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Strong cosmic censorship in T^3-Gowdy spacetimes

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Abstract

Einstein's vacuum equations can be viewed as an initial value problem, and given initial data there is one part of spacetime, the so-called maximal globally hyperbolic development (MGHD), which is uniquely determined up to isometry. Unfortunately, it is sometimes possible to extend the spacetime beyond the MGHD in inequivalent ways. Consequently, the initial data do not uniquely determine the spacetime, and in this sense the theory is not deterministic. It is then natural to make the strong cosmic censorship conjecture, which states that for generic initial data, the MGHD is inextendible. Since it is unrealistic to hope to prove this conjecture in all generality, it is natural to make the same conjecture within a class of spacetimes satisfying some symmetry condition. Here, we prove strong cosmic censorship in the class of T^3-Gowdy spacetimes. In a previous paper, we introduced a set $\mathcal{G}_{i,c}$ of smooth initial data and proved that it is open in the $C^1 \times C^0$-topology. The solutions corresponding to initial data in $\mathcal{G}_{i,c}$ have the following properties. First, the MGHD is C^2-inextendible. Second, following a causal geodesic in a given time direction, it is either complete, or a curvature invariant, the Kretschmann scalar, is unbounded along it (in fact the Kretschmann scalar is unbounded along any causal curve that ends on the singularity). The purpose of the present paper is to prove that $\mathcal{G}_{i,c}$ is dense in the C^∞-topology.

1. Introduction

1.1. *Motivation and background.* In [10], Yvonne Choquet-Bruhat showed that it is possible to formulate the Einstein vacuum equations as an initial value problem. Later, Choquet-Bruhat and Geroch [4] proved that, given vacuum initial data, there is a maximal globally hyperbolic development (MGHD) of the data, and that this development is unique up to isometry. There are however examples for which it is possible to extend the MGHD in inequivalent ways [6]. Consequently, it is not possible to predict what spacetime one is in simply by looking at initial data. This naturally leads to the strong cosmic censorship conjecture, stating that for generic initial data, the MGHD is inextendible. The statement is rather vague, as it does not specify exactly what is meant by generic, and since it does not give a precise definition of inextendibility; a spacetime can be extendible in one differentiability class but inextendible in another. In order to have a precise statement, one has to give a clear definition of these concepts. To prove the conjecture in general is not feasible at this time. For this reason it is tempting to consider the following related problem. Consider a class of initial data satisfying a given set of symmetry conditions. Is it possible to show that the MGHD is inextendible for initial data that are generic in this class? Note that, strictly speaking, this problem is unrelated to the original one, since a class of initial data satisfying symmetry conditions is a nongeneric class in the full set of initial data. However, this is the problem that will be addressed in this paper.

One way of proving that a spacetime is inextendible is to prove that, given a causal geodesic, there are two possible outcomes in a given time direction; either the geodesic is complete, or it is incomplete but the curvature is unbounded along it. Note that the natural associated inextendibility concept is that of C^2-inextendibility. Note also that it is of course conceivable that one could get away with proving less and still getting inextendibility. In this paper, we are concerned with the T^3-Gowdy spacetimes, and for these spacetimes it is known that in one time direction, the causal geodesics are always complete, cf. [18], and in the other, they are always incomplete. One is thus interested in proving that for generic initial data, the curvature is unbounded in the incomplete direction of every causal geodesic. This ties together the strong cosmic censorship conjecture with the problem of trying to understand the structure of singularities in cosmological spacetimes. By the singularity theorems, cosmological spacetimes typically have a singularity in the sense of causal geodesic incompleteness. However, it is of interest to know that one generically also has a singularity in the sense of curvature blow-up.

The fact that T^3-Gowdy spacetimes are future causally geodesically complete ensures inextendibility to the future. By a recent result of Dafermos and Rendall [9], this can also be achieved by another argument, which is shorter, but yields less information concerning the asymptotics and does not prove future causal geodesic completeness.

To our knowledge, the only result concerning strong cosmic censorship in an inhomogeneous cosmological setting is contained in [7]. This paper is concerned with polarized Gowdy spacetimes and contains a proof of the statement that there is an open and dense set of initial data for which the MGHD is inextendible. Note however that the authors do not restrict themselves to T^3 topology; all topologies compatible with Gowdy symmetry are allowed. In our setting, polarized T^3-Gowdy corresponds to setting $Q = 0$ in (2), (3); (see below) i.e. one gets a linear PDE for one unknown function. To analyze the asymptotic behaviour of this linear equation is of course easier, but the freedom one has when perturbing the initial data is more restricted. In other words, not all aspects of the problem are simplified by considering the polarized subcase.

Finally, let us note that a weaker form of strong cosmic censorship can be obtained by combining the results of [8], [21] and [20]. The weaker statement is that there is a dense G_δ set of initial data (in other words a countable intersection of open sets which is also dense) with respect to the C^∞-topology such that the corresponding maximal globally hyperbolic developments are C^2-inextendible. On the other hand, one obtains essentially no information concerning the asymptotic behaviour of the corresponding solutions. In this paper we obtain a complete characterization of the asymptotic behaviour of the solutions for a set of initial data which is open with respect to the $C^2 \times C^1$-topology and dense with respect to the C^∞-topology.

1.2. *Objects of study.* The Gowdy spacetimes were first introduced in [11] (see also [5]), and in [14] the fundamental questions concerning global existence were answered. We shall take the Gowdy vacuum spacetimes on $\mathbb{R} \times T^3$ to be metrics of the form (1), (see below), but let us briefly motivate this choice by giving a geometric characterization. The reader interested in the details is referred to [11] and [5]. The following conditions can be used to define a Gowdy spacetime:

- It is an orientable maximal globally hyperbolic vacuum spacetime.
- It has compact spatial Cauchy surfaces.
- There is a smooth effective group action of $U(1) \times U(1)$ on the Cauchy surfaces under which the metric is invariant.
- The twist constants vanish.

Let us explain the terminology. A group action of a Lie group G on a manifold M is effective if $gp = p$ for all $p \in M$ implies $g = e$. Due to the existence of the symmetries we get two Killing fields. Let us call them X and Y. The *twist constants* are defined by

$$\kappa_X = \varepsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta \quad \text{and} \quad \kappa_Y = \varepsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma Y^\delta.$$

The fact that these objects are constants is due to the field equations. By the existence of the effective group action, one can draw the conclusion that the spatial Cauchy surfaces have topology T^3, S^3, $S^2 \times S^1$ or a Lens space. In all the cases except T^3, the twist constants have to vanish. However, in the case of T^3 this need not be true, and the condition that they vanish is the most unnatural of the ones on the list above. There is however a reason for separating the two cases. Considering the case of T^3 spatial Cauchy surfaces, numerical studies indicate that the Gowdy case is convergent [1] and the general case is oscillatory [2]. Analytically analyzing the case with nonzero twist constants can therefore reasonably be expected to be significantly more difficult than the Gowdy case. We shall here consider the T^3-Gowdy case. In this case the above conditions almost, but not quite, imply the form (1); see [5, pp. 116–117]; we have set some constants to zero. However, the discrepancy can be eliminated by a coordinate transformation which is local in space. Combining this observation with domain-of-dependence arguments we hope will convince the reader that nothing essential is lost by considering metrics of the form (1). Let

$$g = e^{(\tau-\lambda)/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}[e^P d\sigma^2 + 2e^P Q\, d\sigma\, d\delta + (e^P Q^2 + e^{-P})d\delta^2]. \tag{1}$$

Here, $\tau \in \mathbb{R}$ and (θ, σ, δ) are coordinates on T^3. The functions P, Q and λ only depend on τ and θ. Consequently, translations in σ and δ constitute isometries, so that we have a T^2-group of isometries acting on the spacetime. The Einstein vacuum equations become

$$P_{\tau\tau} - e^{-2\tau}P_{\theta\theta} - e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2) = 0, \tag{2}$$

$$Q_{\tau\tau} - e^{-2\tau}Q_{\theta\theta} + 2(P_\tau Q_\tau - e^{-2\tau}P_\theta Q_\theta) = 0, \tag{3}$$

and

$$\lambda_\tau = P_\tau^2 + e^{-2\tau}P_\theta^2 + e^{2P}(Q_\tau^2 + e^{-2\tau}Q_\theta^2), \tag{4}$$

$$\lambda_\theta = 2(P_\theta P_\tau + e^{2P}Q_\theta Q_\tau). \tag{5}$$

Obviously, (2), (3) do not depend on λ, so the idea is to solve these equations and then find λ by integration. There is however one obstruction to this; the integral of the right-hand side of (5) has to be zero. This is a restriction to be imposed on the initial data for P and Q, which is then preserved by the equations. In the end, the equations of interest are however the two nonlinear coupled wave equations (2), (3). In the above parametrization, the singularity corresponds to $\tau \to \infty$, and essentially all the work in this paper concerns the asymptotic behavior of solutions to (2), (3) in this time direction. Note that $P = \tau$, $Q = 0$ and $\lambda = \tau$ is a solution to (2)–(5). The Riemann curvature tensor of the corresponding metric is identically zero.

The equations (2), (3) constitute a wave map equation with hyperbolic space as a target; cf. [21]. The representation of hyperbolic space naturally associated with the equations is

$$g_R = dP^2 + e^{2P}dQ^2 \tag{6}$$

on $\mathbb{R}^2$. The map taking (Q, P) to (Q, e^{-P}) defines an isometry from $(\mathbb{R}^2, g_R)$ to the upper half-plane model. By the wave map structure, isometries of hyperbolic space map solutions to solutions. One particular isometry which we shall need in order to state the results is the inversion, defined by

$$\operatorname{Inv}(Q, P) = \left[\frac{Q}{Q^2 + e^{-2P}}, P + \ln(Q^2 + e^{-2P})\right]. \tag{7}$$

The reason for the name is that it corresponds to an inversion in the unit circle with center at the origin in the upper half-plane model. Given a solution to (2), (3), we shall speak of the associated kinetic and potential energy densities, given respectively by

$$\mathcal{K} = P_\tau^2 + e^{2P} Q_\tau^2, \quad \mathcal{P} = e^{-2\tau}(P_\theta^2 + e^{2P} Q_\theta^2).$$

1.3. *Previously obtained results.* Let us state some results that were proved in [21]. The main result of that paper is that the concept of an asymptotic velocity makes sense. Given a solution to (2), (3), the limit $\lim_{\tau\to\infty} \mathcal{K}(\tau, \theta)$ exists for every θ. We define the asymptotic velocity to be the nonnegative square root of this limit, and denote it by $v_\infty(\theta)$. If we wish to refer to the specific solution $x = (Q, P)$ with respect to which it is defined, we shall use the notation $v_\infty[x]$. There is another perspective on this quantity which is of interest. Let d_R be the topological metric induced by the Riemannian metric (6) and let $(Q_0, P_0) \in \mathbb{R}^2$ be some reference point. Given a solution to (2), (3), we define

$$\rho(\tau, \theta) = d_R\{[Q(\tau, \theta), P(\tau, \theta)], [Q_0, P_0]\}.$$

Note that this is the hyperbolic distance from the reference point to the solution at (τ, θ). We are interested in the limit $\rho(\tau, \theta)/\tau$ as $\tau \to \infty$. Note that if this limit exists, it is independent of the base point (Q_0, P_0). Furthermore, if we apply an isometry of the hyperbolic plane to the solution, the limit is the same for the resulting solution.

THEOREM 1. *Consider a solution to* (2), (3) *and let* $\theta_0 \in S^1$. *Then*

$$\lim_{\tau\to\infty} \frac{\rho(\tau, \theta_0)}{\tau} = v_\infty(\theta_0).$$

Furthermore, v_∞ *is upper semi continuous in the sense that given* θ_0, *there is for every* $\varepsilon > 0$ *a* $\delta > 0$ *such that for all* $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$

$$v_\infty(\theta) \le v_\infty(\theta_0) + \varepsilon.$$

In [21], we showed that v_∞ has several important properties. For instance, if $0 < v_\infty(\theta_0) < 1$, then v_∞ is smooth in a neighborhood of θ_0. If $v_\infty(\theta_0) > 1$ and v_∞ is continuous at θ_0, then it is smooth in a neighborhood. Finally, if $1 < v_\infty(\theta_0) < 2$, then $(1 - v_\infty)^2$ is smooth in a neighborhood of θ_0. In this paper, we show that

v_∞^2 is smooth in a neighborhood of a point at which it is zero, cf. the comments following Lemma 7. As a consequence of the above theorem, one can prove that for $z = \phi_{RD} \circ (Q, P)$, the limit

$$v(\theta) = \lim_{\tau\to\infty} \left[\frac{z}{|z|}\frac{\rho}{\tau}\right](\tau,\theta) \tag{8}$$

always exists; cf. [21]. Note here that ϕ_{RD}, defined in (19), is an isometry from the PQ-plane to the disc model and that $\rho/|z|$ is a real analytic function from the open unit disc to the real numbers if ρ is the hyperbolic distance from the origin of the unit disc to the solution; cf. (21). It would perhaps be more natural to refer to v as the asymptotic velocity, since it gives not only the rate at which the solution tends to the boundary of hyperbolic space, but also the point of the boundary to which it converges. From a geometric point of view, the most important property of v_∞ is however that if $v_\infty(\theta_0) \neq 1$, then the Kretschmann scalar, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, is unbounded along every causal curve ending on θ_0. Note that the special solution $P = \tau$, $Q = 0$ has the property that $v_\infty = 1$. In other words, the curvature need not blow-up if $v_\infty(\theta_0) = 1$.

The type of arguments used to prove the existence of the asymptotic velocity can also be used to prove statements concerning the asymptotic behavior of the first derivatives of P and Q; cf. [21]. Let us use the notation $\mathcal{D}_{\theta_0,\tau} = [\theta_0 - e^{-\tau}, \theta_0 + e^{-\tau}]$.

PROPOSITION 1. *Consider a solution to (2), (3) and let $\theta_0 \in S^1$. Then*

$$\lim_{\tau\to\infty} \||P_\tau(\tau,\cdot)| - v_\infty(\theta_0)\|_{C^0(\mathcal{D}_{\theta_0,\tau},\mathbb{R})} = 0, \quad \lim_{\tau\to\infty} \|(e^P Q_\tau)(\tau,\cdot)\|_{C^0(\mathcal{D}_{\theta_0,\tau},\mathbb{R})} = 0$$

and

$$\lim_{\tau\to\infty} \|\mathcal{P}(\tau,\cdot)\|_{C^0(\mathcal{D}_{\theta_0,\tau},\mathbb{R})} = 0.$$

In particular, $P_\tau(\tau,\theta_0)$ converges to $v_\infty(\theta_0)$ or to $-v_\infty(\theta_0)$. If $P_\tau(\tau,\theta_0) \to -v_\infty(\theta_0)$, then $(Q_1, P_1) = \mathrm{Inv}(Q, P)$ has the property that $P_{1\tau}(\tau,\theta_0) \to v_\infty(\theta_0)$. Furthermore, if $v_\infty(\theta_0) > 0$, then $Q_1(\tau,\theta_0)$ converges to 0.

One important property of the asymptotic velocity is that it can be used as a criterion for the existence of expansions. The following proposition was essentially proved in [17]; see [21] for the details.

PROPOSITION 2. *Let (Q, P) be a solution to (2), (3) and assume $0 < v_\infty(\theta_0) < 1$. If $P_\tau(\tau,\theta_0)$ converges to $v_\infty(\theta_0)$, then there is an open interval I containing θ_0, $v_a, \phi, q, r \in C^\infty(I,\mathbb{R})$, $0 < v_a < 1$, polynomials Ξ_k and a T such that for all $\tau \geq T$*

$$\|P_\tau(\tau,\cdot) - v_a\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{9}$$

$$\|P(\tau,\cdot)-p(\tau,\cdot)\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{10}$$

$$\left\|e^{2p(\tau,\cdot)}Q_\tau(\tau,\cdot)-r\right\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{11}$$

$$\left\|e^{2p(\tau,\cdot)}[Q(\tau,\cdot)-q]+\frac{r}{2v_a}\right\|_{C^k(I,\mathbb{R})} \leq \Xi_k e^{-\alpha\tau}, \tag{12}$$

where $p(\tau,\cdot) = v_a \cdot \tau + \phi$ *and* $\alpha > 0$. *If* $P_\tau(\tau,\theta_0)$ *converges to* $-v_\infty(\theta_0)$, *then* $\mathrm{Inv}(Q,P)$ *has expansions of the above form in a neighborhood of* θ_0.

In order to relate (9)–(12) to the form of the expansions given by earlier authors, let $\tilde{w}$ be the expression appearing inside the norm in (12). Then

$$Q = q + e^{-2p}\left[-\frac{r}{2v_a}+\tilde{w}\right].$$

This clarifies the relation between (10), (12) and the standard way of writing the expansions:

$$P(\tau,\theta) = v_a(\theta)\tau+\phi(\theta)+u(\tau,\theta) \tag{13}$$

$$Q(\tau,\theta) = q(\theta)+e^{-2v_a(\theta)\tau}[\psi(\theta)+w(\tau,\theta)], \tag{14}$$

where $w, u \to 0$ as $\tau \to \infty$ and $0 < v_a(\theta) < 1$. Note that (13), (14) strictly speaking do not say anything about the first time derivatives of P and Q. This is the reason for including the estimates (9) and (11). Given the equations, (9)–(12) are however sufficient for computing the asymptotic behavior of higher order time derivatives. The idea of finding expansions started with the paper [12] by Grubišić and Moncrief, and the first analysis proving the existence of solutions with expansions of the form (13), (14) is contained in [13] and [15]. In these articles, the authors proved that, given v_a, ϕ, q, ψ with $0 < v_a < 1$ of a suitable degree of differentiability, there are unique solutions to the equations with asymptotics of the form (13), (14). In [13], the regularity requirement was that of real analyticity, a condition which was relaxed to smoothness in [15]. Conditions on initial data yielding asymptotic expansions were first given in [19]; see also [17] and [3].

In order to be able to extract the maximum amount of information from the above results, we need to define the Gowdy to Ernst transformation; see [21] for the basic facts needed in this paper. Consider a solution (Q,P) to (2), (3) with $\theta \in \mathbb{R}$ instead of S^1. Then the conditions

$$P_1 = \tau - P, \quad Q_{1\tau} = -e^{2(P-\tau)}Q_\theta, \quad Q_{1\theta} = -e^{2P}Q_\tau \tag{15}$$

determine a solution to the equations on $\mathbb{R}^2$, up to a constant translation in Q. We shall write $(Q_1,P_1) = \mathrm{GE}_{q_0,\tau_0,\theta_0}(Q,P)$, where the role of the constants q_0, τ_0, θ_0 is to specify that $Q_1(\tau_0,\theta_0) = q_0$. It is important to keep in mind that the Gowdy to Ernst transformation does not preserve periodicity in general. However, we shall apply the transformation to solutions with $\theta \in S^1$. What we mean by this is that we apply it to the naturally associated 2π-periodic solution and the outcome is a solution with $\theta \in \mathbb{R}$, which is not necessarily periodic. Using Proposition 1 and 2 together with the Gowdy to Ernst transformation and inversions (7), we can reduce the general situation to one in which $v_\infty < 1$. The reason is the following; cf. [21] for more details. Assume $v_\infty(\theta_0) \geq 1$. By performing an inversion, if necessary, cf. Proposition 1, we can assume that $P_\tau(\tau, \theta_0)$ converges to $v_\infty(\theta_0)$. Performing a Gowdy to Ernst transformation and then an inversion, one obtains a solution $x_2 = (Q_2, P_2)$ with $v_\infty[x_2](\theta_0) = v_\infty(\theta_0) - 1$; cf. (15) and Proposition 1. This procedure can then be repeated until one obtains a solution x_{2k} with $v_\infty[x_{2k}](\theta_0) < 1$. If $v_\infty[x_{2k}](\theta_0) > 0$, we are in a position to use Proposition 2 in order to obtain expansions. One can then trace the solution backward in order to be able to say something about the original solution, but it should be emphasized that it is not in general trivial to do so. However, if $v_\infty(\theta_0)$ is an integer, one cannot apply Proposition 2. On the other hand, the points at which $v_\infty = 1$ are the most important ones, since the curvature need not necessarily become unbounded along causal curves ending on them. Note that by the above procedure, we can transform a solution x_1 with the property $v_\infty[x_1](\theta_0) = 1$ to a solution x_2 with the property that $v_\infty[x_2](\theta_0) = 0$. In fact, all one needs to do is to first apply an inversion, if necessary, and then the Gowdy to Ernst transformation (15). In this way one can translate the problem of perturbing away from $v_\infty = 1$, which is of interest when proving curvature blow-up for generic initial data, to the problem of perturbing away from zero velocity. The main contribution of this paper is to prove that one can perturb away from zero velocity; in fact most of the paper is devoted to proving this fact.

1.4. *Density of the generic solutions.* In order to be able to define the generic set of solutions, we need to define the concepts of true and false spikes. The reader interested in a more detailed discussion of these concepts is referred to [16].

Definition 1. Let $\mathcal{S}_p$ denote the set of smooth solutions to (2), (3) on $\mathbb{R} \times S^1$, and let $\mathcal{S}_{p,c}$ denote the subset of $\mathcal{S}_p$ obeying

$$\int_{S^1} (P_\tau P_\theta + e^{2P} Q_\tau Q_\theta) d\theta = 0. \tag{16}$$

Remark. The left-hand side of (16) is independent of τ due to the equations.

Definition 2. Let $(Q, P) \in \mathcal{S}_p$. Assume $0 < v_\infty(\theta_0) < 1$ for some $\theta_0 \in S^1$ and

$$\lim_{\tau \to \infty} P_\tau(\tau, \theta_0) = -v_\infty(\theta_0).$$

Let $(Q_1, P_1) = \mathrm{Inv}(Q, P)$. By Proposition 2, (Q_1, P_1) has smooth expansions in a neighborhood I of θ_0. In particular, Q_1 converges to a smooth function q_1 in I, and the convergence is exponential in any C^k-norm. By Proposition 1, $q_1(\theta_0) = 0$. If $\partial_\theta q_1(\theta_0) \neq 0$, then θ_0 is called a *nondegenerate false spike*.

We refer the reader to [21] for an interpretation of false spikes in terms of different representations of hyperbolic space. In the above setting, $0 < v_\infty(\theta) < 1$ in a neighborhood of θ_0, and in a punctured neighborhood of θ_0, $\lim_{\tau\to\infty} P_\tau(\tau,\theta) = v_\infty(\theta)$; cf. [21]. The reason for calling θ_0 a spike is that the limit of P_τ makes a jump there. The reason for calling it a false spike is that it disappears if one applies an isometry of hyperbolic space. In other words, it is not geometric.

Let us make some observations in preparation for the definition of nondegenerate true spikes. Assume that $(Q,P) \in \mathcal{S}_p$, $1 < v_\infty(\theta_0) < 2$ and that $P_\tau(\tau,\theta_0) \to v_\infty(\theta_0)$. Let $(Q_1,P_1) = \mathrm{GE}_{q_0,\tau_0,\theta_0}(Q,P)$. By (15), we see that $P_{1\tau}(\tau,\theta_0) \to 1 - v_\infty(\theta_0)$. Since the limit is negative, we can apply an inversion to change the sign; cf. Proposition 1. In other words, $(Q_2,P_2) = \mathrm{Inv}(Q_1,P_1)$ has the property that $P_{2\tau}(\tau,\theta_0) \to v_\infty(\theta_0) - 1$ and $Q_2(\tau,\theta_0) \to 0$. By Proposition 2, we get the conclusion that (Q_2,P_2) have smooth expansions in a neighborhood I of θ_0. In particular, Q_2 converges to a smooth function q_2, and the convergence is exponential in any C^k-norm. By the above, $q_2(\theta_0) = 0$.

Definition 3. Let $(Q,P) \in \mathcal{S}_p$. Assume $1 < v_\infty(\theta_0) < 2$ for some $\theta_0 \in S^1$ and

$$\lim_{\tau\to\infty} P_\tau(\tau,\theta_0) = v_\infty(\theta_0).$$

Let $(Q_2,P_2) = \mathrm{Inv} \circ \mathrm{GE}_{q_0,\tau_0,\theta_0}(Q,P)$. By the observations made prior to the definition, (Q_2,P_2) has smooth expansions in a neighborhood I of θ_0. In particular Q_2 converges to a smooth function q_2 in I and the convergence is exponential in any C^k-norm. If $\partial_\theta q_2(\theta_0) \neq 0$, then θ_0 is called a *nondegenerate true spike*.

