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Abstract

This is the first of two papers where we address and partially confirm a conjecture of Deser and Schwimmer, originally postulated in high energy physics. The objects of study are scalar Riemannian quantities constructed out of the curvature and its covariant derivatives, whose integrals over compact manifolds are invariant under conformal changes of the underlying metric. Our main conclusion is that each such quantity that locally depends only on the curvature tensor (without covariant derivatives) can be written as a linear combination of the Chern-Gauss-Bonnet integrand and a scalar conformal invariant.

1. Introduction

1.1. *Outline of the problem.* Consider any Riemannian manifold (M^n, g^n). The basic local objects that describe the geometry of the metric g^n are the curvature tensor R_{ijkl} and the Levi-Civita connection ∇_{g^n}. We are interested in *intrinsic scalar quantities* $P(g^n)$. These scalar quantities, as defined by Weyl (see also [14] and [4]), are polynomials in the components of the tensors $R_{ijkl}, \dots, \nabla^m_{r_1 \dots r_m} R_{ijkl}$, $\dots$ and g^{ij}, with two basic features: The values of these polynomials must be invariant under changes of the coordinate system (or isometries), and there must also be a number K so that under the re-scaling $g^n \longrightarrow t^2 g^n$ $(t \in \mathbb{R}_+)$, we have $P(t^2 g^n) = t^K P(g^n)$. We then say that $P(g^n)$ is a *scalar Riemannian invariant* of weight K.

It is a classical result, implied in Weyl's work [22], that any such Riemannian invariant $P(g^n)$ of weight K can be written as a linear combination

$$P(g^n) = \sum_{l \in L} a_l C^l(g^n) \tag{1}$$

of complete contractions $C^l(g^n)$ in the form:

$$\text{contr}(\nabla^{m_1}_{r_1\dots r_{m_1}} R_{i_1 j_1 k_1 l_1} \otimes \dots \otimes \nabla^{m_r}_{t_1\dots t_{m_r}} R_{i_r j_r k_r l_r}), \tag{2}$$

for which $C^l(t^2 g^n) = t^K C^l(g^n)$.

This notion of *intrinsic* extends to vector fields. We define an intrinsic vector field $T^a(g^n)$ (a is the free index) of weight K to be a polynomial in the components of the tensors $R_{ijkl}, \dots, \nabla^m_{r_1\dots r_m} R_{ijkl}, \dots$ and g^{ij}, with the property that under changes of coordinates (isometries) that map the coordinate functions $x^1, \dots, x^n$ to the coordinate functions $y^1, \dots, y^n$, $T_a(g^n)$ must satisfy the transformation law:

$$T'^{\alpha}(g^n) = T^i(g^n)\frac{\partial y^\alpha}{\partial x^i},$$

here T'^{α} stands for the vector field expressed in the new coordinate system. Moreover, we say that $T^a(g^n)$ has weight K if $T^a(t^2 g^n) = t^{K-1} T^a(g^n)$.'

By Weyl's work, an intrinsic vector field of weight K must be a linear combination of partial contractions with one free index, each in the form:

$$\text{pcontr}(\nabla^{m_1}_{r_1\dots r_{m_1}} R_{ijkl} \otimes \dots \otimes \nabla^{m_r}_{t_1\dots t_{m_r}} R_{i_r j_r k_r l_r}). \tag{3}$$

We recall that under general conformal re-scalings $\hat{g}^n = e^{2\phi(x)} g^n$ the volume form re-scales by the formula $dV_{\hat{g}^n} = e^{n\phi(x)} dV_{g^n}$; in particular for any constant t we have $dV_{t^2 g^n} = t^n dV_{g^n}$. Thus, for any scalar Riemannian invariant $P(g^n)$ of weight $-n$, $\int_{M^n} P(g^n) dV_{g^n}$ is scale-invariant for all compact and orientable manifolds M^n.

The problem we address in this paper and in [2] is to find all the Riemannian invariants $P(g^n)$ of weight $-n$ for which the integral:

$$\int_{M^n} P(g^n) dV_{g^n} \tag{4}$$

is invariant under conformal re-scalings of the metric g^n on any M^n compact without boundary.

In other words, we are requiring that for every real-valued function $\phi(x) \in C^\infty(M^n)$ we must have that for $\hat{g}^n(x) = e^{2\phi(x)} g^n$:

$$\int_{M^n} P(\hat{g}^n) dV_{\hat{g}^n} = \int_{M^n} P(g^n) dV_{g^n}. \tag{5}$$

This question was originally raised by Deser and Schwimmer in [11] (see also [20] and [5]) in the context of understanding "conformal anomalies". On the other hand, an answer to this question would also shed light on the structure of Q-curvature in high dimensions. The problem, as posed in [11], is the following:

Conjecture 1 (Deser-Schwimmer). Consider a Riemannian scalar $P(g^n)$ of weight $-n$, for some even n. Suppose that for any compact manifold (M^n, g^n) the quantity

$$\int_{M^n} P(g^n) dV_{g^n} \tag{6}$$

is invariant under any conformal change of metric $\hat{g}^n(x) = e^{2\phi(x)} g^n(x)$. Then $P(g^n)$ must be a linear combination of three"obvious candidates", namely:

$$P(g^n) = W(g^n) + \mathrm{div}_i T_i(g^n) + c \cdot \mathrm{Pfaff}(R_{ijkl}). \tag{7}$$

1. $W(g^n)$ is a scalar conformal invariant of weight $-n$; in other words it satisfies $W(e^{2\phi(x)} g^n)(x) = e^{-n\phi(x)} W(g^n)(x)$ for every $\phi \in C^\infty(M^n)$ and every $x \in M^n$.
2. $T^i(g^n)$ is a Riemannian vector field of weight $-n+1$, since for any compact M^n we have $\int_{M^n} \mathrm{div}_i T_i(g^n) dV_{g^n} = 0$.
3. $\mathrm{Pfaff}(R_{ijkl})$ stands for the Pfaffian of the curvature R_{ijkl}, since for any compact Riemannian (M^n, g^n),
$$\int_{M^n} \mathrm{Pfaff}(R_{ijkl}) dV_{g^n} = \frac{2^n \pi^{\frac{n}{2}} (\frac{n}{2}-1)!}{2(n-1)!} \chi(M^n).$$

In this paper and in [2] we show:

THEOREM 1. *Conjecture* 1 *is true*, *in the following restricted version*:

Suppose that (6) *holds*, *and additionally that* $P(g^n)$ *locally depends only on the curvature tensor* R_{ijkl} *and not its covariant derivatives* $\nabla^m R_{ijkl}$ (*meaning that* $P(g^n)$ *is a linear combination of contractions in the form* (2) *with* $m_1 = \cdots = m_r = 0$). *Then*, *there exists a scalar conformal invariant* $W(g^n)$ *of weight* $-n$ *that locally depends only on the Weyl tensor*, *and also a constant* c *so that*:

$$S(g^n) = W(g^n) + c \cdot \mathrm{Pfaff}(R_{ijkl}) \tag{8}$$

where $\mathrm{Pfaff}(R_{ijkl})$ *stands for the Pfaffian of the curvature* R_{ijkl}.

1.2. *Geometric applications of the Deser-Schwimmer conjecture*: Q*-curvature and re-normalized volume.* Q-curvature is a Riemannian scalar quantity introduced by Branson for each even dimension n (see [6]). In dimension 2, $Q^2(g^2) = R(g^2)$ (the scalar curvature), while in dimension 4 its structure is well-understood and has been extensively studied. Its fundamental property is that $Q^n(g^n)$ has weight $-n$ in dimension n and that the integral $\int_{M^n} Q^n(g^n) dV_{g^n}$ over compact manifolds M^n is invariant under conformal charges of the underlying metric g^n. Thus, if one proves Conjecture 1 in full strength, one would derive that $Q^n(g^n)$ can be decomposed as in the right-hand side of (7), in fact with $c \neq 0$.

This fact is all the more interesting due to the *nice* transformation law of Q-curvature under conformal changes $\hat{g}^n = e^{2\phi(x)} g^n$. One then has that $e^{n\phi(x)} Q^n(\hat{g}^n)$

$= Q^n(g^n) + P^n_{g^n}(\phi)$, where $P^n_{g^n}(\phi)$ is a *conformally invariant differential operator*, originally constructed in [16]. Thus, prescribing the Q-curvature can be informally interpreted as prescribing a modified version of the Chern-Gauss-Bonnet integrand Pfaff(R_{ijkl}). This *modified Pfaffian* enjoys a nice transformation law under conformal re-scalings, rather than the messy transformation that governs Pfaff(R_{ijkl}).

This understanding of the structure of Q-curvature in any even dimension raises the question whether the strong results of Chang, Yang, Gursky, Qing et al. in dimension 4 (see for example [8], [9], [19]), have analogues in higher dimensions. Moreover, a proof of Conjecture 1 in full strength will lead to a better understanding of the notion of *re-normalized volume* for conformally compact Einstein manifolds.

Conformally compact Einstein manifolds have been the focus of much research in recent years; see [9], [18], [21], [24], to name just a few. What follows is a very brief discussion of the objects of study, largely reproduced from [18].

We consider manifolds with boundary, (X^{n+1}, g^{n+1}), $\partial X^{n+1} = M^n$, where the boundary M^n carries a conformal structure $[h^n]$. We consider a defining function x for ∂X^{n+1} in X:

$$x|_{\mathring{X}} > 0, \quad x|_{\partial X} = 0, \quad dx|_{\partial X} \neq 0.$$

We then say that g^{n+1} is a *conformally compact* metric on X^{n+1} with *conformal infinity* $[h^n]$, if there exists a smooth metric $\overline{g}^{n+1}$ on $\overline{X}^{n+1}$ so that in $\mathring{X}^{n+1}$:

$$g^{n+1} = \frac{\overline{g}^{n+1}}{x^2}, \quad \overline{g}^{n+1}|_{\partial X^{n+1}} \in [h^n].$$

A conformally compact metric is *asymptotically hyperbolic*, in the sense that its sectional curvatures approach -1 as x approaches 0. We notice that since we can pick different defining functions, the metric g^{n+1} in the interior $\mathring{X}^{n+1}$ defines a conformal class $[h^n]$ on the boundary. In the rest of this discussion, we will be considering conformally compact manifolds (X^{n+1}, g^{n+1}) which in addition are Einstein.

Conformally compact Einstein manifolds are studied as models for the Anti-de-Sitter/Conformal Field Theory (AdS-CFT) correspondence in string theory. In order to compute the partition function for the conformal field theory in the supergravity approximation, one must evaluate the gravitational action $\int_{X^{n+1}} R dV_{g^n}$ for the metric g^{n+1}, which in the case at hand is proportional to the volume of (X^{n+1}, g^{n+1}). Since this volume is clearly infinite (g^{n+1} is asymptotically hyperbolic) one regularizes it through re-normalization, thus introducing the *re-normalized volume*. We briefly discuss this re-normalization procedure and its relation to Q-curvature below. For a more detailed discussion we refer the reader to [15], [17], [23] and the references therein.

It is known that each choice of metric $h \in [h^n]$ on the boundary M^n uniquely determines a defining function x in a collar neighborhood of ∂X^{n+1} in X^{n+1}, say $\partial X^{n+1} \times [0, \varepsilon]$, so that g^{n+1} takes the form:

$$g^{n+1} = x^{-2}(dx^2 + h_x),\ h_0 = h, \tag{9}$$

where h_x is a 1-parameter family of metrics on ∂X^{n+1}. We then consider the volume of the region $R_\varepsilon = \{x > \varepsilon\}$ in (X^{n+1}, g^{n+1}), expanded out in powers of ε, and let $\varepsilon \to 0$. Given that g^{n+1} is Einstein, it follows that if n is odd:

$$\mathrm{vol}_{g^{n+1}}(\{x > \varepsilon\}) = c_0\varepsilon^{-n} + c_2\varepsilon^{-n+2} + \cdots + c_{n-1}\varepsilon^{-1} + V + o(1) \tag{10}$$

whereas if n is even:

$$\mathrm{vol}_{g^{n+1}}(\{x > \varepsilon\}) = c_0\varepsilon^{-n} + c_2\varepsilon^{-n+2} + \cdots + c_{n-1}\varepsilon^{-2} + L\log\left(\frac{1}{\varepsilon}\right) + V + o(1). \tag{11}$$

Moreover, if n is odd and since g^{n+1} is Einstein, then (see [18]) V is independent of the choice of metric h^n in the conformal class $[h^n]$. (Recall that this choice was used in order to write out g^{n+1} in the form (9), and hence also in defining the region R_ε; therefore V depends apriori on the choice $h^n \in [h^n]$). For n odd, V is called the *re-normalized volume* of (X^{n+1}, g^{n+1}).

For n even, V is *not* independent of the choice of metric h^n in the conformal class $[h^n]$. In this case it is the quantity L that demonstrates this invariance. This quantity L represents the failure of defining the re-normalized volume independently of the defining function x. It is therefore called the "conformal anomaly" in the physics literature. Moreover, Graham-Zworski have shown that $L = c_n \cdot \int_{M^n} Q(h^n) dV_{h^n}$, where h^n is an arbitrary metric in the conformal class $[h^n]$. Hence, a proof of Conjecture 1 would immediately imply that L can be written out as:

$$L = \int_{M^n} W(h^n) dV_{h^n} + (\mathrm{Const}) \cdot \chi(M^n) \tag{12}$$

where $W(h^n)$ is a scalar conformal invariant of weight $-n$ and $M^n = \partial X^{n+1}$, while $\chi(M^n)$ stands for the Euler characteristic of M^n and (Const) $\neq 0$.

Another significant result has recently been obtained by Chang, Qing and Yang, [10], relating the re-normalized volume V with the Q-curvature of g^{n+1} and hence with the Euler characteristic of the manifold X^{n+1}. They show that if Conjecture 1 is true, then for n odd one can express the re-normalized volume of (X^{n+1}, g^{n+1}) via the Q-curvature:

$$R.V.[(X^{n+1}, g^{n+1})] = \int_{X^{n+1}} W(g^{n+1}) dV_{g^{n+1}} + (\mathrm{const})_{n+1} \cdot \chi(X^{n+1}) \tag{13}$$

where $(\mathrm{Const})_{n+1}$ is a nonzero dimensional constants and $W(g^{n+1})$ is a scalar conformal invariant of weight $-n-1$. Here the left-hand side stands for the re-normalized volume of the manifold (X^{n+1}, g^{n+1}). Hence, it follows that the re-normalized volume explicitly depends on the topology of X^{n+1}, via its Euler characteristic. A result related to (13) has been independently established (by an entirely different method) by Albin in [1].

This identity raises the question of whether one can adapt the powerful techniques developed for the study of Q-curvature to the study of conformally compact Einstein manifolds. Strong results have already been obtained in dimension 4; see [9]. For higher dimensions one might try to extend the work of Brendle [7] to this setting. Another question would be whether one can obtain expressions analogous to (12) and (13) for the re-normalized areas and conformal anomalies of submanifolds, defined by Graham and Witten in [17].

1.3. *Outline of the paper.* Our theorem is a structure result for $P(g^n)$. We use the "global" conformal invariance under integration of $P(g^n)$ to derive information on its algebraic expression.

In this paper we introduce the main tool that will show Theorem 1, the so-called super divergence formula. For each $P(g^n)$ that satisfies (5), we define an operator $I_{g^n}(\phi)$ that measures the "non-conformally invariant part" of $P(g^n)$ (see (26) below). We then use the property (27) of $I_{g^n}(\phi)$ to derive an explicit local formula which expresses $I_{g^n}(\phi)$ as a divergence of a vector field. This formula, which in our opinion is also of independent interest, thus provides us with an understanding of the algebraic structure of $I_{g^n}(\phi)$. In the sequel to this paper, [2], we will use the super divergence formula to derive information on the algebraic structure of $P(g^n)$ and prove Theorem 1.

The super divergence formula is proven in a number of steps. A more primitive version is the "simple divergence formula" in Section 5. This is then refined three times in Section 6 and we obtain the super divergence formula in subsection 6.3. The only background material needed for all this work is a slight extension of Theorem B.4 in [3], which itself is a generalization of a classical theorem of Weyl in [22]. This extension is discussed in Section 3. Roughly, Theorem B.4 in [3] and our Theorem 2 below assert that a linear identity involving complete contractions which holds for all values we can give to the tensors in those contractions, must then also hold *formally*.

2. **Background material**

2.1. *Definitions and identities.* Whenever we refer to a manifold M^n, we will be assuming it to be *compact* and *orientable*. Moreover, n will be a *fixed*, *even* dimension throughout this paper. We begin by recalling a few definitions and formulas.

Definition 1. In this paper, we will be dealing with *complete contractions of tensors* and their *linear combinations*. *Any complete contraction*:

$$C = \mathrm{contr}((A^1)_{i_1\dots i_s}\otimes\dots\otimes(A^t)_{j_1\dots j_q})$$

will be seen as a formal expression. Each factor $(A^l)_{i_1\dots i_s}$ is an ordered set of slots. Given the factors $(A^1)_{i_1\dots i_s},\dots,(A^t)_{j_1\dots j_q}$, a *complete contraction* is then seen as a set of pairs of slots $(a_1,b_1),\dots,(a_w,b_w)$, with the following properties: if $k\neq l$, $\{a_l,b_l\}\bigcap\{a_k,b_k\}=\varnothing$, $a_k\neq b_k$, $\bigcup_{i=1}^{w}\{a_i,b_i\}=\{i_1,\dots,j_q\}$. Each pair corresponds to a particular contraction.

Two complete contractions

$$\mathrm{contr}((A^1)_{i_1\dots i_s}\otimes\dots\otimes(A^t)_{j_1\dots j_w}) \text{ and } \mathrm{contr}((B^1)_{f_1\dots f_q}\otimes\dots\otimes(B^{t'})_{v_1\dots v_z})$$

will be *identical* if $t=t'$, $(A^l)=(B^l)$ and if the μ^{th} index in A^l contracts against the ν^{th} index in A^r, then the μ^{th} index in B^l contracts against the ν^{th} index in B^r. For any complete contraction, we define its *length* to stand for the number of its factors.

We can also consider linear combinations of complete contractions:

$$\sum_{l\in L} a_l(C_1)^l \text{ and } \sum_{r\in R} b_r(C_2)^r.$$

Two linear combinations as above are considered *identical* if $R=L$ and $a_l=b_l$ and $(C_1)^l=(C_2)^l$. A linear combination of complete contractions as above is identically zero if for every $l\in L$ we have that $a_l=0$.

For any complete contraction C, we will say a factor $(A)_{r_1\dots r_{s_l}}$ has an *internal contraction* if two indices in $(A)_{r_1\dots r_{s_l}}$ are contracting between themselves.

All the above definitions extend to partial contractions and their linear combinations.

We also introduce two language conventions: For any linear combination of complete contractions $\sum_{l\in L} a_l C^l$, when we speak of a *sublinear combination*, we will mean some linear combination $\sum_{l\in L'} a_l C^l$ where $L'\subset L$. Also, when we say that an identity between linear combinations of complete contractions:

$$\sum_{r\in R} a_r C^r = \sum_{t\in T} a_t C^t \tag{14}$$

holds modulo complete contractions of length $\geq\lambda$, we will mean that we have an identity:

$$\sum_{r\in R} a_r C^r = \sum_{t\in T} a_t C^t + \sum_{u\in U} a_u C^u \tag{15}$$

where each C^u has at least λ factors.

Definition 2. Now, for each tensor $T_{ab\dots d}$ and each subset $\{d,e,\dots f\} \subset \{a,b,\dots,d\}$, we define the *symmetrization of the tensor* $T_{ab\dots d}$ *over the slots* $d,e,\dots,f$:

Let Π stand for the set of permutations of the ordered set $\{d,e,\dots,f\}$. For each $\pi \in \Pi$, we define $\pi T_{ab\dots f}$ to stand for the *tensor* that arises from $T_{ab\dots f}$ by permuting the slots $d,e,\dots,f$ according to the permutation π. We then define the *symmetrization of the tensor* $T_{ab\dots d}$ *over the slots* $d,e,\dots,f$ to be:

$$\sum_{\pi\in\Pi} \frac{1}{|\Pi|}\cdot \pi T_{ab\dots d}.$$

If $\{d,e,\dots,f\}=\{a,b,\dots,d\}$, we will denote that *symmetrization by* $T_{(ab\dots d)}$.

We recall a few basic facts from Riemannian geometry. Consider any Riemannian manifold (M^n,g^n) and any $x_0 \in M^n$. We pick any coordinate system $x^1,\dots,x^n$ and denote by X^i the coordinate vector fields, i.e. the vector fields $\frac{\partial}{\partial x^i}$. We will write ∇_i instead of ∇_{X^i}.

The curvature tensor R_{ijkl} of g^n is given by the formula:

$$[\nabla_i\nabla_j - \nabla_j\nabla_i]X_k = R_{ijkl}X^l. \tag{16}$$

In a coordinate system, we can also express it in terms of the Christoffel symbols:

$$R^l_{ijk} = \partial_j\Gamma^l_{ik} - \partial_k\Gamma^l_{ij} + \sum_m(\Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ik}\Gamma^l_{mk}). \tag{17}$$

Moreover, the Ricci tensor Ric_{ik} is then:

$$\mathrm{Ric}_{ik} = R_{ijkl}g^{jl}. \tag{18}$$

We recall the two Bianchi identities:

$$R_{ABCD} + R_{CABD} + R_{BCAD} = 0, \tag{19}$$

$$\nabla_A R_{BCDE} + \nabla_C.R_{ABDE} + \nabla_B R_{CADE} = 0. \tag{20}$$

We also recall how the basic geometric objects transform under the conformal change $\hat{g}^n(x) = e^{2\phi(x)}g^n(x)$. These formulas come from [12].

$$\begin{aligned} R^{\hat{g}^n}_{ijkl} = e^{2\phi(x)}[&R^{g^n}_{ijkl} + \nabla_{il}\phi g_{jk} + \nabla_{jk}\phi g_{il} - \nabla_{ik}\phi g_{jl} - \nabla_{jl}\phi g_{ik} \\ &+\nabla_i\phi\nabla_k\phi g_{jl} + \nabla_j\phi\nabla_l\phi g_{ik} - \nabla_i\phi\nabla_l\phi g_{jk} - \nabla_j\phi\nabla_k\phi g_{il} \\ &+|\nabla\phi|^2 g_{il}g_{jk} - |\nabla\phi|^2 g_{ik}g_{lj}], \end{aligned} \tag{21}$$

$$\mathrm{Ric}^{\hat{g}^n}_{\alpha\beta} = \mathrm{Ric}^{g^n}_{\alpha\beta} + (2-n)\nabla^2_{\alpha\beta}\phi - \Delta\phi g^n_{\alpha\beta} + (n-2)(\nabla_\alpha\phi\nabla_\beta\phi - |\nabla\phi|^2 g_{\alpha\beta}), \tag{22}$$

while the transformation law for the Levi-Civita connection is:

$$\nabla_k^{\hat{g}^n}\eta_l = \nabla_k^{g^n}\eta_l - \nabla_k\phi\eta_l - \nabla_l\phi\eta_k + \nabla^s\phi\eta_s g_{kl}. \tag{23}$$

We now focus on complete contractions $C(g^n)$ in the form (2). We still think of these objects both as formal expressions and also as functions of the metric g^n. Thus, for complete contractions in the form (2), contracting two lower indices ${}_{a},{}_{b}$ will mean that we multiply by g^{ab} and then sum over a, b. We have that under the rescaling $\hat{g}^n = t^2 g^n$ the tensors $\nabla^m R_{ijkl}$ and $(g^n)^{ij}$ transform by $\nabla^m_{r_1\dots r_m} R_{ijkl}(t^2 g^n) = t^2 \nabla^m_{r_1\dots r_m} R_{ijkl}(g^n)$, $(g^n)^{ij}(t^2 g^n) = t^{-2}(g^n)^{ij}(g^n)$. (We will sometimes write $\nabla^m R_{ijkl}$ instead of $\nabla^m_{r_1\dots r_m} R_{ijkl}$, for brevity.) Thus, for each $C(g^n)$ in the form (2), if we define $K = -\sum_{i=1}^r (m_i+2)$, we will have that $C(t^2 g^n) = t^K C(g^n)$. We define K to be the *weight* of this complete contraction.

For future reference, we will consider more general complete contractions defined on manifolds (M^n, g^n) and define their weight.

Definition 3. We consider any *complete contraction* $C_{g^n}(V^1, \dots, V^x)$ in the form:

$$\mathrm{contr}(\nabla^{m_1} R_{ijkl} \otimes \dots \otimes \nabla^{m_r} R_{ijkl} \otimes V^1_{a_1\dots a_{f_1}} \otimes \dots \otimes V^x_{b_1\dots b_{f_x}}) \tag{24}$$

defined for any $x_0 \in M^n$. Here the tensors $V^y_{a_1\dots a_{f_y}}$ are *auxiliary tensors* (all of whose indices are lowered) that have a scaling property under re-scalings of the metric: $V^y_{a_1\dots a_{f_y}}(t^2 g^n) = t^{C_y} V^y_{a_1\dots a_{f_y}}(g^n)$. (An example for a tensor $V^y_{a_1\dots a_{f_y}}$ would be the y^{th} iterated covariant derivative of a function ψ, in which case $C_y = 0$). Moreover, all the tensors here are over $TM^n|_{x_0}$. The particular contractions of any two lower indices will be with respect to the quadratic form $(g^n)^{ij}(x_0)$.

We then define the *weight* of such a complete contraction to be

$$W = -\sum_{i=1}^r (m_i + 2) - \sum_{i=1}^x (f_i - C_y).$$

As for the previous case, we then have that:

$$C_{t^2 g^n}(V^1, \dots, V^x) = t^W C_{g^n}(V^1, \dots, V^x).$$

In this whole paper, when we write a complete contraction and include the metric g^n in the notation, we will imply that the contraction is defined on manifolds (and possibly also depending on additional auxiliary objects, for example scalar functions) and will have a weight, as defined above. Unless otherwise stated, all complete contractions will have weight $-n$.

2.2. *The operator $I_{g^n}(\phi)$ and its polarizations.* For this paper and in [2], we will consider $P(g^n)$ as a linear combination in the form:

$$P(g^n) = \sum_{l\in L} a_l C^l(g^n) \tag{25}$$

where each $C^l(g^n)$ is in the form (2) and has weight $-n$. We assume that $P(g^n)$ satisfies (5).

