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Abstract

We show that a finite number of commuting diffeomorphisms with simultaneously Diophantine rotation numbers are smoothly conjugated to rotations. This solves a problem raised by Moser.

1. **Introduction**

In this paper, we show that if a finite number of commuting smooth circle diffeomorphisms have simultaneously Diophantine rotation numbers (arithmetic condition (1) below), then the diffeomorphisms are smoothly (and simultaneously) conjugated to rotations (see Theorem 1 below).

The problem of smooth linearization of commuting circle diffeomorphisms was raised by Moser in [11] in connection with the holonomy group of certain foliations with codimension 1. Using the rapidly convergent Nash-Moser iteration scheme he proved that if the rotation numbers of the diffeomorphisms satisfy a simultaneous Diophantine condition and if the diffeomorphisms are in some C^∞ neighborhood of the corresponding rotations (the neighborhood being imposed by the constants appearing in the arithmetic condition, as usual in perturbative KAM theorems) then they are C^∞-linearizable, that is, C^∞-conjugated to rotations.

In terms of small divisors, the latter result presented a new and striking phenomenon: if d is the number of commuting diffeomorphisms, the rotation numbers of some or of all the diffeomorphisms may well be non-Diophantine, but still, the full $\mathbb{Z}^d$-action is smoothly linearizable due to the absence of simultaneous resonances. Further, Moser showed in his paper that this new phenomenon is a *genuine* one in the sense that the problem cannot be reduced to that of a single diffeomorphism with a Diophantine frequency. Indeed, it is shown that there exist numbers $\theta_1, \ldots, \theta_d$ that are simultaneously Diophantine but such that for all linearly independent vectors

$a, b \in \mathbb{Z}^{d+1}$, the ratios

$$\frac{a_0 + a_1\theta_1 + \cdots + a_d\theta_d}{b_0 + b_1\theta_1 + \cdots + b_d\theta_d}$$

are Liouville numbers. In this case, the theory for individual circle maps does not suffice to conclude smooth linearization..

According to Moser, the problem of linearizing commuting circle diffeomorphisms could be regarded as a model problem where KAM techniques can be applied to an overdetermined system (due to the commutation relations). This assertion could again be confirmed by the recent work [2] where local rigidity of some higher rank abelian groups was established using a KAM scheme for an overdetermined system.

At the time Moser was writing his paper, the *global* theory of linearization for circle diffeomorphisms (Herman's theory) was already known. A highlight result is that a diffeomorphism with a Diophantine rotation number is smoothly linearizable (without a *local* condition of closeness to a rotation). The proof of the first global smooth linearization theorem given by Herman [5], as well as all subsequent proofs and generalizations ([12], [13], [9], [10], [8], [7]), extensively used the Gauss algorithm of continued fractions that yields the best rational approximations for a real number.

As pointed out in Moser's paper, one of the reasons why the related global problem for a commuting family of diffeomorphisms with rotation numbers satisfying a simultaneous Diophantine condition is difficult to tackle, is precisely the absence of an analogue of the one dimensional continuous fractions algorithm in the case of simultaneous approximations of several numbers (by rationals with the same denominator). Although in a certain sense such algorithms were later developed and even used in the KAM setting, our approach is based on different ideas.

Moser asked *under which conditions on the rotation numbers of n smooth commuting circle diffeomorphisms can one assert the existence of a smooth invariant measure* μ? *In particular is the simultaneous Diophantine condition sufficient*? Here, we answer this question positively (Theorem 1, the existence of a smooth invariant measure being an equivalent statement to smooth conjugacy). On the other hand, it is not hard to see that the same arithmetic condition is optimal (even for the local problem) in the sense given by Remark 1.

Before we state our results and discuss the plan of the proof, we give a brief summary of the linearization theory of single circle diffeomorphisms on which our proof relies.

We denote the circle by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and by $\mathrm{Diff}^r_+(\mathbb{T})$, $r \in [0, +\infty] \cup \{\omega\}$, the group of orientation-preserving diffeomorphisms of the circle of class C^r or real analytic. We represent the lifts of these diffeomorphisms as elements of $D^r(\mathbb{T})$, the group of C^r-diffeomorphisms $\tilde{f}$ of the real line such that $f - \mathrm{Id}_{\mathbb{R}}$ is $\mathbb{Z}$-periodic.

Following Poincaré, one can define the rotation number of a circle homeomorphism f as the uniform limit

$$\rho_f = \lim_{j\to\infty} \frac{\tilde{f}^j(x) - x}{j} \mathrm{mod}[1],$$

where $\tilde{f}^j$ ($j \in \mathbb{Z}$) denote the iterates of a lift of f. A rotation map of the circle with angle θ, denoted by $R_\theta : x \mapsto x + \theta$, has clearly a rotation number equal to θ. Poincaré raised the problem of comparing the dynamics of a homeomorphism of the circle with rotation number θ to the simple rotation R_θ.

A classical result of Denjoy (1932) asserts that if $\rho_f = \theta$ is irrational (not in $\mathbb{Q}$) and if f is of class C^1 with the derivative Df of bounded variations then f is topologically conjugated to R_θ, i.e., there exists a circle homeomorphism h such that $h \circ f \circ h^{-1} = R_\theta$.

Considering the linearized version of the conjugation equation $H(x+\theta) - H(x) = F(x)$ where H and F are real $\mathbb{Z}$-periodic functions defined on $\mathbb{R}$ and where F is assumed to have zero mean, one sees easily (with Fourier analysis, due to the presence of the *small divisors* $|1 - e^{i2\pi n\theta}|$) that the existence of a smooth solution H, is guaranteed for all functions F with zero mean if and only if θ satisfies a Diophantine condition, i.e., if there exist $C > 0$ and $\tau > 0$ such that for any $k \in \mathbb{Z}$, $\|k\theta\| \geq C|k|^{-\tau}$. Nonetheless, when F is in some finite class of differentiability and the linearized equation has a solution, this solution in general is of lower regularity than F. This is the so-called *loss of regularity* phenomenon.

The first result asserting regularity of the conjugation of a circle diffeomorphism to a rotation was obtained by Arnold [1] in the real analytic case: if the rotation number of a real analytic diffeomorphism satisfies certain Diophantine conditions and if the diffeomorphism is sufficiently close to a rotation, then the conjugation is analytic. This result has been proven using the KAM approach. The general idea, due to Kolmogorov, is to use a quadratic Newton approximation method to show that if we start with a map sufficiently close to the rotation it is possible to compose successive conjugations and get closer and closer to the rotation while the successive conjugating maps tend rapidly to the identity. The Diophantine condition is used to control the loss of differentiability in the linearized equation which allows us to compensate this loss at each step of the algorithm due to its quadratic convergence. Applying the same Newton scheme in the C^∞ setting is essentially due to Moser.

At the same time, Arnold also gave examples of real analytic diffeomorphisms with irrational rotation numbers for which the conjugating maps are not even absolutely continuous, thus showing that the small divisors effect was inherent to the regularity problem of the conjugation. Herman also showed that there exist "pathological" examples for any non-Diophantine irrational (Liouville) rotation number (see [5, Chap. XI]; see also [4]).

