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The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz

By EVGENY MUKHIN, VITALY TARASOV, and ALEXANDER VARCHENKO

Abstract

We prove the B. and M. Shapiro conjecture that if the Wronskian of a set of polynomials has real roots only, then the complex span of this set of polynomials has a basis consisting of polynomials with real coefficients. This, in particular, implies the following result:

If all ramification points of a parametrized rational curve $\phi : \mathbb{CP}^1 \to \mathbb{CP}^r$ lie on a circle in the Riemann sphere $\mathbb{CP}^1$, then ϕ maps this circle into a suitable real subspace $\mathbb{RP}^r \subset \mathbb{CP}^r$.

The proof is based on the Bethe ansatz method in the Gaudin model. The key observation is that a symmetric linear operator on a Euclidean space has real spectrum.

In Appendix A, we discuss properties of differential operators associated with Bethe vectors in the Gaudin model. In particular, we prove a statement, which may be useful in complex algebraic geometry; it claims that certain Schubert cycles in a Grassmannian intersect transversally if the spectrum of the corresponding Gaudin Hamiltonians is simple.

In Appendix B, we formulate a conjecture on reality of orbits of critical points of master functions and prove this conjecture for master functions associated with Lie algebras of types A_r, B_r and C_r.

1. **The B. and M. Shapiro conjecture**

1.1. *Statement of the result.* Fix a natural number $r \geqslant 1$. Let $V \subset \mathbb{C}[x]$ be a vector subspace of dimension $r+1$. The space V is called *real* if it has a basis consisting of polynomials in $\mathbb{R}[x]$.

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For a given V, there exists a unique linear differential operator

$$D = \frac{d^{r+1}}{dx^{r+1}} + \lambda_1(x)\frac{d^r}{dx^r} + \cdots + \lambda_r(x)\frac{d}{dx} + \lambda_{r+1}(x),$$

whose kernel is V. This operator is called *the fundamental differential operator of* V. The coefficients of the operator are rational functions in x. The space V is real if and only if all coefficients of the fundamental operator are real rational functions.

The Wronskian of functions $f_1, \dots, f_i$ in x is the determinant

$$\mathrm{Wr}(f_1, \dots, f_i) = \det \begin{pmatrix} f_1 & f_1^{(1)} & \cdots\cdots & f_1^{(i-1)} \\ f_2 & f_2^{(1)} & \cdots\cdots & f_2^{(i-1)} \\ \vdots & \vdots & \cdots\cdots & \vdots \\ f_i & f_i^{(1)} & \cdots\cdots & f_i^{(i-1)} \end{pmatrix}.$$

Let $f_1, \dots, f_{r+1}$ be a basis of V. The Wronskian of the basis does not depend on the choice of the basis up to multiplication by a number. The monic representative is called *the Wronskian of* V and denoted by Wr_V.

THEOREM 1.1. *If all roots of the polynomial* Wr_V *are real*, *then the space* V *is real.*

This statement is the B. and M. Shapiro conjecture formulated in 1993. The conjecture is proved in [EG02b] for $r = 1$; see a more elementary proof also for $r = 1$ in [EG05]. The conjecture, its supporting evidence, and applications are discussed in [EG02b], [EG02a], [EG05], [EGSV06], [ESS06], [KS03], [RSSS06], [Sot97a], [Sot97b], [Sot99], [Sot00b], [Sot03], [Sot00a] and [Ver00].

1.2. *Parametrized rational curves with real ramification points.* For a projective coordinate system $(v_1 : \cdots : v_{r+1})$ on the complex projective space $\mathbb{CP}^r$, the subset of points with real coordinates is called *the real projective subspace* and is denoted by $\mathbb{RP}^r$.

Let $\phi : \mathbb{CP}^1 \to \mathbb{CP}^r$ be a parametrized rational curve. If $(u_1 : u_2)$ are projective coordinates on $\mathbb{CP}^1$ and $(v_1 : \cdots : v_{r+1})$ are projective coordinates on $\mathbb{CP}^r$, then ϕ is given by the formula

$$\phi : (u_1 : u_2) \mapsto (\phi_1(u_1, u_2) : \cdots : \phi_{r+1}(u_1, u_2)),$$

where ϕ_i are homogeneous polynomials of the same degree. We assume that at any point of $\mathbb{CP}^1$ at least one component ϕ_i is nonzero. Choose the local affine coordinate $u = u_1/u_2$ on $\mathbb{CP}^1$ and local affine coordinates $v_1/v_{r+1}, \dots, v_r/v_{r+1}$ on $\mathbb{CP}^r$. In these coordinates, the map ϕ takes the form

$$f : u \mapsto \left(\frac{f_1(u)}{f_{r+1}(u)}, \dots, \frac{f_r(u)}{f_{r+1}(u)}\right), \quad \text{where } f_i(u) = \phi_i(u, 1). \tag{1.1}$$

The map ϕ is said to be *ramified* at a point of $\mathbb{CP}^1$ if its first r derivatives at this point do not span $\mathbb{CP}^r$ [KS03]. More precisely, a point u is *a ramification point* if the vectors $f^{(1)}(u), \dots, f^{(r)}(u)$ are linearly dependent.