We define a differential operator, which will depend both on the metric g^n and the auxiliary $\phi \in C^\infty(M^n)$:

$$I_{g^n}(\phi)(x) = e^{n\phi(x)} P(e^{2\phi(x)} g^n)(x) - P(g^n)(x). \tag{26}$$

We then have by (5) that:

$$\int_{M^n} I_{g^n}(\phi) dVg^n = 0 \tag{27}$$

for every compact manifold (M^n, g^n) and any function $\phi \in C^\infty(M^n)$. Then, using the transformation laws (21) and (23) we see that $I_{g^n}(\phi)$ is a differential operator acting on the function ϕ. In particular, we can pick any $A > 0$ functions $\psi_1(x), \dots, \psi_A(x)$, and choose:

$$\phi(x) = \sum_{l=1}^{A} \psi_l(x).$$

Hence, we have a differential operator $I_{g^n}(\psi_1, \dots, \psi_A)(x)$, so that, by (27):

$$\int_{M^n} I_{g^n}(\psi_1, \dots, \psi_A) dV_{g^n} = 0$$

for any (M^n, g^n), M^n compact and any functions $\psi_1(x), \dots, \psi_A(x) \in C^\infty(M^n)$.

Now, for any given functions $\psi_1(x), \dots, \psi_A(x)$, we can consider re-scalings:

$$\lambda_1\psi_1(x), \dots, \lambda_A\psi_A(x).$$

Hence, as above we will have the equation:

$$\int_{M^n} I_{g^n}(\lambda_1\psi_1, \dots, \lambda_A\psi_A) dV_{g^n} = 0. \tag{28}$$

We can then see $\int_{M^n} I_{g^n}(\lambda_1\psi_1, \dots, \lambda_A\psi_A) dV_{g^n}$ as a polynomial in the factors $\lambda_1, \dots, \lambda_A$. Call this polynomial $\Pi(\lambda_1, \dots, \lambda_A)$.

But then relation (28) gives us that this polynomial Π is identically zero. Hence, each coefficient of each monomial in the variables $\lambda_1, \dots, \lambda_A$ must be zero. We want to pick out a particular such monomial. Pick out any integer $1 \le Z \le A$. Then in $I_{g^n}(\lambda_1\psi_1, \dots, \lambda_A\psi_A)$ (seen as a multi-variable polynomial in $\lambda_1, \dots, \lambda_A$) consider the coefficient of the monomial $\lambda_1 \cdot \dots \cdot \lambda_Z$. This will be a differential operator in the functions $\psi_1, \dots, \psi_Z$, which we will denote by $I^Z_{g^n}(\psi_1, \dots, \psi_Z)$. By elementary properties of polynomials and by the definition of $I_{g^n}(\phi)$ in (26) we have:

$$\begin{aligned} &I^Z_{g^n}(\psi_1, \dots, \psi_Z) \\ &= \partial_{\lambda_1} \dots \partial_{\lambda_Z} [e^{n(\lambda_1\psi_1 + \dots + \lambda_Z\psi_Z)} P(e^{2(\lambda_1\psi_1 + \dots + \lambda_Z\psi_Z)} g^n)]|_{\lambda_1=0,\dots,\lambda_Z=0}. \end{aligned} \tag{29}$$

The precise form of $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$, given $P(g^n)$, can be calculated using the transformation laws in the previous section. We do this in [2]. For the time being, just note that by (28) we have the equation:

LEMMA 1.

$$\int_{M^n} I^Z_{g^n}(\psi_1,\dots,\psi_Z)dV_{g^n}=0 \tag{30}$$

for every compact (M^n,g^n) *and any* $\psi_1,\dots,\psi_Z\in C^\infty(M^n)$.

Proof. This is straightforward from relation (28) and the equation (29). □

From all the above, it is easy to see that $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$ is a linear combination of complete contractions of weight $-n$ in the form:

$$\begin{aligned}\operatorname{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1}\otimes\dots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}\\ \otimes\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\psi_1\otimes\dots\otimes\nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\psi_Z\big).\end{aligned} \tag{31}$$

For the rest of this paper, we will only be using the fact that $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$ satisfies (30) and that it is a linear combination of complete contractions in the form (31).

3. The trans-dimensional isomorphisms

The aim of this section is to establish a natural isomorphism of linear combinations of complete contractions in the form (31) of weight $-n$, between dimensions N and n, if $N\ge n$. In order to make this statement precise and to provide a proof, we will recall some terminology and facts from the appendices in [3]. The main "known fact" to be used is Theorem 2 in the next subsection. This theorem is a slight generalization of Theorem B.4 in [3], and it can be proven using the same ideas. The appendices in [3] generalize classical theorems that can be found in [22].

3.1. *Known facts.* The appendices of [3] deal with identities involving linear combinations of complete contractions. The main assertion there is that under certain hypotheses, when a linear identity involving complete contractions holds "by substitution", it must then also hold "formally". We will be explaining these notions in this subsection. For more details, we refer the reader to [3].

We introduce the "building blocks" of our complete contractions. Firstly, we consider symmetric tensors. Let us consider a family of sets of symmetric tensors $\{T^\alpha=\{T^\alpha_0,T^\alpha_i,\dots,T^\alpha_{i_1\dots i_s},\dots\}\}_{\alpha\in A}$ (T^α_0 is just a scalar, i.e. a tensor of rank zero), defined over the vector space $\mathbb{R}^n$. Here each $\alpha\in A$ is not a free index of the tensors $T^\alpha_{i_1\dots i_s}$. It just is a label that serves to distinguish the tensors $T^{\alpha_1}_{i_1\dots i_s}$ and $T^{\alpha_2}_{i_1\dots i_s}$ when $\alpha_1\neq\alpha_2$.

Our second building block will be a list of tensors that resemble the covariant derivatives of the curvature tensor:

Definition 4. A *set of linearized curvature tensors* is defined to be a list of tensors $R = \{R_{ijkl}, \dots, R_{f_1\dots f_s,ijkl}, \dots\}$ defined over $\mathbb{R}^n$, so that each $R_{x_1\dots x_s,ijkl}$ satisfies the following identities:

1. $R_{x_1\dots x_s,ijkl}$ is *symmetric* in $x_1, \dots, x_s$,
2. $R_{x_1\dots[x_s,ij]kl} = 0$,
3. $R_{x_1\dots x_s,[ijk]l} = 0$,
4. $R_{x_1\dots x_s,ijkl} = -R_{x_1\dots x_s,jikl}$, $R_{x_1\dots x_s,ijkl} = -R_{x_1\dots x_s,ijlk}$,

where in general, $T_{r_1\dots r_m[i_1i_2i_3]f_1\dots f_d}$ will stand for the sum over all the cyclic permutations of the indices i_1, i_2, i_3 (in the case where two of the indices i_1, i_2, i_3 are antisymmetric).

Our third building block is the following set:

Definition 5. Consider a set of tensors $\Xi = \{\Xi^{k_1}_i, \dots \Xi^{k_s}_{i_1\dots i_s}, \dots\}$, where the free indices are $i_1, \dots, i_s, k_s$. Assume that each tensor $\Xi^{k_s}_{i_1\dots i_s}$ is symmetric in the indices $i_1, \dots, i_s$. We call any such tensor a *special tensor*. Any such set Ξ will be called a *set of special tensors*.

We can then form complete contractions of tensors that belong to the sets $\bigcup_{\alpha\in A}\{T^\alpha\}$, R, Ξ. They will be in the form:

$$\mathrm{contr}(u^{l_1} \otimes \cdots \otimes u^{l_Z} \otimes R^{r_1} \otimes \cdots \otimes R^{r_m} \otimes \Xi^{z_1} \otimes \cdots \otimes \Xi^{z_x}) \tag{32}$$

where each tensor u^{l_i} belongs to the set $\bigcup_{\alpha\in A}\{T^\alpha\}$, each tensor R_{r_j} belongs to the set $R = \{R_{ijkl}, \dots, R^s_{f_1\dots f_s,ijkl}, \dots\}$ and each tensor Ξ^z belongs to the set $\Xi = \{\Xi^k_i, \dots \Xi^k_{i_1\dots i_s}, \dots\}$. A particular contraction of two lower indices will be with respect to the Kronecker δ^{ij}, while for an upper and lower index we will be using the Einstein summation convention. We can consider linear combinations of such complete contractions: $\Lambda(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi) = \sum_{l\in L} a_l C^l(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi)$.

For each complete contraction $C(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi)$ that contains a factor $t = R_{i_1\dots i_s,ijkl}$, we will say that we apply the third identity in Definition 4 to the indices i, j, k (or that we permute indices according to the third identity) if we replace the complete contraction $C(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi)$, by

$$-C_1\Big(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi\Big) - C_2\Big(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi\Big),$$

where $C_1(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi)$ is obtained from $C(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi)$ by replacing t by $R_{i_1\dots i_s,kijl}$ and $C_2(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi)$ is obtained from $C(\bigcup_{\alpha\in A}\{T^\alpha\}, R, \Xi)$ by replacing t by $R_{i_1\dots i_s,jkil}$. We similarly define what it means to apply the second identity in Definition 4. It is clear what is meant by applying the first and fourth identities (or by permuting indices according to the first and fourth identities).

Definition 6. Such a *linear combination of complete contractions* vanishes formally if we can can make the linear combination zero using the following list of operations:

By permuting factors in the complete contractions, by permuting indices in the factors in $\bigcup_{\alpha\in A}\{T^\alpha\}$, by using the identities of the factors in R, by permuting the indices $i_1,\dots,i_s$ in the factors $\Xi^{k_s}_{i_1\dots i_s}$ and by applying the distributive rule

$$a\cdot C^l\Big(\bigcup_{\alpha\in A}\{T^\alpha\},R,\Xi\Big)+b\cdot C^l\Big(\bigcup_{\alpha\in A}\{T^\alpha\},R,\Xi\Big)\\ =(a+b)\cdot C^l\Big(\bigcup_{\alpha\in A}\{T^\alpha\},R,\Xi\Big).$$

Also, we will say that the linear combination $\Lambda(\bigcup_{\alpha\in A}\{T^\alpha\},R,\Xi)$ *vanishes upon substitution* if for each set of tensors $\bigcup_{\alpha\in A}\{T^\alpha\}$, R and Ξ that have the above properties, the value of $\Lambda(\bigcup_{\alpha\in A}\{T^\alpha\},R,\Xi)$ is zero.

The following theorem is then an extension of Theorem B.4 in [3] and it follows by the same ideas.

THEOREM 2. *Let us consider a linear combination of complete contractions $\Lambda(\bigcup_{\alpha\in A}\{T^\alpha\},R,\Xi)=\sum_{l\in L}a_lC^l(\bigcup_{\alpha\in A}\{T^\alpha\},R,\Xi)$ as above. For each complete contraction C^l, we denote by $Z^\sharp_l$ the number of symmetric tensors of rank ≥ 1. We also recall that m_l is the number of linearized curvature tensors and x_l the number of special tensors. We assume that for each C^l the sum $Z^\sharp_l+2m_l+2x_l$ is less than or equal to n.*

We then have that if $\Lambda(\bigcup_{\alpha\in A}\{T^\alpha\},R,\Xi)$ vanishes upon substitution in dimension n, it must also vanish formally.

We note that the theorem above also applies when there are no factors from the set Ξ in our linear combination.

3.2. *Corollaries of Theorem* 2. We derive two corollaries of Theorem 2. We will now be considering complete contractions on manifolds.

Consider an auxiliary list of symmetric tensors $\Omega=\{\Omega_{i_1},\dots,\Omega_{i_1\dots i_s},\dots\}$. We impose the condition that these tensors must remain invariant under re-scalings of the metric g^n, i.e. $\Omega_{i_1\dots i_s}(t^2g^n)=\Omega_{i_1\dots i_s}(g^n)$. We then focus our attention on complete contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ of the form:

$$\text{contr}(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{ijkl}\otimes\dots\otimes\nabla^{m_s}_{t_1\dots t_{m_s}}R_{ijkl}\\ \otimes\nabla^{p_1}_{a_1\dots a_{p_1}}\psi_1\otimes\dots\otimes\nabla^{p_Z}_{b_1\dots b_{p_Z}}\psi_Z\otimes\Omega_{i_1\dots i_{h_1}}\otimes\dots\otimes\Omega_{u_1\dots u_{h_y}}). \tag{33}$$

We assume that $y\geq 0$ (in other words, there may also be no factors $\Omega_{i_1\dots i_s}$). If we write $C^l_{g^r}(\psi_1,\dots,\psi_Z,\Omega)$ (replacing g^n by g^r), we will be referring to a complete contraction as above, but defined on an r-dimensional manifold. We will call this the *re-writing* of the complete contraction $C^l_{g^n}(\psi_1, \dots, \psi_Z, \Omega)$ in dimension r. Also, when we speak of the value of $C^l_{g^r}(\psi_1, \dots, \psi_Z, \Omega)(x_0)$, we will mean the value of the above complete contraction at a point x_0 on a manifold (M^r, g^r), for functions $\psi_1, \dots, \psi_Z$ defined around $x_0 \in M^r$ and for symmetric tensors $\Omega_{i_1 \dots i_s}$ defined at x_0. This terminology extends to linear combinations.

Finally, a note about the weight of the complete contractions: By our definition of weight, if $C^l_{g^r}(\psi_1, \dots, \psi_Z, \Omega)$ has weight $-n$, then in the notation of (33):

$$\sum_{i=1}^{s}(m_i + 2) + \sum_{i=1}^{Z} p_i + \sum_{i=1}^{y} h_i = n. \tag{34}$$

Thus, if we have $Z^\sharp$ factors $\nabla^{p_i}\psi_i$ with $p_i \geq 1$, the above implies that:

$$Z^\sharp + 2s + y \leq n. \tag{35}$$

Definition 7. A relation between complete contractions in the form (33):

$$\sum_{l \in L} a_l C^l_{g^n}(\psi_1, \dots, \psi_Z, \Omega) = 0$$

will hold *formally* if we can make the above sum identically zero by performing the following operations: We may permute factors in any complete contraction $C^l_{g^n}(\psi_1, \dots, \psi_Z)$ and also permute indices within the factors $\Omega_{i_1 \dots i_s}$. Furthermore, for each factor $\nabla^p_{r_1 \dots r_p}\psi_h$, with $p = 2$ we may permute r_1, r_2, while for $p > 2$, we may apply the identity:

$$[\nabla_A \nabla_B - \nabla_B \nabla_A] X_C = R_{ABCD} X^D \tag{36}$$

and for each factor $\nabla^m R_{ijkl}$, we may apply the identities:

1. $\nabla^m_{r_1 \dots r_m} R_{ijkl} = -\nabla^m_{r_1 \dots r_m} R_{jikl} = -\nabla^m_{r_1 \dots r_m} R_{ijlk}$.
2. $\nabla^m_{r_1 \dots [r_m} R_{ij]kl} = 0$.
3. $\nabla^m_{r_1 \dots r_m} R_{[ijk]l} = 0$.
4. The identity (36) above.

The application of the second and third identities above has been defined. To apply the fourth identity to a factor $\nabla^p \psi_h$ or $\nabla^m R_{ijkl}$ means that for each complete contraction $C_{g^n}(\psi_1, \dots, \psi_Z, \Omega)$ of the form (32), for each factor $\nabla^p_{r_1 \dots r_p}\psi_h$ or $\nabla^m_{r_1 \dots r_m} R_{ijkl}$ in $C_{g^n}(\psi_1, \dots, \psi_Z, \Omega)$ and each pair of consecutive derivative indices r_{s-1}, r_s we may write:

$$C_{g^n}(\psi_1, \dots, \psi_Z, \Omega) = C'_{g^n}(\psi_1, \dots, \psi_Z, \Omega) + \sum_{h \in H} a_h C^h_{g^n}(\psi_1, \dots, \psi_Z, \Omega)$$

where $C'_{g^n}(\psi_1, \dots, \psi_Z, \Omega)$ is obtained from $C_{g^n}(\psi_1, \dots, \psi_Z, \Omega)$ by replacing the factor $\nabla^p_{r_1 \dots r_p}\psi_h$ or $\nabla^m_{r_1 \dots r_m} R_{ijkl}$ by $\nabla^p_{r_1 \dots r_s r_{s-1} \dots r_p}\psi_h$ or $\nabla^m_{r_1 \dots r_s r_{s-1} \dots r_m} R_{ijkl}$, respectively, and $\sum_{h\in H} a_h C^h_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ is obtained from $C_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ by replacing the factor $\nabla^p_{r_1\dots r_p}\psi_h\ \nabla^m_{r_1\dots r_m}R_{ijkl}$ by one of the summands in the following expressions, respectively, and then summing again:

$$\begin{aligned}(37)\qquad &\textstyle\sum_{\{a_1,\dots a_x\},\{b_1,\dots b_{s-2-x}\}\subset\{r_1,\dots r_{s-2}\},\{a_1,\dots a_x\}\bigcap\{b_1,\dots b_{s-1-x}\}=\varnothing}\\ &\quad(\nabla^x_{a_1\dots a_x}R_{r_{s-1}r_sr_{s+1}}{}^d)(\nabla^{s-1-x}_{b_1\dots b_{s-1-x}}\nabla^{m-s-1}_{dr_{s+2}\dots r_p}\psi_h\\ &\qquad+\cdots+(\nabla^x_{a_1\dots a_x}R_{r_{s-1}r_sr_p}{}^d)(\nabla^{s-1-x}_{b_1\dots b_{s-1-x}}\nabla^{m-s-1}_{r_{s+1}\dots d}\psi_h);\end{aligned}$$

$$\begin{aligned}(38)\qquad &\textstyle\sum_{\{a_1,\dots a_x\},\{b_1,\dots b_{s-2-x}\}\subset\{r_1,\dots r_{s-2}\},\{a_1,\dots a_x\}\bigcap\{b_1,\dots b_{s-2-x}\}=\varnothing}\\ &\quad(\nabla^x_{a_1\dots a_x}R_{r_{s-1}r_sr_{s+1}}{}^d)(\nabla^{s-1-x}_{b_1\dots b_{s-2-x}}\nabla^{m-s-1}_{dr_{s+2}\dots r_m}R_{ijkl})\\ &\qquad+\cdots+(\nabla^x_{a_1\dots a_x}R_{r_{s-1}r_sl}{}^d)(\nabla^{s-1-x}_{b_1\dots b_{s-1-x}}\nabla^{m-s-1}_{r_{s+1}\dots r_m}R_{ijkd}).\end{aligned}$$

Now, our first corollary of Theorem 2:

LEMMA 2. *Consider complete contractions* $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$, *each in the form* (33) *and with weight* $-n$, *so that the identity*:

$$F_{g^n}(\psi_1,\dots,\psi_Z,\Omega)=\sum_{l\in L}a_lC^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)=0 \tag{39}$$

holds at any point x_0 *for any metric* g^n *and any functions* $\psi_1,\dots,\psi_Z$ *defined around* x_0 *and any symmetric tensors* $\Omega_{i_1\dots i_s}$ *defined over* $TM^n|_{x_0}$. *Then the above identity must hold formally.*

Proof. We consider the minimum length τ, among all the complete contractions in (39). Next, we index the complete contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ of length τ in the set $L^\tau\subset L$. Suppose we can show that, applying the above operations, we can make $\sum_{l\in L^\tau}a_lC^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ formally equal to a linear combination $\sum_{r\in R}a_rC^r_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$, where each complete contraction $C^r_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ has length $\geq\tau+1$.

If we can prove the above claim then using a finite number of iterations we will have proven our lemma. This is true since there is obviously a number T, so that all the complete contractions that arise by iteratively applying the identities of Definition 7 to the complete contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$, $l\in L$, must have length $\leq T$. This follows just by the finiteness of the index set L. The rest of this proof will focus on showing that claim.

In order to accomplish this, we begin with a definition. For any complete contraction $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$, let $\operatorname{lin}C^l(R,\Psi_1,\dots,\Psi_Z,\Omega)$ stand for the complete contraction between linearized curvature tensors and symmetric tensors that is obtained from $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ by replacing each factor $\nabla^m_{t_1\dots t_m}R_{ijkl}$ by a linearized curvature tensor $R_{t_1\dots t_m,ijkl}$, and each factor $\nabla^p_{r_1\dots r_p}\psi_h$ by a symmetric p-tensor $\Psi^h_{r_1\dots r_p}$. We will prove a fact to be used many times in the future.

LEMMA 3. *In the above notation, given* (39), *we have, formally,*

$$\sum_{l\in L^\tau} a_l \operatorname{lin} C^l(R,\Psi_1,\dots,\Psi_Z,\Omega)=0.$$

Proof. We recall the following fact, which follows from the proof of Theorem 2.6 in [14]: Given any set R of linearized curvature tensors $R_{t_1\dots t_m,ijkl}(x_0)$, there is a Riemannian metric defined around x_0 so that for any m:

$$(\nabla^m_{t_1\dots t_m}R_{ijkl})^{g^n}(x_0)=R_{t_1\dots t_m,ijkl}(x_0)+C(R)_{t_1\dots t_m,ijkl} \tag{40}$$

where $C(R)_{t_1\dots t_m,ijkl}$ stands for a polynomial in the components of the linearized curvature tensors. This polynomial depends only on m and the indices $t_1,\dots,t_m$, i,j,k,l. Furthermore, each monomial in that polynomial will have degree at least 2.

For any set R of linearized curvature tensors, we call the metric g^n for which (40) holds the *associated* metric. Now, for any choice of symmetric tensors

$$\{T^1_0,T^1_i,\dots,T^1_{i_1\dots i_s},\dots\},\dots,\{T^Z_0,T^Z_i,\dots,T^Z_{i_1\dots i_s},\dots\},$$

there are functions $\psi_1,\dots,\psi_Z$ defined around x_0 so that: $\nabla^s_{i_1\dots i_s}\psi_l(x_0)=T^l_{i_1\dots i_s}$ (for some arbitrary ordering of the indices $i_1,\dots,i_s$ on the left hand side), and also for each permutation $\pi(i_1\dots i_s)$ of the indices $i_1,\dots,i_s$:

$$\nabla^p_{\pi(i_1\dots i_s)}\psi_h(x_0)=\nabla^p_{i_1\dots i_s}\psi_h(x_0)+C(R,\psi_h)_{i_1\dots i_s} \tag{41}$$

where $C(R,T^h)_{i_1\dots i_s}$ stands for a polynomial in the components of the linearized curvature tensors and of one component of a tensor from the set T^h (of rank ≥ 1). This polynomial depends only on p and the indices $i_1,\dots,i_s$. Furthermore, each monomial in that polynomial will have degree at least 2.

For any choice of symmetric tensors $T^l_{i_1\dots i_s}$, we define the functions ψ_l to be their *associated* functions.

Now, we pick any set R of linearized curvature tensors and any set T of symmetric tensors and consider the value of $F_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ for the associated metric g^n and the associated functions ψ_l. By virtue of our remarks, we see that there is a fixed polynomial $\Pi(T,R,\Omega)$ in the vector space of components of the sets T and R, so that for any given set R of linearized curvature tensors and any set T of symmetric tensors at x_0,

$$\Pi(T,R,\Omega)=F_{g^n}(\psi_1,\dots,\psi_Z,\Omega)=0.$$

Furthermore, we observe from (40) that each monomial in $\Pi(T,R,\Omega)$ has degree at least τ. Finally, if $\Pi(T,R,\Omega)|_\tau$ stands for the sublinear combination of monomials of degree τ in $\Pi(T,R,\Omega)$, then

$$\Pi(T,R,\Omega)|_\tau = 0$$

for every set R of linearized curvature tensors and all sets T, Ω of symmetric tensors. But given equations (40) and (41) we see that:

$$\Pi(T,R,\Omega)|_\tau = \sum_{l\in L^\tau} a_l \operatorname{lin} C^l(R,\psi_1,\dots,\psi_Z,\Omega) = 0. \tag{42}$$

Hence, in view of Theorem 2, we have that (42) must hold formally. □

So, for each $\operatorname{lin} C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ there is a sequence of permutations for the factors $\Psi^l_{t_1\dots t_a}$, $,\Omega_{i_1\dots i_s}$ and of applications of the identities of a linearized curvature tensor to the factors $R_{t_1\dots t_m,ijkl}(x_0)$ so that (42) will hold by virtue of the identity $a\cdot C(\bigcup_{i=1}^Z\{T^i\},R,\Omega)+b\cdot C(\bigcup_{i=1}^Z\{T^i\},R,\Omega) = (a+b)\cdot C(\bigcup_{i=1}^Z\{T^i\},R,\Omega)$.

We then repeat these operations to the sublinear combination

$$\sum_{l\in L^\tau} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega).$$

The only difference is that the indices $t_1,\dots,t_m$ in each factor $\nabla^m_{t_1\dots t_m}R_{ijkl}(x_0)$ and the indices $i_1,\dots,i_p$ in each factor $\nabla^s_{i_1\dots i_s}\psi_h$ are not symmetric. Nonetheless, we may permute the indices $i_1,\dots,i_s$ in each factor $\nabla^s_{i_1\dots i_s}\psi_h$ and the indices $t_1,\dots,t_m$ in each factor $\nabla^m_{t_1\dots t_m}R_{ijkl}$ and introduce correction terms, which are complete contractions in the form (43) of length $\geq \tau+1$. Hence, repeating the permutations which made (42) identically zero, we derive our claim. □

We now make a note about the notation we used: We have considered complete contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Omega)$ in the general form (33), and we have explained that there may also be no factors $\Omega_{i_1\dots i_s}$. We make the extra convention that if we refer to a complete contraction $C^l_{g^n}(\psi_1,\dots,\psi_Z)$, we will imply that it is in the form (33) *and* has no factors $\Omega_{i_1\dots i_s}$. Therefore, it will be in the form:

$$\operatorname{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{ijkl}\otimes\dots\otimes\nabla^{m_s}_{t_1\dots t_{m_s}}R_{ijkl}\otimes\nabla^{\nu_1}_{a_1\dots a_{\nu_1}}\psi_1\otimes\dots\otimes\nabla^{\nu_Z}_{b_1\dots b_{\nu_Z}}\psi_Z\big). \tag{43}$$

Our next lemma is another corollary of Theorem 2. We must again introduce a definition.

