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By Hidekazu Furusho

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Abstract

The author will prove that Drinfel'd's pentagon equation implies his two hexagon equations in the Lie algebra, pro-unipotent, pro-l and pro-nilpotent contexts.

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0. Introduction

In his celebrated papers on quantum groups [Dri87], [Dri89], [Dri90] Drinfel'd came to the notion of quasitriangular quasi-Hopf quantized universal enveloping algebra. It is a topological algebra which differs from a topological Hopf algebra in the sense that the coassociativity axiom and the cocommutativity axiom is twisted by an associator and an R-matrix satisfying a pentagon axiom and two hexagon axioms. One of the main theorems in [Dri90] is that any quasitriangular quasi-Hopf quantized universal enveloping algebra modulo twists (in other words gauge transformations [Kas95]) is obtained as a quantization of a pair (called its classical limit) of a Lie algebra and its symmetric invariant 2-tensor. Quantizations are constructed by universal associators. The set of group-like universal associators forms a pro-algebraic variety, denoted M. Its nonemptiness is another of his main theorems (reproved in [BN98]). Our first theorem is on the defining equations of M.

Let us fix notation and conventions: Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure and $U\mathfrak{F}_2 = k\langle\langle X, Y\rangle\rangle$ a noncommutative formal power series ring with two variables X and Y. Its element $\varphi = \varphi(X, Y)$ is called *group-like* if it satisfies $\Delta(\varphi) = \varphi \otimes \varphi$ with $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$ and its constant term is equal to 1. Its coefficient of XY is denoted by $c_2(\varphi)$. For any k-algebra homomorphism $\iota : U\mathfrak{F}_2 \to S$ the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(X), \iota(Y))$. Let $\mathfrak{a}_4$ be the completion (with respect to the natural grading) of the Lie algebra over k with generators t_{ij} ($1 \leqslant i, j \leqslant 4$) and defining relations $t_{ii} = 0$, $t_{ij} = t_{ji}$, $[t_{ij}, t_{ik} + t_{jk}] = 0$ (i,j,k: all distinct) and $[t_{ij}, t_{kl}] = 0$ (i,j,k,l: all distinct).

THEOREM 1. *Let $\varphi = \varphi(X, Y)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies Drinfel'd's pentagon equation*:

(1) $$\varphi(t_{12}, t_{23}+t_{24})\varphi(t_{13}+t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\varphi(t_{12}+t_{13}, t_{24}+t_{34})\varphi(t_{12}, t_{23}).$$

Then there exists an element (*unique up to signature*) $\mu \in \bar{k}$ *such that the pair* (μ, φ) *satisfies his two hexagon equations*:

(2) $$\exp\left\{\frac{\mu(t_{13}+t_{23})}{2}\right\} = \varphi(t_{13}, t_{12}) \exp\left\{\frac{\mu t_{13}}{2}\right\}\varphi(t_{13}, t_{23})^{-1} \exp\left\{\frac{\mu t_{23}}{2}\right\}\varphi(t_{12}, t_{23}),$$

(3) $$\exp\left\{\frac{\mu(t_{12}+t_{13})}{2}\right\} = \varphi(t_{23}, t_{13})^{-1} \exp\left\{\frac{\mu t_{13}}{2}\right\}\varphi(t_{12}, t_{13}) \exp\left\{\frac{\mu t_{12}}{2}\right\}\varphi(t_{12}, t_{23})^{-1}.$$

Actually this μ is equal to $\pm(24c_2(\varphi))^{\frac{1}{2}}$.

It should be noted that we need to use an (actually quadratic) extension of a field k in order to obtain the hexagon equations from the pentagon equation. The associator set $\underline{M}$ is the pro-algebraic variety whose set of k-valued points consists of pairs (μ, φ) satisfying (1), (2) and (3) and M is its open subvariety defined by $\mu \neq 0$. The theorem says that the pentagon equation is essentially a single defining equation of the associator set. The Drinfel'd associator $\Phi_{KZ} \in \mathbf{R}\langle\langle X, Y\rangle\rangle$ is a group-like series constructed by solutions of the KZ-equation [Dri90]. It satisfies (1), (2) and (3) with $\mu = \pm 2\pi\sqrt{-1}$. Its coefficients are expressed by multiple zeta values [LM96] (and [Fur03]). The theorem also says that the two hexagon equations do not provide any new relations under the pentagon equation.

The category of representations of a quasitriangular quasi-Hopf quantized universal enveloping algebra forms a quasitensored category [Dri90], in other words, a braided tensor category [JS93]; its associativity constraint and its commutativity constraint are subject to one pentagon axiom and two hexagon axioms. The Grothendieck-Teichmüller pro-algebraic group GT is introduced in [Dri90] as a group of deformations of the category which change its associativity constraint and its commutativity constraint keeping all three axioms. It is also conjecturally related to the motivic Galois group of $\mathbf{Z}$ (explained in [And04]). Relating to the absolute Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ of $\mathbf{Q}$ its profinite group version $\widehat{\mathrm{GT}}$ is discussed in [Iha91], [Sch97]. Our second theorem is on defining equations of GT.

