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In memoriam Atle Selberg

Abstract

Assuming the Riemann hypothesis, we obtain an upper bound for the moments of the Riemann zeta function on the critical line. Our bound is nearly as sharp as the conjectured asymptotic formulae for these moments. The method extends to moments in other families of L-functions.

1. Introduction

An important problem in analytic number theory is to gain an understanding of the moments

$$M_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt.$$

For positive real numbers k, it is believed that $M_k(T) \sim C_k T(\log T)^{k^2}$ for a positive constant C_k. A precise value for C_k was conjectured by Keating and Snaith [11] based on considerations from random matrix theory. Subsequently, an alternative approach, based on multiple Dirichlet series and producing the same conjecture, was given by Diaconu, Goldfeld and Hoffstein [4]. Recent work by Conrey et al [1] gives a more precise conjecture, identifying lower order terms in an asymptotic expansion for $M_k(T)$.

Despite many attempts, asymptotic formulae for $M_k(T)$ have been established only for $k = 1$ (due to Hardy and Littlewood; see [22]) and $k = 2$ (due to Ingham; see [22]). However we do have the lower bound $M_k(T) \gg_k T(\log T)^{k^2}$. This was established by Ramachandra [13] for positive integers $2k$, by Heath-Brown [6] for all positive rational numbers k, and, assuming the truth of the Riemann hypothesis (RH), by Ramachandra [12] for all positive real numbers k. See also the elegant note [2] giving such a bound assuming RH, and [20] for the best known constants

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implicit in these lower bounds. Analogous conjectures exist (see [1], [4], [10]) for moments of central values of L-functions in families, and in many cases lower bounds of the conjectured order are known (see [16] and [17]).

Here we study the problem of obtaining upper bounds for $M_k(T)$. When $0 \le k \le 2$, Ramachandra, in [13] and [14], and Heath-Brown, in [6] and [7], showed, assuming RH, that $M_k(T) \ll T(\log T)^{k^2}$. The Lindelöf hypothesis is equivalent to the estimate $M_k(T) \ll_{k,\varepsilon} T^{1+\varepsilon}$ for all natural numbers k. Thus, for k larger than 2, it seems difficult to make unconditional progress on bounding $M_k(T)$. If we assume RH, then a classical bound of Littlewood (see [22]) gives that (for $t \ge 10$ and some positive constant C)

$$|\zeta(\tfrac{1}{2}+it)| \ll \exp\Big(C\frac{\log t}{\log\log t}\Big), \tag{1}$$

and therefore $M_k(T) \ll T\exp(2kC\log T/\log\log T)$. We improve upon this, obtaining an upper bound of nearly the conjectured order of magnitude.

COROLLARY A. *Assume* RH. *For every positive real number* k, *and every* $\varepsilon > 0$, *we have*

$$T(\log T)^{k^2} \ll_k M_k(T) \ll_{k,\varepsilon} T(\log T)^{k^2+\varepsilon}.$$

Our proof suggests that the dominant contribution to the $2k$-th moment comes from t such that $|\zeta(\frac{1}{2}+it)|$ has size $(\log T)^k$, and this set has measure about $T/(\log T)^{k^2}$.

COROLLARY B. *Assume* RH. *Let* $k \ge 0$ *be a fixed real number. For large* T *we have*

$$\operatorname{meas}\{t \in [0,T] : |\zeta(\tfrac{1}{2}+it)| \ge (\log T)^k\} = T(\log T)^{-k^2+o(1)}.$$

We will deduce these corollaries by finding an upper bound on the frequency with which large values of $|\zeta(\frac{1}{2}+it)|$ can occur. Throughout we define

$$\mathcal{S}(T,V) = \{t \in [T,2T] : \log|\zeta(\tfrac{1}{2}+it)| \ge V\},$$

and observe that

$$\begin{aligned}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}dt &= -\int_{-\infty}^{\infty} e^{2kV}\, d\operatorname{meas}(\mathcal{S}(T,V)) \\ &= 2k\int_{-\infty}^{\infty} e^{2kV}\operatorname{meas}(\mathcal{S}(T,V))dV.\end{aligned} \tag{2}$$

