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# Nilpotency, almost nonnegative curvature, and the gradient flow on Alexandrov spaces

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**Abstract**

We show that almost nonnegatively curved $m$-manifolds are, up to finite cover, nilpotent spaces in the sense of homotopy theory and have $C(m)$-nilpotent fundamental groups. We also show that up to a finite cover almost nonnegatively curved manifolds are fiber bundles with simply connected fibers over nilmanifolds.

## 1. **Introduction**

Almost nonnegatively curved manifolds were introduced by Gromov in the late 70s [Gro80], with the most significant contributions to their study made by Yamaguchi in [Yam91] and Fukaya and Yamaguchi in [FY92]. Building on their ideas, in the present article we establish several new properties of these manifolds that yield, in particular, new topological obstructions to almost nonnegative curvature. Our techniques also provide simplified proofs of many results from [FY92].

A closed smooth manifold is said to be almost nonnegatively curved if it can Gromov-Hausdorff converge to a single point under a lower curvature bound. By rescaling, this definition is equivalent to the following one, which we will employ throughout this article.

*Definition* 1.0.1. A closed smooth manifold $M$ is called almost nonnegatively curved if it admits a sequence of Riemannian metrics $\{g_n\}_{n\in\mathbb{N}}$ whose sectional curvatures and diameters satisfy

$$\sec(M, g_n) \geqslant -1/n \quad \text{and} \quad \operatorname{diam}(M, g_n) \leqslant 1/n.$$

Almost nonnegatively curved manifolds generalize almost flat as well as nonnegatively curved manifolds. One main source of examples comes from a theorem

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of Fukaya and Yamaguchi. It states that if $F \to E \to B$ is a fiber bundle over an almost nonnegatively curved manifold $B$ whose fiber $F$ is compact and admits a nonnegatively curved metric that is invariant under the structure group, then the total space $E$ is almost nonnegatively curved [FY92]. Further examples are given by closed manifolds that admit cohomogeneity one actions of compact Lie groups (compare [ST04]).

In this work we combine collapsing techniques with a nonsmooth analogue of the gradient flow of concave functions on Alexandrov spaces. This notion is based on the construction of gradient curves of $\lambda$-concave functions used in [PP96] and bears many similarities to the Sharafutdinov retraction [Sha78]. In more general settings, the gradient flow was studied in [Lyt05], [Oht09] and [Sav07]. The gradient flow on Alexandrov spaces plays a key role in the proofs of two of the three main results in this paper, and we believe that it should also prove useful for dealing with other problems related to collapsing under a lower curvature bound.

1.1. To put the main theorems of the present work into perspective, let us first briefly recall some previously known results:

Let $M = M^m$ be an almost nonnegatively curved $m$-manifold.

- Gromov proved in [Gro78] that the minimal number of generators of the fundamental group $\pi_1(M)$ of $M$ can be estimated by a constant $C_1(m)$ depending only on $m$, and in [Gro81] that the sum of Betti numbers of $M$ with respect to any field of coefficients does not exceed some uniform constant $C_2 = C_2(m)$.
- Yamaguchi showed that, up to a finite cover, $M$ fibers over a flat $b_1(M;\mathbb{R})$-dimensional torus and $M^m$ is diffeomorphic to a torus if $b_1(M;\mathbb{R}) = m$ [Yam91].
- Fukaya and Yamaguchi proved that $\pi_1(M)$ is almost nilpotent, i.e., contains a nilpotent subgroup of finite index, and also that $\pi_1(M)$ is $C_3(m)$-solvable, i.e., contains a solvable subgroup of index at most $C_3(m)$ [FY92].
- If a closed manifold has negative Yamabe constant, then it cannot volume collapse with scalar curvature bounded from below; see [Sch89], [LeB01]. In particular, no such manifold can be almost nonnegatively curved.
- The $\hat{A}$-genus of a closed spin manifold $X$ of almost nonnegative Ricci curvature satisfies the inequality $\hat{A}(X) \leqslant 2^{\dim(X)/2}$; see [Gro82], [Gal83].

Let us now state the main results of this article.

1.2. Our first result concerns the hitherto unexplored relation between curvature bounds and the actions of the fundamental group on the higher homotopy groups.

An action by automorphisms of a group $G$ on an abelian group $V$ is called nilpotent if $V$ admits a finite sequence of $G$-invariant subgroups

$$V = V_0 \supset V_1 \supset \cdots \supset V_k = 0$$

such that the induced action of $G$ on $V_i / V_{i+1}$ is trivial for any $i$. A connected CW-complex $X$ is called *nilpotent* if $\pi_1(X)$ is a nilpotent group that operates nilpotently on $\pi_k(X)$ for every $k \geqslant 2$.

Nilpotent spaces play an important role in topology since they enjoy some of the best homotopy-theoretic properties of simply connected spaces, like a Whitehead theorem or reasonable Postnikov towers. Furthermore, unlike the category of simply connected spaces, the category of nilpotent ones is closed under many constructions such as the based loop space functor or the formation of function spaces, and group-theoretic functors, like localization and completion, have topological extensions in this category.

THEOREM A (Nilpotency Theorem). *Let $M$ be a closed almost nonnegatively curved manifold. Then a finite cover of $M$ is a nilpotent space.*

It would be interesting to know whether the order of this covering can be estimated solely in terms of the dimension of $M$.

*Example* 1.2.1. Let $h : S^3 \times S^3 \to S^3 \times S^3$ be defined by

$$h : (x, y) \mapsto (xy, yxy).$$

This map is a diffeomorphism with inverse given by

$$h^{-1} : (u, v) \mapsto (u^2 v^{-1}, vu^{-1}).$$

The induced map $h_*$ on $\pi_3(S^3 \times S^3)$ is given by the matrix $A_h = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Notice that the eigenvalues of $A_h$ are different from 1 in absolute value. Let $M$ be the mapping cylinder of $h$. Clearly, $M$ has the structure of a fiber bundle $S^3 \times S^3 \to M \to S^1$, and the action of $\pi_1(M) \cong \mathbb{Z}$ on $\pi_3(M) \cong \mathbb{Z}^2$ is generated by $A_h$. In particular, $M$ is not a nilpotent space and hence, by Theorem A, it does not admit almost nonnegative curvature. This fact does not follow from any previously known results.

1.3. Our next main result provides an affirmative answer to a conjecture of Fukaya and Yamaguchi [FY92, Conj. 0.15].

THEOREM B ($C$-Nilpotency Theorem for $\pi_1$). *Let $M$ be an almost nonnegatively curved $m$-manifold. Then $\pi_1(M)$ is $C(m)$-nilpotent*, i.e., *$\pi_1(M)$ contains a nilpotent subgroup of index at most $C(m)$.*

Notice that Theorem B is new even for manifolds of nonnegative curvature.

*Example* 1.3.1. For any $C > 0$ there exist prime numbers $p > q > C$ and a finite group $G_{pq}$ of order $pq$ that is solvable but not nilpotent. In particular, $G_{pq}$ does not contain any nilpotent subgroup of index less than or equal to $C$.

Whereas none of the results mentioned so far excludes $G_{pq}$ from being the fundamental group of some almost nonnegatively curved $m$-manifold, Theorem B shows that for $C > C(m)$ none of the groups $G_{pq}$ can be realized as the fundamental group of such a manifold.

1.4. In [FY92], Fukaya and Yamaguchi also conjectured that a finite cover of an almost nonnegatively Ricci curved manifold $M$ fibers over a nilmanifold with a fiber that has nonnegative Ricci curvature and whose fundamental group is finite. This conjecture was later refuted by Anderson [And92].

It is, on the other hand, very natural to consider this conjecture in the context of almost nonnegative *sectional* curvature. In fact, here Yamaguchi's fibration theorem (see [Yam91]) and the results of [FY92] easily imply that a finite cover of an almost nonnegatively curved manifold admits a map onto a nilmanifold whose homotopy fiber is a simply connected closed manifold.

From mere topology, it is, however, not clear whether this homotopy fibration can actually always be made into a genuine fiber bundle. Our next result shows that this is indeed true, and that for manifolds of almost nonnegative sectional curvature Fukaya and Yamaguchi's original conjecture essentially does hold.

THEOREM C (Fibration Theorem). *Let $M$ be an almost nonnegatively curved manifold. Then a finite cover $\widetilde{M}$ of $M$ is the total space of a fiber bundle*

$$F \to \widetilde{M} \to N$$

*over a nilmanifold $N$ with a simply connected fiber $F$. Moreover, the fiber $F$ is almost nonnegatively curved in the sense of the following definition.*

*Definition* 1.4.1. A closed smooth manifold $M$ is called almost nonnegatively curved in the generalized sense if for some nonnegative integer $k$ there exists a sequence of complete Riemannian metrics $g_n$ on $M \times \mathbb{R}^k$ and points $p_n \in M \times \mathbb{R}^k$ such that

(1) the sectional curvatures of the metric balls of radius $n$ around $p_n$ satisfy

$$\sec(B_n(p_n)) \geqslant -1/n;$$

(2) for $n \to \infty$ the pointed Riemannian manifolds $((M \times \mathbb{R}^k, g_n), p_n)$ converge in the pointed Gromov-Hausdorff distance to $(\mathbb{R}^k, 0)$;

(3) the regular fibers over 0 are diffeomorphic to $M$ for all large $n$.

Due to Yamaguchi's fibration theorem [Yam91], manifolds that are almost nonnegatively curved in the generalized sense play the same central role in collapsing under a lower curvature bound as almost flat manifolds do in the Cheeger-Fukaya-Gromov theory of collapsing with bounded curvature; see [CFG92].

It is not yet known whether all manifolds that are almost nonnegatively curved in the generalized sense are almost nonnegatively curved. Clearly if $k = 0$, this definition reduces to the standard one. Moreover, it is easy to see that all results of the present article, as well as all results about almost nonnegatively curved manifolds mentioned earlier (except possibly for the ones concerning the $\hat{A}$-genus and Yamabe constant), hold for manifolds that are almost nonnegatively curved in the sense of Definition 1.4.1.

1.5. Let us now describe the structure of the rest of this article.

In Section 2, after providing some necessary background from Alexandrov geometry, we introduce the gradient flow of the square of a distance function. It serves as one of the main technical tools in the proofs of Theorem A and Theorem B.

In Section 3, we prove Theorem A by a direct application of the gradient flow technique.

In Section 4, we prove Theorem B. The proof is also based on the gradient flow, but is more involved and employs further technical tools such as "limit fundamental groups" of Alexandrov spaces.

In Section 5, we prove Theorem C. This section is completely independent from the rest of the article.

In Section 6, we discuss some further open questions related to our results.

## 2. **Alexandrov geometry and the gradient flow**

This section provides necessary background in Alexandrov geometry. The results of Sections 2.1–2.3 are mostly duplicated from [PP96], [Pet95] and [Pet07]. See [BGP92] for a general reference on Alexandrov spaces.

2.1. *$\lambda$-concave functions.*

*Definition* 2.1.1 (for a space without boundary). Let $A$ be an Alexandrov space without boundary. A Lipschitz function $f\colon A \to \mathbb{R}$ is called *$\lambda$-concave* if for any unit speed minimizing geodesic $\gamma$ in $A$, the function

$$f \circ \gamma(t) - \lambda t^2/2 \quad \text{is concave.}$$

If $A$ is an Alexandrov space with boundary, then its double $\tilde{A}$ is also an Alexandrov space; see [Per91, §5.2]. Let $\mathrm{p} : \tilde{A} \to A$ be the canonical map. Given a function $f$ on $A$, set $\tilde{f} = f \circ \mathrm{p}$.

*Definition* 2.1.2 (for a space with boundary). Let $A$ be an Alexandrov space with boundary. A Lipschitz function $f : A \to \mathbb{R}$ is called $\lambda$-*concave* if for any unit speed minimizing geodesic $\gamma$ in $\tilde{A}$, the function

$$\tilde{f} \circ \gamma(t) - \lambda t^2/2 \quad \text{is concave.}$$

*Remark* 2.1.3. Notice that the restriction of a linear function on $\mathbb{R}^n$ to a ball is not 0-concave in this sense.

