j\to\infty} f_j(t) \geq c > 0,$$

where c depends on ε. To see this, consider the sets $Z_2^n = \{t \in Z_2 \mid \liminf f_j(t) \geq 1/n\}$. This is an increasing family of sets whose union is Z_2, so by countable additivity of Lebesgue measure, $|Z_2^n| \to |Z_2|$. In the same spirit, there is a subset G_ε of H_ε, whose measure is at least $|Z_2| - 2\varepsilon$, where this statement is uniform in j; in particular, there exists $N = N(\varepsilon)$ such that

$$t \in G_\varepsilon,\ j \geq N \implies f_j(t) \geq \frac{c}{2}.$$

Now we want to consider, for $t \in G = G_\varepsilon$, the number of eigenvalues $E_j(t)$ in the interval $[n^2 - a, n^2 + a]$. For a fixed t, it seems very difficult to improve on the bound $O(n)$ from the remainder estimate in Weyl's law. However, as we see below, one does much better by averaging in t. Thus, we shall give a good estimate on

$$\int_{G_\varepsilon} \big(N_t(n^2 + a) - N_t(n^2 - a)\big)\, dt \tag{10}$$

for large n, where N_t is the eigenvalue counting function for Δ_t. This integral can be calculated by considering how much 'time' t each eigenvalue $E_j(t)$ spends in the interval $[n^2 - a, n^2 + a]$. By Weyl's Law, we have $\gamma j \leq E_j(t) \leq \Gamma j$ for $t \in [1, 2]$, with γ, Γ independent of t. Therefore, taking n large enough so that $a \leq n^2/2$, we only need consider j such that $n^2/2\Gamma \leq j \leq 3n^2/2\gamma$. Thus, (10) is equal to

$$\sum_{j=n^2/2\Gamma}^{3n^2/2\gamma} \Big|\{t \in G_\varepsilon \mid E_j(t) \in [n^2 - a, n^2 + a)\}\Big|. \tag{11}$$

Next, we replace $G = G_\varepsilon$ by an open set containing G. On G we have $f_j(t) \geq c/2$ for $j \geq N$. Then the open set

$$O_n = \{t \mid f_j(t) > c/4 \text{ for } N \leq j \leq 3n^2/2\gamma\}$$

contains G. Then for $t \in O_n$, and $n^2 \geq 2\Gamma N$, by (5)

$$-\dot{E}_j(t) \geq c E_j(t)/4, \quad \frac{n^2}{2\Gamma} \leq j \leq \frac{3n^2}{2\gamma}. \tag{12}$$

Integrating this, we find that for $t_1 < t_2$ in the same component of O_n, and $n^2/2\Gamma \leq j \leq 3n^2/2\gamma$,

$$E_j(t_1) - E_j(t_2) \geq \frac{c}{4} E_j(t_2)(t_2 - t_1) \implies t_2 - t_1 \leq \frac{4}{c}\frac{E_j(t_1) - E_j(t_2)}{E_j(t_2)}. \tag{13}$$

Since S_t is an increasing sequence of domains, the Dirichlet eigenvalues $E_j(t)$ are nonincreasing in t. Therefore (13) on each component of O_n implies that the quantity (11), and hence (10), can be bounded above for $n^2 \geq \max(2a, 2\Gamma N)$ by

$$\sum_{j=n^2/2\Gamma}^{3n^2/2\gamma} 2a \cdot \frac{4}{c} \cdot \frac{1}{n^2-a} \leq \sum_{j=1}^{3n^2/2\gamma} 2a \cdot \frac{4}{c} \cdot \frac{1}{n^2/2} = \frac{24a}{c\gamma}. \tag{14}$$

Therefore, on a set $A_n \subset G$ of measure at least $|G| - \varepsilon \geq |Z_2| - 3\varepsilon$, we can assert that $N_t(n^2+a) - N_t(n^2-a)$ is at most ε^{-1} times the right hand side of (14). That is, for sufficiently large n, there is a set A_n of measure at least $|Z_2| - 3\varepsilon$ on which

$$N_t(n^2+a) - N_t(n^2-a) \leq \frac{24a}{c\gamma\varepsilon},$$

which is a bound manifestly independent of n.