THEOREM 2. *Let $F_2(k)$ be the the free pro-unipotent algebraic group with two variables x and y. Suppose that its element $f = f(x, y)$ satisfies Drinfel'd's pentagon equation*:

(4)
$$f(x_{12}, x_{23}x_{24}) f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34}) f(x_{12}x_{13}, x_{24}x_{34}) f(x_{12}, x_{23})$$

in $K_4(k)$. Then there exists an element (*unique up to signature*) $\lambda \in \overline{k}$ *such that the pair* (λ, f) *satisfies his hexagon equations* (3- *and* 2-*cycle relation*):

(5) $$f(z, x) z^m f(y, z) y^m f(x, y) x^m = 1 \text{ with } xyz = 1 \text{ and } m = \frac{\lambda - 1}{2},$$

(6) $$f(x, y) f(y, x) = 1.$$

Actually this λ is equal to $\pm(24c_2(f)+1)^{\frac{1}{2}}$ where $c_2(f)$ stands for $c_2(f(e^X, e^Y))$.

Here $K_4(k)$ stands for the unipotent completion of the pure braid group $K_4 = \ker\{B_4 \to \mathfrak{S}_4\}$ of four strings (B_4: the Artin braid group and $\mathfrak{S}_4$: the symmetric group) with standard generators x_{ij} $(1 \leqslant i, j \leqslant 4)$.

It should be noted again that we need to use an (actually quadratic) extension of a field k in order to obtain the hexagon equations from the pentagon equation. The set of pairs (λ, f) satisfying (4), (5) and (6) determines a pro-algebraic variety $\underline{\mathrm{GT}}$ and GT is its open subvariety defined by $\lambda \neq 0$. The product structure on $\mathrm{GT}(k)$ is given by $(\lambda_1, f_1) \circ (\lambda_2, f_2) := (\lambda, f)$ with $\lambda = \lambda_1\lambda_2$ and $f(x, y) = f_1(f_2 x^{\lambda_2} f_2^{-1}, y^{\lambda_2}) f_2$. The theorem says that the pentagon equation is essentially a single defining equation of GT.

The construction of the paper is as follows. Section 1 is a crucial part of the paper. The implication of the pentagon equation is proved for Lie series. In Section 2 we give a proof of Theorem 1 by using Drinfel'd's gadgets. Section 3 gives a proof of Theorem 2 and its analogue in the pro-l group and pro-nilpotent group setting.

1. Lie algebra case

In this section we prove the Lie algebra version of Theorem 1 in a rather combinatorial argument.

Let $\mathfrak{F}_2$ be the set of Lie-like elements φ in $U\mathfrak{F}_2$ (i.e. $\Delta(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi$).

THEOREM 3. *Let φ be a commutator Lie-like element*[1] *with $c_2(\varphi) = 0$. Suppose that φ satisfies the pentagon equation* (5-*cycle relation*):

(7) $$\varphi(X_{12}, X_{23}) + \varphi(X_{34}, X_{45}) + \varphi(X_{51}, X_{12}) + \varphi(X_{23}, X_{34}) + \varphi(X_{45}, X_{51}) = 0$$

[1] In this paper we call a series $\varphi = \varphi(X, Y)$ *commutator Lie-like* if it is Lie-like and its coefficient of X and Y are both 0, in other words $\varphi \in \mathfrak{F}_2' := [\mathfrak{F}_2, \mathfrak{F}_2]$.

in $\hat{\mathfrak{P}}_5$. *Then it also satisfies the hexagon equations* (3- *and* 2-*cycle relation*):

$$\varphi(X,Y)+\varphi(Y,Z)+\varphi(Z,X)=0 \text{ with } X+Y+Z=0, \tag{8}$$

$$\varphi(X,Y)+\varphi(Y,X)=0. \tag{9}$$

Here $\hat{\mathfrak{P}}_5$ stands for the completion (with respect to the natural grading) of the pure sphere braid Lie algebra $\mathfrak{P}_5$ [Iha91] with five strings; the Lie algebra generated by X_{ij} $(1 \leqslant i, j \leqslant 5)$ with clear relations $X_{ii}=0$, $X_{ij}=X_{ji}$, $\sum_{j=1}^{5} X_{ij}=0$ $(1 \leqslant i, j \leqslant 5)$ and $[X_{ij}, X_{kl}]=0$ if $\{i,j\}\cap\{k,l\}=\varnothing$. It is a quotient of $\mathfrak{a}_4$ (cf. §2).

Proof. There is a projection from $\hat{\mathfrak{P}}_5$ to the completed free Lie algebra $\mathfrak{F}_2$ generated by X and Y by putting $X_{i5}=0$, $X_{12}=X$ and $X_{23}=Y$. The image of the 5-cycle relation gives the 2-cycle relation.