We focus on complete contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi)$ of the form:

$$\operatorname{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{ijkl}\otimes\dots\otimes\nabla^{m_s}_{t_1\dots t_{m_s}}R_{ijkl}\otimes\nabla^{p_1}_{a_1\dots a_{p_1}}\psi_1\otimes\dots \\ \dots\otimes\nabla^{p_Z}_{b_1\dots b_{p_Z}}\psi_Z\otimes\Xi^{k_1}_{i_1\dots i_s}\otimes\dots\otimes\Xi^{k_f}_{j_1\dots j_t}\big). \tag{44}$$

In the manifold context, we impose the re-scaling condition $\Xi^{k_1}_{i_1\dots i_s}(t^2g^n)=\Xi^{k_1}_{i_1\dots i_s}(g^n)$ on the special tensors. When we wish to apply the theorem to a particular case of special tensors, we will easily see that this condition holds.

Definition 8. *A relation between complete contractions in the form* (44):

$$\sum_{l\in L} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi)=0$$

will hold *formally* if we can make the above sum identically zero by performing the following operations: We may interchange factors in any complete contraction $C^l_{g^n}(\psi_1,\dots,\psi_Z)$ and also permute the indices $i_1,\dots,i_s$ among the factors $\Xi^k_{i_1\dots i_s}$. Furthermore, for each factor $\nabla^m R_{ijkl}$, we may apply the identities:

1. $\nabla^m_{r_1\dots r_m} R_{ijkl}=-\nabla^m_{r_1\dots r_m} R_{jikl}=-\nabla^m_{r_1\dots r_m} R_{ijlk}$.
2. $\nabla^m_{r_1\dots[r_m} R_{ij]kl}=0$.
3. $\nabla^m_{r_1\dots r_m} R_{[ijk]l}=0$.
4. We may permute the indices $r_1,\dots,r_m$ by applying of the formula:

$$[\nabla_A\nabla_B-\nabla_B\nabla_A]X_C=R_{ABCD}X^D,$$

as defined in the previous definition;

and for any factor $\nabla^p_{i_1\dots i_p}\psi_h$ we may permute the factors i_1,i_2 if $p=2$ and apply the identity $[\nabla_A\nabla_B-\nabla_B\nabla_A]X_C=R_{ABCD}X^D$, as defined in the previous definition if $p>2$.

We then have:

LEMMA 4. *Consider complete contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi)$, each in the form* (44) *and with weight $-n$, so that the identity*:

$$\sum_{l\in L} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi)=0 \tag{45}$$

holds at any point x_0, for any metric g^n, any functions $\psi_1,\dots,\psi_Z$ defined around x_0 and any special tensors $\Xi^k_{i_1\dots i_s}(x_0)$ defined at x_0. Assume also that each special tensor in each C^l has rank at least 4. *Then the above identity must hold formally.*

Proof. We prove this corollary by using Theorem 2, in the same way that we proved Lemma 2 using Theorem 2.

We only need to observe that for each complete contraction in the form (44) with weight $-n$, if r_i stands for the rank of the i^{th} special tensor then:

$$\sum_{i=1}^{s}(m_i+2)+\sum_{i=1}^{Z}p_i+\sum_{i=1}^{f}(r_i-2)=n. \tag{46}$$

For each $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi)$, we again denote by $Z^\sharp$ the number of factors $\nabla^{p_h}\psi_h$ for which $p_h\neq 0$. Thus, assuming that each special factor has rank at least 4, we deduce that for each complete contraction $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi)$:

$$Z^\sharp+2s+2f\leq n. \tag{47}$$

Let τ be the minimum length among all the contractions

$$C^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi),\qquad l\in L.$$

We define the subset $L^\tau\subset L$ to be the index set of all complete contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi)$ with length τ. As before, we define the linear combination of complete contractions involving linearized curvature tensors rather than "genuine" covariant derivatives of the curvature tensor, and also symmetric tensors Ψ^h rather than "genuine" factors $\nabla^p\psi_h$:

$$\sum_{l\in L^\tau}a_l\operatorname{lin}C^l(R,\Psi_1,\dots,\Psi_Z,\Xi)$$

and we show that

$$\sum_{l\in L^\tau}a_l\operatorname{lin}C^l(R,\Psi_1,\dots,\Psi_Z,\Xi)=0$$

formally. We then deduce that an equation:

$$\sum_{l\in L^\tau}a_lC^l_{g^n}(\psi_1,\dots,\psi_Z,\Xi)=\sum_{r\in R}a_rC^r_{g^n}(\psi_1,\dots,\psi_Z,\Xi) \tag{48}$$

where each $C^r_{g^n}(\psi_1,\dots,\psi_Z,\Xi)$ has length $\geq\tau+1$, will hold formally. By inductive repetition of this argument, we have our lemma. □

These lemmas will prove useful in the future. For now, we note that there are many definitions of an identity holding formally. However, there will be no confusion, since in each of the above cases the complete contractions involve tensors that belong to different categories. Furthermore, in spite of the equivalence that the above theorems and their corollaries imply, whenever we mention an identity in this paper, we will mean (unless we explicitly state otherwise) that it holds at any point and for every metric and set of functions (and maybe special tensors Ξ or symmetric tensors Ω).

3.3. *The isomorphism.* We now conclude that:

PROPOSITION 1. *Suppose that $\{C^r_{g^N}(\psi_1,\dots,\psi_Z)\}_{r\in R}$ are complete contractions in the form* (43) *of weight $-n$. Suppose $N\geq n$. Then*

$$\sum_{r\in R}a_rC^r_{g^N}(\psi_1,\dots,\psi_Z)(x_0)=0$$

for every (M^n, g^n), *every* $x_0 \in M^n$ *and any functions* ψ_l *defined around* x_0 *if and only if*:

$$\sum_{r\in R} a_r C^r_{g^n}(\psi_1, \dots, \psi_Z)(x_0) = 0$$

for every (M^n, g^n), *every* $x_0 \in M^n$ *and any functions* ψ_l *defined around* x_0.

Proof. The above follows by virtue of Lemma 4. □

4. The silly divergence formula

Our aim here is to obtain a formula that expresses $I^Z_{g^n}(\psi_1, \dots, \psi_Z)$ as a divergence of a Riemannian vector field. This first, rather easy, divergence formula is not useful in itself. It will be used, however, in the derivation of the much more subtle *simple divergence formula* in the next section. For now, we claim:

PROPOSITION 2. *Consider any linear combination* $I^Z_{g^n}(\psi_1, \dots, \psi_s)$ *of contractions in the form* (31) *for which* $\int_{M^n} I^Z_{g^n}(\psi_1, \dots, \psi_s) dV_{g^n} = 0$ *for every compact* (M^n, g^n) *and any* $\psi_1, \dots, \psi_s \in C^\infty(M^n)$. *Note that* $I^Z_{g^n}(\psi_1, \dots, \psi_s)$ *defined in* (29) *satisfies this property.*

We then claim that $I^Z_{g^n}(\psi_1, \dots, \psi_Z)$ *is formally equal to the divergence of a Riemannian vector-valued differential operator of weight* $-n + 1$ *in* $\psi_1(x), \dots, \psi_Z(x)$.

Proof. In view of Lemma 2 in the previous subsection, it suffices to show that there is a vector field $T^i_{g^n}(\psi_1, \dots, \psi_Z)$ of weight $-n + 1$ so that:

$$I^Z_{g^n}(\psi_1, \dots, \psi_Z)(x_0) = \mathrm{div}_i T^i_{g^n}(\psi_1, \dots, \psi_Z)(x_0)$$

for any metric g^n and for any functions $\psi_1, \dots, \psi_Z$ around x_0. In order to show this we do the following:

Suppose that

$$I^Z_{g^n}(\psi_1, \dots, \psi_Z) = \sum_{j\in J} a_j C^j_{g^n}(\psi_1, \dots, \psi_Z)$$

where each of the complete contractions $C^j_{g^n}(\psi_1, \dots, \psi_Z)$ is in the form (31). Let us sort out the different values of ν_1 that can appear among the different complete contractions $C^j_{g^n}(\psi_1, \dots, \psi_Z)$. Suppose the set of those different values is the set $L = \{\lambda_1, \dots, \lambda_K\}$ where $0 \le \lambda_1 < \cdots < \lambda_K$.

Let $J_K \subset J$ be the set of the complete contractions $C^j_{g^n}(\psi_1, \dots, \psi_Z)$ with $\nu_1 = \lambda_K$. We then consider the linear combination:

$$F_{g^n}(\psi_1, \dots, \psi_Z) = \sum_{j\in J_K} a_j C^j_{g^n}(\psi_1, \dots, \psi_Z)$$

where each complete contraction $C^j_{g^n}(\psi_1,\dots,\psi_Z)$ is in the form (31) with the same number λ_K of derivatives on ψ_1. Out of $F_{g^n}(\psi_1,\dots,\psi_Z)$, we construct the following vector-valued differential operator:

$$F^i_{g^n}(\psi_1,\dots,\psi_Z)=\sum_{j\in J_K} a_j (C^j)^i_{g^n}(\psi_1,\dots,\psi_Z)$$

where $(C^j)^i_{g^n}(\psi_1,\dots,\psi_Z)$ is made out of $C^j_{g^n}(\psi_1,\dots,\psi_Z)$ by erasing the index χ_1 in (31) and making the index that contracted against it in (31) into a free index.

Let us then observe the following:

LEMMA 5. *The differential operator*

$$\tilde{F}_{g^n}(\psi_1,\dots,\psi_Z)=F_{g^n}(\psi_1,\dots,\psi_Z)-\mathrm{div}_i F^i_{g^n}(\psi_1,\dots,\psi_Z)$$

will be formally equal to a linear combination of complete contractions in the form (31) (*of weight* $-n$), *each of which has* λ_K-1 *derivatives on the function* ψ_1.

Proof. This is straightforward by the construction of the vector-valued operators $(C^j)^i_{g^n}(\psi_1,\dots,\psi_Z)$: Let the derivative ∇_i in the divergence of each $(C^j)^i_{g^n}(\psi_1,\dots,\psi_Z)$ hit the factor $\nabla^{\lambda_K-1}\psi_1$. That summand in the divergence will cancel out the complete contraction $C^j_{g^n}(\psi_1,\dots,\psi_Z)$. Every other complete contraction in $\mathrm{div}_i F^i_{g^n}(\psi_1,\dots,\psi_Z)$ will have λ_K-1 derivatives on ψ_1. This gives our desired conclusion. □

But then repeated application of Lemma 5 gives the following:

We can subtract a divergence of a vector field $L^i_{g^n}(\psi_1,\dots,\psi_Z)$ of weight $-n+1$ from $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$, so that

$$R_{g^n}(\psi_1,\dots,\psi_Z)=I^Z_{g^n}(\psi_1,\dots,\psi_Z)-\mathrm{div}_i L^i_{g^n}(\psi_1,\dots,\psi_Z)$$

is a linear combination of complete contractions in the form (31), each of which has $\nu_1=0$.

We then observe that:

LEMMA 6. *In the above notation*, $R_{g^n}(\psi_1,\dots,\psi_Z)$ *must vanish formally.*

Proof. First observe that for any Riemannian manifold (M^n,g^n) we will have:

$$\int_{M^n} R_{g^n}(\psi_1,\dots,\psi_Z)\,dV_{g^n}=0.$$

This is straightforward, because of Lemma 5 and the definition of $R_{g^n}(\psi_1,\dots,\psi_Z)$; it is obtained from $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$ by subtracting a divergence.

Now, write $R_{g^n}(\psi_1,\dots,\psi_Z)$ as follows:

$$R_{g^n}(\psi_1,\dots,\psi_Z)=\sum_{l\in L} a_l C^l_{g^n}(\psi_2,\dots,\psi_Z)\cdot\psi_1.$$

Then the equation

$$\int_{M^n} R_{g^n}(\psi_1,\dots,\psi_Z)dV_{g^n} = \int_{M^n} \psi_1\cdot[\sum_{l\in L} a_l C^l_{g^n}(\psi_2,\dots,\psi_Z)]dV_{g^n} \tag{49}$$

holds for any function ψ_1, and also the sum $\sum_{l\in L} a_l C^l_{g^n}(\psi_2,\dots,\psi_Z)$ is independent of the function ψ_1. But this shows that $R_{g^n}(\psi_1,\dots,\psi_Z)$ must vanish by substitution. Hence, by Theorem 2, it must vanish formally. □

5. The simple divergence formula

5.1. *The transformation law for $I^Z_{g^N}$ and definitions.* Let $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$ be as in Proposition 2. We then have that $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$ is a divergence of a vector-valued differential operator in $\psi_1(x),\dots,\psi_Z(x)$. This is useful in itself, but we cannot extract information directly from this fact about $P(g^n)$. Nevertheless, it is useful in that we have a relation:

$$I^Z_{g^n}(\psi_1,\dots,\psi_Z) = \mathrm{div}_i L^i_{g^n}(\psi_1,\dots,\psi_Z) \tag{50}$$

which holds formally. But then Proposition 1 tells us the following:

LEMMA 7. *Relation* (50) *holds for any dimension $N \ge n$. That is, considering the complete contractions and the Riemannian vector fields in* (50) *in any dimension $N \ge n$, we have the formula*:

$$I^Z_{g^N}(\psi_1,\dots,\psi_Z) = \mathrm{div}_i L^i_{g^N}(\psi_1,\dots,\psi_Z). \tag{51}$$

Proof. This is straightforward from Propositions 1 and 2. □

Therefore, we will have that for any (M^N, g^N) and any $\psi_1,\dots,\psi_Z \in C^\infty(M^N)$:

$$\int_{M^N} I^Z_{g^N}(\psi_1,\dots,\psi_Z)dV_{g^N} = 0. \tag{52}$$

Now, equation (52) is *not* scale-invariant. This can be used to our advantage in the following way: Pick out any point $x_0 \in M^N$. Pick out a small geodesic ball around x_0, of radius ε. From now on, we will assume the functions $\psi_1,\dots,\psi_Z$ to be compactly supported in $B(x_0,\varepsilon)$. Then we can pick any coordinate system around x_0 and write out $I^Z_{g^N}(\psi_1,\dots,\psi_Z)$ in that coordinate system

$$\int_{\mathbb{R}^N} I^Z_{g^N}(\psi_1,\dots,\psi_Z)dV_{g^n} = 0. \tag{53}$$

Now, let our coordinate system around x_0 be $\{x_1,\dots,x_N\}$. For that coordinate system, we will denote each point in $B(x_0,\varepsilon)$ by $\vec{x}$. Let also $\vec{\xi}$ be an arbitrary vector in $\mathbb{R}^N$. We then consider the following conformal change of metric in $B(x_0,\varepsilon)$:

$$\hat{g}^N(x) = e^{2\vec{\xi}\cdot\vec{x}} g^N(x).$$

We have that (53) must also hold for this metric. The volume form will re-scale as follows:

$$dV_{\hat{g}^N}(x) = e^{N\vec{\xi}\cdot\vec{x}}dV_{g^n}(x).$$

Now, we have that $I^Z_{g^N}(\psi_1,\dots,\psi_Z)(x)$ is a linear combination of complete contractions in the form (31). So, in order to find how any given complete contraction in the form (31) transforms under the above conformal change, it suffices to find how each of its factors will transform. In order to do that, we can employ the identities of the first section.

The transformation law of Ricci curvature, for this special conformal transformation, is given by equation (22), replacing ϕ by $\vec{x}\cdot\vec{\xi}$. Recall that $\nabla_i(\vec{\xi}\cdot\vec{x}) = \vec{\xi}_i$, therefore:

$$\begin{aligned}\mathrm{Ric}^{\hat{g}^N}_{\alpha\beta}(x) = {}&\mathrm{Ric}^{g^N}_{\alpha\beta}(x) + (2-N)\nabla^2_{\alpha\beta}(\vec{\xi}\cdot\vec{x})\\ &-\Delta_{g^N}(\vec{\xi}\cdot\vec{x})g^N_{\alpha\beta} + (N-2)(\vec{\xi}_\alpha\vec{\xi}_\beta - \vec{\xi}^k\vec{\xi}_k g^N_{\alpha\beta}).\end{aligned} \tag{54}$$

The scalar curvature will transform as:

$$R^{\hat{g}^N}(x) = e^{-2\vec{\xi}\cdot\vec{x}}[R^{g^N} + 2(1-N)\Delta_{g^N}(\vec{\xi}\cdot\vec{x}) - (N-1)(N-2)\vec{\xi}^k\vec{\xi}_k], \tag{55}$$

and the full curvature tensor:

$$\begin{aligned}R^{\hat{g}^N}_{ijkl}(x) = e^{2\vec{\xi}\cdot\vec{x}}\Big\{&R^g_{ijkl}(x) + [\vec{\xi}_i\vec{\xi}_k g^N_{jl} - \vec{\xi}_i\vec{\xi}_l g^N_{jk} + \vec{\xi}_j\vec{\xi}_l g^N_{ik} - \vec{\xi}_j\vec{\xi}_k g^N_{il}]\\ &- \nabla^2_{ik}(\vec{\xi}\cdot\vec{x})g^N_{jl} - \nabla^2_{jl}(\vec{\xi}\cdot\vec{x})g^N_{ik} + \nabla^2_{jk}(\vec{\xi}\cdot\vec{x})g^N_{il}\\ &+ \nabla^2_{il}(\vec{\xi}\cdot\vec{x})g^N_{jk} + |\vec{\xi}|^2 g^N_{il}g^N_{jk} - |\vec{\xi}|^2 g^N_{ik}g^N_{lj}]\Big\}.\end{aligned} \tag{56}$$

Hence, in order to find the transformation laws for the covariant derivatives of the full curvature tensor, the Ricci curvature tensor and of the factors $\nabla^p\psi_h$, we will need the transformation law for the Levi-Civita connection in the case at hand:

$$(\nabla_k\eta_l)^{\hat{g}^N}(x) = (\nabla_k\eta_l)^{g^N} - \vec{\xi}_k\eta_l - \vec{\xi}_l\eta_k + \vec{\xi}^s\eta_s g^N_{kl}. \tag{57}$$

These relations show that in (53), under the re-scaling $g^N(x) \longrightarrow \hat{g}^N(x) = e^{2\vec{\xi}\cdot\vec{x}}g^N(x)$, the integrand $I^Z_{g^N}(\psi_1,\dots,\psi_Z)(x)$ undergoes a transformation as follows:

$$\begin{aligned}&I^Z_{\hat{g}^N}(\psi_1,\dots,\psi_Z)(x)\\ &\qquad = e^{-n\vec{\xi}\cdot\vec{x}}[I^Z_{g^N}(\psi_1,\dots,\psi_Z)(x) + S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})(x)],\end{aligned} \tag{58}$$

where $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is obtained by applying the transformation laws is a linear combination of complete contractions, each of which described above to each factor in every complete contraction in $I^Z_{g^N}(\psi_1,\dots,\psi_Z)$. $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ depends on $\vec{\xi}$. Hence equation (53) will give, for the metric $\hat{g}^N$:

$$\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}[I^Z_{g^N}(\psi_1,\dots,\psi_Z)+S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]dV_{g^n}=0. \tag{59}$$

Roughly speaking, our goal for this subsection will be to perform integrations by parts for the complete contractions in $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})(x)$ in order to reduce equation (53) to the form:

$$\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}[I^Z_{g^N}(\psi_1,\dots,\psi_Z)+L^Z_{g^N}(\psi_1,\dots,\psi_Z)]dV_{g^n}=0$$

where $L^Z_{g^N}(\psi_1,\dots,\psi_Z)(x)$ is independent of $\vec{\xi}$. This will be done and explained rigorously below. Keeping this vaguely outlined strategy in mind, we note the identity:

$$\nabla_s(e^{(N-n)\vec{\xi}\cdot\vec{x}})=(N-n)\vec{\xi}_s(e^{(N-n)\vec{\xi}\cdot\vec{x}}). \tag{60}$$

More generally, we denote by $\partial^m_{s_1\dots s_k}$ the coordinate derivative with respect to our coordinate system. Then, for $k>1$,

$$\partial^k_{s_1\dots s_k}(\vec{\xi}\cdot\vec{x})=0 \tag{61}$$

for every $x\in B(x_0,\varepsilon)$.

Let us consider the Christoffel symbols Γ^k_{ij} with respect to our arbitrary coordinate system. Let

$$S\nabla^m_{s_1\dots s_m}\vec{\xi}_j \text{ stand for } \nabla^m_{(s_1\dots s_m}\vec{\xi}_{j)} \quad\text{and}\quad S\nabla^p_{r_1\dots r_p}\Gamma^k_{ij} \text{ stand for } \nabla^p_{(r_1\dots r_p}\Gamma^k_{ij)}.$$

Write $I^Z_{g^N}(\psi_1,\dots,\psi_Z)$ as a linear combination of complete contractions in the following form:

$$\begin{aligned}\operatorname{contr}\big(&\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1}\otimes\cdots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}\otimes\nabla^{p_1}_{t_1\dots t_{p_1}}\mathrm{Ric}_{\alpha_1\beta_1}\\&\otimes\cdots\otimes\nabla^{p_q}_{z_1\dots z_{p_q}}\mathrm{Ric}_{\alpha_q\beta_q}\otimes\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\psi_1\otimes\cdots\otimes\nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\psi_Z\big)\end{aligned} \tag{62}$$

where each of the factors $\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1},\dots,\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}$ has no two of the indices i,j,k,l contracting against each other in (62).

Now, in dimension N, we can apply the identities (56), (54),(55) and (57) to write $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ as a linear combination of complete contractions in the following two forms:

$$\begin{aligned}(63)\quad &\mathrm{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}} R_{i_1 j_1 k_1 l_1} \otimes\dots\otimes \nabla^{m_s}_{v_1\dots v_{m_s}} R_{i_s j_s k_s l_s} \otimes \nabla^{p_1}_{t_1\dots t_{p_1}} \mathrm{Ric}_{\alpha_1\beta_1}\\ &\otimes\dots\otimes \nabla^{p_q}_{z_1\dots z_{p_q}} \mathrm{Ric}_{\alpha_q\beta_q} \otimes \nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}} \psi_1\\ &\otimes\dots\otimes \nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}} \psi_Z \otimes \vec{\xi} \otimes\dots\otimes \vec{\xi}\big),\end{aligned}$$

$$\begin{aligned}(64)\quad &\mathrm{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}} R_{i_1 j_1 k_1 l_1} \otimes\dots\otimes \nabla^{m_s}_{v_1\dots v_{m_s}} R_{i_s j_s k_s l_s} \otimes \nabla^{p_1}_{t_1\dots t_{p_1}} \mathrm{Ric}_{\alpha_1\beta_1}\\ &\otimes\dots\otimes \nabla^{p_q}_{z_1\dots z_{p_q}} \mathrm{Ric}_{\alpha_q\beta_q} \otimes \nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}} \psi_1 \otimes\dots\otimes \nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}} \psi_Z \otimes \vec{\xi}\\ &\otimes\dots\otimes \vec{\xi} \otimes S[\nabla^{w_1}_{u_1\dots u_{w_1}} \vec{\xi}] \otimes\dots\otimes S[\nabla^{w_l}_{q_1\dots q_{w_l}} \vec{\xi}]\big),\end{aligned}$$

where each $w_a \geq 1$. We also let k stand for the number of factors $\vec{\xi}$ and l for the number of factors $S\nabla^{w}\vec{\xi}$.

We will call complete contractions in the above two forms $\vec{\xi}$-contractions. In order to see that we can indeed write $S^Z_{g^N}(\psi_1,\dots,\psi_Z)$ as a linear combination of complete contractions in the above form, we only need the equation:

$$\begin{aligned}(65)\qquad \nabla_a S\nabla^m_{r_1\dots r_m}\vec{\xi}_j = S\nabla^m_{ar_1\dots r_m}\vec{\xi}_j + C_{m-1}\cdot S^*\nabla^{m-1}_{r_1\dots r_{m-1}} R_{aijd}\vec{\xi}^d\\ + \sum_{u\in U^m} a_u \mathrm{pcontr}(\nabla^{m'} R_{abcd} S\nabla^{s_u}\vec{\xi}),\end{aligned}$$

where $S^*\nabla^{m-1}_{r_1\dots r_{m-1}} R_{aijd}$ stands for the symmetrization of $\nabla^{m-1}_{r_1\dots r_{m-1}} R_{aijd}$ over the indices $r_1,\dots,r_{m-1},i$ and the symbol

$$\mathrm{pcontr}(\nabla^{m'} R_{abcd} S\nabla^{s_u}\vec{\xi})$$

stands for a partial contraction of at least one factor $\nabla^m R_{ajkl}$ (to one of which the index a belongs) against a factor $S\nabla^{s_u}\vec{\xi}$ with $s_u \geq 1$.

Our next goal is to answer the following: Given a fixed linear combination $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$ and its rewriting $I^Z_{g^N}(\psi_1,\dots,\psi_Z)$ in any dimension $N \geq n$, how does $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ depend upon the dimension N?

In order to answer this question, we will introduce certain definitions. Let us for this purpose treat the function $\vec{\xi}\cdot\vec{x}$ as a function $\omega(x)$. Hence $\vec{\xi}_i = \nabla_i(\vec{\xi}\cdot\vec{x})$ and we can speak of the rewriting of a $\vec{\xi}$-contraction in dimension N. We will consider the complete contraction $C^i_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ together with its rewriting $C^i_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ in every dimension $N \geq n$ and call this sequence a dimension-independent complete contraction.

On the other hand, we define:

Definition 9. Any factor of the form $\vec{\xi}$ or of the form $S\nabla^m\vec{\xi}$, $m \geq 1$, will be called a *$\vec{\xi}$-factor*.

Definition 10. Consider a sequence $\{C_{(g,N)}(\psi_1,\dots,\psi_Z,\vec{\xi})\}$ of complete contractions times coefficients in dimensions $N = n, n+1,\dots$ where the following formula holds: There is a fixed complete contraction, say $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ and a fixed rational function $Q(N)$ so that:

$$C_{(g,N)}(\psi_1,\dots,\psi_Z,\vec{\xi}) = Q(N)\cdot C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

where $C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is the rewriting of $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ in dimension N. In that case, we will say that we have a *dimension-dependent* $\vec{\xi}$*-contraction*. Furthermore, we will say that the three defining numbers of $C_{(g,N)}(\psi_1,\dots,\psi_Z,\vec{\xi})$ are (d,k,l) where d is *the degree of the rational function* $Q(N)$, k is the *number* of factors $\vec{\xi}$ and l is the *number of factors* $S\nabla^m_{i_1\dots i_m}\vec{\xi}_a$, $m \geq 1$.