We assume that a generic point of $\mathbb{CP}^1$ is not a ramification point.

THEOREM 1.2. *If all ramification points of the parametrized rational curve ϕ lie on a circle in the Riemann sphere $\mathbb{CP}^1$, then ϕ maps this circle into a suitable real subspace $\mathbb{RP}^r \subset \mathbb{CP}^r$.*

A *maximally inflected curve* is, by definition [KS03], a parametrized real rational curve whose ramification points are all real. Theorem 1.2 implies the existence of maximally inflected curves for every placement of the ramification points.

Theorem 1.2 follows from Theorem 1.1. Indeed, if all ramification points lie on a circle, then after a linear change of coordinates $(u_1 : u_2)$, we may assume that the ramification points lie on the real line $\mathbb{RP}^1$ and that the point $(0 : 1)$ is not a ramification point. After a linear change of coordinates $(v_1 : \cdots : v_{r+1})$ on $\mathbb{CP}^r$, we may assume that ϕ_{r+1} is not zero at any of the ramification points. Let us use the affine coordinates $u = u_1/u_2$ and $v_1/v_{r+1}, \dots, v_r/v_{r+1}$, and formula (1.1). Then the determinant of coordinates of the vectors $f^{(1)}(u), \dots, f^{(r)}(u)$ is equal to

$$\mathrm{Wr}\left(\frac{f_1}{f_{r+1}}, \dots, \frac{f_r}{f_{r+1}}, 1\right)(u) = \frac{1}{(f_{r+1})^{r+1}} \, \mathrm{Wr}(f_1, \dots, f_r, f_{r+1})(u).$$

Hence the vectors $f^{(1)}(u), \dots, f^{(r)}(u)$ are linearly dependent if and only if the Wronskian of $f_1, \dots, f_{r+1}$ at u is zero. Since not all points of $\mathbb{CP}^1$ are ramification points, the complex span V of polynomials $f_1, \dots, f_{r+1}$ is an $(r+1)$-dimensional space. By assumptions of Theorem 1.2, all zeros of the Wronskian of V are real. By Theorem 1.1, the space V is real. This means that there exist projective coordinates on $\mathbb{CP}^r$ in which all polynomials $f_1, \dots, f_{r+1}$ are real. Theorem 1.2 is deduced from Theorem 1.1.

1.3. *Reduction of Theorem* 1.1 *to a special case.*

THEOREM 1.3. *If all roots of the Wronskian are real and simple*, *then V is real.*

We deduce Theorem 1.1 from Theorem 1.3. Indeed, let V_0 be an $(r+1)$-dimensional space of polynomials whose Wronskian has real roots only. Let d be the degree of a generic polynomial in V_0.

- Let $\mathbb{C}_d[x]$ be the space of polynomials of degree not greater than d.
- Let $G(r+1, d)$ be the Grassmannian of $(r+1)$-dimensional vector subspaces in $\mathbb{C}_d[x]$.

- Let $\mathbb{P}(\mathbb{C}_{(r+1)(d-r)}[x])$ be the projective space associated with the vector space $\mathbb{C}_{(r+1)(d-r)}[x]$.

The varieties $G(r+1,d)$ and $\mathbb{P}(\mathbb{C}_{(r+1)(d-r)}[x])$ have the same dimension. The assignment $V \mapsto \mathrm{Wr}_V$ defines a finite morphism

$$\pi : G(r+1,d) \to \mathbb{P}(\mathbb{C}_{(r+1)(d-r)}[x]);$$

see for example [Sot97b] and [EG02b]. The space V_0 is a point of $G(r+1,d)$.

Since π is finite and V_0 has Wronskian with real roots only, there exists a continuous curve $\epsilon \mapsto V_\epsilon \in G(r+1,d)$ for $\epsilon \in [0,1)$ such that the Wronskian of V_ϵ for $\epsilon > 0$ has simple real roots only. By Theorem 1.3, the space V_ϵ is real for $\epsilon > 0$. Hence, the fundamental differential operator of V_ϵ has real coefficients. Therefore, the fundamental differential operator of V_0 has real coefficients and the space V_0 is real. Theorem 1.1 is deduced from Theorem 1.3.