To prove Corollaries A and B, we desire estimates for the measure of $\mathcal{S}(T,V)$ for large T and all $V \ge 3$. To place our main theorem in context, let us recall the beautiful result of Selberg that as t varies in $[T,2T]$, the distribution of $\log|\zeta(\frac{1}{2}+it)|$ is approximately Gaussian with mean 0 and variance $\frac{1}{2}\log\log T$. Precisely, Selberg's

theorem (see [18] and [19]) shows that for any fixed $\lambda \in \mathbb{R}$, and as $T \to \infty$,

$$\operatorname{meas}\Big(\mathcal{S}(T, \lambda\sqrt{\tfrac{1}{2}\log\log T})\Big) = T\Big(\frac{1}{\sqrt{2\pi}}\int_{\lambda}^{\infty} e^{-x^2/2}dx + o(1)\Big).$$

Although Selberg's result holds only for V of size $\sqrt{\log\log T}$, we may speculate that in a much larger range for V a similar estimate holds:

$$\operatorname{meas}(\mathcal{S}(T, V)) \ll T\frac{\sqrt{\log\log T}}{V}\exp\Big(-\frac{V^2}{\log\log T}\Big). \tag{3}$$

Such an estimate would lead, via (2), to the bound $M_k(T) \ll_k T(\log T)^{k^2}$. In our main theorem we establish that a weaker form of (3) holds in the range $3 \le V = o((\log\log T)\log_3 T)$, where throughout we write $\log_3$ for log log log. In the application to moments, the crucial range is when V is of size about $k\log\log T$, and our theorem shows that a version of (3) holds for such V. For larger values of V, we obtain an upper bound of the form

$$\operatorname{meas}(\mathcal{S}(T, V)) \ll T\exp(-\tfrac{1}{129}V\log V)$$

(at least when $\frac{1}{2}(\log\log T)\log_3 T < V$), and the shape of this estimate is in keeping with Littlewood's bound (1).

THEOREM. *Assume* RH. *Let T be large, let $V \ge 3$ be a real number, and let $\mathcal{S}(T, V)$ be as defined above. If $10\sqrt{\log\log T} \le V \le \log\log T$, then*

$$\operatorname{meas}(\mathcal{S}(T, V)) \ll T\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{\log\log T}\Big(1 - \frac{4}{\log_3 T}\Big)\Big);$$

if $\log\log T < V \le \frac{1}{2}(\log\log T)\log_3 T$, then

$$\operatorname{meas}(\mathcal{S}(T, V)) \ll T\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{\log\log T}\Big(1 - \frac{7V}{4(\log\log T)\log_3 T}\Big)^2\Big);$$

and, finally, if $\frac{1}{2}(\log\log T)\log_3 T < V$, then

$$\operatorname{meas}(\mathcal{S}(T, V)) \ll T\exp\Big(-\tfrac{1}{129}V\log V\Big).$$

In the limited range $0 \le V \le \log\log T$, Jutila [8] had previously shown that

$$\operatorname{meas}(\mathcal{S}(T, V)) \ll T\exp\Big(-\frac{V^2}{\log\log T}\Big(1 + O\Big(\frac{V}{\log\log T}\Big)\Big)\Big).$$

I have shown recently in [21] that in the range $3 \le V \le \frac{1}{5}\sqrt{\log T/\log\log T}$ one has

$$\operatorname{meas}(\mathcal{S}(T, V)) \gg \frac{T}{(\log T)^4}\exp\Big(-10\frac{V^2}{\log\frac{\log T}{8V^2\log V}}\Big).$$

In contrast to our theorem above, these results are unconditional.

As mentioned already, we build on Selberg's work on the distribution of $\log\zeta(\frac{1}{2}+it)$. Selberg computed the moments of the real and imaginary parts of $\log\zeta(\frac{1}{2}+it)$. To achieve this, he found an ingenious expression for $\log\zeta(\frac{1}{2}+it)$ in terms of primes. His ideas work very well for the imaginary part of the logarithm, but are more complicated for the real part of the logarithm owing to zeros of the zeta function lying very close to $\frac{1}{2}+it$. One novelty in our work is the realization that if we seek an upper bound for $\log|\zeta(\frac{1}{2}+it)|$ (which is what is needed for our theorem), then the effect of the zeros very near $\frac{1}{2}+it$ is actually benign. This is our main proposition given below, from which we will deduce our theorem. We should comment that Selberg's work is unconditional, and uses zero-density results which put most of the zeros near the critical line. It would be interesting to see how much of our work can be recovered unconditionally.