(Given a rational function $Q(N) = P(N)/L(N)$, we define the *degree of* $Q(N)$, $\deg[Q(N)] = \deg[P(N)] - \deg[L(N)]$. We also define the leading order coefficient of $Q(N)$ to be a_P/a_L, where a_P is the *leading order coefficient* of $P(N)$ and a_L is the leading order coefficient of $L(N)$).

Given a fixed set of numbers $\{a_i\}, i \in I$, and a set of dimension-dependent $\vec{\xi}$-contractions $C^i_{(g,N)}(\psi_1,\dots,\psi_Z,\vec{\xi})$, we can form in each dimension $N \geq n$ the linear combination:

$$L_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi}) = \sum_{i\in I} a_i C^i_{(g,N)}(\psi_1,\dots,\psi_Z,\vec{\xi}).$$

Hence we obtain in this way a sequence of linear combinations, where the index set for the sequence is the set $\mathbf{N} = \{n, n+1, \dots\}$.

Definition 11. A *sequence of linear combinations* as above is *dimension-dependent* and is *suitable* if for each of the $\vec{\xi}$-contractions $C^i_{(g,N)}(\psi_1,\dots,\psi_Z,\vec{\xi})$ its three defining numbers satisfy: $k+l \geq d$.

We then have:

LEMMA 8. $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ *is a suitable linear combination of* $\vec{\xi}$*-contractions of the form* (63) *and* (64), *with* $k+l \geq d \geq 1$.

Proof. We write

$$I^Z_{g^N}(\psi_1,\dots,\psi_Z) = \sum_{i\in I} a_i C^i_{g^N}(\psi_1,\dots,\psi_Z)$$

where each $C^i_{g^N}(\psi_1,\dots,\psi_Z)$ is in the form (31) and has weight $-n$.

We introduce some further terminology. We call the tensors

$$(\nabla^m_{r_1\dots r_m} R_{ijkl})^{g^N}, \quad (\nabla^p_{r_1\dots r_p}\psi_l)^{g^N}, \quad (S\nabla^m_{r_1\dots r_m}\vec{\xi}_a)^{g^N}, \quad \vec{\xi}_i, \quad \text{and} \quad g^N_{ij}$$

the *free* tensors. We call partial contractions of those tensors the *extended* free tensors. (Recall that a *partial contraction* means a tensor product with some pairs of indices contracting against each other.)

We see that $e^{-2\vec{\xi}\cdot\vec{x}}(\nabla^m_{r_1\dots r_m} R_{ijkl})^{\hat{g}^N}$, $(\nabla^p_{r_1\dots r_p}\psi_l)^{\hat{g}^N}$ can be written as linear combinations of extended free tensors, after applying the identity (65), if necessary.

Now, consider any complete contraction $C^i_{g^N}(\psi_1,\dots,\psi_Z)$ (in the form (31)) in $I^Z_{g^N}(\psi_1,\dots,\psi_Z)$ and do the following: For each of its factors $\nabla^m_{r_1\dots r_m}R_{ijkl}$ or $\nabla^p_{r_1\dots r_p}\psi_l$, calculate:

$$e^{-2\vec{\xi}\cdot\vec{x}}(\nabla^m_{r_1\dots r_m}R_{ijkl})^{\hat{g}^N} = \sum_{j\in J'} a_j T^j_{r_1\dots r_m ijkl}$$
$$(\nabla^p_{r_1\dots r_p}\psi_l)^{\hat{g}^N} = \sum_{j\in J} a_j T^j_{r_1\dots r_p}$$

where each $T^r_{i_1\dots i_s}$ is an extended free tensor. We then replace each $\nabla^m_{r_1\dots r_m}R_{ijkl}$ by one $e^{2\vec{\xi}\cdot\vec{x}}a_j T^j_{r_1\dots r_m ijkl}$ and each $\nabla^p_{r_1\dots r_p}\psi_l$ by one $a_j T^j_{r_1\dots r_p}$. After this, we perform the same contractions of indices as in $C^i_{g^N}(\psi_1,\dots,\psi_Z)$, with respect to the metric (g^N). We do this according to the following algorithm: Suppose we are contracting two indices α,β. If none of them belongs to a tensor g^N_{ij}, we just take that particular contraction. If α but not β belongs to a factor $g^N_{\alpha\gamma}$, we cross out the index β in the other factor and replace it by γ, and then omit the $g^N_{\alpha\gamma}$. Finally, if both the indices α,β belong to the same factor $g^N_{\alpha\beta}$, we cross out that factor and bring out a factor of N. Adding over all those substitutions, we then obtain $e^{n\vec{\xi}\cdot\vec{x}}C^i_{\hat{g}^N}(\psi_1,\dots,\psi_Z)$.

Thus, $e^{n\vec{\xi}\cdot\vec{x}}C^i_{\hat{g}^N}(\psi_1,\dots,\psi_Z)$ is a dimension-dependent linear combination. It follows from this that $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is a dimension-dependent linear combination, in the form:

$$S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi}) = \sum_{l\in L} N^{b_l} C^l_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

where each complete contraction $C^l_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is in the form:

$$\begin{aligned}(66)\quad \mathrm{contr}\big(&\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1}\otimes\dots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}\otimes\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\psi_1\\ &\otimes\dots\otimes\nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\psi_Z\otimes\vec{\xi}\otimes\dots\otimes\vec{\xi}\otimes S\nabla^{w_1}_{u_1\dots u_{w_1}}\vec{\xi}\otimes\dots\otimes S\nabla^{w_l}_{q_1\dots q_{w_l}}\vec{\xi}\big)\end{aligned}$$

where $l\ge 0$ and the factors $\nabla^m R_{ijkl}$ are allowed to have internal contractions.

Therefore, what remains to be checked is that each dimension-dependent $\vec{\xi}$-contraction $N^{b_i}C^i_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ in $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ with $|\vec{\xi}|\ge 1$ satisfies the identity $k+l\ge d$.

In order to see this, let us consider any summand $T^j_{r_1\dots r_m ijkl}$ or $T^j_{r_1\dots r_p}$ and denote by $|g|$ the number of its factors g^N_{ij} and by $|\vec{\xi}|$ the number of its factors $\vec{\xi}_i$ or $S\nabla^m_{r_1\dots r_m}\vec{\xi}_a$. It follows, from identities (56) and (57) that for each $T^j_{r_1\dots r_m ijkl}$ or $T^j_{r_1\dots r_p}$ we have $|\vec{\xi}|\ge|g|$.

By virtue of that inequality, the formula (65) (which shows us that if we write a complete contraction in the form (66) as a linear combination of complete contractions in the forms (63), (64), the number of $\vec{\xi}$-factors remains unaltered) and the algorithm outlined above, we observe that for each dimension-dependent

$\vec{\xi}$-contraction $N^{b_i} C^i_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})$, we will have that b_i is less than or equal to the number of factors $\vec{\xi}$ or $S\nabla^m\vec{\xi}$ in $C^i_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})$. □

Definition 12. Consider any complete contraction $C^Z_{g^N}(\psi_1, \ldots, \psi_Z)$, in the form (62). Consider the quantity:

$$e^{n\vec{\xi}\cdot\vec{x}} C^Z_{e^{2\vec{\xi}\cdot\vec{x}} g^N}(\psi_1, \ldots, \psi_Z)(x) - C^Z_{g^N}(\psi_1, \ldots, \psi_Z)(x)$$

which can be computed by applying the identities (56), (54) (55), (57), (65) to each factor in $C^Z_{g^N}(\psi_1, \ldots, \psi_Z)$. We write:

$$e^{n\vec{\xi}\cdot\vec{x}} C^Z_{e^{2\vec{\xi}\cdot\vec{x}} g^N}(\psi_1, \ldots, \psi_Z)(x) - C^Z_{g^N}(\psi_1, \ldots, \psi_Z)(x) = \sum_{t\in T} a_t N^{b_t} C^t_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})(x)$$

where each *dimension-dependent* $\vec{\xi}$*-contraction* $N^{b_t} C^t_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})(x)$ satisfies $k + l \geq b_t$. Here $C^t_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})(x)$ stands for the rewriting of $C^t_{g^n}(\psi_1, \ldots, \psi_Z, \vec{\xi})(x)$ in dimension N.

There are many expressions as above for

$$e^{n\vec{\xi}\cdot\vec{x}} C^Z_{e^{2\vec{\xi}\cdot\vec{x}} g^N}(\psi_1, \ldots, \psi_Z)(x) - C^Z_{g^N}(\psi_1, \ldots, \psi_Z)(x)$$

that are equal by substitution but not identical. Once we pick one such expression, we will call each *dimension-dependent* $\vec{\xi}$*-contraction* $N^{b_t} C^t_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})(x)$ a descendant of $C^Z_{g^N}(\psi_1, \ldots, \psi_Z)(x)$.

We are now near the point where we can integrate by parts in the relation (59). At this stage, we will distinguish between descendants of the complete contractions in $I^Z_{g^N}(\psi_1, \ldots, \psi_Z)$.

Definition 13. For any complete contraction $C^i_{g^N}(\psi_1, \ldots, \psi_Z)$ in $I^Z_{g^N}(\psi_1, \ldots, \psi_Z)$, we will call one of its descendants *easy* if $d < l + k$.

A descendant in the form (63) will be called *good* if $d = k > 0$ and $l = 0$. A descendant in the form (64) will be called *undecided* if $d = k + l$ and $k, l > 0$. (That is, it contains at least one factor of the form $S\nabla^p\vec{\xi}_i$ with $p \geq 1$ and at least one factor of the form $\vec{\xi}$).

Finally, a *descendent in the form* (64) with $d = k + l$ will be called *hard* if $k = 0, l > 0$ (that is, if all its $\vec{\xi}$-factors are of the form $S\nabla^m\vec{\xi}_j$, with $m \geq 1$).

Thus, given (59) in any dimension N, we have $S^Z_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})$ written out as a linear combination of good, easy, undecided and hard complete contractions.

5.2. *The integrations by parts for* $S^Z_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})$. We want to perform integrations by parts in equation (59). We will treat the four cases above separately.

Let us first treat the easy $\vec{\xi}$-contractions. Using (17), we write out each factor of the form $S\nabla^m\vec{\xi}_s$ as a linear combination of partial contractions of the Christoffel symbols and their derivatives (with respect to our arbitrarily chosen coordinate system) and also of the vector $\vec{\xi}$. We also write out each of the tensors $\nabla^m R_{ijkl}$ as a linear combination of partial contractions of Christoffel symbols and their derivatives. Hence, given an *easy* $\vec{\xi}$-contraction $P(N)\cdot C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$, we express it in our coordinate system as:

$$\text{contr}(\partial^{m_1}\Gamma^k_{ij}\otimes\cdots\otimes\partial^{m_s}\Gamma^k_{ij}\otimes\nabla^{p_1}\psi_1\otimes\cdots\otimes\nabla^{p_Z}\psi_Z\otimes\vec{\xi}\otimes\cdots\otimes\vec{\xi}). \tag{67}$$

Hence we will have the following identity:

$$\begin{aligned}\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} P(N)\cdot C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}\\ = \int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} P(N)\cdot \sum_{l\in L} a_l \text{Contr}_l(\partial^m\Gamma,\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n},\end{aligned} \tag{68}$$

where the degree of the polynomial $P(N)$ is strictly less than the number of factors $\vec{\xi}$ in the contraction $\text{Contr}_l(\partial^m\Gamma,\psi_1,\dots,\psi_Z,\vec{\xi})$. Now, we use the identity (60) in order to replace one factor $\vec{\xi}_i$ in the complete contraction by the factor $\nabla_i e^{(N-n)\vec{\xi}\cdot\vec{x}}/(N-n)$. We then integrate by parts with respect to the derivative ∇_i and note here that this integration by parts is with respect to the Riemannian connection ∇_i.

We get the following:

$$\begin{aligned}\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} P(N)\cdot C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}\\ = -\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} \frac{P(N)}{N-n} \sum_{k\in K} a_k \text{Contr}_k(\partial^m\Gamma,\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}.\end{aligned} \tag{69}$$

Each complete contraction $\text{Contr}_k(\partial^m\Gamma,\psi_1,\dots,\psi_Z\vec{\xi})$ is in the form (67). Also, the number of factors $\vec{\xi}$ in each contraction $\text{Contr}_k(\partial^m\Gamma,\psi_1,\dots,\psi_Z,\vec{\xi})$ is by one less than the number of such factors in the complete contraction $C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$. Hence, inductively repeating the above procedure we obtain:

$$\begin{aligned}\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} P(N)\cdot C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}\\ = \int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} \frac{P(N)}{(N-n)^w} \sum_{k\in K} a_k \text{Contr}_k(\partial^m\Gamma,\psi_1,\dots,\psi_Z)dV_{g^n},\end{aligned} \tag{70}$$

where we will have $\deg[P(N)]=d<w$.

The good $\vec{\xi}$-contractions. Let us now deal with the good complete contractions in $S^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$. In this case it is useful not to write things out in terms of

Christoffel symbols but to work intrinsically on the Riemannian manifold. We have a good $\vec{\xi}$-contraction $P(N)\cdot C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ in the form (63) and we want to perform integration by parts in the integral:

$$\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} P(N)\cdot C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}. \tag{71}$$

We will again use the identity (60). Let us arbitrarily pick out one of the $k=d$ factors $\vec{\xi}$ in $C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$. Now, use the identity (60) in order to replace the factor $\vec{\xi}_i$ in the complete contraction by the factor

$$\frac{\nabla_i[e^{(N-n)\vec{\xi}\cdot\vec{x}}]}{N-n}.$$

We then integrate by parts with respect to the derivative ∇_i. Let us again note that this integration by parts is with respect to the Riemannian connection ∇_i.

Now, if the $\vec{\xi}$-contraction $C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ in the form (63) has L factors (including the k factors $\vec{\xi}$), the integration by parts will produce a sum of $L-1$ complete contractions. Explicitly, we will have:

$$\begin{aligned}\int_{\mathbb{R}^N} & e^{(N-n)\vec{\xi}\cdot\vec{x}} P(N)\cdot C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n} \\ &= -\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} \frac{P(N)}{N-n}\cdot\sum_{\alpha=1}^{L-1} C^\alpha_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}.\end{aligned} \tag{72}$$

We separate these $\vec{\xi}$-contractions $C^\alpha_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ into two categories: A $\vec{\xi}$-contraction belongs to the first category if the derivative ∇_i has hit one of the factors $\nabla^m R_{ijkl}$, $\nabla^p\text{Ric}$ or $\nabla^p\psi_k$. Hence, we see that

$$\frac{P(N)}{N-n}\cdot C^\alpha_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

is a linear combination of $\vec{\xi}$-contractions in the form (63) with $k-1$ factors $\vec{\xi}$. If $k=1$, each will be in the form (62). Otherwise, each of them will be a good $\vec{\xi}$-contraction.

On the other hand, a $\vec{\xi}$-contraction $\frac{P(N)}{N-n}\cdot C^\alpha_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ belongs to the second category if the derivative ∇_i hit one of the $k-1$ factors $\vec{\xi}$. In that case, we get a $\vec{\xi}$-contraction in the form (64) with $k-2$ factors $\vec{\xi}$ and one factor $\nabla_i\vec{\xi}$. It will be an *undecided* or a *hard* $\vec{\xi}$-contraction.

Now, we can repeat the above intrinsic integration by parts for each of the good $\vec{\xi}$-contractions $\frac{P(N)}{N-n}\cdot C^\alpha_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$, each of the form (63) with $k-1$ factors $\vec{\xi}$. Each of these integrations by parts will give a sum of $\vec{\xi}$-contractions, $L-k+1$ of which are in the form (63) with $k-2$ factors $\vec{\xi}$ and $k-2$ of them will be of the form (64) (either undecided or hard). Hence, we can form a procedure of k steps, starting from $C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ and integrating by parts one factor $\vec{\xi}$ at a time. At each stage we get a sum of good and of undecided or hard $\vec{\xi}$-contractions out of this integration by parts. We then focus on the good $\vec{\xi}$-contractions that we have obtained and we repeat the integration by parts. Thus, after this sequence of integrations by parts we will have:

$$\begin{aligned}(73)\quad &\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} P(N)\cdot C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}\\ &=\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}[\frac{P(N)}{(N-n)^k}\sum_{j\in J} a_j C^j_{g^N}(\psi_1,\dots,\psi_Z)\\ &\qquad+\sum_{h\in H}\frac{P_h(N)}{(N-n)^{s_h}}C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]dV_{g^n}\end{aligned}$$

where the complete contractions $C^j_{g^N}(\psi_1,\dots,\psi_Z)$ are in the form (62) (they are independent of the variable $\vec{\xi}$) and the $\vec{\xi}$-contractions $C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ are in the form (64) and are undecided or hard. Each of the undecided $\vec{\xi}$-contractions will have at most $k-1$ $\vec{\xi}$-factors. For each complete contraction $C^j_{g^N}(\psi_1,\dots,\psi_Z)$ we have that $\deg[P(N)]=k$. For each complete contraction $C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$, with l_h factors $\nabla\vec{\xi}$ and k_h factors $\vec{\xi}$, we have $k_h+l_h+s_h=\deg[P_h(N)]$.

The undecided $\vec{\xi}$-contractions. We now proceed to integrate by parts the undecided $\vec{\xi}$-contractions. Let $C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ be an undecided $\vec{\xi}$-contraction in the form (64). We will perform integrations by parts in the integral:

$$\int_{R^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}\frac{P_h(N)}{(N-n)^{m_h}}C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})(x)dV_{g^n}.$$

Let us suppose that the length of the $\vec{\xi}$-contraction (including the k factors $\vec{\xi}$ and the l factors $S\nabla^m\vec{\xi}$) is L. We will first integrate by parts the factors $\vec{\xi}$. We pick one at random and integrate by parts as before, using the familiar formula (60). We then get a sum of $\vec{\xi}$-contractions as follows:

$$\begin{aligned}(74)\quad &\int_{R^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}\frac{P_h(N)}{(N-n)^{m_h}}\cdot C^Z_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}\\ &=-\int_{R^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}\frac{P_h(N)}{(N-n)^{m_h+1}}\cdot\sum_{\alpha=1}^{L-1} C^{h,\alpha}_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})dV_{g^n}.\end{aligned}$$

We sort out the complete contractions according to what sort of factor was hit by the derivative ∇_i. If $C^{h,\alpha}_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ arises when the covariant derivative ∇_i hits a factor of the form $\nabla^m R_{ijkl}$ or $\nabla^p\mathrm{Ric}$ or $\nabla^p\psi_l$, we get a $\vec{\xi}$-contraction with $k-1$ factors $\vec{\xi}$ and l factors $S\nabla^m\vec{\xi}$, $m\geq 1$. If $C^{h,\alpha}_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ arises when ∇_i hits a factor $\vec{\xi}$, we get another $\vec{\xi}$-contraction with $k-2$ factors $\vec{\xi}$ and $l+1$ factors $S\nabla^m\vec{\xi}$ where $m \geq 1$.

Finally, if $C^{h,\alpha}_{\vec{g}^N}(\psi_1, \dots, \psi_Z, \vec{\xi})$ arises when ∇_i hits a factor $S\nabla^m\vec{\xi}$, we get a factor $\nabla_i S\nabla^m\vec{\xi}$. We then decompose that factor according to equation (65). In either case, we have reduced by 1 the number of $\vec{\xi}$-factors.

The good $\vec{\xi}$-contractions we have already seen how to treat. Finally, if we get an undecided $\vec{\xi}$-contraction, we have reduced the number of $\vec{\xi}$-factors.

The hard $\vec{\xi}$-contractions. Suppose that $\frac{P(N)}{(N-n)^m} C^j_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ is a hard $\vec{\xi}$-contraction in the form (64) with $k = 0$. We pick out one of the l factors $S\nabla^m_{r_1\dots r_m}\vec{\xi}_j$ and write it as

$$S\nabla^{m-1}_{(r_1\dots r_{m-1}}\Gamma^k_{r_m j)}\vec{\xi}_k.$$

We then integrate by parts the factor $\vec{\xi}_k$ and obtain a formula:

$$\int_{R^N} [e^{(N-n)\vec{\xi}\cdot\vec{x}} \frac{P(N)}{(N-n)^m} C^j_{g^n}(\psi_1, \dots \psi_Z, \vec{\xi})] dV_{g^n} \tag{75}$$
$$= -\int_{R^N} [e^{(N-n)\vec{\xi}\cdot\vec{x}} \sum_{h\in H^j} \frac{P(N)}{(N-n)^{m+1}} C^h_{g^n}(\psi_1, \dots \psi_Z, \vec{\xi})] dV_{g^n}$$

where each complete contraction $C^h_{g^n}(\psi_1, \dots \psi_Z, \vec{\xi})$ is either in the form (63) or in the form (64) or in the form:

$$\begin{aligned}\text{contr}\big(&\nabla^{m_1}_{r_1\dots r_{m_1}} R_{i_1 j_1 k_1 l_1} \otimes \cdots \otimes \nabla^{m_s}_{v_1\dots v_{m_s}} R_{i_s j_s k_s l_s} \\ &\otimes \nabla^{p_1}_{t_1\dots t_{p_1}} \text{Ric}_{\alpha_1\beta_1} \otimes \cdots \otimes \nabla^{p_q}_{z_1\dots z_{p_q}} \text{Ric}_{\alpha_q\beta_q} \otimes \nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\psi_1 \\ &\otimes \cdots \otimes \nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\psi_Z \otimes S\nabla^{z_1}\Gamma^k_{ij} \otimes S\nabla^{w_1}\vec{\xi} \otimes \cdots \otimes S\nabla^{w_a}\vec{\xi} \otimes (\vec{\xi})\big),\end{aligned} \tag{76}$$

where the symbol $(\vec{\xi})$ means that there may or may not be a factor $\vec{\xi}$.

We see that each $C^h_{g^n}(\psi_1, \dots \psi_Z, \vec{\xi})$ can be taken to be in the form (76), by the following reasoning: If the covariant derivative ∇_k hits a factor $\nabla^m R_{ijkl}$ or $\nabla^p\text{Ric}$ or $\nabla^\nu\psi_l$, then we will get a $\vec{\xi}$-contraction in the form (76). If it hits a factor $S\nabla^m\vec{\xi}_j$, we apply the formula (65) and get a linear combination of $\vec{\xi}$-contractions in the form (76). Finally, if it hits the factor $S\nabla^m\Gamma^k_{ij}$, we will get a complete contraction as in (76) with $l-1$ factors $S\nabla^m\vec{\xi}$, and with an extra factor $\nabla_k S\nabla^{m-1}\Gamma^k_{r_m j}$. We then apply the formula:

$$\begin{aligned}\nabla_a S\nabla^m_{r_1\dots r_m}\Gamma^k_{ij} = {}& S\nabla^{m+1}_{ar_1\dots r_m}\Gamma^k_{ij} + C_m \cdot S^*\nabla^m_{r_1\dots r_m} R_{aij}{}^k \\ &+ \sum_{u\in U^m} a_u \text{pcontr}(\nabla^{m'} R_{fghj}, S\nabla^{x_u}\Gamma^k_{bc})\end{aligned} \tag{77}$$

where the symbol $\text{pcontr}(\nabla^{m'} R_{fghj}, S\nabla^{x_u}\Gamma^k_{bc})$ (we call that sublinear combination the *correction terms*) stands for a partial contraction of at least one factor

$\nabla^{m'} R_{fghj}$ against a factor $S\nabla^{x_u}\Gamma^k_{bc}$ *or* a partial contraction of $a \geq 2$ factors $\nabla^{m''} R_{f'g'h'j'}$. We recall that $S^*\nabla^m_{r_1\dots r_m} R_{aijk}$ stands for the symmetrization of the tensor $\nabla^m_{r_1\dots r_m} R_{aijk}$ over the indices $r_1, \dots, r_m, i, j$.

Furthermore, we have that in each such partial contraction, the index a appears in a factor $\nabla^{m'} R_{fghj}$.

In order to check that in each correction term there can be at most one factor $S\nabla^p\Gamma$, we only have to observe that in order to symmetrize a tensor $\nabla_k S\nabla^{m-1}\Gamma^k_{r_m j}$, we only introduce correction terms by virtue of the formula $[\nabla_a\nabla_b - \nabla_b\nabla_a]X_c = R_{abcd}X^d$, and the formula $\nabla_a\Gamma^k_{bc} - \nabla_b\Gamma^k_{ac} = R_{abc}{}^k$. Hence, for each application of the above formulas, we may decrease the number of factors $\nabla^p\Gamma$, but we cannot increase it.

Thus we see that our $\vec{\xi}$-contraction will be a linear combination of $\vec{\xi}$-contractions in the form (76) or (64).

So, in general, we must integrate by parts expressions of the following form:

$$\int_{R^N} [e^{(N-n)\vec{\xi}\cdot\vec{x}} \frac{P(N)}{(N-n)^m} C^j_{g^n}(\psi_1, \dots \psi_Z, \vec{\xi})] dV_{g^n}$$

where the complete contraction $C^j_{g^n}(\psi_1, \dots \psi_Z, \vec{\xi})$ is in the form:

$$\begin{aligned}(78)\quad &\mathrm{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}} R_{i_1 j_1 k_1 l_1} \otimes \dots \otimes \nabla^{m_s}_{v_1\dots v_{m_s}} R_{i_s j_s k_s l_s} \otimes \nabla^{p_1}_{t_1\dots t_{p_1}} \mathrm{Ric}_{\alpha_1\beta_1} \\ &\otimes \dots \otimes \nabla^{p_q}_{z_1\dots z_{p_q}} \mathrm{Ric}_{\alpha_q\beta_q} \otimes \nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}} \psi_1 \otimes \dots \otimes \nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}} \psi_Z \\ &\otimes S\nabla^{z_1}\Gamma^k_{ij} \otimes \dots \otimes S\nabla^{z_v}\Gamma^k_{ij} \otimes S\nabla^{u_1}\vec{\xi} \otimes \dots \otimes S\nabla^{u_d}\vec{\xi} \otimes \vec{\xi} \otimes \dots \otimes \vec{\xi}\big).\end{aligned}$$

The integration by parts of such complete contractions can be done as before: If there is a factor $\vec{\xi}$ then we integrate by parts using it, and symmetrize and anti-symmetrize as will be explained below. If there is no factor $\vec{\xi}$, we pick out one factor $S\nabla^m_{r_1\dots r_m}\vec{\xi}_j$ and write it as

$$S\nabla^{m-1}_{r_1\dots r_{m-1}} \Gamma^k_{r_m j} \vec{\xi}_k.$$

We then integrate by parts with respect to the factor $\vec{\xi}_k$, using the formula (60). If the derivative ∇_k hits a factor $\nabla^m R_{ijkl}$, or $\nabla^p \psi_l$, or $\nabla^p\mathrm{Ric}$, we leave them as they are. If it hits a factor $S\nabla^x\Gamma^k_{ij}$ or a factor $S\nabla^m\vec{\xi}$, we apply the formulas, (65), (77) respectively.