For our convenience we denote $\varphi(X_{ij}, X_{jk})$ $(1 \leqslant i, j, k \leqslant 5)$ by φ_{ijk}. Then the 5-cycle relation can be read as

$$\varphi_{123}+\varphi_{345}+\varphi_{512}+\varphi_{234}+\varphi_{451}=0.$$

We denote the left-hand side by P. Let σ_i $(1 \leqslant i \leqslant 4)$ be elements of $\mathfrak{S}_5$ defined as follows: $\sigma_1(12345)=(12345)$, $\sigma_2(12345)=(54231)$, $\sigma_3(12345)=(13425)$ and $\sigma_4(12345)=(43125)$. Then

$$\begin{aligned}\sum_{i=1}^{4}\sigma_i(P) = {} & \varphi_{123}+\varphi_{345}+\varphi_{512}+\varphi_{234}+\varphi_{451}\\ & +\varphi_{542}+\varphi_{231}+\varphi_{154}+\varphi_{423}+\varphi_{315}\\ & +\varphi_{134}+\varphi_{425}+\varphi_{513}+\varphi_{342}+\varphi_{251}\\ & +\varphi_{431}+\varphi_{125}+\varphi_{543}+\varphi_{312}+\varphi_{254}.\end{aligned}$$

By the 2-cycle relation, $\varphi_{ijk}=-\varphi_{kji}$ $(1 \leqslant i, j, k \leqslant 5)$. This gives

$$\begin{aligned}\sum_{i=1}^{4}\sigma_i(P) = {} & (\varphi_{123}+\varphi_{231}+\varphi_{312})+(\varphi_{512}+\varphi_{125}+\varphi_{251})\\ & +(\varphi_{234}+\varphi_{342}+\varphi_{423})+(\varphi_{542}+\varphi_{425}+\varphi_{254}).\end{aligned}$$

By $[X_{23}, X_{12}+X_{23}+X_{31}]=[X_{31}, X_{12}+X_{23}+X_{31}]=[X_{12}, X_{12}+X_{23}+X_{31}]=0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi_{231}=\varphi(X_{23}, X_{31})=\varphi(X_{23}, -X_{12}-X_{23})$ and $\varphi_{312}=\varphi(X_{31}, X_{12})=\varphi(-X_{12}-X_{23}, X_{12})$.

By $[X_{51}, X_{12}+X_{25}+X_{51}]=[X_{12}, X_{12}+X_{25}+X_{51}]=[X_{25}, X_{12}+X_{25}+X_{51}]=0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi_{512}=\varphi(X_{51}, X_{12})=\varphi(-X_{12}-X_{25}, X_{12})$ and $\varphi_{251}=\varphi(X_{25}, X_{51})=\varphi(X_{25}, -X_{12}-X_{25})$.

By $[X_{23}, X_{42}+X_{23}+X_{34}]=[X_{34}, X_{42}+X_{23}+X_{34}]=[X_{42}, X_{42}+X_{23}+X_{34}]=0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi_{234}=\varphi(X_{23}, X_{34})=\varphi(X_{23}, -X_{42}-X_{23})$ and $\varphi_{342}=\varphi(X_{34}, X_{42})=\varphi(-X_{42}-X_{23}, X_{42})$.

By $[X_{54}, X_{42}+X_{25}+X_{54}] = [X_{42}, X_{42}+X_{25}+X_{54}] = [X_{25}, X_{42}+X_{25}+X_{54}] = 0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi_{542} = \varphi(X_{54}, X_{42}) = \varphi(-X_{42}-X_{25}, X_{42})$ and $\varphi_{254} = \varphi(X_{25}, X_{54}) = \varphi(X_{25}, -X_{42}-X_{25})$.

Let $R(X,Y) = \varphi(X,Y) + \varphi(Y, -X-Y) + \varphi(-X-Y, X)$. Then

$$\sum_{i=1}^{4} \sigma_i(P) = R(X_{21}, X_{23}) + R(X_{21}, X_{25}) + R(X_{24}, X_{23}) + R(X_{24}, X_{25}).$$

The elements X_{21}, X_{23}, X_{24} and X_{25} generate a completed Lie subalgebra $\mathfrak{F}_3$ of $\hat{\mathfrak{P}}_5$ which is free of rank 3 and whose set of relations is given by $X_{21}+X_{23}+X_{24}+X_{25}=0$. It contains $\sum_{i=1}^{4}\sigma_i(P)$. Let $q_1 : \mathfrak{F}_3 \to \mathfrak{F}_2$ be the projection sending $X_{21} \mapsto X$, $X_{23} \mapsto Y$ and $X_{24} \mapsto X$. Then

$$q_1\left(\sum_{i=1}^{4} \sigma_i(P)\right) = R(X,Y) + R(X,-2X-Y) + R(X,Y) + R(X,-2X-Y).$$

Since $P=0$, we have $R(X,-2X-Y) = -R(X,Y)$. Let $q_2 : \mathfrak{F}_3 \to \mathfrak{F}_2$ be the projection sending $X_{21} \mapsto X$, $X_{23} \mapsto X$ and $X_{24} \mapsto Y$. Then

$$q_2\left(\sum_{i=1}^{4} \sigma_i(P)\right) = R(X,X) + R(X,-2X-Y) + R(Y,X) + R(Y,-2X-Y).$$