In the above setting, the choice of constants is of no importance, $0 < v_\infty(\theta) < 1$ in a punctured neighborhood of θ_0 and $\lim_{\tau\to\infty} P_\tau(\tau,\theta) = v_\infty(\theta)$ in a neighborhood of θ_0; cf. [21]. Again, the reason for calling θ_0 a spike is that the limit of P_τ makes a jump there. Since v_∞ makes a jump in this case, the discontinuity in the limit of P_τ does however remain after having applied an isometry. This justifies the name true spike.

Definition 4. Let $\mathcal{G}_{l,m}$ be the set of $(Q,P) \in \mathcal{S}_p$ with l nondegenerate true spikes $\theta_1,\dots,\theta_l$ and m nonegenerate false spikes $\theta_1',\dots,\theta_m'$ such that

$$\lim_{\tau\to\infty} P_\tau(\tau,\theta) = v_\infty(\theta),$$

for all $\theta \notin \{\theta'_1, \dots, \theta'_m\}$ and $0 < v_\infty(\theta) < 1$ for all $\theta \notin \{\theta_1, \dots, \theta_l\}$. Let $\mathcal{G}_{l,m,c} = \mathcal{G}_{l,m} \cap \mathcal{S}_{p,c}$. Finally

$$\mathcal{G} = \bigcup_{l=0}^{\infty} \bigcup_{m=0}^{\infty} \mathcal{G}_{l,m}, \quad \mathcal{G}_c = \bigcup_{l=0}^{\infty} \bigcup_{m=0}^{\infty} \mathcal{G}_{l,m,c}.$$

Let $x \in \mathcal{G}$. By Proposition 2, we have smooth expansions of the form (9)–(12) in a neighborhood of all points except for a finite number of nondegenerate true and false spikes. In a neighborhood of the nondegenerate false spikes, $\mathrm{Inv}x$ does however have expansions of this form. Finally, $\mathrm{Inv} \circ \mathrm{GE}_{q_0,\tau_0,\theta_0} x$ has smooth expansions of the form (9)–(12) in a neighborhood of the nondegenerate true spikes. Consequently, the generic solutions are quite well understood. We refer the reader to [16] for more details concerning the behavior of solutions in a neighborhood of true and false spikes. In [21], we proved the following.

PROPOSITION 3. *$\mathcal{G}_{l,m}$ is open in the $C^2 \times C^1$-topology on initial data and $\mathcal{G}_{l,m,c}$, considered as a subset of $\mathcal{S}_{p,c}$, is open with respect to the $C^2 \times C^1$-topology on initial data.*

PROPOSITION 4. *Given $x \in \mathcal{G}_{l,m}$, there is an open neighborhood O of x in the $C^1 \times C^0$-topology on initial data such that for each $\hat{x} \in O$, $v_\infty[\hat{x}](\theta) \in (0,1) \cup (1,2)$ for all $\theta \in S^1$.*

Remark. Note that the solutions in O have the property that the curvature blows up everywhere on the singularity; cf. [21].

The purpose of the present paper is to prove that $\mathcal{G}$ and $\mathcal{G}_c$ are dense in $\mathcal{S}_p$ and $\mathcal{S}_{p,c}$ respectively.

THEOREM 2. *$\mathcal{G}$ and $\mathcal{G}_c$ are dense in $\mathcal{S}_p$ and $\mathcal{S}_{p,c}$ respectively with respect to the C^∞-topology on initial data.*

The proof is to be found at the end of the paper.

Definition 5. Let (M, g) be a connected Lorentz manifold which is at least C^2. Assume there is a connected C^2 Lorentz manifold $(\widehat{M}, \hat{g})$ of the same dimension as M and an isometric embedding $i : M \to \widehat{M}$ such that $i(M) \neq \widehat{M}$. Then M is said to be C^2-*extendible*. If (M, g) is not C^2-extendible, it is said to be C^2-*inextendible*.

Finally, we are able to give a precise statement of strong cosmic censorship in the class of T^3-Gowdy spacetimes.

COROLLARY 1. *Consider the set of smooth, periodic initial data $\mathcal{S}_{i,p,c}$ of (2), (3) satisfying (16). There is a subset $\mathcal{G}_{i,c}$ of $\mathcal{S}_{i,p,c}$ with the following properties*:

- *$\mathcal{G}_{i,c}$ is open with respect to the $C^1 \times C^0$-topology on $\mathcal{S}_{i,p,c}$,*
- *$\mathcal{G}_{i,c}$ is dense with respect to the C^∞-topology on $\mathcal{S}_{i,p,c}$,*

- *every spacetime corresponding to initial data in $\mathcal{G}_{i,c}$ has the property that in one time direction, it is causally geodesically complete, and in the opposite time direction, the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded along every inextendible causal curve,*
- *for every spacetime corresponding to initial data in $\mathcal{G}_{i,c}$, the maximal globally hyperbolic development is C^2-inextendible.*

Remark. All T^3-Gowdy spacetimes have the property that every causal geodesic is complete to the future and incomplete to the past; cf. [18].

Proof. Let $\mathcal{G}_{i,c}$ be the union of the open neighborhoods constructed in Proposition 4 intersected with $\mathcal{S}_{i,p,c}$. The result then follows from Theorem 2 and [21]. □

1.5. *Perturbing away from zero velocity.* The contribution of the present paper is Theorem 2. The main tool needed to obtain this result is the ability to perturb away from zero velocity. As was pointed out at the end of Section 1.3, solutions which have zero velocity at some point are of special importance. Let us consider such a solution. By the continuity properties of the asymptotic velocity and domain-of-dependence arguments, we can assume that the velocity is small everywhere and zero at some points. The objective is then to prove that given such a solution x, there is a sequence of solutions x_k, converging to x in the C^∞-topology on initial data, which is such that x_k never has zero velocity. The sequence x_k is obtained by perturbing the initial data of x at a later and later time. One is left with two problems. First, the velocity of the perturbed solution is supposed to be nonzero everywhere and second, the initial data of x_k at a fixed hypersurface, say $\tau = 0$, have to converge to the initial data of x. Obviously, the two criteria are in conflict with each other. We want the perturbation to be large in order to achieve nonzero velocity, and we want it to be small in order for the initial data for the different solutions to converge on a fixed Cauchy surface. Furthermore, at first sight it might seem unpleasant to compare the initial data for x_k and x at a fixed Cauchy surface, since this involves comparing the solutions in an interval whose length tends to infinity. There are however scaling reasons why the above argument should work. Consider the polarized Gowdy equation, i.e. (2) with $Q = 0$,

$$P_{\tau\tau} - e^{-2\tau}P_{\theta\theta} = 0. \tag{17}$$

Define the energies

$$\mathcal{E}_k = \frac{1}{2}\int_{S^1}[(\partial_\theta^k\partial_\tau P)^2 + e^{-2\tau}(\partial_\theta^{k+1}P)^2]d\theta.$$

They are all monotonically decaying, so that $\partial_\theta^k\partial_\tau P$ are all bounded to the future by Sobolev embedding. Integrating this bound, we obtain the conclusion that the $\partial_\theta^k P$ do not grow faster than linearly. Inserting this information into (17), we conclude that $\partial_\theta^k \partial_\tau P$ converges to its limit with an error of the form $O(\tau e^{-2\tau})$. Say that P_τ converges to zero. Then the perturbation in P_τ necessary to achieve a nonzero velocity is of the order of magnitude $O(\tau e^{-2\tau})$. Let us try to get a feeling for how much we can perturb the initial data at late times in order to get convergence at $\tau = 0$. Since $\mathscr{E}_k' \geq -2\mathscr{E}_k$, we have

$$\mathscr{E}_k(0) \leq e^{2\tau} \mathscr{E}_k(\tau). \tag{18}$$

Making a perturbation of the order of magnitude $O(\tau e^{-2\tau})$ in $\partial_\theta^k \partial_\tau P$ at τ and letting $\mathscr{E}_k$ denote the energy of the difference between the solution we started with and the perturbed solution, we conclude that $\mathscr{E}_k(\tau)$ is of the order of magnitude $O(\tau^2 e^{-4\tau})$. We see that this yields convergence at $\tau = 0$ due to (18). Observe that one cannot in general perturb away from zero velocity if one restricts one's attention to solutions of (17). The reason is associated with the problem of finding suitable perturbations, a problem which is easier when one considers the full Gowdy equations instead of only the polarized case. In the nonlinear setting, the situation is of course much more complicated. First, we need estimates for how fast the kinetic energy density converges to the square of the asymptotic velocity. In this step it is very important to get more or less optimal estimates for different quantities; in particular it is important to get polynomial growth estimates for certain quantities instead of exponential growth with an arbitrarily small exponent. The reason is that in the nonlinear setting these quantities will appear as factors, and when a large number of factors multiply each other there is a big difference between the two types of estimates. Second, we need to prove convergence to the solution we started with with respect to the C^∞-topology on initial data. The last step may seem to be unpleasant, but it is not so bad for the following reason. In the linear setting, the energy of the difference between the actual solution and the perturbed solution, $\mathscr{E}_k(\tau)$, should obey $e^{2\tau}\mathscr{E}_k(\tau) \to 0$ in order for the difference to converge at $\tau = 0$. In the nonlinear setting we get basically the same result. The reason is that the nonlinear terms are always of higher order and involve objects that can be bounded by the velocity, which can be assumed to be arbitrarily small. The nonlinear terms in other words do not really play an important role, if one has the estimates already mentioned.

1.6. *Outline of the paper.* In the first part of the paper, we prove that it is possible to perturb away from zero velocity proceeding as described above. The first task is to get good bounds on how fast the kinetic energy density converges to the square of the asymptotic velocity. This is the subject of Sections 3 and 4. How to find a suitable perturbation is sorted out in Section 5. The convergence to the solution one started with in the C^∞-topology on initial data is then proved in

Sections 6–8. The remaining sections are concerned with using the tools developed in order to prove the density result.

2. **Notation and monotonic quantities**

2.1. *Equations in the disc model.* As has already been discussed in [17], there are problems associated with the PQ-plane as a model for hyperbolic space. In solutions to (2), (3), false spikes typically appear asymptotically, and they require special attention. In the disc model however, they do not appear. This is related to the fact that if the solution has nonzero velocity at a spatial point, then it tends to the boundary of hyperbolic space at that spatial point. In the disc model, the boundary is a circle, and there is no distinguished boundary point. When going from the disc model to the upper half-plane, one rips open the boundary circle into a line, and in this way one obtains a distinguished point on the boundary, namely the point at infinity. At a nondegenerate false spike, the solution tends to infinity, but at points in a punctured neighborhood, it tends to the real line. We refer the reader to [17] and [21] for a more technical discussion of this aspect. There is another problem associated with the PQ-plane. The concept of velocity as we have defined it above is one dimensional, and it may seem strange that we should be able to perturb away from zero velocity. In the disc model, the asymptotic velocity however becomes a two-dimensional object in a natural way, cf. (8), and so it becomes clearer why it should be possible to perturb away from zero velocity. Finally, the problem of false spikes is always present if one is close to zero velocity. For these reasons, the arguments concerning perturbing away from zero velocity are made in the disc model.

Let us discuss some different representations of hyperbolic space. Define

$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\}, \quad g_H = \frac{dx^2 + dy^2}{y^2}, \quad \phi_{RH}(Q, P) = (Q, e^{-P}).$$

Then (H, g_H) is the upper half-plane model of hyperbolic space, and ϕ_{RH} is an isometry between $(\mathbb{R}^2, g_R)$ and (H, g_H). Define

$$D = \{z \in \mathbb{C} : |z| < 1\}, \quad g_D = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}, \quad \phi_{HD} = \frac{z - i}{z + i}.$$

Then (D, g_D) is the disc model of hyperbolic space, and ϕ_{HD} is an isometry between (H, g_H) and (D, g_D). Finally, what we shall refer to as *the canonical map*,

$$\phi_{RD}(Q, P) = \frac{Q + i(e^{-P} - 1)}{Q + i(e^{-P} + 1)} \tag{19}$$

defines an isometry between $(\mathbb{R}^2, g_R)$ and (D, g_D). The inverse is given by

$$(Q,P)=\left[-\frac{2\mathrm{Im} z}{|1-z|^2}, -\ln(1-|z|^2)+2\ln|1-z|\right]. \tag{20}$$

Let us define

$$\rho=\ln\frac{1+|z|}{1-|z|}, \tag{21}$$

i.e. ρ is the distance from the origin to z with respect to the hyperbolic metric. Combining the last two equations, we get

$$P=\rho-2\ln(1+|z|)+2\ln|1-z|. \tag{22}$$

Let us derive the Gowdy equations in the disc model by considering the associated action. In the disc model it takes the form

$$\int_{\mathbb{R}}\int_{S^1}\left[\frac{2|z_\tau|^2}{(1-|z|^2)^2}-e^{-2\tau}\frac{2|z_\theta|^2}{(1-|z|^2)^2}\right]d\theta d\tau,$$

where $z\in C^\infty(\mathbb{R}\times S^1, D)$. The corresponding Euler-Lagrange equations are, after some reformulation,

$$\partial_\tau\left(\frac{z_\tau}{1-|z|^2}\right)-e^{-2\tau}\partial_\theta\left(\frac{z_\theta}{1-|z|^2}\right)=\frac{2}{(1-|z|^2)^2}\mathfrak{Q}(z,\partial z). \tag{23}$$

If we use the convention that for $\xi,\zeta\in\mathbb{C}$, $\xi\zeta$ denotes ordinary complex multiplication and $\xi\cdot\zeta$ denotes the inner product of ξ and ζ viewed as vectors in $\mathbb{R}^2$, then

$$\mathfrak{Q}(w,\partial z)=|z_\tau|^2w-(w\cdot z_\tau)z_\tau-e^{-2\tau}(|z_\theta|^2w-(w\cdot z_\theta)z_\theta), \tag{24}$$

where we have used ∂z as a shorthand for $(z_\tau, e^{-\tau}z_\theta)$. Note that for a fixed ∂z, τ and θ, $a(w)=\mathfrak{Q}[w,\partial z(\tau,\theta)]$ defines a linear function in w over the real numbers. Observe that ϕ_{RD} defined in (19) constitutes a bijective map from solutions of (2), (3) to solutions of (23). If, given a solution x of (2), (3), we suddenly speak of a solution z of (23), we shall take it to be understood that $z=\phi_{RD}\circ x$, and vice versa. In fact, we shall use the notation $z\in\mathcal{S}_p$, meaning that $\phi_{RD}^{-1}z\in\mathcal{S}_p$ and similarly for $\mathcal{S}_{p,c}$. Note that the left-hand side of (16) equals $c_0[z]$ as defined in (92) if $z=\phi_{RD}(Q,P)$.

2.2. *Notation and monotonic quantities.* Let us define the potential and kinetic energy densities by

$$\mathcal{P}=\frac{4e^{-2\tau}|z_\theta|^2}{(1-|z|^2)^2} \tag{25}$$

$$\mathcal{K}=\frac{4|z_\tau|^2}{(1-|z|^2)^2}. \tag{26}$$

Note that these concepts make geometric sense, since they are defined using only the metric of hyperbolic space, and that they coincide with the earlier definitions, when $z = \phi_{RD}(Q, P)$. If $I = [a, b]$ is a subinterval of $\mathbb{R}$, let

$$\mathcal{D}_I = \{(\tau, \theta) \in \mathbb{R}^2 : \theta \in [a - e^{-\tau}, b + e^{-\tau}]\}.$$

The definition if I is an open interval is similar. If I only consists of the point θ_0, we shall also write $\mathcal{D}_{\theta_0}$. Let

$$\mathcal{D}_{I,\tau} = [a - e^{-\tau}, b + e^{-\tau}].$$

We shall often use the above notation in situations where $\theta \in S^1$. We shall then take it to be understood that we mean the image of the above objects under the map that identifies spatial points that are at a distance $k2\pi$, $k \in \mathbb{Z}$, apart. Let us define

$$\mathcal{A}_{k,\pm} = 2e^{\tau} \left| \partial_\theta^k \left(\frac{z_\tau \pm e^{-\tau} z_\theta}{1 - |z|^2} \right) \right|^2, \tag{27}$$

and, for notational convenience,

$$a_l = \partial_\theta^l \left(\frac{z_\tau}{1 - |z|^2} \right), \quad b_l = e^{-\tau} \partial_\theta^l \left(\frac{z_\theta}{1 - |z|^2} \right). \tag{28}$$

For $k = 0$, we shall use the notation $\mathcal{A}_\pm$ instead of $\mathcal{A}_{0,\pm}$. In order to be able to obtain estimates, we require the following definition,

$$F_{I,k} = \|\mathcal{A}_{k,+}\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})} + \|\mathcal{A}_{k,-}\|_{C^0(\mathcal{D}_{I,\tau},\mathbb{R})}.$$

If $k = 0$, we shall speak of F_I, and if $I = S^1$, we shall speak of F_k rather than of $F_{S^1,k}$. Finally, $F = F_0$. Compute

$$\begin{aligned}(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{A}_{k,\pm} &= 2e^{\tau}\{|a_k|^2 - |b_k|^2\} \\ &\quad + 8e^{\tau}\partial_\theta^k \left[\frac{\mathcal{Q}(z, \partial z) \pm e^{-\tau}\{(z \cdot z_\tau) z_\theta - (z \cdot z_\theta) z_\tau\}}{(1 - |z|^2)^2} \right] \cdot [a_k \pm b_k].\end{aligned} \tag{29}$$

Note that

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{A}_\pm = \frac{1}{2} e^{\tau}(\mathcal{K} - \mathcal{P}) = \frac{1}{2}(\mathcal{A}_+ + \mathcal{A}_-) - e^{\tau}\mathcal{P}. \tag{30}$$

The most basic and important estimate which holds for solutions to (23) is the following.

LEMMA 1. *Consider a solution to (23) and let I be a subinterval of S^1. Then for all $\tau \geq \tau_0$,*

$$e^{-\tau} F_I(\tau) \leq e^{-\tau_0} F_I(\tau_0).$$

Proof. Let us estimate, for $\tau \geq \tau_0$ and $\theta \in \mathcal{D}_{I,\tau}$,

$$\begin{aligned}\mathcal{A}_{\pm}(\tau,\theta) &= \mathcal{A}_{\pm}(\tau_0, \theta \pm e^{-\tau_0} \mp e^{-\tau}) \\ &\quad + \int_{\tau_0}^{\tau} [(\partial_\tau \mp e^{-s}\partial_\theta)\mathcal{A}_{\pm}](s, \theta \pm e^{-s} \mp e^{-\tau}) ds \\ &\leq \|\mathcal{A}_{\pm}\|_{C^0(\mathcal{D}_{I,\tau_0},\mathbb{R})} + \frac{1}{2}\int_{\tau_0}^{\tau} F_I(s) ds.\end{aligned}$$

Taking the supremum over $\theta \in \mathcal{D}_{I,\tau}$ and adding the two estimates, we get the conclusion

$$F_I(\tau) \leq F_I(\tau_0) + \int_{\tau_0}^{\tau} F_I(s) ds.$$

The statement follows by Grönwall's lemma. □

The following lemma was essentially proved in [17]. It is a starting point for the estimates of the rate at which the kinetic energy density converges to the square of the asymptotic velocity. Since we are interested in the behaviour of families of solutions, it is very important to keep track of the dependence of different constants on the initial data.

LEMMA 2. *Consider a solution z to (23). Assume that $\rho(\tau,\theta) \leq \tau - 2$ for all $(\tau,\theta) \in [T,\infty) \times S^1$. Then, there is a $v \in C^0(S^1,\mathbb{R}^2)$ such that for all $\tau \geq T$,*

$$\begin{aligned}\left\| \frac{1}{\tau}\frac{z(\tau,\cdot)}{|z(\tau,\cdot)|}\rho(\tau,\cdot) - v \right\|_{C^0(S^1,\mathbb{R}^2)} + \left\| \frac{2z_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} - v \right\|_{C^0(S^1,\mathbb{R}^2)} \\ + e^{-\tau}\left\| \frac{2z_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} \right\|_{C^0(S^1,\mathbb{R}^2)} \leq 6G^{1/2}(T)\frac{T}{\tau},\end{aligned}$$

where G is as defined in (31).

Remark. Since ρ is nonnegative, it is implicitly assumed in the statement of the above lemma that $\tau \geq 2$. We shall make this implicit assumption throughout in what follows.

Proof. Consider the proof of Lemma 5 in [17]. Let

$$G = \frac{1}{2}\sum_{\pm} \left\| \frac{2z_\tau}{1-|z|^2} - \frac{\rho}{\tau}\frac{z}{|z|} \pm \frac{2e^{-\tau}z_\theta}{1-|z|^2} \right\|^2_{C^0(S^1,\mathbb{R}^2)}. \tag{31}$$

In the above mentioned proof it is shown that, under the assumptions of the lemma,

$$G(\tau) \leq G(\tau_0)\left(\frac{\tau_0}{\tau}\right)^2$$

for all $\tau \geq \tau_0 \geq T$. As argued in the proof, we have

$$\left(\rho_\tau - \frac{\rho}{\tau}\right)^2 + \sinh^2\rho \left|\partial_\tau\left(\frac{z}{|z|}\right)\right|^2 \leq G(\tau_0)\left(\frac{\tau_0}{\tau}\right)^2,$$

assuming $|z| > 0$. Define g by

$$g = \frac{\rho}{\tau}\frac{z}{|z|}.$$

Note that $\rho/|z|$ is a real analytic function from D to the real numbers if one defines the value at the origin appropriately. We get, for $|z| > 0$,

$$|\partial_\tau g| \leq \frac{1}{\tau}\left|\rho_\tau - \frac{\rho}{\tau}\right| + \frac{1}{\tau}\left|\partial_\tau\left(\frac{z}{|z|}\right)\right|\rho \leq 2G^{1/2}(\tau_0)\frac{\tau_0}{\tau^2},$$

since $\rho \leq \sinh\rho$. By the arguments in the mentioned lemma, we get the same estimate if $z = 0$. We conclude that

$$\|g(\tau_2, \cdot) - g(\tau_1, \cdot)\|_{C^0(S^1,\mathbb{R}^2)} \leq 2G^{1/2}(\tau_0)\frac{\tau_0}{\tau_1},$$

assuming $\tau_2 \geq \tau_1 \geq \tau_0$. Thus there is a $v \in C^0(S^1, \mathbb{R}^2)$ such that

$$\|g(\tau, \cdot) - v\|_{C^0(S^1,\mathbb{R}^2)} \leq 2G^{1/2}(\tau_0)\frac{\tau_0}{\tau}.$$

The lemma follows. □

In the following, C will denote any *numerical* constant, which may be indexed by an integer, but which is independent of the particular solution. If the constant depends on the particular solution, through objects such as $G(\tau_0)$, we shall use the notation K, and note what parameters it depends upon. Under the assumptions of the above lemma, we conclude that

$$\left\|\frac{2z_\tau}{1-|z|^2} - \frac{z}{|z|}v_\infty\right\|_{C^0(S^1,\mathbb{R}^2)} \leq CG^{1/2}(T)\frac{T}{\tau}. \tag{32}$$

In principle, there is of course a problem with this estimate if $z = 0$. However, if we define $z/|z|$ to be zero when $z = 0$, the estimate is still valid.

3. **Estimates for the corrections**

The purpose of this section and the next is to obtain estimates that tell us how fast the kinetic energy density converges to its final value, given that the velocity is smaller than one. It is very important to get more or less optimal estimates in order to be able to perturb away from zero velocity. It should be possible to get growth estimates of the form $e^{\varepsilon\tau}$ for some small ε for the norms of interest without any greater effort. However, in the nonlinear setting, when we wish to prove that the sequence of perturbed solutions converges to the original one in the C^∞-topology on initial data, we have to deal with terms with an arbitrarily large number of such factors, and then we loose control. If we have polynomial growth estimates instead, we are in a better position. We shall also need to keep track of how the estimates depend on the particular solution, since we want to have estimates for sequences of solutions converging to a fixed one. For this reason, the following analysis is unfortunately rather technical.