*Remark* 2.1.4. In the above definitions, the Lipschitz condition is only technical. With some extra work, all results of this section can be extended to continuous functions.

2.2. *Tangent cone and differential.* Given a point $p$ in an Alexandrov space $A$, we denote by $T_p = T_p(A)$ the tangent cone at $p$.

If $d$ denotes the metric of an Alexandrov space $A$, let us denote by $\lambda A$ the space $(A, \lambda d)$. Let $i_\lambda : \lambda A \to A$ be the canonical map. The limit of $(\lambda A, p)$ for $\lambda \to \infty$ is the tangent cone $T_p$ at $p$; see [BGP92, §7.8.1].

*Definition* 2.2.1. For any function $f : A \to \mathbb{R}$ the function $d_p f : T_p \to \mathbb{R}$ such that

$$d_p f = \lim_{\lambda \to \infty} \lambda(f \circ i_\lambda - f(p))$$

is called the differential of $f$ at $p$.

It is easy to see that for a $\lambda$-concave function $f$, the differential $d_p f$ is defined everywhere, and that $d_p f$ is a 0-concave function on the tangent cone $T_p$.

*Definition* 2.2.2. Given a $\lambda$-concave function $f : A \to \mathbb{R}$, a point $p \in A$ is called critical point of $f$ if $d_p f \leqslant 0$.

2.3. *Gradient curves.* With a slight abuse of notation we will call elements of the tangent cone $T_p$ the "tangent vectors" at $p$. The origin of $T_p$ plays the role of the zero vector and is denoted by $o = o_p$. For a tangent vector $v$ at $p$ we define its absolute value $|v|$ as the distance $|ov|$ in $T_p$. For two tangent vectors $u$ and $v$ at $p$ we can define their "scalar product"

$$\langle u, v \rangle = (|u|^2 + |v|^2 - |uv|^2)/2 = |u| \cdot |v| \cos \alpha,$$

where $\alpha = \angle uov$ in $T_p$.

For two points $p, q \in A$ we define $\log_p q$ to be a tangent vector $v$ at $p$ such that $|v| = |pq|$ and such that the direction of $v$ coincides with a direction from $p$ to $q$ (if such a direction is not unique, we choose any one of them). Given a curve $\gamma(t)$ in $A$, we denote by $\gamma^+(t)$ the right and by $\gamma^-(t)$ the left tangent vectors to $\gamma(t)$, where, respectively,

$$\gamma^\pm(t) \in T_{\gamma(t)} \quad \text{and} \quad \gamma^\pm(t) = \lim_{\varepsilon \to +0} \frac{\log_{\gamma(t)} \gamma(t \pm \varepsilon)}{\varepsilon}.$$

For a real function $f(t)$ with $t \in \mathbb{R}$ we denote by $f^+(t)$ its right derivative and by $-f^-(t)$ its left derivative. Note that our sign convention (which is chosen to agree with the notion of right and left derivatives of curves) is not quite standard. For example,

$$\text{if } f(t) = t, \quad \text{then} \quad f^+(t) \equiv 1 \quad \text{and} \quad f^-(t) \equiv -1.$$

*Definition* 2.3.1. For $f$ a $\lambda$-concave function on $A$, a vector $g \in T_p(A)$ is called a gradient of $f$ at $p \in A$ (in short: $g = \nabla_p f$) if

(i) $d_p f(x) \leqslant \langle g, x\rangle$ for any $x \in T_p$, and

(ii) $d_p f(g) = \langle g, g\rangle$.

It is easy to see that any $\lambda$-concave function has a uniquely defined gradient vector field. Moreover, if $d_p f(x) \leqslant 0$ for all $x \in T_p$, then $\nabla_p f = o$ (here $o$ denotes the origin of the tangent cone $T_p$); otherwise

$$\nabla_p f = d_p f(\xi)\xi,$$

where $\xi$ is the (necessarily unique) unit vector for which the function $d_p f$ attains its maximum.

Moreover, for any minimizing geodesic $\gamma : [a, b] \to U$ parametrized by arc length, the following inequality holds:

$$\langle \gamma^+(a), \nabla_{\gamma(a)} f\rangle + \langle \gamma^-(b), \nabla_{\gamma(b)} f\rangle \geqslant -\lambda(b-a). \tag{2.3.1}$$

Indeed,

$$\begin{aligned}\langle \gamma^+(a), \nabla_{\gamma(a)} f\rangle + \langle \gamma^-(b), \nabla_{\gamma(b)} f\rangle &\geqslant d_{\gamma(a)} f(\gamma^+(a)) + d_{\gamma(b)} f(\gamma^-(b)) \\ &= (f \circ \gamma)^+|_a + (f \circ \gamma)^-|_b \geqslant -\lambda(b-a).\end{aligned}$$

*Definition* 2.3.2. A curve $\alpha \colon [a, b] \to A$ is called an *$f$-gradient curve* if

$$\alpha^+(t) = \nabla_{\alpha(t)} f \quad \text{for any } t \in [a, b].$$

PROPOSITION 2.3.3. *Given a $\lambda$-concave function $f \colon A \to \mathbb{R}$ and a point $p \in A$, there is a unique gradient curve $\alpha \colon [0, \infty) \to A$ such that $\alpha(0) = p$.*

*Moreover, if $\alpha$ and $\beta$ are two $f$-gradient curves, then*

$$|\alpha(t_1)\beta(t_1)| \leqslant |\alpha(t_0)\beta(t_0)| \exp(\lambda(t_1 - t_0)) \quad \textit{for all } t_1 \geqslant t_0.$$

The gradient curve can be constructed as a limit of broken geodesics, made up of short segments with directions close to the gradient. The convergence, uniqueness, as well as the last inequality in Proposition 2.3.3 follow from inequality (2.3.1) above, while Corollary 2.3.5 below guarantees that the limit is indeed a gradient curve, having a unique right tangent vector at each point.

LEMMA 2.3.4. *Let $A_n \xrightarrow{\text{GH}} A$ be a sequence of Alexandrov spaces with curvature $\geqslant k$ which Gromov-Hausdorff converges to an Alexandrov space $A$.*

*Let $f_n \to f$, where $f_n : A_n \to \mathbb{R}$ is a sequence of $\lambda$-concave functions converging to $f : A \to \mathbb{R}$. Let $p_n \to p$, where $p_n \in A_n$ and $p \in A$. Then*

$$|\nabla_p f| \leqslant \liminf_{n\to\infty} |\nabla_{p_n} f_n|.$$

COROLLARY 2.3.5. *Given a $\lambda$-concave function $f$ on $A$ and a sequence of points $p_n \in A$ such that $p_n \to p$, we have*

$$|\nabla_p f| \leqslant \liminf_{n\to\infty} |\nabla_{p_n} f|.$$

*Proof of Lemma* 2.3.4. Fix an $\varepsilon > 0$ and choose $q$ near $p$ such that

$$(f(q) - f(p))/|pq| > |\nabla_p f| - \varepsilon.$$

Now choose $q_n \in A_n$ such that $q_n \to q$. If $|pq|$ is sufficiently small and $n$ is sufficiently large, the $\lambda$-concavity of $f_n$ then implies that

$$\liminf_{n\to\infty} \frac{d_{p_n} f_n(v_n)}{|v_n|} \geqslant |\nabla_p f| - 2\varepsilon \quad \text{for } v_n = \log_{p_n}(q_n) \in T_{p_n}(A_n).$$

Therefore,

$$\liminf_{n\to\infty} |\nabla_{p_n} f_n| \geqslant |\nabla_p f| - 2\varepsilon \quad \text{for any } \varepsilon > 0,$$

i.e.,

$$\liminf_{n\to\infty} |\nabla_{p_n} f_n| \geqslant |\nabla_p f|. \qquad \square$$

LEMMA 2.3.6. *Let $f$ be a $\lambda$-concave function, $\lambda \geqslant 0$ and $\alpha(t)$ be an $f$-gradient curve, and let $\bar\alpha(s)$ be its reparametrization by arc length. Then $f \circ \bar\alpha$ is $\lambda$-concave.*

*Proof.*

$$\begin{aligned}(f\circ\bar\alpha)^+(s_0) = |\nabla_{\bar\alpha(s_0)} f| &\geqslant \frac{d_{\bar\alpha(s_0)} f(\log_{\bar\alpha(s_0)}(\bar\alpha(s_1))}{|\bar\alpha(s_1)\,\bar\alpha(s_0)|} \\ &\geqslant \frac{f(\bar\alpha(s_1)) - f(\bar\alpha(s_0)) - \lambda|\bar\alpha(s_1)\bar\alpha(s_0)|^2/2}{|\bar\alpha(s_1)\bar\alpha(s_0)|} \\ &\geqslant \frac{f(\bar\alpha(s_1)) - f(\bar\alpha(s_0))}{s_1 - s_0} - \lambda|s_1 - s_0|/2.\end{aligned}$$

Since $|\bar\alpha(s_1)\,\bar\alpha(s_0)|/(s_1 - s_0)$ goes to 1 as $s_1$ goes to $s_0+$, it follows that $f \circ \bar\alpha$ is $\lambda$-concave. $\square$

2.4. *The gradient flow on Alexandrov spaces.* Let $f$ be a $\lambda$-concave function on an Alexandrov space $A$. Consider the map $\Phi_f^T : A \to A$ defined as follows: $\Phi_f^T(x) = \alpha_x(T)$, where $\alpha_x : [0,\infty) \to A$ is the $f$-gradient curve with $\alpha_x(0) = x$. The map $\Phi_f^T$ is called the *$f$-gradient flow at time $T$*. From Proposition 2.3.3, it is clear that $\Phi_f^T$ is an $\exp(\lambda T)$-Lipschitz map. Next we want to prove that this map behaves nicely under Gromov-Hausdorff-convergence.

THEOREM 2.4.1. *Let $A_n \xrightarrow{\text{GH}} A$ be a sequence of Alexandrov spaces with curvature $\geqslant k$ that converges to an Alexandrov space $A$.*

*Let $f_n \to f$, where $f_n : A_n \to \mathbb{R}$ is a sequence of $\lambda$-concave functions and $f : A \to \mathbb{R}$.*

*Then $\Phi_{f_n}^T \to \Phi_f^T$.*

Theorem 2.4.1 immediately follows from the following lemma:

LEMMA 2.4.2. *Let $A_n \xrightarrow{\text{GH}} A$ be a sequence of Alexandrov spaces with curvature $\geqslant k$ that converges to an Alexandrov space $A$.*

*Let $f_n \to f$, where $f_n : A_n \to \mathbb{R}$ is a sequence of $\lambda$-concave functions and $f : A \to \mathbb{R}$. Let $\alpha_n : [0,\infty) \to A_n$ be the sequence of $f_n$-gradient curves with $\alpha_n(0) = p_n$, and let $\alpha : [0,\infty) \to A$ be the $f$-gradient curve with $\alpha(0) = p$.*

*Then $\alpha_n \to \alpha$.*

*Proof.* We may assume without loss of generality that $f$ has no critical points. (Otherwise consider instead the sequence $A_n' = A_n \times \mathbb{R}$ with $f_n'(a \times x) = f_n(a) + x$.)

Let $\bar\alpha_n(s)$ denote the reparametrization of $\alpha_n(t)$ by arc length. Since all $\bar\alpha_n$ are 1-Lipschitz, we can choose a converging subsequence from any subsequence of $\bar\alpha_n$. Let $\bar\beta : [0,\infty) \to A$ be its limit.

Clearly, $\bar\beta$ is also 1-Lipschitz and hence $|\bar\beta^+| \leqslant 1$. Hence by Lemma 2.3.4,

$$\begin{aligned}\lim_{n\to\infty} f_n \circ \bar\alpha_n|_a^b &= \lim_{n\to\infty} \int_a^b |\nabla_{\bar\alpha_n(s)} f_n| ds \\ &\geqslant \int_a^b |\nabla_{\bar\beta(s)} f| \geqslant \int_a^b d_{\beta(t)} f(\beta^+(t)) = f \circ \beta|_a^b.\end{aligned}$$

On the other hand, since $\bar\alpha_n \to \bar\beta$ and $f_n \to f$, we have $f_n \circ \bar\alpha_n|_a^b \to f \circ \bar\beta|_a^b$. Therefore, in both of these inequalities in fact equality holds.