By $\varphi \in \mathfrak{F}_2'$, $R(X,X)=0$. By definition, $R(Y,-2X-Y) = R(2X,Y)$. Since $P=0$, $-R(X,Y) + R(Y,X) + R(Y,2X) = 0$. The 2-cycle relation gives $R(X,Y) = -R(Y,X)$. Therefore $2R(X,Y) = R(2X,Y)$. Expanding this equation in terms of a linear basis, such as the Hall basis, we see that $R(X,Y)$ must be of the form $\sum_{m=1}^{\infty} a_m (adY)^{m-1}(X)$ with $a_m \in k$. Since it satisfies $R(X,Y) = -R(Y,X)$, we have $a_1 = a_3 = a_4 = a_5 = \cdots = 0$. By our assumption $c_2(\varphi)=0$, a_2 must be 0 also. Therefore $R(X,Y)=0$, which is the 3-cycle relation. □

We note that the assumption $c_2(\varphi)=0$ is necessary: e.g. the element $\varphi = [X,Y]$ satisfies the 5-cycle relation but it does not satisfy the 3-cycle relation.

Remark 4. There is partially a geometric picture in the proof: We have a de Rham fundamental groupoid [Del89] (see also [Fur07]) of the moduli $\mathcal{M}_{0,n} = \{(x_1 : \cdots : x_n) \in (\mathbf{P}^1)^n | x_i \neq x_j (j \neq j)\}/\mathrm{PGL}(2)$ for $n \geqslant 4$, its central extension given by the normal bundle of $\mathcal{M}_{0,n-1}$ inside its stable compactification $\overline{\mathcal{M}}_{0,n}$ and maps between them. An automorphism of the system is determined by considering what happens to the canonical de Rham path from '0' to '1' (*loc. cit.*) in $\mathcal{M}_{0,4} = \mathbf{P}^1 \backslash \{0,1,\infty\}$. Equation (7) reflects the necessary condition that such an automorphism must keep the property that the image of the composite of the path, the boundaries of the fundamental pentagon $\mathcal{B}_5$ [Iha91] formed by the divisors $x_i = x_{i+1}$ ($i \in \mathbf{Z}/5\mathbf{Z}$) in $\overline{\mathcal{M}}_{0,5}(\mathbf{R})$, must be a trivial loop. Each $\sigma_i(\mathcal{B}_5)$ ($1 \leqslant i \leqslant 4$) is a connected component of $\mathcal{M}_{0,5}(\mathbf{R})$. The sum of four 5-cycles $\sum_{i=1}^{4}\sigma_i(P)$ corresponds to a path following the (oriented) boundaries of the four real pentagonal regions $\sigma_i(\mathcal{B}_5)$ of $\mathcal{M}_{0,5}(\mathbf{R})$. The four 3-cycles correspond to four loops around the four boundary divisors $x_4=x_5$, $x_3=x_4$, $x_5=x_1$ and $x_1=x_3$ in $\overline{\mathcal{M}}_{0,5}(\mathbf{R})$. The author expects that the geometric interpretation might help to adapt our proof to the pro-finite context (cf. Question 14).

The equations (7), (8) and (9) are defining equations of Ihara's stable derivation (Lie-)algebra [Iha91]. Its Lie bracket is given by $\langle\varphi_1,\varphi_2\rangle := [\varphi_1,\varphi_2]+D_{\varphi_2}(\varphi_1)-D_{\varphi_1}(\varphi_2)$ where D_φ is the derivation of $\mathfrak{F}_2$ given by $D_\varphi(X)=[\varphi,X]$ and $D_\varphi(B)=0$. We note that its completion with respect to degree is equal to the graded Lie algebra $\mathfrak{grt}_1$ of the Grothendieck-Teichmüller group GT in [Dri90]. Our theorem says that the pentagon equation is its single defining equation and two hexagon equations are needless for its definition when $\deg\varphi\geqslant 3$.

2. Proof of Theorem 1

This section is devoted to a proof of Theorem 1. Between the Lie algebra $\mathfrak{a}_4$ in Theorem 1 and $\hat{\mathfrak{P}}_5$ in Theorem 3 there is a natural surjection $\tau:\mathfrak{a}_4\to\hat{\mathfrak{P}}_5$ sending t_{ij} to X_{ij} $(1\leqslant i,j\leqslant 4)$. Its kernel is generated by $\Omega=\sum_{1\leqslant i<j\leqslant 4}t_{ij}$. We also denote its induced morphism $U\mathfrak{a}_4\to U\hat{\mathfrak{P}}_5$ by τ. On the pentagon equation we have

LEMMA 5. *Let φ be a group-like element. Giving the pentagon equation (1) for φ is equivalent to showing that φ is commutator group-like*[2] *and φ satisfies the 5-cycle relation in $U\hat{\mathfrak{P}}_5$:*

$$\varphi(X_{12},X_{23})\varphi(X_{34},X_{45})\varphi(X_{51},X_{12})\varphi(X_{23},X_{34})\varphi(X_{45},X_{51})=1. \tag{10}$$