1.4. *The upper bound for the number of complex vector spaces with the same exponents at infinity and the same Wronskian.* Let $f_1, \dots, f_{r+1}$ be a basis of V such that $\deg f_i = d_i$ for some sequence $\boldsymbol{d} = \{d_1 < \dots < d_{r+1}\}$. We say that V has exponents $\boldsymbol{d}$ at infinity. If V has exponents $\boldsymbol{d}$ at infinity, then $\deg \mathrm{Wr}_V = n$, where $n = \sum_{i=1}^{r+1}(d_i - i + 1)$. Let $T = \prod_{s=1}^{n}(x - z_s)$ be a polynomial in x with simple (complex) roots $z_1. \dots, z_n$. Then the upper bound for the number of complex vector spaces V with exponents $\boldsymbol{d}$ at infinity and Wronskian T is given by the number $N(\boldsymbol{d})$ defined as follows.

Consider the Lie algebra $\mathfrak{sl}_{r+1}$ with Cartan decomposition

$$\mathfrak{sl}_{r+1} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

and simple roots $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$. Fix the invariant inner product on $\mathfrak{h}^*$ by the condition $(\alpha_i, \alpha_i) = 2$. For any integral dominant weight $\Lambda \in \mathfrak{h}^*$, denote by L_Λ the irreducible $\mathfrak{sl}_{r+1}$-module with highest weight Λ. Let $\omega_r \in \mathfrak{h}^*$ be the last fundamental weight.

For $i = 1, \dots, r$, introduce the numbers

$$l_i = \textstyle\sum_{j=1}^{i}(d_j - j + 1)$$

and the integral dominant weight

$$\Lambda(\boldsymbol{d}) = n\omega_r - \textstyle\sum_{i=1}^{r} l_i \alpha_i . \tag{1.2}$$

Set $N(\boldsymbol{d})$ to be the multiplicity of the module $L_{\Lambda(\boldsymbol{d})}$ in the n-fold tensor product

$$L_{\omega_r}^{\otimes n} = L_{\omega_r} \otimes \dots \otimes L_{\omega_r} .$$

According to Schubert calculus, the number of complex $(r+1)$-dimensional vector spaces V with exponents $\boldsymbol{d}$ at infinity and Wronskian T is not greater than the number $N(\boldsymbol{d})$. This is a standard statement of Schubert calculus; see for example [MV04, §5].

Thus, in order to prove Theorem 1.3, it is enough to prove this:

THEOREM 1.4. *For generic real $z_1, \dots, z_n$, there exist exactly $N(\boldsymbol{d})$ distinct real vector spaces V with exponents $\boldsymbol{d}$ at infinity and with Wronskian $T = \prod_{s=1}^{n}(x - z_s)$.*

1.5. *Structure of the paper.* In Section 2, for generic complex $z_1, \dots, z_n$, we construct exactly $N(\boldsymbol{d})$ distinct complex vector spaces V with exponents $\boldsymbol{d}$ at infinity and Wronskian T. In Section 3, we show that all of these vector spaces are real if $z_1, \dots, z_n$ are real. This proves Theorem 1.4.

The constructions of Sections 2 and 3 are the Bethe ansatz constructions for the Gaudin model on $L_{\omega_r}^{\otimes n}$.

In Appendix A, we discuss properties of differential operators associated with the Bethe vectors in the Gaudin model and give applications of the Bethe ansatz constructions of Section 3. In particular, we prove Corollary A.3, which may be useful in complex algebraic geometry; it claims that certain Schubert cycles in a Grassmannian intersect transversally if the spectrum of the corresponding Gaudin Hamiltonians is simple; cf. [EH83] and [MV04].

In Appendix B, we formulate a conjecture on reality of orbits of critical points of master functions and prove this conjecture for master functions associated with Lie algebras of types A_r, B_r, C_r.

2. Construction of spaces of polynomials

2.1. *Construction of (not necessarily real) spaces with exponents d at infinity and with Wronskian $T = \prod_{s=1}^{n}(x - z_s)$ having simple roots.* Write $\boldsymbol{z} = (z_1, \dots, z_n)$. Introduce a function of $l_1 + \cdots + l_r$ variables

$$\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}, \dots, t_1^{(r)}, \dots, t_{l_r}^{(r)})$$

by the formula

$$\Phi_{\boldsymbol{d}}(\boldsymbol{t};\boldsymbol{z}) = \prod_{j=1}^{l_r}\prod_{s=1}^{n}(t_j^{(r)} - z_s)^{-1}\prod_{i=1}^{r}\prod_{1\leqslant j<s\leqslant l_i}(t_j^{(i)} - t_s^{(i)})^2 \times \prod_{i=1}^{r-1}\prod_{j=1}^{l_i}\prod_{k=1}^{l_{i+1}}(t_j^{(i)} - t_k^{(i+1)})^{-1}. \tag{2.1}$$

The function $\Phi_{\boldsymbol{d}}$ is a rational function of $\boldsymbol{t}$, depending on parameters $\boldsymbol{z}$. The function is called *the master function*.