In the end, we will have the following formula for the integration by parts of a hard $\vec{\xi}$-contraction $C_{g^N}(\psi_1, \dots, \psi_Z, \vec{\xi})$:

$$\begin{aligned}(79)\quad &\int_{R^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} \frac{P(N)}{(N-n)^m} C_{g^N}(\psi_1, \dots, \psi_Z, \vec{\xi}) dV_{g^n} \\ &\qquad = \int_{R^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} \frac{P(N)}{(N-n)^{m'}} \sum_{s\in S} a_s C^s_{g^N}(\psi_1, \dots, \psi_Z) dV_{g^n}\end{aligned}$$

where the degree of the rational function $\frac{P(N)}{(N-n)^{m'}}$ is zero and the complete contractions $C^s_{g^N}(\psi_1,\dots,\psi_Z)$ are in the general form:

$$\begin{aligned}(80)\quad \mathrm{contr}\big(&\nabla^{m_1}_{r_1\dots r_{m_1}} R_{i_1 j_1 k_1 l_1}\otimes\dots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}} R_{i_s j_s k_s l_s}\\ &\otimes\nabla^{p_1}_{t_1\dots t_{p_1}}\mathrm{Ric}_{\alpha_1\beta_1}\otimes\dots\otimes\nabla^{p_q}_{z_1\dots z_{p_q}}\mathrm{Ric}_{\alpha_q\beta_q}\\ &\otimes\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\psi_1\otimes\dots\otimes\nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\psi_Z\otimes S\nabla^{x_1}\Gamma^{k_1}_{ij}\otimes\dots\otimes S\nabla^{x_u}\Gamma^{k_u}_{ij}\big)\end{aligned}$$

where $u\ge 0$. Therefore, by virtue of (77), we see that if $C_{g^n}(\psi_1,\dots,\psi_Z)$ is hard, then the integrand on the right-hand side of (79) may apriori contain complete contractions in the form (31). We accept this for the time being, although we will later show, in Lemma 14 that, in fact, there will be cancellation among such complete contractions.

5.3. *The simple divergence formula.* Therefore, after a series of integrations by parts, the relation (59) can be brought into the form:

$$\begin{aligned}(81)\quad \int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}\Big[I^Z_{g^N}(\psi_1,\dots,\psi_Z)-\sum_{a\in A}\alpha_a\frac{P_a(N)}{(N-n)^{r_a}}C^a_{g^N}(\psi_1,\dots,\psi_Z)\\ -\sum_{b\in B}\beta_b\frac{P_b(N)}{(N-n)^{r_b}}C^b_{g^N}(\psi_1,\dots,\psi_Z)\Big]dV_{g^n}=0\end{aligned}$$

where $\deg[P_a(N)]=r_a$ and $\deg[P_b(N)]<r_b$. The complete contractions

$$\frac{P_a(N)}{(N-n)^{r_a}}C^{Z,a}_{g^N}(\psi_1,\dots,\psi_Z)(x)$$

have arisen from iterated integrations by parts of the good, the hard and the undecided complete contractions. They are in the form (62) or (80). We may assume with no loss of generality that the leading order coefficient of each of the polynomials $P_a(N)$ is 1, incorporating it in α_a.

The complete contractions $C^b_{g^N}(\psi_1,\dots,\psi_Z)(x)$ have arisen from the easy complete contractions. All of the complete contractions in the formula (81) have arisen according to the procedure we outlined in the previous subsection.

Now, relation (81) shows us that the quantity between brackets is zero for every $x\in B(\tilde{x}_0,\varepsilon)$. In particular,

$$\begin{aligned}(82)\quad I^Z_{g^N}(\psi_1,\dots,\psi_Z)(\tilde{x}_0)-\sum_{a\in A}\alpha_a\frac{P_a(N)}{(N-n)^{r_a}}C^a_{g^N}(\psi_1,\dots,\psi_Z)(\tilde{x}_0)\\ -\sum_{b\in B}\beta_b\frac{P_b(N)}{(N-n)^{r_b}}C^b_{g^N}(\psi_1,\dots,\psi_Z)(\tilde{x}_0)=0\end{aligned}$$

for every Riemannian manifold (M^N,g^N), any functions $\psi_1,\dots,\psi_Z$ around $\tilde{x}_0\in M^N$ and any coordinate system around $\tilde{x}_0\in M^N$. Now pick any (M^n,g^n), any

$x_0 \in M^n$ and any coordinate system around x_0. We define $M^N = M^n \times S^1 \times \cdots \times S^1$ (S^1 has the standard flat metric and g^N is the product metric). We pick $\tilde{x}_0 = (x_0, 0, \dots, 0)$ and consider the induced coordinate system around $\tilde{x}_0$. Hence

$$(83) \quad I^Z_{g^n}(\psi_1, \dots, \psi_Z)(x_0) - \sum_{a\in A} \alpha_a \frac{P_a(N)}{(N-n)^{r_a}} C^a_{g^n}(\psi_1, \dots, \psi_Z)(x_0) \\ - \sum_{b\in B} \beta_b \frac{P_b(N)}{(N-n)^{r_b}} C^b_{g^n}(\psi_1, \dots, \psi_Z)(x_0) = 0$$

for every Riemannian manifold (M^n, g^n), any functions $\psi_1, \dots, \psi_Z$ around $x_0 \in M^n$ and any coordinate system around $x_0 \in M^n$.

In equation (83), N is just a free variable. Hence, we can take the limit as $N \longrightarrow \infty$ in (83) and obtain the *simple divergence formula*:

$$(84) \qquad I^Z_{g^n}(\psi_1, \dots, \psi_Z)(x_0) - \sum_{a\in A} \alpha_a C^a_{g^n}(\psi_1, \dots, \psi_Z)(x_0) = 0.$$

So we have disposed of the integrations by parts of the easy complete contractions.

6. **The three refinements of the simple divergence formula**

6.1. *The first refinement*: *Separating intrinsic from un-intrinsic complete contractions.* We recall from the previous section that some of the complete contractions in (84) will be in the form (62). On the other hand, we have also found that there will be complete contractions in the general form (80), with $u \geq 1$. Accordingly, we introduce the following dichotomy:

Definition 14. Complete contractions in the form (31) or (62) will be called *intrinsic*. Complete contractions in the general form (80) with $u > 0$ will be called *un-intrinsic*.

We consider, in (84), the two sub-linear combinations of the *intrinsic* and of the *un-intrinsic* complete contractions. Written that way, (84) will be:

$$(85) \quad I^Z_{g^n}(\psi_1, \dots, \psi_Z) - \sum_{l\in L} \alpha_l C^l_{g^n}(\psi_1, \dots, \psi_Z) - \sum_{r\in R} \alpha_r C^r_{g^n}(\psi_1, \dots, \psi_Z) = 0$$

where the complete contractions $C^l_{g^n}(\psi_1, \dots, \psi_Z)(x)$ are the intrinsic ones and the complete contractions $C^r_{g^n}(\psi_1, \dots, \psi_Z)(x)$ are the un-intrinsic ones. We have, of course, that $L \cup R = A$.

Our next goal is to show that:

$$(86) \qquad I^Z_{g^n}(\psi_1, \dots, \psi_Z)(x) - \sum_{l\in L} \alpha_l C^l_{g^n}(\psi_1, \dots, \psi_Z)(x) = 0$$

which is equivalent to proving:

$$(87) \qquad \sum_{r\in R} \alpha_r C^r_{g^n}(\psi_1, \dots, \psi_Z)(x) = 0.$$

So let us focus on showing (86). We treat the value of the left-hand side of (84) as a function of the coordinate system wanting to show, roughly speaking, that the tensors $S\nabla^m\Gamma^k_{ij}(x_0)$ are not independent of the coordinate system in which they are expressed. In other words, they are not intrinsic tensors of the Riemannian manifold (M^n, g^n).

LEMMA 9. (86) *holds.*

Proof. We consider the tensors $S\nabla^m_{s_1\dots s_m}\Gamma^k_{ij}(x_0)$, $\Gamma^k_{ij}(x_0)$, written out in any coordinate system. We want to see what their values can be, given our metric g^n around x_0.

We need to recall the following fact from [13]: Consider a coordinate transformation around the point $x_0 \in M^n$. Let us say we had coordinates $\{x^1, \dots, x^n\}$ and now we have coordinates $\{y^1, \dots, y^n\}$. Then the Christoffel symbols Γ^k_{ij} will transform as follows:

$$\tilde{\Gamma}^\lambda_{\mu\nu}\frac{\partial x^l}{\partial y^\lambda}(x_0) = \Gamma^l_{ij}(x_0)\frac{\partial x^i}{\partial y^\mu}\frac{\partial x^j}{\partial y^\nu} + \frac{\partial^2 x^l}{\partial y^\mu \partial y^\nu}(x_0) \tag{88}$$

(where $\tilde{\Gamma}^\lambda_{\mu\nu}(x_0)$ stands for the Christoffel symbols in the new coordinate system).

Now, the tensors $\nabla^m R_{ijkl}$ are intrinsic tensors of the Riemannian manifold. That means that they satisfy the intrinsic transformation law under coordinate changes, as in [13].

We will need the following lemma:

LEMMA 10. *Consider a point $x_0 \in M^n$ and a coordinate system $\{x^1, \dots, x^n\}$ around x_0 for which $g^n_{ij}(x_0) = \delta_{ij}$. Then, given any list of special tensors $T^k_{r_1\dots r_{p+2}}$, which are symmetric in the indices $r_1, \dots r_{p+2}$, there is a coordinate system $\{y^1, \dots \dots, y^n\}$ around $x_0 \in M^n$ so that the tensors $S\nabla^p_{r_1\dots r_p}\Gamma^k_{r_{p+1}r_{p+2}}$ have the values of the arbitrarily chosen tensors $T^k_{r_1\dots r_{p+2}}$ at x_0 and furthermore we have that $[\frac{\partial y}{\partial x}](x_0) = \mathrm{Id}^{n\times n}$ and $g^n_{ij} = \delta_{ij}$ (with respect to the new coordinate system).*

Proof. We observe that by [13] when we change the coordinate system $\{x^1, \dots \dots, x^n\}$ into $\{y^1, \dots, y^n\}$, the tensors $\nabla^p_{r_1\dots r_p}\Gamma^l_{r_{p+1}r_{p+2}}$ will transform as follows:

$$\begin{aligned}\nabla^p_{r'_1\dots r'_p}\tilde{\Gamma}^\lambda_{r'_{p+1}r'_{p+2}}\frac{\partial x^l}{\partial y^\lambda}(x_0) &= \nabla^p_{r_1\dots r_p}\Gamma^l_{r_{p+1}r_{p+2}}\frac{\partial x^{r_1}}{\partial y^{r'_1}}\cdots\frac{\partial x^{r_{p+2}}}{\partial y^{r'_{p+2}}}(x_0) \\ &\quad + \frac{\partial^{p+2}x^l}{\partial y^{r'_1}\dots\partial y^{r'_{p+2}}}(x_0) + \sum\left(\partial^f\Gamma, \frac{\partial^h x}{\partial^h y}\right)(x_0)\end{aligned} \tag{89}$$

where $\tilde{\Gamma}$ stands for the Christoffel symbols in the new coordinate system and $\sum(\partial^f\Gamma, \partial^h x/\partial^h y)$ stands for a linear combination of partial contractions of factors against factors $\partial^h x/\partial^h y$ with $h < p+2$.

Now, we can prescribe

$$\frac{\partial^{p+2} x^l}{\partial y^{r'_1} \dots \partial y^{r'_{p+2}}}(x_0)$$

to have any symmetric value in the indices $r_1, \dots, r_{p+2}$. Therefore, if we write out the transformation law for $S\nabla^p_{r_1 \dots r_p} \Gamma^k_{r_{p+1} r_{p+2}}(x_0)$ under coordinate changes, then the linearized part of its transformation law will be precisely

$$\frac{\partial^{p+2} y^l}{\partial x^{r_1} \dots \partial x^{r_{p+2}}}(x_0).$$

Hence, by induction on p, we have our lemma. □

We call these arbitrary tensors $T^k_{r_1 \dots r_{p+2}}$ the *un-intrinsic free variables*. By construction, they satisfy $T^k_{r_1 \dots r_{p+2}}(t^2 g^n) = T^k_{r_1 \dots r_{p+2}}(g^n)$. Thus, they are special tensors.

But then it is straightforward to check Lemma 9. We can break equation (85) into two summands: the left-hand side of (86) plus the left-hand side of (87). We may then pick any $\lambda \in \mathbb{R}$ and a new coordinate system so that $(S\nabla^p \Gamma^k_{ij})'(x_0) = \lambda \cdot (S\nabla^p \Gamma^k_{ij})(x_0)$. (Here $(S\nabla^p \Gamma^k_{ij})'(x_0)$ stands for the value of $S\nabla^p \Gamma^k_{ij}(x_0)$ with respect to the new coordinate system). We can then see the left-hand side of (85) as a polynomial in λ, $\Pi(\lambda)$. We have that the constant term of $\Pi(\lambda)$ must be zero. Also, the constant term of $\Pi(\lambda)$ is precisely the left-hand side of (86). We have shown our lemma. □

6.2. *The second refinement*: *An intrinsic divergence formula.* We begin this subsection with one more convention. Given an equation of the form:

$$\sum_{l \in L} a_l C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi}) = 0 \tag{90}$$

where each $C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ is a complete contraction in the form (64), we will be thinking of the factors $S\nabla^m_{r_1 \dots r_m} \vec{\xi}_j$ ($m \geq 0$) as symmetric $(m+1)$-tensors in the indices $r_1, \dots, r_m, j$ so that

$$S\nabla^m_{r_1 \dots r_m} \vec{\xi}_j(t^2 g^n) = S\nabla^m_{r_1 \dots r_m} \vec{\xi}_j(g^n).$$

This condition trivially holds since $\nabla^m_{r_1 \dots r_m} \vec{\xi}_j = \nabla^{m+1}_{r_1 \dots r_m j}(\vec{x} \cdot \vec{\xi})$. Moreover, we imply that the above equation holds for every $x_0 \in (M^n, g^n)$ (g^n can be any Riemannian metric), any functions $\psi_1, \dots, \psi_Z$ defined around x_0, any vector $\vec{\xi} \in \mathbb{R}^n$ and any coordinate system defined around x_0.

Now, we define $C^l_{g^n}(\psi_1, \dots, \psi_Z, \Omega)$ to stand for complete contraction that arises from $C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ by replacing each factor $S\nabla^m_{r_1 \dots r_m} \vec{\xi}_j$ by an auxiliary symmetric tensor $\Omega_{r_1 \dots r_m j}$. We claim:

LEMMA 11. *Assuming* (90) (*as explained above*), *we have that*:

$$\sum_{l \in L} a_l C^l_{g^n}(\psi_1, \ldots, \psi_Z, \Omega) = 0 \tag{91}$$

will hold for every $x_0 \in (M^n, g^n)$ (g^n *can be any Riemannian metric*), *any functions* $\psi_1, \ldots, \psi_Z$ *defined around* x_0 *and any symmetric tensors* $\Omega_{i_1 \ldots i_s}$.

Proof. First, we observe that for every sequence $\Omega_{i_1}, \ldots, \Omega_{i_1 \ldots i_s}, \ldots$ of symmetric tensors for which $\Omega_{i_1} \neq 0$, we have that there is vector $\vec{\xi} \in \mathbb{R}^n$ and also a coordinate system around $x_0 \in M^n$ so that:

$$\vec{\xi}_{i_1} = \Omega_{i_1}, \ldots, S\nabla^m_{r_1 \ldots r_m} \vec{\xi}_j = \Omega_{r_1 \ldots r_m j}, \ldots .$$

This is clear by virtue of the formula $S\nabla^m_{r_1 \ldots r_m} \vec{\xi}_j = S\nabla^{m-1}_{r_1 \ldots r_{m-1}} \Gamma^k_{r_m j} \vec{\xi}_k$ and by Lemma 10.

Now, for any sequence $\Omega_{i_1}, \ldots, \Omega_{i_1 \ldots i_s}, \ldots$ where $\Omega_{i_1} = 0$, we only have to consider any vector $\vec{\varepsilon}_i$ where $|\vec{\varepsilon}_i|$ is small. We then have that there is a coordinate system so that $\vec{\xi}_i = \vec{\varepsilon}_i$ and $S\nabla^m_{r_1 \ldots r_m} \vec{\xi}_j = \Omega_{r_1 \ldots r_m j}$, for every $m \geq 1$. Letting $\vec{\varepsilon}_i \longrightarrow 0$, we obtain our lemma. $\square$

Now, the aim of this subsection is to further refine Lemma 9. We will need certain preliminary observations. Notice the following: Let us pick out one $\vec{\xi}$-contraction $Q(N) \cdot C^l_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi})$ of the form (64) with $k + l \leq |\vec{\xi}|$. We have then treated the integrals

$$\int_{R^N} e^{(N-n)\vec{\xi} \cdot \vec{x}} Q(N) \cdot C^l_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi}) dV_{g^n}$$

and performed integrations by parts, obtaining a relation

$$\int_{R^N} e^{(N-n)\vec{\xi} \cdot \vec{x}} Q(N) \cdot C^l_{g^N}(\psi_1, \ldots, \psi_Z, \vec{\xi}) dV_{g^n} = \int_{R^N} e^{(N-n)\vec{\xi} \cdot \vec{x}} \sum_{s \in S^l} Q_s(N) C^{l,s}_{g^N}(\psi_1, \ldots, \psi_Z) dV_{g^n} \tag{92}$$

where the degree of the rational function $Q_s(N)$ is zero. Adding up all the integrations by parts, writing things in dimension n and taking the limit $N \longrightarrow \infty$ gives us the formula (84). We call this procedure by which we integrate by parts one $\vec{\xi}$-factor at a time the *iterative procedure* of integrating by parts.

After all the integrations by parts for a $\vec{\xi}$-contraction as in (92), we will call the quantity:

$$\lim_{N \longrightarrow \infty} \sum_{s \in S} Q_s(N) C^{l,s}_{g^n}(\psi_1, \ldots, \psi_Z)$$

the *final outcome* of the iterative integration by parts. This is denoted by

$$F[Q(N) C^l_{g^n}(\psi_1, \ldots, \psi_Z, \vec{\xi})].$$

Recall that we are assuming the leading order coefficient of $Q_s(N)$ to be 1. We make a further notational convention: When we write out the good or undecided or hard $\vec{\xi}$-contractions and also when we integrate by parts, we will be omitting the dimensional rational function $Q_i(N)$. This is justified by the fact that we eventually take a limit $N \longrightarrow \infty$. So all the formulas that appear in this section will be true after we take the limit $N \longrightarrow \infty$. We refer to this notational convention as the N-cancelled notation.

As an example of this notational convention, we apply the third summand on the right-hand side of the formula (57) to the pair $({}^m, {}_m)$ in $\nabla^m R_{mjkl}$ and bring out $\vec{\xi}^m R_{mjkl}$ instead of saying that we bring out $N\vec{\xi}^m R_{mjkl}$. Also, we replace a factor Ric_{ij} by $-\nabla_i \vec{\xi}_j$ or a factor R by $-|\vec{\xi}|^2$ (instead of $-N\nabla_i \vec{\xi}_j$ or $-N^2|\vec{\xi}|^2$ respectively).

Observation 1. The formal expression for $F[C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})]$ depends on the order in which we perform the integrations by parts. In general, whenever we make reference to the integrations by parts, we assume that we arbitrarily pick an order in which to perform integrations by parts, subject to the restrictions imposed in the corresponding section or any extra restrictions we wish to impose.

We need some conventions to state and prove our Lemma for this subsection:

Definition 15. In N-cancelled notation: Consider any good or undecided $\vec{\xi}$-contraction $C_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$, in the form (63) or (64), with $\vec{\xi}$-factors in $C_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ is $k+l$. We perform the iterative integrations by parts, subject to the following restriction: In each step of the iterative integration by parts, suppose we start off with X $\vec{\xi}$-factors. We integrate by parts with respect to a factor $\vec{\xi}_i$ and obtain a linear combination of $\vec{\xi}$-contractions (each in the form (63) or (64)), each with $X-1$ $\vec{\xi}$-factors. In that linear combination we cross out the hard $\vec{\xi}$-contractions. We then pick out one of the $\vec{\xi}$-contractions remaining (it will either be good or undecided) and again integrate by parts with respect to a factor $\vec{\xi}$. After $k+l$ steps, this procedure will terminate and there remains an expression:

$$\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} \sum_{h\in H} a_h Q_h(N) C^h_{g^N}(\psi_1, \dots, \psi_Z) dV_{g^N}.$$

Each complete contraction $C^h_{g^N}(\psi_1, \dots, \psi_Z)$ is in the form (62) and the rational function $Q_h(N)$ has degree 0 and leading order coefficient 1.

Define $\sum_{h\in H} a_h C^h_{g^n}(\psi_1, \dots, \psi_Z)$ to be the *outgrowth* of $C_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$; we denote it by $O[C_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})]$.

We claim:

PROPOSITION 3. *Consider the sublinear combination* $\tilde{S}^Z_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ *of* $S^Z_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ *which consists of the good and the undecided* $\vec{\xi}$*-contractions. If in our* N*-cancelled notation*

$$\tilde{S}^Z_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}) = \sum_{l\in L} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

then we claim:

$$I^Z_{g^n}(\psi_1,\dots,\psi_Z) + \sum_{l\in L} a_l O[C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] = 0. \tag{93}$$

Proof of Proposition 3. Suppose that the linear combination of the hard $\vec{\xi}$-contractions encountered along the iterative integration by parts of $C^l_{g^n}(\psi_1,\dots$ $\dots,\psi_Z,\vec{\xi})$ is $\sum_{b\in B^l} a_b C^b_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. We can then write:

$$\begin{aligned} F[C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] = O[C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] \\ + \sum_{b\in B^l} a_b F[C^b_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]. \end{aligned} \tag{94}$$

Now, let us note the following fact for the final outcome of the iterative integration by parts of a hard $\vec{\xi}$-contraction $C^u_{g^n}(\phi,\vec{\xi})$, in the form (64) with $k=0$, $l>0$.

LEMMA 12. *Suppose that*:

$$F[C^u_{g^n}(\phi,\vec{\xi})] = \sum_{y\in Y^u} a_y C^y_{g^n}(\phi).$$

Then, there will be one complete contraction $a_y C^y_{g^n}(\phi)$ (*along with its coefficient*) *which is obtained from* $C^u_{g^n}(\phi,\vec{\xi})$ *by replacing each of the* l *factors* $S\nabla^m_{r_1\dots r_m}\vec{\xi}_j$ *by* $-S\nabla^m_{dr_1\dots r_{m-1}}\Gamma^d_{r_m j}$. *That complete contraction arises when each derivative* ∇^d, *in the integration by parts of* $S\nabla^m_{r_1\dots r_m}\vec{\xi}_j = S\nabla^{m-1}_{r_1\dots r_{m-1}}\Gamma^d_{r_m j}\vec{\xi}_d$, *hits the factor* $S\nabla^{m-1}_{r_1\dots r_{m-1}}\Gamma^d_{r_m j}$, *and then we symmetrize using* (77). *We denote this complete contraction by* $DF[C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$.

Furthermore, *other complete contractions* $a_y C^y_{g^n}(\phi)$ *in* $F[C^u_{g^n}(\phi,\vec{\xi})]$ *will have strictly less than* l *un-intrinsic free variables* $S\nabla^m_{r_1\dots r_m}\Gamma^d_{ij}$ *for which* d *contracts against one of the indices* $r_1,\dots,r_m,i,j$ *in* $C^y_{g^n}(\phi)$.

Proof. This follows from the procedure by which we integrate by parts and also from the formula (77). □

Next, we consider the sublinear combination of good, hard and undecided $\vec{\xi}$-contractions in $S^Z_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$:

$$\sum_{l\in L} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}).$$

We break up the index set L as follows: $l \in L^1$ if and only if $C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ is good or undecided; $l \in L^2$ if and only if $C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ is hard.

For any hard $\vec{\xi}$-contraction $C_{g^n}(\phi, \vec{\xi})$, we break up the linear combination $F[C_{g^n}(\phi, \vec{\xi})]$ into the sublinear combination $F^{\mathrm{Intr}}[C_{g^n}(\phi, \vec{\xi})]$ of intrinsic complete contractions and the sublinear combination $F^{\mathrm{UnIntr}}[C_{g^n}(\phi, \vec{\xi})]$ of unintrinsic complete contractions.

We can then rewrite (86) as:

$$
\begin{aligned}
(95)\qquad I^Z_{g^n}(\psi_1, \dots, \psi_Z) &+ \sum_{l \in L^1} a_l O[C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] \\
&+ \sum_{l \in L^1} a_l \sum_{b \in B^l} a_b F^{\mathrm{Intr}}[C^b_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] \\
&+ \sum_{l \in L^2} a_l F^{\mathrm{Intr}}[C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] = 0
\end{aligned}
$$

and also (87) as:

$$
\begin{aligned}
(96)\qquad \sum_{l \in L^1} a_l \sum_{b \in B^l} a_b F^{\mathrm{UnIntr}}[C^b_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})]& \\
+ \sum_{l \in L^2} a_l F^{\mathrm{UnIntr}}[C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] &= 0.
\end{aligned}
$$

We then claim:

LEMMA 13. *We have that*:

$$
\begin{aligned}
\sum_{l \in L^1} a_l \sum_{b \in B^l} a_b F^{\mathrm{Intr}}[C^b_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})]& \\
+ \sum_{l \in L^2} a_l F^{\mathrm{Intr}}[C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] &= 0
\end{aligned}
$$

for every (M^n, g^n), *every* $\psi_1, \dots, \psi_Z$.