PROPOSITION. *Assume* RH. *Suppose T is large, let $t\in[T,2T]$, and let $2\le x\le T^2$. Let $\lambda_0=0.4912\ldots$ denote the unique positive real number satisfying $e^{-\lambda_0}=\lambda_0+\lambda_0^2/2$. For all $\lambda\ge\lambda_0$, we have the estimate*

$$\log|\zeta(\tfrac{1}{2}+it)|\le \operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)}{n^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log n}\frac{\log(x/n)}{\log x}+\frac{(1+\lambda)}{2}\frac{\log T}{\log x}+O\Big(\frac{1}{\log x}\Big).$$

Taking $x=(\log T)^{2-\varepsilon}$ in our proposition, and estimating the sum over n trivially, we obtain the following explicit form of Littlewood's bound (1), which improves upon the previous estimate obtained by Ramachandra and Sankaranarayanan [15]. There is certainly some scope to improve our Corollary C below, and it may be instructive to understand what the limit of the method would be (in a way analogous to the elegant treatment of Im $\log\zeta(\frac{1}{2}+it)$ given by Goldston and Gonek [5]).

COROLLARY C. *Assume* RH. *For all large t, we have*

$$|\zeta(\tfrac{1}{2}+it)|\le\exp\Big(\Big(\frac{1+\lambda_0}{4}+o(1)\Big)\frac{\log t}{\log\log t}\Big)\le\exp\Big(\frac{3}{8}\frac{\log t}{\log\log t}\Big).$$

The method developed here is robust and applies equally well to moments in families of L-functions; we discuss this briefly in Section 4 below. We end the introduction by deriving Corollaries A and B.

Proof of Corollaries A *and* B. As mentioned earlier, the lower bound for $M_k(T)$ in Corollary A is due to Ramachandra [12]. The upper bound follows upon inserting the bounds of our theorem into (2). In performing this computation, it is convenient to use our theorem in the crude form

$$\begin{aligned}&\operatorname{meas}(\mathcal{S}(T,V))\ll T(\log T)^{o(1)}\exp(-V^2/\log\log T)\quad\text{for } 3\le V\le 4k\log\log T,\\&\operatorname{meas}(\mathcal{S}(T,V))\ll T(\log T)^{o(1)}\exp(-4kV)\qquad\qquad\text{for } V>4k\log\log T.\end{aligned}$$

From our theorem, we may see that the contribution to $M_k(T)$ from those $t \in [0,T]$ with $|\zeta(\frac{1}{2}+it)| > (\log T)^{k+\varepsilon}$ or with $|\zeta(\frac{1}{2}+it)| < (\log T)^{k-\varepsilon}$ is $o(T(\log T)^{k^2})$. Combining this with the lower bound $M_k(T) \gg_k T(\log T)^{k^2}$ we obtain the lower bound for the measure implicit in Corollary B. The upper bound implicit there follows from our theorem. □

2. **Proof of the main proposition**

In proving our proposition we may suppose that t does not coincide with the ordinate of a zero of $\zeta(s)$. Letting $\rho = \frac{1}{2} + i\gamma$ run over the nontrivial zeros of $\zeta(s)$, we define

$$F(s) = \operatorname{Re} \sum_{\rho} \frac{1}{s-\rho} = \sum_{\rho} \frac{\sigma - 1/2}{(\sigma-1/2)^2 + (t-\gamma)^2}.$$

Visibly $F(s)$ is nonnegative in the half-plane $\sigma \geq 1/2$. Recall Hadamard's factorization formula which gives (see [3, (8) and (11) of Ch. 12])

$$\operatorname{Re} \frac{\zeta'}{\zeta}(s) = -\operatorname{Re} \frac{1}{s-1} + \tfrac{1}{2}\log \pi - \tfrac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma}(\tfrac{1}{2}s+1) + F(s),$$

so that for $t \in [T, 2T]$ an application of Stirling's formula yields

$$-\operatorname{Re} \frac{\zeta'}{\zeta}(s) = \tfrac{1}{2}\log T + O(1) - F(s). \tag{4}$$