The master functions arise in the hypergeometric solutions of the KZ equations [SV91], [Var95] and in the Bethe ansatz method for the Gaudin model [RV95], [SV03], [MV00], [MV04], [MV05], [Var06]. For more general master functions, see Appendix B. In particular, the master function (2.1) corresponds to the collection $(\omega_r, \dots, \omega_r)$ of integral dominant sl_{r+1} weights and the integral dominant weight $\Lambda(\boldsymbol{d})$; see (1.2).

The product of symmetric groups $\Sigma_{\boldsymbol{l}} = \Sigma_{l_1} \times \dots \times \Sigma_{l_r}$ acts on the variables $\boldsymbol{t}$ by permuting the coordinates with the same upper index. The master function is $\Sigma_{\boldsymbol{l}}$-invariant.

We call a point $\boldsymbol{t}$ with complex coordinates a *critical point* of $\Phi_{\boldsymbol{d}}(\,\cdot\,;\boldsymbol{z})$ if

$$\Bigl(\Phi_{\boldsymbol{d}}^{-1}\frac{\partial \Phi_{\boldsymbol{d}}}{\partial t_j^{(i)}}\Bigr)(\boldsymbol{t};\boldsymbol{z}) = 0 \quad \text{for } i = 1, \dots, r \text{ and } j = 1, \dots, l_i.$$

In other words, a point $\boldsymbol{t}$ will be called a critical point if the system

$$\begin{aligned}
0 &= -\sum_{s=1, s\neq j}^{l_1} \frac{2}{t_j^{(1)} - t_s^{(1)}} + \sum_{s=1}^{l_2} \frac{1}{t_j^{(1)} - t_s^{(2)}}, \\
0 &= -\sum_{s=1, s\neq j}^{l_i} \frac{2}{t_j^{(i)} - t_s^{(i)}} + \sum_{s=1}^{l_{i-1}} \frac{1}{t_j^{(i)} - t_s^{(i-1)}} + \sum_{s=1}^{l_{i+1}} \frac{1}{t_j^{(i)} - t_s^{(i+1)}}, \\
0 &= \sum_{s=1}^{n} \frac{1}{t_j^{(r)} - z_s} - \sum_{s=1, s\neq j}^{l_r} \frac{2}{t_j^{(r)} - t_s^{(r)}} + \sum_{s=1}^{l_{r-1}} \frac{1}{t_j^{(r)} - t_s^{(r-1)}}
\end{aligned} \tag{2.2}$$

of $l_1 + \dots + l_r$ equations is satisfied, where $j = 1, \dots, l_1$ in the first group of equations, $i = 2, \dots, r-1$ and $j = 1, \dots, l_i$ in the second group of equations, and $j = 1, \dots, l_r$ in the last group of equations. We require that all denominators in these equations are not equal to zero.

In the Gaudin model, the equations (2.2) are called *the Bethe ansatz equations*.

The set of critical points of $\Phi_{\boldsymbol{d}}(\,\cdot\,;\boldsymbol{z})$ is $\Sigma_{\boldsymbol{l}}$-invariant.

For a critical point $\boldsymbol{t}$, define the tuple $\boldsymbol{y}^{\boldsymbol{t}} = (y_1, \dots, y_r)$ of polynomials in variable x by

$$y_i(x) = \prod_{j=1}^{l_i} (x - t_j^{(i)}) \quad \text{for } i = 1, \dots, r. \tag{2.3}$$

Consider the $(r+1)$-st order linear differential operator

$$D_{\boldsymbol{t}} = \Bigl(\frac{d}{dx} - \ln'\Bigl(\frac{T}{y_r}\Bigr)\Bigr)\Bigl(\frac{d}{dx} - \ln'\Bigl(\frac{y_r}{y_{r-1}}\Bigr)\Bigr)\cdots\Bigl(\frac{d}{dx} - \ln'\Bigl(\frac{y_2}{y_1}\Bigr)\Bigr)\Bigl(\frac{d}{dx} - \ln'(y_1)\Bigr),$$

where $\ln'(f)$ denotes $(df/dx)/f$ for any f. Denote by $V_{\boldsymbol{t}}$ the kernel of $D_{\boldsymbol{t}}$.

Call $D_{\boldsymbol{t}}$ *the fundamental operator of the critical point* $\boldsymbol{t}$, and call $V_{\boldsymbol{t}}$ *the fundamental space of the critical point* $\boldsymbol{t}$.

