A nonhomogeneous orbit closure of a diagonal subgroup

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Abstract

Let $G = \text{SL}(n, \mathbb{R})$ with $n \geq 6$. We construct examples of lattices $\Gamma \subset G$, subgroups $A$ of the diagonal group $D$ and points $x \in G/\Gamma$ such that the closure of the orbit $Ax$ is not homogeneous and such that the action of $A$ does not factor through the action of a one-parameter nonunipotent group. This contradicts a conjecture of Margulis.

1. Introduction

1.1. Topological rigidity and related questions. Let $G$ be a real Lie group, $\Gamma$ a lattice in $G$, meaning a discrete subgroup of finite covolume, and $A$ a closed connected subgroup. We are interested in the action of $A$ on $G/\Gamma$ by left multiplication; we will restrict ourselves to the topological properties of these actions, referring the reader to [6] and [3] for references and recent developments on related measure theoretical problems.

Two linked questions arise when one studies continuous actions of topological groups: what are the closed invariant sets, and what are the orbit closures?

In the homogeneous action setting we are considering, there is a class of closed sets that admit a simple description: a closed subset $X \subset G$ is said to be homogeneous if there exists a closed connected subgroup $H \subset G$ such that $X = Hx$ for some (and hence every) $x \in X$. Let us say that the action of $A$ on $G/\Gamma$ is topologically rigid if for any $x \in G/\Gamma$, the closure $\overline{Ax}$ of the orbit $Ax$ is homogeneous.

The most basic example of a topologically rigid action is when $G = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$, $A$ and any vector subspace of $G$. It turns out that the behavior of elements of $A$ for the adjoint action on the Lie algebra $\mathfrak{g}$ of $G$ plays an important role in our problem. Recall that an element $g \in G$ is said to be $\text{Ad}$-unipotent if $\text{Ad}(g)$ is unipotent, and $\text{Ad}$-split over $\mathbb{R}$ if $\text{Ad}(g)$ is diagonalizable over $\mathbb{R}$. If the closed,
connected subgroup $A$ of $G$ is generated by $\text{Ad}$-unipotent elements, a celebrated theorem of Ratner [13] asserts that the action of $A$ is always topologically rigid, settling a conjecture due to Raghunathan.

When $A$ is generated by elements which are $\text{Ad}$-split over $\mathbb{R}$, much less is known. Consider the model case of $G = \text{SL}(n, \mathbb{R})$ and $A$ the group of diagonal matrices with nonnegative entries. If $n = 2$, it is easy to produce nonhomogeneous orbit closures (see e.g. [7]); more generally, a similar phenomenon can be observed when $A$ is a one-parameter subgroup of the diagonal group (see [6, 4.1]). However, for $A$ the full diagonal group, if $n \geq 3$, to the best of our knowledge, the only nontrivial example of a nonhomogeneous $A$-orbit closure is due to Rees, later generalized in [7]. In an unpublished preprint, Rees exhibited a lattice $\Gamma$ of $G = \text{SL}(3, \mathbb{R})$ and a point $x \in G/\Gamma$ such that for the full diagonal group $A$, the orbit closure $A\overline{x}$ is not homogeneous. Her construction was based on the following property of the lattice: there exists a $\gamma \in \Gamma \cap A$ such that the centralizer $C_G(\gamma)$ of $\gamma$ is isomorphic to $\text{SL}(2, \mathbb{R}) \times \mathbb{R}^*$, and such that $C_G(\gamma) \cap \Gamma$ is, in this product decomposition and up to finite index, $\Gamma_0 \times \langle \gamma \rangle$, where $\Gamma_0$ is a lattice in $\text{SL}(2, \mathbb{R})$ (see [4], [7]). Thus in this case the action of $A$ on $C_G(\gamma)/C_G(\gamma) \cap \Gamma$ factors to the action of a 1-parameter nonunipotent subgroup on $\text{SL}(2, \mathbb{R})/\Gamma_0$, which, as we saw, has many nonhomogeneous orbits.

Rees’ example shows that factor actions of 1-parameter non-$\text{Ad}$-unipotent groups are obstructions to the topological rigidity of the action of diagonal subgroups. The following conjecture of Margulis [8, Conj. 1.1] (see also [6, 4.4.11]) essentially states that these are the only ones:

**Conjecture 1.** Let $G$ be a connected Lie group, $\Gamma$ a lattice in $G$, and $A$ a closed, connected subgroup of $G$ generated by $\text{Ad}$-split over $\mathbb{R}$ elements. Then for any $x \in G/\Gamma$, one of the following holds:

(a) $A\overline{x}$ is homogeneous, or

(b) There exists a closed connected subgroup $F$ of $G$ and a continuous epimorphism $\phi$ of $F$ onto a Lie group $L$ such that

- $A \subseteq F$,
- $Fx$ is closed in $G/\Gamma$,
- $\phi(F_x)$ is closed in $L$, where $F_x$ denotes the stabilizer $\{g \in F | gx = x\}$,
- $\phi(A)$ is a one-parameter subgroup of $L$ containing no nontrivial $\text{Ad}_L$-unipotent elements.

A first step toward this conjecture has been done by Lindenstrauss and Weiss [7], who proved that in the case $G = \text{SL}(n, \mathbb{R})$ and $A$ is the full diagonal group, if the closure of a $A$-orbit contains a compact $A$-orbit that satisfies some irrationality conditions, then this closure is homogeneous. See also [15]. Recently, using an approach based on measure theory, Einsiedler, Katok and Lindenstrauss proved
that if moreover $\Gamma = \text{SL}(n, \mathbb{Z})$, then the set of bounded $A$-orbits has Hausdorff dimension $n - 1$ [3, Th. 10.2].