A crucial conjecture was that, to the contrary, the hypothesis of closeness to rotations should not be necessary for smooth linearization, that is, any smooth diffeomorphism of the circle with a Diophantine rotation number must be smoothly conjugated to a rotation. This *global* statement was finally proved by Herman in [5] for almost every rotation number, and later on by Yoccoz in [12] for all Diophantine numbers. In the end of the 80's two different approaches to the Herman theory were developed by Khanin, Sinai ([9], [10]) and Katznelson, Ornstein ([8], [7]). These approaches give sharp results on the smoothness of the conjugacy in the case of diffeomorphisms of finite and low smoothness. In principle all three approaches can be used to study the case of commuting diffeomorphisms. In the present paper we focus on the C^∞ and C^ω case and use the classical Herman-Yoccoz approach.

Herman and Yoccoz developed powerful machinery giving sharp estimates on derivatives growth for the iterates of circle diffeomorphisms, the essential criterion for the C^r regularity of the conjugation of a C^k diffeomorphism f, $k \geq r \geq 1$, being the fact that the family of iterates (f^n) should be bounded in the C^r topology. The Herman-Yoccoz estimates on the growth of derivatives of the iterates of f will be crucial for us in all the paper.

2. **Results**

For $\theta \in \mathbb{T}$ and $r \in [1, +\infty] \cup \{\omega\}$, we denote by $\mathfrak{D}^r_\theta$ the subset of $\mathrm{Diff}^\infty_+(\mathbb{T})$ of diffeomorphisms having rotation number θ.

Let $d \in \mathbb{N}$, $d \geq 2$, and assume that $(\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$ are such that there exist $\nu > 0$ and $C > 0$ such that for each $k \in \mathbb{Z}^*$,

$$\max(\|k\theta_1\|, \ldots, \|k\theta_d\|) \geq C|k|^{-\nu}. \tag{1}$$

Finally, we say that a family of circle diffeomorphisms $(f_1, \ldots, f_d)$ is *commuting* if $f_i \circ f_j = f_j \circ f_i$ for all $1 \leq i \leq j \leq d$. Here we assume that $f_i \in \mathfrak{D}^r_{\theta_i}$, $1 \leq i \leq d$. Note that if h is a homeomorphism of the circle such that $h \circ f_1 \circ h^{-1} = R_{\theta_1}$, then for every $j \leq d$ we have that $h \circ f_j \circ h^{-1}$ commutes with R_{θ_1}, from which it is easy to see that $h \circ f_j \circ h^{-1} = R_{\theta_j}$. Hence, for $r \geq 2$, Denjoy theory gives a homeomorphism that conjugates every f_j to the corresponding rotation. Here, we prove the following.

THEOREM 1. *Assume that* $\theta_1, \ldots, \theta_d$ *satisfy* (1) *and consider* $f_i \in \mathfrak{D}^\infty_{\theta_i}$, *for* $i = 1, \ldots, d$. *If the family* $(f_1, \ldots, f_d)$ *is commuting, there exists* $h \in \mathrm{Diff}^\infty_+(\mathbb{T})$, *such that* $h \circ f_i \circ h^{-1} = R_{\theta_i}$ *for each* $1 \leq i \leq d$.

Remark 1. This sufficient arithmetic condition is also necessary to guarantee some regularity on the conjugating homeomorphism h (essentially unique, up to translation). As in the case of individual maps (see for example [5, Chap. XI] and [4]) there is indeed a sharp dichotomy in the statement of Theorem 1 in case the arithmetic condition (1) is not satisfied: *Assume that* $\theta_1, \ldots, \theta_d$ *do not satisfy* (1); *then there exist* $f_i \in \mathcal{D}^{\infty}_{\theta_i}$, $i = 1, \ldots, d$ *such that a family* $(f_1, \ldots, f_d)$ *is commuting and such that the conjugating homeomorphism of the maps* f_i *to the rotations* R_{θ_i} *is not absolutely continuous.* For the reader's convenience, we briefly describe how the construction of a single diffeomorphism with a non absolutely continuous invariant measure can be applied to the construction of a commuting family. We first recall the general scheme of the construction based on successive conjugations (for more details, see for example [6]). Given a sequence of fast converging rationals $\alpha_n = p_n/q_n$, a diffeomorphism f is constructed as

$$f = \lim H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1},$$

where

$$H_n = h_1 \circ \cdots \circ h_n \quad \text{and} \quad h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n.$$

The convergence of the construction in the C^{∞} category is guaranteed by the fact that $|\alpha_{n+1} - \alpha_n|$ is much smaller than $1/\|H_n\|_{C^n}$, while the fact that the limit diffeomorphism does not preserve an absolutely continuous measure is guaranteed by a specific increasingly strong distortion of the successive conjugations h_n. The "wildness" of h_n translates into its C^n-norms growing faster than any power of q_n, $q_n^{n^2}$ being sufficient for the purpose at hand. The consistency between the above requirements is insured by the fast convergence of the sequence α_n; for example if $\|H_n\|_{C^n}$ is of the order of $q_n^{n^2}$, it is sufficient for the convergence of the scheme that $|\alpha_{n+1} - \alpha_n| \le q_n^{-n^3}$. Within this scheme, given any Liouville number α, the sequence α_n can be chosen and the construction can be made so that the limit diffeomorphism has rotation number α. In fact, if $\tilde{\alpha}_n$ is an equally fast converging sequence as α_n, and has the same denominators q_n, then the same construction with the same conjugacies h_n but with α_n replaced by $\tilde{\alpha_n}$ would converge to a diffeomorphism $\tilde{f}$ with rotation number $\tilde{\alpha} = \lim \tilde{\alpha}_n$. By construction $\tilde{f}$ commutes with the diffeomorphism f. Finally, the existence of the sequences α_n and $\tilde{\alpha}_n$ is exactly equivalent to the condition that α and $\tilde{\alpha}$ are not simultaneously Diophantine. It is easy to see that the argument we have illustrated in the case of two diffeomorphisms is valid for an arbitrary number of frequencies.

As a corollary of Theorem 1 and of the local theorem (on commuting diffeomorphisms) of Moser in the real analytic category [11] we have by the same techniques as in [5, Chap. XI. 6]:

COROLLARY 1. *Assume that* $\theta_1, \ldots, \theta_d$ *satisfy* (1) *and consider* $f_i \in \mathcal{D}^{\omega}_{\theta_i}$, *for* $i = 1, \ldots, d$. *If* $(f_1, \ldots, f_d)$ *is commuting, there exists* $h \in \mathrm{Diff}^{\omega}_{+}(\mathbb{T})$, *such that* $h \circ f_i \circ h^{-1} = R_{\theta_i}$ *for each* $1 \le i \le d$.

In the analytic setting the condition (1) is not optimal although it is necessary to impose some arithmetic condition. It is possible to show that in the case when the rotation numbers $(\theta_1, \dots, \theta_d) \in \mathbb{T}^p$ are such that there exist $a \in (0,1)$ and infinitely many $k \in \mathbb{N}$ satisfying

$$\max(\|k\theta_1\|, \dots, \|k\theta_d\|) \leq a^k$$

then it is possible to construct a commuting family $(f_1, \dots, f_d) \in \mathcal{D}^\omega_{\theta_1} \times \cdots \times \mathcal{D}^\omega_{\theta_d}$ such that the conjugating homeomorphism of the maps f_i to the rotations R_{θ_i} is not absolutely continuous.

It is a delicate problem however to find the optimal arithmetic condition under which any commuting family of real analytic diffeomorphisms will be linearizable in the real analytic category. For a single real analytic diffeomorphism, the optimal condition was obtained by Yoccoz in [13].