THEOREM 2.1 [MV04, §5]. *The fundamental space* $V_{\boldsymbol{t}}$ *is an* $(r+1)$*-dimensional space of polynomials with exponents* $\boldsymbol{d}$ *at infinity and Wronskian* T. *The tuple* $\boldsymbol{y}^{\boldsymbol{t}}$ *can be recovered from the fundamental space* $V_{\boldsymbol{t}}$ *as follows. Let* $f_1, \dots, f_{r+1}$ *be a basis of* $V_{\boldsymbol{t}}$ *consisting of polynomials with* $\deg f_i = d_i$ *for all* i. *Then* $y_1, \dots, y_r$ *are respective scalar multiples of the polynomials*

$$f_1, \quad \mathrm{Wr}(f_1, f_2), \quad \mathrm{Wr}(f_1, f_2, f_3), \quad \dots, \mathrm{Wr}(f_1, \dots, f_r).$$

Thus distinct orbits of critical points define distinct $(r+1)$-dimensional spaces V with exponents $\boldsymbol{d}$ at infinity and Wronskian T.

THEOREM 2.2 [MV05, Th. 6.1]. *For generic complex* $z_1, \dots, z_n$, *the master function* $\Phi_{\boldsymbol{d}}(\,\cdot\,; \boldsymbol{z})$ *has* $N(\boldsymbol{d})$ *distinct orbits of critical points.*

Therefore, by Theorems 2.1 and 2.2, we constructed $N(\boldsymbol{d})$ distinct spaces of polynomials with Wronskian T. All these spaces are fundamental spaces of critical points of the master function $\Phi_{\boldsymbol{d}}(\,\cdot\,; \boldsymbol{z})$.

3. Bethe vectors

3.1. *Generators.* Let $E_{i,j}$ for $i, j = 1, \dots, r+1$ be the standard generators of $\mathfrak{gl}_{r+1}$. The elements $E_{i,j}$ for $i \neq j$ and $H_i = E_{i,i} - E_{i+1,i+1}$ for $i = 1, \dots, r$ are the standard generators of $\mathfrak{sl}_{r+1}$. We have $\mathfrak{sl}_{r+1} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where

$$\mathfrak{n}_+ = \bigoplus_{i<j} \mathbb{C} \cdot E_{i,j}, \qquad \mathfrak{h} = \bigoplus_{i=1}^{r} \mathbb{C} \cdot H_i, \qquad \mathfrak{n}_- = \bigoplus_{i>j} \mathbb{C} \cdot E_{i,j}.$$

3.2. *Construction of Bethe vectors.* For $\mu \in \mathfrak{h}^*$, denote by $L_{\omega_r}^{\otimes n}[\mu]$ the vector subspace of $L_{\omega_r}^{\otimes n}$ of vectors of weight μ and by $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu]$ the vector subspace of singular vectors of weight μ, that is,

$$L_{\omega_r}^{\otimes n}[\mu] = \{v \in L_{\omega_r}^{\otimes n} \mid hv = \langle \mu, h \rangle v \text{ for any } h \in \mathfrak{h}\},$$
$$\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu] = \{v \in L_{\omega_r}^{\otimes n} \mid \mathfrak{n}_+ v = 0,\ hv = \langle \mu, h \rangle v \text{ for any } h \in \mathfrak{h}\}.$$

For a given $\boldsymbol{l} = (l_1, \dots, l_r)$, set $l = l_1 + \dots + l_r$ and $\mu = n\omega_r - \sum_{i=1}^{r} l_i \alpha_i$. Let $\mathbb{C}^l$ be the space with coordinates $t_j^{(i)}$ for $i = 1, \dots, r$ and $j = 1, \dots, l_i$, and let $\mathbb{C}^n$ be the space with coordinates $z_1, \dots, z_n$. We construct a rational map

$$\omega : \mathbb{C}^l \times \mathbb{C}^n \to L_{\omega_r}^{\otimes n}[\mu]$$

called *the universal weight function.*

Let $P(\boldsymbol{l}, n)$ be the set of sequences $I = (i_1^1, \dots, i_{k_1}^1; \dots; i_1^n, \dots, i_{k_n}^n)$ of integers in $\{1, \dots, r\}$ such that for all $i = 1, \dots, r$, the integer i appears in I precisely

l_i times. For $I \in P(\boldsymbol{l}, n)$, the l positions in I are partitioned into subsets $I_1, \dots, I_r$, where I_i consists of positions of the integer i. To every position $\binom{a}{b}$ in I, assign an integer j_b^a such that $\{j_b^a \mid \binom{a}{b} \in I_i\} = \{1, \dots, l_i\}$. For $\sigma = (\sigma_1, \dots, \sigma_r) \in \Sigma_{\boldsymbol{l}}$, denote by $t_I(\binom{a}{b}; \sigma)$ the variable $t^{(i)}_{\sigma_i(j)}$, where $i = i_b^a$ and $j = j_b^a$ and $\sigma_i(j)$ denotes the image of j under the permutation σ_i. For a given σ, the assignment of this variable to a position establishes a bijection of l positions of I and the set $\{t_1^{(1)}, \dots, t_{l_1}^{(1)}, \dots, t_1^{(r)}, \dots, t_{l_r}^{(r)}\}$.