This section is concerned with estimates for what we shall refer to as corrections. For technical reasons, it is not enough to consider objects of the form $\mathcal{A}_{k,\pm}$ defined in (27); one has to add certain corrections to them in order to get good estimates. The reason is roughly as follows. Consider (17). Carrying out estimates similar to the ones obtained in Lemma 1, and observing that any spatial derivatives of P satisfy the same equation, one obtains the result that $\partial_\theta^k \partial_\tau P$ and $e^{-\tau}\partial_\theta^{k+1} P$ are bounded for any k. Consider (13). Clearly, the estimate obtained for $\partial_\theta^k \partial_\tau P$ is optimal, but the estimate for $e^{-\tau}\partial_\theta^{k+1} P$ is essentially worthless. One can obtain linear growth for $\partial_\theta^k P$ by simply integrating the bound for $\partial_\theta^k \partial_\tau P$, and this estimate is optimal, as can be seen from (13). However, integrating the bound for $\partial_\theta^k \partial_\tau P$ involves the cost of one derivative, a price one can certainly pay in a linear setting but not in a nonlinear one. When obtaining estimates for $k+1$ derivatives, it is essential to have better estimates for k spatial derivatives than one has from the estimates for k derivatives. The solution is to add a term to $\mathcal{A}_{k,\pm}$ involving k spatial derivatives and to obtain an improvement for the estimate of expressions involving k spatial derivatives simultaneously with the estimates for $k+1$ derivatives. The question is then what factor we should choose in front of the term involving k spatial derivatives. We have found the following correction to yield acceptable results

$$\mathscr{C}_k = 2\tau^{-2} e^{\tau}(\rho^4 + 1)|\partial_\theta^k z|^2.$$

An assumption we shall typically be making in the following lemmas is that

$$e^{-\tau}\sum_{k=1}^{l}[\sup_{\theta\in S^1}\mathcal{A}_{k,+} + \sup_{\theta\in S^1}\mathcal{A}_{k,-} + \sup_{\theta\in S^1}\mathscr{C}_k] \le K_l \tau^{m_l}, \tag{33}$$

for all $\tau \ge T$, where K_l and m_l are some constants.

LEMMA 3. *Consider a solution to (23), and assume that $\rho(\tau,\theta) \le \tau - 2$ for all $(\tau,\theta) \in [T,\infty)\times S^1$. Then, for all $\tau \ge T$,*

$$(\partial_\tau \pm e^{-\tau}\partial_\theta)\mathscr{C}_1 \le \mathscr{C}_1 + C[1 + G^{1/2}(T)T]\tau^{-1}(\mathcal{A}_{1,+} + \mathcal{A}_{1,-} + \mathscr{C}_1), \tag{34}$$

where C is a numerical constant. Furthermore, if (33) holds for all $\tau \ge T$ and some $l \ge 1$, then

$$\begin{aligned}(\partial_\tau \pm e^{-\tau}\partial_\theta)\mathscr{C}_{l+1} &\le \mathscr{C}_{l+1}\\ &\quad + C[1 + G^{1/2}(T)T]\tau^{-1}(\mathcal{A}_{l+1,+} + \mathcal{A}_{l+1,-} + \mathscr{C}_{l+1}) + e^{\tau}\Pi_{l+1}(\tau)\end{aligned} \tag{35}$$

for some polynomial Π_{l+1} satisfying the estimate

$$\Pi_{l+1}(\tau) \le C_l(1 + K_l^3)\tau^{3m_l+7}. \tag{36}$$

Remark. Note that the constant C in (35) does not depend on l.

Proof. Let us compute

$$(\partial_\tau \pm e^{-\tau}\partial_\theta)\mathscr{C}_k = \mathscr{C}_k - 2\tau^{-1}\mathscr{C}_k + 2\tau^{-2}e^\tau[(\partial_\tau \pm e^{-\tau}\partial_\theta)\rho^4]|\partial_\theta^k z|^2 \\ +4\tau^{-2}e^\tau(1+\rho^4)(\partial_\theta^k\partial_\tau z \pm e^{-\tau}\partial_\theta^{k+1}z)\cdot\partial_\theta^k z.$$

We consider $(\partial_\tau \pm e^{-\tau}\partial_\theta)\rho^4$. If $\rho(\tau,\theta)=0$, then this expression is zero at the point (τ,θ), so that we assume $\rho \neq 0$. Observe that under this assumption,

$$e^{-\tau}|\rho_\theta| \le \frac{2e^{-\tau}|z_\theta|}{1-|z|^2} \quad \text{and} \quad |\rho_\tau| \le \frac{2|z_\tau|}{1-|z|^2},$$

since

$$\rho_\tau^2 + \sinh^2\rho\left|\partial_\tau\left(\frac{z}{|z|}\right)\right|^2 = \frac{4|z_\tau|^2}{(1-|z|^2)^2} \tag{37}$$

and similarly for the θ derivative. Thus

$$|(\partial_\tau \pm e^{-\tau}\partial_\theta)\rho^4| \le 4\rho^3\left[\frac{2|z_\tau|}{1-|z|^2} + e^{-\tau}\frac{2|z_\theta|}{1-|z|^2}\right].$$

Note that

$$\frac{2|z_\tau|}{1-|z|^2} + e^{-\tau}\frac{2|z_\theta|}{1-|z|^2} \le C[1+G^{1/2}(T)T]\tau^{-1}(1+\rho),$$

by Lemma 2, so that

$$(1+\rho^4)^{-1}|(\partial_\tau \pm e^{-\tau}\partial_\theta)\rho^4| \le C[1+G^{1/2}(T)T]\tau^{-1}.$$

Consider

$$\tau^{-2}e^\tau(1+\rho^4)(\partial_\theta^k\partial_\tau z \pm e^{-\tau}\partial_\theta^{k+1}z)\cdot\partial_\theta^k z.$$

Note that

$$z_{\tau\theta} = (1-|z|^2)\partial_\theta\left(\frac{z_\tau}{1-|z|^2}\right) - \frac{2(z\cdot z_\theta)z_\tau}{1-|z|^2}. \tag{38}$$

Since $1-|z| = 2/(1+e^\rho)$, we have

$$(\rho^4+1)^{1/2}(1-|z|^2) \le \frac{4(\rho^4+1)^{1/2}}{e^\rho+1} \le C.$$

Thus

$$\tau^{-2}e^\tau(1+\rho^4)(1-|z|^2)\partial_\theta\left(\frac{z_\tau}{1-|z|^2}\right)\cdot z_\theta \\ \le C\tau^{-1}e^\tau[\tau^{-1}(1+\rho^4)^{1/2}|z_\theta|]\left|\partial_\theta\left(\frac{z_\tau}{1-|z|^2}\right)\right| \le C\tau^{-1}[\mathscr{A}_{1,+}+\mathscr{A}_{1,-}+\mathscr{C}_1],$$

where we have used the inequality $ab \leq (a^2+b^2)/2$ in the last step. Consider

$$\begin{aligned}
&-\tau^{-2}e^{\tau}(1+\rho^4)\frac{2(z\cdot z_\theta)(z_\tau\cdot z_\theta)}{1-|z|^2}\\
&\quad = -\tau^{-2}e^{\tau}(1+\rho^4)(z\cdot z_\theta)\left[\left(\frac{2z_\tau}{1-|z|^2}-\frac{z}{|z|}\frac{\rho}{\tau}+\frac{z}{|z|}\frac{\rho}{\tau}\right)\cdot z_\theta\right]\\
&\quad \leq C[1+G^{1/2}(T)T]\tau^{-1}\mathscr{C}_1.
\end{aligned}$$

Note that the sign is crucial in this inequality. Similarly to the above, we have

$$\pm\tau^{-2}e^{\tau}(1+\rho^4)e^{-\tau}z_{\theta\theta}\cdot z_\theta = \pm\tau^{-2}e^{\tau}(1+\rho^4)(1-|z|^2)\partial_\theta\left(\frac{e^{-\tau}z_\theta}{1-|z|^2}\right)\cdot z_\theta$$

$$\mp 2\tau^{-2}e^{\tau}(1+\rho^4)\frac{e^{-\tau}(z\cdot z_\theta)|z_\theta|^2}{1-|z|^2} \leq C[1+G^{1/2}(T)T]\tau^{-1}[\mathscr{A}_{1,+}+\mathscr{A}_{1,-}+\mathscr{C}_1].$$

This proves the estimate for $\mathscr{C}_1$. Consider (38). Let us differentiate this equality l times. Due to the assumptions, we get

$$\partial_\theta^{l+1}\partial_\tau z = (1-|z|^2)\partial_\theta^{l+1}\left(\frac{z_\tau}{1-|z|^2}\right)-\frac{2(z\cdot\partial_\theta^{l+1}z)z_\tau}{1-|z|^2}+\mathscr{R}_{1,l+1},$$

where $\mathscr{R}_{1,l+1}$ can be bounded by a polynomial. In fact, the estimate (33) and the structure of (38) yield

$$|\mathscr{R}_{1,l+1}| \leq C_l(1+K_l^3)^{1/2}\tau^{3m_l/2+2}, \tag{39}$$

where the $+2$ in the exponent is due to the factor τ^{-2} contained in $\mathscr{C}_k$. Similarly,

$$e^{-\tau}\partial_\theta^{l+2}z = (1-|z|^2)e^{-\tau}\partial_\theta^{l+1}\left(\frac{z_\theta}{1-|z|^2}\right)-\frac{2(z\cdot\partial_\theta^{l+1}z)e^{-\tau}z_\theta}{1-|z|^2}+\mathscr{R}_{2,l+1},$$

where $\mathscr{R}_{2,l+1}$ can be bounded by a polynomial, and we have an estimate similar to (39). Note that

$$\tau^{-2}e^{\tau}(1+\rho^4)(1-|z|^2)\partial_\theta^{l+1}\left(\frac{z_\tau}{1-|z|^2}\right)\cdot\partial_\theta^{l+1}z \leq C\tau^{-1}(\mathscr{A}_{l+1,+}+\mathscr{A}_{l+1,-}+\mathscr{C}_{l+1}),$$

as above. The other terms, except for $\mathscr{R}_{i,l+1}$, $i=1,2$, can also be dealt with in the same way we handled $\mathscr{C}_1$, which is why we get the same constant (independent of l). Finally, consider

$$\begin{aligned}
\tau^{-2}e^{\tau}(1+\rho^4)|\mathscr{R}_{i,l+1}\cdot\partial_\theta^{l+1}z| &\leq \tau^{-2}e^{\tau}(1+\rho^4)\frac{1}{2}[\tau^{-1}|\partial_\theta^{l+1}z|^2+\tau\mathscr{R}_{i,l+1}^2]\\
&\leq C\tau^{-1}\mathscr{C}_{l+1}+e^{\tau}\Pi_{l+1}''(\tau),
\end{aligned}$$

for some polynomial Π''_{l+1}, since $\rho \le \tau$. Using (39) and the similar estimate for $\mathcal{R}_{2,l+1}$, we get the conclusion that we can choose

$$\Pi''_{l+1}(\tau) \le C_l(1+K_l^3)\tau^{3m_l+7}.$$

The lemma follows. □

4. Main estimates

Let us turn to the estimates for the derivative of $\mathcal{A}_{l,\pm}$. By (29), the relevant expression to consider is

$$4\partial_\theta^l\left[\frac{\mathfrak{Q}(z,\partial z)\pm e^{-\tau}\{(z\cdot z_\tau)z_\theta-(z\cdot z_\theta)z_\tau\}}{(1-|z|^2)^2}\right]\cdot[a_l\pm b_l],$$

where we have used the terminology of (28). We define this expression to be the sum of three terms, $\mathcal{D}_{i,l,\pm}$, $i=1,2,3$, where, cf. the definition (24),

$$\mathcal{D}_{1,l,\pm}=4\partial_\theta^l\left[\frac{|z_\tau|^2z-(z\cdot z_\tau)z_\tau}{(1-|z|^2)^2}\right]\cdot[a_l\pm b_l],$$

$$\mathcal{D}_{2,l,\pm}=4\partial_\theta^l\left[\frac{-e^{-2\tau}\{|z_\theta|^2z-(z\cdot z_\theta)z_\theta\}}{(1-|z|^2)^2}\right]\cdot[a_l\pm b_l],$$

$$\mathcal{D}_{3,l,\pm}=\pm4\partial_\theta^l\left[\frac{e^{-\tau}(z\cdot z_\tau)z_\theta-e^{-\tau}(z\cdot z_\theta)z_\tau}{(1-|z|^2)^2}\right]\cdot[a_l\pm b_l].$$

LEMMA 4. *Consider a solution to (23) and assume that $\rho(\tau,\theta)\le\tau-2$ for all $\tau\ge T$ and $\theta\in S^1$. Then*

$$\mathcal{D}_{2,1,\pm}\le C[1+G^{1/2}(T)T]\tau^{-1}e^{-\tau}(\mathcal{A}_{1,+}+\mathcal{A}_{1,-}). \tag{40}$$

Furthermore, if (33) holds for all $\tau\ge T$ and some $l\ge1$, then

$$\mathcal{D}_{2,l+1,\pm}\le C[1+G^{1/2}(T)T]\tau^{-1}e^{-\tau}(\mathcal{A}_{l+1,+}+\mathcal{A}_{l+1,-})+\Pi_{l+1}, \tag{41}$$

where C is a constant and

$$\Pi_{l+1}\le C_{l+1}(1+K_l^3)\tau^{3m_l+3}. \tag{42}$$

Proof. Let us start by computing

(43)

$$4\partial_\theta\left[\frac{-e^{-2\tau}\{|z_\theta|^2z-(z\cdot z_\theta)z_\theta\}}{(1-|z|^2)^2}\right]=-4e^{-2\tau}\left[2z\left\{\frac{z_\theta}{1-|z|^2}\cdot\partial_\theta\left(\frac{z_\theta}{1-|z|^2}\right)\right\}\right.$$
$$\left.-\left\{z\cdot\partial_\theta\left(\frac{z_\theta}{1-|z|^2}\right)\right\}\frac{z_\theta}{1-|z|^2}-\frac{z\cdot z_\theta}{1-|z|^2}\partial_\theta\left(\frac{z_\theta}{1-|z|^2}\right)\right].$$

Since

$$\frac{e^{-\tau}|z_\theta|}{1-|z|^2} \le C[1+G^{1/2}(T)T]\tau^{-1},$$

we get (40). In the general case we differentiate (43) l times and get the estimate

$$\left|4\partial_\theta^{l+1}\left[\frac{-e^{-2\tau}\{|z_\theta|^2 z-(z\cdot z_\theta)z_\theta\}}{(1-|z|^2)^2}\right]\right| \le Ce^{-2\tau}\frac{|z_\theta|}{1-|z|^2}\left|\partial_\theta^{l+1}\left(\frac{z_\theta}{1-|z|^2}\right)\right| + \Pi''_{l+1},$$

where C is independent of l and Π''_{l+1} satisfies the estimate

$$\Pi''_{l+1} \le C_l(1+K_l^3)^{1/2}\tau^{3m_l/2+1}.$$

When estimating $\mathfrak{D}_{2,l+1,\pm}$, the polynomial term can be dealt with in the same way it was handled in the proof of the estimates for the correction term. We conclude that (41) and (42) hold. □

LEMMA 5. *Consider a solution to* (23), *and assume that* $\rho(\tau,\theta) \le \tau-2$ *for all* $\tau \ge T$ *and* $\theta \in S^1$. *Then*

$$\mathfrak{D}_{1,1,\pm} \le 2\frac{v_\infty}{|z|}[(a_1\cdot z)z-|z|^2 a_1]\cdot(a_1 \pm b_1) + C[1+G^{1/2}(T)T]^2\tau^{-1}e^{-\tau}(\mathcal{A}_{1,+}+\mathcal{A}_{1,-}+\mathcal{C}_1), \tag{44}$$

with the notation defined in (28). *Furthermore*, *if* (33) *holds for all* $\tau \ge T$ *and some* $l \ge 1$, *then*

$$\mathfrak{D}_{1,l+1,\pm} \le 2\frac{v_\infty}{|z|}[(a_{l+1}\cdot z)z-|z|^2 a_{l+1}]\cdot(a_{l+1}\pm b_{l+1}) + C[1+G^{1/2}(T)T]^2\tau^{-1}e^{-\tau}(\mathcal{A}_{l+1,+}+\mathcal{A}_{l+1,-}+\mathcal{C}_{l+1}) + \Pi_{l+1},$$

where

$$\Pi_{l+1} \le C_{l+1}(1+K_l^3)\tau^{3m_l+3}.$$

Proof. We compute

$$2\partial_\theta^{l+1}\left[\frac{|z_\tau|^2 z-(z\cdot z_\tau)z_\tau}{(1-|z|^2)^2}\right] = 4\left[a_{l+1}\cdot\frac{z_\tau}{1-|z|^2}\right]z + 2\frac{|z_\tau|^2}{(1-|z|^2)^2}\partial_\theta^{l+1}z$$

$$-2\frac{\partial_\theta^{l+1}z\cdot z_\tau}{(1-|z|^2)^2}z_\tau - 2(z\cdot a_{l+1})\frac{z_\tau}{1-|z|^2} - 2\frac{z\cdot z_\tau}{1-|z|^2}a_{l+1} + \mathcal{R}_{3,l+1},$$

where $\mathcal{R}_{3,1}=0$ and $\mathcal{R}_{3,l+1}$ satisfies an estimate

$$|\mathcal{R}_{3,l+1}| \le C_{l+1}(1+K_l^3)^{1/2}\tau^{3m_l/2+1}.$$

Estimate

$$\left|2\frac{|z_\tau|^2}{(1-|z|^2)^2}\partial_\theta^{l+1}z-2\frac{\partial_\theta^{l+1}z\cdot z_\tau}{(1-|z|^2)^2}z_\tau\right| \\ \leq C[1+G^{1/2}(T)T]^2\tau^{-2}(1+\rho^4)^{1/2}|\partial_\theta^{l+1}z|.$$

The resulting terms can be dealt with as in earlier lemmas. The polynomial term is also not a problem. In the remaining terms, we can replace $2z_\tau/(1-|z|^2)$ with $v_\infty z/|z|$, with an acceptable error term, since we have (32). Thus, we only need to consider

$$\frac{v_\infty}{|z|}[2(a_{l+1}\cdot z)z-(z\cdot a_{l+1})z-|z|^2a_{l+1}]=\frac{v_\infty}{|z|}[(a_{l+1}\cdot z)z-|z|^2a_{l+1}].$$

The lemma follows. □

LEMMA 6. *Consider a solution to (23), and assume that $\rho(\tau,\theta)\leq\tau-2$ for all $\tau\geq T$ and $\theta\in S^1$. Then*

$$\mathcal{D}_{3,1,\pm}\leq\pm2\frac{v_\infty}{|z|}[-(b_1\cdot z)z+|z|^2b_1]\cdot(a_1\pm b_1) \\ +C[1+G^{1/2}(T)T]^2\tau^{-1}e^{-\tau}(\mathcal{A}_{1,+}+\mathcal{A}_{1,-}+\mathcal{C}_1). \tag{45}$$

Furthermore, if (33) holds for all $\tau\geq T$ and some $l\geq1$, then

$$\mathcal{D}_{3,l+1,\pm}\leq\pm2\frac{v_\infty}{|z|}[-(b_{l+1}\cdot z)z+|z|^2b_{l+1}]\cdot(a_{l+1}\pm b_{l+1}) \\ +C[1+G^{1/2}(T)T]^2\tau^{-1}e^{-\tau}(\mathcal{A}_{l+1,+}+\mathcal{A}_{l+1,-}+\mathcal{C}_{l+1})+\Pi_{l+1},$$

where

$$\Pi_{l+1}\leq C_{l+1}(1+K_l^3)\tau^{3m_l+3}.$$

Proof. We need to consider

$$2\partial_\theta^{l+1}\left[\frac{e^{-\tau}(z\cdot z_\tau)z_\theta-e^{-\tau}(z\cdot z_\theta)z_\tau}{(1-|z|^2)^2}\right]=2\frac{(\partial_\theta^{l+1}z\cdot z_\tau)e^{-\tau}z_\theta}{(1-|z|^2)^2} \\ +2(z\cdot a_{l+1})\frac{e^{-\tau}z_\theta}{1-|z|^2}+2\frac{z\cdot z_\tau}{1-|z|^2}b_{l+1}-2e^{-\tau}\frac{(\partial_\theta^{l+1}z\cdot z_\theta)z_\tau}{(1-|z|^2)^2} \\ -2(z\cdot b_{l+1})\frac{z_\tau}{1-|z|^2}-2e^{-\tau}\frac{z\cdot z_\theta}{1-|z|^2}a_{l+1}+\mathcal{R}_{4,l+1},$$

where $\mathcal{R}_{4,l+1}$ satisfies the same sort of estimate as $\mathcal{R}_{3,l+1}$ in the previous lemma. Furthermore, $\mathcal{R}_{4,1}=0$, a conclusion which does not depend on any assumptions. Due to estimates of the form

$$\left|2\frac{(\partial_\theta^{l+1}z\cdot z_\tau)e^{-\tau}z_\theta}{(1-|z|^2)^2}\right|\leq C[1+G^{1/2}(T)T]^2\tau^{-2}(1+\rho^4)^{1/2}|\partial_\theta^{l+1}z|$$

and

$$\frac{e^{-\tau}|z_\theta|}{1-|z|^2} \leq C[1+G^{1/2}(T)T]\tau^{-1},$$

the only terms that cannot be dealt with by arguments already presented are the ones that contain b_{l+1}. In the case of these terms, we replace $2z_\tau/(1-|z|^2)$ with $v_\infty z/|z|$ similarly to the proof of the previous lemma. The relevant terms are then

$$\frac{v_\infty}{|z|}[|z|^2 b_{l+1} - (z\cdot b_{l+1})z].$$

The lemma follows. □

COROLLARY 2. *Consider a solution to (23), and assume that $\rho(\tau,\theta) \leq \tau - 2$ for all $\tau \geq T$ and for all $\theta \in S^1$. Then*

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{A}_{1,\pm} \leq \frac{1}{2}(\mathcal{A}_{1,+}+\mathcal{A}_{1,-}) + C[1+G^{1/2}(T)T]^2\tau^{-1}(\mathcal{A}_{1,+}+\mathcal{A}_{1,-}+\mathcal{C}_1). \tag{46}$$

Furthermore, if (33) holds for all $\tau \geq T$ and some $l \geq 1$, then

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\mathcal{A}_{l+1,\pm} \leq \frac{1}{2}(\mathcal{A}_{l+1,+}+\mathcal{A}_{l+1,-}) + C[1+G^{1/2}(T)T]^2\tau^{-1}(\mathcal{A}_{l+1,+}+\mathcal{A}_{l+1,-}+\mathcal{C}_{l+1}) + e^\tau \Pi_{l+1}(\tau) \tag{47}$$

where

$$\Pi_{l+1} \leq C_{l+1}(1+K_l^3)\tau^{3m_l+3}. \tag{48}$$

Proof. Consider (40), (44) and (45). We need to compute

$$[(a_1\cdot z)z - |z|^2 a_1 \mp (b_1\cdot z)z \pm |z|^2 b_1]\cdot(a_1 \pm b_1) = (a_1\cdot z)^2 - |z|^2|a_1|^2 - (z\cdot b_1)^2 + |z|^2|b_1|^2 \leq |z|^2|b_1|^2.$$

We conclude that

$$\sum_{i=1}^3 \mathfrak{D}_{i,1,\pm} \leq 2v_\infty|z||b_1|^2 + C[1+G^{1/2}(T)T]^2\tau^{-1}e^{-\tau}(\mathcal{A}_{1,+}+\mathcal{A}_{1,-}+\mathcal{C}_1)$$
$$\leq 2|b_1|^2 + C[1+G^{1/2}(T)T]^2\tau^{-1}e^{-\tau}(\mathcal{A}_{1,+}+\mathcal{A}_{1,-}+\mathcal{C}_1),$$

since $v_\infty \leq 1$ and $|z| \leq 1$. By (29), we get the first conclusion of the corollary. The second statement follows by a similar argument. □

Before stating the next corollary, let us introduce

$$\mathcal{A}^c_{k,\pm} = \mathcal{A}_{k,\pm} + \mathcal{C}_k, \quad F^c_{k,\pm}(\tau) = \|\mathcal{A}^c_{k,\pm}(\tau,\cdot)\|_{C^0(S^1,\mathbb{R})}, \quad F^c_k = F^c_{k,+} + F^c_{k,-}.$$