To finish the proof we show that there is a set of measure at least $|Z_2| - 4\varepsilon$ that is contained in A_n for infinitely many n. That is, defining

$$B_k = \{t \in Z_2 \mid t \in A_n \text{ for at least } k \text{ distinct values of } n\}, \tag{15}$$

we show that $|\cap_k B_k| \geq |Z_2| - 4\varepsilon$. To show this consider the sets

$$D_k = \{t \in Z_2 \mid t \in A_n \text{ for at least } k \text{ distinct values of } n \text{ in the range } k \leq n < 5k\}.$$

Since $D_k \subset B_k$ and B_k is a decreasing family of sets, it suffices to show that $|D_k| \geq |Z_2| - 4\varepsilon$ for every k. To see this, on one hand

$$\sum_{n=k}^{5k-1} |A_n| \geq 4k(|Z_2| - 3\varepsilon).$$

On the other hand, by the definition of D_k,

$$\sum_{n=k}^{5k-1} |A_n| \leq 4k|D_k| + k(|Z_2| - |D_k|).$$

Putting these together we obtain

$$|D_k| \geq |Z_2| - 4\varepsilon,$$

as required.

We have now shown that for a subset of Z_2 of measure at least $|Z_2| - 4\varepsilon$, there is a sequence of integers n_j (depending on t) for which (3) holds, and therefore the mass statement in Theorem 4 holds for all such t using the argument from the introduction. Thus the conclusion of Theorem 4 holds for all $t \in Z_1$ and all $t \in Z_2$ except on a set of measure at most 4ε. This completes the proof.

5. **Appendix.**

by Andrew Hassell and Luc Hillairet

In this appendix, we show how the proof above for the stadium domain can be adapted to partially rectangular domains X_t, and other boundary conditions, thereby proving Theorem 1 in full generality. Again we prove a stronger version which gives more information about non-Liouville quantum limits on X_t. To state this result, we denote by BB the union of the bouncing-ball trajectories in S^*X_t, by TT the union of billiard trajectories that do not enter the rectangle ('trapped trajectories'), and by ET the excluded trajectories in [25]. The set ET, only relevant when X_t has boundary, consists of the billiard trajectories that either (i) hit a non-smooth point of the boundary at some time, (ii) reflect from the boundary infinitely often in finite time, or (iii) touch ∂X_t tangentially at some time[3]. All these sets have measure zero; that TT has measure zero follows from ergodicity, while that ET has measure zero is shown in [25].

THEOREM 5. *Let X_t be a partially rectangular domain, and let Δ_t be the Dirichlet, Neumann or Robin Laplacian on X_t. For every $\varepsilon > 0$ there exists a subset $B_\varepsilon \subset [1, 2]$ of measure at least $1 - 4\varepsilon$, and a strictly positive constant $m(\varepsilon)$ with the following property. For every $t \in B_\varepsilon$, there exists a quantum limit of Δ_t that either has mass at least $m(\varepsilon)$ on* BB, *or else concentrates entirely on* BB ∪ TT ∪ ET.

Remark 6. Since BB ∪ TT ∪ ET has measure zero, this implies that Δ_t is non-QUE.

The main task is to replace the boundary formula (5) for the variation of eigenvalues with an interior formula. Let X be a partially rectangular domain with rectangular part $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$, and let X_t be the domain with the rectangle replaced by $[-t\pi/2, t\pi/2] \times [-\pi/2, \pi/2]$, where $t \in [1, 2]$. Let Δ_t denote the Laplacian on X_t with either the Dirichlet boundary condition or the Robin boundary condition $d_n u = bu$, $b \in \mathbb{R}$ constant (which of course includes the Neumann condition as the special case $b = 0$).