1.2. Statement of the results. In this article we exhibit some counterexamples to the above conjecture when $G = \text{SL}(n, \mathbb{R})$ for $n \geq 6$ and $A$ is some strict subgroup of the diagonal group of matrices with nonnegative entries. Let $D$ be the diagonal subgroup of $G$; note that $D$ has dimension $n - 1$. Our main result is:

**Theorem 1.** Assume $n \geq 6$.

1. There exist a $(n - 3)$ dimensional closed and connected subgroup $A$ of $D$, and a point $x \in \text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$ such that the closure of the $A$-orbit of $x$ satisfies neither condition (a) nor condition (b) of the conjecture.

2. There exist a lattice $\Gamma$ of $\text{SL}(n, \mathbb{R})$, an $(n - 2)$ dimensional closed and connected subgroup $A$ of $D$ and a point $x \in \text{SL}(n, \mathbb{R})/\Gamma$ such that the closure of the $A$-orbit of $x$ satisfies neither condition (a) nor condition (b) of the conjecture.

It will be clear from the proofs that these examples however satisfy a third condition:

(c) There exist a closed connected subgroup $F$ of $G$ and two continuous epimorphisms $\phi_1, \phi_2$ of $F$ onto Lie groups $L_1, L_2$ such that

- $A \subset F$,
- $F x$ is closed in $G/\Gamma$,
- For $i = 1, 2$, $\phi_i(F x)$ is closed in $L_i$,
- $(\phi_1, \phi_2) : F \to L_1 \times L_2$ is surjective
- $(\phi_1, \phi_2) : A \to \phi_1(A) \times \phi_2(A)$ is not surjective.

Construction of these examples is the subject of Section 2, whereas the proof that they satisfy the required properties is postponed to Section 3.

1.3. Toral endomorphisms. To conclude this introduction, we would like to mention that the idea behind this construction can also be used to yield examples of ‘nonhomogeneous’ orbits for diagonal toral endomorphisms.

Let $1 < p_1 < \cdots < p_q$, with $q \geq 2$, be integers generating a multiplicative nonlacunary semigroup of $\mathbb{Z}$ (that is, the $Q$-subspace $\bigoplus_{1 \leq i \leq q} Q \log(p_i)$ has dimension at least 2). We consider the abelian semigroup $\Omega$ of endomorphisms of the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ generated by the maps $z \mapsto p_i z \mod \mathbb{Z}^n$, $1 \leq i \leq q$.

In the one-dimensional situation, described by Furstenberg [5], every $\Omega$-orbit is finite or dense. If $n \geq 2$, Berend [1] showed that minimal sets are the finite orbits of rational points, but there are other obvious closed $\Omega$-invariant sets, namely the orbits of rational affine subspaces. Meiri and Peres [10] showed that closed invariant sets have integral Hausdorff dimension.
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Note that the study of the orbit of a point lying in a proper rational affine subspace reduces to the study of finitely many orbits in lower dimensional tori, although some care must be taken about the pre-periodic part of the rational affine subspace (for example, if \( q = n = 2 \), and if \( \alpha \in T^1 \) is irrational with nondense \( p_1 \)-orbit, the orbit closure of the point \((\alpha, 1/p_2) \in T^2\) is the union of a horizontal circle and a finite number of strict closed infinite subsets of some horizontal circles).

With this last example in mind, Question 5.2 of [10] can be re-formulated: is a proper closed invariant set necessarily a subset of a finite union of rational affine tori? Or, equivalently, if a point is outside any rational affine subspace, does it necessarily have a dense orbit? It turns out that this is not the case at least for \( n \geq 2q \), as the following example shows.

**Theorem 2.** Let \( N \) be an integer greater than \( q \log \frac{p_2}{p_1} \), and let \( z \) be the point in the \( 2q \)-dimensional torus \( T^{2q} \) defined by the coordinates modulo 1:

\[
z = (z_1, \ldots, z_{2q}) = \left( \sum_{k \geq 1} p_1^{-N^{2k}}, \ldots, \sum_{k \geq 1} p_q^{-N^{2k}}, \sum_{k \geq 1} p_1^{-N^{2k}+1}, \ldots, \sum_{k \geq 1} p_q^{-N^{2k}+1} \right).
\]

Then the point \( z \in T^{2q} \) is not contained in any rational affine subspace, but its orbit \( \Omega z \) is not dense.

The proof of Theorem 2 will be the subject of Section 4.

2. **Sketch of proof of Theorem 1**

2.1. *The direct product setup.* We now describe how these examples are built. Choose two integers \( n_1 \geq 3, n_2 \geq 3 \), such that \( n_1 + n_2 = n \). For \( i = 1, 2 \), let \( \Gamma_i \) be a lattice in \( G_i = \text{SL}(n_i, \mathbb{R}) \).

Let \( g_i \) be an element of \( G_i \) such that \( g_i \Gamma_i g_i^{-1} \) intersects the diagonal subgroup \( D_i \) of \( \text{SL}(n_i, \mathbb{R}) \) in a lattice; in other words \( g_i \Gamma_i \) has a compact \( D_i \)-orbit; such elements exist; see [11]. In fact, we will need an additional assumption on \( g_i \), namely that the tori \( g_i^{-1}D_ig_i \) are irreducible over \( \mathbb{Q} \). The precise definition of this property and the proof of the existence of such a \( g_i \), a consequence of a theorem of Prasad and Rapinchuk [12, Th. 1], will be the subject of Section 3.1.