Integrating (4) as $\sigma = \operatorname{Re}(s)$ varies from $\frac{1}{2}$ to $\sigma_0 (> \frac{1}{2})$, we obtain, setting $s_0 = \sigma_0 + it$,

$$\begin{aligned} &\log|\zeta(\tfrac{1}{2}+it)| - \log|\zeta(s_0)| \\ &\qquad = \big(\tfrac{1}{2}\log T + O(1)\big)(\sigma_0 - \tfrac{1}{2}) - \int_{1/2}^{\sigma_0} F(\sigma + it)\, d\sigma \\ &\qquad = (\sigma_0 - \tfrac{1}{2})\big(\tfrac{1}{2}\log T + O(1)\big) - \frac{1}{2}\sum_{\rho} \log \frac{(\sigma_0 - \frac{1}{2})^2 + (t-\gamma)^2}{(t-\gamma)^2}. \end{aligned}$$

Since $\log(1+x^2) \geq x^2/(1+x^2)$, we deduce that

$$\log|\zeta(\tfrac{1}{2}+it)| - \log|\zeta(s_0)| \leq (\sigma_0 - \tfrac{1}{2})\big(\tfrac{1}{2}\log T + O(1) - \tfrac{1}{2}F(s_0)\big). \tag{5}$$

LEMMA 1. *Unconditionally, for any s not coinciding with 1 or a zero of $\zeta(s)$ and for any $x \geq 2$, we have*

$$\begin{aligned} -\frac{\zeta'}{\zeta}(s) = \sum_{n \leq x} \frac{\Lambda(n)}{n^s} \frac{\log(x/n)}{\log x} + \frac{1}{\log x}\Big(\frac{\zeta'}{\zeta}(s)\Big)' + \frac{1}{\log x}\sum_{\rho} \frac{x^{\rho - s}}{(\rho - s)^2} \\ - \frac{x^{1-s}}{(1-s)^2 \log x} + \frac{1}{\log x}\sum_{k=1}^{\infty} \frac{x^{-2k-s}}{(2k+s)^2}. \end{aligned}$$

Proof. This is similar to an identity of Selberg; see Titchmarsh [22, Th. 14.20]. With $c = \max(1, 2-\sigma)$, we consider

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(s+w)\frac{x^w}{w^2}dw = \sum_{n\le x}\frac{\Lambda(n)}{n^s}\log(x/n),$$

which follows from integrating term by term using the Dirichlet series expansion of $-(\zeta'/\zeta)(s+w)$. On the other hand, moving the line of integration to the left and calculating residues, this equals

$$-\frac{\zeta'}{\zeta}(s)\log x - \Big(\frac{\zeta'}{\zeta}(s)\Big)' - \sum_\rho \frac{x^{\rho-s}}{(\rho-s)^2} + \frac{x^{1-s}}{(1-s)^2} - \sum_{k=1}^{\infty}\frac{x^{-2k-s}}{(2k+s)^2}.$$

Equating these two expressions we obtain the lemma. □

Take $s = \sigma + it$ in Lemma 1, extract the real parts of both sides, and integrate over σ from σ_0 to ∞. Thus, for $2 \le x \le T^2$,

$$\log|\zeta(s_0)| = \mathrm{Re}\Big(\sum_{n\le x}\frac{\Lambda(n)}{n^{s_0}\log n}\frac{\log(x/n)}{\log x} - \frac{1}{\log x}\frac{\zeta'}{\zeta}(s_0) + \frac{1}{\log x}\sum_\rho \int_{\sigma_0}^{\infty}\frac{x^{\rho-s}}{(\rho-s)^2}d\sigma + O\Big(\frac{1}{\log x}\Big)\Big).$$

Using (4), and observing that

$$\sum_\rho\Big|\int_{\sigma_0}^{\infty}\frac{x^{\rho-s}}{(\rho-s)^2}d\sigma\Big| \le \sum_\rho\int_{\sigma_0}^{\infty}\frac{x^{1/2-\sigma}}{|s_0-\rho|^2}d\sigma = \sum_\rho\frac{x^{1/2-\sigma_0}}{|s_0-\rho|^2\log x} = \frac{x^{1/2-\sigma_0}F(s_0)}{(\sigma_0-1/2)\log x},$$

we deduce that

$$\log|\zeta(s_0)| \le \mathrm{Re}\sum_{n\le x}\frac{\Lambda(n)}{n^{s_0}\log n}\frac{\log(x/n)}{\log x} + \frac{\log T}{2\log x} - \frac{F(s_0)}{\log x} + \frac{x^{1/2-\sigma_0}F(s_0)}{(\sigma_0-1/2)\log^2 x} + O\Big(\frac{1}{\log x}\Big). \tag{6}$$