Proving this will also show Proposition 3. We will in fact prove a stronger statement than Lemma 13:

LEMMA 14. *We have*

$$
(97)\qquad \sum_{l \in L^1} a_l \sum_{b \in B^l} a_b C^b_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi}) + \sum_{l \in L^2} a_l C^l_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi}) = 0
$$

for any point x_0, *any metric* g^n *around* x_0, *any functions* $\psi_1, \dots, \psi_Z$ *and any coordinate system.*

Proof that Lemma 13 *follows from Lemma* 14. Consider the linear combination

$$
\begin{aligned}
\sum_{l \in L^1} a_l \sum_{b \in B^l} a_b N^{p_b} Q_b(N) C^b_{g^N}(\psi_1, \dots, \psi_Z, \vec{\xi})& \\
+ \sum_{l \in L^2} a_l N^{p_l} Q_l(N) C^l_{g^N}(\psi_1, \dots, \psi_Z, \vec{\xi})&
\end{aligned}
$$

of the hard $\vec{\xi}$-contractions put aside, *without* the N-cancelled notation. Here, the rational functions $Q_b(N)$, $Q_l(N)$ have degree zero and leading order coefficient 1. Moreover, if we denote by $|\vec{\xi}|_b$, $|\vec{\xi}|_l$ the number of $\vec{\xi}$-factors in $C^b_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ respectively, we will have that $p_b=|\vec{\xi}|_b$ and $p_l=|\vec{\xi}|_l$.

For the purposes of this proof, we will consider any hard or easy $\vec{\xi}$-contraction $N^a Q_a(N) C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$, where $Q_a(N)$ has degree zero and leading order coefficient one. We perform the iterative integrations by parts, as explained in the previous subsection, and obtain a relation:

$$\int_{\mathbb{R}} e^{(N-n)\vec{\xi}\cdot\vec{x}} N^a Q_a(N) C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi}) dV_{g^n} \\ = \int_{\mathbb{R}} e^{(N-n)\vec{\xi}\cdot\vec{x}} [\sum_{u\in U} a_u \frac{N^a}{(N-n)^{p_u}} Q_a(N) C^u_{g^N}(\psi_1,\dots,\psi_Z)] dV_{g^n} \tag{98}$$

where either $a=p_u$ for every $u\in U$ or $a<p_u$ for every $u\in U$, depending whether $C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is hard or easy, respectively. We then denote the expression between brackets by $E[N^a Q_a(N) C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]$.

As before, we break $E[N^a Q_a(N) C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]$ into two sublinear combinations

$$E^{\mathrm{Intr}}[N^a Q_a(N) C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})],$$
$$E^{\mathrm{Unintr}}[N^a Q_a(N) C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})],$$

that consist of the intrinsic and un-intrinsic complete contractions, respectively.

Now, in view of Lemma 14, it follows that:

$$\sum_{l\in L^1} a_l \sum_{b\in B^l} a_b N^{p_b} Q_b(N) C^b_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi}) \\ + \sum_{l\in L^2} a_l N^{p_l} Q_l(N) C^l_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi}) \\ = \sum_{w\in W} a_w N^{p_w} Q_w(N) C^w_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi}) \tag{99}$$

where each $\vec{\xi}$-contraction $N^{p_w} Q_w(N) C^w_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is easy, and moreover the rational function $Q_w(N)$ has degree zero and leading order coefficient one. We deduce that:

$$\sum_{l\in L^1} a_l \sum_{b\in B^l} a_b E[N^{p_b} Q_b(N) C^b_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})] \\ + \sum_{l\in L^2} a_l E[N^{p_l} Q_l(N) C^l_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})] \\ = \sum_{w\in W} a_w E[N^{p_w} Q_w(N) C^w_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})] \tag{100}$$

and therefore:

$$\sum_{l\in L^1} a_l \sum_{b\in B^l} a_b E^{\mathrm{Intr}}[N^{p_b} Q_b(N) C^b_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})] + \sum_{l\in L^2} a_l E^{\mathrm{Intr}}[N^{p_l} Q_l(N) C^l_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})] = \sum_{w\in W} a_w E^{\mathrm{Intr}}[N^{p_w} Q_w(N) C^w_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]. \tag{101}$$

We then define a new operation Oplim that acts on linear combinations

$$\sum_{h\in H} a_h E[N^{p_l} Q_l(N) C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]$$

(where $N^{p_l} Q_l(N) C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ may be either hard or undecided), by re-writing them in dimension n (thus the coefficients N are now independent of the dimension n) and letting $N \longrightarrow \infty$. We act on the linear combinations on the left and right hand sides of the above by the operation Oplim and deduce:

$$\sum_{l\in L^1} a_l \sum_{b\in B^l} a_b F^{\mathrm{Intr}}[C^b_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})] + \sum_{l\in L^2} a_l F^{\mathrm{Intr}}[C^l_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})] = \sum_{w\in W} a_w \mathrm{Oplim}\{E^{\mathrm{Intr}}[N^{p_w} Q_w(N) C^w_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]\} = 0. \tag{102}$$

Thus, we indeed have that Lemma 13 follows from Lemma 14. □

Proof of Lemma 14. We rewrite (97) in the form:

$$\sum_{l\in L} a_l (C^l)_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}) = 0.$$

Now, $l \in L_\mu$ if and only if $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ has $\mu > 0$ factors $S\nabla^a\vec{\xi}$. We prove the following: Suppose that for some $M > 0$ and every $\mu > M$:

$$\sum_{l\in L_\mu} a_l (C^l)_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}) = 0.$$

We will then show that:

$$\sum_{l\in L_M} a_l (C^l)_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}) = 0. \tag{103}$$

If we can show the above claim, our proof will follow by induction. Now, recall that if for some linear combination of hard $\vec{\xi}$-contractions we have (in N-cancelled notation) that $\sum_{r\in R} a_r C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}) = 0$, then, by the argument above, it follows that:

$$\sum_{r\in R} a_r F^{\mathrm{UnIntr}}[C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] = 0.$$

Therefore, in view of our induction hypothesis,

$$\sum_{\mu>M} \sum_{l\in L_\mu} a_l F^{\mathrm{UnIntr}}[(C^l)_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] = 0. \tag{104}$$

Therefore, for the proof of our inductive statement we may assume that the above sublinear combination from (96) has been crossed out when we refer to (96).

In order to show (103), we will initially show:

$$\sum_{l\in L_M} a_l DF[(C^l)_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]=0. \tag{105}$$

First, for any complete contraction in the form (79), we call a factor $S\nabla^{\nu}_{r_1\dots r_\nu}\Gamma^k_{ij}$ where the index k contracts against one of the indices $r_1,\dots,j$ a *useful* factor. Any complete contraction in (96) which does not belong to the sublinear combination (103) will have strictly less than M useful factors. This follows from our definition of the index set L_M and Lemma 12.

Now, denote by Special($\sum_{l\in L_M}$) the sublinear combination in (105) that consists of complete contractions all of whose factors in the form $S\nabla^p\Gamma^k_{ij}$ satisfy $p\geq 1$. It follows that:

$$\text{Special}\Big(\sum_{l\in L_M}\Big)=0 \tag{106}$$

by substitution. But then, in view of Lemma 4 we have that (106) holds formally. Then notice that under all the permutation identities in Definition 8, the number of factors $S\nabla^p_{f_1\dots f_p}\Gamma^k_{ij}$ where the index k contracts against one of the indices $f_1,\dots,j$ remains invariant. Hence, since the left-hand side of (105) is the sublinear combination in (96) with the maximum number of useful factors, (105) follows.

But then (105) holds formally (again by Lemma 4). Hence, we imitate the permutations of factors in (105) to make it formally zero for the $\vec{\xi}$-contractions in (103). We only have to observe that if we can permute the indices of two tensors $S\nabla^{m+1}_{dr_1\dots r_m}\Gamma^d_{ij}(x_0)$, $S\nabla^{m+1}_{dr'_1\dots r'_m}\Gamma^d_{i'j'}(x_0)$ to make them formally identical, we can then also permute the indices of the tensors

$$S\nabla^m_{r_1\dots r_m}\Gamma^k_{ij}(x_0)\vec{\xi}_k,\ \ S\nabla^m_{r'_1\dots r'_m}\Gamma^k_{i'j'}(x_0)\vec{\xi}_k$$

to make them formally identical. This shows Lemma 13 and thus Proposition 3. □

6.3. *The third refinement*: *The super divergence formula.* We begin this subsection with a few definitions.

Definition 16. A $\vec{\xi}$-contraction $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ will be called *stigmatized* if it is in the form (64) and each of its factors $\vec{\xi}$ contracts against another factor $\vec{\xi}$. We note that $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is allowed to contain factors $S\nabla^m\vec{\xi}$, $m\geq 1$.

Now, consider any good or undecided $\vec{\xi}$-contraction $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ and consider its iterative integration by parts.

Definition 17. We define the *pure outgrowth* of $C_{g^n}(\psi_1,\dots,\psi_Z)$ to be the sublinear combination of the outgrowth by discarding additionnal terms and imposing additionnal restrictions on the integration by parts:First, whenever we encounter a *hard* $\vec{\xi}$*-contraction* we discard it. Second, whenever we encounter a *stigmatized* $\vec{\xi}$*-contraction* we also discard it. Lastly, if we have a $\vec{\xi}$-contraction which is neither hard nor stigmatized, we will choose to integrate by parts with respect to a factor $\vec{\xi}$ that does not contract against another factor $\vec{\xi}$.

In the end, we will be left with a *linear combination*:

$$\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}} \sum_{h\in H} Q_h(N)\cdot a_h C^h_{g^N}(\psi_1,\dots,\psi_Z) dV_{g^N}.$$

Each complete contraction $C^h_{g^N}(\psi_1,\dots,\psi_Z)$ is in the form (62) and the rational function $Q_h(N)$ has degree 0 and leading order coefficient 1.

We define: $PO[C_{g^n}(\psi_1,\dots,\psi_Z)] = \sum_{h\in H} a_h C^h_{g^n}(\psi_1,\dots,\psi_Z)$.

Our goal for this subsection will be to show:

PROPOSITION 4. *If the sublinear combination of good and undecided* $\vec{\xi}$*-contractions in* $S^Z_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ *is* $\sum_{l\in L} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, *then*:

$$I^Z_{g^n}(\psi_1,\dots,\psi_Z) + \sum_{l\in L} a_l PO[C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] = 0. \tag{107}$$

Before proving this proposition, we will need some preliminary lemmas.

LEMMA 15. *Consider a good or undecided* $\vec{\xi}$*-contraction* $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, *in the form* (63) *or* (64). *Suppose that* $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ *has* α *factors* $|\vec{\xi}|^2$ *and* β *factors* R (*scalar curvature*). *Consider the iterative integration by parts* (*as in the previous subsection*) *of* $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. *Then, at each step along the iterative integration by parts of* $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, *the number of factors* $|\vec{\xi}|^2$ *and the number of factors* R *does not increase.*

Proof. The proof is by induction, following the iterative integration by parts. □

We also define:

Definition 18. Given any $\vec{\xi}$-contraction $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ in the general form (64), let A be the number of its factors $\nabla^m R_{ijkl}$, $\nabla^p\mathrm{Ric}$, Z be the number of factors $\nabla^p\psi_l$, C the number of its factors $S\nabla^m\vec{\xi}$ (with $m\geq 1$), D the number of its factors $|\vec{\xi}|^2$ and E the number of its factors $\vec{\xi}$ that do not contract against another factor $\vec{\xi}$. We then define the $\vec{\xi}$*-length of* $C_{g^n}(\psi_1,\dots,\psi_Z)$ to be $A+Z+C+D$. For any partial contraction in the form (63) or (64), or any $\vec{\xi}$-contraction with factors $\nabla^u\vec{\xi}$ (non-symmetrized), we define its $\vec{\xi}$*-length* in the same way.

We now seek to understand how any given complete contraction $C_{g^n}(\psi_1,\dots$ $\dots,\psi_Z)$ in the form (62) can give rise to good, undecided or hard $\vec{\xi}$-contractions under the re-scaling

$$\hat{g}^N = e^{2\vec{\xi}\cdot\vec{x}} g^N.$$

Definition 19. Let us consider any dimension-dependent complete contraction $Q(N)\cdot C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$, where $C_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is in the form (64), with factors $\nabla^m\vec{\xi}$ instead of $S\nabla^m\vec{\xi}$. We will call such a $\vec{\xi}$-contraction *de-symmetrized*. Recall that $|\vec{\xi}|$ stands for the number of $\vec{\xi}$-factors. We will call such a dimension-dependent $\vec{\xi}$-contraction *acceptable* if $\deg[Q(N)]=|\vec{\xi}|$ and *unacceptable* if $\deg[Q(N)] < |\vec{\xi}|$.

Now, consider any $C_{g^n}(\psi_1,\dots,\psi_Z)$, which is in the form (62). We want to understand how the sublinear combination of acceptable complete contractions arises in $e^{n\vec{\xi}\cdot\vec{x}}C_{e^{2\vec{\xi}\cdot\vec{x}}g^N}(\psi_1,\dots,\psi_Z)$. We need one small convention before making our definition: Whenever we have a factor $\nabla^m_{r_1\dots r_m}\mathrm{Ric}_{ij}$ with $m\ge 1$, we will assume that i,j are not contracting between themselves. This can be done with no loss of generality by virtue of the formula $\nabla_a R = 2\nabla^b\mathrm{Ric}_{ab}$. Thus, we think of our complete contraction as being in the form:

$$\mathrm{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{ijkl}\otimes\cdots\otimes\nabla^{m_s}_{t_1\dots t_{m_s}}R_{ijkl}\otimes\nabla^{p_1}_{r_1\dots r_{p_1}}\mathrm{Ric}_{ij}\otimes\cdots\otimes\nabla^{p_q}_{t_1\dots t_{p_q}}\mathrm{Ric}_{ij}\otimes R^{\alpha}\otimes\nabla^{p_1}_{a_1\dots a_{p_1}}\psi_1\otimes\cdots\otimes\nabla^{p_Z}_{b_1\dots b_{p_Z}}\psi_Z\big) \tag{108}$$

where the factors $\nabla^m R_{ijkl}$ do not have internal contractions between the indices i,j,k,l, the factors $\nabla^p\mathrm{Ric}_{ij}$ do not have internal contractions between the indices i,j. We are now ready for our definition.

Definition 20. We consider internally contracted tensors in one of the following forms: $\nabla^p_{r_1\dots r_p}\psi_l$, $\nabla^p_{r_1\dots r_p}\mathrm{Ric}_{ij}$, $\nabla^p_{r_1\dots r_p}\vec{\xi}_j$ or $\nabla^p_{r_1\dots r_p}R_{ijkl}$. The indices that are not internally contracted are considered to be free.

We will call a pair of internally contracting indices, at least one of which is a derivative index, *an internal derivative contraction*. We now want to define the good substitutions of each tensor above.

For the tensor $\nabla^p_{r_1\dots r_p}\psi_l$, we denote the pairs of internal contractions by $(r_{a_1},r_{b_1}),\dots(r_{a_l},r_{b_l})$. The ordering of the indices r_a,r_b in (r_a,r_b) is arbitrarily chosen. We define the set of good substitutions of the tensor $\nabla^p_{r_1\dots r_p}\psi_l$ as follows: For any subset $\{w_1,\dots,w_j\}\subset\{1,\dots,l\}$ (including the empty set) the tensor

$$\vec{\xi}^{r_{b_{w_1}}}\dots\vec{\xi}^{r_{b_{w_j}}}\nabla^{p-j}_{r_1\dots\hat{r}_{a_{w_1}}\dots r_p}\psi_l$$

is a *good substitution* of $\nabla^p_{r_1\dots r_p}\psi_l$. We similarly define the *set of good substitutions of any tensor* $\nabla^p_{r_1\dots r_p}\mathrm{Ric}_{r_{p+1}r_{p+2}}$, $\nabla^p_{r_1\dots r_p}\vec{\xi}_{r_{p+1}}$ or $\nabla^p_{r_1\dots r_p}R_{r_{p+1}r_{p+2}r_{p+3}r_{p+4}}$ (this last is allowed to have internal contractions, but not among the set $r_{p+1},\dots,r_{p+4}$): For any tensor above, let the *set of pairs of internal derivative contractions* be $(r_{a_1},r_{b_1}),\dots,(r_{a_l},r_{b_l})$. The order of r_a,r_b in (r_a,r_b) is arbitrarily chosen, but r_a must be a derivative index. Also, for the factor $\nabla^p_{r_1\dots r_p}\mathrm{Ric}_{r_{p+1}r_{p+2}}$, if $p\ge 1$, we assume that the indices r_{p+1},r_{p+2} do not contract against each other.

Then, we define the *set of good substitutions of any tensor* as above as follows: For any subset $\{w_1, \dots, w_j\} \subset \{1, \dots, l\}$ (including the empty set), the tensor

$$\vec{\xi}^{r_{b_{w_1}}} \dots \vec{\xi}^{r_{b_{w_j}}} \nabla^{p-j}_{r_1 \dots \hat{r}_{a_{w_1}} \dots r_p} \mathrm{Ric}_{r_{p+1} r_{p+2}} \qquad \text{or}$$

$$\vec{\xi}^{r_{b_{w_1}}} \dots \vec{\xi}^{r_{b_{w_j}}} \nabla^{p-j}_{r_1 \dots \hat{r}_{a_{w_1}} \dots r_p} \vec{\xi}_{r_{p+1}} \qquad \text{or}$$

$$e^{-2\vec{\xi}\cdot\vec{x}} \vec{\xi}^{b_{w_1}} \dots \vec{\xi}^{b_{w_j}} \nabla^{p-j}_{r_1 \dots \hat{r}_{a_{w_1}} \dots r_p} R_{r_{p+1} r_{p+2} r_{p+3} r_{p+4}},$$

respectively, is a *good substitution*.

We define any partial contraction $C^{i_1 \dots i_s}_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ in the form (63) or (64) to be *nice* if in no factor $\vec{\xi}_i$ is the index i free and no factor $\vec{\xi}$ contracts against another factor $\vec{\xi}$ in $C^{i_1 \dots i_s}_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$.

We are now ready for the lemma on acceptable descendants. We want to study the transformation law of any $C_{g^n}(\psi_1, \dots, \psi_Z)$ in the form (62) under the re-scaling $g^N \longrightarrow \hat{g}^N = e^{2\vec{\xi}\cdot\vec{x}} g^N$. We do this in steps: Pick out any factor $T^s_{a_1 \dots a_j}$ in $C_{g^n}(\psi_1, \dots, \psi_Z)$ and make the indices a_i that contract against any other factor in $C_{g^n}(\psi_1, \dots, \psi_Z)$ into free indices. Thus we obtain a factor $(T^s_{a_1 \dots a_j})_{a_{h_1} \dots a_{h_l}}$, which we will call the *liberated form* of the factor $T^s_{a_1 \dots a_j}$. We view $C_{g^n}(\psi_1, \dots, \psi_Z)$ as a complete contraction among those tensors $T^s_{a_{h_1} \dots a_{h_l}}$ and then consider each tensor

$$(T^s_{a_{h_1} \dots a_{h_l}})^{\hat{g}^N}.$$

It will be a tensor of rank l. It follows that if we replace each $(T^s_{a_{h_1} \dots a_{h_l}})^{g^N}$ by $(T^s_{a_{h_1} \dots a_{h_l}})^{\hat{g}^N}$ and take the same contractions of indices as for $C_{g^n}(\psi_1, \dots, \psi_Z)$, with respect to the metric (g^N), we will obtain $e^{n\vec{\xi}\cdot\vec{x}} C_{e^{2\vec{\xi}\cdot\vec{x}} g^N}(\psi_1, \dots, \psi_Z)$.

LEMMA 16 (The acceptable descendants). *Given a complete contraction* $C_{g^n}(\psi_1, \dots, \psi_Z)$ *in the form* (108), *the sublinear combination of the acceptable* $\vec{\xi}$*-contractions in* $e^{n\vec{\xi}\cdot\vec{x}} C_{e^{2\vec{\xi}\cdot\vec{x}} g^N}(\psi_1, \dots, \psi_Z)$ *can be described as folows, in* N*-cancelled notation*:

Each of its liberated factors $(T^s_{a_1 \dots a_j})^{g^N}_{a_{h_1} \dots a_{h_l}}$ *can be replaced according to the pattern*:

1. *Any factor of the form* $\nabla^m_{r_1 \dots r_m} R_{ijkl}$ (*where the indices* i, j, k, l *do not contract between themselves*) *can be replaced by a good substitution of* $\nabla^m_{r_1 \dots r_m} R_{ijkl}$ *or by a nice partial contraction of* $\vec{\xi}$*-length* ≥ 2.
2. *Any factor* $\nabla^p \psi_l$ *can be replaced by a good substitution of* $\nabla^p \psi_l$ *or by a nice partial contraction of* $\vec{\xi}$*-length* ≥ 2.
3. *Any factor* $\nabla^p_{r_1 \dots r_p} \mathrm{Ric}_{ij} \neq R$ *can be substituted either by a good substitution of* $\nabla^p_{r_1 \dots r_p} \mathrm{Ric}_{ij}$ *or a good substitution of* $-\nabla^{p+1}_{r_1 \dots r_p i} \vec{\xi}_j$ *or by a nice partial contraction of* $\vec{\xi}$*-length* ≥ 2.

4. *Any factor R can be left unaltered or be substituted by $-2\nabla^i\vec{\xi}_i$ or by $-|\vec{\xi}|^2$.*

The sublinear combination of acceptable $\vec{\xi}$-contractions in

$$e^{n\vec{\xi}\cdot\vec{x}}C_{e^{2\vec{\xi}\cdot\vec{x}}g^N}(\psi_1,\dots,\psi_Z)$$

arises by substituting each liberated factor $(T_{a_1\dots a_j})^{g^N}_{a_{h_1}\dots a_{h_l}}$ in $C_{g^n}(\psi_1,\dots,\psi_Z)$ as explained above and then performing the same particular contractions among the liberated factors as in $C_{g^n}(\psi_1,\dots,\psi_Z)$, with respect to the metric g^N.

Proof. The proof of this lemma is a matter of applying formulas (57), (55), (54) and (56) as well as (65).

Consider any sequence of tensors times coefficients: $a(N)\cdot(T_{i_1\dots i_j})^{g^N}$, where $N=n,n+1,\dots$ and the tensors $(T_{i_1\dots i_j})^{g^n}$ are partial contractions of the form:

$$\text{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1}\otimes\cdots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}\\ \otimes\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\psi_l\otimes\nabla^m_{u_1\dots u_m}\vec{\xi}_z\otimes\cdots\otimes\nabla^m_{u_1\dots u_m}\vec{\xi}_z\otimes g^N_{ij}\otimes\cdots\otimes g^N_{ij}\big) \tag{109}$$

where there is at least one factor $\nabla^\nu\psi_l$ or $\nabla^m R_{ijkl}$ or $\nabla^m\vec{\xi}$, but not necessarily one of each kind; $a(N)$ is a rational function in N and $(T_{i_1\dots i_j})^{g^N}$ is the rewriting of $(T_{i_1\dots i_j})^{g^n}$ in dimension N.

For any such partial contraction let $|g|$ stand for the number of factors g^N_{ij}, $|\vec{\xi}|$ stand for the number of factors $\nabla^m\vec{\xi}$ and $\deg[a(N)]$ stand for the degree of the rational function $a(N)$.

We also consider linear combinations:

$$\sum_{t\in T}a_t(N)(T^t_{i_1\dots i_s})^{g^N} \tag{110}$$

where each sequence $a_t(N)(T^t_{i_1\dots i_s})^{g^N}$ is as above. From now on we will just speak of the partial contraction $a_t(N)(T^t_{i_1\dots i_s})^{g^N}$, rather than the sequence of partial contractions times coefficients.

We say that such a partial contraction is *useful* if $|g|=0$, $|\vec{\xi}|=\deg[a_t(N)]$ and the index k in each factor $\vec{\xi}_k$ is not free and there are no factors $|\vec{\xi}|^2$. We will call a partial contraction *useless* if $\deg[a_t(N)]+|g|<|\vec{\xi}|$ or if $\deg[a_t(N)]+|g|=|\vec{\xi}|$ and $|g|>0$. Note that “useless” is not the negation of “useful”.

Consider any tensor $(\nabla^m_{r_1\dots r_m}R_{ijkl})^{g^N}$ or $(\nabla^p\psi_l)^{g^N}$ or $(\nabla^p_{t_1\dots t_p}\text{Ric}_{ij})^{g^N}$ with internal contractions. Suppose that the free indices are $i_1,\dots i_s$. We will write those tensors out as $(\nabla^m_{r_1\dots r_m}R_{ijkl})^{g^N}_{i_1,\dots i_s}$, $(\nabla^p\psi_l)^{g^N}_{i_1,\dots i_s}$, $(\nabla^p_{t_1\dots t_p}\text{Ric}_{ij})^{g^N}_{i_1\dots i_s}$.

We claim that any tensor

$$e^{-2\vec{\xi}\cdot\vec{x}}(\nabla^m_{r_1\dots r_m}R_{ijkl})^{\hat{g}^N}_{i_1\dots i_s}$$

or $(\nabla^p\psi_l)_{i_1\dots i_s}{}^{\hat{g}^N}$ or $(\nabla^p_{t_1\dots t_p}\mathrm{Ric}_{ij})i_1\dots i_s{}^{\hat{g}^N}$ (where i,j,k,l in the first case and i,j in the second do not contract against each other) is a linear combination of useful and useless tensors, as in (110). Furthermore, we claim that each useful partial contraction of $\vec{\xi}$-length 1 in the expression for

$$(\nabla^m_{r_1\dots r_m}R_{ijkl})^{\hat{g}^N}_{i_1\dots i_s} \quad \text{or} \quad (\nabla^p\psi_l)^{\hat{g}^N}_{i_1\dots i_s} \quad \text{or} \quad (\nabla^p_{t_1\dots t_p}\mathrm{Ric}_{ij})^{\hat{g}^N}_{i_1\dots i_s}$$

will be one of the good substitutions described in Definition 20. We refer to this as claim A.