Let \( \pi_i : G_i \to G_i / \Gamma_i \) be the canonical quotient map. Define for \( i = 1, 2 \):

\[
y_i = \pi_i \begin{pmatrix}
1 & 0 & \cdots & 0 & 1 \\
0 & 1 & & & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 0 \\
0 & \cdots & 0 & 1 
\end{pmatrix} g_i.
\]
The $D_i$-orbit of $y_i$ is dense, by the following argument. It is easily seen that the closure of $D_i y_i$ contains the compact $D_i$-orbit $\mathcal{T}_i = \pi_i(D_i g_i)$. The $\mathbb{Q}$-irreducibility of $\mathcal{T}_i$ is sufficient to show that the assumptions of the theorem of Lindenstrauss and Weiss [7, Th. 1.1] are satisfied (Lemma 3.1); thus, by this theorem, we obtain that there exists a group $H_i < G_i$ such that $H_i y_i = D_i y_i$.

Again because of $\mathbb{Q}$-irreducibility, the group $H_i$ is necessarily the full group; i.e., $H_i = G_i$ (proof of Lemma 3.2).

Let $A_1$ be the $(n - 3)$ dimensional subgroup of $G_1 \times G_2$ given by:

$$A_1 = \left\{ (\text{diag}(a_1, \ldots, a_{n_1}), \text{diag}(b_1, \ldots, b_{n_2})) : \prod_{i=1}^{n_1} a_i = \prod_{j=1}^{n_2} b_j = \frac{a_1 b_1}{a_{n_1} b_{n_2}} = 1, a_i > 0, b_j > 0 \right\}.$$  

Then the $A_1$-orbit of $(y_1, y_2)$ is not dense in $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ (Lemma 3.3), but $G_1 \times G_2$ is the smallest, closed, connected subgroup $F$ of $G_1 \times G_2$ such that $A_1(y_1, y_2) \subseteq F(y_1, y_2)$ (Lemma 3.7).

This yields a counterexample to Conjecture 1 which can be summarized as follows:

**Proposition 1.** For $i = 1, 2$, let $n_i \geq 3$ and $\Gamma_i$ be a lattice in $G_i = \text{SL}(n_i, \mathbb{R})$. For $A_1$, $y_1, y_2$ depicted as above, the $A_1$-orbit of $(y_1, y_2)$ in $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ satisfies neither condition (a) nor condition (b) of Conjecture 1.

2.2. **Proof of Theorem 1, part (1).** In order to obtain the first part of Theorem 1, choose $\Gamma_i = \text{SL}(n_i, \mathbb{Z}), \Gamma = \text{SL}(n, \mathbb{Z})$ and consider the embedding of $G_1 \times G_2$ in $G$, where matrices are written in blocks:

$$\Psi : (M_{n_1,n_1}, N_{n_2,n_2}) \mapsto \begin{bmatrix} M_{n_1,n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & N_{n_2,n_2} \end{bmatrix}.$$  

This embedding gives rise to an embedding $\overline{\Psi}$ of $G_1 \times G_2/\Gamma_1 \times \Gamma_2$ into $G/\Gamma$.

Let $y_1, y_2$ be two points as above, let $x = \overline{\Psi}(y_1, y_2)$ and take $A = \Psi(A_1)$. We claim that this point $x$ and this group $A$ satisfy Theorem 1, part (1). In fact, since the image of $\overline{\Psi}$ is a closed, connected $A$-invariant subset of $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$, everything takes place in this direct product.

2.3. **Proof of Theorem 1, part (2).** The second part of Theorem 1 is obtained as follows. Let $\sigma$ be the nontrivial field automorphism of the quadratic extension

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1The reader only interested in the case $n = 6$ and $\Gamma = \text{SL}(6, \mathbb{Z})$ might note that when $\Gamma_1 = \text{SL}(3, \mathbb{Z}), \Gamma_2 = \text{SL}(3, \mathbb{Z}), [7, \text{Cor. 1.4}]$ can be used directly in the proof of Lemma 3.2; then the notion of $\mathbb{Q}$-irreducibility becomes unnecessary, and the entire Section 3.1 can be skipped.
$\mathbb{Q}(\sqrt{2})/\mathbb{Q}(\sqrt{2})$. Consider for any $m \geq 1$:

$$\text{SU}(m, \mathbb{Z}[\sqrt{2}], \sigma) = \left\{ M \in \text{SL}(m, \mathbb{Z}[\sqrt{2}]) : (t M^o) M = I_m \right\}.$$ 

Then $\text{SU}(m, \mathbb{Z}[\sqrt{2}], \sigma)$ is a lattice in $\text{SL}(m, \mathbb{R})$, as will be proved in Section 3.5 (see [4, Appendix] for $m = 3$). Define for $i = 1, 2$, $\Gamma_i = \text{SU}(n_i, \mathbb{Z}[\sqrt{2}], \sigma)$, and $\Gamma = \text{SU}(n, \mathbb{Z}[\sqrt{2}], \sigma)$. Now consider the map:

$$\varphi : G_1 \times G_2 \times \mathbb{R} \to G, \quad (X, Y, t) \mapsto \begin{bmatrix} e^{nt} X & 0 \\ 0 & e^{-nt} Y \end{bmatrix}.$$