Fix a highest weight vector v_{ω_r} in L_{ω_r}. To every $I \in P(\boldsymbol{l}, n)$, assign the vector

$$E_I v = E_{i_1^1+1, i_1^1} \cdots E_{i_{k_1}^1+1, i_{k_1}^1} v_{\omega_r} \otimes \cdots \otimes E_{i_1^n+1, i_1^n} \cdots E_{i_{k_n}^n+1, i_{k_n}^n} v_{\omega_r}$$

in $L_{\omega_r}^{\otimes n}[\mu]$ and scalar functions $\omega_{I,\sigma}$ labeled by $\sigma = (\sigma_1, \dots, \sigma_r) \in \Sigma_{\boldsymbol{l}}$, where

$$\omega_{I,\sigma} = \omega_{I,\sigma,1}(z_1) \dots \omega_{I,\sigma,n}(z_n),$$

$$\omega_{I,\sigma,j}(z_j) = \frac{1}{(t_I(\binom{j}{1};\sigma) - t_I(\binom{j}{2};\sigma)) \cdots (t_I(\binom{j}{k_j-1};\sigma) - t_I(\binom{j}{k_j};\sigma))\,(t_I(\binom{j}{k_j};\sigma) - z_j)}.$$

We set

$$\omega(\boldsymbol{t}; \boldsymbol{z}) = \sum_{I \in P(\boldsymbol{l}, n)} \sum_{\sigma \in \Sigma_{\boldsymbol{l}}} \omega_{I,\sigma} E_I v. \tag{3.1}$$

The universal weight function is invariant with respect to the $\Sigma_{\boldsymbol{l}}$-action on variables $t_j^{(i)}$.

The universal weight function was introduced in [Mat90] and [SV91] to solve the KZ equations. The other formulas for the universal weight function can be found in [RSV05].

Examples. If $n = 2$ and $\boldsymbol{l} = (1, 1, 0, \dots, 0)$, then

$$\begin{aligned}\omega(\boldsymbol{t}; \boldsymbol{z}) = {} & \frac{E_{2,1}E_{3,2}v_{\omega_r} \otimes v_{\omega_r}}{(t_1^{(1)} - t_1^{(2)})(t_1^{(2)} - z_1)} + \frac{E_{3,2}E_{2,1}v_{\omega_r} \otimes v_{\omega_r}}{(t_1^{(2)} - t_1^{(1)})(t_1^{(1)} - z_1)} \\ & + \frac{E_{2,1}v_{\omega_r} \otimes E_{3,2}v_{\omega_r}}{(t_1^{(1)} - z_1)(t_1^{(2)} - z_2)} + \frac{E_{3,2}v_{\omega_r} \otimes E_{2,1}v_{\omega_r}}{(t_1^{(2)} - z_1)(t_1^{(1)} - z_2)} \\ & + \frac{v_{\omega_r} \otimes E_{2,1}E_{3,2}v_{\omega_r}}{(t_1^{(1)} - t_1^{(2)})(t_1^{(2)} - z_2)} + \frac{v_{\omega_r} \otimes E_{3,2}E_{2,1}v_{\omega_r}}{(t_1^{(2)} - t_1^{(1)})(t_1^{(1)} - z_2)}.\end{aligned}$$

If $\boldsymbol{l} = (2, 0, \dots, 0)$, then

$$\begin{aligned}\omega(\boldsymbol{t}; \boldsymbol{z}) = {} & \Big(\frac{1}{(t_1^{(1)} - t_2^{(1)})(t_2^{(1)} - z_1)} + \frac{1}{(t_2^{(1)} - t_1^{(1)})(t_1^{(1)} - z_1)}\Big) E_{2,1}^2 v_{\omega_r} \otimes v_{\omega_r} \\ & + \Big(\frac{1}{(t_1^{(1)} - z_1)(t_2^{(1)} - z_2)} + \frac{1}{(t_2^{(1)} - z_1)(t_1^{(1)} - z_2)}\Big) E_{2,1}v_{\omega_r} \otimes E_{2,1}v_{\omega_r} \\ & + \Big(\frac{1}{(t_1^{(1)} - t_2^{(1)})(t_2^{(1)} - z_2)} + \frac{1}{(t_2^{(1)} - t_1^{(1)})(t_1^{(1)} - z_2)}\Big) v_{\omega_r} \otimes E_{2,1}^2 v_{\omega_r}.\end{aligned}$$

The values of the universal weight function at the critical points of the master function are called the *Bethe vectors*.