3. Plan of the proof of Theorem 1

As in the global theory of circle diffeomorphisms, we will start by proving the C^1 regularity of the conjugation and then we will derive from it, by Hadamard convexity inequalities and bootstrap techniques, the C^∞ regularity. In each of these two moments of the proof the commutativity of the diffeomorphisms in question will be used differently.

The first step in the proof is a simple arithmetic observation for which we need the following definition: given an angle θ we say that a sequence of successive denominators of θ, $q_l, q_{l+1}, \dots, q_n$, is a *Diophantine string of exponent* $\tau > 1$ if for all $s \in [l, n-1]$, $q_{s+1} \leq q_s^\tau$. The observation is that if we consider a sufficiently large number of angles $\theta_1, \dots, \theta_p$ such that each d-tuple satisfies (1) then we can find Diophantine strings of the same exponent τ (function of ν and d) for different θ_j's, such that these strings overlap (with a margin that can be made arbitrarily large when the number of angles considered is increasing). In other words, one can follow successive denominators along a Diophantine string i until its end, say at some q_{j_i,n_i}, where it is possible to switch to the next string $i+1$ starting from a denominator $q_{j_{i+1},l_{i+1}}$ that is much smaller than q_{j_i,n_i} ($q_{j_{i+1},l_{i+1}} \leq q^\xi_{j_i,n_i}$, ξ as small as desired if the number p increases). The next elementary but crucial observation is that given $f_1, \dots, f_d$ with rotation numbers $\theta_1, \dots, \theta_d$ satisfying (1), it is possible, by consideration of compositions of these diffeomorphisms, to obtain as many diffeomorphisms as desired in such a way that any d-tuple formed by their rotation numbers will satisfy (1). Sections 4 and 5 deal with these results on the alternated configuration of Diophantine strings.

With this configuration in hand the proof of C^1-conjugacy goes as follows. First, to simplify the notation we consider only the case $d = 2$ (the proof for $d \geq 3$

is exactly the same) and assume that the Diophantine strings of $\theta = \rho_{f_1}$ and $\beta = \rho_{f_2}$ themselves form an alternated configuration (Conditions (4)–(6)). Using notation m_n and M_n for the minimum and the maximum on the circle of $|x - f^{q_n}(x)|$ (where q_n denotes the denominators of the convergents of θ, and with similar notations $\tilde{q}_n$, $\tilde{m}_n$, and $\tilde{M}_n$ for β and g), we see that a criterion for C^1-conjugacy of f to a rotation is that the ratio M_n/m_n be bounded. It is known that $m_n \leq \theta_n \leq M_n$ where $\theta_n = |q_n\theta - p_n|$ and the goal is to show that eventually both m_n and M_n become comparable to θ_n up to a multiplicative constant. In [12] a crucial recurrence relation between these quantities at the steps n and $n+1$ is exhibited that allows us to show that the quantities m_n and M_n end up having the same order, provided that a Diophantine condition on θ is satisfied. The latter recurrence relation is obtained as a result of the analysis of the growth of the Schwartzian derivatives of the iterates of f.

Here we will rely on the same recurrence relation but use it only along the Diophantine strings and try to propagate the improvement of estimates when we switch strings using the commutation relation between f and g. Actually this will work efficiently once M_s is not too big compare with θ_s, namely, if M_s for q_s in some Diophantine string for θ is less than $\theta_s^{1-\sigma}$ for some fixed $\sigma > 0$ that depends on τ (it is possible to take $\sigma = 1/(2\tau^2)$). This can be interpreted as a "local" result that yields C^1 conjugation for diffeomorphisms that are close to rotations (see Proposition 5).

The existence of very long Diophantine strings (which corresponds to one of the angles being super-Liouville) presents the simplest case illustrating how the local situation can indeed be reached using only one string (see §6.3).

In general however, before reaching the local situation, switching from one string to a consecutive one may in fact lead to a loss of control in the estimates (see the first equation in the proof of Lemma 3), so that a different strategy must be adopted. Keeping in mind that the objective is to show that $u_s \to 1$ where u_s is defined by $M_s = \theta_s^{u_s}$ (with $\tilde{M}_s$, $\tilde{u}_s$, and β_s for β and g), the idea is to use each angle alone to study "the dynamics" of u_s: after we measure the gain in the exponent u when we pass through a Diophantine string, we jump to the beginning of the successive string of the same angle. In this operation we can readily bound the loss in the exponent u as function of the size of the jump (which in turn is less than the size of the overlapping Diophantine string of the other angle). Repeating these two steps inductively, we get a dynamics on the exponent u_i measured at the exit of the i^{th} Diophantine string (of the same angle, see Lemma 4). Doing so for each angle we see that at least for one of them, namely the one with the overall longest Diophantine strings (in the sense given by (18) or (19)), the sequence u_i (or $\tilde{u}_i$) eventually becomes larger than $1-\sigma$.

The idea for proving higher regularity is to use convexity arguments as in [5], [12] to bound the derivatives of the iterates of f and g. However, in our case we will only seek to bound these derivatives for iterates f^u and g^v at *Diophantine times* u and v that are (respectively) linear combinations of multiples of denominators q_s and $\tilde{q}_s$ that belong to Diophantine strings (each q_s is as usual multiplied by at most q_{s+1}/q_s). Due to the overlapping of strings, this will be sufficient for proving regularity of the conjugation (see §7.1).

Given a denominator q_s in a Diophantine string, the fact that the ratio q_{s+1}/q_s is bounded by a fixed power of q_s is naturally crucial in the control of the derivatives of the diffeomorphisms $f^{aq_s}, a \le q_{s+1}/q_s$. Although in the Herman-Yoccoz theory for circle diffeomorphisms with Diophantine rotation number, the control of the derivatives of f^{q_s} is obtained using the Diophantine condition on the diffeomorphism's rotation number (see the computations in [12, §8]), one can show (see §7.2 below), that the existence of a sufficiently long sequence of Diophantine strings before and up to some denominator q_s, combined with the existence of a C^1-conjugacy to a rotation, allows us to prove a bound on the derivatives of f^{q_s} which is enough for our purpose.

Thus, in addition to the alternation of Diophantine strings used for C^1 regularity we must make sure that there is enough Diophantine "margin" before q_{l_i}. This is done (in Proposition 2) through the use of even greater number of angles θ_i, which amounts to considering more diffeomorphisms of the form $f^i \circ g$. In a sense, we use more and more relations in the commuting group of diffeomorphisms as we want to improve the regularity of the conjugation.

The rest of the proof of higher regularity is inspired by the bootstrap calculations of [12].

Nowhere in our proof of Theorem 1, neither in the proof of the existence of C^1-conjugation nor in that of its higher regularity, did we try to optimize our *use of derivatives* of the diffeomorphism f, that is assumed to be of class C^∞. The case of commuting families of finite and low smoothness will be considered elsewhere ([3]). The class of regularity on commuting diffeomorphisms that would guarantee C^1-conjugation under a given simultaneous Diophantine condition is an interesting problem that is not addressed in this paper.

4. **Preliminary: Diophantine strings**

We recall that for every irrational number θ we can uniquely define an increasing sequence of integers q_n such that $q_1 = 1$ and

$$\|k\theta\| > \|q_n\theta\| \quad \text{for all } k < q_{n+1}, k \neq q_n.$$

This sequence is called the sequence of denominators of the best rational approximations, or convergents, of α.