Define $M$ to be the image of $\varphi$. This time, $\varphi$ factors into a finite covering $\overline{\varphi}$ of homogeneous spaces:

$$\overline{\varphi} : G_1 \times G_2 \times \mathbb{R}/\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbb{Z} \to M/M \cap \Gamma \subset G/\Gamma,$$

where $\alpha = (3 + 2\sqrt{2}) + \sqrt{2}(2 + 2\sqrt{2})$ satisfies $\alpha^{-1} = \sigma(\alpha)$. Consider the points $y_i$ constructed above, and let $x = \overline{\varphi}(y_1, y_2, 0)$. Choose:

$$A = \left\{ \text{diag}(a_1, \ldots, a_n) \mid \prod_{i=1}^{n} a_i = \frac{a_1 a_{n1+1}}{a_n a_n} = 1, \ a_i > 0 \right\} \subset \text{SL}(n, \mathbb{R}).$$

We claim that this lattice $\Gamma$, this point $x$ and this group $A$ satisfy Theorem 1, part (2). What happens here is that the $A$-orbit of $x$ is a circle bundle over an $A_1$-orbit (up to the finite cover $\overline{\varphi}$), as in Rees’ example.

### 3. Proof of Theorem 1

3.1. **$\mathbb{Q}$-irreducible tori.** Fix $i \in \{1, 2\}$. Recall that $\Gamma_i$ is a lattice in $G_i = \text{SL}(n_i, \mathbb{R})$. Since $n_i \geq 3$, by Margulis’s arithmeticity Theorem [16, Th. 6.1.2], there exists a semisimple algebraic $\mathbb{Q}$-group $H_i$ and a surjective homomorphism $\theta$ from the connected component of identity of the real points of this group $H_i^0(\mathbb{R})$ to $\text{SL}(n_i, \mathbb{R})$, with compact kernel, such that $\theta(H_i(\mathbb{Z}) \cap H_i^0(\mathbb{R}))$ is commensurable with $\Gamma_i$.

Following Prasad and Rapinchuk, we say that a $\mathbb{Q}$-torus $T \subset H_i$ is $\mathbb{Q}$-irreducible if it does not contain any proper subtorus defined over $\mathbb{Q}$. By [12, Th. 1(ii)], there exists a maximal $\mathbb{Q}$-anisotropic $\mathbb{Q}$-torus $T_i \subset H_i$, which is $\mathbb{Q}$-irreducible. Because any two maximal $\mathbb{R}$-tori of $\text{SL}(n_i, \mathbb{R})$ are $\mathbb{R}$-conjugate, there exists $g_i \in G_i$ such that $\theta(T_i^0(\mathbb{R})) = g_i^{-1} D_i g_i$. The subgroup $T_i(\mathbb{Z})$ is a cocompact lattice in $T_i(\mathbb{R})$ since $T_i$ is $\mathbb{Q}$-anisotropic [2, Th. 8.4 and Def. 10.5]. Because $\theta(H_i(\mathbb{Z}) \cap H_i^0(\mathbb{R}))$ and $\Gamma_i$ are commensurable and $\theta$ has compact kernel, it follows that both $\Gamma_i \cap g_i^{-1} D_i g_i$ and $\theta(T_i^0(\mathbb{Z})) \cap \Gamma_i \cap g_i^{-1} D_i g_i$ are also cocompact lattices.
in $g_i^{-1} D_i g_i$. The resulting topological torus $\pi_i(D_i g_i) \subset G_i / \Gamma_i$ will be denoted $\mathcal{T}_i$. Write $z_i = \pi_i(g_i)$, so that $\mathcal{T}_i = D_i z_i$.

For every $1 \leq k < l \leq n_i$, define as in [7]:

$$N^{(i)}_{k,l} = \left\{ \text{diag}(a_1, \ldots, a_{n_i}) : \prod_{s=1}^{n_i} a_s = 1, \ a_k = a_l, \ a_s > 0 \right\} \subset D_i,$$

Of interest to us amongst the consequences of $\mathbb{Q}$-irreducibility is the fact that an element of $\mathcal{T}_i \cap g_i^{-1} D_i g_i$ lying in a wall of a Weyl chamber is necessarily trivial. This is expressed in the following form:

**Lemma 3.1.** For every $1 \leq k < l \leq n_i$, and any closed connected subgroup $L$ of positive dimension of $N^{(i)}_{k,l}$, the $L$-orbit of $z_i$ is not compact.

**Proof.** Assume the contrary; that is, $Lz_i$ is compact. This implies that $g_i^{-1} L g_i \cap \mathcal{T}_i$ is a uniform lattice in $g_i^{-1} L g_i$, so that $g_i^{-1} L g_i \cap \theta(H_i(\mathbb{Z}))$ is also a uniform lattice. Since $L$ is nontrivial, there exists an element $\gamma \in H_i(\mathbb{Z}) \cap H_i^0(\mathbb{R})$ of infinite order, such that $g_i \theta(\gamma) g_i^{-1}$ is in $L$. Note that since $\theta$ has compact kernel, $T_i(\mathbb{Z})$ is a lattice in $\theta^{-1}(\theta(T_i^0(\mathbb{R})))$ and is then a subgroup of finite index in $H_i(\mathbb{Z}) \cap H_i^0(\mathbb{R}) \cap \theta^{-1}(\theta(T_i^0(\mathbb{R})))$, so there exists $n > 0$ such that $\gamma^n$ belongs to $T_i(\mathbb{Z})$. Consider the representation:

$$\rho : H_i^0(\mathbb{R}) \to GL(sl(n_i, \mathbb{R})).$$

$$x \mapsto \text{Ad}(g_i \theta(x) g_i^{-1}).$$

Recall that $\chi(\text{diag}(a_1, \ldots, a_{n_i})) = a_k / a_l$ is a weight of $\text{Ad}$ with respect to $D_i$, so that $\chi$ is a weight of $\rho$ with respect to $T_i$. By [12, Prop. 1(iii)], the $\mathbb{Q}$-irreducibility of $T_i$ implies that $\chi(\gamma^n) \neq 1$, but this contradicts the fact that $\theta(\gamma^n) \in g_i^{-1} N^{(i)}_{k,l} g_i$. \hfill $\square$