Hence, $|\nabla_{\bar\beta(s)} f| = \lim_{n\to\infty} |\nabla_{\bar\alpha_n(s)} f_n|$, $|\bar\beta^+(s)| = 1$ and the directions of $\bar\beta^+(s)$ and $\nabla_{\bar\beta(s)} f$ coincide almost everywhere. This implies that $\bar\beta(s)$ is a gradient curve reparametrized by arc length. In other words, if $\bar\alpha(s)$ denotes the reparametrization of $\alpha(t)$ by arc length, then $\bar\beta(s) = \bar\alpha(s)$ for all $s$. It only remains to show that the original parameter $t_n(s)$ of $\alpha_n$ converges to the original parameter $t(s)$ of $\alpha$.

Notice that $|\nabla_{\bar\alpha_n(s)} f_n| dt_n = ds$ or $dt_n/ds = ds/d(f_n \circ \bar\alpha_n)$. Likewise, $dt/ds = ds/d(f \circ \bar\alpha)$. Then the convergence $t_n \to t$ follows from the $\lambda$-concavity of $f_n \circ \bar\alpha_n$ (see Lemma 2.3.6) and the convergence $f_n \circ \bar\alpha_n \to f \circ \bar\alpha$. □

2.5. *Gradient balls.* Let $A$ be an Alexandrov space, and let $S \subset A$ be a subset of $A$. A function $f : A \to \mathbb{R}$ that can be represented as

$$f = \sum_i \theta_i \tfrac{1}{2} \operatorname{dist}^2_{a_i}, \quad \text{with} \quad \theta_i \geqslant 0, \ \sum_i \theta_i = 1 \ \text{ and } \ a_i \in S$$

will be called a *cocos-function with respect to* $S$ (where "cocos" stands for **co**nvex **co**mbination of **s**quares of distance functions). A broken gradient curve for a collection of such functions will be called cocos-curve with respect to $S$.

For $p \in A$ and $T, r \in \mathbb{R}_+$, let us define $\beta^r_T(p)$, the gradient ball with center $p$ and radius $T$ with respect to $B_r(p)$, as the set of all end points of cocos-curves with respect to $B_r(p)$ that start at $p$ with total time $\leqslant T$.

LEMMA 2.5.1. (I) *There exists a* $T = T(m) \in \mathbb{R}_+$ *such that for any* $m$*-dimensional Alexandrov space* $A$ *with curvature* $\geqslant -1$ *and any* $q \in A$ *there is a point* $p \in A$ *such that* (i) $|pq| \leqslant 1$ *and* (ii) $B_1(p) \subset \beta^1_T(p)$.

(II) *There exists a* $T' = T'(m) \in \mathbb{R}_+$ *such that the following holds. Let* $A$ *be an Alexandrov space that is a quotient* $A = \tilde{A}/\Gamma$ *of an* $m$*-dimensional Alexandrov space* $\tilde{A}$ *with curvature* $\geqslant -1$ *by a discrete action of a group of isometries* $\Gamma$. *Let* $q \in A$ *and* $p = p(q) \in A$ *be as in part* (I) *above.*

*Then for any lift* $\tilde{p} \in \tilde{A}$ *of* $p$, *one has that* $B_1(\tilde{p}) \subset \beta^1_{T'}(\tilde{p})$.

*Proof.* The proof is similar to the construction of a strained point in an Alexandrov space; see [BGP92].

Set $\delta = 10^{-m}$. Take $a_1 = q$, and take $b_1$ to be a farthest point from $a_1$ in the closed ball $\overline{B}_1(a_1)$. Take $a_2$ to be a midpoint of $a_1 b_1$, and let $b_2$ be a farthest point from $a_2$ such that $|a_1 b_2| = |a_1 a_2|$ and $|a_2 b_2| \leqslant \delta |a_1 b_1|$, etc. On the $k$-th step we have to take $a_k$ to be a midpoint of $a_{k-1} b_{k-1}$ and $b_k$ to be a farthest point from $a_k$ such that $|a_i b_k| = |a_i a_k|$ for all $i < k$ and $|a_k b_k| \leqslant \delta |a_{k-1} b_{k-1}|$.

After $m$ steps, take $p$ to be a midpoint of $a_m b_m$. We only have to check that we can find a $T = T(m)$ such that $\beta^1_T(p) \supset B_1(p)$.

Let $t_i$ be the minimal time such that $B_{|a_i b_i|/\delta^m}(p) \subset \beta^1_{t_i}(p)$. Then one can take $T = t_1$. Therefore it is enough to give estimates for $t_m$ and $t_{k-1}/t_k$ only in terms of $\delta$ and $m$. Looking at the ends of broken gradient curves starting at $p$ for the functions $\operatorname{dist}^2_p/2$, $\operatorname{dist}^2_{a_i}/2$ and $\operatorname{dist}^2_{b_i}/2$, we easily see that $t_n \leqslant 1/\delta^m$. Now, looking at the ends of broken gradient curves starting at $B_{|a_{k-1} b_{k-1}|/\delta^m}(p)$ for the functions $\operatorname{dist}^2_p/2$, $\operatorname{dist}^2_{a_i}/2$ and $\operatorname{dist}^2_{b_i}/2$, we have that $t_{k-1}/t_k \leqslant 1/\delta^m$. Therefore $t_1 \leqslant 1/\delta^{m2} = 10^{-m3}$. This finishes the proof of part (I).

For part (II), notice that

(a) for any $r, t > 0$ we have $\beta^r_t(p) \subset B_{re^t}(p)$;

(b) if $\beta^r_t(p) \supset B_\rho(p)$, then $\beta^r_{t+\tau}(p) \supset B_{\rho e^\tau}(p)$;

(c) if $\rho = |px|$ and $x \in \beta^{r+\rho}_t(p)$, then $\beta^r_\tau(x) \subset \beta^{r+\rho}_{t+\tau}(p)$.

Take $\varepsilon = e^{-T}/4$ and apply part (I) of the lemma to $\tilde{A}/\varepsilon$ to find a point $p' \in \tilde{A}$ such that $|\tilde{p}p'| \leqslant \varepsilon$ and $B_\varepsilon(p') \subset \beta^\varepsilon_T(p') \subset \tilde{A}$. Then for some deck transformation $\gamma$, we have $\gamma p' \in \beta^\varepsilon_T(p) \subset B_{\varepsilon e^T}(p)$. Therefore $\gamma p' \in B_{1/2}(\tilde{p})$. Hence, taking $T' = 2T + 1/\varepsilon = 2T + 4e^T$, we obtain

$$\beta^1_{T'}(p) \supset \beta^\varepsilon_{T+1/\varepsilon}(\gamma p') \supset B_1(\tilde{p}). \qquad \square$$

2.6. *Short basis.* We will use the following construction due to Gromov.

Given an Alexandrov space $A$ with a marked point $p \in A$, and a group $\Gamma$ acting discretely on $A = (A, d)$, one can define a short basis of the action of $\Gamma$ at $p$ as follows:

For $\gamma \in \Gamma$, define the norm of $\gamma$ by the formula $|\gamma| = d(p, \gamma(p))$. Choose $\gamma_1 \in \Gamma$ with the minimal norm in $\Gamma$. Then choose $\gamma_2$ to have minimal norm in $\Gamma \backslash \langle \gamma_1 \rangle$. On the $n$-th step, choose $\gamma_n$ to have minimal norm in $\Gamma \backslash \langle \gamma_1, \gamma_2, \dots, \gamma_{n-1} \rangle$. The sequence $\{\gamma_1, \gamma_2, \dots\}$ is called a *short basis* of $\Gamma$ at $p$. In general, the number of elements of a short basis can be finite or infinite. In the special case of the action of the fundamental group $\pi_1(A, p)$ on the universal cover of $A$, one speaks of the short basis of $\pi_1(A, p)$.

It is easy to see that for a short basis $\{\gamma_1, \gamma_2, \dots\}$ of the fundamental group of an Alexandrov space $A$ the following is true:

(1) If $A$ has diameter $d$, then $|\gamma_i| \leqslant 2d$.

(2) If $A$ is compact, then $\{\gamma_i\}$ is finite.

(3) For any $i > j$, we have $|\gamma_i| \leqslant |\gamma_j^{-1}\gamma_i|$.

The third property implies that if $\tilde{p} \in \tilde{A}$ is in the preimage of $p$ in the universal cover $\tilde{A}$ of $A$ and $\tilde{p}_i = \gamma_i(\tilde{p})$, then

$$|\tilde{p}_i \tilde{p}_j| \geqslant \max\{|\tilde{p}\tilde{p}_i|, |\tilde{p}\tilde{p}_j|\}.$$

As Gromov observed, if $A$ is an Alexandrov space with curvature $\geqslant \kappa$ and diameter $\leqslant d$, the last inequality implies that $\angle \tilde{p}_i \tilde{p} \tilde{p}_j > \delta = \delta(\kappa, d) > 0$. This yields an upper bound on the number of elements of a short basis in terms of $\kappa$, $d$ and the dimension of $A$.

## 3. **Nilpotency of almost nonnegatively curved manifolds**

In this section we prove Theorem A.

3.1. *Preliminary lemmas.* Suppose $M$ is an almost nonnegatively curved manifold. Let us denote by $M_n = (M, g_n)$ for $n \in \mathbb{N}$ a sequence of Riemannian metrics on $M$ such that $\sec(M_n) \geqslant -1/n$ and $\operatorname{diam}(M_n) \leqslant 1/n$. Let us denote by $\widetilde{M}$ the universal covering of $M$, and by $\widetilde{M}_n \to M_n$ the universal Riemannian covering of $M_n$ (i.e., $\widetilde{M}_n$ is $\widetilde{M}$ equipped with the pullback of the Riemannian metric $g_n$).

KEY LEMMA 3.1.1. *Given* $\varepsilon > 0$ *and* $r_2 > r_1 > 0$, *suppose that* $\widetilde{M}_n \supset B_{r_2}(p_n) \supset B_{r_1}(p_n)$. *Then, for n sufficiently large, there is a* $(1+\varepsilon)$*-Lipschitz map* $\Phi_n : B_{r_2}(p_n) \to B_{r_1}(p_n)$ *that is homotopic to the identity on* $B_{r_2}(p_n)$.

*Proof.* Fix $R \gg r_2$ (here $R > 1000(1 + 1/\varepsilon)r_2$ will suffice). Note that

$$B_R(p_n) \xrightarrow{\text{GH}} B_R \subset \mathbb{R}^q \quad \text{as } n \to \infty.$$

Choose a finite $R/1000$-net $\{a_i\}$ of $\partial B_R \subset \mathbb{R}^q$. Let $a_{i,n} \in M_n$ be sequences such that $a_{i,n} \to a_n$. Consider the sequence of functions $f_n : M_n \to \mathbb{R}$ with $f_n = \min_i \operatorname{dist}^2_{a_{i,n}}$.

For large $n$, the functions $f_n$ are 2-concave in $B_R(p_n)$, so that, in particular, the gradient flows $\Phi^T_{f_n}|_{B_{r_2}(p_n)}$ are $e^{2T}$-Lipschitz. Moreover, if $\xi_x$ denotes the starting vector of a unit speed shortest geodesic from $x$ to $p_n$, then for any $x \in B_{r_2}(p_n) \backslash B_{r_1}(p_n)$ we have $\langle \xi_x, \nabla f \rangle \geqslant R/2$. Therefore,

$$\Phi^T_{f_n}(B_{r_2}(p_n)) \subset B_{r_1}(p_n) \quad \text{if } T = 2r_2/R.$$

Thus $\Phi_n = \Phi^{2r_2/R}_{f_n}$ provides a $4r_2/R$-Lipschitz map $B_{r_2}(p_n) \to B_{r_1}(p_n)$, and it is $(1+\varepsilon)$-Lipschitz if one chooses $R$ sufficiently large. □

For $\gamma \in \pi_1(M)$, set $|\gamma|_n = |p\,\gamma(p)|_{\widetilde{M}_n}$; see Section 2.6.