The Bethe vectors of critical points of the same $\Sigma_{\boldsymbol{l}}$-orbit coincide, since both the critical point equations and the universal weight function are $\Sigma_{\boldsymbol{l}}$-invariant.

The universal weight function takes values in $L_{\omega_r}^{\otimes n}[\mu]$. But if $\boldsymbol{t}$ is a critical point of the master function, then the Bethe vector $\omega(\boldsymbol{t};\boldsymbol{z})$ belongs to the subspace of singular vectors $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu] \subset L_{\omega_r}^{\otimes n}[\mu]$; see [RV95] and comments on this fact in [MV05, §2].

By Theorem 2.2, the master function $\Phi_{\boldsymbol{d}}(\,\cdot\,;\boldsymbol{z})$ has $N(\boldsymbol{d})$ distinct orbits of critical points for generic $\boldsymbol{z}$. Form a list $\boldsymbol{t}^1,\dots,\boldsymbol{t}^{N(\boldsymbol{d})}$ of representatives in each of the orbits. These critical points define a collection of Bethe vectors $\omega(\boldsymbol{t}^1;\boldsymbol{z}),\dots,$ $\omega(\boldsymbol{t}^{N(\boldsymbol{d})};\boldsymbol{z})$ belonging to $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu]$. The space $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu]$ has dimension $N(\boldsymbol{d})$.

THEOREM 3.1 [MV05, Th. 6.1]. *For generic $\boldsymbol{z}$, the Bethe vectors form a basis in* $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu]$.

3.3. *The Gaudin model.* The Gaudin Hamiltonians on $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu]$ are certain linear operators acting on $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu]$ and (rationally) depending on a complex parameter x. We use the construction of the Gaudin Hamiltonians suggested in [Tal04] and [CT04]; see also [MTV06]. We consider the $\mathfrak{sl}_{r+1}$-module L_{ω_r} as the $\mathfrak{gl}_{r+1}$-module of highest weight $(0,\dots,0,-1)$.

To define the Gaudin Hamiltonians consider the differential operators

$$X_{i,j}(x) = \delta_{i,j}\frac{d}{dx} - \sum_{s=1}^{n}\frac{E_{j,i}^{(s)}}{x-z_s} \quad \text{for all } i,j=1,\dots,r+1,$$

where $\delta_{i,j}$ is the Kronecker symbol and $E_{j,i}^{(s)} = 1^{\otimes(s-1)}\otimes E_{j,i}\otimes 1^{\otimes(n-s)}$. These differential operators act on $L_{\omega_r}^{\otimes n}$-valued functions in x. The order of X_{ij} is one if $i=j$ and is zero otherwise.

Set

$$\boldsymbol{M} = \sum_{\sigma\in\Sigma_{r+1}} (-1)^{\sigma} X_{1,\sigma(1)}(x)X_{2,\sigma(2)}(x)\dots X_{r+1,\sigma(r+1)}(x), \tag{3.2}$$

where $(-1)^{\sigma}$ denotes the sign of the permutation. The operator M is the row-determinant of the matrix (X_{ij}).

For example, for $r=1$, we have

$$\boldsymbol{M} = \Big(\frac{d}{dx} - \sum_{s=1}^{n}\frac{E_{1,1}^{(s)}}{x-z_s}\Big)\Big(\frac{d}{dx} - \sum_{s=1}^{n}\frac{E_{2,2}^{(s)}}{x-z_s}\Big) - \Big(\sum_{s=1}^{n}\frac{E_{2,1}^{(s)}}{x-z_s}\Big)\Big(\sum_{s=1}^{n}\frac{E_{1,2}^{(s)}}{x-z_s}\Big).$$

Write

$$\boldsymbol{M} = \frac{d^{r+1}}{dx^{r+1}} + M_1(x)\frac{d^r}{dx^r} + \cdots + M_{r+1}(x),$$

where $M_i(x) : L_{\omega_r}^{\otimes n} \to L_{\omega_r}^{\otimes n}$ are linear operators depending on x. The coefficients $M_1(x), \ldots, M_{r+1}(x)$ are called the *Gaudin Hamiltonians*.

LEMMA 3.2. *The Gaudin Hamiltonians commute*: $[M_i(u), M_j(v)] = 0$ *for all* i, j, u *and* v. *The Gaudin Hamiltonians commute with the* $\mathfrak{gl}_{r+1}$*-action on* $L_{\omega_r}^{\otimes n}$; *in particular*, *they preserve* $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu]$.

The first statement is seen, for example, in [KS82], [Tal04], [CT04], and [MTV06, Prop. 8.2]. The second statement is seen, for example, in [KS82] and in [MTV06, Prop. 8.3].