### 3.2. Contraction and expansion.

For real $s$, denote by $a_i(s)$ the following $n_i \times n_i$-matrix:

$$a_i(s) = \text{diag}(e^{s/2}, 1, \ldots, 1, e^{-s/2}),$$

and write simply $N_i$ for $N^{(i)}_{1,n_i}$. Write also:

$$h_i(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & t \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix}.$$
Then the following commutation relation holds:

\[ a_i(s)h_i(t) = h_i(e^s t)a_i(s); \]

that is, the direction \( h_i \) is expanded for positive \( s \); note that both \( h_i \) and \( a_i \) commute with elements of \( N_i \). It is easy to check from equation (1) that

\[ A_1 = \{ (a_1(s)d_1, a_2(-s)d_2) : s \in \mathbb{R}, d_i \in N_i, i = 1, 2 \}. \]

Recall that \( y_i = h_i(1)z_i \).

**Lemma 3.2.** (1) If \( s \leq 0 \), for any \( d \in N_i \) the point \( a_i(s)dy_i \) lies in the compact set \( K_i = h_i([0, 1])z_i \).

(2) The \( D_i \)-orbit of \( y_i \) is dense in \( G_i / \Gamma_i \).

(3) The set \( \{ a_i(s)dy_i : s \geq 0, d \in N_i \} \) is dense in \( G_i / \Gamma_i \).

**Proof.** The first statement is clear from the commutation relation. It also implies that \( D_i y_i \) contains the compact torus \( z_i \) in its closure.

To prove the second point, we rely heavily on the paper of Lindenstrauss and Weiss. [7, Th. 1.1] applies here, since the hypothesis of their theorem is precisely the conclusion of Lemma 3.1 for \( L = N_{k,l}^{(i)} \). So the following holds: there exists a reductive subgroup \( H_i \), containing \( D_i \), such that \( \overline{D_i y_i} = H_i y_i \), and \( H_i \cap \Gamma_i \) is a lattice in \( H_i \). Write \( L = D_i \cap C_{G_i}(H_i) \).

Since \( D_i y_i \) is not closed, \( H_i \neq D_i \), so there exists a nontrivial root relatively to \( D_i \) for the adjoint representation of \( H_i \) on its Lie algebra, which is a subalgebra of \( \mathfrak{s}l(n_i, \mathbb{R}) \). Thus there exist \( k, l \) such that \( L \subset N_{k,l}^{(i)} \). By [7, Step 4.1 of Lemma 4.2], \( L z_i \) is compact, so that by Lemma 3.1, \( L \) is trivial. By [7, Prop. 3.1], \( H_i \) is the connected component of the identity of \( C_{G_i}(L) \), so that \( H_i = G_i \), as desired.

The third claim follows from the first and second claim together with the fact that \( K_i \) has empty interior. \( \square \)

### 3.3. Topological properties of the \( A_1 \)-orbit.

**Lemma 3.3.** The \( A_1 \)-orbit of \( (y_1, y_2) \) is not dense in \( G_1 \times G_2 / \Gamma_1 \times \Gamma_2 \).

**Proof.** Consider the open set \( U = K_1^c \times K_2^c \). We claim that the \( A_1 \)-orbit of \( (y_1, y_2) \) does not intersect \( U \). Indeed, if \( (a_1(s)d_1, a_2(-s)d_2) \in A_1 \) with \( s \in \mathbb{R} \) and \( d_i \in N_i \), the previous lemma implies that if \( s \geq 0 \), \( a_2(-s)d_2y_2 \in K_2 \), and if \( s \leq 0 \), \( a_1(s)d_1y_1 \in K_1 \).

The following elementary result will be useful:

**Lemma 3.4.** Let \( p_i : G_1 \times G_2 \to G_i \) be the first (resp. second) coordinate morphism. If \( F \subset G_1 \times G_2 \) is a subgroup such that \( p_i(F) = G_i \) for \( i = 1, 2 \), and \( A_1 \subset F \), then \( F = G_1 \times G_2 \).
Proof. Let $F_1 = \ker(p_1) \cap F$. Since $F_1$ is normal in $F$, $p_2(F_1)$ is normal in $p_2(F) = G_2$. Note that $N_2 \subset p_2(A_1 \cap \ker(p_1)) \subset p_2(F_1)$ is not finite, and that $G_2$ is almost simple; consequently the normal subgroup $p_2(F_1)$ of $G_2$ is equal to $G_2$. When $(a, b) \in G_1 \times G_2$, by assumption there exists $f \in F$ such that $p_1(f) = a$. Let $f_1 \in F_1$ be such that $p_2(f_1) = b p_2(f)^{-1}$; then $(a, b) = f_1 f \in F$. \hfill \square

We will have to apply several times the two following well-known lemmas:

**Lemma 3.5.** Let $L$ be a Lie group, $\Lambda \subset L$ a lattice, $M, N$ two closed, connected subgroups of $L$, such that for some $w \in L/\Lambda$, $Mw$ and $Nw$ are closed. Then $(M \cap N)w$ is closed.