Adding the inequalities (5) and (6), we obtain that

$$\log|\zeta(\tfrac12+it)| \le \mathrm{Re}\sum_{n\le x}\frac{\Lambda(n)}{n^{s_0}\log n}\frac{\log x/n}{\log x} + \tfrac12\log T\Big(\sigma_0-\tfrac12+\frac{1}{\log x}\Big) + F(s_0)\Big(\frac{x^{1/2-\sigma_0}}{(\sigma_0-1/2)\log^2 x} - \frac{1}{\log x} - \tfrac12(\sigma_0-\tfrac12)\Big) + O\Big(\frac{1}{\log x}\Big). \tag{7}$$

We choose $\sigma_0 = \frac{1}{2} + \lambda/\log x$, where $\lambda \geq \lambda_0$. This restriction on λ ensures that the term involving $F(s_0)$ in (7) makes a negative contribution, and may therefore be omitted. The proposition follows. □

3. **Proof of the theorem**

Our proof of the theorem rests upon our main proposition. We begin by showing that the sum over prime powers appearing there may be restricted just to primes.

LEMMA 2. *Assume* RH. *Let* $T \leq t \leq 2T$, *let* $2 \leq x \leq T^2$, *and let* $\sigma \geq \frac{1}{2}$. *Then*

$$\Big| \sum_{\substack{n \leq x \\ n \neq p}} \frac{\Lambda(n)}{n^{\sigma+it} \log n} \frac{\log x/n}{\log x} \Big| \ll \log\log\log T + O(1).$$

Proof. The terms $n = p^k$ for $k \geq 3$ clearly contribute an amount $\ll 1$, and it remains to handle the terms $n = p^2$. By following closely the explicit formula proof of the prime number theorem (see [3, §§17 and 18]) we obtain that, on RH,

$$\sum_{p \leq z} (\log p) p^{-2it} \ll z/T + \sqrt{z}(\log zT)^2.$$

By partial summation, using this estimate when $z \geq (\log T)^4$ and the trivial $\ll z$ for smaller z, we deduce that for $\sigma \geq \frac{1}{2}$

$$\sum_{p \leq \sqrt{x}} \frac{1}{p^{2\sigma+2it}} \frac{\log(\sqrt{x}/p)}{\log \sqrt{x}} \ll \log\log\log T. \qquad \square$$

We also need a standard mean value estimate whose proof we include for completeness.

LEMMA 3. *Let* T *be large, and let* $2 \leq x \leq T$. *Let* k *be a natural number such that* $x^k \leq T/\log T$. *For any complex numbers* $a(p)$ *we have*

$$\int_T^{2T} \Big| \sum_{p \leq x} \frac{a(p)}{p^{1/2+it}} \Big|^{2k} dt \ll Tk! \Big(\sum_{p \leq x} \frac{|a(p)|^2}{p} \Big)^k.$$

Proof. Write

$$\Big(\sum_{p \leq x} \frac{a(p)}{p^{1/2+it}} \Big)^k = \sum_{n \leq x^k} \frac{a_{k,x}(n)}{n^{1/2+it}},$$

where $a_{k,x}(n) = 0$ unless n is the product of k (not necessarily distinct) primes, all below x. In that case, if we write the prime factorization of n as $n = \prod_{i=1}^r p_i^{\alpha_i}$,

then $a_{k,x}(n) = \binom{k}{\alpha_1,\dots,\alpha_r} \prod_{i=1}^r a(p_i)^{\alpha_i}$. Now