We now compute the variation of the eigenvalues of Δ_t with respect to t. To state this result, we introduce some notation. Let g_t and L_t be as in Section 2, and let I_t denote the isometry from (X_1, g_t) to X_t. Let M_t denote the multiplication operator $(1 + (t - 1)\phi(x))^{-1/2}$, and let ∂_x^t denote $M_t \partial_x M_t$. Then the domain of L_t, under any of the boundary conditions above is independent of t, and $L_t = -(\partial_x^t)^2 - \partial_y^2$ on its domain. Let ϕ_t denote the function $\phi M_t^2 \circ I_t^{-1}$ on X_t.

[3] Here we do not exclude the trajectories that do not meet the boundary forwards or backwards in time, as is done in [25].

PROPOSITION 7. *Let $u(t)$ be a real eigenfunction of Δ_t, L^2-normalized on X_t, with eigenvalue $E(t)$, depending smoothly on t. Let Q be the operator*

$$Q = -4\partial_x \phi_t \partial_x + [\partial_x, [\partial_x, \phi_t]] \tag{16}$$

acting on functions on X_t. Then

$$\dot{E}(t) = -\frac{1}{2}\langle Qu(t), u(t)\rangle \tag{17}$$

and there exists C, depending only on the function ϕ, such that

$$\dot{E}(t) \leq C, \quad t \in [1,2]. \tag{18}$$

Proof. Let $v(t)$ be the eigenfunction of L_t corresponding to $u(t)$. Then

$$E(t) = \langle L_t v(t), v(t)\rangle.$$

Since $\dot{v}(t) \in \operatorname{dom}(L_t)$ is orthogonal to $v(t)$, while $L_t v(t)$ is a multiple of $v(t)$, we have

$$\dot{E}(t) = \langle \dot{L}_t v(t), v(t)\rangle. \tag{19}$$

Using the expression for L_t above,

$$\dot{L}_t = \partial_t (\partial_x^t)^2,$$

and since

$$\partial_t \partial_x^t = -\frac{1}{2}\Big(\phi M_t^2 \partial_x^t + \partial_x^t M_t^2 \phi\Big),$$

we obtain

$$\dot{L}_t = -\frac{1}{2}\Big(\phi M_t^2 (\partial_x^t)^2 + 2\partial_x^t \phi M_t^2 \partial_x^t + (\partial_x^t)^2 \phi M_t^2\Big).$$

Substituting this into (19) gives an expression for $\dot{E}(t)$ in terms of $v(t)$. Writing this in terms of $u(t)$ on X_t gives the equivalent expression

$$\dot{E}(t) = -\frac{1}{2}\Big\langle \Big(\phi_t \partial_x^2 + 2\partial_x \phi_t \partial_x + \partial_x^2 \phi_t\Big) u(t), u(t)\Big\rangle \tag{20}$$

which can be rearranged to (17). To prove (18), we observe that $-4\partial_x \phi_t \partial_x$ is a positive operator, while $[\partial_x, [\partial_x, \phi_t]]$ is a multiplication operator by a smooth function of x and t, and hence bounded as an operator on L^2 by a constant independent of t for $t \in [1,2]$. □

Remark 8. The boundary condition plays a very limited role in this proof. It enters only in deriving $\dot{E} = \langle \dot{L}v, v\rangle$. This requires that $\langle L\dot{v}, v\rangle = \langle \dot{v}, Lv\rangle = 0$, which is implied by $\dot{v} \in \operatorname{dom}(L)$, which in turn follows from the t-independence of $\operatorname{dom}(L_t)$. This condition rules out Robin boundary conditions of the form $d_n = bu$ where b is nonconstant along the horizontal sides of the rectangle. However, 'nonconstant Robin' or even more general boundary conditions could be permitted on other parts of the boundary of X_t, provided that they are t-independent.