Note that

$$e^{-\tau}\sum_{k=1}^{l}[\sup_{\theta\in S^1}\mathcal{A}_{k,+}+\sup_{\theta\in S^1}\mathcal{A}_{k,-}+\sup_{\theta\in S^1}\mathcal{C}_k]\leq C\sum_{k=1}^{l}e^{-\tau}F_k^c.$$

COROLLARY 3. *Consider a solution to (23), and assume that $\rho(\tau,\theta)\leq\tau-2$ for all $\tau\geq T$ and $\theta\in S^1$. Then, for all $\tau\geq T$,*

$$e^{-\tau}F_1^c(\tau)\leq e^{-T}F_1^c(T)\tau^{m_1} \tag{49}$$

where $m_1=C[1+G^{1/2}(T)T]^2$. In general,

$$e^{-\tau}F_{l+1}^c(\tau)\leq K_{l+1}\tau^{m_{l+1}}, \tag{50}$$

where K_{l+1} is a polynomial in $e^{-T}F_{j+1}^c(T)$, $j=0,\dots,l$, and

$$m_{l+1}=C_{l+1}[1+G^{1/2}(T)T]^2.$$

Proof. Due to (34) and (46), we get

$$(\partial_\tau\mp e^{-\tau}\partial_\theta)\mathcal{A}_{1,\pm}^c\leq\frac{1}{2}(\mathcal{A}_{1,+}^c+\mathcal{A}_{1,-}^c)+\frac{1}{2}m_1\tau^{-1}(\mathcal{A}_{1,+}^c+\mathcal{A}_{1,-}^c).$$

where $m_1=C[1+G^{1/2}(T)T]^2$. Thus

$$\begin{aligned}\mathcal{A}_{1,\pm}^c(\tau,\theta\pm e^{-\tau})&=\mathcal{A}_{1,\pm}^c(\tau_0,\theta\pm e^{-\tau_0})\\&\quad+\int_{\tau_0}^{\tau}[(\partial_\tau\mp e^{-u}\partial_\theta)\mathcal{A}_{1,\pm}^c](u,\theta\pm e^{-u})du\\&\leq F_{1,\pm}^c(\tau_0)+\int_{\tau_0}^{\tau}\left(\frac{1}{2}+\frac{m_1}{2u}\right)F_1^c(u)du.\end{aligned}$$

Taking the supremum over θ and adding the two estimates, we get

$$F_1^c(\tau)\leq F_1^c(\tau_0)+\int_{\tau_0}^{\tau}\left(1+\frac{m_1}{u}\right)F_1^c(u)du.$$

Grönwall's lemma then yields

$$F_1^c(\tau)\leq F_1^c(\tau_0)e^{\tau-\tau_0}\left(\frac{\tau}{\tau_0}\right)^{m_1}.$$

We get (49) if we insert $\tau_0=T$ in the above estimate and observe that $T\geq 2$. This result constitutes the zeroth step in an induction process. Let us assume that we have an estimate of the form (33) for some $l \geq 1$. Then

$$\begin{aligned}\mathscr{A}^c_{l+1,\pm}(\tau, \theta \pm e^{-\tau}) = \mathscr{A}^c_{l+1,\pm}(\tau_0, \theta \pm e^{-\tau_0}) \\ + \int_{\tau_0}^{\tau} [(\partial_\tau \mp e^{-u}\partial_\theta)\mathscr{A}^c_{l+1,\pm}](u, \theta \pm e^{-u}) du \\ \leq F^c_{l+1,\pm}(\tau_0) + \int_{\tau_0}^{\tau} \left[\left(\frac{1}{2} + \frac{m}{2u} \right) F^c_{l+1}(u) + e^u \Pi_{l+1} \right] du,\end{aligned}$$

where $m = C[1 + G^{1/2}(T)T]^2$, due to Lemma 3 and Corollary 2. Taking the supremum in θ and adding the two estimates, we get

$$F^c_{l+1}(\tau) \leq F^c_{l+1}(\tau_0) + \int_{\tau_0}^{\tau} \left[\left(1 + \frac{m}{u}\right) F^c_{l+1}(u) + e^u \Pi_{l+1} \right] du. \tag{51}$$

Let us denote the right-hand side by h. Then

$$h' \leq \left(1 + \frac{m}{\tau}\right) h + e^{\tau} \Pi_{l+1},$$

so that

$$\partial_\tau [e^{-\tau} \tau^{-m} h] \leq \tau^{-m} \Pi_{l+1}. \tag{52}$$

Note here that $m = C[1 + G^{1/2}(T)T]^2$, where C is independent of l. Thus there is no restriction in the assumption that $m \leq m_1 \leq m_2 \ldots$. Since Π_{l+1} satisfies an estimate of the form (36), we conclude that

$$\int_{\tau_0}^{\tau} u^{-m} \Pi_{l+1}(u) du \leq C_l (1 + K_l^3) \tau^{-m} \tau^{3m_l+8}.$$

Thus, (52), the definition of h and (51) yield

$$e^{-\tau} F^c_{l+1}(\tau) \leq e^{-\tau_0} F^c_{l+1}(\tau_0) \left(\frac{\tau}{\tau_0} \right)^m + C_l (1 + K_l^3) \tau^{3m_l+8}.$$

If we let $\tau_0 = T$, an induction argument leads to the conclusion that

$$e^{-\tau} F^c_{l+1}(\tau) \leq K_{l+1} \tau^{m_{l+1}},$$

where K_{l+1} is a polynomial in $e^{-T} F^c_{j+1}(T)$, $j = 0, \ldots, l$, and

$$m_{l+1} = C_{l+1}[1 + G^{1/2}(T)T]^2.$$

The corollary follows. $\square$

It is in fact possible to improve these estimates slightly. In the formulation of the next lemma, it will be convenient to use the notation

$$c_k(\tau) = \left\| \partial_\theta^k \left(\frac{z \cdot z_\theta}{1-|z|^2} \right)(\tau, \cdot) \right\|_{C^0(S^1,\mathbb{R})}, \tag{53}$$

$$d_k(\tau) = \left\| (1-|z|^2)\partial_\theta^k \left(\frac{z_\theta}{1-|z|^2} \right)(\tau, \cdot) \right\|_{C^0(S^1,\mathbb{R})}. \tag{54}$$

COROLLARY 4. *Consider a solution to (23), and assume that $\rho(\tau,\theta) \le \tau - 2$ for all $\tau \ge T$ and $\theta \in S^1$. Then for each $k \ge 0$ and $\tau \ge T$,*

$$d_k(\tau) + c_k(\tau) \le L_k \tau^{m_k}, \tag{55}$$

where $m_k = C_k[1 + G^{1/2}(T)T]^2$ and L_k is a polynomial in $e^{-T}F^c_{j+1}(T)$ and $c_j(T)$, $j = 0, \dots, k$.

Proof. Compute

$$\begin{aligned}\partial_\tau \partial_\theta^k \left(\frac{z \cdot z_\theta}{1-|z|^2} \right) &= \partial_\theta^k \left[z_\theta \cdot \frac{z_\tau}{1-|z|^2} + z \cdot \partial_\theta \left(\frac{z_\tau}{1-|z|^2} \right) \right] \\ &= \sum_{i+j=k+1} a_{ij} \partial_\theta^i z \cdot \partial_\theta^j \left(\frac{z_\tau}{1-|z|^2} \right).\end{aligned}$$

By the previous corollary, we get the conclusion that

$$c_k(\tau) \le c_k(T) + K_k \tau^{m_k}, \tag{56}$$

where $m_k = C_k[1 + G^{1/2}(T)T]^2$, and K_k is a polynomial in $e^{-T}F^c_{j+1}(T)$, $j = 0, \dots, k$. Note that

$$\partial_\theta^k \left(\frac{z_\theta}{1-|z|^2} \right) = \sum a_{i,j_1,\dots,j_m} \frac{\partial_\theta^i z}{1-|z|^2} \partial_\theta^{j_1} \left(\frac{z \cdot z_\theta}{1-|z|^2} \right) \cdots \partial_\theta^{j_m} \left(\frac{z \cdot z_\theta}{1-|z|^2} \right).$$

Multiplying this equation by $1-|z|^2$, we can bound the right-hand side as in the statement of the lemma due to the previous corollary and (56). □

Let us try to say something concerning the optimality of the estimates. Note that by [13] and [15], it is possible to construct solutions to the Gowdy equations with the asymptotics (13), (14) as long as $0 < v_a < 1$ and $v_a, \phi, q, \psi \in C^\infty(S^1, \mathbb{R})$. The functions u and w tend to zero as $\tau \to \infty$. Note that if $z = \phi_{RD} \circ (Q, P)$, where ϕ_{RD} is as defined in (19), then

$$\frac{4|z_\theta|^2}{(1-|z|^2)^2} = P_\theta^2 + e^{2P} Q_\theta^2.$$

Furthermore

$$1 - |z| \le 1 - |z|^2 \le 2(1-|z|), \quad e^{-\rho} \le 1 - |z| \le 2e^{-\rho}$$

so that $e^{-\rho} \leq 1-|z|^2 \leq 4e^{-\rho}$. Thus (49) implies $e^{2P}Q_\theta^2 \leq \Pi e^{2v_\infty \tau}$, where Π is a polynomial, since $\rho = v_\infty \tau + O(1)$. On the other hand, there is a point θ_0 at which $q_\theta(\theta_0) \neq 0$. Then

$$e^{2P(\tau,\theta_0)}Q_\theta^2(\tau,\theta_0) \approx c_0 e^{2v_\infty(\theta_0)\tau},$$

where $c_0 \neq 0$, since we have (13), (14) and $P = v_\infty \tau + O(1)$. We see that the only way the estimate $\|z_\theta\|_{C^0(S^1,\mathbb{R}^2)} \leq \Pi$ can be improved lies in the degree of the polynomial.

LEMMA 7. *Consider a solution to (23) and assume that $\rho(\tau,\theta) \leq \tau - 2$ for all $\tau \geq T$ and $\theta \in S^1$. Assume furthermore that $v_\infty \leq 1/4$. Then there are constants L_1, L_1', m_2 of the form*

$$L_1' = L_1 \exp\{C[1+G^{1/2}(T)T]\}, \quad m_2 = C[1+G^{1/2}(T)T]^2,$$

where L_1 is a polynomial in $e^{-T}F_{j+1}^c(T)$ and $c_j(T)$, $j=0,1$, such that if $\tau \geq m_2$ and $\theta \in S^1$, then

$$\left|\frac{2|z_\tau(\tau,\theta)|}{1-|z(\tau,\theta)|^2} - v_\infty(\theta)\right| \leq 2L_1' \exp[-2\tau + 2v_\infty(\theta)\tau]\tau^{m_2}.$$

Proof. Let us take the scalar product of (23) with $z_\tau/(1-|z|^2)$. We get

$$\left|\frac{z_\tau}{1-|z|^2}\cdot \partial_\tau\left(\frac{z_\tau}{1-|z|^2}\right)\right| \leq e^{-2\tau}\left[\left|\partial_\theta\left(\frac{z_\theta}{1-|z|^2}\right)\right| + 4\frac{|z_\theta|^2}{(1-|z|^2)^2}\right]\frac{|z_\tau|}{1-|z|^2},$$

where we have used the fact that $|z| \leq 1$. Let us introduce the function

$$f = \frac{|z_\tau|^2}{(1-|z|^2)^2}.$$

Due to the Corollaries 3 and 4, we get the conclusion that

$$|\partial_\tau f| \leq L_1 e^{-2\tau}(1-|z|^2)^{-2}\tau^{m_2} f^{1/2},$$

where L_1 is a polynomial in $e^{-T}F_{j+1}^c(T)$ and $c_j(T)$, $j=0,1$, and m_2 is as in the statement of the lemma. Note that

$$(1-|z|^2)^{-2} \leq e^{2\rho} \leq \exp[2v_\infty \tau + 12 G^{1/2}(T)T],$$

where we have used (21) and Lemma 2. Consequently

$$|\partial_\tau f| \leq L_1' e^{-2\tau + 2v_\infty \tau}\tau^{m_2} f^{1/2}, \tag{57}$$

where L_1' is as in the statement of the lemma. Let us assume that $v_\infty \leq 1/4$. Since

$$\partial_\tau(e^{-\tau}\tau^{m_2}) \leq 0$$

if $\tau \geq m_2$, we then get

$$\int_\tau^\infty e^{-2s+2v_\infty s}s^{m_2}ds \leq e^{-\tau}\tau^{m_2}\int_\tau^\infty e^{-s+2v_\infty s}ds \leq 2e^{-2\tau+2v_\infty\tau}\tau^{m_2}.$$

Using this estimate together with (57) and the fact that $f^{1/2}$ converges to $v_\infty/2$, we get the conclusion that

$$|2f^{1/2}(\tau,\theta) - v_\infty(\theta)| \le 2L_1' \exp[-2\tau + 2v_\infty(\theta)\tau]\tau^{m_2},$$

assuming $\tau \ge m_2$. □

Note that by arguments similar to ones given in the proof of the lemma,

$$\partial_\tau \partial_\theta^k \left[\frac{|z_\tau|^2}{(1-|z|^2)^2}\right]$$

converges to zero exponentially, when we assume $v_\infty \le 1-\gamma$ for some $\gamma > 0$, so that v_∞^2 is smooth under these assumptions. Using this observation, domain-of-dependence arguments and the fact that the velocity is continuous in a neighborhood of every point where it is zero, we get the conclusion that v_∞^2 is smooth in a neighborhood of every point where it is zero; cf. Lemma 14.

5. Perturbations of the initial data

Given a solution whose asymptotic velocity is not always positive, we wish to perturb the initial data at some late time T_1 in such a way that the perturbed solution never has zero velocity at the singularity. Furthermore, we wish to prove that if one lets T_1 tend to infinity in this construction, the perturbed solution converges to the solution one perturbed around, assuming the distance is measured in the C^∞ topology of initial data on some fixed Cauchy surface. The purpose of this section is to produce a candidate perturbation, and in later sections we prove that it has the properties we desire.

As a preparation for the construction, we make the following observation.

LEMMA 8. *Consider $\sigma \in C^1([a,b],\mathbb{R}^2)$. Let $\varepsilon > 0$ and define*

$$T_\varepsilon[\sigma] = \bigcup_{s\in[a,b]} B_\varepsilon[\sigma(s)],$$

where $B_\varepsilon(p)$ denotes the open ball with center p and radius ε. If μ denotes the Lebesgue measure on $\mathbb{R}^2$, then

$$\mu\{T_\varepsilon[\sigma]\} \le 4\pi\varepsilon l[\sigma] + 8\pi\varepsilon^2, \quad \textit{where} \quad l[\sigma] = \int_a^b |\sigma'(s)|ds. \tag{58}$$

Remark. The estimate is hardly optimal, but it will do for our purposes.

Proof. Define a sequence $s_0 \le s_1 \le \cdots \le s_k$ by the conditions:

$$s_0 = a, \quad \int_{s_i}^{s_{i+1}} |\sigma'(s)|ds = \varepsilon, \quad i = 0,\ldots,k-1, \quad \int_{s_k}^b |\sigma'(s)|ds \le \varepsilon.$$

Note that k could equal zero. We shall also denote b by s_{k+1}. Define

$$S_\varepsilon = \bigcup_{j=0}^{k+1} B_{2\varepsilon}[\sigma(s_j)].$$

Note that $T_\varepsilon[\sigma] \subseteq S_\varepsilon$. We get

$$\mu\{T_\varepsilon[\sigma]\} \le \mu[S_\varepsilon] \le (k+2)4\pi\varepsilon^2 \le 4\pi\varepsilon l[\sigma] + 8\pi\varepsilon^2,$$

since $k\varepsilon \le l[\sigma]$. The lemma follows. □

LEMMA 9. *Consider a solution to* (23) *with* $\rho(\tau,\theta) \le \tau - 2$ *for* $(\tau,\theta) \in [T,\infty)\times S^1$. *Let* $\alpha = 19/10$ *and* $\beta = 11/10$. *Then there is a* $T' \ge T$ *such that for any* $\tau \ge T'$, *there is a point* $p_0 \in \mathbb{R}^2$ *satisfying*

$$|p_0| \le e^{-\beta\tau} \quad \text{and} \quad \inf_{\theta\in S^1}\left|\frac{z_\tau(\tau,\theta)}{1-|z(\tau,\theta)|^2} - p_0\right| \ge e^{-\alpha\tau}.$$

In terms of data at T, *it is sufficient if*

$$T' = C\ln K + C[1+G^{1/2}(T)T]^4,$$

where K *is a polynomial in* $e^{-T}F^c_{j+1}(T)$, $j = 0,1$.

Proof. For the sake of brevity, let us introduce the notation

$$\gamma = \frac{z_\tau}{1-|z|^2}.$$

Due to the estimates (50), we have

$$\|\gamma(\tau,\cdot)\|_{C^2(S^1,\mathbb{R}^2)} \le K\tau^m, \tag{59}$$

for all $\tau \ge T$, where $m = C[1+G^{1/2}(T)T]^2$ and K is a polynomial in $e^{-T}F^c_{j+1}(T)$, $j = 0,1$. We wish to find a $p_0 \in \mathbb{R}^2$ such that

$$p_0 \in B_{r_\beta}(0) \quad \text{and} \quad B_{r_\alpha}(p_0) \cap \{\gamma(\tau,\theta) : \theta \in S^1\} = \emptyset, \tag{60}$$

where $r_\beta = e^{-\beta\tau}$ and $r_\alpha = e^{-\alpha\tau}$. Let us introduce the notation

$$A_\beta(\tau) = \{\theta \in S^1 : \gamma(\tau,\theta) \in B_{2r_\beta}(0)\},$$
$$A_{\alpha,\beta}(\tau) = \bigcup_{\theta\in A_\beta(\tau)} B_{r_\alpha}[\gamma(\tau,\theta)].$$

We wish to prove that

$$\mu[A_{\alpha,\beta}(\tau)] < \mu[B_{r_\beta}(0)]. \tag{61}$$

This would then immediately imply the existence of a $p_0 \in B_{r_\beta}(0) - A_{\alpha,\beta}(\tau)$. That p_0 has the first of the desired properties in (60) is clear. To prove that it has the second, let us assume the opposite. Then there is a θ such that $|p_0 - \gamma(\tau,\theta)| < r_\alpha$. Since $r_\alpha < r_\beta$, we conclude that

$$\gamma(\tau,\theta) \in B_{2r_\beta}(0),$$

which implies that $\theta \in A_\beta(\tau)$, and thus that $p_0 \in A_{\alpha,\beta}(\tau)$. We get a contradiction, and thus p_0 has the desired properties (60).

Note that in the estimate (58), there is a "boundary" term $8\pi\varepsilon^2$, which is a nuisance. The reason is the following. Say that $A_\beta(\tau)$ can be written as the union of intervals $I_1,\dots,I_k$, and say that we apply (58) to each of the intervals I_j. Then the first term in the estimate, $4\pi\varepsilon l[\sigma]$, is insensitive to the number k since it has nice additive properties, but the boundary term certainly is sensitive to how many times we enter $B_{2r_\beta}(0)$. There is a technical way around this. Consider only subintervals I of $[0,2\pi]$ such that the solution has to travel from $\partial B_{3r_\beta}(0)$ to $\partial B_{2r_\beta}(0)$ in the interval, and apply (58) to I. This leads to the conclusion that $l[\gamma(\tau,\cdot)|_I] \geq r_\beta$, and since we wish to use (58) with $\varepsilon = r_\alpha$, we see that the boundary term in this case is insignificant in comparison with the first term. Let us be more precise. Fix τ. Given a θ such that $|\gamma(\tau,\theta)| \leq 2r_\beta$, let I_θ be the maximal interval such that $|\gamma(\tau,\theta')| \leq 3r_\beta$ for all $\theta' \in I_\theta$. By continuity, $|\gamma(\tau,\theta')| = 3r_\beta$ on the boundary of I_θ, or $I_\theta = S^1$. The set N_β of points where $|\gamma(\tau,\theta)| \leq 2r_\beta$ is compact, and the interiors of the I_θ constitute an open covering. Let the interiors of $I_i = I_{\theta_i}$ $i = 1,\dots,k$ constitute a finite subcovering. Note that by maximality, if two intervals intersect each other, they have to coincide; otherwise the union would be the maximum interval. In other words, we can assume that the I_i have empty intersection. Note that if N_β is empty, $A_{\alpha,\beta}(\tau)$ is empty, which is an unproblematic special case. Let us therefore assume that $k \geq 1$. The set

$$A'_{\alpha,\beta}(\tau) = \bigcup_{j=1}^{k} T_{r_\alpha}[\gamma(\tau,\cdot)|_{I_j}]$$

contains $A_{\alpha,\beta}(\tau)$, and we shall estimate its measure. Note that if $\gamma(\tau,\cdot)$ never leaves $B_{3r_\beta}(0)$, then $k = 1$ and $I_1 = S^1$. If it does leave, we have the estimate

$$l[\gamma(\tau,\cdot)|_{I_j}] \geq r_\beta$$

for all j. Since $r_\alpha \leq r_\beta$, we thus get

$$\mu\{T_{r_\alpha}[\gamma(\tau,\cdot)|_{I_j}]\} \leq 12\pi r_\alpha l[\gamma(\tau,\cdot)|_{I_j}].$$

Consequently

$$\mu[A_{\alpha,\beta}(\tau)] \leq \mu[A'_{\alpha,\beta}(\tau)] \leq 12\pi r_\alpha \sum_{j=1}^{k} l[\gamma(\tau,\cdot)|_{I_j}]. \tag{62}$$

What remains to be estimated is

$$\int_{S(\tau)} |(\partial_\theta \gamma)(\tau, \theta)| d\theta, \quad \text{where} \quad S(\tau) = \bigcup_{j=1}^{k} I_j.$$

Let $\delta = \beta/3$ and define

$$S_{\delta,1}(\tau) = \{\theta \in S(\tau) : |(\partial_\theta \gamma)(\tau, \theta)| \le r_\delta\}, \quad S_{\delta,2}(\tau) = \{\theta \in S(\tau) : |(\partial_\theta \gamma)(\tau, \theta)| \ge r_\delta\}$$

where $r_\delta = e^{-\delta\tau}$. Since

$$\int_{S_{\delta,1}(\tau)} |(\partial_\theta \gamma)(\tau, \theta)| d\theta \le 2\pi r_\delta, \tag{63}$$

we shall only be concerned with the set $S_{\delta,2}(\tau)$. Consider S^1 to be the interval $[0, 2\pi]$ with the endpoints identified, and let $J = [\phi_1, \phi_2] \subseteq S_{\delta,2}(\tau)$ be maximal; i.e. any larger interval will contain a point in the complement of $S_{\delta,2}(\tau)$. Let

$$\phi_3 = \phi_1 + \frac{r_\delta}{4K\tau^m},$$

where K and m are the constants that appear in (59), and define $v_1 = \partial_\theta \gamma(\tau, \phi_1)$. By assumption, $|v_1| \ge r_\delta$, and by the bound on the second derivative of γ, (59), we get the conclusion that for $\theta \in [\phi_1, \phi_3]$,

$$|(\partial_\theta \gamma)(\tau, \theta) - v_1| \le \frac{1}{4}|v_1|. \tag{64}$$

Let us estimate the distance the curve γ has carried out in the direction $\hat{v}_1 = v_1/|v_1|$ during an interval $[\phi_1, \phi] \subseteq [\phi_1, \phi_3]$. Using (64), we get the conclusion that

$$[\gamma(\tau, \phi) - \gamma(\tau, \phi_1)] \cdot \hat{v}_1 \ge \frac{3}{4}(\phi - \phi_1)|v_1|.$$

Note that if $(\phi - \phi_1)|v_1| \ge 9r_\beta$, then $\phi \notin S_{\delta,2}(\tau)$. This inequality holds if $\phi \ge \phi_4$, where $\phi_4 = \phi_1 + 9e^{-2\delta\tau}$. We assume that τ is great enough that

$$9e^{-2\delta\tau} \le \frac{e^{-\delta\tau}}{4K\tau^m}. \tag{65}$$

Note that $J \subseteq [\phi_1, \phi_4]$ and that $[\phi_4, \phi_3] \cap S_{\delta,2}(\tau) = \emptyset$. In particular,

$$\frac{|\phi_2 - \phi_1|}{|\phi_3 - \phi_1|} \le CK\tau^m e^{-\delta\tau}.$$

For every maximal interval J in $S_{\delta,2}(\tau)$, except for possibly the last one, there is thus an interval $J \subseteq \hat{J}$, whose left boundary point coincides with that of J, such that if μ_1 is the Lebesgue measure on $\mathbb{R}$,

$$\frac{\mu_1[\hat{J} \cap S_{\delta,2}(\tau)]}{\mu_1[\hat{J}]} \le CK\tau^m e^{-\delta\tau}.$$