**Proof.** This is a weaker form of [14, Lemma 2.2]. \hfill \square

**Lemma 3.6.** Let $L$ be a connected Lie group, $\Lambda \subset L$ a discrete subgroup, $M, N$ two subgroups of $L$, such that $M$ is closed and connected, and $N$ is a countable union of closed sets. For any $w \in L/\Lambda$, if $Mw \subset Nw$, then $M \subset N$.

**Proof.** Up to changing $\Lambda$ by one of its conjugates in $L$, one can assume that $w = \Lambda \in L/\Lambda$. By assumption, $M \Lambda \subset N \Lambda$ so that $M \subset N \Lambda \subset L$. Recall that $M$ is closed, that $\Lambda$ is countable, and that $N$ is a countable union of closed sets, so Baire’s category theorem applies, and there exist $\lambda \in \Lambda$ and an open set $U$ of $M$ such that $U \subset N \lambda$, so that $UU^{-1} \subset N$. Since $M$ is a connected subgroup, $UU^{-1}$ generates $M$, and so $M \subset N$. \hfill \square

The following lemma will be useful both for proving that the closure of $A_1(y_1, y_2)$ is not homogeneous, and for proving it does not fiber over a 1-parameter group orbit.

**Lemma 3.7.** Let $F$ be a closed connected subgroup of $G_1 \times G_2$ such that $F(y_1, y_2)$ contains the closure of $A_1(y_1, y_2)$. Then $F = G_1 \times G_2$.

**Proof.** By Lemma 3.2, the set of first coordinates of the set

$$\{(a(s)d_1 y_1, a(-s)d_2 y_2) : s \geq 0, d_1 \in N_1\}$$

is dense in $G_1/\Gamma_1$ and the second coordinates lies in the compact set $K_2$, so the closure of $A_1(y_1, y_2)$ contains points of arbitrary first coordinate with their second coordinate in $K_2$. Consequently, the set of first coordinates of $F(y_1, y_2)$ is the whole $G_1/\Gamma_1$, and similarly for the set of second coordinates. For $i = 1, 2$, Lemma 3.6 now applies to $L = M = G_i$, $\Lambda = \Gamma_i$, $N = p_i(F)$, which is a countable union of closed sets because $G_1 \times G_2$ is $\sigma$-compact, and $w = y_i$, and so $p_i(F) = G_i$.

In order to apply Lemma 3.4 and finish the proof, we have to show that $A_1 \subset F$. Again, this follows from a direct application of Lemma 3.6 to $L = G_1 \times G_2$, $\Lambda = \Gamma_1 \times \Gamma_2$, $M = A_1$, $N = F$, $w = (y_1, y_2)$. \hfill \square
3.4. Proof of Theorem 1, part (1). We now proceed to proving Theorem 1, part (1). The proof of Proposition 1 is similar and is omitted.

Recall that in this case, we fixed $A = \Psi(A_1)$ and $x = \overline{\Psi}(y_1, y_2)$.

Assume $\overline{Ax}$ is homogeneous; that is, $\overline{Ax} = Fx$ for a closed connected subgroup $F$ of $G$. Since $Ax \subseteq \overline{\Psi}(G_1 \times G_2/\Gamma_1 \times \Gamma_2)$, which is closed in $G/\Gamma$, Lemma 3.6 implies that $F \subseteq \Psi(G_1 \times G_2)$. By Lemma 3.7, $F = \Psi(G_1 \times G_2)$, so that $F \times = G/\Gamma$ and $Ax$ is dense in $\overline{\Psi}(G_1 \times G_2)$, which is a contradiction.

Now assume $\overline{Ax}$ fibers over the orbit of a one-parameter subgroup. Let $F$ be a closed connected subgroup, $L$ a Lie group and $\phi : F \rightarrow L$ a continuous epimorphism satisfying (b) of Conjecture 1. Let $F' = F \cap \Psi(G_1 \times G_2)$, we have $A \subseteq F'$. By Lemma 3.5, $F'$ is closed in $F \times \overline{\Psi}(G_1 \times G_2)$, and so is closed in $G/\Gamma$. By Lemma 3.7, $F' = \Psi(G_1 \times G_2)$ necessarily. Let $H = \text{Ker}(\phi \circ \psi) \subset G \times G_2$, so that $A_1/(A_1 \cap H)$ is a one-parameter group by assumption (b) of the conjecture.

The subgroup $H$ is a normal subgroup of the semisimple group $G_1 \times G_2$, which has only four kinds of normal subgroups: finite, $G_1 \times G_2$, $G_1 \times \text{finite}$ and $\text{finite} \times G_2$. None of these possible normal subgroups has the property that it intersects $A_1$ in a codimension 1 subgroup; so this is a contradiction.