COROLLARY 3.1.2. *Let* $M$ *be almost nonnegatively curved manifold. Let*

$$h : \pi_1(M) \to \operatorname{Aut}(H^*(\widetilde{M}, \mathbb{Z})/\operatorname{tor}$$

*be the natural action of* $\pi_1(M)$ *on* $H^*(\widetilde{M}, \mathbb{Z})$. *Then there is a sequence of norms* $\|*\|_n$ *on* $H^*(\widetilde{M}, \mathbb{Z})/\operatorname{tor}$ *such that the following holds. Given any* $\varepsilon > 0$, *there is* $n \in \mathbb{Z}_+$ *such that for any* $\gamma \in \pi_1(M)$ *with* $|\gamma|_n \leqslant 2\operatorname{diam}(M_n)$, *we have* $\|h(\gamma)\|_n \leqslant 1 + \varepsilon$.

*Proof.* [FY92, Th. 0.1] and Yamaguchi's fibration theorem [Yam91] imply that if $n$ is sufficiently large, for any fixed $r \in \mathbb{R}_+$ we have that for any $p_n \in \widetilde{M}_n$ the inclusion map $B_r(p_n) \to \widetilde{M}_n$ is a homotopy equivalence.

Let $\|*\|_{n,r}$ denote the $L_\infty$-norm on differential forms on $B_r(p_n) \subset \widetilde{M}_n$.

Fix $r_2 > r_1 > 0$. If $\omega$ is a differential form on $B_{r_1}(p_n) \subset M_n$ and $n$ is sufficiently large, Key Lemma 3.1.1 implies that

$$\|\Phi^*_n(\omega)\|_{n,r_2} \leqslant (1+\varepsilon)\|\omega\|_{n,r_1} \quad \text{and} \quad 2\operatorname{diam}(M_n) \leqslant r_2 - r_1.$$

If now $\omega$ is a form on $B_{r_2}(p_n) \in \widetilde{M}_n$ and $\gamma \in \pi_1(M)$ is such that

$$|\gamma|_n = |p_n\,\gamma(p_n)| \leqslant 2\operatorname{diam}(M_n) \leqslant r_2 - r_1,$$

then $B_{r_1}(p_n) \subset B_{r_2}(\gamma(p_n)) \subset \widetilde{M}_n$, whence

$$\|\Phi^*_n(\gamma^*(\omega))\|_{n,r_2} \leqslant (1+\varepsilon)\|\gamma^*(\omega)\|_{n,r_1} \leqslant (1+\varepsilon)\|\omega\|_{n,r_2}.$$

Thus, for the induced norms on the de Rham cohomology of $\widetilde{M}$ (and on its integral subspace $H^*(\widetilde{M}, \mathbb{Z})/\operatorname{tor})$), we have

$$\|[\gamma^*(\omega)]\|_{n,r_2} \leqslant (1+\varepsilon)\|[\omega]\|_{n,r_2}.$$

Therefore the sequence of norms $\|*\|_n = \|*\|_{n,r_2}$ satisfies the conditions of the corollary. $\square$

LEMMA 3.1.3. *There exists a constant $N = N(n,k) \in \mathbb{Z}_+$ such that the following holds. If $G$ is a subgroup of* $\mathrm{GL}(n,\mathbb{Z})$ *and $S$ is a set of generators of $G$ with $\#(S) \leqslant k$ such that the eigenvalues of each element of $S^N$ are all equal to* 1 *in absolute value, then the same is true for the eigenvalues of all elements of $G$.*

*Proof.* Let $B$ be the set of all matrices in $\mathrm{GL}(n,\mathbb{Z})$ whose eigenvalues are all equal to 1 in absolute value. Since the characteristic polynomials of such matrices are uniformly bounded and have integer coefficients, there are only finitely many of them. Let $\overline{B}$ be the Zariski closure of $B$ in the set of all real $n \times n$ matrices. By the above, all eigenvalues of elements of $\overline{B}$ also have absolute value 1.

Consider now the space $V = \mathbb{R}^{kn^2}$ of $k$-tuples of real $n \times n$ matrices.

Consider a collection of matrices $(M_1, M_2, \dots, M_k) \in V$, where $M_i \in \mathrm{GL}(n,\mathbb{R})$. Let $F_k$ be a free group on $k$ generators, generated by $S = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$, and let $h : F_k \to \mathrm{GL}(n,\mathbb{R})$ be the homomorphism defined by $h(\gamma_i) = M_i$. The property that $h(\gamma)$ is an element of $\overline{B}$ for any $\gamma \in F_k$ then describes an algebraic subset $A_\gamma \subset V$.

The intersection $A = \bigcap_{\gamma \in F_k} A_\gamma$ is also algebraic, and therefore there is a finite number $N = N(n,k)$ such that $A = \bigcap_{\gamma \in S^N} A_\gamma$ for $S^N \subset F_k$. $\square$

LEMMA 3.1.4. *Let $\Gamma$ be a subgroup of* $\mathrm{GL}(n,\mathbb{Z})$ *such that the eigenvalues of each element of $\Gamma$ are equal to* 1 *in absolute value. Then $\Gamma$ contains a subgroup $\Gamma'$ of finite index whose elements' eigenvalues are all equal to* 1.

*Proof.* Let $G$ denote the Zariski closure of $\Gamma$ in $\mathrm{GL}(n,\mathbb{R})$ . Then $G$, being an algebraic group, is a Lie group with finitely many components. Let $G_\circ$ be the identity component of $G$. By the same argument as in the proof of the previous lemma, the set of all characteristic polynomials of the elements of $G$ is finite. Therefore the characteristic polynomial of any element of $G_\circ$ is identically equal to $(x-1)^n$.

Thus the subgroup $\Gamma' = \Gamma \cap G_\circ$ satisfies all conditions of the lemma. $\square$

*Remark* 3.1.5. As was pointed out to us by Yu. Zarkhin, one can alternatively take $\Gamma'$ to be the kernel of the composition of the homomorphisms $\Gamma \to \mathrm{GL}(n,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{Z}/3\mathbb{Z})$. In this way one obtains a bound $[\Gamma : \Gamma'] \leqslant 3^{n^2}$. To see that $\Gamma'$ satisfies the conclusion of Lemma 3.1.4, one should notice that every element of $\Gamma$ is a quasi-unipotent matrix, since all its eigenvalues are roots of unity. The desired result then follows from the so-called Minkowski lemma. Apply, for instance, [SZ96, Th. 7.2] for $n = 3$ and $k = 1$ (so $R(1, 3) = 1$), where we take $\mathbb{O}$ to be the ring of $n \times n$ integer matrices.

3.2. *Proof of Theorem* A. Let $M$ be an almost nonnegatively curved manifold. As usual, denote by $M_n = (M, g_n)$ for $n \in \mathbb{N}$ a sequence of Riemannian metrics on $M$ such that $\sec(M_n) \geqslant -1/n$ and $\operatorname{diam}(M_n) \leqslant 1/n$, by $\widetilde{M}$ the universal covering of $M$, and by $\widetilde{M}_n \to M_n$ the universal Riemannian covering of $M_n$.

After passing to a finite cover of $M$, by [FY92] we may assume that $\pi_1(M)$ is nilpotent.

Fix $p \in M$, and let $\{\gamma_{i,n}\}$ be a short basis of $\pi_1(M_n, p)$; see Section 2.6. Then, if $n$ is sufficiently large, the short basis $\{\gamma_{i,n}\}$ has at most $k = k(\dim M)$ elements and its elements satisfy $|\gamma_{i,n}|_n \leqslant 2/n$ for every $i$. Also, Corollary 3.1.2 implies that given $\varepsilon > 0$, for all large $n$ and every $i$ we have $\|h(\gamma_{i,n})\|_n < 1 + \varepsilon$ and $\|h(\gamma_{i,n}^{-1})\|_n < 1 + \varepsilon$.

Take $N = N(k, m)$ as in Lemma 3.1.3. One can choose $\varepsilon > 0$ so small that if $p$ is a polynomial with integer coefficients whose roots have absolute values lying between $1/(1+\varepsilon)^N$ and $(1+\varepsilon)^N$, then all roots of $p$ have absolute values equal to 1. This follows from the fact that the total number of integer polynomials whose roots are all contained in a fixed bounded region is finite.

Set $S_n := \{\gamma_{i,n}\}$. Then for any $\gamma \in S_n^N$ we have

$$\|h(\gamma)\|_n < (1+\varepsilon)^N \quad \text{and} \quad \|h(\gamma^{-1})\|_n < (1+\varepsilon)^N.$$

Therefore the absolute values of all eigenvalues lie between $1/(1+\varepsilon)^N$ and $(1+\varepsilon)^N$. Since the characteristic polynomial of $h(\gamma)$ has integer coefficients, the absolute values of all the eigenvalues of $h(\gamma)$ are in fact equal to 1.

Apply now Lemma 3.1.3. It follows that the absolute values of all eigenvalues of $h(\gamma)$ are equal to 1 for any $\gamma \in \pi_1(M)$. Then Lemma 3.1.4 implies that after passing to a finite cover $M'$ of $M$, all eigenvalues of $h(\gamma)$ are equal to 1 for any $\gamma \in \pi_1(M')$. By Engel's theorem, one can choose an integral basis of $H^*(\widetilde{M}, \mathbb{R})$ such that the action of $\pi_1(M)$ on $H^*(\widetilde{M}, \mathbb{Z})/\operatorname{tor}$ is given by upper triangular matrices. Therefore, by passing to a finite cover $M''$ of $M'$, we can assume that the action of $\pi_1(M'')$ on $H^*(\widetilde{M}, \mathbb{Z})$ (and on $H_*(\widetilde{M}, \mathbb{Z})$) is nilpotent.

Recall (see, e.g., [HMR75, 2.19]) that a connected CW-complex with nilpotent fundamental group is nilpotent if and only if the action of its fundamental group on the homology of its universal cover is nilpotent. Thus $M''$ is a nilpotent space, which completes the proof of Theorem A. $\square$

## 4. C-nilpotency of the fundamental group

4.1. In this section we will prove Theorem B. It will follow from the following somewhat stronger result.

THEOREM 4.1.1. *For any integer $m$ there exist constants $\epsilon(m) > 0$ and $C(m) > 0$ such that the following holds. If $M^m$ is a closed, smooth $m$-manifold that admits a Riemannian metric $g$ with* $\sec(M^m, g) > -\epsilon(m)$ *and* $\operatorname{diam}(M^m, g) < 1$, *then the fundamental group of $M^m$ is $C(m)$-nilpotent, i.e., $\pi_1(M^m)$ contains a nilpotent subgroup of index $\leqslant C(m)$.*

*Remark* 4.1.2. The proofs of Theorems A and C show that corresponding versions of those results also hold when these theorems are reformulated in a fashion similar to Theorem 4.1.1.

By an argument by contradiction, Theorem 4.1.1 follows from the following statement: Given a sequence of Riemannian $m$-manifolds $(M_n, g_n)$ with diameters $\operatorname{diam}(M_n, g_n) \leqslant 1/n$ and sectional curvatures $\sec(g_n) \geqslant -1/n$, one can find $C \in \mathbb{R}$ such that $\pi_1(M_n)$ is $C$-nilpotent for all sufficiently large $n$.

4.2. *Algebraic lemmas.* Recall that the group of outer automorphisms $\operatorname{Out}(G)$ of a group $G$ is defined as the quotient of its automorphism group $\operatorname{Aut}(G)$ by the subgroup of inner automorphisms $\operatorname{Inn}(G)$.