We will check this by induction on m or p, respectively. For $m=0$ or $p=1$, the fact is straightforward from (54) and (55). So, assume we know that fact for $p=K$ or $m=K$ and let us show it for $p=K+1$ or $m=K+1$. Consider first the case of a tensor $(\nabla^{K+1}_{r_1\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_1\dots i_s}$. We inquire whether the index r_1 is free. If so, we then use our inductive hypothesis for $p=K$ knowing that the tensor $(\nabla^K_{r_2\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_2\dots i_s}$ satisfies the induction hypothesis. We now use this to find the tensor $(\nabla^{K+1}_{r_1\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_1\dots i_s}$, writing

$$(\nabla^K_{r_2\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_2\dots i_s} = \sum_{t\in T_1} a_t(N)T^t_{g^N}(\psi_l,\vec{\xi})_{i_2\dots i_s} + \sum_{t\in T_2} a_t(N)T^t_{g^N}(\psi_l,\vec{\xi})_{i_2\dots i_s} \tag{111}$$

where the first sublinear combination stands for the useful tensors and the second stands for the useless tensors.

We only have to apply the transformation law (57) to each pair $(r_1,i_2),\dots$ $\dots,(r_1,i_s)$. We easily observe that if any summand in the expression of

$$(\nabla^K_{r_2\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_2\dots i_s}$$

is useless, then any application of the identity (57) to any pair of indices $(r_1,i_2),\dots$ $\dots,(r_1,i_s)$ will give rise to a useless partial contraction. On the other hand, consider any factor $T^t_{g^N}(\psi_l,\vec{\xi})_{i_2\dots i_s}$ in $(\nabla^K_{r_2\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_2\dots i_s}$ which is useful. Then observe that when we apply any of the last three summands in (57) to any pair of indices $(r_1,i_2),\dots,(r_1,i_s)$ and bring out a factor $\vec{\xi}$, we obtain a useless partial contraction. Finally, substituting $(\nabla_{r_1}X_{i_l})^{\hat{g}^N}$ by $(\nabla_{r_1}X_{i_l})^{(g^N)}$ (the first summand on the right hand side of (23)), we get a linear combination of useful $\vec{\xi}$-contractions, by applying the rule

$$\nabla_i[A_{k_1\dots k_s}\otimes B_{u_1\dots u_h}] = \nabla_i A_{k_1\dots k_s}\otimes B_{u_1\dots u_h} + A_{k_1\dots k_s}\otimes\nabla_i B_{u_1\dots u_h}.$$

Furthermore, if a partial contraction $a_t(N)T^t_{g^N}(\psi_l,\vec{\xi})$ in $(\nabla^K_{r_2\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_2\dots i_s}$ contains no factors $\vec{\xi}_k$ where the index k is free, nor factors $|\vec{\xi}|^2$, then we will have no such factors in $\nabla_{r_1}a_t(N)T^t_{g^N}(\psi_l,\vec{\xi})$ either.

Finally, any partial contraction in $\nabla_{r_1} a_t(N) T^t_{g^N}(\psi_l, \vec{\xi})$ of $\vec{\xi}$-length 1 will arise if T^t has $\vec{\xi}$-length 1 and provided the derivative does not hit any factor $\vec{\xi}_k$. So, by our inductive hypothesis, any useful partial contraction in $(\nabla^{K+1}_{r_1\dots r_{K+1}} \psi_l)_{i_1\dots i_s}{}^{\hat{g}^N}$ of $\vec{\xi}$-length 1 is a good substitution.

Next, we consider the case where the index r_1 in $(\nabla^{K+1}_{r_1\dots r_{K+1}} \psi_l)^{\hat{g}^N}_{i_1\dots i_s}$ is not a free index, supposing that r_1 contracts against r_j. We consider the tensor

$$(\nabla^{K}_{r_2\dots r_{K+1}} \psi_l)^{\hat{g}^N}_{i_2\dots r_j\dots i_s}$$

obtained from $(\nabla^{K+1}_{r_1\dots r_{K+1}} \psi_l)_{i_1\dots i_s}{}^{\hat{g}^N}$ by erasing the derivative ∇_{r_1} and making the index r_j into a free index. We consider the transformation law for

$$(\nabla^{K}_{r_2\dots r_{K+1}} \psi_l)^{\hat{g}^N}_{i_1\dots r_j\dots i_s}.$$

Our inductive hypothesis applies. So, in order to determine $(\nabla^{K+1}_{r_1\dots r_{K+1}} \psi_l)^{\hat{g}^N}_{i_1\dots i_s}$, we have to apply (57) to each pair

$$(r_1, i_2), \dots, (r_1, i_s), (r_1, r_j)$$

and then contract r_1 and r_j. If we consider any useless partial contraction in

$$(\nabla^{K}_{r_2\dots r_{K+1}} \psi_l)^{\hat{g}^N}_{i_1\dots r_j\dots i_s},$$

then any application of the law (57) to any pair above will give us a useless partial contraction.

Now, let us consider any useful partial contraction $a_t(N) \cdot T^t_{i_1\dots r_j\dots i_s}$ in

$$(\nabla^{K}_{r_2\dots r_{K+1}} \psi_l)^{\hat{g}^N}_{i_1\dots r_j\dots i_s}.$$

If we apply the identity (57) to any pair of indices $(r_1, i_2), \dots, (r_1, i_s), (r_1, r_j)$ without bringing out a factor $\vec{\xi}$ (meaning that we apply the first summand on the right-hand side of (57)), then by the same reasoning as before we have our claim. On the other hand, if we apply the identity (57) to any pair of indices $(r_1, i_1), \dots, (r_1, i_s)$ and bring out a factor $\vec{\xi}$, then after contracting r_1, r_j we will obtain a useless partial contraction. Also, if we apply the transformation law (57) to the pair (r_1, r_j) and bring out a factor $\vec{\xi}$ but not a factor g_{ij}, then after contracting r_1, r_j we will again obtain a useless partial contraction. Finally, if we apply the transformation law (57) to $(\nabla_{r_1} X_{r_j})^{\hat{g}^N}$ and bring out $g^N_{r_1 r_j} \vec{\xi}^s X_s$, then after contracting r_1, r_j we bring out a factor N. We thus obtain another useful $\vec{\xi}$-contraction.

Finally, notice that if $a_t(N) \cdot T^t_{i_1\dots r_j\dots i_s}$ had $\vec{\xi}$-length 1, then by our inductive hypothesis it was a good substitution of $(\nabla^p \psi_l)_{i_1\dots r_j\dots i_s}$. Hence, for each such good substitution, we now have the option of either substituting $\nabla^{r_1} X_{r_j}$ by $N \vec{\xi}^s X_s$ or leaving it unaltered. Therefore, the set of useful $\vec{\xi}$-contractions of $\vec{\xi}$-length 1 in

$(\nabla^{K+1}_{r_1\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_1\dots i_s}$ is indeed contained in the set of good substitutions of

$$(\nabla^{K+1}_{r_1\dots r_{K+1}}\psi_l)_{i_1\dots i_s}.$$

Moreover, since no useful tensor $a_t(N)T^t_{i_1\dots r_j\dots i_s}(\psi_l,\vec{\xi})$ in $(\nabla^K_{r_2\dots r_{K+1}}\psi_l)^{\hat{g}^N}_{i_1\dots r_j\dots i_s}$ has factors $\vec{\xi}$ or $|\vec{\xi}|^2$, there will be no such factors in either $a_t(N)\nabla^{r_j}T^t_{i_1\dots r_j\dots i_s}(\psi_l,\vec{\xi})$ or $a_t(N)\vec{\xi}^{r_j}T^t_{i_1\dots r_j\dots i_s}(\psi_l,\vec{\xi})$. Hence, we have completely shown our inductive step.

The case of the tensors $(\nabla^m_{r_1\dots r_m}R_{ijkl})^{\hat{g}^N}_{i_1\dots i_s}$ and $(\nabla^p_{r_1\dots r_p}R_{ij})^{\hat{g}^N}_{i_1\dots i_s}$ is proven by the same argument: The cases $m=0$, $p=0$ follow by equations (56), (54) and then the inductive argument still applies, since it is only an iterative application of the formula (57). We have proven claim A.

Now, in order to complete the proof of Lemma 16, we only have to observe that if we substitute any liberated factor $T\neq R$ from $C_{g^n}(\psi_1,\dots,\psi_Z)$ by a useless partial contraction, and then proceed to replace the other factors by either useful or useless partial contractions and then perform the same contractions for those replacements as for $C_{g^N}(\psi_1,\dots,\psi_Z)$, we will obtain an unacceptable complete contraction in $e^{n\vec{\xi}\cdot\vec{x}}C_{e^{2\vec{\xi}\cdot\vec{x}}g^N}(\psi_1,\dots,\psi_Z)$. This follows by the same reasoning as for Lemma 8. Regarding the substitutions of scalar curvature, we can replace it by either a factor $-(2-N)\nabla^a\vec{\xi}_a$ (in which case $\deg[-(2-N)]=1$ and $|\vec{\xi}|=1$) or by $-(N-1)(N-2)|\vec{\xi}|^2$ (in which case $\deg[-(N-1)(N-2)]=2$ and $|\vec{\xi}|=2$). □

Let us now state a corollary of Lemma 16 regarding the linear combination of good, hard and undecided descendants of a complete contraction $C_{g^n}(\psi_1,\dots,\psi_Z)$, in the form (62).

We consider any complete contraction $C_{g^n}(\psi_1,\dots,\psi_Z)$, in the form (62), and write it in the form (108). We then consider the sublinear combination of its acceptable descendants, in the form:

(112)
$$\operatorname{contr}\big(\nabla^{m_1}_{r_1\dots r_{m_1}}R_{ijkl}\otimes\dots\otimes\nabla^{m_s}_{t_1\dots t_{m_s}}R_{ijkl}\otimes\nabla^{p_1}_{r_1\dots r_{p_1}}\mathrm{Ric}_{ij}\otimes\dots\otimes\nabla^{p_q}_{t_1\dots t_{p_q}}\mathrm{Ric}_{ij}\\ \otimes R^\alpha\otimes\nabla^{p_1}_{a_1\dots a_{p_1}}\psi_1\otimes\dots\otimes\nabla^{p_Z}_{b_1\dots b_{p_Z}}\psi_Z\otimes\nabla^{b_1}\vec{\xi}\otimes\dots\otimes\nabla^{b_v}\vec{\xi}\big).$$

Then, by repeated application of formula (65), we write each such de-symmetrized descendant as a linear combination of good, hard and undecided $\vec{\xi}$-contractions of the form (63) or (64).

We then claim the following:

LEMMA 17. *Given any complete contraction $C_{g^n}(\psi_1,\dots,\psi_Z)$ in the form (62), of length L, then each of its good or hard or undecided descendants constructed above will have $\vec{\xi}$-length $\geq L$.*

Furthermore, if the complete contraction $C_{g^n}(\psi_1,\dots,\psi_Z)$ has no factors R, then none of its descendants will contain a factor $-|\vec{\xi}|^2$. On the other hand, if $C_{g^n}(\psi_1,\dots,\psi_Z)$ contains $A>0$ factors R, then we can write the sublinear combination of its good, undecided and hard descendants as follows:

$$\sum_{l\in L} a_l[C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})+\sum_{r\in R^l} C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] \tag{113}$$

where each $\vec{\xi}$-contraction $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ arises from $\sum C_{g^n}(\psi_1,\dots,\psi_Z)$ by doing all the substitutions explained in Lemma 16 *but leaving all the factors R unaltered, while $\sum_{r\in R^l} C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ arises from $\sum C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by substituting a nonzero number of factors R by either $-2\nabla^i\vec{\xi}_i$ or $-|\vec{\xi}|^2$ and then summing over all those different substitutions.*

Proof. This lemma follows straightforwardly from Lemma 16: We only have to make note that $\vec{\xi}$-length is additive and that the correction terms that we introduce in the symmetrization of factors $\nabla^p\vec{\xi}$ (using (65)) may increase the $\vec{\xi}$-length but not decrease it. So, since we are substituting each factor in $C_{g^n}(\psi_1,\dots,\psi_Z)$ by a tensor of $\vec{\xi}$-length ≥ 1, the first claim of our lemma will follow.

Our second claim will follow from the transformation law (55), provided we can show that no factors $|\vec{\xi}|^2$ arise when we symmetrize and anti-symmetrize the factors $\nabla^p\vec{\xi}$ and then repeat the same particular contractions as for $C_{g^n}(\psi_1,\dots$ $\dots,\psi_s)$. In order to see this, we only have to observe that for each factor of the form $\nabla^p\vec{\xi}_j$, $p\geq 1$, none of the correction terms in its symmetrization involve a factor$\vec{\xi}_a$ with the index a being free.

This follows because in order to symmetrize the factor $\nabla^p\vec{\xi}_j$ we only use the identities $[\nabla_a\nabla_b-\nabla_b\nabla_a]\vec{\xi}_j=R_{abjd}\vec{\xi}^d$ and, if $k\geq 1$:

$$\nabla^u\{[\nabla_a\nabla_b-\nabla_b\nabla_a]\nabla^k\vec{\xi}\}=\sum(\nabla^t R\nabla^y\vec{\xi})$$

where $\sum(\nabla^t R\nabla^y\vec{\xi})$ stands for a linear combination of partial contractions of the form $\nabla^\alpha R_{ijkl}\nabla^y\vec{\xi}$, where $1\leq y<k+u+2$. $\square$

Proof of Proposition 4. (This lasts through page 1302.) Recall that we have defined a stigmatized $\vec{\xi}$-contraction to be in the form:

$$\begin{aligned}\operatorname{contr}\big(&\nabla^{m_1}_{r_1\dots r_{m_1}}R_{i_1j_1k_1l_1}\otimes\dots\otimes\nabla^{m_s}_{v_1\dots v_{m_s}}R_{i_sj_sk_sl_s}\\&\otimes\nabla^{p_1}_{t_1\dots t_{p_1}}\mathrm{Ric}_{\alpha_1\beta_1}\otimes\dots\otimes\nabla^{p_q}_{z_1\dots z_{p_q}}\mathrm{Ric}_{\alpha_q\beta_q}\otimes\nabla^{\nu_1}_{\chi_1\dots\chi_{\nu_1}}\psi_1\otimes\dots\\&\otimes\nabla^{\nu_Z}_{\omega_1\dots\omega_{\nu_Z}}\psi_Z\otimes S\nabla^{\mu_1}\vec{\xi}_{j_1}\otimes\dots\otimes S\nabla^{\mu_r}\vec{\xi}_{j_s}\otimes|\vec{\xi}|^2\otimes\dots\otimes|\vec{\xi}|^2\big)\end{aligned} \tag{114}$$

where each $\mu_i \geq 1$ and there are r factors $S\nabla^\nu \vec{\xi}$ and $s > 0$ factors $|\vec{\xi}|^2$. If $r = 0$, we will call the the above $\vec{\xi}$-contraction stigmatized of type 1 and if $r > 0$, we will call it stigmatized of type 2.

Let us, for each good or undecided $\vec{\xi}$-contraction $C^l_{g^n}(\phi, \vec{\xi})$ (with X $\vec{\xi}$-factors) break up its outgrowth $O[C^l_{g^n}(\phi, \vec{\xi})]$ as follows: After each integration by parts of a factor $\vec{\xi}$, we discard any hard $\vec{\xi}$-contractions that arise, but moreover, when we encounter any stigmatized complete contractions of type 1 or type 2 we put them aside. We denote by $\sum_{k \in K^l_1} a_k C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ the sublinear combination of $\vec{\xi}$-contractions that we are left with after $X - 1$ integrations by parts, after we have discarded all the hard $\vec{\xi}$-contractions we encounter and after we have put aside all the stigmatized $\vec{\xi}$-contractions we encounter. We also denote by $\sum_{k \in K^l_2} a_k C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$, $\sum_{k \in K^l_3} a_k C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ the sublinear combinations of stigmatized $\vec{\xi}$-contractions of types 1 and 2, respectively, that we have put aside along our iterative integrations by parts.

We will then have:

$$O[C^l_{g^n}(\phi, \vec{\xi})] = \sum_{k \in K^l_1} a_k O[C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] + \sum_{k \in K^l_2} a_k O[C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] + \sum_{k \in K^l_3} a_k O[C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})].$$

We observe that the $\vec{\xi}$-contractions $C^k_{g^n}(\psi_1, \dots, \psi_Z)$, $k \in K^l_1$ are good, in the form (63) with one factor $\vec{\xi}$. Hence, we can rewrite (93) as follows:

$$I^Z_{g^n}(\phi) + \sum_{l \in L} a_l \Big\{ \sum_{k \in K^1_l} a_k O[C^k(\phi, \vec{\xi})] + \sum_{k \in K^2_l} a_k O[C^k(\phi, \vec{\xi})] + \sum_{k \in K^3_l} a_k O[C^k(\phi, \vec{\xi})] \Big\} = 0. \tag{115}$$

Our Proposition 4 will follow from the following equation:

$$\sum_{l \in L} a_l \Big\{ \sum_{k \in K^l_2} a_k O[C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] + \sum_{k \in K^l_3} a_k O[C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] \Big\} = 0. \tag{116}$$

In fact, we will show that:

$$\sum_{l \in L} a_l \Big\{ \sum_{k \in K^l_2} a_k C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi}) \Big\} = 0 \tag{117}$$

and

$$\sum_{l \in L} a_l \Big\{ \sum_{k \in K^l_3} a_k C^k_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi}) \Big\} = 0. \tag{118}$$

We see that (116) follows from the above two equations by the same reasoning by which Lemma 13 follows from Lemma 14. We first show (118); to do this we define a procedure called the *sieving* integration by parts.

Definition 21. Consider any good or undecided $\vec{\xi}$-contraction $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ and its iterative integrations by parts. We impose the following rules: Whenever along the iterative integration by parts we encounter a hard $\vec{\xi}$-contraction, we erase it and put it in the linear combination $H[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$. Whenever we encounter a $\vec{\xi}$-contraction which is stigmatized of type 2, we erase it and put it in the linear combination $\mathrm{Stig}^2[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$. Also, whenever encountering a stigmatized $\vec{\xi}$-contraction of type 1, we erase it and put it in the linear combination $\mathrm{Stig}^1[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$.

Furthermore, having any complete contraction $C_{g^n}(\psi_1,\dots,\psi_Z)$ which is in the form (62), we consider the linear combination of its good or undecided or hard descendants, say $\sum_{d\in D} a_d C^d_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. We define

$$
\begin{aligned}
H[C_{g^n}(\psi_1,\dots,\psi_Z)] &= \sum_{d\in D} a_d H[C^d_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})],\\
PO[C_{g^n}(\psi_1,\dots,\psi_Z)] &= \sum_{d\in D} a_d PO[C^d_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})],\\
\mathrm{Stig}^2[C_{g^n}(\psi_1,\dots,\psi_Z)] &= \sum_{d\in D} a_d \mathrm{Stig}^2[C^d_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})],\\
\mathrm{Stig}^1[C_{g^n}(\psi_1,\dots,\psi_Z)] &= \sum_{d\in D} a_d \mathrm{Stig}^1[C^d_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})].
\end{aligned}
$$

LEMMA 18. *With any complete contraction $C_{g^n}(\psi_1,\dots,\psi_Z)$ in the form* (62) *of weight $-n$, there is a way to perform our sieving integration by parts, so that we can express the four quantities just defined as follows*:

$$
C_{g^n}(\psi_1,\dots,\psi_Z)+PO[C_{g^n}(\psi_1,\dots,\psi_Z)] = \sum_{v\in V} a_v C^v_{g^n}(\psi_1,\dots,\psi_Z)R^{\alpha_v} \tag{119}
$$

where each $C^v_{g^n}(\psi_1,\dots,\psi_Z)$ is of weight $-n+2\alpha_v$, in the form (62), *with no factors R* (*they are pulled out on the right*);

$$
\begin{aligned}
H[C_{g^n}(\psi_1,\dots,\psi_Z)] = &\sum_{v\in V} a_v C^v_{g^n}(\psi_1,\dots,\psi_Z)\cdot G(R,\alpha_v,-2\nabla^i\vec{\xi}_i)\\
&+\sum_{f\in F} a_f C^f_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})\cdot R^{\alpha_f}\\
&+\sum_{f\in F} a_f C^f_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})\cdot G(R,\alpha_f,-2\nabla^i\vec{\xi}_i)
\end{aligned} \tag{120}
$$

where each $C^f_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is of weight $-n+2\alpha_f$, in the form (64) *with $k=0$ and with no factors R* (*they are pulled out on the right*), *and where $G(R,\lambda,B)$*

stands for the sum over all the possible substitutions of λ factors R by a factor B, so that we make at least one such substitution; *and finally*

$$\text{(121)}\quad \mathrm{Stig}^1[C_{g^n}(\psi_1,\ldots,\psi_Z)] = \sum_{v\in V} a_v C^v_{g^n}(\psi_1,\ldots,\psi_Z)\cdot G(R,\alpha_v,-|\vec{\xi}|^2);$$

$$\text{(122)}\quad \mathrm{Stig}^2[C_{g^n}(\psi_1,\ldots,\psi_Z)] = \sum_{v\in V} a_v C^v_{g^n}(\psi_1,\ldots,\psi_Z)\cdot T^*(\alpha_v,R,-2\nabla^i\vec{\xi}_i,-|\vec{\xi}|^2) + \sum_{f\in F} a_f C^f_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})\cdot T(\alpha_f,R,-2\nabla^i\vec{\xi}_i,-|\vec{\xi}|^2)$$

where $T(j,R,-2\nabla^i\vec{\xi}_i,-|\vec{\xi}|^2)$ stands for the sum over all the possible ways to substitute a nonzero number of factors in R^j by either $-2\nabla^i\vec{\xi}_i$ or $-|\vec{\xi}|^2$, *so that at least one factor is substituted by $-|\vec{\xi}|^2$ and $T^*(j,R,-2\nabla^i\vec{\xi}_i,-|\vec{\xi}|^2)$ stands for the same thing*, *with the additional restriction that at least one factor R must be substituted by $-2\nabla^i\vec{\xi}_i$.*

Proof. We consider the linear combination of $C_{g^n}(\psi_1,\ldots,\psi_Z)$, together with its good, undecided and hard descendants in $e^{N\vec{\xi}\cdot\vec{x}}C_{e^{2\vec{\xi}\cdot\vec{x}}g^n}(\psi_1,\ldots,\psi_Z)$, grouped as in (113). Given any $l\in L$, we pick any $C^r_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$, $r\in R^l$, and identify any factor T in $C^r_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$ distinct from $-2\nabla^i\vec{\xi}_i$ and $-|\vec{\xi}|^2$ with a factor in $C^l_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$. We say that such a factor in $C^r_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$, $r\in R^l$, *corresponds* to a factor in $C^l_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$.

We will now perform integrations by parts among the sublinear combinations of good, hard and undecided descendants in

$$\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}C^a_{e^{2\vec{\xi}\cdot\vec{x}}g^N}(\psi_1,\ldots,\psi_Z)dV_{g^N},$$

so that after any number of integrations by parts we will be left with an integrand of $\vec{\xi}$-contractions as in (113):

For any $C^l_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$, we pick out a factor $\vec{\xi}_i$ (which does not contract against another $\vec{\xi}$) and perform an integration by parts. We will obtain a formula:

$$\text{(123)}\quad \int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}Q(N)C^l_{g^N}(\psi_1,\ldots,\psi_Z,\vec{\xi})dV_{g^N} = \int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}\frac{Q(N)}{N-n}[\sum_{\alpha=1}^{L} C^{l,\alpha}_{g^N}(\psi_1,\ldots,\psi_Z,\vec{\xi})]dV_{g^N}.$$

Consider any $\vec{\xi}$-contraction $C^{l,\alpha}_{g^N}(\psi_1,\ldots,\psi_Z,\vec{\xi})$ which arises when ∇_i hits a factor T in $C^l_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$ with $T\neq R$. Then consider any $\vec{\xi}$-contraction $C^r_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$, $r\in R^l$, and integrate by parts the corresponding factor $\vec{\xi}_i$. Consider the $\vec{\xi}$-contraction $C^{r,\alpha}_{g^n}(\psi_1,\ldots,\psi_Z,\vec{\xi})$ which arises when ∇_i hits the corresponding factor T as before. It is then clear that each linear combination

$$C^{l,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})+\sum_{r\in R^l}C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

is of the form of equation (113). Notice that by Observation 1 we are free to impose this restriction on the order of integrations by parts of the factors $\vec{\xi}$ in $\sum_{r\in R^l}C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. We note that the order in which we integrate by parts is consistent with our rules on dropping $\vec{\xi}$-contractions into the sublinear combinations $PO[\dots]$, $H[\dots]$, $\mathrm{Stig}^1[\dots]$, $\mathrm{Stig}^2[\dots]$. This will follow from the arguments below.

Now we consider any $\vec{\xi}$-contraction that arises in the integration by parts of $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ when ∇_i hits a factor $T=R$. We restrict our attention to the $\vec{\xi}$-contractions $C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, $r\in R^l$, which arise from $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ when we leave the factor $T(=R)$ unaltered. Suppose their index set is $R^l_{\alpha,+}$. We then observe that the linear combination

$$C^{l,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})+\sum_{r\in R^l_{\alpha,+}}C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

is of the form (113).

Finally, consider the $\vec{\xi}$-contractions $C^{r_1}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, $C^{r_2}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ which arise from $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by substitution of the factor $T(=R)$ by $-2\nabla^i\vec{\xi}_i$ and $-|\vec{\xi}|^2$ respectively. Also, define $R^l_1,R^l_2\subset R^l$ to be the index sets of all the $\vec{\xi}$-contractions $C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ which arise from $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by substitution of the factor $T(=R)$ by $-2\nabla^i\vec{\xi}_i$ and $-|\vec{\xi}|^2$, respectively, and by substitution of at least one more factor R. We then consider the $\vec{\xi}$-contractions $C^{r_1,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ and $C^{r_2,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ which arise from integration by parts of $C^{r_1}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ and $C^{r_2}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, respectively, when ∇_i hits the factors $-2\nabla^i\vec{\xi}_i$, $-|\vec{\xi}|^2$, respectively. We also consider the $\vec{\xi}$-contractions

$$C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})\ \text{ and }\ C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

in the integration by parts of each $C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, $r\in R^l_1$ or $r\in R^l_2$ when ∇_i hits the factors $-2\nabla^i\vec{\xi}_i$ $-|\vec{\xi}|^2$, respectively, which correspond to the factors $-2\nabla^i\vec{\xi}_i$ or $-|\vec{\xi}|^2$ in $C^{r_1}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ and $C^{r_2}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. It follows by construction that the sublinear combinations

$$C^{r_1,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})+\sum_{r\in R^l_1}C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})\quad\text{and}$$

$$C^{r_2,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})+\sum_{r\in R^l_2}C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

are in the form of equation (113).