Proof. Assume (1). Denote the abelianization of $\varphi(X,Y)\in k\langle\langle X,Y\rangle\rangle$ by $\varphi^{\mathrm{ab}}\in k[[X,Y]]$. The series φ is group-like, so φ^{ab} is as well, i.e. $\Delta(\varphi^{\mathrm{ab}})=\varphi^{\mathrm{ab}}\otimes\varphi^{\mathrm{ab}}$. Therefore φ^{ab} must be of the form $\exp\{\alpha X+\beta Y\}$ with $\alpha,\beta\in k$. Equation (1) gives $\alpha X_{12}+\beta X_{34}=0$. Hence $\alpha=\beta=0$ which means that φ is commutator group-like. Therefore

$$\varphi(X_{12},X_{51})=\varphi(X_{12},-X_{21}-X_{52})=\varphi(X_{12},X_{23}+X_{24})$$

by $[X_{12},X_{51}+X_{21}+X_{52}]=[X_{51},X_{51}+X_{21}+X_{52}]=0$,

$$\varphi(X_{45},X_{34})=\varphi(-X_{43}-X_{53},X_{34})=\varphi(X_{13}+X_{23},X_{34})$$

by $[X_{45},X_{45}+X_{43}+X_{53}]=[X_{34},X_{45}+X_{43}+X_{53}]=0$ and

$$\begin{aligned}\varphi(X_{45},X_{51})&=\varphi(-X_{14}-X_{15},X_{51})=\varphi(-X_{14}-X_{15},-X_{14}-X_{45})\\&=\varphi(X_{12}+X_{13},X_{24}+X_{34})\end{aligned}$$

[2] In this paper we call a series $\varphi=\varphi(X,Y)$ *commutator group-like* if it is group-like and its coefficient of X and Y are both 0.

by $[X_{45}, X_{45}+X_{14}+X_{51}] = [X_{51}, X_{45}+X_{14}+X_{51}] = 0$ and $[X_{14}+X_{15}, X_{51}+X_{14}+X_{45}] = [X_{51}, X_{51}+X_{14}+X_{45}] = 0$. (N.B. If φ is commutator group-like, $\varphi(A+C, B) = \varphi(A, B+C) = \varphi(A, B)$ with $[A, C] = [B, C] = 0$.) So the image of (1) by τ is

$$\varphi(X_{12}, X_{51})\varphi(X_{45}, X_{34}) = \varphi(X_{23}, X_{34})\varphi(X_{45}, X_{51})\varphi(X_{12}, X_{23}). \tag{11}$$

Lemma 6 gives (10).

Conversely, assume (10) and the commutator group-likeness for φ. Lemma 6 gives equality (11). Whence we say (1) modulo $\ker\tau$. That is, the quotient of the left-hand side of (1) by the right-hand side of (1) is expressed as $\exp\gamma\Omega$ for some $\gamma \in k$. Since both sides of (1) are commutator group-like, $\exp\gamma\Omega$ must be as well. Therefore γ must be 0, which gives (1). □

LEMMA 6. *Let φ be a group-like element. If φ is commutator group-like and it satisfies the 5-cycle relation (10), it also satisfies the 2-cycle relation*:

$$\varphi(X, Y)\varphi(Y, X) = 1. \tag{12}$$

Furthermore, if φ satisfies the pentagon equation (1), it also satisfies (12).

Proof. There is a projection $U\mathfrak{P}_5^\wedge \to U\mathfrak{F}_2$ by putting $X_{i5} = 0$ $(1 \leqslant i \leqslant 5)$, $X_{12} = X$ and $X_{23} = Y$. The image of (10) is (12) by the commutator group-likeness.

As was shown in Lemma 6, equation (1) for φ in $U\mathfrak{a}_4$ implies its commutator group-likeness and (11) in $U\hat{\mathfrak{P}}_5$. The image of (11) by the projection gives equation (12). □

In [IM95], the equivalence between (1) and (10) is shown, assuming the commutatativity and the 2-cycle relation in the pro-finite group setting. But by the above argument the latter assumption can be excluded.

As for the hexagon equations we also have

LEMMA 7. *Let φ be a group-like element. Giving two hexagon equations (2) and (3) for φ is equivalent to giving the 2-cycle relation (12) and the 3-cycle relation*:

$$e^{\frac{\mu X}{2}}\varphi(Z, X)e^{\frac{\mu Z}{2}}\varphi(Y, Z)e^{\frac{\mu Y}{2}}\varphi(X, Y) = 1 \textit{ with } X + Y + Z = 0. \tag{13}$$

Proof. We review the proof in [Dri90]. The Lie subalgebra generated by t_{12}, t_{13} and t_{23} is the direct sum of its center, generated by $t_{12}+t_{23}+t_{13}$, and the free Lie algebra generated by $X = t_{12}$ and $Y = t_{23}$. The projections of (2) and (3) to the first component are both tautologies but the projections to the second component are

$$e^{\frac{\mu X}{2}}\varphi(Z, X)e^{\frac{\mu Z}{2}}\varphi(Z, Y)^{-1}e^{\frac{\mu Y}{2}}\varphi(X, Y) = 1$$

and

$$e^{\frac{\mu X}{2}}\varphi(Z,X)e^{\frac{\mu Z}{2}}\varphi(Z,Y)^{-1}e^{\frac{\mu Y}{2}}\varphi(Y,X)^{-1}=1.$$

They are equivalent to (12) and (13). □

The following are keys to prove Theorem 1.