LEMMA 4.2.1 (a characterization of $C$-nilpotent groups). *Let*

$$\{1\} = G_\ell \subseteq \cdots \subseteq G_1 \subseteq G_0 = G$$

*be a sequence of groups satisfying the following properties: For any $i$,*

(i) *$G_i \trianglelefteq G$ is normal in $G$;*

(ii) *the image of the conjugation homomorphism $h_i : G \to \operatorname{Out}(G_i/G_{i+1})$ is finite of order at most $C_i$;*

(iii) *$G_i/G_{i+1}$ contains an abelian subgroup $E_i$ of index $\leqslant c_i$.*

*Then $G$ contains a nilpotent subgroup $N$ of index at most*

$$C = C(c_1, \ldots, c_\ell, C_1, \ldots, C_\ell),$$

*where $N$ is of nilpotency class $\leqslant \ell$.*

*Proof.* First of all, notice that property (i) assures that the objects described in parts (ii) and (iii) of the lemma are well defined.

Set $\Gamma_i := G_i/G_{i+1}$.

Let $H = \bigcap \ker h_i$. Notice that $[G : H] \leqslant \prod_i C_i$ and that the image of $H$ under the conjugation homomorphism $f_i : G \to \operatorname{Aut}(\Gamma_i)$ lies in $\operatorname{Inn}(\Gamma_i)$, i.e., $f_i|_H : H \to \operatorname{Inn}(\Gamma_i)$.

By passing to a subgroup, we can assume $E_i \trianglelefteq \Gamma_i$ is normal of index $\leqslant C(c_i)$ (we can take $C(c_i) = c_i!$).

By increasing $E_i$ if necessary we can assume $E_i$ contains the center of $\Gamma_i$.

Let $Z_i$ be the image of $E_i$ under the projection map $\pi : \Gamma_i \to \mathrm{Inn}(\Gamma_i)$. Clearly $[\mathrm{Inn}(\Gamma_i) : Z_i] \leqslant c_i$, and $Z_i \trianglelefteq \mathrm{Inn}(\Gamma_i)$ is normal.

Let $N = H \cap (\bigcap_i f_i^{-1}(Z_i))$ and $N_i = N \cap G_i$. Then

$$[G : N] \leqslant C = C(c_1, \dots, c_\ell, C_1, \dots, C_\ell)$$

and $N$ satisfies these properties for any $i$: (i′) $N_i \trianglelefteq N$ is normal in $N$, and (ii′) $N_i/N_{i+1}$ is in the center of $N/N_{i+1}$. That is, $N$ is nilpotent of nilpotency length $\leqslant \ell$.

Condition (i′) is obvious so we only need to check (ii′).

To see (ii′) observe that by construction the image of the conjugation action $N \to \mathrm{Aut}(\Gamma_i)$ lies in $\mathrm{Inn}(\Gamma_i)$. In fact, it lies in $\pi(A_i)$ and as such it acts trivially on $E_i$. Lastly observe that $N_i/N_{i+1} \subset E_i$.

Indeed, by construction, for any $g \in N_i/N_{i+1} \subset \Gamma_i$ there is $a \in E_i$ such that $\pi(g) = \pi(a)$. Therefore, $g = az$ for some $z$ in the center of $\Gamma_i$. By our assumption on $E_i$ this means that $g \in E_i$.

Thus $N$ acts trivially on $N_i/N_{i+1}$, which means that $N$ is nilpotent and $G$ is $C$-nilpotent. □

TRIVIAL LEMMA 4.2.2 (a characterization of finite actions). *If $S$ is a finite set of generators of a group $G$ with $S^{-1} = S$, and $h : G \to H$ is a homomorphism with $|h(S^n)| < n$ for some $n > 0$, then $h(S^n) = h(G)$ and, in particular, $|h(G)| < n$.*

Let now $\Gamma$ be a group which acts discretely by isometries on an Alexandrov space $A$ with curvature $\geqslant -1$. Choose a marked point $p \in A$. Assume that $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is a finite short basis of $\Gamma$ at $p$ (see Section 2.6), and that $\theta \leqslant |\gamma_i| \leqslant 1$, where $|\gamma| := |p\gamma(p)|$. Let $\#(R)$ denote the number of elements $\gamma \in \Gamma$ with $|\gamma| \leqslant R$. The Bishop–Gromov inequality implies that

$$\#(R) \leqslant v^m_{-1}(R)/v^m_{-1}(\theta),$$

where $m = \dim A$ and $v^m_{-1}(r)$ is the volume of the ball of radius $r$ in the $m$-dimensional simply connected space form of curvature $-1$. Therefore, if $\#\#(L)$ denotes the number of homomorphisms $h : \Gamma \to \Gamma$ with norm $\leqslant L$ (i.e., the number of homomorphisms for which $|h(\gamma)| \leqslant L|\gamma|$) for any $\gamma \in \Gamma$), then

$$\#\#(L) \leqslant \#(L)^n \leqslant \left[\frac{v^m_{-1}(L)}{v^m_{-1}(\theta)}\right]^n . \tag{4.2.1}$$

4.3. *The blow-up construction.* As $n \to \infty$, the manifolds $M_n$ clearly converge to a point, $A_0$.

Set $M_{n,1} := M_n$ and $\vartheta_{n,1} := \operatorname{diam} M_{n,1}$.

Rescale now $M_{n,1}$ by $1/\vartheta_{n,1}$ so that $\operatorname{diam}(M_{n,1}/\vartheta_{n,1}) = 1$. Passing to a subsequence if necessary, one has that the manifolds $(1/\vartheta_{n,1})M_{n,1}$ converge to $A_1$, where $A_1$ is a compact nonnegatively curved Alexandrov space with diameter 1.

Now choose a regular point $\overline{p}_1 \in A_1$, and consider distance coordinates around $\overline{p}_1 \in U_1 \to \mathbb{R}^{k_1}$, where $k_1$ is the dimension of $A_1$. The distance functions can be lifted to $U_{n,1} \subset (1/\vartheta_{n,1})M_{n,1}$.

Let $M_{n,2}$ be the level set of $U_{n,1} \to \mathbb{R}^{k_1}$ that corresponds to $\overline{p}_1$. Clearly, $M_{n,2}$ is a compact submanifold of codimension $k_1$. Set $\vartheta_{n,2} := \operatorname{diam} M_{n,2}$. Passing again to a subsequence if necessary, one has that the sequence $(1/\vartheta_{n,2})M_{n,2}$ converges to some Alexandrov space $A_2$. As before, $A_2$ is a compact nonnegatively curved Alexandrov space with diameter 1. Define $k_2 := k_1 + \dim A_2$. If one now chooses a marked point in $M_{n,2}$, then, as $n \to \infty$, $M_n/\vartheta_{n,2}$ converges to $A_2 \times \mathbb{R}^{k_1}$, which is of some dimension $k_2 > k_1$.

We repeat this procedure until, at some step, $k_\ell = m$.

As a result one obtains a sequence $\{A_i\}$ of compact nonnegatively curved Alexandrov spaces with diameter 1 that satisfies

$$\dim A_i = k_i - k_{i-1}, \quad \text{so that} \ \sum_{i=1}^{\ell} \dim A_i = m.$$

We also obtain a sequence of rescaling factors $\vartheta_{n,i} = \operatorname{diam} M_{n,i}$, and a nested sequence of submanifolds

$$\{p_n\} = M_{n,\ell} \subset \cdots \subset M_{n,2} \subset M_{n,1} = M_n,$$

which in turn induces a sequence of homomorphisms

$$\{1\} = \pi_1(M_{n,\ell}) \xrightarrow{\iota} \cdots \xrightarrow{\iota} \pi_1(M_{n,2}) \xrightarrow{\iota} \pi_1(M_{n,1}) = \pi_1(M_n).$$

Let $G_i := G_i(n) := \iota^i \pi_1(M_{n,i})$.

For $n$ sufficiently large, the subgroups $G_i(n)$ are those that are generated by elements of norm $\leqslant 3\vartheta_{n,i}$. Equivalently, if one takes a short basis $\{\gamma_i\}$ of $G(n)$, then $G_i$ is the subgroup generated by all elements $\gamma_i$ such that $|\gamma_i| \leqslant 3\vartheta_{n,i}$.

4.4. *Limit fundamental groups of Alexandrov spaces.* We will now define the limit, or $L$-fundamental groups, of the Alexandrov spaces $A_i$ constructed above. This notion is similar to the notion of the fundamental group of an orbifold. However, we note in advance that the construction of the $L$-fundamental group does not only depend on the spaces $A_i$, but also on the chosen rescaled subsequence of $M_n$. In fact, the following construction shows that the limit fundamental group $\pi_1^L(A_i)$ of $A_i$ is isomorphic to $\pi_1(M_{n,i}, M_{n,i+1})$ for all large enough $n$. But, unlike $\pi_1(M_{n,i})$, the groups $\pi_1^L(A_i)$ will not depend on $n$.

*The limit fundamental groups of $A_i$.* Consider the converging sequence

$$(M_n/\vartheta_{n,i}, p_n) \xrightarrow{\mathrm{GH}} (A_i \times \mathbb{R}^{k_{i-1}}, \overline{p}_i \times 0)$$

(here the interesting case is collapsing). Recall that $\overline{p}_i \in A_i$ is a regular point. Fix $\varepsilon > 0$ such that $\mathrm{dist}_{\overline{p}_i}$ on $A_i$ does not have critical values in $(0, 2\varepsilon)$. Take a sequence $R_n$ that converges very slowly to infinity (here we will need $R_n \vartheta_{n,i}/\vartheta_{n,i-1} \to 0$ and $R_n \to \infty$).

Consider then a sequence of Riemannian coverings

$$\Pi : (\widetilde{B}_n, \tilde{p}_n) \to (B_{R_n}(p_n), p_n) \quad \text{of } \; B_{R_n}(p_n) \subset M_n/\vartheta_{n,i}$$

with $\pi_1(\widetilde{B}_n, \tilde{p}_n) = \pi_1(B_\varepsilon(p_n), p_n)$, where $B_\varepsilon(p_n) \subset M_n/\vartheta_{n,i}$.

After passing to a subsequence if necessary, the sequence $(\widetilde{B}_n, \tilde{p}_n)$ converges to a nonnegatively curved Alexandrov space $\tilde{A}_i \times \mathbb{R}^{k_{i-1}}$, where the space $\tilde{A}_i$ has the same dimension as $A_i$. Indeed, by construction it contains an isometric copy of $B_\varepsilon(p_{n,i})$, and therefore

$$\dim \tilde{A}_i + k_{i-1} = \dim \lim_{i\to\infty} B_\varepsilon(p_{n,i}) = \dim A_i + k_{i-1}.$$

Let us show that for all sufficiently large $n$,

$$\imath(\pi_1(M_{n,i+1})) \trianglelefteq \pi_1(M_{n,i}).$$

Assume that $\Pi(\tilde{q}_n) = \tilde{p}_n$ and that $\tilde{q}_n \to \bar{q}_n \in \tilde{A}_i$. Connect $\overline{p}_n$ and $\bar{q}_n$ by a geodesic, which, by [Pet98], only passes through regular points. Note that in a small neighborhood of this geodesic in $M_n$ we have two copies of $M_{n,i+1}$, near $\tilde{p}_n$ and $\tilde{q}_n$. Therefore, applying Yamaguchi's fibration theorem in a small neighborhood of this geodesic, we can construct a diffeomorphism from $M_{n,i+1}$ to itself. This implies that for any loop $\gamma$ that after lifting connects $\tilde{p}_n$ and $\tilde{q}_n$, we have $\gamma^{-1}\imath\pi_1(M_{n,i+1})\gamma \subset \imath\pi_1(M_{n,i+1})$, i.e., $\imath\pi_1(M_{n,i+1}) \triangleleft \pi_1(M_{n,i})$ (for an alternative argument see also [FY92]).