$$\int_T^{2T} \Big|\sum_{p\le x} \frac{a(p)}{p^{1/2+it}}\Big|^{2k} dt = \sum_{m,n\le x^k} \frac{a_{k,x}(m)\overline{a_{k,x}(n)}}{\sqrt{mn}} \int_T^{2T} \Big(\frac{n}{m}\Big)^{it} dt$$
$$= T \sum_{n\le x^k} \frac{|a_{k,x}(n)|^2}{n} + O\Big(\sum_{\substack{m,n\le x^k \\ m\ne n}} \frac{|a_{k,x}(m)a_{k,x}(n)|}{\sqrt{mn}\,|\log(m/n)|}\Big)$$

upon separating the diagonal terms $m = n$ and the off-diagonal terms $m \ne n$. Since $2\big|a_{k,x}(m)a_{k,x}(n)/\sqrt{mn}\big| \le |a_{k,x}(m)|^2/m + |a_{k,x}(n)|^2/n$, we see that the off-diagonal terms above contribute

$$\ll \sum_{n\le x^k} \frac{|a_{k,x}(n)|^2}{n} \sum_{\substack{m\le x^k \\ m\ne n}} \frac{1}{|\log(m/n)|}$$
$$\ll x^k \log(x^k) \sum_{n\le x^k} \frac{|a_{k,x}(n)|^2}{n} \ll T \sum_{n\le x^k} \frac{|a_{k,x}(n)|^2}{n},$$

recalling that $x^k \le T/\log T$. The lemma follows upon noting that

$$\sum_{n\le x^k} \frac{|a_{k,x}(n)|^2}{n}$$
$$= \sum_{p_1<\cdots<p_r\le x} \sum_{\substack{\alpha_1,\dots,\alpha_r\ge 1 \\ \sum\alpha_i = k}} \binom{k}{\alpha_1,\dots,\alpha_r}^2 \frac{|a(p_1)|^{2\alpha_1}\cdots|a(p_r)|^{2\alpha_r}}{p_1^{\alpha_1}\cdots p_r^{\alpha_r}}$$
$$\le k! \sum_{p_1<\cdots<p_r\le x} \sum_{\substack{\alpha_1,\dots,\alpha_r\ge 1 \\ \sum\alpha_i = k}} \binom{k}{\alpha_1,\dots,\alpha_r} \frac{|a(p_1)|^{2\alpha_1}\cdots|a(p_r)|^{2\alpha_r}}{p_1^{\alpha_1}\cdots p_r^{\alpha_r}}$$
$$= k! \Big(\sum_{p\le x} \frac{|a(p)|^2}{p}\Big)^k. \qquad \square$$

In proving our theorem, we may assume that

$$10\sqrt{\log\log T} \le V \le \tfrac{3}{8}\log T/\log\log T.$$

We also keep in mind that T is large. We define a parameter A by

$$A = \begin{cases} \frac{1}{2}\log_3 T & \text{if } V \le \log\log T, \\ \frac{\log\log T}{2V}\log_3 T & \text{if } \log\log T < V \le \frac{1}{2}(\log\log T)\log_3 T, \\ 1 & \text{if } V > \frac{1}{2}(\log\log T)\log_3 T. \end{cases}$$

We further set $x = T^{A/V}$ and $z = x^{1/\log\log T}$. Using Lemma 2 and our proposition, we find that

$$\log|\zeta(\tfrac{1}{2}+it)| \le S_1(t) + S_2(t) + \frac{1+\lambda_0}{2A}V + O(\log\log\log T),$$

where

$$S_1(t) = \Big|\sum_{p\le z} \frac{1}{p^{\frac{1}{2}+\frac{\lambda_0}{\log x}+it}} \frac{\log(x/p)}{\log x}\Big|,$$

$$S_2(t) = \Big|\sum_{z<p\le x} \frac{1}{p^{\frac{1}{2}+\frac{\lambda_0}{\log x}+it}} \frac{\log(x/p)}{\log x}\Big|.$$

If $t \in \mathcal{S}(T,V)$ then we must either have

$$S_2(t) \ge \frac{V}{8A} \qquad \text{or} \qquad S_1(t) \ge V\Big(1-\frac{7}{8A}\Big) =: V_1.$$

By Lemma 3 we see that for any natural number $k \le V/A - 1$, we have

$$\int_T^{2T} |S_2(t)|^{2k}\,dt \ll Tk!\Big(\sum_{z<p\le x}\frac{1}{p}\Big)^k \ll T\big(k(\log\log\log T + O(1))\big)^k.$$