LEMMA 8. *Let φ_1 and φ_2 be commutator group-like elements. Put $\varphi_3 = \varphi_2 \circ \varphi_1(X,Y) := \varphi_2(\varphi_1 X \varphi_1^{-1}, Y)\cdot\varphi_1$. Assume that φ_1 satisfies (10), (12) and*

$$\varphi(Z,X)\varphi(Y,Z)\varphi(X,Y)=1 \text{ with } X+Y+Z=0. \tag{14}$$

Then φ_2 satisfies (10) if and only if φ_3 satisfies (10).

Proof. By the arguments in [Sch97, §1.2], φ_1 determines an automorphism of $U\mathfrak{P}_5^\wedge$ sending

$$X_{12}\mapsto X_{12},\quad X_{23}\mapsto \varphi_1(X_{12},X_{23})^{-1}X_{23}\varphi_1(X_{12},X_{23}),$$
$$X_{34}\mapsto \varphi_1(X_{34},X_{45})X_{34}\varphi_1(X_{34},X_{45})^{-1},\quad X_{45}\mapsto X_{45}$$

and

$$X_{51}\mapsto \varphi_1(X_{12},X_{23})^{-1}\varphi_1(X_{45},X_{51})^{-1}X_{51}\varphi_1(X_{45},X_{51})\varphi_1(X_{12},X_{23}).$$

The direct calculation shows that the left-hand side of (10) for φ_2 maps to the left-hand side of (10) for $\varphi_3(X,Y)$. This gives the claim. □

LEMMA 9. *Let φ be a commutator group-like element with $c_2(\varphi)=0$. Suppose that φ satisfies (10). Then it also satisfies (14).*

Proof. The proof is given by induction. Suppose that we have (14) mod $\deg n$. The element φ satisfies the commutator group-likeness, (10), (12) and (14) mod $\deg n$, in other words, it is an element of algebraic group $\mathrm{GRT}_1^{(n)}(k)$ [Dri90, §5]. Denote its corresponding Lie element by ψ. It is an element of the Lie algebra $\mathfrak{grt}_1^{(n)}(k)$ (*loc. cit.*), that means, it is expressed by $\psi=\sum_{i=3}^{n-1}\psi^{(i)}\in k\langle\langle X,Y\rangle\rangle$ where $\psi^{(i)}$ is a homogeneous Lie element with $\deg\psi^{(i)}=i$ and satisfies (7), (8) and (9). The Lie algebra $\mathfrak{grt}_1(k)=\varprojlim\mathfrak{grt}_1^{(n)}(k)$ is graded by degree and ψ also determines an element (denoted by the same symbol ψ) of $\mathfrak{grt}_1(k)$. Let $\mathrm{Exp}:\mathfrak{grt}_1(k)\to\mathrm{GRT}_1(k)=\varprojlim\mathrm{GRT}_1^{(n)}(k)$ be the exponential morphism. Put $\varphi_1=\mathrm{Exp}\,\psi$. It is commutator group-like and it satisfies (10), (12), (14) and $\varphi\equiv\varphi_1 \mod \deg n$ (*loc. cit.*). Let φ_2 be a series defined by $\varphi=\varphi_2\circ\varphi_1$. Then φ_2 is commutator group-like and it satisfies (10) by Lemma 8. By $\varphi\equiv\varphi_1 \mod \deg n$, $\varphi_2\equiv 1 \mod \deg n$. Denote the degree n-part of φ_2 by $\psi^{(n)}$. Because $\varphi_2\equiv 1+\psi^{(n)} \mod \deg n+1$, (10) for φ_2 yields (7) for $\psi^{(n)}$ and the group-likeness for φ_2 yields the Lie-likeness for $\psi^{(n)}$. By Theorem 3, $\psi^{(n)}$ satisfies (8) and (9), which means $\psi^{(n)}\in\mathfrak{grt}_1(k)$. Since $\mathrm{Exp}\,\psi^{(n)}\in\mathrm{GRT}_1(k)$ and $\varphi_2\equiv\mathrm{Exp}\,\psi^{(n)} \mod \deg n+1$, φ_2 belongs to $\mathrm{GRT}_1^{(n+1)}(k)$. Since φ_1 also determines an element of $\mathrm{GRT}_1^{(n+1)}(k)$, φ must belong to $\mathrm{GRT}_1^{(n+1)}(k)$. This means that φ satisfies (14) mod $\deg n+1$. □