This easily yields that $A_i = \tilde{A}_i/\Gamma_i$, where $\Gamma_i$ is a group of isometries that acts discretely on $\tilde{A}_i$. The group $\Gamma_i$ is denoted by $\pi_1^L(A_i)$ (the *limit* or *L-fundamental group* of $A_i$). This group is clearly isomorphic to

$$\pi_1(M_{n,i}, M_{n,i+1}) = \pi_1(M_{n,i})/\imath(\pi_1(M_{n,i+1}))$$

for all sufficiently large $n$, and the space $\tilde{A}_i$ will be called the *universal covering* of $A_i$.

Because $\tilde{A}_i$ is nonnegatively curved and $A_i = \tilde{A}_i/\pi_1^L(A_i)$ is compact, Toponogov's splitting theorem says $\tilde{A}_i$ isometrically splits as $\tilde{A}_i = K_i \times \mathbb{R}^{s_i}$, where $K_i$ is a compact Alexandrov space with curv $\geqslant 0$. Since $\pi_1^L(A_i)$ is a group of isometries that acts discretely on $\tilde{A}_i$, it follows that $\pi_1^L(A_i)$ is a virtually abelian group.

4.5. *Final steps.* Consider now the corresponding series

$$\{1\} = G_\ell(n) \subset \cdots \subset G_1(n) \subset G_0(n) = \pi_1(M_n).$$

The theorem then follows from this lemma:

LEMMA 4.5.1. *For all sufficiently large $n$, the series*

$$\{1\} = G_\ell(n) \subset \cdots \subset G_1(n) \subset G_0(n)$$

*constructed above satisfies the assumptions of Lemma* 4.2.1 *for numbers $C_i$ and $c_i$ not depending on $n$.*

We first prove the following.

SUBLEMMA 4.5.2. *Each subgroup $G_i(n)$ is normal in $G(n)$.*

*Proof.* We will show by reverse induction on $k$ that $G_i(n) \trianglelefteq G_k(n)$ for any $k \leqslant i$. Let us assume that we already know that $G_i(n) \trianglelefteq G_{k+1}(n)$. Since

$$\imath\pi_1(M_{n,k+1}) \trianglelefteq \pi_1(M_{n,k}),$$

we know that $G_{k+1}(n) \trianglelefteq G_k(n)$. Consider the covering

$$\Pi_{k+1} : (\widetilde{M}_{n,k+1}, \tilde{p}_{n,k+1}) \to (M_n, p_n)$$

with covering group $\Gamma_{k+1}(n)$.

Clearly $(\widetilde{M}_{n,k+1}, \tilde{p}_{n,k+1}) \xrightarrow{\text{GH}} \mathbb{R}^{s_i}$ for some integer $s_i$. From Lemma 2.5.1, it follows that for any $a \in G$ with $|a| < 1$ there is a cocos-curve $\gamma$ in $\widetilde{M}_{n,k+1}$ with total time $T$ connecting $\tilde{p}_n$ and $a(\tilde{p}_n)$ in $\widetilde{M}_{n,k+1}$. Then clearly $\gamma \sim ga$ for some $g \in G_{k+1}(n)$. Let us denote by $\Phi^T : \widetilde{M}_{n,i} \to \widetilde{M}_{n,i}$ the gradient flow corresponding to $\gamma$.

Let $\gamma_j$ be a loop from the short basis of $G_i(n)$. As was mentioned in Section 4.3, if $n$ is large, then length $\gamma_j \leqslant 3\vartheta_{n,i}$. Let us denote by $\tilde{\gamma}_j$ a lift of $\gamma_j$ to $\widetilde{M}_{n,i}$. Let $\tilde{p}_{n,j} \in \widetilde{M}_{n,i}$ be its starting point. Since $[\gamma_j] \in G_i(n)$, we have that $\tilde{\gamma}_j$ is a loop in $\widetilde{M}_{n,i}$. Consider then the loop $\gamma'_j = \Pi \circ \Phi^T \circ \tilde{\gamma}_j$. Clearly,

$$[\gamma_j] = a^{-1}g^{-1}[\gamma'_j]ga, \quad \text{or} \quad [\gamma'_j] = ga[\gamma_j]a^{-1}g^{-1}.$$

Proposition 2.3.3 implies that $\operatorname{length}(\gamma'_j) \leqslant \exp(2T)\operatorname{length}(\gamma_j)$. Therefore, $ga[\gamma_j]a^{-1}g^{-1} \in G_i(n)$ for sufficiently large $n$, and since $g \in G_i(n) \trianglelefteq G_{k+1}(n)$ it follows that $a[\gamma_j]a^{-1} \in G_i(n)$, i.e., $G_i(n) \trianglelefteq G_k(n)$. □

*Proof of Lemma* 4.5.1. The group

$$\pi_1^L(A_i) = \pi_1(M_{n,i}, M_{n,i+1}) = \pi_1(M_{n,i})/\imath(\pi_1(M_{n,i}))$$

is virtually abelian. Let $d_i$ be the minimal index of an abelian subgroup of $\pi_1^L(A_i)$. The epimorphism $\imath^i : \pi_1(M_{n,i}) \to G_i$ induces an epimorphism $\pi_1^L(A_i) \to G_i(n)/G_{i+1}(n)$. Hence $G_i(n)/G_{i+1}(n)$ is $d_i$-abelian for all large $n$.

Consider the covering $\Pi_i : \widetilde{M}_{n,i} \to M_n$ with covering group $G_i(n)$, and let $\tilde{p}_{n,i}$ be a preimage of $p_n$. Clearly

$$(\widetilde{M}_{n,i}, \tilde{p}_{n,i}) \xrightarrow{\text{GH}} \mathbb{R}^{s_i} \quad \text{for some integer } s_i.$$

Applying Lemma 2.5.1, it follows that for any $a \in G(n)$ with $|a| < 1$ there is a cocos-curve $\gamma$ in $\widetilde{M}_{n,i}$ that connects $p$ and $a(p)$. Then clearly $\gamma \sim ga$ for some $g \in G_i(n)$. Let us denote by $\Phi^T : \widetilde{M}_{n,i} \to \widetilde{M}_{n,i}$ the gradient flow corresponding to $\gamma$.

Let $b \in G_i(n)$ and $\beta$ be a loop representing $b$. Let us denote by $\tilde{\beta}$ a lift of $\beta$ to $\widetilde{M}_{n,i}$. Let $\tilde{p}_{n,i} \in \widetilde{M}_{n,i}$ be its starting point. Since $[\beta] \in G_i(n)$, we have that $\tilde{\beta}$ is a loop in $\widetilde{M}_{n,i}$.

Consider now the loop $\beta' = \Pi \circ \Phi^T \circ \tilde{\beta}$. Clearly,

$$b = [\beta] = a^{-1} g^{-1} [\beta'] ga, \quad \text{or} \quad [\beta'] = gaba^{-1}g^{-1}.$$

Proposition 2.3.3 then implies that $\operatorname{length}(\beta') \leqslant \exp(2T) \operatorname{length}(\beta)$. Therefore, if

$$h_a : G_i(n)/G_{i+1}(n) \to G_i(n)/G_{i+1}(n)$$

is induced by the conjugation $b \to aba^{-1}$, then for any $a \in G(n)$ there is a $g \in G_i(n)$ such that $|h_{ga}| \leqslant \exp(2T)$.

Let now $\delta_i$ be the minimal norm of the elements of $\pi_1^L(A_i)$, where $\pi_1^L(A_i)$ acts on $\tilde{A}_i$. Then (4.2.1) implies that the image of the action of $G(n)$ by conjugation in $\operatorname{Out}(G_i(n)/G_{i+1}(n))$ is $C_i$-finite, where $C_i$ depends only on $c_i$, $T$, and $\delta_i$. □

4.6. *Remark on nonfree actions.* Theorem 4.1.1 can be reformulated:

There exists a constant $\epsilon(m) > 0$ such that if $N^m$ is a Riemannian manifold that admits a free discrete isometric action by a group $\Gamma$ such that

$$\sec(N) > -\epsilon(m) \quad \text{and} \quad \operatorname{diam}(N/\Gamma) < 1,$$

then $\Gamma$ is $C(m)$-nilpotent.

B. Wilking pointed out to us that one can actually easily remove from this reformulation of Theorem 4.1.1 the assumption that the $\Gamma$ action be free.

COROLLARY 4.6.1. *There exists a constant $\epsilon(m) > 0$ such that if $N^m$ is a Riemannian manifold that admits a discrete isometric action by a group $\Gamma$ such that* $\sec(N) > -\epsilon(m)$ *and* $\operatorname{diam}(N/\Gamma) < 1$, *then $\Gamma$ is $C(m)$-nilpotent.*

*Proof.* Let $\epsilon = \epsilon(m)$ be as provided by Theorem 4.1.1, and suppose $N$ satisfies the assumptions of the corollary for this $\epsilon$. Let $F$ be the frame bundle of $N$. Then the action of $\Gamma$ on $N$ lifts to a free isometric action on $F$. As was observed in [FY92], using Cheeger's rescaling trick, $F$ can be equipped with a $\Gamma$-invariant metric satisfying $\sec(F) > -\epsilon(m)$ and $\operatorname{diam}(F/\Gamma) < 1$. Since the induced action of $\Gamma$ on $F$ is free, the claim now follows from Theorem 4.1.1. □

## 5. Proof of the fibration theorem

5.1. Let $M$ be an almost nonnegatively curved manifold. Let us denote by $M_n = (M, g_n)$ a sequence of Riemannian metrics on $M$ such that $\sec(M_n) \geqslant -1/n$ and $\operatorname{diam}(M_n) \leqslant 1/n$.

Let us denote by $\widetilde{M}$ the universal cover of $M$ and by $\widetilde{M}_n \to M_n$ the universal Riemannian covering of $M_n$ (i.e., $\widetilde{M}$ equipped with the pull back of the metric $g_n$ on $M$).

By [FY92], passing to a finite cover we may assume that $\Gamma = \pi_1(M)$ is a nilpotent group without torsion. Hence, to prove the topological part of Theorem C, it is enough to show the following:

THEOREM 5.1.1. *Suppose $M$ is a closed almost nonnegatively curved $m$-manifold such that $\Gamma = \pi_1(M)$ is a nilpotent group without torsion. Then $M$ is the total space of a fiber bundle*

$$F \to M \to N,$$

*where the base $N$ is a nilmanifold and the fiber $F$ is simply connected.*

The assumption on $\Gamma$ implies that we can fix a series

$$\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \Gamma_2 \rhd \cdots \rhd \Gamma_\ell = \{1\}$$

such that $\Gamma_i$ is normal in $\Gamma$ and $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$.

Let us first give an informal proof.

5.2. *An informal proof of Theorem* 5.1.1. We use induction to construct the bundles

$$F_i \longrightarrow M \xrightarrow{f_i} N_i,$$

where each $N_i$ is a nilmanifold with $\pi_1(N_i) = \Gamma/\Gamma_i$ and $\pi_1(F_i) \cong \Gamma_i$. Since the base of induction is trivial, we are only interested in the induction step.

Fix $p \in N_i$, and let $F_i(p)$ be the fiber over $p$. For any sufficiently large $n$, choose a subgroup $G_i = G_i(n)$ such that $\Gamma_i \lhd G_i \lhd \Gamma_{i+1}$ and $[\Gamma_i : G_i]$ is finite, but large enough so that the cover $\overline{F}_i(p)$ of $F_i(p)$ corresponding to $G_i$ is Hausdorff close to a unit circle $S^1$.

Construct now a bundle map $\varphi_p : \overline{F}_i(p) \to S^1$ by lifting distance functions from $S^1$. (This can be done by a slight generalization of a construction in [FY92] and [BGP92].) Let $\omega_p = d\varphi_p$.

Then $\omega_p$ is a closed integral nondegenerate one-form on $F_i(p)$. Since deck transformations are isometries, after averaging by $\mathbb{Z}_a$, where $a = [\Gamma_i : G_i]$, we can assume that $\omega_p$ is $\mathbb{Z}_a$-invariant. Thus $\omega_p$ descends to a form on $F_i(p)$, which when integrated gives a bundle map $F_i(p)$ onto a small $S^1$.

Note that although this bundle is defined only up to rotations of $S^1$, its fibers are well defined.