Now we indicate how the proof in Section 4 may be modified to prove Theorem 5. We redefine $f_j(t)$ by

$$f_j(t) = E_j^{-1}\langle Qu_j, u_j\rangle \tag{21}$$

so that $\dot{E}_j(t) = -E_j(t) f_j(t)$ as above, and partition the t-interval $[1,2]$ into $Z_1 \cup Z_2$ as before. Consider any $t \in Z_1$. Then there is an increasing sequence j_k of integers such that $f_{j_k}(t) \to 0$. In this case, we can construct an operator properly supported in the interior of X_t for which (1) fails to hold for $j = j_k \to \infty$. Choose a function $\zeta(y)$ taking values between 0 and 1 which is equal to 1 near $y = 0$ and vanishes near $|y| = \pi/2$. Then in view of (21) and (16),

$$\begin{aligned} \|\zeta(y)(\phi^t)^{1/2}(h\partial_x)u_{j_k}\|_2^2 &\le \|(\phi^t)^{1/2}(h\partial_x)u_{j_k}\|_2^2 \\ &= -\langle h^2 Qu_{j_k}, u_{j_k}\rangle + O(h) \to 0, \quad h = h_{j_k} = E_{j_k}^{-1/2}. \end{aligned} \tag{22}$$

Therefore, defining $A_h = \zeta(y)^2(h\partial_x)\phi^t(x)(h\partial_x)$,

$$\begin{aligned} \lim_{k\to\infty}\langle A_h u_{j_k}, u_{j_k}\rangle = \|\zeta(y)(\phi^t)^{1/2}(h\partial_x)u_{j_k}\|_2^2 \to 0 \\ \neq \frac{1}{S^*X_t}\int_{S^*X_t}\sigma(A_h), \quad h = h_{j_k}, \end{aligned}$$

since $\sigma(A_h) \ge 0$, and is > 0 on a set of positive measure. Thus X_t is not QUE. Moreover, using the pseudodifferential calculus, this implies that $\langle B_{h_{j_k}} u_{j_k}, u_{j_k}\rangle \to 0$ for any B_h microsupported where $\sigma(A_h) > 0$. Then, parametrix constructions for the wave operator microlocally near rays that reflect nontangentially at the boundary (see for example [6]) show that $\langle B'_{h_{j_k}} u_{j_k}, u_{j_k}\rangle \to 0$ for any B'_h with symbol supported close to any $q' \in S^*X_t^\circ$ enjoying the property that it is obtained from a point q such that $\sigma(A_h)(q) > 0$ by following the billiard flow through a finite number of nontangential reflections at smooth points of ∂X_t. This property is true for any $q' \notin \mathrm{BB} \cup \mathrm{TT} \cup \mathrm{ET}$ (for a suitable choice of ζ depending on q'), since the symbol of A_h is positive on all unit covectors lying over $\operatorname{supp}\phi^t \times \operatorname{supp}\zeta$ which are not vertical. It follows that the sequence u_{j_k} concentrates away from all such points, which is to say that it concentrates at $\mathrm{BB} \cup \mathrm{TT} \cup \mathrm{ET}$.

The argument for $t \in Z_2$ continues exactly as before, except that instead of the nonincreasing condition $\dot{E}_j(t) \le 0$, we only have the weaker condition $\dot{E}_j(t) \le C$ thanks to (18). We modify the argument below equation (13) as follows: Define $E_j^*(t) = E_j(t) - Ct$. Then $\dot{E}_j^*(t) \le 0$ and (12) is valid for $E_j^*(t)$; hence we obtain using the method of Section 4

$$\sum_{j=n^2/2\Gamma}^{3n^2/2\gamma} \Big|\{t \in G_\varepsilon \mid E_j^*(t) \in [n^2 - a^*, n^2 + a^*)\}\Big| \le \frac{24a^*}{c\gamma}. \tag{23}$$

Now we use the observation

$$E_j(t) \in [n^2 - a, n^2 + a) \implies E_j^*(t) \in [n^2 - a - 2C, n^2 + a + 2C)$$

to deduce the estimate (23) for $E_j(t)$ with a^* replaced by a on the left-hand side and by $a + 2C$ on the right-hand side. The rest of the argument from Section 4 can now be followed to its conclusion.

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