Choosing k to be the largest integer below $V/A-1$, we obtain that the measure of $t \in [T,2T]$ with $S_2(t) \ge V/(8A)$ is

$$\ll T\Big(\frac{8A}{V}\Big)^{2k}(2k\log\log\log T)^k \ll T\exp\Big(-\frac{V}{2A}\log V\Big). \tag{8}$$

Next we consider the measure of the set of $t \in [T,2T]$ with $S_1(t) \ge V_1$. By Lemma 3, we see that for any natural number $k \le \log(T/\log T)/\log z$,

$$\int_T^{2T} |S_1(t)|^{2k}\,dt \ll Tk!\Big(\sum_{p\le z}\frac{1}{p}\Big)^k \ll T\sqrt{k}\Big(\frac{k\log\log T}{e}\Big)^k,$$

so that the measure of $t \in [T,2T]$ with $S_1(t) \ge V_1$ is

$$\ll T\sqrt{k}\Big(\frac{k\log\log T}{eV_1^2}\Big)^k.$$

When $V \le (\log\log T)^2$, we choose k as the greatest integer less than $V_1^2/\log\log T$, and when $V > (\log\log T)^2$ we choose $k = [10V]$. It then follows that the measure of $t \in [T,2T]$ with $S_1(t) \ge V_1$ is

$$\ll T\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V_1^2}{\log\log T}\Big) + T\exp(-4V\log V). \tag{9}$$

Our theorem follows upon combining the estimates (8) and (9).

4. **Moments of L-functions in families**

We briefly sketch here the modifications needed to obtain bounds for moments of L-functions in families. Throughout we assume the generalized Riemann hypothesis for the appropriate L-functions under consideration. If q is a large prime, then Rudnick and Soundararajan [16] showed that, for positive rational numbers k,

$$\sum_{\chi \pmod q}^{*} |L(\tfrac{1}{2}, \chi)|^{2k} \gg_k q(\log q)^{k^2},$$

where the sum is over primitive characters χ. The argument given here carries over to obtain the upper bound $\ll_{k,\varepsilon} q(\log q)^{k^2+\varepsilon}$ for all positive real k. The only difference is that one uses the orthogonality relations of the characters $\pmod q$ to treat nondiagonal terms in the analogue of Lemma 3.

More interesting is the case of quadratic Dirichlet L-functions. In [17] Rudnick and Soundararajan showed that for rational numbers $k \geq 1$

$$\sum_{|d|\leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \gg_k X(\log X)^{k(k+1)/2},$$

where the sum is over fundamental discriminants d, and χ_d denotes the associated primitive quadratic character. To obtain an upper bound, we seek to bound the frequency of large values of $L(\frac{1}{2}, \chi_d)$. Analogously to our proposition, we find that[1], for any $x \geq 2$ and with λ_0 as in our proposition,

$$\log L(\tfrac{1}{2}, \chi_d) \leq \sum_{2\leq n\leq x} \frac{\Lambda(n)\chi_d(n)}{n^{\frac{1}{2}+\frac{\lambda_0}{\log x}}\log n} \frac{\log(x/n)}{\log x} + \frac{(1+\lambda_0)}{2}\frac{\log|d|}{\log x} + O\Big(\frac{1}{\log x}\Big).$$

Notice that, in contrast to Lemma 2, the contribution of the prime squares in our sum is $\sim \frac{1}{2}\log\log x$, since $\chi_d(p^2) = 1$ for all $p \nmid d$. Taking this key difference into account, we may argue as in Section 4, using now quadratic reciprocity and the Pólya-Vinogradov inequality to develop the analogue of Lemma 3. Thus we obtain that the number of d with $|d| \leq X$ and $\log L(\frac{1}{2}, \chi_d) \geq V + \frac{1}{2}\log\log X$ is

$$\ll X \exp\Big(-\frac{V^2}{2\log\log X}(1+o(1))\Big)$$

when $\sqrt{\log\log X} \leq V = o((\log\log X)\log_3 X)$; when $V \geq (\log\log X)\log_3 X$ this number is $\ll X\exp(-cV\log V)$ for some positive constant c. From these estimates we deduce that

$$\sum_{|d|\leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \ll_{k,\varepsilon} X(\log X)^{k(k+1)/2+\varepsilon}.$$

[1]Since we are assuming GRH we know that $L(\frac{1}{2}, \chi_d) \geq 0$. If $L(\frac{1}{2}, \chi_d) = 0$ we interpret $\log L(\frac{1}{2}, \chi_d)$ as $-\infty$ so that the claimed inequality is still valid.