Due to this estimate and the fact that one maximal interval does not add more than $e^{-2\delta\tau}$ to the measure, we have

$$\mu_1[S_{\delta,2}(\tau)] \le CK\tau^m e^{-\delta\tau}.$$

Using the estimate (59) again, we have

$$\begin{aligned}\int_{S(\tau)} |(\partial_\theta\gamma)(\tau,\theta)|d\theta &\le \int_{S_{\delta,1}(\tau)} |(\partial_\theta\gamma)(\tau,\theta)|d\theta + \int_{S_{\delta,2}(\tau)} |(\partial_\theta\gamma)(\tau,\theta)|d\theta \\ &\le 2\pi r_\delta + CK^2\tau^{2m}e^{-\delta\tau} \le CK^2\tau^{2m}e^{-\delta\tau}.\end{aligned}$$

By (62), we conclude that

$$\mu[A_{\alpha,\beta}(\tau)] \le CK^2\tau^{2m}e^{-(\alpha+\delta)\tau}.$$

In order to obtain (61), we require

$$CK^2\tau^{2m}e^{-(\alpha+\delta)\tau} < \pi e^{-2\beta\tau}. \tag{66}$$

This inequality is satisfied for τ large enough if $\alpha+\delta > 2\beta$, i.e. if $\alpha > 5\delta$. However, $\alpha - 5\delta = 1/15$. Both (65) and (66) follow from $\tau \ge C\ln K + Cm\ln\tau$, which follows from $\tau \ge C\ln K$ and $\tau \ge Cm\ln\tau$. The last of these inequalities follows from $\tau^{1/2} \ge Cm$ and the fact that $\tau^{1/2} \ge \ln\tau$. Also, the last of these inequalities holds if $\tau \ge 4$. The lemma follows. □

6. **Perturbations, basic identities**

Let z and $\tilde{z}$ be two solutions to (23), and let $\hat{z} = z - \tilde{z}$. Define

$$\hat{a}_k = \partial_\theta^k\left(\frac{\hat{z}_\tau}{1-|z|^2}\right), \quad \hat{b}_k = e^{-\tau}\partial_\theta^k\left(\frac{\hat{z}_\theta}{1-|z|^2}\right), \quad \hat{\mathscr{A}}_{k,\pm} = 2e^\tau|\hat{a}_k \pm \hat{b}_k|^2.$$

Let us compute

$$\begin{aligned}(\partial_\tau \mp e^{-\tau}\partial_\theta)\hat{\mathscr{A}}_{k,\pm} = 2e^\tau\Big\{&|\hat{a}_k|^2 - |\hat{b}_k|^2 + 2\partial_\theta^k\Big[\partial_\tau\hat{a}_0 - e^{-\tau}\partial_\theta\hat{b}_0 \\ &\pm e^{-\tau}\partial_\tau\left(\frac{\hat{z}_\theta}{1-|z|^2}\right) \mp e^{-\tau}\partial_\theta\left(\frac{\hat{z}_\tau}{1-|z|^2}\right)\Big]\cdot(\hat{a}_k \pm \hat{b}_k)\Big\}.\end{aligned}$$

Furthermore,

$$\partial_\tau\hat{a}_0 - e^{-\tau}\partial_\theta\hat{b}_0 = I_1 + I_2,$$

where, by the definition (24),

$$I_1 = \frac{2\mathfrak{Q}(z,\partial z)}{(1-|z|^2)^2} - \frac{2\mathfrak{Q}(\tilde{z},\partial\tilde{z})}{(1-|z|^2)(1-|\tilde{z}|^2)}$$

and

$$I_2 = -\frac{\tilde{z}_\tau}{1-|\tilde{z}|^2}\left\{-2\frac{\tilde{z}\cdot\tilde{z}_\tau}{1-|z|^2} + 2(z\cdot z_\tau)\frac{1-|\tilde{z}|^2}{(1-|z|^2)^2}\right\}$$
$$+e^{-2\tau}\frac{\tilde{z}_\theta}{1-|\tilde{z}|^2}\left\{-2\frac{\tilde{z}\cdot\tilde{z}_\theta}{1-|z|^2} + 2(z\cdot z_\theta)\frac{1-|\tilde{z}|^2}{(1-|z|^2)^2}\right\}.$$

Finally, let

$$I_3 = e^{-\tau}\partial_\tau\left(\frac{\hat{z}_\theta}{1-|z|^2}\right) - e^{-\tau}\partial_\theta\left(\frac{\hat{z}_\tau}{1-|z|^2}\right)$$
$$= \frac{e^{-\tau}\hat{z}_\theta}{1-|z|^2}\frac{2z\cdot z_\tau}{1-|z|^2} - \frac{\hat{z}_\tau}{1-|z|^2}\frac{2e^{-\tau}z\cdot z_\theta}{1-|z|^2}.$$

With this notation,

$$(\partial_\tau \mp e^{-\tau}\partial_\theta)\widehat{\mathscr{A}}_{k,\pm} = 2e^\tau\left\{|\hat{a}_k|^2 - |\hat{b}_k|^2 + 2\partial_\theta^k(I_1+I_2\pm I_3)\cdot(\hat{a}_k\pm\hat{b}_k)\right\}. \tag{67}$$

Consider, for some $\varepsilon > 0$,

$$\widehat{\mathscr{C}}_k = \frac{1}{2}\varepsilon^2 e^\tau|\hat{c}_k|^2, \quad \text{where} \quad \hat{c}_k = \partial_\theta^k\left(\frac{\hat{z}}{1-|z|^2}\right).$$

Now,

$$(\partial_\tau \pm e^{-\tau}\partial_\theta)\widehat{\mathscr{C}}_k = \widehat{\mathscr{C}}_k + \varepsilon^2 e^\tau\partial_\theta^k\left(\frac{\hat{z}_\tau \pm e^{-\tau}\hat{z}_\theta}{1-|z|^2}\right)\cdot\hat{c}_k$$
$$+\varepsilon^2 e^\tau\partial_\theta^k\left[\frac{\hat{z}}{1-|z|^2}\frac{2z\cdot(z_\tau\pm e^{-\tau}z_\theta)}{1-|z|^2}\right]\cdot\hat{c}_k. \tag{68}$$

7. **Perturbations, convergence**

We consider a solution to (23), and assume that

$$\left\|\frac{z_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\right\|_{C^0(S^1,\mathbb{R}^2)} + \left\|\frac{e^{-\tau}z_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\right\|_{C^0(S^1,\mathbb{R}^2)} \le \varepsilon, \tag{69}$$

and $\rho(\tau,\theta) \le \tau - 2$ for $\tau \ge T$, where $\varepsilon > 0$. We are interested in modifying the initial data at $T_1 \ge T$, by letting

$$\left(\frac{\hat{z}_\tau}{1-|z|^2}\right)(T_1,\cdot) = c_{T_1}, \quad \tilde{z}(T_1,\cdot) = z(T_1,\cdot), \tag{70}$$

where c_{T_1} is a constant satisfying

$$|c_{T_1}| \le e^{-\beta T_1}, \tag{71}$$

for some $\beta > 1$. In fact we shall take c_{T_1} to be the point p_0 whose existence is guaranteed by Lemma 9, and so, in particular, we can take $\beta = 11/10$. Note that

(70) and (71) lead to the conclusion that $\widehat{\mathscr{C}}_k(T_1,\cdot)=0$ for all k, that $\widehat{\mathscr{A}}_{k,\pm}(T_1,\cdot)=0$ for all $k\geq 1$, and that

$$|\widehat{\mathscr{A}}_{0,\pm}(T_1,\cdot)|\leq 2e^{(1-2\beta)T_1}. \tag{72}$$

Note that we shall keep β fixed and let T_1 tend to infinity. Let us fix k and make the following *bootstrap assumptions*:

$$\left\|\frac{\hat{z}(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\right\|_{C^0(S^1,\mathbb{R}^2)}\leq\varepsilon, \tag{73}$$

$$\left\|\frac{e^{-\tau}\hat{z}_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\right\|_{C^0(S^1,\mathbb{R}^2)}+\left\|\frac{\hat{z}_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\right\|_{C^0(S^1,\mathbb{R}^2)}\leq\varepsilon, \tag{74}$$

$$\left\|\frac{e^{-\tau}\hat{z}_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\right\|_{C^k(S^1,\mathbb{R}^2)}+\left\|\frac{\hat{z}_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\right\|_{C^k(S^1,\mathbb{R}^2)}\leq 1, \tag{75}$$

$$\left\|\frac{\hat{z}(\tau,\cdot)}{1-|z(\tau,\cdot)|^2}\right\|_{C^k(S^1,\mathbb{R}^2)}\leq 1. \tag{76}$$

Note that for T_1 great enough, the bootstrap assumptions are satisfied in a neighborhood of $\tau=T_1$. We shall assume that the above inequalities are satisfied in the interval $[T_2,T_1]$ for some $T_2\in[T,T_1]$. We shall then use the assumptions to prove that for a fixed β, ε small enough and T_1 large enough, we obtain an improvement of the estimates as a conclusion. This then implies the validity of the bootstrap assumptions on the entire interval $[T,T_1]$. It is perhaps of some interest to point out that in the end, ε is only required to be smaller than a *numerical* constant *independent* of the solution. Let us introduce some notation.

Definition 6. Let z be a solution to (23) with the property that (69) holds for some $0<\varepsilon\leq 1/4$ and all $\tau\geq T$, and $\rho(\tau,\theta)\leq\tau-2$ for all $\tau\geq T$ and all $\theta\in S^1$. Then z is said to be an ε,T-*solution*. Given an ε,T-solution z, let $\tilde{z}$ be a solution to (23) defined by (70), where c_{T_1} is some constant satisfying (71), where $\beta=11/10$ and $T_1\geq T$. Then $\tilde{z}$ is said to be a T_1,z-*solution*. Given an ε,T-solution z, a constant K_k which is a polynomial in $e^{-T}F_j^c[z](T)$, $j=1,\dots,k$ is called a $K_k[z]$-*constant*, a constant m_k of the form $C_k[1+G^{1/2}[z](T)T]^2$ is referred to as an $m_k[z]$-*constant* and a constant L_k which is a polynomial in $e^{-T}F_{j+1}^c[z](T)$ and $c_j[z](T)$ for $j=0,\dots,k-1$ is called an $L_k[z]$-*constant*.

Let us write down some *consequences of the bootstrap assumptions*. We shall always assume $\varepsilon\leq 1/4$, so that (73) implies

$$\frac{1}{2}\leq\frac{1-|\tilde{z}|^2}{1-|z|^2}\leq\frac{3}{2}. \tag{77}$$

Combining (69) with (74), we conclude that

$$\left\| \frac{e^{-\tau}\tilde{z}_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} \right\|_{C^0(S^1,\mathbb{R}^2)} + \left\| \frac{\tilde{z}_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} \right\|_{C^0(S^1,\mathbb{R}^2)} \leq 2\varepsilon. \tag{78}$$

Combining (75) with (50), we conclude that

$$\left\| \frac{e^{-\tau}\tilde{z}_\theta(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} \right\|_{C^k(S^1,\mathbb{R}^2)} + \left\| \frac{\tilde{z}_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} \right\|_{C^k(S^1,\mathbb{R}^2)} \leq K_k \tau^{m_k}, \tag{79}$$

where K_k is a $K_k[z]$-constant and m_k is an $m_k[z]$-constant. Note that z is a *fixed solution*. Compute

$$\partial_\theta^j \tilde{z} = -\partial_\theta^j \hat{z} + \partial_\theta^j z = -\sum_{l=0}^{j} \binom{j}{l} \partial_\theta^l \left(\frac{\hat{z}}{1-|z|^2} \right) \partial_\theta^{j-l}(1-|z|^2) + \partial_\theta^j z.$$

Using (50) and (76), we conclude that

$$\|\tilde{z}\|_{C^k(S^1,\mathbb{R})} \leq K_k \tau^{m_k}, \tag{80}$$

where K_k and m_k have the same structure as above. Consider

$$\frac{\hat{z}_\theta}{1-|z|^2} = \partial_\theta \left(\frac{\hat{z}}{1-|z|^2} \right) - 2\frac{z\cdot z_\theta}{1-|z|^2}\frac{\hat{z}}{1-|z|^2}.$$

Using this identity together with (76) and (55), we conclude that

$$\left\| \frac{\hat{z}_\theta}{1-|z|^2} \right\|_{C^{k-1}(S^1,\mathbb{R}^2)} \leq L_k \tau^{m_k}, \tag{81}$$

where L_k is an $L_k[z]$-constant. Since

$$\frac{z\cdot z_\theta - \tilde{z}\cdot\tilde{z}_\theta}{1-|z|^2} = \frac{\hat{z}\cdot z_\theta + \tilde{z}\cdot\hat{z}_\theta}{1-|z|^2},$$

we conclude that

$$\left\| \frac{\tilde{z}\cdot\tilde{z}_\theta}{1-|z|^2} \right\|_{C^{k-1}(S^1,\mathbb{R})} \leq L_k \tau^{m_k}, \tag{82}$$

where L_k and m_k are of the same form as above. Finally,

$$\partial_\theta \left(\frac{1-|z|^2}{1-|\tilde{z}|^2} \right) = -\frac{2z\cdot z_\theta}{1-|z|^2}\frac{1-|z|^2}{1-|\tilde{z}|^2} + \frac{(1-|z|^2)^2}{(1-|\tilde{z}|^2)^2}\frac{2\tilde{z}\cdot\tilde{z}_\theta}{1-|z|^2}.$$

Using this identity and the above inequalities, we inductively conclude that

$$\left\| \frac{1-|z|^2}{1-|\tilde{z}|^2} \right\|_{C^k(S^1,\mathbb{R})} \leq L_k \tau^{m_k}. \tag{83}$$

7.1. *Notation.* Let us introduce the notation

$$\widehat{\mathscr{A}}^c_{k,\pm} = \widehat{\mathscr{A}}_{k,\pm} + \widehat{\mathscr{C}}_k, \quad \widehat{F}^c_{k,\pm}(\tau) = \sup_{\theta\in S^1} \widehat{\mathscr{A}}^c_{k,\pm}(\tau,\theta), \quad \widehat{F}^c_k = \widehat{F}^c_{k,+} + \widehat{F}^c_{k,-}.$$

Note that

$$2e^\tau[|\hat{a}_k|^2 + |\hat{b}_k|^2] + \widehat{\mathscr{C}}_k \le \frac{1}{2}[\widehat{\mathscr{A}}^c_{k,+} + \widehat{\mathscr{A}}^c_{k,-}] \le \frac{1}{2}\widehat{F}^c_k.$$

7.2. *The zeroth order.* We consider the consequences of the bootstrap assumptions in the case $k = 0$.

LEMMA 10. *Let z be an ε, T-solution and $\tilde{z}$ a T_1, z-solution. Assume furthermore that z and $\tilde{z}$ satisfy the bootstrap assumptions* (73), (74) *in an interval* $[T_2, T_1]$. *Then, for $\tau \in [T_2, T_1]$,*

$$\widehat{F}^c_0(\tau) \le \widehat{F}^c_0(T_1) + \int_\tau^{T_1} (1 + C\varepsilon)\widehat{F}^c_0(s)ds. \tag{84}$$

Proof. Let us estimate $|I_i|$, $i = 1, 2, 3$. Consider I_1. We exchange one factor $(1-|z|^2)^{-1}$ in the first term with $(1-|\tilde{z}|^2)^{-1}$. To this end, we use (77) to estimate

$$\left|\frac{1}{1-|z|^2} - \frac{1}{1-|\tilde{z}|^2}\right| \le \frac{2|\hat{z}|}{(1-|z|^2)(1-|\tilde{z}|^2)} \le \frac{4|\hat{z}|}{(1-|z|^2)^2}.$$

Using (69), (78) and this sort of estimate, we conclude that

$$|I_i| \le C\varepsilon^2|\hat{c}_0| + C\varepsilon(|\hat{a}_0| + |\hat{b}_0|)$$

if $i = 1$. In fact, the same type of estimate holds if $i = 2, 3$. Using (67), we conclude that

$$\left|(\partial_\tau \mp e^{-\tau}\partial_\theta)\widehat{\mathscr{A}}_{0,\pm}\right| \le 2e^\tau\left\{|\hat{a}_0|^2 + |\hat{b}_0|^2 + C\varepsilon(|\hat{a}_0|^2 + |\hat{b}_0|^2) + C\varepsilon^3|\hat{c}_0|^2\right\}.$$

Using (68), we get

$$\left|(\partial_\tau \pm e^{-\tau}\partial_\theta)\widehat{\mathscr{C}}_0\right| \le \widehat{\mathscr{C}}_0 + C\varepsilon e^\tau\left[|\hat{a}_0|^2 + |\hat{b}_0|^2 + \varepsilon^2|\hat{c}_0|^2\right].$$

Let $\tau \in [T_2, T_1]$ and estimate

$$\begin{aligned}\widehat{\mathscr{A}}^c_{0,\pm}(\tau, \theta \pm e^{-\tau}) &= \widehat{\mathscr{A}}^c_{0,\pm}(T_1, \theta \pm e^{-T_1}) \\ &\quad - \int_\tau^{T_1}\left[(\partial_\tau \mp e^{-\tau}\partial_\theta)\widehat{\mathscr{A}}^c_{0,\pm}\right](s, \theta \pm e^{-s})ds \\ &\le \widehat{F}^c_{0,\pm}(T_1) + \int_\tau^{T_1}\left[\frac{1}{2} + C\varepsilon\right]\widehat{F}^c_0(s)ds.\end{aligned}$$

Taking the supremum over θ and adding the two estimates, we get (84). □

7.3. *Higher orders.* To be able to deal with the higher orders, we shall in the end have to carry out an induction argument. In preparation for this, we prove the following lemma.

LEMMA 11. *Let z be an ε, T-solution and $\tilde{z}$ a T_1, z-solution. Assume furthermore that z and $\tilde{z}$ satisfy the bootstrap assumptions (73)–(76) in an interval $[T_2, T_1]$. Then, for $\tau \in [T_2, T_1]$ and $j = 1, \dots, k$,*

$$\widehat{F}_j^c(\tau) \le \int_\tau^{T_1} [(1 + C\varepsilon)\widehat{F}_j^c(s) + e^{s/2} R_j(s)(\widehat{F}_j^c)^{1/2}(s)]ds, \tag{85}$$

where C is a numerical constant independent of j and R_j satisfies the estimate

$$R_j(\tau) \le \varepsilon^{-1} L_j \tau^{m_j} e^{-\tau/2} \sum_{i=0}^{j-1} (\widehat{F}_i^c)^{1/2}(\tau) \tag{86}$$

for all $\tau \in [T_2, T_1]$. Here L_j is an $L_j[z]$-constant and m_j is an $m_j[z]$-constant.

Proof. Let us consider $\partial_\theta^j I_i$ for $i = 1, 2, 3$. Let us divide I_1 into a sum of I_{11} and I_{12}, where

$$I_{11} = \frac{2}{(1-|z|^2)^2}\left[\mathfrak{Q}(z, \partial z) - \mathfrak{Q}(\tilde{z}, \partial\tilde{z})\right], \quad I_{12} = \frac{2(|z|^2 - |\tilde{z}|^2)}{(1-|z|^2)^2(1-|\tilde{z}|^2)}\mathfrak{Q}(\tilde{z}, \partial\tilde{z}).$$

It is convenient to divide I_{11} into the sum of I_{111} and I_{112}, where

$$I_{111} = \frac{2}{(1-|z|^2)^2}\left[\mathfrak{Q}(z, \partial z) - \mathfrak{Q}(z, \partial\tilde{z})\right], \quad I_{112} = \frac{2}{(1-|z|^2)^2}\mathfrak{Q}(\hat{z}, \partial\tilde{z}).$$

Most of the terms that appear when computing the derivative can be estimated by

$$L_j \tau^{m_j} \sum_{i=0}^{j-1} (|\hat{a}_i| + |\hat{b}_i| + |\hat{c}_i|) \le \varepsilon^{-1} L_j \tau^{m_j} e^{-\tau/2} \sum_{i=0}^{j-1} (\widehat{F}_i^c)^{1/2}(\tau). \tag{87}$$

We shall denote terms that can be estimated in this fashion by $\mathfrak{R}$, possibly with some suitable index. Let us consider the jth derivative of a representative term in I_{111}, namely

$$\begin{aligned}
\partial_\theta^j\left(\frac{2(|z_\tau|^2 - |\tilde{z}_\tau|^2)z}{(1-|z|^2)^2}\right) &= \partial_\theta^j\left(\frac{2(z_\tau \cdot \hat{z}_\tau + \tilde{z}_\tau \cdot \hat{z}_\tau)z}{(1-|z|^2)^2}\right) \\
&= 2\left[\partial_\theta^j\left(\frac{\hat{z}_\tau}{1-|z|^2}\right) \cdot \frac{z_\tau}{1-|z|^2}\right. \\
&\qquad \left. + \partial_\theta^j\left(\frac{\hat{z}_\tau}{1-|z|^2}\right) \cdot \frac{\tilde{z}_\tau}{1-|z|^2}\right] z + \mathfrak{R}.
\end{aligned}$$

In order to arrive at this conclusion, we of course have to use the bootstrap assumptions (73)–(76) and their consequences (77)–(83). We shall use these inequalities without further comment in the following. For the remaining terms in I_{111}, we have similar expressions, and we obtain the estimate

$$|\partial_\theta^j I_{111}| \le C\varepsilon(|\hat{a}_j| + |\hat{b}_j|) + |\mathcal{R}_{111,j}|.$$

Note that C does not depend on j. Let us consider I_{112}. Due to the definition of the energy and of the corrections, it is convenient to pair together z_τ, $e^{-\tau} z_\theta$ and $\hat{z}$ with factors $(1-|z|^2)^{-1}$. This leaves one factor $1-|z|^2$. Considering a representative term in I_{112}, we get

$$\partial_\theta^j \left(\frac{2|\tilde{z}_\tau|^2 \hat{z}}{(1-|z|^2)^2} \right) = \frac{2|\tilde{z}_\tau|^2}{1-|z|^2} \partial_\theta^j \left(\frac{\hat{z}}{1-|z|^2} \right) + \mathcal{R}. \tag{88}$$

The arguments for the other terms are similar, and we conclude that

$$|\partial_\theta^j I_{11}| \le C\varepsilon(|\hat{a}_j| + |\hat{b}_j| + \varepsilon|\hat{c}_j|) + |\mathcal{R}_{11,j}|.$$

Consider I_{12}. Note that we can write it as

$$I_{12} = \frac{2(z\cdot\hat{z} + \tilde{z}\cdot\hat{z})}{1-|z|^2} \frac{1-|z|^2}{1-|\tilde{z}|^2} \frac{\mathfrak{Q}(\tilde{z}, \partial\tilde{z})}{(1-|z|^2)^2}.$$

When differentiating, we pair $\hat{z}$ with $(1-|z|^2)^{-1}$ in the first factor, and in the third factor, we pair together each derivative with a factor $(1-|z|^2)^{-1}$. The important terms that result after differentiation are the ones in which all the derivatives hit $\hat{z}/(1-|z|^2)$. We have

$$|\partial_\theta^j I_{12}| \le C\varepsilon^2 |\hat{c}_j| + |\mathcal{R}_{12,j}|.$$

Let us consider I_2. It is convenient to write it as the sum of two terms, I_{21} and I_{22}, where

$$\begin{aligned} I_{21} &= -\frac{\tilde{z}_\tau}{1-|\tilde{z}|^2} \left\{ -2\frac{\tilde{z}\cdot\tilde{z}_\tau}{1-|z|^2} + 2(z\cdot z_\tau)\frac{1-|\tilde{z}|^2}{(1-|z|^2)^2} \right\} \\ &= -\frac{\tilde{z}_\tau}{1-|z|^2} \frac{1-|z|^2}{1-|\tilde{z}|^2} \left\{ 2\frac{\hat{z}\cdot z_\tau + \tilde{z}\cdot\hat{z}_\tau}{1-|z|^2} + 2\frac{z\cdot z_\tau}{1-|z|^2} \frac{\hat{z}\cdot z + \tilde{z}\cdot\hat{z}}{1-|z|^2} \right\}. \end{aligned}$$

When differentiating, a derivative should always be paired together with a factor of $(1-|z|^2)^{-1}$, and similarly for $\hat{z}$. Finally, the quotient $(1-|z|^2)/(1-|\tilde{z}|^2)$ should be viewed as one unit. In particular, before differentiating, we write

$$\frac{\hat{z}\cdot z_\tau}{1-|z|^2} = (1-|z|^2)\frac{\hat{z}}{1-|z|^2} \cdot \frac{z_\tau}{1-|z|^2}.$$

Again, the only terms that cannot be estimated as in (87) arise when all the derivatives hit the terms involving $\hat{z}$ or $\hat{z}_\tau$. The argument concerning I_{22} is practically identical, and we get

$$|\partial_\theta^j I_2| \le C\varepsilon(|\hat{a}_j| + |\hat{b}_j| + \varepsilon|\hat{c}_j|) + |\mathcal{R}_{2,j}|.$$

Finally, we can treat I_3 similarly to the above expressions, and we obtain

$$|\partial_\theta^j I_3| \leq C\varepsilon(|\hat{a}_j| + |\hat{b}_j|) + |\mathcal{R}_{3,j}|.$$

Adding up, we get

$$4e^\tau |\partial_\theta^j (I_1 + I_2 \pm I_3) \cdot (\hat{a}_j \pm \hat{b}_j)| \\ \leq C\varepsilon e^\tau (|\hat{a}_j|^2 + |\hat{b}_j|^2 + \varepsilon^2 |\hat{c}_j|^2) + e^\tau |\mathcal{R}_j|(|\hat{a}_j| + |\hat{b}_j|).$$

Combining this with (67) and (68), we conclude that

$$|(\partial_\tau \mp e^{-\tau}\partial_\theta)\widehat{\mathcal{A}}^c_{j,\pm}| \leq \left(\frac{1}{2} + C\varepsilon\right)(\widehat{\mathcal{A}}^c_{j,+} + \widehat{\mathcal{A}}^c_{j,-}) + e^{\tau/2}|\mathcal{R}_j|(\widehat{\mathcal{A}}^c_{j,+} + \widehat{\mathcal{A}}^c_{j,-})^{1/2}.$$

We can argue as in the case $j = 0$ in order to obtain

$$\begin{aligned} \widehat{F}^c_j(\tau) &\leq \widehat{F}^c_j(T_1) + \int_\tau^{T_1} [(1 + C\varepsilon)\widehat{F}^c_j(s) + e^{s/2}\|\mathcal{R}_j(s,\cdot)\|_{C^0(S^1,\mathbb{R})}(\widehat{F}^c_j)^{1/2}(s)]ds \\ &= \int_\tau^{T_1} [(1 + C\varepsilon)\widehat{F}^c_j(s) + e^{s/2}\|\mathcal{R}_j(s,\cdot)\|_{C^0(S^1,\mathbb{R})}(\widehat{F}^c_j)^{1/2}(s)]ds, \end{aligned}$$

since $\widehat{F}^c_j(T_1) = 0$ by definition. The lemma follows. □

7.4. *Induction argument.* We are now in a position to put together the previous two lemmas in order to control the size of $\hat{z}$ and $\hat{z}_\tau$ at $\tau = T$.