3.5. The arithmetic lattice. Here we prove that $\text{SU}(n, \mathbb{Z}[\sqrt{2}], \sigma)$ is a lattice in $\text{SL}(n, \mathbb{R})$. Let $P, Q$ be the polynomials with coefficients in $\mathbb{Q}(\sqrt{2})$ such that for any $X, Y \in M_n(\mathbb{C})$

$$\det(X + \sqrt{2}Y) = P(X, Y) + \sqrt{2}Q(X, Y).$$

For an integral domain $A \subset \mathbb{C}$, consider the set of pairs of matrices:

$$G(A) = \{(X, Y) \in M_n(A)^2 : \text{tr}XX - \sqrt{2}YY = I_n, \text{tr}XY - YX = 0, P(X, Y) = 1, Q(X, Y) = 0\},$$

which implies that $(\text{tr}X - \sqrt{2}Y)(X + \sqrt{2}Y) = I_n$ and $\det(X + \sqrt{2}) = 1$ for all $(X, Y) \in G(A)$. Endow $G(A)$ with the multiplication given by

$$(X, Y)(X', Y') = (XX' + \sqrt{2}YY', YY' + YX'),$$

which is such that the map $\phi : G(A) \rightarrow \text{SL}(n, C), (X, Y) \mapsto X + i\sqrt{2}Y$ is a morphism. With this structure, $G$ is an algebraic group, which is clearly defined over $\mathbb{Q}(\sqrt{2})$. Let $\tau$ be the nontrivial field automorphism of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$; it can be checked that the map $\phi$ is an isomorphism between $G(\mathbb{R})$ and $\text{SL}(n, \mathbb{R})$, and that moreover $\phi' : G(\mathbb{R}) \rightarrow \text{SL}(n, \mathbb{C}), (X, Y) \mapsto X + i\sqrt{2}Y$ is an isomorphism onto $\text{SU}(n)$. Let $H = \text{Res}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}G = G \times G^\tau$. Then $H$ is defined over $\mathbb{Q}$ (see for example [16, 6.1.3], for the definition and properties of the restriction of scalars functor). It follows from a theorem of Borel and Harish-Chandra [16, Th. 3.1.7] that $H(\mathbb{Z})$ is a lattice in $H(\mathbb{R})$. Since $\text{SU}(n)$ is compact, it follows that the projection of $H(\mathbb{Z})$
onto the first factor of $G(\mathbb{R}) \times G^*(\mathbb{R})$ is again a lattice. Using the isomorphism between $G(\mathbb{R})$ and $\text{SL}_n(\mathbb{R})$, this projection can be identified with

$$G(\mathbb{Z}[\sqrt{2}]) = \text{SU}(n, \mathbb{Z}[\sqrt{2}] + \sqrt{2}\mathbb{Z}[\sqrt{2}], \sigma) = \text{SU}(n, \mathbb{Z}[\sqrt{2}], \sigma).$$

3.6. **Proof of Theorem 1, part (2).** Note that, as stated implicitly in Section 2.3,

$$\varphi(\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbb{Z}) \subset \Gamma \cap M,$$

so that $\Gamma \cap M$ is a lattice in $M$, and $M/(M \cap \Gamma)$ is a closed, $A$-invariant subset of $G/\Gamma$. Notice also that the map $\Psi$ defined by equation (2) defines an embedding $\Psi : G_1 \times G_2 / \Gamma_1 \times \Gamma_2 \rightarrow G/\Gamma$.

Assume $Ax$ is homogeneous, that is $Ax = Fx$ for a closed connected subgroup $F$ of $G$. Since $Ax \subset M/(M \cap \Gamma)$, which is closed in $G/\Gamma$, Lemma 3.6 applied twice gives that $A \subset F \subset M$. When $F' = F \cap \Psi(G_1 \times G_2)$, again by Lemma 3.5. $F'x$ is a closed subset of $\text{Im}(\Psi)$. Since $A_1 \subset F'$, $\Psi(A_1)x \subset F'x$ and Lemma 3.7 implies that $F'' = \Psi(G_1 \times G_2)$. Since $A$ contains $\varphi(e, e, t)$ for all $t \in \mathbb{R}$, we have $M = AF' \subset F$ so that $F = M$ necessarily.

By Lemma 3.3, the $A_1$-orbit of $(y_1, y_2)$ is not dense; the topological transitivity of the action of $A_1$ on $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$ implies that moreover the closure of this orbit has empty interior. Thus, the $A_1 \times \mathbb{R}$-orbit of $(y_1, y_2, 0)$ is also nowhere dense in $G_1 \times G_2 \times \mathbb{R} / \Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbb{Z}$. The map $\overline{\varphi}$ being a finite covering, the $A$-orbit of $x$ is nowhere dense. This is a contradiction with $F = M$.

Now assume $Ax$ fibers over the orbit of a one-parameter non-$\text{Ad}$-unipotent subgroup. Let $F$ be a closed connected subgroup, $L$ a Lie group and $\phi : F \rightarrow L$ a continuous epimorphism satisfying the (b) of the conjecture. Letting $F' = F \cap \Psi(G_1 \times G_2)$ and $F'' = F \cap M$, we have $A_1 \subset F'$ and $A \subset F''$. Similarly, $F'x$ and $F''x$ are closed in $G/\Gamma$. Again, by Lemma 3.7, $F'' = \Psi(G_1 \times G_2)$ necessarily, and like before, $AF' \subset F'' \subset M$ so that $F'' = M$.