Hence, if we start with a linear combination of $\vec{\xi}$-contractions in the form (113), then for each integration by parts in any $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, we can consider the corresponding integrations by parts of each $C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, $r\in R^l$, and at the next step we will be left with a linear combination of $\vec{\xi}$-contractions in the form (113).

If at any stage $C^{l,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is a complete contraction in the form (62), we put it into $PO[C_{g^n}(\psi_1,\dots,\psi_Z)]$. Also, the $\vec{\xi}$-contraction in

$$\sum_{r\in R^l} C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

which arises from $C^{l,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by substituting factors R only by $-|\vec{\xi}|^2$ is stigmatized of type 1, and it is put into $\mathrm{Stig}^1[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$. The $\vec{\xi}$-contraction in $\sum_{r\in R^l} C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ which arises from $C^{l,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by substituting factors R only by $-2\nabla^i\vec{\xi}_i$ is a hard $\vec{\xi}$-contraction and we put it into $H[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$. Finally, any $\vec{\xi}$-contraction in $\sum_{r\in R^l} C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ which arises from $C^{l,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by substituting at least one factor R by $-|\vec{\xi}|^2$ and at least another factor R by $-2\nabla^i\vec{\xi}_i$ is stigmatized of type 2 and we put it into $\mathrm{Stig}^2[C_{g^n}(\psi_1,\dots,\psi_Z)]$.

Let us also note that the $\vec{\xi}$-contraction $C^{r_1,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ will always be undecided (it contains a factor $\nabla_i\vec{\xi}_k\vec{\xi}^k$). For the $\vec{\xi}$-contraction $C^{r_2,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ (and also for its followers), we decompose the factor $\nabla_{ik}\vec{\xi}^k$ into $S\nabla_{ik}\vec{\xi}^k$ and $\mathrm{Ric}_{ik}\vec{\xi}^k$. We notice that substituting the factor $\nabla_{ik}\vec{\xi}^k$ by $\mathrm{Ric}_{ik}\vec{\xi}^k$ will give either a good or an undecided $\vec{\xi}$-contraction.

Now, we suppose that the $\vec{\xi}$-contraction $C^{r_1,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ (after the symmetrization $-2(\nabla_i\nabla^k\vec{\xi}_k)\to -2(S\nabla_{ik}\vec{\xi}^k)$) is hard. We then observe that the $\vec{\xi}$-contraction in $\sum_{r\in R^l_1} C^{r,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ which arises from $C^{r_1,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by performing an integration by parts of $\vec{\xi}_i$ and hitting $-2\nabla^k\vec{\xi}_k$ and symmetrizing by $-2(\nabla_i\nabla^k\vec{\xi}_k)\to -2(S\nabla_{ik}\vec{\xi}^k)$ and then by substituting factors R only by $-2\nabla^i\vec{\xi}_i$ is also hard. Furthermore, any $\vec{\xi}$-contraction which arises from $C^{l,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ or from $C^{r_1,\alpha}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, by substitution of factors R only by $-|\vec{\xi}|^2$, is stigmatized of type 2.

So we notice that for each $\vec{\xi}$-contraction that are put in $PO[C_{g^n}(\psi_1,\dots,\psi_Z)]$ or $H[C_{g^n}(\psi_1,\dots,\psi_Z)]$, the $\vec{\xi}$-contractions put into $\mathrm{Stig}^1[C_{g^n}(\psi_1,\dots,\psi_Z)]$ or $\mathrm{Stig}^2[C_{g^n}(\psi_1,\dots,\psi_Z)]$ will be of the form described in (121) and (122). Lemma 18 is proven. □

We now want to apply the above lemma in order to prove equations (117) and (118) making a notational convention: Given any contraction $C^z_{g^n}(\psi_1,\dots,\psi_Z)$ in the form (62), let us write it as $C^{z'}{}_{g^n}(\psi_1,\dots,\psi_Z)\cdot R^\alpha$, where $C^{z'}{}_{g^n}(\psi_1,\dots,\psi_Z)$ does not contain factors R. We then define:

$$\sum_{r\in R^z} C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}) = C^{z'}{}_{g^n}(\psi_1,\dots,\psi_Z)\cdot[G(R,\alpha,-2\nabla^i\vec{\xi}_i)+T(R,\alpha,-2\nabla^i\vec{\xi}_i,-|\vec{\xi}|^2)]. \tag{124}$$

Here each summand on the right-hand side arises from one of the substitutions described in the definitions of $G(R,\alpha,-2\nabla^i\vec{\xi}_i)$ and $T(R,\alpha,-2\nabla^i\vec{\xi}_i,-|\vec{\xi}|^2)$. Also, given any hard $\vec{\xi}$-contraction $C^h_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ in the form (64), we write it as $C^{h'}{}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})\cdot R^\alpha$, where $C^{h'}{}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is in the form (64) and does not contain factors R. We then define:

$$\sum_{w\in W^h} C^w_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}) = C^{h'}{}_{g^n}(\psi_1,\dots,\psi_Z)\cdot[G(R,\alpha,-2\nabla^i\vec{\xi}_i)+T(R,\alpha,-2\nabla^i\vec{\xi}_i,-|\vec{\xi}|^2)]. \tag{125}$$

Here each summand on the right-hand side arises from one of the substitutions described in the definition of

$$G(R,\alpha,-2\nabla^i\vec{\xi}_i) \text{ and } T(R,\alpha,-2\nabla^i\vec{\xi}_i,-|\vec{\xi}|^2).$$

We now prove equations (117) and (118) through an inductive argument. We first recall the terminology and notation used in Lemma 18. Consider

$$I^Z_{g^n}(\psi_1,\dots,\psi_Z)=\sum_{a\in A} b_a C^a_{g^n}(\psi_1,\dots,\psi_Z).$$

For any complete contraction $C^a_{g^n}(\psi_1,\dots,\psi_Z)$ consider the sublinear combination of its good, hard or undecided descendants, say $\sum_{x\in X^a} c_x C^x_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. We perform integrations by parts in the expression

$$\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}\Big(\sum_{a\in A} b_a\Big[C^a_{g^n}(\psi_1,\dots,\psi_Z)+\sum_{x\in X^a} c_x C^x_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})\Big]\Big)dV_{g^N} = 0$$

as explained in Lemma 18. Whenever we encounter hard or stigmatized $\vec{\xi}$-contractions, we stop (and do not discard). In the end, we are left with a linear combination of sums of complete contractions:

$$\begin{aligned}\int_{\mathbb{R}^N} e^{(N-n)\vec{\xi}\cdot\vec{x}}\Big(&\sum_{z\in Z} Q^z(N)a_z C^z_{g^N}(\psi_1,\dots,\psi_Z)\\ &+\sum_{h\in H} Q^h(N)a_h C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})\\ &+\sum_{z\in Z} Q^z(N)a_z\sum_{r\in R^Z} C^r_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})\\ &+\sum_{h\in H}\Big[Q^h(N)a_h\sum_{w\in W^h} C^w_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})\Big]\Big)dV_{g^N}=0.\end{aligned} \tag{126}$$

Here each rational function has degree zero and leading order coefficient equal to 1. Moreover, $\sum_{z\in Z} Q^z(N)a_z C^z_{g^N}(\psi_1,\dots,\psi_Z)$ is the sublinear combination that is dropped into $PO[\dots]$, while $\sum_{h\in H} Q^h(N)a_h C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is the sublinear combination that arises by summing over all the sublinear combinations of hard $\vec{\xi}$-contractions in the form $\sum_{f\in F} a_f C^f_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ on the right-hand side of (120). Then

$$\sum_{r\in R^Z} Q^z(N)C^r_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi}) \quad \text{and} \quad \sum_{w\in W^h} Q^h(N)C^w_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$$

are the sublinear combinations of hard and stigmatized (of both types) $\vec{\xi}$-contractions that arise from $\sum_{z\in Z} a_z C^z_{g^N}(\psi_1,,\psi_Z)$ and $\sum_{h\in H} a_h C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ respectively, when we perform the substitutions for the factors $\hat{R}$ that are explained in (124), (125).

Our inductive assumption is the following: For any T, We define $Z^T\subset Z$ to be the index set of complete contractions $C^z_{g^N}(\psi_1,\dots,\psi_Z)$ with T factors R. Furthermore, we define $Z^{|T}$ to be the index set of complete contractions $C^z_{g^N}(\psi_1,\dots,\psi_Z)$ with more than T factors R and also define $H^T\subset Z$ to be the index set of complete contractions $C^z_{g^N}(\psi_1,\dots,\psi_Z)$ with T factors R. Also, $H^{|T}\subset H$ is the index set of $\vec{\xi}$-contractions $C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ with more than T factors R. We now inductively assume that for some T:

$$\sum_{z\in Z^{|T}} a_z C^z_{g^N}(\psi_1,\dots,\psi_Z)=0 \tag{127}$$

and

$$\sum_{h\in H^{|T}} a_h C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})=0. \tag{128}$$

We furthermore assume that:

$$\sum_{z\in Z^{|T}} a_z[\sum_{r\in R^z} C^r_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]=0 \tag{129}$$

and also that:

$$\sum_{h\in H^{|T}} a_h[\sum_{w\in W^z} C^w_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})]=0. \tag{130}$$

Our goal will be to prove:

$$\sum_{z\in Z^T} a_z C^z_{g^N}(\psi_1,\dots,\psi_Z)=0, \tag{131}$$

$$\sum_{h\in H^T} a_h C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})=0 \tag{132}$$

and furthermore:

$$\sum_{z\in Z^T}\Big[a_z\sum_{r\in R^z} C^r_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})\Big]=0, \tag{133}$$

$$\sum_{h\in H^T}\Big[a_z\sum_{r\in R^h} C^r_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})\Big]=0. \tag{134}$$

We first state and prove a lemma that will be useful for this purpose:

LEMMA 19. *Suppose there is a set of hard $\vec{\xi}$-contractions*

$$\{C^l_{g^n}(\psi_1,\dots,\psi)_Z,\vec{\xi})\}_{l\in L},$$

each in the form (64) *with $k=0$ (meaning no factors $\vec{\xi}$) and of weight $-n$. Now, suppose that*:

$$\sum_{l\in L} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})=0 \tag{135}$$

for every (M^n,g^n), for every $\psi_1,\dots,\psi_Z\in C^\infty(M^n)$ and every coordinate system.

We define the subsets $L^m\subset L$ as follows: $l\in L^m$ if and only if $C^l_{g^n}(\psi_1,\dots$ $\dots,\psi_Z,\vec{\xi})$ has m factors R. For each L^m for which $L^m\neq\varnothing$:

$$\sum_{l\in L^m} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})=0. \tag{136}$$

The same result is true if there are complete contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z)$ instead of hard $\vec{\xi}$-contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$.

Proof. We will think of the $\vec{\xi}$-contractions $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ as being in the form (44). Any $\vec{\xi}$-contraction in the form (64) with m factors R will give rise to $\vec{\xi}$-contractions in the form (44) with m factors R.

For some $M>0$ and for each $\mu>M$:

$$\sum_{l\in L^\mu} a_l C^l_{g^n}(\psi_1,\dots,\psi_s,\vec{\xi})=0.$$

Notice that if we can prove that:

$$\sum_{l\in L^M} a_l C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})=0 \tag{137}$$

then the whole lemma will follow by induction. In view of our induction hypothesis, we erase the sublinear combination $\sum_{\mu>M}\sum_{l\in L^\mu} a_l C^l_{g^n}(\psi_1,\dots,\psi_s,\vec{\xi})$ from (135).

Recall that (135) holds for any Riemannian metric, any functions $\psi_1,\dots\psi_Z$, any coordinate system and any $\vec{\xi}$. Hence, equation (135) must hold *formally*.

If we can prove that the number of factors R in a complete contraction of the form (44) remains invariant under the permutations of Definition 7, we will have our lemma.

For any complete contraction $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ of the form (64), we will call one of its factors $\nabla^m_{r_1\dots r_m}R_{ijkl}$ *connected* if one of the indices $r_1,\dots,l$ contracts against another factor in $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. From the identities in Definition 7, we see that any permutation of indices in any connected factor $\nabla^m_{r_1\dots r_m}R_{ijkl}$ in $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ will give rise to a complete contraction $C^{l'}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, which is obtained from $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by substituting its factor $\nabla^m_{r_1\dots r_m}R_{ijkl}$ by a number of factors $\nabla^p R_{ijkl}$, each connected in $C^{l'}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$.

For any complete contraction of the form $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, we will call one of its factors $\nabla^m_{r_1\dots r_m}R_{ijkl}$ m-self-contained if all the indices $r_1,\dots,l$ contract against another index in $\nabla^m_{r_1\dots r_m}R_{ijkl}$. Any application of the identities of Definition 7 to a factor $\nabla^m_{r_1\dots r_m}R_{ijkl}$ will give rise to a complete contraction $C^{l'}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, which is obtained from $C^l_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by substitution of its factor $\nabla^m_{r_1\dots r_m}R_{ijkl}$ by a number of factors $\nabla^p R_{ijkl}$, each of which is either m-self-contained or connected in $C^{l'}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$.

Hence we have shown our lemma. □

We now prove (131) observing that if a complete contraction $C^z_{g^N}(\psi_1,\dots,\psi_Z)$ has γ factors R, then each $\vec{\xi}$-contraction $C^r_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$ with $r\in R^z$, has strictly less than γ factors R. Furthermore, if $C^h_{g^N}(\psi_1,\dots,\psi_Z,\vec{\xi})$, has ε factors R then each $C^w_{g^N}(\psi_1,\dots,\psi_Z)$, $w\in W^h$ has strictly less than ε factors R. Finally, we notice that along the iterative integrations by parts the number of factors R either decreases or remains the same; it cannot increase. Now, we apply Lemma 14 and (97) to the case at hand. For any $\vec{\xi}$-contraction $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, we have defined $O[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$ to stand for its outgrowth. We also define $H[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$ to stand for the sublinear combination of the hard $\vec{\xi}$-contractions that arise along its iterative integration by parts. We then re-express the equation in Proposition 3 as follows:

$$\sum_{m=0}^{T}\Big(\sum_{z\in Z^m}a_z C^z_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})+\sum_{r\in R^z}O[C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]\Big)+\sum_{m=0}^{T}\Big(\sum_{h\in H^m}a_h\sum_{w\in W^h}O[C^w_{g^n}(\psi_1,\dots,\psi_Z)]\Big)=0. \tag{138}$$

Let us consider the sublinear combination of complete contractions in (138) with T factors R. It follows from our reasoning above and from Lemma 15 that it is precisely the left-hand side of (131). Hence, invoking Lemma 19, we derive (131). Therefore, by the construction of $\sum_{r\in R^z}C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, we obtain (133).

Furthermore, we re-express (97) as follows:

$$\sum_{m=0}^{T}\Big(\sum_{h\in H^m}a_h C^h_{g^n}(\psi_1,\dots,\psi_Z)+\sum_{w\in W^h}H[C^w_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]\Big)+\sum_{m=0}^{T}\Big(\sum_{z\in Z^m}a_z\sum_{r\in R^z}H[C^r_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]\Big)=0. \tag{139}$$

Now, we consider the sublinear combination of $\vec{\xi}$-contractions in the above equation with T factors R. From our reasoning above, from Lemma 15 and also from equation (131), we have that this sublinear combination is precisely the left-hand side (132). Hence, invoking Lemma 19, we have (132). Finally, (134) follows from (132) and from its definition. Hence, in view of (131) and Lemma 18, we obtain (117), (118). This completes the proof of our Proposition 4. $\square$

We now state a fact that illustrates its usefulness.

LEMMA 20. *Consider a good or undecided or hard $\vec{\xi}$-contraction $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, of $\vec{\xi}$-length L. Then $PO[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]$ will consist of complete contractions of length greater than or equal to L, or $PO[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]=0$.*

Consider the hard or the stigmatized $\vec{\xi}$-contractions that arise along the iterative integrations by parts. Any such $\vec{\xi}$-contraction has $\vec{\xi}$-length $\geq L$.

Proof. The proof is by induction. Initially, to make things easier, consider the case where there are no factors $|\vec{\xi}|^2$ in $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. Think of $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ as being in the form (64) with $\vec{\xi}$-length M and with E factors $\vec{\xi}$ and C factors $S\nabla^m\vec{\xi}$. We will perform induction on $C+E$.

Initially suppose $C+E=1$. Then if $C=1$, $E=0$, our $\vec{\xi}$-contraction is hard, so that $PO[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]=0$. If $E=1, C=0$, the proof is the same as for the inductive step:

Suppose we know the claim is true for $E+C=p$ and we want to prove it for $E+C=p+1$. Pick out a factor $\vec{\xi}_i$ and do an integration by parts with respect to it. If ∇_i hits a factor $\nabla^m R_{ijkl}$ or $\nabla^p \mathrm{Ric}_{ij}$ or $\nabla^p\psi_k$, we get a $\vec{\xi}$-contraction in the form (63) or (64) with $E+C=p$ and $\vec{\xi}$-length M. If ∇_i hits a factor $\vec{\xi}$, we get a $\vec{\xi}$-contraction in the form (63) or (64) with $E+C=p$ and $\vec{\xi}$-length $M+1$. If it hits a factor $S\nabla^m\vec{\xi}$ ($m\geq 1$), then after applying identity (65), we obtain a linear combination of complete contractions in the form (63) or (64) with $C+E=p$ and $\vec{\xi}$ length $\geq M$.

Now, suppose we do allow factors $|\vec{\xi}|^2$ in $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$. We again proceed by induction on the number $C+E$. If all the $\vec{\xi}$-factors in $C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ are in the form $|\vec{\xi}|^2$ or $S\nabla^m\vec{\xi}$, we already have a stigmatize $\vec{\xi}$-con- traction. Hence, $PO[C_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]=0$ in that case. Otherwise, there is at least one factor $\vec{\xi}_i$ that does not contract against another factor $\vec{\xi}$. We integrate by parts with respect to it. If ∇_i hits a factor $\nabla^m R_{ijkl}$ or $\nabla^a \mathrm{Ric}_{ij}$ or $\vec{\xi}$ or $|\vec{\xi}|^2$, we fall under our induction hypothesis with $\vec{\xi}$-length M or $M+1$. If it hits a factor $S\nabla^m\vec{\xi}$, we apply (65) and obtain a a linear combination of $\vec{\xi}$-contractions that fall under our induction hypothesis, by the same reasoning as above. This completes the proof. $\square$

6.4. *Conclusion*: *The algorithm for the super divergence formula.* We apply Proposition 4 to see how it can provide us with a divergence formula for $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$. Here is the algorithm:

Write

$$I^Z_{g^n}(\psi_1,\dots,\psi_Z)=\sum_{r\in R}a_r C^r_{g^n}(\psi_1,\dots,\psi_Z),$$

where each complete contraction $C^r_{g^n}(\psi_1,\dots,\psi_Z)$ is in the form (62).

For each complete contraction $C^r_{g^n}(\psi_1,\dots,\psi_Z)$ we consider the set of its good or undecided descendants, along with their coefficients (see Definition 12), say $a_b C^{r,b}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}), b\in B^r$. So each $C^{r,b}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ is in the form (63) or (64) and has S_b $\vec{\xi}$-factors (see Definition 9).

We then integrate by parts each $\vec{\xi}$-contraction $C^{r,b}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, and introduce the following convention: Whenever along this iterative integration by parts we obtain a hard or a stigmatized $\vec{\xi}$-contraction (see Definition 16), we discard it. For each $\vec{\xi}$-contraction $C^{r,b}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$, consider the $\vec{\xi}$-contractions $a_x C^{r,b,x}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}), x\in X^b$, that we are left with after S_b-1 integrations by parts (along with their coefficients). They are in the form (63) with one factor $\vec{\xi}$.

We then construct a vector field $(C^{r,b,x}_{g^n})^j(\psi_1,\dots,\psi_Z)$ out of any given $C^{r,b,x}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ by crossing out the factor $\vec{\xi}_j$ and making the index that contracted against j into a free index. By virtue of Proposition 4, we have:

$$I^Z_{g^n}(\psi_1,\dots,\psi_Z)=\sum_{r\in R}a_r\sum_{b\in B^r}a_b\sum_{x\in X^b}\mathrm{div}_j a_x (C^{r,b,x}_{g^n})^j(\psi_1,\dots,\psi_Z). \tag{140}$$

We will refer to this equation as the *super divergence formula* and denote it by $\mathrm{supdiv}[I^Z_{g^n}(\psi_1,\dots,\psi_Z)]=0$. We note that there are many such formulas, since at each stage we pick a factor $\vec{\xi}$ to integrate by parts (subject to the restrictions that we have imposed because of Observation 1).

Now, we establish a notational convention and make two observations: First, for any complete contraction $C^r_{g^n}(\psi_1,\dots,\psi_Z)$, define:

$$\begin{aligned}\mathrm{Tail}[C^r_{g^n}(\psi_1,\dots,\psi_Z)]\\ =C^r_{g^n}(\psi_1,\dots,\psi_Z)+\sum_{b\in B_r}a_b PO[C^{r,b}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})].\end{aligned} \tag{141}$$

Then, notice that if the complete contraction $C^r_{g^n}(\psi_1,\dots,\psi_Z)$ has length L, then each complete contraction in its tail will have length $\geq L$. This follows from Lemmas 20 and 16.

Furthermore, the super divergence formula holds for any $I^Z_{g^n}(\psi_1,\dots,\psi_Z)=\sum_{r\in R}a_r C^r_{g^n}(\psi_1,\dots\psi_Z)$ where each complete contraction is in the form (62) with weight $-n$, for which $\int_{M^n}I^Z_{g^n}(\psi_1,\dots,\psi_Z)dV_{g^n}=0$ for every compact

Riemannian (M^n, g^n) and any $\psi_1, \dots, \psi_Z \in C^\infty(M^n)$. In other words, the super divergence formula does not depend on the fact that $I^Z_{g^n}(\psi_1, \dots, \psi_Z)$ arises from a polarization of the transformation law of $P(g^n)$.

6.5. *The shadow formula.* We will draw another conclusion from Lemma 14 and Proposition 4.

As before, write $I^Z_{g^n}(\psi_1, \dots, \psi_Z) = \sum_{r\in R} a_r C^r_{g^n}(\psi_1, \dots, \psi_Z)$, where each complete contraction $C^r_{g^n}(\psi_1, \dots, \psi_Z)$ is in the form (62).

For each complete contraction $C^r_{g^n}(\psi_1, \dots, \psi_Z)$ we consider the set of its good or undecided or hard descendants, along with their coefficients (see Definition 12), say $a_b C^{r,b}_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi}), b \in B^r$. So each $C^{r,b}_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$ is in the form (63) or (64) and has S_b $\vec{\xi}$-factors.

We then begin to integrate by parts each $\vec{\xi}$-contraction $C^{r,b}_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})$, in the order explained in Definition 17 making the following convention:

Whenever we encounter a hard or a stigmatized $\vec{\xi}$-contraction, we put it aside. Whenever we encounter a good $\vec{\xi}$-contraction with $k = 1$ (and $l = 0$), we discard it.

We then consider the set of the hard or stigmatized $\vec{\xi}$-contractions, along with their coefficients, that are left after this procedure. Suppose that set is

$$\{a_t C^t_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})\}_{t\in T}.$$

We then have the *shadow formula* for $I^Z_{g^n}(\psi_1, \dots, \psi_Z)$:

$$\sum_{t\in T} a_t C^t_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi}) = 0 \tag{142}$$

for every (M^n, g^n), every $\psi_1, \dots, \psi_Z \in C^\infty(M^n)$, any coordinate system and any $\vec{\xi} \in \mathbb{R}^n$. We will denote this equation by $\mathrm{Shad}[I^Z_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] = 0$. It follows, as for the super divergence formula, that the shadow equation holds for any $I^Z_{g^n}(\psi_1, \dots, \psi_Z)$ that integrates to zero on any (M^n, g^n), for any $\psi_1, \dots, \psi_Z$ and for any coordinate system and any $\vec{\xi} \in \mathbb{R}^n$. It does not depend on the fact that $I^Z_{g^n}(\psi_1, \dots, \psi_Z)$ is the polarized transformation law for some $P(g^n)$.

Recalling the notation of Definition 21, we additionally define:

$$\begin{aligned}\mathrm{Tail}^{\mathrm{Shad}}[C^r_{g^n}(\psi_1, \dots, \psi_Z)] = \sum_{b\in B^r} a_b \big\{ & H[C^b_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] \\ & + \mathrm{Stig}^1[C^b_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})] + \mathrm{Stig}^2[C^b_{g^n}(\psi_1, \dots, \psi_Z, \vec{\xi})]\big\}.\end{aligned} \tag{143}$$

We may then re-express the shadow formula as:

$$\sum_{r\in R} a_r \mathrm{Tail}^{\mathrm{Shad}}[C^r_{g^n}(\psi_1, \dots, \psi_Z)] = 0. \tag{144}$$

The above equation follows straightforwardly from Lemma 14 and also from equation (118). Moreover, for future reference we define:

$$O^{\mathrm{Shad}}[C^b_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] = H[C^b_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] + \mathrm{Stig}^1[C^b_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})] + \mathrm{Stig}^2[C^b_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})]. \tag{145}$$

We furthermore show the following: For any $m \geq 0$, let T^m stand for the sublinear combination in (142) with m factors $|\vec{\xi}|^2$. Also,

$$\sum_{t\in T^m} a_t C^t_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi}) = 0 \tag{146}$$

for every (M^n, g^n), every $\psi_1,\dots,\psi_Z \in C^\infty(M^n)$, any coordinate system and any $\vec{\xi} \in \mathbb{R}^n$.

This follows since (142) must hold formally and the number of factors $\vec{\xi}$ that contract against another factor $\vec{\xi}$ is invariant under the permutations of Definition 7.

Furthermore it follows, from Lemma 16 and also from Lemma 20, that if a complete contraction $C^l_{g^n}(\psi_1,\dots,\psi_Z)$ of length L in $I^Z_{g^n}(\psi_1,\dots,\psi_Z)$ gives rise to a hard or stigmatized $\vec{\xi}$-contraction $C^{l,z}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ in (142), by the procedure outlined above, then $C^{l,z}_{g^n}(\psi_1,\dots,\psi_Z,\vec{\xi})$ will have $\vec{\xi}$-length $\geq L$.

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