Let $p \in \mathbb{N}$, and $\theta_1, \ldots, \theta_p$ be irrational numbers. For $1 \le j \le p$, we denote by $(q_{j,n})$ the sequence of denominators of the convergents of θ_j. For $\tau > 0$, we define

$$\mathcal{A}_\tau(\theta_j) = \{s \in \mathbb{N} \ / \ q_{j,s+1} \le q_{j,s}^{\tau}\}.$$

A Diophantine string (with exponent τ) for a number θ_i is then a sequence $l, l+1, \ldots, n-1 \in \mathcal{A}_\tau(\theta_i)$.

We will prove in this section the main arithmetic result related to the simultaneous Diophantine property (1) to be used to prove Theorem 1. Given $\nu > 0$ we introduce the sequence (τ_s) given by $\tau_0 = \nu$ and $\tau_s = 2\tau_{s-1} + 3$, for $s \ge 1$.

PROPOSITION 1. *Let $\nu > 0$, $K > 0$ and $d \in \mathbb{N}$, $d \ge 2$. There exists $p \in \mathbb{N}$ such that: if $\theta_1, \ldots, \theta_p$ are numbers for which there exists $C > 0$ such that each $d-$tuple (of disjoint numbers) $(\theta_{i_1}, \ldots, \theta_{i_d})$ satisfies (1); if $U > 0$ is sufficiently large and if $U \le V \le U^K$, then there exists $k \in \{1, \ldots, p\}$, with a Diophantine string $l, \ldots, n-1 \in \mathcal{A}_{\tau_{d-1}}(\theta_k)$ with*

$$q_{k,l} \le U \le V \le q_{k,n}.$$

Definition 1. For $\tau > 0$, $C > 0$, $d \in \mathbb{N}^*$, and an interval $I \subset \mathbb{R}$ we define

$$D_{d,\tau,C}(I) = \{(\theta_1, \ldots, \theta_d) \in \mathbb{R}^d \ / \ \sup_{1 \le i \le d} \|k\theta_i\| \ge C k^{-\tau}, \ \forall k \in I \cap \mathbb{N}\}.$$

For $C = 1$, we use the simplified notation $D_{d,\tau}(I) := D_{d,\tau,1}(I)$.

We will need the following elementary but crucial argument.

LEMMA 1. *Let $\nu > 0, C > 0, d \in \mathbb{N}, d \ge 2$. Define $\epsilon = 1/(2\nu+2)$. There exists U_0 such that if $V \ge U \ge U_0$, and if $\theta_1, \ldots, \theta_d$ are numbers such that*

$$(\theta_1, \ldots, \theta_d) \in D_{d,\nu,C}([U, V]),$$

and if an integer $s \in [U, V^{1/2}]$ satisfies $\|s\theta_d\| \le s^{-(2\nu+3)}$, then

$$(\theta_1, \ldots, \theta_{d-1}) \in D_{d-1,2\nu+3}([s, e])$$

with $e = \min(V^{1/2}, \|s\theta_d\|^{-\epsilon})$.

Proof. If $k \in [s, e]$ satisfies

$$\sup_{i \le d-1} \|k\theta_i\| \le k^{-(2\nu+3)},$$

we claim that the number $ks \in [U, V]$ satisfies

$$\sup_{i \le d} \|ks\theta_i\| \le (ks)^{-(\nu+\frac{1}{2})},$$

which violates $(\theta_1, \ldots, \theta_d) \in D_{d,\nu,C}([U, V])$, if s is sufficiently large. To prove the claim, observe that for $i \le d-1$,

$$\|ks\theta_i\| \le sk^{-(2\nu+3)} \le k^{-2(\nu+1)} \le (ks)^{-(\nu+\frac{1}{2})},$$

while, for $i = d$,

$$\|ks\theta_d\| \le k^{-(\nu+\frac{1}{2})} k^{\nu+\frac{3}{2}} \|s\theta_d\| \le k^{-(\nu+\frac{1}{2})} \|s\theta_d\|^{1-\epsilon(\nu+\frac{3}{2})}$$
$$\le k^{-(\nu+\frac{1}{2})} s^{-(\nu+\frac{1}{2})}. \qquad \square$$

Because $\eta = (2\nu+3)/(2\nu+2) > 1$, Lemma 1 has the following immediate consequence.

COROLLARY 2. *Let $\nu > 0$, $K > 0$ and $d \in \mathbb{N}$, $d \ge 2$. There exists $N \in \mathbb{N}$ such that: for each $C > 0$, there exists $U_0 > 0$, such that if $U \ge U_0$ and $U \le V \le U^K$, and if $p \ge N + d - 1$ and $\theta_1, \dots, \theta_p$ are numbers such that for each d−tuple (of disjoint indices) $i_1, \dots, i_d$, $(\theta_{i_1}, \dots, \theta_{i_d}) \in D_{d,\nu,C}([U, V])$, then there exist $j_1, \dots, j_N \le p$ such that any $(d-1)$-tuple (of disjoint indices), $i_1, \dots, i_{d-1} \in \{1, \dots, p\} - \{j_1, \dots, j_N\}$, satisfies $(\theta_{i_1}, \dots, \theta_{i_{d-1}}) \in D_{d-1,2\nu+3}([U, V^{1/2}])$.*

Proof. We can in fact take $N = [\ln K/\ln \eta] + 2$. Let $p \ge N + d - 1$ and let $k_1 \in \mathbb{N}$, $k_1 \ge U$, be the smallest integer (if it exists) such that $\|k_1\theta_i\| \le {k_1}^{-(2\nu+3)}$ for some $i \in \{1, \dots, p\}$. Denote by θ_{j_1} the corresponding angle. Take $\epsilon = 1/(2\nu+2)$, as in Lemma 1. Then, define $k_2 \ge \|k_1\theta_i\|^{-\epsilon}$ to be the smallest integer (if it exists) such that $\|k_2\theta_i\| \le {k_1}^{-(2\nu+3)}$ for some $i \in \{1, \dots, p\} - \{j_1\}$ and denote by θ_{j_2} the corresponding angle. We can thus construct sequences $k_1, \dots, k_N$, $j_1, \dots, j_N$, and $\theta_{j_1}, \dots, \theta_{j_N}$ such that

- $k_{i+1} \ge \|k_i\theta_{j_i}\|^{-\epsilon}$;
- $\|k_i\theta_i\| \le {k_i}^{-(2\nu+3)}$;
- for every $l \in \{1, \dots, p\} - \{j_1, \dots, j_N\}$, and for every

  $$k \in [U, k_1) \cup (\|k_1\theta_{j_1}\|^{-\epsilon}, k_2) \cup \cdots \cup (\|k_{N-1}\theta_{j_{N-1}}\|^{-\epsilon}, k_N),$$

  we have $\|k\theta_l\| \ge k^{-(2\nu+3)}$.