LEMMA 12. *There is an* $0 < \varepsilon_0 \leq 1/200$ *such that the following holds. Let* z *be an* ε_0, T*-solution and* $\tilde{z}$ *a* T_1, z*-solution. Fix* k*. Then there is a* $T_{1,k}$*, depending continuously on* $e^{-T} F^c_{j+1}[z](T)$*,* $c_j[z](T)$ *for* $j = 0, \ldots, k-1$ *and* $G^{1/2}[z](T)T$*, such that if* $T_1 \geq T_{1,k}$*,* $j = 0, \ldots, k$ *and* $\tau \in [T, T_1]$*, then*

$$e^{-\tau}\widehat{F}^c_j(\tau) \leq \varepsilon_0^{-2j} L_j T_1^{m_j} \exp[-(\beta - 1)T_1 - (2 + \kappa_0\varepsilon_0)\tau], \tag{89}$$

where κ_0 *is a positive numerical constant,* L_j *is an* $L_j[z]$*-constant and* m_j *is an* $m_j[z]$*-constant.*

Remark. The condition that $\varepsilon_0 \leq 1/200$ will be needed in the proof of Theorem 3. We take it to be understood that $\varepsilon = \varepsilon_0$ in the definition of $\mathcal{C}_j$, and thus in the definition of $\widehat{F}^c_j$.

Proof. Before proceeding to the proof, let us make some preliminary observations. Note that the constant C appearing in (85) is independent of j so that we can assume it coincides with the constant C appearing in (84). We denote the common constant by κ_0. Let us define ε_0 by

$$\varepsilon_0 = \min\left\{\varepsilon_{0,1}, \frac{1}{200}\right\}, \quad \text{where} \quad (\kappa_0 + 1)\varepsilon_{0,1} = \beta - 1 = \frac{1}{10}.$$

Let us assume that the bootstrap assumptions (73)–(76) are satisfied in $[T_2, T_1]$. As long as T_1 is large enough, depending only on β and ε_0 (i.e. on numerical constants), the bootstrap assumptions are fulfilled in a neighborhood of T_1. Thus we know that $[T_2, T_1]$ is nonempty. What remains to be shown is that, assuming T_1 to be large enough, depending on the objects mentioned in the lemma, T_2 can be taken to equal T. This will follow if we can prove that the bootstrap assumptions imply an improvement of themselves.

Let us first prove (89) for $j = 0$. By (84) and a Grönwall's lemma type argument, we get

$$\hat{F}_0^c(\tau) \le \hat{F}_0^c(T_1) \exp\{(1+\kappa_0\varepsilon_0)(T_1-\tau)\}.$$

Due to the comments made in connection with (72) and the definition of $\hat{F}_0^c$, we conclude that

$$\begin{aligned} e^{-\tau}\hat{F}_0^c(\tau) &\le C\exp[(2+\kappa_0\varepsilon_0-2\beta)T_1-(2+\kappa_0\varepsilon_0)\tau] \\ &\le C\exp[-(\beta-1)T_1-(2+\kappa_0\varepsilon_0)\tau], \end{aligned}$$

since $\kappa_0\varepsilon_0 \le \beta - 1$. In other words, (89) holds for $j = 0$ with L_0 a numerical constant and $m_0 = 0$. For T_1 large enough, we get the conclusion that the right-hand side is less than $\varepsilon_0^4/16$. This reproduces (73) and (74) with a margin.

Assume inductively that (89) is true for $j-1$, where $j \ge 1$. Due to (86) and the inductive assumption, we get

$$R_j(\tau) \le \varepsilon_0^{-j} L_j T_1^{m_j} \exp\left[-\frac{1}{2}(\beta-1)T_1 - \left(1+\frac{1}{2}\kappa_0\varepsilon_0\right)\tau\right], \tag{90}$$

where we used the fact that $\tau \le T_1$. Let us denote the right-hand side of (85) by h, and define $g = h\exp[(1+\kappa_0\varepsilon_0)\tau]$. Estimate, using (85) and (90),

$$g' \ge -\varepsilon_0^{-j} L_j T_1^{m_j} \exp\left[-\frac{1}{2}(\beta-1)T_1\right] g^{1/2}.$$

Integrating this inequality yields, since $T \ge 0$,

$$(\hat{F}_j^c)^{1/2}(\tau) \le \varepsilon_0^{-j} L_j T_1^{m_j+1} \exp\left[-\frac{1}{2}(\beta-1)T_1 - \frac{1}{2}(1+\kappa_0\varepsilon_0)\tau\right],$$

which implies the induction hypothesis with $j-1$ replaced with j. Again, for T_1 great enough, we have no problem producing improvements of the bootstrap assumptions. The lemma follows. □

8. Perturbing away from zero velocity

Finally, we are in a position to prove that we can perturb away from zero velocity.

THEOREM 3. *Consider a solution z to (23) and assume that $\rho(\tau,\theta)\le\tau-3$ and (69) hold for all $\tau\ge T\ge 4$ and $\theta\in S^1$, with ε in (69) replaced by ε_0, which ε_0 is the constant appearing in the statement of Lemma 12. Then there is a sequence of solutions z_l to (23) such that the z_l converge to z in the C^∞ topology on initial data for $\tau=T$, and $v_\infty[z_l]>0$.*

Proof. Consider Lemma 12, for a fixed k, and Lemma 7. Choose a sequence $T_l\ge T_{1,k},T'$, where $T_{1,k}$ is the constant mentioned in the statement of Lemma 12 and T' is the constant mentioned in Lemma 9, such that $T_l\to\infty$. For each T_l, choose a $p_{0,l}$ as in the statement of Lemma 9, and define z_l to be the solution to (23) defined by specifying initial data at T_l by (70), where c_{T_1} should be replaced with $p_{0,l}$, T_1 should be replaced by T_l and $\tilde{z}$ by z_l. Then z_l is a T_l,z-solution. Note that Lemma 12 is applicable to the solutions z_l and that (89) holds for z_l with T_1 replaced with T_l. Consequently, the distance between z and z_l converges to zero when measured in the $C^{k+1}\times C^k$-norm on initial data at $\tau=T$. Let us prove that the asymptotic velocity of z_l is nonzero for l great enough. In order to do this we need to prove that Lemma 7 is applicable to z_l for l large enough. Combining (69) and (74), we conclude that $e^{-T}F[z_l](T)$ is bounded by $16\varepsilon_0^2$. Consequently, by Lemma 1,

$$\left\|\frac{2z_{l,\tau}(\tau,\cdot)}{1-|z_l(\tau,\cdot)|^2}\right\|_{C^0(S^1,\mathbb{R})}\le 4\varepsilon_0\le\frac{1}{50}\tag{91}$$

for all $\tau\ge T$. In particular $v_\infty[z_l]\le 1/50$. Furthermore $\rho_l(T,\theta)\le T-2$ for l large enough, where ρ_l is ρ defined with respect to z_l. Since $\rho_{l,\tau}$ is dominated by the left-hand side of (91), we conclude that $\rho_l(\tau,\theta)\le\tau-2$ for all $\tau\ge T$ and $\theta\in S^1$. Assuming k is at least 2, we conclude that for l large enough, we can use the same constants as in the statement of Lemma 7 if we increase the numerical constants involved. By construction and Lemma 7,

$$\left|\frac{2z_{l,\tau}(T_l,\cdot)}{1-|z_l(T_l,\cdot)|^2}\right|\ge e^{-\alpha T_l},$$
$$\left|\frac{2|z_{l,\tau}(T_l,\theta)|}{1-|z_l(T_l,\theta)|^2}-v_\infty[z_l](\theta)\right|\le 2L_1'\exp\{-2T_l+2v_\infty[z_l](\theta)T_l\}T_l^{m_2},$$

where $\alpha=19/10$ and the constants L_1' and m_2 are independent of l. We conclude that for T_l large enough $v_\infty[z_l]$ is never zero, since $-2+2v_\infty[z_l]\le -49/25$. To conclude, what we have proved is that for any k and any $\eta>0$, there is a solution $\tilde{z}$ to (23) such that the asymptotic velocity corresponding to $\tilde{z}$ is never zero, and the distance between z and $\tilde{z}$, when measured in the $C^{k+1}\times C^k$-norm of initial data for $\tau=T$ is less than η. The theorem follows. □

9. Velocity identically equal to zero

Consider a periodic solution z to (23). In order to get an actual solution to Einstein's equations, we need an integral condition to be satisfied, namely $c_0[z] = 0$, where

$$c_0[z] = \int_{S^1} \frac{4z_\tau \cdot z_\theta}{(1-|z|^2)^2} d\theta. \tag{92}$$

Note that $c_0[z]$ is independent of τ due to (23). Furthermore, $c_0[z]$ coincides with the integral appearing on the left-hand side of (16). Let us consider a solution for which the asymptotic velocity is identically zero, and try to perturb away from that, preserving $c_0[z] = 0$. Note that by Lemma 7 and Lemma 1, if v_∞ is identically zero, then $c_0[z] = 0$.

THEOREM 4. *Consider a solution z to (23) and assume that $v_\infty[z] \equiv 0$. Then there is a sequence of solutions z_l to (23), with $v_\infty[z_l] > 0$ and $c_0[z_l] = 0$ such that z_l converges to z in the C^∞-topology on initial data.*

Proof. Using Lemma 7 and the fact that the velocity is identically zero, we conclude that

$$\left\| \frac{z_\tau(\tau,\cdot)}{1-|z(\tau,\cdot)|^2} \right\|_{C^0(S^1,\mathbb{R})} \le L_1' \tau^{m_2} e^{-2\tau}.$$

Thus, we do not need Lemma 9 in order to prove the existence of p_0 satisfying the conditions of the statement of Lemma 9. In fact, at a late enough time, any p_0 satisfying $|p_0| = e^{-\beta\tau}$ will do. The argument to prove that there is a sequence of solutions z_l converging to z_l with $v_\infty[z_l] > 0$ is as in the proof of Theorem 3. What remains is to prove that we can choose p_0 such that $c_0[z_l] = 0$. We perturb as in (70), with $c_{T_1} = p_0$ and $\hat{z} = z - \tilde{z}$. Since $c_0[z] = 0$, we have

$$\int_{S^1} \left[\frac{4\tilde{z}_\tau \cdot \tilde{z}_\theta}{(1-|\tilde{z}|^2)^2} \right] (T_1,\theta) d\theta = -p_0 \cdot \int_{S^1} \left[\frac{4z_\theta}{1-|z|^2} \right] (T_1,\theta) d\theta.$$

By letting p_0 be orthogonal to

$$\int_{S^1} \left[\frac{z_\theta}{1-|z|^2} \right] (T_1,\theta) d\theta,$$

we conclude that $c_0[\tilde{z}] = 0$ (if the integral is zero, we are of course free to choose p_0 arbitrarily). □

10. Density of generic solutions

10.1. *Perturbation and localization tools.* Due to how the domain-of-dependence looks, two different spatial points are outside each other's domain of influence at a late enough time, when looking in the direction toward the singularity. This allows us to focus our attention on limited regions of the singularity. On a formal level, the most convenient way to do this is to modify the initial data outside the region one wishes to study so that the behaviour outside is simple in some sense. One lemma that will be needed in the process is the following, it was proved in [21].

LEMMA 13. *Consider a solution z to (23), where $\theta \in \mathbb{R}$, and let $z_l \to z$ in the $C^1 \times C^0$-topology on initial data. Assume $v_\infty[z](\theta) < 1$ for all $\theta \in I = [\theta_1, \theta_2]$. Then $v[z]$ is continuous in I, as well as $v[z_l]$ for l large enough, and*

$$\lim_{l\to\infty} \|v[z] - v[z_l]\|_{C^0(I,\mathbb{R})} = 0.$$

Remark. We defined v in (8) and the $C^1 \times C^0$-topology on initial data for solutions with $\theta \in \mathbb{R}$ was defined in [21].

We shall also need the following results from [21].

PROPOSITION 5. *Let (Q, P) be a solution to (2), (3) and assume $v_\infty = 0$ in a compact interval K with nonempty interior. Then there are $q, \phi \in C^\infty(K, \mathbb{R})$, polynomials Ξ_k and a T such that for all $\tau \geq T$*

$$\|P_\tau(\tau, \cdot)\|_{C^k(K,\mathbb{R})} + \|P(\tau, \cdot) - \phi\|_{C^k(K,\mathbb{R})} \leq \Xi_k e^{-2\tau}, \tag{93}$$

$$\|Q_\tau(\tau, \cdot)\|_{C^k(K,\mathbb{R})} + \|Q(\tau, \cdot) - q\|_{C^k(K,\mathbb{R})} \leq \Xi_k e^{-2\tau}. \tag{94}$$

PROPOSITION 6. *Let (Q, P) solve (2), (3). Then there is a subset $\mathscr{E}$ of S^1 which is open and dense, and for each $\theta_0 \in \mathscr{E}$, there is an open neighborhood of θ_0 such that either (Q, P) or* $\mathrm{Inv}(Q, P)$ *has expansions of the form (93), (94) or (9)–(12). If $v_\infty(\theta_0) \geq 1$, then the q appearing in the expansions is a constant and α can be taken equal to 2.*

Remark. A result of this form was already obtained in [3].

The following lemma gives one way of modifying the initial data in order to achieve the objective alluded to above.

LEMMA 14. *Consider a solution z to (23) where $\theta \in \mathbb{R}$. Let $I = [\theta_1, \theta_2]$ and assume that $v_\infty[z](\theta) \leq \alpha$ for all $\theta \in I$ and some $\alpha \in \mathbb{R}$. For every $\varepsilon,\ \eta > 0$, there is a solution $\tilde{z}$ to (23) and a T, both depending on $\varepsilon,\ \eta$ and z, such that*

- *$\tilde{z}$ coincides with z in $[T, \infty) \times I$,*
- *$\tilde{z}(\tau, \theta) = 0$ for $\tau \geq T$ outside of $[T, \infty) \times [\theta_1 - \eta, \theta_2 + \eta]$,*
- *$v_\infty[\tilde{z}](\theta) \leq \alpha + \varepsilon$ for all $\theta \in \mathbb{R}$.*

Remark. We shall refer to $\tilde{z}$ as an *ε, η-cutoff of z around I*, and we shall call T the *cutoff time*.

Proof. Let $\varepsilon > 0$. For each $i = 1, 2$, there is a closed interval I_i containing θ_i in its interior and a T_i such that

$$e^{-\tau} F_{I_i}[z](\tau) \leq \left(\alpha + \frac{1}{2}\varepsilon\right)^2 \tag{95}$$

for all $\tau \geq T_i$. This follows from continuity and monotonicity. Let $\eta > 0$ be small enough that $[\theta_1 - \eta, \theta_1] \subseteq I_1$ and $[\theta_2, \theta_2 + \eta] \subseteq I_2$. Due to Proposition 6, there are closed intervals I_i', $i = 1, 2$, with nonempty interiors, such that $I_1' \subseteq (\theta_1 - \eta, \theta_1)$ and $I_2' \subseteq (\theta_2, \theta_2 + \eta)$ with the property that we have asymptotic expansions of the form (9)–(12) or of the form (93), (94) in I_i', after applying an inversion, if necessary. If $v_\infty[z](\theta_0) \geq 1$ for some $\theta_0 \in I_i'$, we get expansions with q equal to a constant, say q_0, and $\alpha = 2$. Since the arguments are essentially the same for the two I_i', we consider only I_1'. Let $\eta_1 = |I_1'|$ and let θ_1' be the center of I_1'. Let $T \geq T_1, T_2$ be large enough that $e^{-T} \leq \eta_1/4$.

Let $\phi_i \in C^\infty(\mathbb{R}, [0, 1])$, $i = 1, 2$ have the properties that ϕ_1 equals 1 in $[\theta_1' + \eta_1/4, \infty)$ and 0 in $(-\infty, \theta_1']$, and ϕ_2 equals 1 in $[\theta_1', \infty)$ and 0 in $(-\infty, \theta_1' - \eta_1/4]$. After having applied ϕ_{RD}^{-1}, plus possibly an inversion, we obtain a solution (Q, P) to (2), (3) with expansions. In particular Q converges to q in I_1'. Modify the initial data at T according to

$$\widetilde{P} = \phi_1 P, \quad \widetilde{Q} = \phi_2[\phi_1 Q + (1 - \phi_1)q], \quad \widetilde{P}_\tau = \phi_1 P_\tau, \quad \widetilde{Q}_\tau = \phi_1 Q_\tau.$$

Note that $\phi_2(1 - \phi_1)$ has support in I_1', so that $\phi_2(1 - \phi_1)q$ is well defined. Since the isometry maps the origin of the PQ-plane to the origin of the disc model, the first statement and half of the second statement of the lemma follow. Note that

$$e^{-\tau} \widetilde{P}_\theta = e^{-\tau}[\phi_{1\theta} P + \phi_1 P_\theta]$$

can be assumed to be arbitrarily small in I_1' by demanding that τ be great enough, since P and P_θ do not grow faster than linearly due to the existence of the expansions, and since $\phi_{1\theta}$ has a bound only depending on η_1. Note that $\phi_{2\theta} \neq 0$ implies $\phi_1 = 0$ and that $\phi_1\phi_2 = \phi_1$. Compute

$$\begin{aligned} e^{\widetilde{P} - \tau} \widetilde{Q}_\theta &= e^{\widetilde{P} - \tau}\{\phi_{2\theta}[\phi_1 Q + (1 - \phi_1)q] + \phi_2[\phi_{1\theta}(Q - q) + \phi_1 Q_\theta + (1 - \phi_1)q_\theta]\} \\ &= e^{-\tau}\phi_{2\theta} q + e^{\widetilde{P} - \tau}\phi_{1\theta}(Q - q) + \phi_1 e^{\widetilde{P} - \tau} Q_\theta + e^{\widetilde{P} - \tau}\phi_2(1 - \phi_1)q_\theta. \end{aligned}$$

If $v_\infty = 0$ in I_1', it is clear that this expression converges to zero there. In the remaining cases, $v_\infty > 0$ in I_1' and we can assume that T is great enough that P is positive in I_1'. Consequently, $\widetilde{P} \leq P$. The first term can be assumed to be arbitrarily small by assuming τ to be great enough, since $\phi_{2\theta}$ has a bound only depending on η_1. The two middle terms can be assumed to be arbitrarily small due to the existence of the expansions. For the last term there are two cases. If $v_\infty < 1$, it converges to zero. Otherwise, q had to be a constant to start with, so that the term does not exist in that case. We conclude that

$$e^{-2\tau}[\widetilde{P}_\theta^2 + e^{2\widetilde{P}}\widetilde{Q}_\theta^2]$$

can be assumed to be arbitrarily small in I_1' for τ great enough, due to the existence of the expansions. Since

$$\widetilde{P}_\tau^2 + e^{2\widetilde{P}}\widetilde{Q}_\tau^2 \le P_\tau^2 + e^{2P}Q_\tau^2$$

in I_1', we can assume that $e^{-\tau}F_{I_1}(\tau) \le (\alpha+\varepsilon)^2$ for τ large enough, yielding half of the third statement. In order to arrive at this conclusion, we just noted that to the left of I_1', $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{P}_\tau$, $\widetilde{Q}_\tau$ are zero and to the right, they coincide with the corresponding objects for P and Q. □

LEMMA 15. *Consider a solution z to (23), where $\theta \in \mathbb{R}$ and let $I = (\theta_1, \theta_2)$. Assume there are a T and a sequence z_l of solutions to (23), where $\theta \in \mathbb{R}$, such that $[z_l(\tau,\cdot), z_{l,\tau}(\tau,\cdot)]$ converge to $[z(\tau,\cdot), z_\tau(\tau,\cdot)]$ in the C^∞ topology on I for every $\tau \ge T$. Then for any $0 < \delta < |I|/2$, there are a sequence $\tilde{z}_l$ of solutions to (23), where $\theta \in \mathbb{R}$, and a T' such that*

- *$\tilde{z}_l$ converges to z in the C^∞ topology on initial data,*
- *$\tilde{z}_l$ coincides with z for $\tau \ge T'$ outside of $[T',\infty) \times I$,*
- *$\tilde{z}_l$ coincides with z_l in $[T',\infty) \times [\theta_1+\delta, \theta_2-\delta]$.*

Remark. We shall refer to $\tilde{z}_l$ as a *δ-interpolation of z and z_l in I*.

Proof. Let $\psi \in C_0^\infty(\mathbb{R},[0,1])$ satisfy $\psi = 1$ in $[\theta_1 + 3\delta/4, \theta_2 - 3\delta/4]$ and $\psi = 0$ outside $(\theta_1 + \delta/4, \theta_2 - \delta/4)$. Assume also that $\exp(-T') \le \delta/4$. Define

$$\tilde{z}_l(T',\cdot) = \psi z_l(T',\cdot) + (1-\psi)z(T',\cdot), \quad \tilde{z}_{l,\tau}(T',\cdot) = \psi z_{l,\tau}(T',\cdot) + (1-\psi)z_\tau(T',\cdot).$$

All the desired properties follow. □

Consider a solution z to (23), where $\theta \in \mathbb{R}$. Say that the asymptotic velocity is small in some interval $I = [\theta_1, \theta_2]$, but it is nonzero on the boundary of I. It will be convenient to know that it is possible to find a sequence z_l of solutions converging to z with the properties that for some T, z_l coincides with z for $\tau \ge T$ outside of a set of the form $[T,\infty) \times I$, and z_l has nonzero asymptotic velocity in I.