Let $H = \text{Ker}(\phi \circ \varphi) \subset G_1 \times G_2 \times \mathbb{R}$, so that $A_1 \times \mathbb{R} / (A_1 \times \mathbb{R} \cap H)$ is a one-parameter group. This time, possibilities for the closed normal subgroup $H$ are: finite $\times \Lambda$, $G_1 \times G_2 \times \Lambda$, $G_1 \times \text{finite} \times \Lambda$ and finite $\times G_2 \times \Lambda$, where $\Lambda$ is a closed subgroup of $\mathbb{R}$. Of all these possibilities, only $G_1 \times G_2 \times \Lambda$, where $\Lambda$ is discrete, has the required property that $A_1 \times \mathbb{R} / (A_1 \times \mathbb{R} \cap H)$ is a one-parameter group. This proves that $\Psi(G_1 \times G_2) \subset \text{Ker}(\phi)$, and so $F \subset N_G(\Psi(G_1 \times G_2))$. However, the normalizer of $\Psi(G_1 \times G_2)$ in $G$ is the group of block matrices having, for connected component of the identity, the group $M$. So by connectedness of $F$, $F \subset M$, and since $M = F'' \subset F$, we have $F = M$. Thus $L = F / \text{Ker}(\phi) = \mathbb{R} / \Lambda$ is abelian, and a fortiori every element of $L$ is unipotent; this contradicts (b).

4. **Proof of Theorem 2**

The proof of Theorem 2 is divided in two independent lemmas.
LEMMA 4.1. The family \((z_1, \ldots, z_{2q}, 1)\) is linearly independent over \(\mathbb{Q}\).

Proof. Consider a linear combination:

\[
\sum_{i=1}^{q} a_i z_i + b_i z_{i+q} = c.
\]

We can assume that \(a_i, b_i\) and \(c\) are integers. Letting \(k_0 \geq 1\), write

\[
(3) \quad \left( \prod_{i=1}^{q} p_i \right)^{N^{2k_0+1}} \left( \sum_{i=1}^{q} \sum_{k=1}^{k_0} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} - c \right)
\]

\[
= - \left( \prod_{i=1}^{q} p_i \right)^{N^{2k_0+1}} \left( \sum_{i=1}^{q} \sum_{k \geq k_0+1} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} \right).
\]

It is clear that the left-hand side is an integer. Since \(1 < p_1 < \cdots < p_q\), the right-hand side is less in absolute value than

\[
p_q^{qN^{2k_0+1}} 2q \sup_i(|a_i|, |b_i|) \sum_{k \geq 0} \left( p_1^{-N^{2k_0+2}} \right)^{N^{2k}} \leq 4q \sup_i(|a_i|, |b_i|) p_q^{qN^{2k_0+1}} p_1^{-N^{2k_0+2}} \leq 4q \sup_i(|a_i|, |b_i|) \exp(N^{2k_0+1} (q \log p_q - N \log p_1)).
\]

Since \(N > q \log(p_2) / \log(p_1)\), the last expression tends to zero. This proves the right-hand side of (3) is zero for large enough \(k_0\); so for all large \(k\),

\[
\sum_{i=1}^{q} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} = 0.
\]

The \(p_i\) being distincts, this implies that for \(i \in \{1, \ldots, q\}\), \(a_i = b_i = 0\). \(\square\)

The following lemma implies easily that the orbit of \(z\) under \(\Omega\) cannot be dense.

LEMMA 4.2. For all \(\varepsilon > 0\) there exists \(L > 0\), such that for all \(n_1, \ldots, n_q \geq 0\) with \(\sum_{i=1}^{q} n_i \geq L\), there exists \(j \in \{1, \ldots, 2q\}\) such that \(p_1^{n_1} \cdots p_q^{n_q} z_j\) lies in the interval \([0, \varepsilon]\) modulo 1.

Proof. Let \(s \in \{1, \ldots, q\}\) be such that for all \(r \in \{1, \ldots, q\}\), \(p_s^{n_s} \geq p_r^{n_r}\). Let \(k_0\) be the integer part of \(\log(n_s) / 2 \log(N)\); then either \(N^{2k_0} \leq n_s \leq N^{2k_0+1}\), or \(N^{2k_0+1} \leq n_s \leq N^{2k_0+2}\). In the first case, take \(j = s\); then:

\[
p_1^{n_1} \cdots p_q^{n_q} z_j = p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq 1} p_s^{-N^{2k}} = p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \mod 1.
\]
We have
\[ \sum_{k \geq k_0 + 1} p_s^{-N^{2k}} \leq 2 p_s^{-N^{2k_0 + 2}}; \]
so, using the fact that for all \( r \in \{1, \ldots, q\} \), \( p_r^{n_r} \leq p_s^{n_s} \), we obtain:
\[ p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq k_0 + 1} p_s^{-N^{2k}} \leq 2 p_s^q N^{2k_0 + 1 - N^{2k_0 + 2}} \leq 2 p_s^{N^{2k_0 + 1}(q-N)}. \]
but by hypothesis we have \( N > q \log(p_q) / \log(p_1) > q \), so that the preceding bound is small whenever \( k_0 \) is large. Because of the definition of \( k_0 \), we have
\[ k_0 \geq \frac{\log \sum_{i=1}^q n_i \log p_i}{2 \log N} \geq \frac{\log \frac{L \log p_1}{q \log p_q}}{2 \log N}, \]
so that \( k_0 \) is arbitrarily large when \( L \) is large.

In the second case \( N^{2k_0 + 1} \leq n_s \leq N^{2k_0 + 2} \), and one can proceed similarly with \( j = s + q \).

\[ \Box \]

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E-mail address: francois.maucourant@univ-rennes1.fr
Université Rennes 1, Institut de Recherche Mathématique de Rennes,
Campus de Beaulieu, F-35042 Rennes, France,
http://perso.univ-rennes1.fr/francois.maucourant/