Since $\Gamma_{i+1}$ is normal in $\Gamma$, the choice of the covering $\overline{F}_i(p)$ of $F_i(p)$ is unambiguous for all $p \in N_i$. By using a partition of unity on $N_i$ we can glue the forms $\omega_p$ into a global 1-form on $M$ which satisfies the properties that

(a) $\omega|_{F(p)}$ is closed and integral for any $p$;

(b) $\omega|_{F(p)}$ is nondegenerate.

Integrating $\omega$ over the various $F(p)$, we construct a continuous family of bundles $F_p \to S^1$. The level sets partition each $F(p)$ and hence the whole $M$ into fibers of a fiber bundle, whose quotient space is then a circle bundle $N_{i+1}$ over $N_i$. $\approx \square$

This gives a good idea of the proof. However, to make it precise some extra work has to be done. In particular, one has to be careful with the construction of $\omega$. To make this construction possible we have to keep track of how $F(p)$ was obtained. Namely, we have to use that the fiber $F(p)$ was obtained by a construction as in Yamaguchi's fibration theorem; see [Yam91] or [BGP92]. This makes the induction proof quite technical.

We now proceed with the rigorous proof of Theorem 5.1.1.

5.3. *Proof of Theorem* 5.1.1. Let us denote by $\widetilde{M}_{n,i}$ the Riemannian covering of $M_n$ with respect to $\Gamma_i$.

For any choice of marked points $p_n$ we have

$$(\widetilde{M}_{n,i}, p_n, \pi_1(M)) \xrightarrow{\text{GH}} (\mathbb{R}^i, 0, \mathbb{R}^i)$$

in equivariant Gromov-Hausdorff convergence, where $\mathbb{R}^i$ acts on itself by translations. Indeed, the limit space must be a nonnegatively curved simply connected Alexandrov space, and since diam $M_n \to 0$, it possesses a transitive group action by a nilpotent group. Then Euclidean space, acting as a group of translations, is here the only choice, and it is easy to see that the dimension of the limit must be equal to $i$.

Therefore $(\widetilde{M}_n, p_n, \pi_1(M)) \xrightarrow{\text{GH}} (\mathbb{R}^\ell, 0, \mathbb{R}^\ell)$, and we may also assume that

$$(\widetilde{M}_n, p_n, \Gamma_i) \xrightarrow{\text{GH}} (\mathbb{R}^\ell, 0, \mathbb{R}^{\ell-i}) \quad \text{for each } i,$$

where $\mathbb{R}^{\ell-i}$ is the coordinate subspace of $\mathbb{R}^\ell$ that corresponds to the first $\ell - i$ elements of the standard basis.

Now, let us give a technical definition:

If $R$ is a Riemannian manifold, let us denote by $\widetilde{\operatorname{dist}}_p$ the average of a distance function over a small ball around $p$. This enables us to work with the $C^1$ function $\widetilde{\operatorname{dist}}_p$ instead of the Lipschitz function $\operatorname{dist}_p$.

*Definition* 5.3.1. Suppose $R_n \xrightarrow{\text{GH}} R$ is a sequence of Riemannian $m$-manifolds with curvature $\geqslant \kappa$ that Gromov-Hausdorff converges to a Riemannian $m'$-manifold $R$, where $m' \leqslant m$. A sequence of forms $\omega_n$ on $R_n$ is said to $\varepsilon$-approximate a form $\omega$ on $R$, if

(i) for any point $p \in R$ there is a neighborhood $U \ni p$ that admits a distance chart $f : U \to \mathbb{R}^{m'}$,

$$f(x) = (\mathrm{dist}_{a_1}(x), \mathrm{dist}_{a_2}(x), \dots, \mathrm{dist}_{a_{m'}}(x))$$

that is a smooth regular map, and

(ii) smooth lifts of $f$ to $U_n \subset R_n$ give, for $n$ large enough, regular maps

$$f_n(x) = (\widetilde{\mathrm{dist}}_{a_{1,n}}(x), \widetilde{\mathrm{dist}}_{a_{2,n}}(x), \dots, \widetilde{\mathrm{dist}}_{a_{m',n}}(x))$$

with $a_{i,n} \in M_n$, $a_{i,n} \to a_n$ such that

$$|(f_n \circ f^{-1})^*(\omega) - \omega_n|_{C^0} < \varepsilon \quad \text{for all sufficiently large } n.$$

Theorem 5.1.1 now easily follows from the following lemma:

LEMMA 5.3.2. *Given $\varepsilon > 0$, there is a sequence of one-forms*

$$\{\omega_{1,n}, \omega_{2,n}, \dots, \omega_{k,n}\}$$

*on $\widetilde{M}_n$ with the following properties*:

(i) *For each $i$, $\omega_{i,n}$ is a $\pi_1(M)$-invariant form on $\widetilde{M}_n$.*

(ii) *The forms $\omega_{i,n}$ $\varepsilon$-approximate the coordinate forms $dx_i$ on $\mathbb{R}^k$. In particular, the forms $\{\omega_{i,n}\}$ are nowhere zero and almost orthonormal at each point.*

(iii) *If $\omega_{j,n}(X) = \omega_{j,n}(Y) = 0$ for all $j < i$, then $d\omega_{i,n}(X, Y) = 0$. In particular, for each $i$ and all sufficiently large $n$, the distribution corresponding to the system of equations*

$$\omega_{j,n}(X) = 0 \quad \text{for all } j \leqslant i$$

*defines on $\widetilde{M}_n$ a foliation $\mathcal{F}_{i,n}$.*

(iv) *If $\widetilde{F}_{i,n}(x) \subset \widetilde{M}_n$ denotes the fiber of the foliation $\mathcal{F}_{i,n}$ through the point $x \in \widetilde{M}_n$, then each $\widetilde{F}_{i,n}(x)$ is $\Gamma_i$-invariant, i.e., for any $\gamma \in \Gamma_i$ one has that $\widetilde{F}_{i,n}(x) = \widetilde{F}_{i,n}(\gamma x)$. Moreover, the action of $\Gamma_i$ on $\widetilde{F}_{i,n}(x)$ is cocompact for each $i$. In particular, $\mathcal{F}_{i,n}$ induces on $M_n$ the structure of a fiber bundle.*

*Proof.* We will construct these forms by induction. Assume that we have already constructed one-forms $\omega_1, \omega_2, \dots, \omega_{i-1}$ with all the required properties. They give a $\pi_1(M)$-invariant fibration of $\widetilde{M}_n$ by submanifolds $\widetilde{F}_{i-1,n}(x)$ through each point $x \in \widetilde{M}_n$, with tangent spaces defined by the equations $\omega_j(X) = 0$ for $j = 1, \dots, i-1$.

Denote by $\theta : \mathbb{R} \to [0, 1]$ a smooth monotone function that is equal to 1 before 0 and to 0 after 1. Choose numbers $\delta_n > 0$ slowly converging to 0, and let $\Theta_{i,n} : \widetilde{M}_n \to \mathbb{R}_+$ be the function defined by

$$\Theta_{i,n}(x) = \min_{y \in F_{i-1,n}(x)} \theta(|p_n y|/\delta_n).$$

Clearly $\Theta_{i,n}$ is a continuous $\Gamma_{i-1}$-invariant function, which is constant on each $F_{i-1,n}(x)$. Moreover, for large $n$, $\Theta_{i,n}$ has support in some $C_i\delta_n$-neighborhood of $F_{i-1,n}(p_n)$, and is equal to 1 in some $c_i\delta_n$-neighborhood of $F_{i-1,n}(p_n)$.

Now let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth nondecreasing function that is 0 before $1/2$ and 1 after $3/2$. Consider the form

$$\omega'_{i,n} = \Theta_{i,n} \cdot d(\varphi \circ \widetilde{\operatorname{dist}}_{\Gamma_i a_{i,n}}),$$

where $a_{i,n} \in \widetilde{M}_n$ is a sequence of points converging to $-e_i \in \mathbb{R}^\ell$, and $\widetilde{\operatorname{dist}}_{\Gamma_i a_{i,n}}$ is the average of $\operatorname{dist}_{\Gamma_i x}$ for $x$ in a small ball around $a_{i,n}$. The support of $\omega'_i$ has two components, one that contains $p_n$ (notice here that $p_n \to 0 \in \mathbb{R}^\ell$), and another that does not. (It follows from the construction that the limit of $F_{i-1,n}(p_n)$ is a coordinate plane in $\mathbb{R}^\ell$.)

Set $\omega''_{i,n} := \omega'_{i,n}$ on the component of $p_n$, and let this form be 0 otherwise. Clearly, $\omega''_{i,n}$ is then a continuous $\Gamma_i$-invariant form whose restriction to $\widetilde{F}_{i-1,n}(x)$ is exact. Moreover, each level set of its integral over $\widetilde{F}_{i-1,n}(x)$ is $\Gamma_i$-invariant.

By construction, the form $\omega''/|\omega''|$ is now (in the sense of Definition 5.3.1) close to $dx_i$ at the points where $|\omega''| \neq 0$. Take

$$\omega_{i,n} = c \sum_{\gamma \in \Gamma/\Gamma_i} \gamma\, \omega',$$

where the coefficient $c$ is chosen in such a way that $|\omega_{i,n}(p_n)| = 1$. As $\delta_n$ is a sequence slowly converging to zero, we may assume that $\operatorname{diam}(M_n)/\delta_n \to 0$. Therefore, $\omega_{i,n}$ is the form we need. □

Notice that the proof actually shows that the fibers in Theorem 5.1.1 are almost nonnegatively curved manifolds in the generalized sense with $k = \ell$. Therefore, the proof of Theorem C is complete. □

## 6. **Open questions**

We would like to conclude this work by posing a number of related open questions.

6.1. *Is the torsion contained in the center*? As was noted earlier, Theorem B is new even for manifolds of nonnegative curvature. For such manifolds it is known that their fundamental groups are almost abelian, and Fukaya and Yamaguchi conjectured the following:

CONJECTURE 6.1.1 (see [FY92]). *The fundamental group of a nonnegatively curved $m$-manifold is $C(m)$-abelian.*

In this regard we would like to pose the following two conjectures:

MAIN CONJECTURE 6.1.2. *There is a* $C = C(m)$ *such that if* $M^m$ *is almost nonnegatively curved, then there is a nilpotent subgroup* $N \subset \pi_1(M)$ *of index* $\leqslant C$ *whose torsion is contained in its center (or, at least, whose torsion is commutative).*

CONJECTURE 6.1.3. *If* $M^m$ *is almost nonnegatively curved, then the action of* $\pi_1(M)$ *on* $\pi_2(M)$ *is almost trivial (or maybe even* $C(m)$*-trivial), i.e., there exists a finite index subgroup of* $\pi_1(M)$ *(or, respectively, a subgroup of index* $\leqslant C(m)$*) that acts trivially on* $\pi_2(M)$*.*

Main Conjecture 6.1.2 implies in particular that the fundamental groups of closed positively curved $m$-manifolds are $C(m)$-abelian.

In fact, as was pointed out to us by B. Wilking, the truth of Main Conjecture 6.1.2 would also imply a positive answer to Conjecture 6.1.1. Indeed, if $\sec(M) \geqslant 0$, then the universal cover $\widetilde{M}$ of $M$ is isometric to the product $\mathbb{R}^n \times K$, where $K$ is a compact Riemannian manifold and the $\pi_1(M)$ action on $\mathbb{R}^n \times K$ is diagonal. It follows from [Wil00, Cor. 6.3] that one can deform the metric on $M$ so that its universal cover is still isometric to $\mathbb{R}^n \times K$ and the induced action on $K$ is finite. By passing, as in the proof of Corollary 4.6.1, to the induced action on the frame bundle of $K$, one reduces the statement to Main Conjecture 6.1.2.

We tried to prove these conjectures by studying successive blow-ups of the collapsing sequence $M_n$ as done in Section 4.3.