On the other hand, Lemma 1 implies that for any $s \le N$ and for any $(d-1)$-tuple (of disjoint indices) $i_1, \dots, i_{d-1} \in \{1, \dots, p\} - \{j_1, \dots, j_N\}$, we have $(\theta_{i_1}, \dots, \theta_{i_{d-1}}) \in D_{d-1,2\nu+3}([k_s, \min(\|k_s\theta_{j_s}\|^{-\epsilon}, V^{1/2})])$. Now, the crucial observation is that $k_N \ge k_1^{\eta^N} > V^{1/2}$, which implies

$$(\theta_{i_1}, \dots, \theta_{i_{d-1}}) \in D_{d-1,2\nu+3}([U, V^{1/2}]). \qquad \square$$

Proof of Proposition 1. We assume that p and U are sufficiently large and start by replacing U and V with $U^{1/(2\tau_{d-1})}$ and $V^{2^{d-1}}$. We then apply Corollary 2 $d-1$ times to get that there exists $k \in \{1, \dots, p\}$ such that $\theta_k \in D_{1,\tau_{d-1}}([U^{1/(2\tau_{d-1})}, V])$. We claim that θ_k satisfies the properties required in Proposition 1. Indeed, it is sufficient to prove that θ_k must have a denominator $q_{k,l} \in [U^{1/(2\tau_{d-1})}, U]$. But if this is not so, there is some $q_{k,l} \le U^{1/(2\tau_{d-1})}$ such that $q_{k,l+1} \ge U$, but then

$m = q_{k,l} U^{1/(2\tau_{d-1})} \leq U$ satisfies $\|m\theta_k\| \leq m^{-\tau_{d-1}}$, contradicting the condition $\theta_k \in D_{1,\tau_{d-1}}([U^{1/(2\tau_{d-1})}, V])$. □

5. Alternated configuration of denominators

Definition 2. We say that $\theta_1, \dots, \theta_p$ are in an *alternated configuration* if there exist $\tau > 1$, and two increasing sequences of integers, l_i and n_i such that for each i there exists $j_i \in \{1, \dots, p\}$ with

$$l_i, l_i + 1, l_i + 2, \dots, n_i - 1 \in \mathcal{A}_\tau(\theta_{j_i}), \tag{2}$$

and

$$q_{j_i,l_i}^{\tau^2} \leq q_{j_i,n_i}^{\frac{1}{\tau^2}} \leq q_{j_{i+1},l_{i+1}} \leq q_{j_i,n_i}^{\frac{1}{\tau}}. \tag{3}$$

We shall call $\tau > 1$ the exponent of the alternated configuration. From Proposition 1 it is straightforward to derive the following

PROPOSITION 2. *Let $\nu > 0$, $\xi > 0$, and $d \in \mathbb{N}$, $d \geq 2$. Let $\tau := \tau_{d-1}$. There exists $p \in \mathbb{N}$ such that if $\theta_1, \dots, \theta_p$ are numbers for which there exists $C > 0$ such that each d-tuple (of disjoint numbers) $(\theta_{i_1}, \dots, \theta_{i_d})$ satisfies* (1) *then $\theta_1, \dots, \theta_p$ are in an alternated configuration (with exponent τ) with in addition that for each i there exists l_i' such that $q_{j_i,l_i'} \leq q_{j_i,l_i}^{\xi}$ and such that $l_i', l_i' + 1, \dots, l_i - 1 \in \mathcal{A}_\tau(\theta_{j_i})$.*

In our proof of Theorem 1, we will show that if $f_1, \dots, f_p$ are smooth commuting diffeomorphisms with rotation numbers $\theta_1, \dots, \theta_p$ that are in an alternated configuration, then the diffeomorphisms are C^1-conjugated to rotations. The additional condition, i.e., the existence of long Diophantine strings before q_{l_i}, is then used to prove the higher regularity of the conjugacy; the higher the regularity required, the longer these Diophantine strings should be ($\xi \to 0$).

To adapt Proposition 2 to a family of d commuting diffeomorphisms, we use the following somehow artificial trick:[1] consider $\theta_1, \dots, \theta_d$ satisfying (1) and define for $s \in \mathbb{N}$,

$$\tilde{\theta}_s = \theta_1 + s\theta_2 + \cdots + s^{d-1}\theta_d.$$

Observe that for any $p \geq d$, there exists $C > 0$ such that for any disjoint indices $i_1, \dots, i_d \leq p$, we have that $(\tilde{\theta}_{i_1}, \dots, \tilde{\theta}_{i_d})$ satisfies (1). Proposition 2 can now be applied to $\tilde{\theta}_1, \dots, \tilde{\theta}_p$. On the other hand, given $f_1, \dots, f_d$ as in Theorem 1, the diffeomorphism $\tilde{f}_s = f_1 \circ f_2^s \circ f_3^{s^2} \circ \cdots \circ f_d^{s^{d-1}}$ has rotation number $\tilde{\theta}_s$.

Since it does not alter the proof but only simplifies the notations we will assume for the sequel that $d = 2$ and that θ and β are already in an alternated

[1] We may attribute, as we did in the introduction, the usefulness of this trick to the fact that it exploits the relations in the group, isomorphic to $\mathbb{Z}^d$, of commuting diffeomorphisms.

configuration; that is, there exist $\tau > 1$, and two increasing sequences of integers, l_i and n_i, such that

$$l_{2i}, \dots, n_{2i} - 1 \in \mathcal{A}_\tau(\theta), \tag{4}$$

$$l_{2i+1}, \dots, n_{2i+1} - 1 \in \mathcal{A}_\tau(\beta), \tag{5}$$

and

$$q_{l_{2i}}^{\tau^2} \le q_{n_{2i}}^{1/\tau^2} \le \tilde{q}_{l_{2i+1}} \le q_{n_{2i}}^{1/\tau}, \quad \tilde{q}_{l_{2i+1}}^{\tau^2} \le \tilde{q}_{n_{2i+1}}^{1/\tau^2} \le q_{l_{2i+2}} \le \tilde{q}_{n_{2i+1}}^{1/\tau}, \tag{6}$$

where (q_n) and $(\tilde{q}_n)$ denote respectively the sequences of denominators of the convergents of θ and β.

6. Proof of C^1-conjugation

Given θ and β satisfying (4)–(6) and two commuting diffeomorphisms $f \in \mathcal{D}_\theta$, $g \in \mathcal{D}_\beta$ we will show in this section that f and g are C^1-conjugated to the rotations R_θ and R_β.

6.1. Let

$$\begin{aligned}
\theta_n &= |q_n\theta - p_n|, & \beta_n &= |\tilde{q}_n\beta - \tilde{p}_n|,\\
M_n &= \sup d(f^{q_n}(x), x), & \tilde{M}_n &= \sup d(g^{\tilde{q}_n}(x), x),\\
m_n &= \inf d(f^{q_n}(x), x), & \tilde{m}_n &= \inf d(g^{\tilde{q}_n}(x), x),\\
U_n &= \frac{M_n}{m_n}, & \tilde{U}_n &= \frac{\tilde{M}_n}{\tilde{m}_n}.
\end{aligned}$$

Recall that

$$1/(q_{n+1} + q_n) \le \theta_n \le 1/q_{n+1}, \quad 1/(\tilde{q}_{n+1} + \tilde{q}_n) \le \beta_n \le 1/\tilde{q}_{n+1}. \tag{7}$$

Recall also that since $\int_{\mathbb{T}} |f^{q_n} - \mathrm{id}| d\mu = \theta_n$, (where μ is the unique probability measure invariant by f) then

$$m_n \le \theta_n \le M_n.$$

Herman proved that a diffeomorphism is C^r conjugated to a rotation if and only if its iterates form a bounded sequence in the C^r-topology (see [5, Chap. IV]). Based on the latter observation, the following criterion for C^1 conjugacy was used in [5] and in [12, §7.6]:

PROPOSITION 3. *If there exists* $C > 0$ *such that* $\limsup U_n \le C$, *then* f *is* C^1*-conjugated to* R_θ (*actually* $\liminf U_n \le C$ *is enough*).