As a last example consider the family of quadratic twists of a given elliptic curve E. We write the L-function for E as $L(E,s)=\sum_{n=1}^{\infty}a(n)n^{-s}$, where the $a(n)$ are normalized so that $|a(n)|\le d(n)$ (the number of divisors of n). The methods of [16] and [17] can be used to show that for rational numbers $k\ge 1$,

$$\sum_{|d|\le X}^{\flat} L(E\otimes\chi_d,\tfrac12)^k \gg_k X(\log X)^{k(k-1)/2}.$$

Writing $a(p)=\alpha_p+\beta_p$ with $\alpha_p\beta_p=1$ we obtain analogously to our proposition that, for $x\ge 2$ and λ_0 as before,

$$\log L(E\otimes\chi_d,\tfrac12)\le \sum_{\substack{n=p^\ell\le x\\ \ell\ge 1}}\frac{\chi_d(p^\ell)(\alpha_p^\ell+\beta_p^\ell)}{\ell n^{1/2+\lambda_0/\log x}}\frac{\log(x/p^\ell)}{\log x} + (1+\lambda_0)\frac{\log|d|}{\log x}+O\Big(\frac{1}{\log x}\Big).$$

The contribution of the terms $\ell\ge 3$ above is $O(1)$, and the contribution of $\ell=2$ (the prime squares) is

$$\sum_{\substack{p\le\sqrt{x}\\ p\nmid d}}\frac{a(p^2)-1}{2p^{1+2\lambda_0/\log x}}\frac{\log(x/p^2)}{\log x}\sim -\tfrac12\log\log x.$$

After taking this feature into account, we may develop the analogous argument of Section 4. Thus, the number of d with $|d|\le X$ and $\log L(E\otimes\chi_d,\frac12)\ge V-\frac12\log\log X$ is

$$\ll X\exp\Big(-\frac{V^2}{2\log\log X}(1+o(1))\Big)$$

when $\sqrt{\log\log X}\le V=o((\log\log X)\log_3 X)$; when $V\ge(\log\log X)\log_3 X$ this number is $\ll X\exp(-cV\log V)$ for some positive constant c. This leads to the upper bound

$$\sum_{|d|\le X}^{\flat} L(E\otimes\chi_d,\tfrac12)^k \ll X(\log X)^{k(k-1)/2+\varepsilon}.$$

Our work above is in keeping with conjectures of Keating and Snaith [10] (see (51) and (79) there) that an analogue of Selberg's result holds in families of L-functions. Thus in the unitary family of $\chi \pmod q$, we expect that the distribution of $\log|L(\frac12,\chi)|$ is Gaussian with mean 0 and variance $\sim\frac12\log\log q$. In the symplectic family of quadratic Dirichlet characters, we expect that the distribution of $\log L(\frac12,\chi_d)$ is Gaussian with mean $\sim\frac12\log\log|d|$ and variance $\sim\log\log|d|$. Thus most values of $L(\frac12,\chi_d)$ are quite large. In the orthogonal family of quadratic twists of an elliptic curve, first we must restrict to those twists with positive sign of the functional equation, or else the L-value is 0. In this restricted

class, we expect that the distribution of $\log L(E \otimes \chi_d, \frac{1}{2})$ is Gaussian with mean $\sim -\frac{1}{2}\log\log|d|$ and variance $\sim \log\log|d|$. Thus the values in an orthogonal family tend to be small. With a little more work, the ideas in this paper would show (assuming GRH) that the frequency with which $\log|L|$ exceeds Mean $+ \lambda \cdot \sqrt{\text{Var}}$ is bounded above by $\frac{1}{\sqrt{2\pi}}\int_\lambda^\infty e^{-x^2/2}dx$ for any fixed real number λ. If in addition to GRH, we assume that most of the L-functions under consideration do not have a zero very near[2] $\frac{1}{2}$, then Selberg's techniques would yield these Keating-Snaith analogues.

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