THEOREM 10. *Let φ be a commutator group-like element. Suppose that φ satisfies the 5-cycle relation (10). Then there exists an element (unique up to signature) $\mu \in \overline{k}$ such that the pair (μ, φ) satisfies the 3-cycle relation (13). Actually this μ is equal to $\pm(24c_2(\varphi))^{\frac{1}{2}}$.*

Proof. We may assume $c_2(\varphi) \neq 0$ by Lemma 9. Let μ be a solution of $x^2 = 24c_2(\varphi)$ in $\overline{k}^{\times}$. Let M'_μ (resp. M_μ [Dri90]) be the pro-affine algebraic variety whose $\overline{k}$-valued points are commutator group-like series φ in $\overline{k}\langle\langle X, Y\rangle\rangle$ satisfying (10) and $c_2(\varphi) = \frac{\mu^2}{24}$ (resp. (10), (12) and (13)) for (μ, φ). By calculating the coefficient of XY in (13) for (μ, φ), we get $3c_2(\varphi) - \frac{\mu^2}{8} = 0$. Thus M_μ is a pro-subvariety of M'_μ. To prove $M'_\mu = M_\mu$, it suffices to show this for $\mu = 1$ because we have a replacement $\varphi(A, B)$ by $\varphi(\frac{A}{\mu}, \frac{B}{\mu})$. In a similar way to [Fur06, §6] the regular function ring $\mathcal{O}(M'_1)$ (resp. $\mathcal{O}(M_1)$) is encoded the weight filtration $W = \{W_n\mathcal{O}(M'_1)\}_{n\in\mathbf{Z}}$ (resp. $\{W_n\mathcal{O}(M_1)\}_{n\in\mathbf{Z}}$). The algebra $\mathcal{O}(M'_1)$ (resp. $\mathcal{O}(M_1)$) is generated by x_W's (W: word[3]) and defined by the commutator group-likeness, (10) and $c_2(\varphi) = \frac{1}{24}$ (resp. (10), (12) and (13)) for $\varphi = 1 + \sum_W x_W W$. Set $\deg x_W = \deg W$. Each $W_n\mathcal{O}(M'_1)$ (resp. $W_n\mathcal{O}(M_1)$) is the vector space generated by polynomials whose total degree is less than or equal to n.

The inclusion $M_1 \to M'_1$ gives a projection $\mathcal{O}(M'_1) \twoheadrightarrow \mathcal{O}(M_1)$ which is strictly compatible with the filtrations. It induces a projection $p : \mathrm{Gr}^W_\cdot\mathcal{O}(M'_1) \twoheadrightarrow \mathrm{Gr}^W_\cdot\mathcal{O}(M_1)$ between their associated graded quotients. The graded quotient $\mathrm{Gr}^W_\cdot\mathcal{O}(M_1)$ is isomorphic to $\mathcal{O}(\mathrm{GRT}_1)$ by [Fur06, Th. 6.2.2]. It is the algebra generated by $\bar{x}_W$'s and defined by the commutator group-likeness, (10), (12) and (14) for $\overline{\varphi} = 1 + \sum_W \bar{x}_W W$. On the other hand, the graded quotient $\mathrm{Gr}^W_\cdot\mathcal{O}(M'_1)$ is generated by $\bar{x}_W$'s. These generators especially satisfy the commutator group-likeness, (10) and $c_2(\overline{\varphi}) = 0$ for $\overline{\varphi} = 1 + \sum_W \bar{x}_W W$ among others. By the previous lemmas, $\overline{\varphi}$ must also satisfy (12) and (14). Therefore p should be an isomorphism. This implies $M'_1 = M_1$. □

The combination of this theorem with the previous lemmas completes the proof of Theorem 1.

3. Proof of Theorem 2

In this section we deduce Theorem 2 from Theorem 10 and also show its pro-l group analogue (Corollary 12) and its pro-nilpotent group analogue (Corollary 13).

[3] A *word* means a monic monomial element but 1 in $k\langle\langle X, Y\rangle\rangle$.

Proof of Theorem 2. Let f be an element of $F_2(k)$ satisfying (4). Let λ be a solution of $\frac{x^2-1}{24}=c_2(f)$. Let $\mu\in k^\times$ and $\varphi\in k\langle\langle A,B\rangle\rangle$ be a pair such that φ is commutator group-like and (μ,φ) satisfies (10), (12) and (13). Put $\varphi'=f(\varphi e^{\mu X}\varphi^{-1},e^{\mu Y})\cdot\varphi\in\bar{k}\langle\langle A,B\rangle\rangle$. In the proof of [Dri90, Prop. 5.1] it is shown that giving (4) for f is equivalent to giving (1) for φ'. Hence φ' satisfies (10) by Lemma 5. Put $\mu'=\lambda\mu$. Equation (13) for (μ,φ) gives $c_2(\varphi)=\frac{\mu^2}{24}$. So $c_2(\varphi')=c_2(\varphi)+\mu^2c_2(f)=\frac{\mu'^2}{24}$. Since φ' satisfies (10), Theorem 10 gives (13) for (μ',φ'). Consider the group isomorphism from $F_2(k)$ to the set of group-like elements of $U\mathfrak{F}_2$ which sends x to $e^{\mu X}$ and y to $e^{-\frac{\mu}{2}X}\varphi(Y,X)e^{\mu Y}\varphi(Y,X)^{-1}e^{\frac{\mu}{2}X}$. Consequently z goes to $\varphi(Z,X)e^{\mu Z}\varphi(Z,X)^{-1}$ by (12) and (13) for (μ,φ). The direct calculation shows that the left-hand side of (5) maps to the left-hand side of (13). Therefore giving (5) for (λ,f) is equivalent to giving (13) for (μ',φ'). This completes the proof of Theorem 2. □