Our proof of C^1-conjugacy in Theorem 1 relies on the following central estimate of [12].

PROPOSITION 4. *For any* $f \in \mathcal{D}_\theta$, *for any* $K \in \mathbb{N}$, *there exists* $C = C(f, K)$ *such that*

$$M_n \le M_{n-1} \frac{(\theta_n/\theta_{n-1}) + CM_{n-1}^K}{1 - CM_{n-1}^{1/2}}, \tag{8}$$

$$m_n \ge m_{n-1} \frac{(\theta_n/\theta_{n-1}) - CM_{n-1}^K}{1 + CM_{n-1}^{1/2}}. \tag{9}$$

6.2. The goal of this section is to prove the following "local" result:

PROPOSITION 5. *Let* $\sigma = 1/(2\tau^2)$. *There exists* $i_0 \in \mathbb{N}$ *such that if for some even* (*odd*) *integer* $i \ge i_0$, *we have*

$$M_{n_i-1} \le \frac{1}{q_{n_i}^{1-\sigma}}$$

(*with* $\tilde{M}_{n_i-1}$ *and* $\tilde{q}_{n_i}$ *instead of* M_{n_i-1} *and* q_{n_i} *if* i *is odd*) *then* U_n *and* $\tilde{U}_n$ *are bounded.*

Remark 2. This can be viewed as a local result on C^1-conjugation, since it states that if M_{n_i-1} for i sufficiently large is not too far from what it should be if f were C^1-conjugated to the rotations, then f and g must indeed be C^1-conjugated to the rotations.

Proof of Proposition 5. We will assume that i is even, the other case being similar. Due to the commutation of f and g we have:

LEMMA 2. *For any even integer* i, *let* $L_i = [\beta_{l_{i+1}-1}/\theta_{n_i-1}]$; *now*

$$\tilde{M}_{l_{i+1}-1} \le (1 + L_i) M_{n_i-1}, \tag{10}$$

$$\tilde{m}_{l_{i+1}-1} \ge L_i m_{n_i-1}, \tag{11}$$

$$\tilde{U}_{l_{i+1}-1} \le (1 + \frac{1}{L_i}) U_{n_i-1}. \tag{12}$$

Proof. Notice that for any $x \in \mathbb{T}$,

$$\begin{aligned} \bigcup_{k=0}^{L_i-1} R_\theta^{kq_{n_i-1}}([x, R_\theta^{q_{n_i-1}}(x)]) &\subset [x, R_\beta^{\tilde{q}_{l_{i+1}-1}}(x)] \\ &\subset \bigcup_{k=0}^{L_i} R_\theta^{kq_{n_i-1}}([x, R_\theta^{q_{n_i-1}}(x)]). \end{aligned} \tag{13}$$

Since f and g commute there exists a continuous homeomorphism h that conjugates f to R_θ and g to R_β, and (10)–(12) follow immediately from (13). □

Proposition 5 clearly follows from the next statement:

LEMMA 3. *Let* $\sigma = 1/(2\tau^2)$. *There exists* $i_0 \in \mathbb{N}$, *such that if* $i \geq i_0$ *and* $M_{n_i-1} \leq 1/q_{n_i}^{1-\sigma}$, *then*

$$\tilde{M}_{n_{i+1}-1} \leq \frac{1}{\tilde{q}_{n_{i+1}}^{1-\sigma}}, \tag{14}$$

and

$$\tilde{U}_{n_{i+1}-1} \leq a_i U_{n_i-1} \tag{15}$$

with $a_i \geq 1$, *and* $\Pi_{i \geq i_0} a_i < \infty$.

Proof of Lemma 3. From (10) we have

$$\tilde{M}_{l_{i+1}-1} \leq \left(1 + \frac{\beta_{l_{i+1}-1}}{\theta_{n_i-1}}\right) M_{n_i-1} \leq \left(1 + 2\frac{q_{n_i}}{\tilde{q}_{l_{i+1}}}\right) \frac{1}{{q_{n_i}}^{1-\sigma}};$$

hence (6) implies, for i sufficiently large,

$$\tilde{M}_{l_{i+1}-1} \leq \frac{3}{\tilde{q}_{l_{i+1}}^{1/2}}. \tag{16}$$

Now if we let $K = 2[\tau] + 2$ in Proposition 4, then if $i \geq i_0$, with i_0 sufficiently large, we obtain from (8), (7) and (5) that

$$\tilde{M}_{l_{i+1}} \leq \tilde{M}_{l_{i+1}-1} \frac{\beta_{l_{i+1}}}{\beta_{l_{i+1}-1}} (1 + \tilde{q}_{l_{i+1}}^{-1/5})$$

and by induction

$$\tilde{M}_{n_{i+1}-1} \leq b_i \tilde{M}_{l_{i+1}-1} \frac{\beta_{n_{i+1}-1}}{\beta_{l_{i+1}-1}} \tag{17}$$

with $b_i \geq 1$ and $\Pi_{i \geq i_0} b_i < \infty$. Thus, (14) follows from (6).

By the same token, from (9) in Proposition 4 and (16) we get for $i \geq i_0$, with i_0 sufficiently large

$$\tilde{m}_{n_{i+1}-1} \geq c_i \tilde{m}_{l_{i+1}-1} \frac{\beta_{n_{i+1}-1}}{\beta_{l_{i+1}-1}}$$

with $c_i \leq 1$ and $\Pi_{i \geq i_0} c_i > 0$. Together with (17) this implies that

$$\tilde{U}_{n_{i+1}-1} \leq d_i \tilde{U}_{l_{i+1}-1}$$

with $d_i \geq 1$ and $\Pi d_i < \infty$. This, with (12) and (6), implies (15). □

6.3. *Moving towards the "local" situation. Proof of* C^1*-conjugation.* The main ingredient in improving the bound of M_i towards the "local" condition of Proposition 5 is the following.

Let $A_i \geq \tau^4$ and $B_i \geq \tau^4$ be such that

$$q_{n_{2i}} = q_{l_{2i}}^{A_i}, \quad , \tilde{q}_{n_{2i+1}} = \tilde{q}_{l_{2i+1}}^{B_i}.$$

LEMMA 4. *For any $b \in \mathbb{N}$, there exists i_0 such that if $i \geq i_0$ and $u_i > 0$ is such that $M_{l_{2i}-1} = 1/q_{l_{2i}}^{u_i}$, then*

$$M_{n_{2i}-1} \leq 1/q_{n_{2i}}^{\rho_i}$$

with $\rho_i = \min(1-\sigma, A_i^b u_i)$.

Although not useful to the sequel we can already observe that C^1 conjugacy can be achieved in the *a priori* delicate situation of very Liouville frequencies.

Remark 3. An immediate consequence of Proposition 5 and Lemma 4 is the C^1-conjugacy in the particular case of very long Diophantine strings, namely if there exist $\epsilon > 0$ and a strictly increasing subsequence of the even integers $(i_j)_{\{j \in \mathbb{N}\}}$, such that

$$q_{n_{i_j}} \geq q_{l_{i_j}}^{(\ln q_{l_{i_j}})^\epsilon}.$$

Proof of Lemma 4. We denote $l = l_{2i}$ and $n = n_{2i}$. Let $r \geq 1$ be an integer such that

$$\frac{A_i}{\tau^4} \leq \tau^{4r} \leq A_i.$$

Let $\tilde{K} := ([\tau]+1)^{8b}$, so that $\tilde{K}^r \geq A_i^b$. In Proposition 4 take $K := [4\tau\tilde{K}]$.