LEMMA 16. *Consider a solution z to (23) where $\theta \in \mathbb{R}$. Let $I = [\theta_1, \theta_2]$ be such that $|I| < 2\pi$,*

$$v_\infty(\theta_i) = \varepsilon > 0 \quad \text{and} \quad v_\infty(\theta) \le \varepsilon$$

for all $\theta \in I$, where $\varepsilon \le \varepsilon_0$ and ε_0 is as in the statement of Lemma 12. Then there are a T and a sequence of solutions z_l to (23), where $\theta \in \mathbb{R}$, such that

- *z_l converges to z in the C^∞ topology of initial data,*

- *z_l coincides with z for $\tau \geq T$ outside $[T, \infty) \times I$,*
- *$v_\infty[z_l](\theta) > 0$ for all $\theta \in I$.*

Proof. Let $\eta < (2\pi - |I|)/2$ and perform an $\varepsilon_0/2, \eta$-cutoff of z around I. The resulting solution $\tilde{z}$ has the properties stated in Lemma 14. In particular, $v_\infty[\tilde{z}] \leq 3\varepsilon_0/2$ and we can view it as a solution to (23) with $\theta \in S^1$. For a late enough time, $\tilde{z}$ will thus satisfy the conditions of Theorem 3 due to Lemma 2. Consequently, there is a sequence $\tilde{z}_l$ of periodic solutions to (23) converging to $\tilde{z}$ such that $v_\infty[\tilde{z}_l] > 0$. Let $0 < \delta < |I|/2$ be such that $v_\infty[z] > 0$ in $S_\delta = [\theta_1, \theta_1 + \delta] \cup [\theta_2 - \delta, \theta_2]$ and let z_l be a δ-interpolation of z and $\tilde{z}_l$ in $\mathrm{int} I$. Let us prove the third statement. In $[\theta_1 + \delta, \theta_2 - \delta]$, $v_\infty[z_l] = v_\infty[\tilde{z}_l] > 0$, and in S_δ we can use Lemma 13 to conclude that for l large enough, $v_\infty[z_l] > 0$ there. The remaining statements follow by construction. □

When carrying out perturbations, the condition $c_0[z] = 0$ is not always preserved. The point is then to perturb the perturbed solution so that one achieves $c_0[z] = 0$.

LEMMA 17. *Consider a smooth periodic solution z to (23), satisfying $c_0[z] = 0$, where c_0 is as defined in (92). Assume the following*:

- *z_l are periodic solutions to (23) converging to z in the C^∞ topology on initial data,*
- *$S \subseteq S^1$ is compact, $J = [\theta_3, \theta_4]$ has nonempty interior and $J \cap S = \emptyset$,*
- *there is a T such that z_l coincides with z for $\tau \geq T$ outside of $[T, \infty) \times S$.*

Then there are a T' and a sequence of periodic solutions z'_k to (23) such that

- *z'_k converges to z in the C^∞ topology on initial data,*
- *$c_0[z'_k] = 0$,*
- *$z'_k = z_k$ for $\tau \geq T'$ outside of $[T', \infty) \times J$,*
- *if $0 < v_\infty[z] < 1$ in J, the same is true of z'_k.*

Remark. We shall say that z'_k is an S, J*-correction to* z_l.

Proof. Let $T_1 \geq T$ be large enough that $J' = [\theta'_3, \theta'_4] = [\theta_3 + e^{-T_1}, \theta_4 - e^{-T_1}]$, considered as a subinterval of S^1, has nonempty interior. We have the following two cases.

Case 1. Assume there is a $\theta_0 \in (\theta'_3, \theta'_4)$ such that $z_\theta(T_1, \theta_0) \neq 0$. Let $\phi \in C^\infty(S^1, \mathbb{R})$ have the properties that the support of ϕ is contained in the interior of J', $\phi(\theta_0) = 1$ and $0 \leq \phi \leq 1$. Define, for $\tau = T_1$,

$$z'_l = z_l, \quad \text{and} \quad z'_{l,\tau} = z_{l,\tau} + \varepsilon_l \phi z_\theta,$$

where ε_l has been chosen so that

$$0=\int_{S^1}\frac{4z'_{l,\theta}\cdot z'_{l,\tau}}{(1-|z'_l|^2)^2}d\theta=\int_{S^1}\frac{4z_{l,\theta}\cdot z_{l,\tau}}{(1-|z_l|^2)^2}d\theta+\varepsilon_l\int_{S^1}\frac{4\phi|z_\theta|^2}{(1-|z|^2)^2}d\theta. \tag{96}$$

Note that the integral that ε_l multiplies is a fixed positive number, so that there is an ε_l fulfilling (96). Furthermore, the first integral on the right-hand side of (96) converges to zero, so that the sequence ε_l converges to zero. We conclude that the sequence of solutions z'_l has the property that $c_0[z'_l]=0$, z'_l converges to z in the C^∞ topology on initial data and z'_l coincides with z_l for $\tau\geq T_1$ outside $[T_1,\infty)\times J$. The last statement of the lemma follows from Lemma 13.

Case 2. Assume $z_\theta=0$ in J'. In this case, it will be convenient to consider the problem in the PQ-variables instead of in the disc model. Then P is constant in J', and we shall denote this constant p_0. Let

$$\theta_m=\frac{\theta'_4+\theta'_3}{2},\quad h=\frac{\theta'_4-\theta'_3}{2},\quad J_1=[\theta'_3,\theta_m],\quad J_2=[\theta_m,\theta'_4].$$

Let $\phi\in C^\infty(S^1,\mathbb{R})$ have support in the interior of J_1 and assume that it is not identically zero. Let

$$\phi_1(\theta)=\phi(\theta),\quad \phi_2(\theta)=\phi(\theta-h).$$

Then ϕ_2 has support in the interior of J_2. There are two subcases to consider.

Subcase 1. Let us first assume that for $\tau=T_1$,

$$\int_{S^1}P_\tau(\phi_1-\phi_2)d\theta\neq 0. \tag{97}$$

Define, for $\theta\in J'$,

$$p_\varepsilon(\theta)=p_0+\varepsilon\int_{\theta'_3}^{\theta}[\phi_1(s)-\phi_2(s)]ds.$$

Define, in T_1,

$$P'_l(T_1,\theta)=P_l(T_1,\theta)\ \forall\ \theta\notin J',\quad P'_l(T_1,\theta)=p_{\varepsilon_l}(\theta)\ \forall\ \theta\in J',$$
$$Q'_l=Q_l,\quad P'_{l,\tau}=P_{l,\tau},\quad Q'_{l,\tau}=Q_{l,\tau},$$

where ε_l has been chosen so that

$$\begin{aligned}0&=\int_{S^1}(P'_{l,\theta}P'_{l,\tau}+e^{2P'_l}Q'_{l,\theta}Q'_{l,\tau})d\theta\\&=\int_{S^1}(P_{l,\theta}P_{l,\tau}+e^{2P_l}Q_{l,\theta}Q_{l,\tau})d\theta+\varepsilon_l\int_{S^1}P_\tau(\phi_1-\phi_2)d\theta.\end{aligned}$$

Note that $(P_{l,\theta},Q_{l,\theta})=0$ in J' for all l. The argument can now be finished as in the first case.

Subcase 2. Assume that the left-hand side of (97) is zero. Define, for $\tau = T_1$,

$$P'_l(T_1,\theta) = P_l(T_1,\theta),\ \ P'_{l,\tau}(T_1,\theta) = P_{l,\tau}(T_1,\theta)\ \forall\ \theta \notin J',$$
$$Q'_l = Q_l,\ \ Q'_{l,\tau} = Q_{l,\tau}$$
$$P'_l(T_1,\theta) = p_{\varepsilon_l}(\theta),\ \ P'_{l,\tau}(T_1,\theta) = P_{l,\tau}(T_1,\theta) + |\varepsilon_l|\phi_1(\theta)\ \forall\ \theta \in J',$$

where ε_l has been chosen so that

$$\begin{aligned} 0 &= \int_{S^1} (P'_{l,\theta}P'_{l,\tau} + e^{2P'_l}Q'_{l,\theta}Q'_{l,\tau})d\theta \\ &= \int_{S^1} (P_{l,\theta}P_{l,\tau} + e^{2P_l}Q_{l,\theta}Q_{l,\tau})d\theta + \varepsilon_l|\varepsilon_l|\int_{S^1}\phi_1^2 d\theta. \end{aligned}$$

We can complete the argument as before. □

COROLLARY 5. *Consider $z \in \mathcal{S}_p$ with $v_\infty[z] < 1$. Then there is a sequence of $z_l \in \mathcal{S}_p$ such that z_l converges to z in the C^∞ topology on initial data and $0 < v_\infty[z_l] < 1$. If $c_0[z] = 0$, then $c_0[z_l] = 0$.*

Proof. If the velocity is identically zero, we can apply Theorem 4 and Lemma 13; so let us assume that this is not the case. Let $\theta_0 \in S^1$ be such that $2\delta := v_\infty(\theta_0) > 0$ and let N be the set of points where $v_\infty = 0$. If N is empty we are done, and so we assume it is not. Let $0 < \varepsilon \le \delta, \varepsilon_0$, where ε_0 is the constant appearing in the statement of Lemma 12. For $\theta_1 \in N$, let I_{θ_1} be the largest interval containing θ_1 such that $v_\infty(\theta) \le \varepsilon$ for $\theta \in I_{\theta_1}$. Note that I_{θ_1} is a compact proper subinterval of S^1, since $v_\infty(\theta_0) \ge 2\varepsilon$. Let $\chi_i \in N$, $i = 1, 2$. Either the I_{χ_i} are disjoint or coincide. The reason is the following. Assume $I_{\chi_1} \cap I_{\chi_2}$ is nonempty. Then the union is an interval I, and $v_\infty \le \varepsilon$ in I. By maximality $I_{\chi_i} = I$ for $i = 1, 2$. Since v_∞ is continuous in the present setting, $v_\infty = \varepsilon$ on the boundary of I_{θ_1} and the boundary points of I_{θ_1} are accumulation points of the set where $v_\infty > \varepsilon$. Since $v_\infty \in C^0(S^1, \mathbb{R})$, N is a compact set. For each $\chi \in N$, $\mathrm{int} I_\chi$ is an open set containing χ. Since the corresponding open covering has a finite subcovering, there is a finite number of points $\theta_i \in S^1$, $i = 1, \dots, k$, such that $\mathrm{int} I_{\theta_i}$ is a covering of N. By the above argument, we can assume that the I_{θ_i} are disjoint. For each $i = 1, \dots, k$, we can apply Lemma 16 in order to get a T_i and a $z_{i,l}$ with properties as stated there. Letting $T = \max\{T_1, \dots, T_k\}$, we can define the initial data of z_l to coincide with those of $z_{i,l}$ in I_{θ_i} and with those of z elsewhere. Let $S = \cup I_{\theta_i}$ and let J be a compact interval with nonempty interior in the complement of S. If $c_0[z] = 0$, let z'_k be an S, J-correction of z_l. Otherwise, let $z'_k = z_k$. Then z'_k has the desired properties. □

Consider, for $k \in \mathbb{N}$, $k \ge 1$, the set

$$\mathcal{U}_k = \{z \in C^2(\mathbb{R} \times S^1, D) : z \text{ is a solution to (23)},\ v_\infty[z](\theta) < k\ \forall\ \theta \in S^1\}.$$

LEMMA 18. *The set $\mathcal{U}_k$ is open in the $C^1 \times C^0$-topology of initial data.*

Remark. The topology mentioned was defined in [21].

Proof. Let $z \in \mathcal{U}_k$ and $\theta \in S^1$. Then there are a $T_\theta \in \mathbb{R}$, an $\varepsilon_\theta > 0$ and an interval I_θ, containing θ in its interior, such that for $\tau = T_\theta$, $e^{-\tau} F_{I_\theta}[z](\tau) \leq k^2 - \varepsilon_\theta$. The reason is that the same can be assumed to hold with I_θ replaced by $\{\theta\}$ and ε_θ replaced by $2\varepsilon_\theta$. The statement then follows by continuity. Since $e^{-\tau} F_{I_\theta}[z]$ is monotonically decaying, we conclude that the same holds for all $\tau \geq T_\theta$. Since the interiors of the I_θ form an open covering, there is a finite number of points $\theta_1, \dots, \theta_m$ such that the interiors of the $I_i = I_{\theta_i}$ cover S^1. Let $T = \max\{T_{\theta_1}, \dots, T_{\theta_m}\}$, $\varepsilon = \min\{\varepsilon_{\theta_1}, \dots, \varepsilon_{\theta_m}\}$. We have $e^{-\tau} F_{I_i}[z](\tau) \leq k^2 - \varepsilon$ for all $i = 1, \dots, m$, and $\tau \geq T$. There is an open neighborhood O of z in the $C^1 \times C^0$-topology of initial data at $\tau = T$ such that if $\tilde{z} \in O$, then $e^{-\tau} F_{I_i}[\tilde{z}](\tau) \leq k^2 - \varepsilon/2$ for all $i = 1, \dots, m$ and $\tau = T$. By the monotonicity of the left-hand side for each i, and the fact that the interiors of the I_i cover S^1, we draw the conclusion that $O \subset \mathcal{U}_k$. □

We shall need the following result from [21].

THEOREM 5. *Let (Q, P) solve (2), (3) and assume that $k \leq v_\infty(\theta) < k + 2$ for all $\theta \in K$, where K is a compact interval with nonempty interior and $k \in \mathbb{N}$, $k \geq 1$. Then either (Q, P) has expansions in K of the form (9)–(12) or* $\mathrm{Inv}(Q, P)$ *has such expansions. Furthermore, the q appearing in the expansions is a constant and $\alpha = 2$.*

LEMMA 19. *Consider $z \in \mathcal{U}_{k+1}$, $k \in \mathbb{N}$, $k \geq 1$. Let*

$$\mathcal{V}_z = \{\theta \in S^1 : v_\infty[z](\theta) \geq k\}.$$

Then $\mathcal{V}_z$ is compact. Furthermore, if $I \subseteq \mathcal{V}_z$ is a compact interval with nonempty interior, then $v_\infty[z]$ restricted to I is continuous, and after applying ϕ_{RD}^{-1}, plus possibly an inversion, the solution has smooth expansions in I of the form (9)–(12) with q constant and $\alpha = 2$.

Proof. Since S^1 is compact, all we need to prove is that $\mathcal{V}_z$ is closed. Let $\theta_k \to \theta'$, with $\theta_k \in \mathcal{V}_z$. Assume $v_\infty[z](\theta') < k$. Then this must also be true of $v_\infty[z](\theta_k)$ for k large enough, due to the upper semicontinuity of v_∞; cf. Theorem 1. The remaining part follows from Theorem 5. □

10.2. *Characterizations of true and false spikes.* It will be useful to have a more flexible characterization of the concepts true and false spikes. The following result proves the existence of an object to be used for that purpose.

LEMMA 20. *Consider a solution z to (23) and assume that $0 < v_\infty(\theta) < 1$ for all $\theta \in K$, where K is a compact subinterval of S^1 with nonempty interior. Then* *there is a* $\varphi \in C^\infty(K, \mathbb{R}^2)$ *such that* $|\varphi(\theta)| = 1$ *for all* $\theta \in K$ *and*

$$\|z(\tau, \cdot) - \varphi\|_{C^k(K,\mathbb{R}^2)} \le \Pi_k(\tau) e^{-2\alpha\tau},$$

where $\alpha = \inf_{\theta \in K} v_\infty(\theta)$ *and* Π_k *is a polynomial in* τ.

Remark. It is allowed to take $K = S^1$.

Proof. Due to the section on uniform convergence in [21], we conclude that ρ/τ converges uniformly to v_∞ in K. Using (37), we conclude that $z(\tau, \cdot)/|z(\tau, \cdot)|$ converges uniformly. Finally $|z(\tau, \cdot)|$ converges uniformly to 1. Consequently, $z(\tau, \cdot)$ converges uniformly to a continuous function φ. Let $\theta \in K$. After having performed an inversion if necessary, we can assume that $z(\tau, \theta)$ does not converge to 1; cf. (98). Looking at the solution in the PQ-plane, keeping (22) in mind, we conclude that $P(\tau, \theta)/\tau$ must converge to $v_\infty(\theta)$. Due to Proposition 2, we conclude that there must be smooth expansions in a neighborhood I of θ. Applying ϕ_{RD} to the solution we see that $z(\tau, \cdot)$ has to converge exponentially in every C^k norm on I to a smooth function. Using the compactness of K, we get the global statement of the lemma. $\square$

Note that an inversion in the PQ-plane corresponds to the isometry $-\bar{z}$ in the disc model; i.e.

$$\phi_{RD} \circ \mathrm{Inv} \circ \phi_{RD}^{-1}(z) = -\bar{z}. \tag{98}$$

LEMMA 21. *Let* $(Q, P) \in \mathcal{S}_p$ *and* $z = \phi_{RD} \circ (Q, P)$. *Assume* $0 < v_\infty(\theta_0) < 1$. *Note that then there are an open neighborhood* I_0 *of* θ_0 *and a* $\varphi \in C^\infty(I_0, \mathbb{C})$ *such that* $|\varphi| = 1$ *and* $z(\tau, \cdot)$ *converges to* φ *in any* C^k *norm on* I_0. *The following two statements are equivalent*:

- $\theta_0 \in S^1$ *is a nondegenerate false spike of* (Q, P),
- $\varphi(\theta_0) = 1$ *and* $\varphi_\theta(\theta_0) \neq 0$.

Proof. Let $(Q_1, P_1) = \mathrm{Inv}(Q, P)$ and $z_1 = -\bar{z}$. Regardless of whether θ_0 is a nondegenerate false spike or $\varphi(\theta_0) = 1$, we get the conclusion that (Q_1, P_1) has smooth expansions of the form (9)–(12) in a neighborhood I_0 of θ_0; cf. Proposition 2. Say that Q_1 converges to q_1. Then

$$z_1 = \frac{Q_1 + i(e^{-P_1} - 1)}{Q_1 + i(e^{-P_1} + 1)} = \frac{q_1 - i}{q_1 + i} + \cdots,$$

where $\cdots$ represents terms that converge to zero exponentially in the C^1 norm on I_0. We conclude that

$$\varphi = -\frac{q_1 + i}{q_1 - i}. \tag{99}$$

From this it is clear that the conditions $q_1(\theta_0) = 0$ and $q_{1\theta}(\theta_0) \neq 0$ are equivalent to the conditions $\varphi(\theta_0) = 1$ and $\varphi_\theta(\theta_0) \neq 0$. Note that the fact that $q_1(\theta_0) = 0$ and the fact that we have expansions of the form

$$Q_1 = q_1 + e^{-2v_\infty\tau}[\psi + O(e^{-\varepsilon\tau})], \quad P_1 = v_\infty\tau + \phi + O(e^{-\varepsilon\tau})$$

imply that

$$\lim_{\tau\to\infty} P_\tau(\tau, \theta_0) = -v_\infty(\theta_0).$$

□

It will be convenient to have a different characterization of the concept of a nondegenerate true spike.

LEMMA 22. *Let $(Q, P) \in \mathcal{S}_p$ and assume that*

$$1 < \lim_{\tau\to\infty} P_\tau(\tau, \theta_0) < 2 \tag{100}$$

for some $\theta_0 \in S^1$. Then Q converges to a smooth function q in a neighborhood of θ_0, and the convergence is exponential in any C^k-norm. Furthermore, $q_\theta(\theta_0) = 0$ and the following two statements are equivalent:

- *θ_0 is a nondegenerate true spike,*
- *$q_{\theta\theta}(\theta_0) \neq 0$.*

Proof. Let

$$(Q_2, P_2) = \mathrm{Inv} \circ \mathrm{GE}_{q_0,\tau_0,\theta_0}(Q, P)$$

for some q_0, τ_0, θ_0. Then

$$\lim_{\tau\to\infty} P_{2\tau}(\tau, \theta_0) = v_\infty(\theta_0) - 1.$$

By Proposition 2, there are asymptotic expansions of the form (9)–(12) in a neighborhood of θ_0. Since

$$(Q_1, P_1) = \mathrm{Inv}(Q_2, P_2) \Rightarrow (Q, P) = \mathrm{GE}_{Q(\tau_0,\theta_0),\tau_0,\theta_0}(Q_1, P_1),$$

we can compute

$$Q_\theta = -e^{2P_1}Q_{1\tau} = e^{2P_2}Q_2^2 Q_{2\tau} - Q_{2\tau} - 2P_{2\tau}Q_2.$$

By the existence of the expansions, Q_2 converges to q_2, $e^{2P_2}Q_{2\tau}$ to r_2 and $P_{2\tau}$ converges to v_2. The convergence is exponential in any C^k-norm in a neighborhood of θ_0. Note that $q_2(\theta_0) = 0$. We conclude that Q_θ converges to

$$q_\theta = r_2 q_2^2 - 2v_2 q_2$$

exponentially in any C^k-norm, so that $q_\theta(\theta_0) = 0$. Note that $Q(\tau, \theta_0)$ converges due to the fact that $P_\tau(\tau, \theta_0)$ converges to a positive number and the fact that $e^P Q_\tau$

is bounded. Thus we are allowed to conclude that Q converges to a smooth function q in a neighborhood of θ_0 and that the convergence is exponential in any C^k-norm.

If θ_0 is a nondegenerate true spike, then $q_2(\theta_0)=0$, $q_{2\theta}(\theta_0)\neq 0$ and $v_2(\theta_0)\neq 0$. We conclude that the second characterization holds. Assuming $q_{\theta\theta}(\theta_0)\neq 0$, we get the first characterization, since $q_2(\theta_0)=0$ by construction. □

COROLLARY 6. *Let $(Q,P)\in\mathcal{S}_p$ and assume that $\theta_0\in S^1$ is a nondegenerate true spike. If $Q(\tau,\theta_0)$ converges to a nonzero value, then θ_0 is a nondegenerate true spike of* $(Q_1,P_1)=\mathrm{Inv}(Q,P)$.

Proof. By the second characterization of a true spike given in Lemma 22, we know that Q converges to a function q such that $q_\theta(\theta_0)=0$, but $q_{\theta\theta}(\theta_0)\neq 0$. Since $q(\theta_0)\neq 0$, we know that $P_{1\tau}(\tau,\theta_0)$ converges to $v_\infty(\theta_0)$ so that by Lemma 22, Q_1 has to converge to a smooth function q_1 exponentially in any C^k-norm in a neighborhood around θ_0. Since

$$Q_1=\frac{Q}{Q^2+e^{-2P}},$$

we conclude that $q_1=1/q$. Due to the properties of q, we have that θ_0 is a nondegenerate true spike of (Q_1,P_1). □

LEMMA 23. *Let $(Q,P)\in\mathcal{S}_p$ and $z=\phi_{RD}(Q,P)$. Assume that for all $\theta\in S^1$, $0<[1-v_\infty(\theta)]^2<1$. Then $z(\tau,\cdot)$ converges to a smooth function φ such that $|\varphi|=1$, and the convergence is exponential in any C^k-norm. Furthermore, assuming $1<v_\infty(\theta_0)<2$, $\varphi_\theta(\theta_0)=0$ and the following two statements are equivalent*:

- *θ_0 is a nondegenerate true spike*,
- *$\varphi(\theta_0)\neq 1$ and $\varphi_{\theta\theta}\neq 0$.*

Proof. Using arguments as in the proof of Lemma 20, one sees that in a neighborhood of a point θ where $0<v_\infty(\theta)<1$, $z(\tau,\cdot)$ converges exponentially in any C^k-norm to a function φ. If $1<v_\infty(\theta)<2$ we can apply an inversion, if necessary, in order to obtain the conclusion that $z(\tau,\theta)$ converges to something different from 1. Viewing the solution in the PQ-variables, we have

$$\lim_{\tau\to\infty}P_\tau(\tau,\theta)=v_\infty(\theta).$$

Let

$$(Q_2,P_2)=\mathrm{Inv}\circ\mathrm{GE}_{q_0,\tau_0,\theta_0}(Q,P).$$

Just as in the proof of Lemma 22, we get smooth expansions and the conclusion that Q converges to a smooth function q. Furthermore, the convergence is exponential in any C^k-norm. Using the notation of the proof of Lemma 22, we have

$$e^{-P}=e^{P_1-\tau}=e^{P_2-\tau}(Q_2^2+e^{-2P_2}).$$

Since the asymptotic velocity associated with (Q_2, P_2) is strictly less than one, we conclude that e^{-P} converges to zero exponentially in any C^k-norm. Since

$$z = \frac{Q + i(e^{-P} - 1)}{Q + i(e^{-P} + 1)},$$

we conclude that $z(\tau, \cdot)$ converges to $(q - i)/(q + i)$ exponentially in any C^k-norm. Note that since $q_\theta(\theta) = 0$ by Lemma 22, we obtain $\varphi_\theta(\theta) = 0$. Inverting the solution, if necessary, we conclude that there is a neighborhood of θ such that $z(\tau, \cdot)$ converges to a function φ, exponentially in any C^k norm. Since S^1 is compact, there is a $\varphi \in C^\infty(S^1, \mathbb{C})$ such that $|\varphi| = 1$ and $z(\tau, \cdot)$ converges exponentially to φ in any C^k-norm.