Remark 11. By the same argument as Lemma 5, giving the pentagon equation (4) for f is equivalent to giving that $f(e^X,e^Y)$ is commutator group-like and f satisfies the 5-cycle relation in $P_5(k)$:

$$f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})=1.$$

Here $P_5(k)$ means the unipotent completion of the pure sphere braid group with five strings and x_{ij} means its standard generator. Occasionally, in some of the literature, the formula is used directly instead of (4) in the definition of the Grothendieck-Teichmüller group.

As a corollary, the following pro-l (l: a prime) group and pro-nilpotent group version of Theorem 2 are obtained by the natural embedding from the pro-l completion $F_2^{(l)}$ to $F_2(\mathbf{Q}_l)$ and its associated embedding from the pro-nilpotent completion $F_2^{\mathrm{nil}}=\prod_{l:\text{a prime}}F_2^{(l)}$ to $\prod_l F_2(\mathbf{Q}_l)$.

COROLLARY 12. *Let $f=f(x,y)$ be an element of $F_2^{(l)}$ satisfying (4) in $K_4^{(l)}$ (: the pro-l completion of K_4). Then there exists λ such that the pair (λ,f) satisfies (5) and (6). Actually this λ is equal to $\pm(24c_2(f)+1)^{\frac{1}{2}}$.*

COROLLARY 13. *Let $f=f(x,y)$ be an element of F_2^{nil} satisfying (4) in $K_4^{\mathrm{nil}}=\prod_l K_4^{(l)}$. Then there exists λ such that the pair (λ,f) satisfies (5) and (6). Actually this λ is equal to $\pm(24c_2(f)+1)^{\frac{1}{2}}$.*

It should be noted that though λ might lie on a quadratic extension equation (5) makes sense for such (λ,f). In the pro-unipotent context taking a quadratic extension is necessary. The Drinfel'd associator $\Phi_{KZ}\in\mathbf{R}\langle\langle X,Y\rangle\rangle$ satisfies (2) and (3) with $\mu=\pm2\pi\sqrt{-1}\notin\mathbf{R}^\times$. In the pro-$l$ context the author thinks that it might also happen $\pm(24c_2(f)+1)^{\frac{1}{2}}\notin\mathbf{Z}_l^\times$.

We have a group theoretical definition of $c_2(f)$ (cf. [LS97, Lemma 9]): Let $F_2^{(l)}(1) := [F_2^{(l)}, F_2^{(l)}]$ and $F_2^{(l)}(2) := [F_2^{(l)}(1), F_2^{(l)}(1)]$ where $[\cdot,\cdot]$ means the topological commutator. The quotient group $F_2^{(l)}(1)/F_2^{(l)}(2)$ is cyclic generated by the commutator (x, y). For $f \in F_2^{(l)}(1)$, $c_2(f) \in \mathbf{Z}_l$ is defined by $(x, y)^{c_2(f)} \equiv f$ in this quotient. Posing the following question on a pro-finite group analogue of Theorem 2 might be particularly interesting:

Question 14. Let $f = f(x, y)$ be an element of the pro-finite completion $\widehat{F}_2$ satisfying (4) (hence (6)) in the pro-finite completion $\widehat{K}_4$. Let $c_2(f)$ be an element in $\widehat{\mathbf{Z}}$ defined in a similar way to the above. Assume that there exists λ in $\widehat{\mathbf{Z}}$ such that $\lambda^2 = 24c_2(f) + 1$. Then does the pair (λ, f) satisfy (5)?

Remark 15. Although the pentagon equation (4) implies the two hexagon equations (5) and (6) of GT, it does not mean that the pentagon axiom [Dri90, (1.7)] implies two hexagon axioms, [Dri90, (1.9a) and (1.9b)], of braided tensor categories. The pentagon equation (4) of GT is a consequence of the three axioms of braided tensor categories. GT is interpreted as a group of deformations of braided tensor categories by Drinfel'd in [Dri90, §4]. Equation (4) of GT is read as a condition to keep the pentagon axiom. However it is formulated in terms of the braid group K_4, where its generators x_{ij}'s are subject to the braid relations. In his interpretation the relations are guaranteed by the dodecagon diagram (the Yang-Baxter equation) (see [JS93, Prop. 2.1] and [Kas95, Th. XIII.1.3]) which is deduced from two hexagon axioms.

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E-mail address: furusho@math.nagoya-u.ac.jp
GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA 464-8602, JAPAN