Notice that $q_l^{\tau^{4r}} \leq q_l^{A_i} \leq q_n$. Hence, we can introduce a sequence of integers p_s, $s = 0, \dots, r$, such that $p_0 = l$, and for each $1 \leq s \leq r$

$$q_{p_{s-1}}^{\tau^3} \leq q_{p_s} \leq q_{p_{s-1}}^{\tau^4}.$$

Using the first estimate of Proposition 4, and following the idea of [12, §7.4] we easily construct, for $j \in [l, n]$, positive sequences u_j and $a_j \leq 2$ such that $u_l = 1/\ln q_l$, $a_l = 1$. Also, for $j \in [l-1, n-1]$, we have $M_j \leq a_{j+1}/q_{j+1}^{u_{j+1}}$, where for each $j \in [l, n-1]$ one of the two following alternatives holds:

(i) If $\theta_j/\theta_{j-1} \leq CM_{j-1}^{K/2}$ then $a_{j+1} = a_j$ and $u_{j+1} = \tilde{K}u_j$;

(ii) If $\theta_j/\theta_{j-1} > CM_{j-1}^{K/2}$ then $a_{j+1} = b_j a_j$ and $u_{j+1} = u_j$, with $\Pi b_j \leq 2$. In this case, we actually have $M_j \leq b_j M_{j-1}\theta_j/\theta_{j-1}$.

Now, if there exists $s \in [0, r-1]$ such that for every $j \in [p_s, p_{s+1}-1]$, alternative (ii) holds, then (assuming without loss of generality that $\tau \geq 2$) we have

$$M_{p_{s+1}-1} \leq 2M_{p_s-1}\frac{\theta_{p_{s+1}-1}}{\theta_{p_s-1}} \leq \frac{q_{p_s}}{q_{p_{s+1}}} \leq \frac{1}{q_{p_{s+1}}^{1-\sigma}}$$

after which, and as in the proof of Lemma 3, only alternative (ii) can happen for all $j \in [p_{s+1}-1, n-1]$, so that, arguing again as in Lemma 3, we get $M_{n-1} \leq \frac{1}{q_n^{1-\sigma}}$ and we are done.

Otherwise, we have for every $s \in [0, r-1]$, at least one $j \in [p_s, p_{s+1}-1]$ for which alternative (i) holds; hence $u_{p_{s+1}} \geq \tilde{K} u_{p_s}$. Subsequently, $u_{p_r} \geq \tilde{K}^r u_l \geq A_i^b u_l$. The lemma is thus proved. $\square$

Recall that $A_i \geq \tau^4$ and $B_i \geq \tau^4$ are such that

$$q_{n_{2i}} = q_{l_{2i}}^{A_i}, \quad , \tilde{q}_{n_{2i+1}} = \tilde{q}_{l_{2i+1}}^{B_i}.$$

Then, clearly at least one of the following two limits holds:

$$\limsup \frac{\Pi_{j=1}^i A_j^2}{\Pi_{j=1}^i B_j} = +\infty, \tag{18}$$

$$\limsup \frac{\Pi_{j=1}^i B_j^2}{\Pi_{j=1}^i A_j} = +\infty. \tag{19}$$

We will assume that (18) holds, the other case being similar. We will show how Lemma 4 applied with $b = 2$, implies that eventually the condition of Proposition 5 will be satisfied, thus yielding C^1-conjugacy.

Notice first that $q_{l_{2(i+1)}} \leq q_{n_{2i}}^{B_i}$. Furthermore, $M_{l_{2(i+1)}-1} \leq M_{n_{2i}-1}$ since $q_{l_{2(i+1)}} \geq q_{n_{2i}}$.

Now, if i_0 is some sufficiently large integer, and if at step i_0 we do not have $M_{n_{2i_0}-1} \leq 1/q_{n_{2i_0}}^{1-\sigma}$, we observe as above that

$$M_{l_{2(i_0+1)}-1} \leq M_{n_{2i_0}-1} \leq 1/q_{l_{2(i_0+1)}}^{u_{i_0} A_{i_0}^2 / B_{i_0}}.$$

A continued application of the lemma hence shows that either at some $i \geq i_0 + 1$ the condition of Proposition 5 will be satisfied, or for every $i \geq i_0$ we have

$$M_{l_{2i}-1} \leq 1/q_{l_{2i}}^{u_{i_0} \Pi_{j=i_0}^{i-1} A_j^2 / B_j},$$

which, with our assumption that (18) holds, contradicts the fact that for every i, $M_{l_{2i}-1} \geq 1/(2q_{l_{2i}})$.

Remark 4. In the general situation, the alternated configuration of denominators may require the use of more than two angles, that is more than two diffeomorphisms. Our proof remains quite the same. Indeed, let $\theta_1, \ldots, \theta_p$ be in an alternated configuration as in Definition 2. Define A_i such that $q_{j_i, n_i} = q_{j_i, l_i}^{A_i}$. Then there exists $k \in [1, p]$ such that

$$\limsup_{I \in \mathbb{N}} \frac{\Pi_{j_i = k, i \leq I} A_i^{p+1}}{\Pi_{j_i \neq k, i \leq I} A_i} = +\infty$$

and the proof of C^1-conjugation follows the same lines as above, the only difference being that we would take $b = p + 1$ in application of Lemma 4. As before we take

$\tilde{K} = 2[\tau^{4b+1}]$) and then $K := [4\tau\tilde{K}]$ in Proposition 4 which is possible since the diffeomorphisms considered are of class C^∞. We see here the dramatic increase in our need of differentiability to prove C^1-conjugation as the number d of commuting diffeomorphisms in Theorem 1 increases.

7. **Higher regularity**

We fix $r \geq 2$. Knowing that the diffeomorphisms f and g are C^1-conjugated to the rotations, we will now prove that the conjugacy is in fact of class C^r.

In all the sequel, we fix $k = [(r+2)(2+\tau)] + 2$. And we take $\xi = 1/k$ in Proposition 2.

As in the proof of C^1-conjugation, we will continue to assume for simplicity that we are given θ and β satisfying (4)–(6) with in addition that there exists for each i, l_i' such that if i is even, then

$$q_{l_i'} \leq q_{l_i}^{1/k}, \quad \text{and} \quad l_i', \ldots, l_i - 1 \in \mathcal{A}_\tau(\theta), \tag{20}$$

with a similar property involving β if i is odd.

Given two commuting diffeomorphisms $f \in \mathcal{D}_\theta$, $g \in \mathcal{D}_\beta$ such that f and g are C^1-conjugated, we will show that the conjugacy is actually of class C^r.