We can prove Conjectures 6.1.2 and 6.1.3 in the case where all spaces $A_i$ that appear in the construction in Section 4.3 are closed Riemannian manifolds. Moreover, we believe we have an argument to prove it if all the $A_i$ are Alexandrov spaces without boundary.

It seems that if we would have just a slightly better understanding of collapsing to a ray, then we could prove the conjectures. Here is the simplest related question we cannot solve:

*Question* 6.1.4. Let $M_n = (S^2 \times \mathbb{R}^2, g_n)$ be a sequence of complete Riemannian manifolds with $\sec(M_n) \geqslant -\varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$. Assume $(M_n, p_n) \xrightarrow{\text{GH}} (\mathbb{R}_+, 0)$ for a sequence of points $p_n \in M_n$. Let $q_n \in M_n$ be a sequence of points such that $|p_n q_n| = 1$ and such that there is a sequence of rescalings $\lambda_n \to \infty$ such that $(\lambda_n M_n, q_n) \xrightarrow{\text{GH}} S^2 \times S^1 \times \mathbb{R}$, where the latter space is equipped with the product of the canonical metrics.

(i) Can it happen that $(\lambda_n M_n, p_n) \xrightarrow{\text{GH}} (\mathbb{R}_+, 0)$?

(ii) Is the Gromov-Hausdorff limit of $(\lambda_n M_n, p_n)$ at least 3-dimensional?

(iii) What are the possible limits of $(\lambda_n M_n, p_n)$?

In the case where all $A_i$ are Riemannian manifolds, crucial to the proof is the following topological statement:

THEOREM 6.1.5. *Given manifolds $F_1, F_2, \dots, F_n$ such that each $F_i$ is either $S^1$ or is simply connected, if $E$ is the total space of a tower of fiber bundles*

$$E = E_n \xrightarrow{F_n} E_{n-1} \xrightarrow{F_{n-1}} \cdots \xrightarrow{F_1} E_0 = \{point\}$$

*and each of the bundles $E_k \xrightarrow{F_k} E_{k-1}$ are homotopically trivial over the 1-skeleton, then $\pi_1(E)$ contains a nilpotent subgroup $N$ such that*

$$[\pi_1(E) : N] \leqslant C(F_1, F_2, \dots, F_n) \quad \textit{and} \quad \mathrm{Tor}(N) \subset Z(N).$$

Our current proof of this theorem is surprisingly nontrivial,[1] and finding an easier proof will probably help to prove our conjecture in full generality.

Conjectures 6.1.1 and 6.1.2 are also related to the following conjecture of Rong (cf. [Ron96a], [Ron96b]):

CONJECTURE 6.1.6 (Rong). *Positively curved $m$-manifolds have $C(m)$-cyclic fundamental groups.*

This conjecture has been proved by Rong [Ron96a] under the additional assumption of a uniform upper curvature bound. We also believe that if one could carry out the above program for proving Main Conjecture 6.1.2, one would have a good shot at handling Rong's conjecture as well.

6.2. *The simply connected case.* So far we have only discussed manifolds with nontrivial fundamental groups. However, some of our arguments also work in a more general setting. We hope that it might be possible to use them to obtain new restrictions on simply connected almost nonnegatively curved manifolds as well as on collapsing with a lower curvature bound.

Let us indicate one possible approach to do so.

Let us denote by $\mathcal{M}(F)$ the space of self homotopy equivalences of a manifold $F$. Assume now that $F$ is simply connected and that $\tilde{f} : S^k \times F \to F$ is a map such that $\tilde{f}_u : F \to F$ is homotopic to the identity for some (and therefore *any*) $u \in S^k$. Then $\tilde{f}$ represents an element $\alpha = [\tilde{f}] \in \pi_k(\mathcal{M}(F))$. Let $g$ be a Riemannian metric on $F$. Define

$$\mathrm{dil}_g(\tilde{f}) = \max_{u \in S^k} \mathrm{dil}_g(\tilde{f}_u),$$

where $\mathrm{dil}_g(\tilde{f}_u)$ is the optimal Lipschitz constant of $\tilde{f}_u$ with respect to $g$. For any $\alpha \in \pi_k(\mathcal{M}(F))$ define

$$\mathrm{dil}_g(\alpha) = \inf_{[h]=\alpha} \mathrm{dil}_g(h).$$

Finally, define

$$\mathrm{DIL}(\alpha) = \inf_g \mathrm{dil}_g(\alpha)$$

[1] The note http://www.math.psu.edu/petrunin/papers/almpos/TinZ.pdf gives a sketch of the proof of Theorem 6.1.5.

over all Riemannian metrics on $F$ and

$$\mathrm{DIL}_+(\alpha) = \inf_g \mathrm{dil}_g(\alpha)$$

over all Riemannian metrics on $F$ with diam $\leqslant 1$ and sec $\geqslant -1$.

Clearly, both $\mathrm{DIL}(\alpha)$ and $\mathrm{DIL}_+(\alpha)$ are homotopy invariants of $\alpha$.

Now suppose that $M_n \xrightarrow{\mathrm{GH}} S^{k+1}$ is a sequence of Riemannian manifolds collapsing to a round sphere with $\sec(M_n) \geqslant k$. By Yamaguchi's fibration theorem, $M_n$ is a fiber bundle over $S^{k+1}$ with almost nonnegatively curved fiber $F_n$. This bundle is classified by an element $\alpha$ of $\pi_k(\mathrm{Aut}(F_n))$, and by using our gradient flow technique we can estimate $\mathrm{DIL}_+(\alpha)$ (and hence $\mathrm{DIL}(\alpha)$) from above.

Therefore, if one could find examples of a simply connected $F$ and an $\alpha$ with arbitrary big $\mathrm{DIL}_+(\alpha)$, one would obtain new restrictions on collapsing to a sphere with curvature bounded from below, and probably more restrictions for the topological type of manifolds with lower curvature and upper diameter bounds in general. In fact, $F$ need not be simply connected as long as the total space of the bundle $F \to M \to S^{k+1}$ is.

While we believe that finding examples with arbitrary large $\mathrm{DIL}(\alpha)$) is very difficult (and might even be impossible), we have several candidates to produce large $\mathrm{DIL}_+(\alpha)$.

On the other hand, the problem of finding $\alpha$ with $\mathrm{DIL}(\alpha) > 1$ seems quite interesting in its own right and might have other applications unrelated to collapsing.

Let us next describe some possible sources of examples with $\mathrm{DIL}_+(\alpha) > 1$:

*Example* 6.2.1. Obviously, if $\mathrm{dil}_g(h) = 1$, then $h_u$ is a homotopy of isometries of $(F, g)$. Let $G$ be the isometry group of $F$. Then $G$ can be viewed as a subset of $\mathcal{M}(F)$. Therefore, if $[h] \neq 0$ in $\pi_k(M(F))$, then $[h_u] \neq 0$ in $\pi_k(G)$. Now $G$ is a compact Lie group, in particular, $\pi_2(G) = 0$ (and even more generally $\pi_{2n}(G)$ is finite). On the other hand, there are spaces $F$ such that the space $\mathcal{M}(F)$ might have nontrivial second homotopy; for example, one can take $F = \mathrm{SU}(6)/\mathrm{SU}(3)\times\mathrm{SU}(3)$, for which the canonical metric on $F$ has nonnegative curvature, and it follows from [TO97, Ch. 5, Ex. 4.14] that $\pi_2(M(F)) \otimes \mathbb{Q}$ is nontrivial. Therefore, there is an $\alpha \in \pi_2(M(F))$ such that $\mathrm{dil}_g(\alpha) > 1$ for any metric $g$ on $F$; we believe it should be true that $\mathrm{DIL}_+(\alpha) > 1$. Still, it might happen that $\mathrm{DIL}(\alpha) = 1$.

Another possible source of such manifolds is provided by the following example due to D. Sullivan.

*Example* 6.2.2. Let $N^7$ be the total space of an $S^3$ bundle over $S^4$ with zero Euler class and nontrivial $p_1$. Clearly $N^7$ is rationally equivalent to $S^4 \times S^3$. In particular, its minimal model has no nontrivial derivations of degree $-1$. Therefore, by [Sul77, 13.3], there exists a diffeomorphism $f : N \to N$ that is homotopic to the identity but whose obstruction to being diffeotopic to the identity is a nonzero element of $H^3(N,\mathbb{Z}) \cong \mathbb{Z}$. Let $M^8$ be the mapping cylinder of $f$. Clearly $M$ is homotopy equivalent to $N \times S^1$, and hence it is spin with signature zero. On the other hand, by construction, $p_1^2(M) \neq 0$. Since the signature of $M$ is zero we must necessarily have that $p_2(M) \neq 0$, and hence $\hat{A}(M) \neq 0$. In particular, by the Atiyah-Hirzebruch theorem, $M$ does not admit an $S^1$ action, and hence the corresponding element $\alpha \in \pi_1(\mathcal{M}(M))$ has $\operatorname{dil}_g(\alpha) > 1$ for any metric $g$ on $M$.

*Remark* 6.2.3. As mentioned in the introduction, it is known that a spin manifold $X$ of almost nonnegative Ricci curvature has $\hat{A}(X) \leqslant 2^{\dim(X)/2}$; see [Gro82, p. 41] and [Gal83]. Clearly, a finite cover of the manifold $M$ constructed above violates this restriction and therefore $M$ does not admit almost nonnegative Ricci curvature. However, it could possibly be almost nonnegatively curved in the generalized sense.

6.3. *Further questions.* Recall that a simply connected space $C$ is called *rationally elliptic* if it is homotopy equivalent to a finite CW-complex and

$$\dim \pi_*(C,\mathbb{Q}) < \infty.$$

A conjecture of Grove and Halperin [GH82] says that simply connected nonnegatively curved manifolds are rationally elliptic. This conjecture was extended by Grove to include almost nonnegatively curved manifolds [Gro02]. Later, Totaro has proposed the following definition of rationally elliptic spaces, which covers manifolds with infinite fundamental groups:

A connected topological space $X$ is *rationally elliptic* if it is homotopy equivalent to a finite CW-complex, it has a finite covering that is a nilpotent space, and its universal cover is rationally elliptic in the ordinary sense.

With this definition one can extend Grove's conjecture to manifolds that are not simply connected, as follows:

CONJECTURE 6.3.1. *Any almost nonnegatively curved manifold in the generalized sense is rationally elliptic.*

Theorem A reduces this conjecture to the simply connected case, which is undoubtedly the most difficult part of the problem.

It has been shown in [PP06] that if $M$ is a nilpotent closed manifold that admits a Riemannian metric with zero topological entropy, then its universal cover $\widetilde{M}$ is rationally elliptic. Coupled with Theorem A, this means that to prove Conjecture 6.3.1 it would be sufficient to show that a manifold with almost nonnegative curvature in the generalized sense admits a metric with zero topological entropy. However, we think that this might be wrong in general.

As pointed out in the discussion in Section 1 before Theorem C, it already follows from Yamaguchi's fibration theorem and [FY92] that a finite cover of an almost nonnegatively curved manifold maps onto a nilmanifold with homotopy fiber a simply connected closed manifold. While this is formally weaker than the statement of Theorem C, it would be interesting to have an answer to the following, purely topological, question:

*Question* 6.3.2. Let $M \xrightarrow{f} N$ be a map from a closed manifold $M$ to a nilmanifold $N$ such that the homotopy fiber of $f$ is a simply connected closed manifold. Is it true that after passing to a finite cover, the map $f$ becomes homotopic to a fiber bundle projection?

*Question* 6.3.3. Is it true that manifolds that are almost nonnegatively curved in the generalized sense are almost nonnegatively curved?

In view of Theorems A and B it is also reasonable to pose the following question:

*Question* 6.3.4. Is it true that almost nonnegatively curved $m$-manifolds $M^m$ are $C(m)$-nilpotent spaces?

It is clear from the proof of Theorems A and B that this is true if the universal cover of $M^m$ has torsion free integral cohomology.

In view of Theorem B it is also natural to raise the following question:

*Question* 6.3.5. Can one give an explicit bound on $C(m)$ in Theorem B?

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