Assume that θ_0 is a nondegenerate true spike. Then, as argued above, Q converges to q and e^{-P} converges to zero, and the convergence is exponential in any C^k-norm in a neighborhood of θ_0. Consequently $\varphi = (q - i)/(q + i)$ in a neighborhood of θ_0. Since $q(\theta_0) \in \mathbb{R}$, $q_\theta(\theta_0) = 0$ and $q_{\theta\theta}(\theta_0) \neq 0$, the second characterization holds. Assuming that the second characterization holds, we conclude that $P_\tau(\tau, \theta_0)$ tends to $v_\infty(\theta_0)$, so that Q converges to q, e^{-P} to zero and $\varphi = (q - i)/(q + i)$. We conclude that θ_0 is a nondegenerate true spike using the second characterization of Lemma 22. □

10.3. *Density of the generic solutions.* We prove that the generic solutions are dense in the full set of solutions by an induction argument. The following lemma constitutes the zeroth step.

LEMMA 24. *Let $z \in \mathcal{U}_1 \cap \mathcal{S}_p$. Then there is a sequence of $z_l \in \mathcal{S}_p$ such that*

- *z_l converges to z in the C^∞ topology on initial data,*
- *if $c_0[z] = 0$ then $c_0[z_l] = 0$,*
- *$0 < v_\infty[z_l](\theta) < 1$ for all $\theta \in S^1$,*
- *$z_l(\tau, \cdot)$ converges to $\varphi_l \in C^\infty(S^1, \mathbb{C})$ such that $|\varphi_l| = 1$ and if $\varphi_l(\theta) = 1$, then $\varphi_{l\theta}(\theta) \neq 0$.*

Remark. Note that in particular, the number of θ for which z_l converges to 1 is finite.

Proof. Let the sequence z_l be as in the statement of Corollary 5 and φ_l denote the limit of $z_l(\tau, \cdot)$. By Lemma 20, $\varphi_l \in C^\infty(S^1, \mathbb{C})$, with $|\varphi_l| = 1$. Let $\mathcal{M}_l$ denote the image under φ_l of the set of points where $\varphi_{l\theta} = 0$. By Sard's theorem, the measure of $\mathcal{M}_l$ is zero, and consequently the union of the $\mathcal{M}_l$, say $\mathcal{M}$, has measure zero. We conclude that there is a sequence $\gamma_k \in \mathbb{R}$, $\gamma_k \to 0$ such that if $\varphi_l(\theta) = e^{i\gamma_k}$, then $\varphi_{l\theta}(\theta) \neq 0$. Given l, let us choose a k_l such that $d(z_l', z_l) \leq 1/l$, where $z_l' = e^{-i\gamma_{k_l}} z_l$ and d is a metric reproducing the C^∞-topology on initial data.

Note that the sequence z_l' has the same properties as the sequence z_l. Furthermore, if $z_l'(\tau,\theta) \to 1$, then $\varphi_l(\theta) = e^{i\gamma_{k_l}}$ so that $\varphi_{l\theta}(\theta) \neq 0$. The set of points for which z_l' converges to 1 is thus discrete so that it is finite. □

COROLLARY 7. *$\mathcal{G}$ is dense in $\mathcal{U}_1 \cap \mathcal{S}_p$ and $\mathcal{G}_c$ is dense in $\mathcal{U}_1 \cap \mathcal{S}_{p,c}$*

Proof. The conclusion follows by combination of Lemmas 21 and 24. □

LEMMA 25. *Assume $\mathcal{G}$ is dense in $\mathcal{U}_k \cap \mathcal{S}_p$ for some $k \in \mathbb{N}$, $k \geq 1$. Consider a solution (Q, P) to (2), (3) with $\theta \in \mathbb{R}$ and an interval $I = [\theta_1, \theta_2]$ with $0 < |I| < 2\pi$ such that*

$$-(k-1) + 2\varepsilon \leq \lim_{\tau\to\infty} P_\tau(\tau,\theta) \leq k + 1 - 2\varepsilon$$

for all $\theta \in I$ and some $0 < \varepsilon < 1/2$. Then, given $0 < \delta < |I|/2$, there are a T and a sequence (Q_l, P_l) of solutions to (2), (3) such that

- *(Q_l, P_l) converges to (Q, P) in the C^∞ topology on initial data,*
- *(Q_l, P_l) coincides with (Q, P) for $\tau \geq T$ outside of $[T,\infty) \times I$,*
- *in $[\theta_1+\delta, \theta_2-\delta]$, $P_{l,\tau}(\tau,\theta)$ converges to a number in the interval $(0, 1)$ except for a finite number of points in which the limit belongs to the set $(-1, 2)$,*
- *if $k = 1$, then $P_{l,\tau}(\tau,\theta)$ converges to a number in the interval $(0, 1)$ in $[\theta_1 + \delta, \theta_2 - \delta]$ except for a finite number of nondegenerate true spikes, where $Q_l(\tau,\theta)$ converges to a nonzero number.*

Proof. In the present proof, we shall speak of several different solutions; z, z_2 etc. If we then speak of (Q, P), (Q_2, P_2) etc., we shall take it to be understood that $z = \phi_{RD}(Q, P)$, $z_2 = \phi_{RD}(Q_2, P_2)$ etc. and vice versa. Furthermore, the proof consists of several simple steps. Since there are many of them, we shall however state the simple conclusions of the steps clearly.

Step 1, *definition of z_2*. Let

$$(Q_2, P_2) = \mathrm{Inv} \circ \mathrm{GE}_{q_1,\tau_1,\theta_1}(Q, P),$$

for some choice of q_1, τ_1, θ_1. We get

$$-k + 2\varepsilon \leq \lim_{\tau\to\infty} P_{2\tau}(\tau,\theta) \leq k - 2\varepsilon$$

for all $\theta \in I$.

Step 2, *definition of z'*. Let $\eta < (2\pi - |I|)/2$ and z' be an ε, η-cutoff of z_2 around I with cutoff time T. Note that we can view z' as a 2π-periodic solution to (23), and that $z' \in \mathcal{U}_k \cap \mathcal{S}_p$.

Step 3, *definition of $\tilde{z}$*. Let us consider z' to be a function from $\mathbb{R}^2$ to D. Let $(\widetilde{Q}, \widetilde{P}) = S(Q', P')$, where

$$S = \mathrm{GE}_{Q(T,\theta_1),T,\theta_1} \circ \mathrm{Inv}. \tag{101}$$

Then $\tilde{z} = z$ in $[T, \infty) \times I$. The reason is that if one takes the square of the Gowdy to Ernst transformation, the resulting P, Q_θ and Q_τ are the same as the ones we started with. The only freedom is a constant, which we have set to be the right one in the definition of S.

Step 4, *definition of* z_l'. By assumption, there is a sequence $z_l' \in \mathcal{G}$ converging to z'. By Sard's theorem we can shift each solution an arbitrarily small distance in the Q-direction in order to obtain the following conclusion: if

$$P'_{l,\tau}(\tau, \theta) \to v_\infty[z_l'](\theta) \quad \text{and} \quad Q_l'(\tau, \theta) \to 0, \tag{102}$$

then $v_\infty[z_l'](\theta) < 1$ and $Q'_{l,\theta}(\tau, \theta)$ converges to a nonzero number. The reason is the following. By assumption, z_l' only has a finite number of true spikes. For each true spike $P'_{l,\tau}$ converges to the corresponding $v_\infty[z_l']$, so that Q_l' converges to some value. Let us denote the set of limit values of Q_l' for nondegenerate true spikes by A_l. Note that A_l is finite. Any translation outside of $-A_l$ will ensure that the limit of Q for the resulting solution is nonzero for each nondegenerate true spike. We can thus assume without loss of generality that (102) implies $v_\infty[z_l'](\theta) < 1$. Since A_l is finite this statement is stable under small perturbations. The rest follows by Sard's theorem.

Step 5, *definition of* $\tilde{z}_l$. Let $(\widetilde{Q}_l, \widetilde{P}_l) = S(Q_l', P_l')$, where we view z_l' and $\tilde{z}_l$ to be functions from $\mathbb{R}^2$ to D. Since S is a continuous map with respect to the C^∞-topology on initial data, we conclude that $\tilde{z}_l$ converges to $\tilde{z}$ with respect to this topology. Since $\tilde{z} = z$ in $[T, \infty) \times I$, we conclude that $\tilde{z}_l$ converges to z with respect to the C^∞-topology on initial data on the interval $\{\tau\} \times I$ for all $\tau \geq T$.

Note that in I, $\widetilde{P}_{l,\tau}$ converges to a number in the interval $(0, 1)$ except for a finite set of points in which it converges to an element in $(-1, 2)$. If $k = 1$, and if $\widetilde{P}_{l,\tau}(\tau, \theta)$ does not converge to a number in $(0, 1)$, then θ has to be a nondegenerate true spike by construction. By shifting an arbitrarily small distance in the Q-direction, we can assume that if θ is a nondegenerate true spike, then $\widetilde{Q}_l(\tau, \theta)$ converges to a nonzero number. Letting z_l be a δ-interpolation between z and $\tilde{z}_l$ in I yields the conclusions of the lemma. $\square$

Let us denote by $\mathcal{U}_{k+1,g}$ the set of solutions $z \in \mathcal{U}_{k+1}$ for which there is a $\theta \in \mathbb{R}$ such that $v_\infty[z](\theta) < k$.

LEMMA 26. *Assume that* $\mathcal{G}$ *is dense in* $\mathcal{U}_k \cap \mathcal{S}_p$ *for some* $k \geq 1$. *Then* $\mathcal{U}_{k+1,g} \cap \mathcal{S}_p$ *is dense in* $\mathcal{U}_{k+1} \cap \mathcal{S}_p$ *and* $\mathcal{U}_{k+1,g} \cap \mathcal{S}_{p,c}$ *is dense in* $\mathcal{U}_{k+1} \cap \mathcal{S}_{p,c}$.

Proof. Let $z \in \mathcal{U}_{k+1} \cap \mathcal{S}_p$ but $z \notin \mathcal{U}_{k+1,g} \cap \mathcal{S}_p$. Then $v_\infty[z] \geq k$ for all $\theta \in S^1$. By performing an inversion on z, if necessary, and viewing it in the PQ-variables, we have

$$\lim_{\tau \to \infty} P_\tau(\tau, \theta) = v_\infty(\theta)$$

for all $\theta \in S^1$; cf. Theorem 5. Let I be an interval with $0 < |I| < 2\pi$, $0 < \delta < |I|/2$ and let (Q_I, P_I) be a solution as constructed in Lemma 25. Denote the corresponding solution in the disc model by z_I. By construction, z_I has points in I such that $v_\infty[z_I] < 1$. Let J be a compact subinterval in the complement of I with nonempty interior. If $c_0[z] = 0$, let $\hat{z}_I$ be an I, J correction to z_I. Otherwise, let $\hat{z}_I = z_I$. Then $\hat{z}_I$ has the desired properties, since $\mathcal{U}_{k+1}$ is open by Lemma 18. □

LEMMA 27. *$\mathcal{G}$ is dense in $\mathcal{U}_2 \cap \mathcal{S}_p$ and $\mathcal{G}_c$ is dense in $\mathcal{U}_2 \cap \mathcal{S}_{p,c}$.*

Proof. Let $z \in \mathcal{U}_2 \cap \mathcal{S}_p$. If $v_\infty[z] < 1$, we can apply Corollary 7, so let us assume that this is not the case. Due to Corollary 7 and Lemma 26, we can assume that there is a $\theta_0 \in S^1$ such that $0 < v_\infty[z](\theta_0) < 1$. The lower bound is due to the fact that $(1 - v_\infty[z])^2$ is continuous under the conditions of the present lemma; cf. [21], and the fact that $v_\infty[z] \leq 2 - \varepsilon$ for some $\varepsilon > 0$ due to the semicontinuity of v_∞; cf. Theorem 1. Let I_0 be a closed interval containing θ_0 in its interior such that $0 < v_\infty[z] < 1$ in I_0. Let $\theta \in S^1$ be such that $v_\infty[z](\theta) \geq 1$ and let $I_{\theta,1}$ be the maximal interval containing θ such that $v_\infty[z] \geq 1$ in $I_{\theta,1}$. Considering $\text{Inv}(Q, P)$ instead of (Q, P), if necessary, we can assume that

$$\lim_{\tau \to \infty} P_\tau(\tau, \theta') = v_\infty[z](\theta')$$

in $I_{\theta,1}$; cf. Theorem 5. Then there is a closed interval I_θ containing $I_{\theta,1}$ in its interior, an $\varepsilon_\theta > 0$ and a T_θ such that

$$\frac{1}{2} \sum_{\pm} \|(P_\tau - 1 \pm e^{-\tau} P_\theta)^2 + e^{2P}(Q_\tau \pm e^{-\tau} Q_\theta)^2\|_{C^0(\mathcal{D}_{I_\theta,\tau}, \mathbb{R})} \leq 1 - 2\varepsilon_\theta \tag{103}$$

for all $\tau \geq T_\theta$. Note that the left-hand side is monotonic by [21]. We can assume that I_0 and I_θ are disjoint and that $0 < v_\infty[z] < 1$ on the boundary of I_θ. Since $\mathcal{V}_z$ defined in Lemma 19 with $k = 1$ is compact (due to Lemma 19) and the interiors of the I_θ form an open covering of $\mathcal{V}_z$, we can find $\theta_1, \ldots, \theta_k \in S^1$ such that the interiors of $I_i = I_{\theta_i}$ cover $\mathcal{V}_z$. We can assume that no I_i is contained in the union of the I_j for $j \neq i$. As a consequence, no point in S^1 is contained in the intersection of three different I_i, since the I_i are intervals. For the sake of argument, let us assume that I_1 intersects one of the other intervals. Let $\theta \in I_1$ be such that it does not belong to any other of the intervals. Moving to the right inside I_1, let θ' be the first point belonging to, say, $I_1 \cap I_i$. If there is no such point we are done. Then $\theta' \in \partial I_i$ so that $0 < v_\infty[z](\theta') < 1$. We can then redefine I_1 by letting the right most boundary point be a point θ_1' somewhat to the left of θ'. We can assume that $0 < v_\infty[z](\theta_1') < 1$. We can repeat the argument going to the left. The redefined I_{θ_1} has the same properties as I_{θ_1}, and additionally, it does not intersect any of the other I_i. We can repeat the procedure with all the I_i, and can consequently assume that no two I_i intersect each other.

Let $T = \max\{T_{\theta_1}, \dots, T_{\theta_k}\}$ and $\varepsilon = \min\{\varepsilon_{\theta_1}, \dots, \varepsilon_{\theta_k}\}$. Consider $I_1 = [\theta_a, \theta_b]$. After applying an inversion if necessary, we have (103). We are thus in a position to use Lemma 25, since we have Corollary 7. Let $\delta > 0$ be small enough that $0 < v_\infty[z] < 1$ in $I_{\delta,a} = [\theta_a - \delta, \theta_a + \delta]$, and similarly in $I_{\delta,b}$, defined analogously. Apply Lemma 25 to I_1, δ, with δ as above. We then get a T_1 and a sequence of solutions (Q_l, P_l) with the properties stated in that lemma. By the definition of δ, we know that $v_\infty[z_l]$ belongs to $(0, 1)$ in $I_{\delta,a}$ and $I_{\delta,b}$ for l large enough due to Lemma 13. By Corollary 6, the only exception to $0 < v_\infty < 1$ in $[\theta_a + \delta, \theta_b - \delta]$ is a finite number of nondegenerate true spikes. We may of course have some false spikes. We can repeat the procedure in $I_2, \dots, I_k$. If there are points with $v_\infty[z] = 0$, we can deal with them as in the proof of Corollary 5. Furthermore, we can do the necessary operations while still keeping away from $I_0, \dots, I_k$. Finally, we can arrange $c_0[z_l]$ to be zero by doing a suitable correction, only modifying the solution inside I_0. What remains is then the problem that there can be infinitely many false spikes. Due to Lemma 23, we conclude that $z_l(\tau, \cdot)$ converges to a smooth function φ_l. By Sard's theorem, the measure of the image of the set of points at which $\varphi_{l\theta} = 0$ is zero. We can thus rotate the solution by an arbitrarily small angle in order to obtain a solution with the property that every time $\varphi_{l\theta} = 0$, $\varphi_l \neq 1$. Note that the rotation will map nondegenerate true spikes to nondegenerate true spikes, and that the rotated solution will only have a finite number of nondegenerate false spikes. Finally, beyond the finite number of nondegenerate true and false spikes, P_τ converges to a number in the interval $(0, 1)$. □

Proof of Theorem 2. We proceed by induction. Let us assume that $\mathcal{G}$ is dense in $\mathcal{S}_p \cap \mathcal{U}_k$. Note that this is true for $k = 2$ due to Lemma 27. Let $z \in \mathcal{U}_{k+1} \cap \mathcal{S}_p$. By Lemma 26, we can assume that $z \in \mathcal{U}_{k+1,g} \cap \mathcal{S}_p$. Let I_0 be a compact interval with nonempty interior such that $v_\infty[z] < k$ in I_0. By an argument which is basically identical to the beginning of the proof of Lemma 27, we get intervals $I_1, \dots, I_l$ with the property that $\mathcal{V}_z$, defined in Lemma 19, is contained in the union of the interiors of the I_i. Furthermore, the I_i are disjoint, and there are an $\varepsilon > 0$ and a T such that after applying an inversion if necessary,

$$-(k-1) + 2\varepsilon \leq \lim_{\tau \to \infty} P_\tau(\tau, \theta) \leq k + 1 - 2\varepsilon,$$

for $\theta \in I_i$. Finally, $v_\infty[z] < k$ on the boundary of I_i. We use the notation $I_i = [\theta_{i1}, \theta_{i2}]$, and $I_{ij,\delta} = [\theta_{ij} - \delta, \theta_{ij} + \delta]$. Since $v_\infty[z](\theta_{ij}) < k$, there are, assuming δ to be small enough, a $\xi > 0$ and a T' such that

$$e^{-\tau} F_{I_{ij,\delta}}[z](\tau) \leq (k - 2\xi)^2$$

for all $\tau \geq T'$. We can now apply Lemma 25 to each of the intervals I_i using δ as above. We thus get a sequence of solutions (Q_m, P_m) converging to the original solution and coinciding with the original solution for $\tau \geq T''$ outside of

$[T'', \infty) \times \cup_{i=1}^{l} I_i$, for some T''. For m large enough, we have

$$e^{-\tau} F_{I_{ij,\delta}}[z_m](\tau) \leq (k-\xi)^2$$

for all i, j and $\tau \geq T'''$ for some T'''. By construction we have $v_\infty[z_m] < k$ on S^1 since $k \geq 2$. If we had $c_0[z] = 0$ to start with, we can use I_0 to correct z_m so that we have $c_0[z_m] = 0$. In doing so, we do not violate the condition $v_\infty[z_m] < k$, for m large enough, due to an argument similar to the proof of Lemma 18 with S^1 replaced by I_0. The theorem follows by induction since $z \in \mathcal{S}_p$ implies $z \in \mathcal{U}_k$ for some $k \in \mathbb{N}$, $k \geq 1$. □

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References

[1] B. K. Berger and D. Garfinkle, Phenomenology of the Gowdy universe on $T^3 \times \mathbf{R}$, *Phys. Rev.* D **57** (1998), 4767–4777. MR 99k:83010

[2] B. K. Berger, J. Isenberg, and M. Weaver, Oscillatory approach to the singularity in vacuum spacetimes with T^2 isometry, *Phys. Rev.* D **64** (2001), 084006, 20. MR 2002h:83073

[3] M. Chae and P. T. Chruściel, On the dynamics of Gowdy space-times, *Comm. Pure Appl. Math.* **57** (2004), 1015–1074. MR 2005i:83085 Zbl 1074.83020

[4] Y. Choquet-Bruhat and R. Geroch, Global aspects of the Cauchy problem in general relativity, *Comm. Math. Phys.* **14** (1969), 329–335. MR 40 #3872 Zbl 0182.59901

[5] P. T. Chruściel, On space-times with U(1) × U(1) symmetric compact Cauchy surfaces, *Ann. Physics* **202** (1990), 100–150. MR 91h:83007 Zbl 0727.53078

[6] P. T. Chruściel and J. Isenberg, Nonisometric vacuum extensions of vacuum maximal globally hyperbolic spacetimes, *Phys. Rev.* D **48** (1993), 1616–1628. MR 94f:83007

[7] P. T. Chruściel, J. Isenberg, and V. Moncrief, Strong cosmic censorship in polarized Gowdy spacetimes, *Class. Quantum Grav.* **7** (1990), 1671–1680. MR 91i:83042 Zbl 0703.53081

[8] P. T. Chruściel and K. Lake, Cauchy horizons in Gowdy spacetimes, *Class. Quantum Grav.* **21** (2004), S153–S169. MR 2005d:83092 Zbl 1047.83004

[9] M. Dafermos and A. D. Rendall, Inextendibility of expanding cosmological models with symmetry, *Class. Quantum Grav.* **22** (2005), L143–L147. MR 2006k:83031 Zbl 1087.83050

[10] Y. Fourès-Bruhat, Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires, *Acta Math.* **88** (1952), 141–225. MR 14,756g Zbl 0049.19201

[11] R. H. Gowdy, Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: topologies and boundary conditions, *Ann. Physics* **83** (1974), 203–241. MR 49 #4521 Zbl 0325.53061

[12] B. Grubišić and V. Moncrief, Asymptotic behavior of the $T^3 \times \mathbf{R}$ Gowdy space-times, *Phys. Rev.* D **47** (1993), 2371–2382. MR 94a:83003

[13] S. KICHENASSAMY and A. D. RENDALL, Analytic description of singularities in Gowdy spacetimes, *Class. Quantum Grav.* **15** (1998), 1339–1355. MR 99d:83077 Zbl 0949.83050

[14] V. MONCRIEF, Global properties of Gowdy spacetimes with $T^3 \times \mathbf{R}$ topology, *Ann. Physics* **132** (1981), 87–107. MR 83e:83007

[15] A. D. RENDALL, Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity, *Class. Quantum Grav.* **17** (2000), 3305–3316. MR 2002e:83066 Zbl 0967.83021

[16] A. D. RENDALL and M. WEAVER, Manufacture of Gowdy spacetimes with spikes, *Class. Quantum Grav.* **18** (2001), 2959–2975. MR 2002e:83010 Zbl 0995.83010

[17] H. RINGSTRÖM, Asymptotic expansions close to the singularity in Gowdy spacetimes, *Class. Quantum Grav.* **21** (2004), S305–S322. MR 2005e:83011 Zbl 1040.83032

[18] ———, On a wave map equation arising in general relativity, *Comm. Pure Appl. Math.* **57** (2004), 657–703. MR 2005h:58049 Zbl 1059.83011

[19] ———, On Gowdy vacuum spacetimes, *Math. Proc. Cambridge Philos. Soc.* **136** (2004), 485–512. MR 2005h:83018 Zbl 1050.83020

[20] ———, Curvature blow up on a dense subset of the singularity in T^3-Gowdy, *J. Hyperbolic Diff. Equations* **2** (2005), 547–564. MR 2006f:83081 Zbl 02202331

[21] ———, Existence of an asymptotic velocity and implications for the asymptotic behavior in the direction of the singularity in T^3-Gowdy, *Comm. Pure Appl. Math.* **59** (2006), 977–1041. MR 2007d:83111 Zbl 1114.83010

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