7.1. *The control of the derivatives at alternating "Diophantine times" is sufficient.* We define two sets of integers, the "Diophantine times", as

$$\mathcal{A} = \left\{m \in \mathbb{N} \;/\; m = \sum a_s q_s,\ a_s \leq q_{s+1}/q_s,\ \text{with } s \in [l_{2i}, n_{2i} - 1],\ i \in \mathbb{N}\right\},$$
$$\tilde{\mathcal{A}} = \left\{m \in \mathbb{N} \;/\; m = \sum \tilde{a}_s \tilde{q}_s,\ \tilde{a}_s \leq \tilde{q}_{s+1}/\tilde{q}_s,\ \text{with } s \in [l_{2i+1}, n_{2i+1} - 1],\ i \in \mathbb{N}\right\}.$$

We also define two sets of diffeomorphisms

$$\mathcal{X} = \{f^n \;/\; n \in \mathbb{N}\},$$
$$\mathcal{C} = \{f^u \circ g^v \;/\; u \in \mathcal{A}, v \in \tilde{\mathcal{A}}\}.$$

The following is an elementary lemma due to (6)

LEMMA 5. *If we set*

$$\mathbb{O} = \{u\theta + v\beta \bmod[1] \;/\; u \in \mathcal{A}, v \in \tilde{\mathcal{A}}\}$$

then $\overline{\mathbb{O}} = \mathbb{T}$. As a consequence, $\mathcal{C}$ is dense in $\mathcal{X}$ in the C^0-topology.

Sketch of the proof. Fix i_0 and consider the set $\mathcal{U}_{i_0} = \{m\theta,\ m \leq q_{n_{2i_0}}\}$. It is known that the set $\mathcal{U}_{i_0}$ is $2/q_{n_{2i_0}}$ dense in the circle. Also, the set

$$\left\{\sum b_l \tilde{q}_l \beta,\ b_l \leq \tilde{q}_{l+1}/\tilde{q}_l, l \in [l_{2i_0+1}, n_{2i_0+1} - 1]\right\}$$

is $2/\tilde{q}_{n_{2i_0+1}}$ dense in an interval of size larger than $1/(2\tilde{q}_{2l_{2i_0+1}})$ with extremity 0. Since by (6) $2/q_{n_{2i}} \ll 1/(2\tilde{q}_{l_{2i+1}})$, we conclude that the set

$$\mathcal{V}_{i_0} = \left\{m\theta + \sum b_l \tilde{q}_l \beta,\ b_l \le \tilde{q}_{l+1}/\tilde{q}_l, l \in [l_{2i_0+1}, n_{2i_0+1} - 1],\ m \le q_{n_{2i_0}}\right\}$$

is $2/\tilde{q}_{n_{2i_0+1}}$ dense in the circle. Using (6) again, we get that the set

$$\mathcal{U}_{i_0+1} = \Big\{m\theta + \sum a_s q_s \theta + \sum b_l \tilde{q}_l \beta,\ b_l \le \tilde{q}_{l+1}/\tilde{q}_l, l \in [l_{2i_0+1}, n_{2i_0+1} - 1],$$
$$a_s \le q_{s+1}/q_s, s \in [l_{2i_0+2}, n_{2i_0+2} - 1],\ m \le q_{n_{2i_0}}\Big\}$$

is $2/q_{2n_0+2}$ dense in the circle. We thus prove inductively the first assertion of the lemma. The conclusion that $\mathcal{C}$ is dense in $\mathcal{X}$ then follows from the simultaneous C^0 conjugacy of f and g to R_θ and R_β respectively. □

From the above lemma, it is enough to control the derivatives of the f^u and g^v at the *Diophantine times* $u \in \mathcal{A}$ and $v \in \tilde{\mathcal{A}}$:

COROLLARY 3. *If $\mathcal{C}$ is bounded in the C^{r+1}-topology, then the conjugating diffeomorphism h of f to R_θ is of class C^r.*

Proof. We know that $\frac{1}{n}\sum_{i=0}^{n-1} f^i$ converges in the C^0-topology to h (see [5, Chap. IV]). From Lemma 5, this implies that there exist sequences (u_n) and (v_n) of numbers in $\mathcal{A}$ and $\tilde{\mathcal{A}}$ such that the sequence $\frac{1}{n}\sum_{i=0}^{n-1} f^{u_i} \circ g^{v_i}$ converges in the C^0 topology to h. By our C^{r+1}-boundness assumption, we can extract from the latter sequence a sequence that converges in the C^r-topology, so that necessarily $h \in \mathrm{Diff}^r_+(\mathbb{T})$. □

7.2. It follows from standard computations (see [12, §8.10]) that the assumption of Corollary 3 holds true if we prove

LEMMA 6. *There exists $\nu > 0$ such that, for i (even) sufficiently large, we have for any $s \in [l_i, n_i - 1]$ and for any $0 \le a \le q_{s+1}/q_s$*

$$\|\ln Df^{aq_s}\|_{r+1} \le q_s^{-\nu}$$

(*with g and $\tilde{q}_s$ instead of f and q_s if i is odd*).

Proof. We only work with f since the arguments for g are the same. The proof is based on the estimates of [12, §8] and we start by recalling some facts that were proven there:

For $k \in \mathbb{N}^*$, define for $s \in \mathbb{N}$, $\Delta_s^{(k)} = \|D^{k-1} \ln Df^{q_s}\|_0 + \theta_s$. Then it follows from the C^1-conjugation of f to R_θ (see [12, Lemme 5]) that

$$\Delta_s^{(k)} \le q_s^{(k-1)/2}.$$

We will use this fact with $k = [(r+2)(2+\tau)] + 2$ and use the notation Δ_s for $\Delta_s^{(k)}$.

Observe that for $s \in [l_i', n_i - 1]$, we have (if i is sufficiently large)

$$(\Delta_s q_{s+1})^{1/k} q_s^{-1} \leq q_s^{-1/4}. \tag{21}$$

Hence it follows from [12, Lemme 14 in §8.8] that for any $s \in [l_i', n_i - 1]$, and for any $0 \leq a \leq q_{s+1}/q_s$,

$$\| \ln Df^{aq_s} \|_{r+1} \leq C q_s^{-1} (\Delta_s q_{s+1})^{\rho} \tag{22}$$

where $\rho = (r+2)/k$ and C is some constant.

If we denote

$$\Delta_s' = \mathrm{Sup}\{|(D^{k-1} \ln Df^{q_t} \circ f^m)(Df^m)^{k-1}|_0,\ 0 \leq t \leq s, m \geq 0\},$$

then $\Delta_s \leq C\Delta_s'$ for some constant C provided that f is not a rotation. Observe that since $\|Df^m\|_0$ is bounded, $\Delta_s' \leq C q_s^{(k-1)/2}$ for some constant C. If we denote $V_s = \mathrm{Max}\{\Delta_t'/q_t,\ 0 \leq t \leq s\}$, then due to (21) we have from [12, §8.9] that for $s \in [l_i', n_i - 1]$,

$$V_{s+1} \leq V_s(1 + C q_s^{-1/4})$$

for some constant C. Hence, for $s \in [l_i', n_i - 1]$, we have $V_s \leq 2V_{l_i'} \leq C q_{l_i'}^{(k-3)/2}$. If $s \geq l_i$ this gives

$$\Delta_s' \leq C q_s q_{l_i'}^{(k-3)/2} \leq C q_s^2$$

because we assumed that $q_{l_i'} \leq q_{l_i}^{2/k}$.

Finally, if $s \in [l_i, n_i - 1]$ we have that $(\Delta_s q_{s+1})^{\rho} \leq C q_s^{(2+\tau)(r+2)/k} \leq q_s^{1-1/k}$ and we conclude using (22) that the statement of Lemma 6 holds which ends the proof of higher regularity. □

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