THEOREM 3.3 [MTV06, Th. 9.2]. *For any critical point* $\boldsymbol{t}$ *of the master function* $\Phi_{\boldsymbol{d}}(\cdot\,;\boldsymbol{z})$, *the Bethe vector* $\omega(\boldsymbol{t};\boldsymbol{z})$ *is an eigenvector of* $M_i(x)$ *for* $i = 1, \ldots, r+1$. *The corresponding eigenvalues* $\mu_i(x)$ *are given by the formula*

$$\frac{d^{r+1}}{dx^{r+1}} + \mu_1(x)\frac{d^r}{dx^r} + \cdots + \mu_{r+1}(x) = \\ \Big(\frac{d}{dx} + \ln'(y_1)\Big)\Big(\frac{d}{dx} + \ln'\Big(\frac{y_2}{y_1}\Big)\Big)\cdots\Big(\frac{d}{dx} + \ln'\Big(\frac{y_r}{y_{r-1}}\Big)\Big)\Big(\frac{d}{dx} + \ln'\Big(\frac{T}{y_r}\Big)\Big).$$

Set

$$\begin{aligned}\boldsymbol{K} &= \frac{d^{r+1}}{dx^{r+1}} - \frac{d^r}{dx^r}M_1(x) + \cdots + (-1)^{r+1}M_{r+1}(x)\\ &= \frac{d^{r+1}}{dx^{r+1}} + K_1(x)\frac{d^r}{dx^r} + \cdots + K_{r+1}(x).\end{aligned}$$

This is the differential operator that is formally adjoint to the differential operator $(-1)^{r+1}\boldsymbol{M}$. The coefficients $K_i(x) : L_{\omega_r}^{\otimes n} \to L_{\omega_r}^{\otimes n}$ are linear operators depending on x. These coefficients can be expressed as differential polynomials in $M_1(x), \ldots, M_{r+1}(x)$. For instance,

$$K_1(x) = -M_1(x), \qquad K_2(x) = M_2(x) - r\frac{d}{dx}M_1(x),$$

and so on. Similarly, the operators $M_1(x), \ldots, M_{r+1}(x)$ can be expressed as differential polynomials in $K_1(x), \ldots, K_{r+1}(x)$.

By Lemma 3.2, the operators $K_1(x), \ldots, K_{r+1}(x)$ pairwise commute, that is, $[K_i(u), K_j(v)] = 0$ for all i, j, u and v, and they commute with the $\mathfrak{gl}_{r+1}$-action on $L_{\omega_r}^{\otimes n}$.

The operators $K_1(x), \ldots, K_{r+1}(x)$ will be called the *Gaudin Hamiltonians*, just like the operators $M_1(x), \ldots, M_{r+1}(x)$.

For any critical point $\boldsymbol{t}$ of the master function $\Phi_{\boldsymbol{d}}(\cdot\,;\boldsymbol{z})$, the Bethe vector $\omega(\boldsymbol{t};\boldsymbol{z})$ is an eigenvector of the Gaudin Hamiltonians $K_i(x)$ for $i = 1, \ldots, r+1$ by

Theorem 3.3 and Lemma 3.2. The corresponding eigenvalues $\lambda_i(x)$ are given by the formula

$$\frac{d^{r+1}}{du^{r+1}}+\lambda_1(x)\frac{d^r}{dx^r}+\cdots+\lambda_{r+1}(x)$$
$$=\Big(\frac{d}{dx}-\ln'\Big(\frac{T}{y_r}\Big)\Big)\Big(\frac{d}{dx}-\ln'\Big(\frac{y_r}{y_{r-1}}\Big)\Big)\Big(\frac{d}{dx}-\ln'\Big(\frac{y_2}{y_1}\Big)\Big)\Big(\frac{d}{dx}-\ln'(y_1)\Big).$$

Note that this is the fundamental differential operator $D_{\boldsymbol{t}}$ of the critical point $\boldsymbol{t}$.

COROLLARY 3.4. *For generic* $\boldsymbol{z}$,

- *the Bethe vectors form an eigenbasis of the Gaudin Hamiltonians* $K_i(x)$ *for* $i=1,\dots,r+1$,
- *the operators* $K_1(x),\dots,K_{r+1}(x)$ *have simple joint spectrum, that is, their eigenvalues separate the basis Bethe eigenvectors.*

The first statement follows from Theorem 3.1 and Theorem 3.3.

Let us prove the second statement. If two Bethe vectors have the same eigenvalues, then they have the same fundamental operators, hence the same fundamental spaces. The fundamental space of a critical point uniquely determines the orbit of the critical point by Theorem 2.1. Hence the two Bethe vectors correspond to the same orbit of critical points and hence are equal.

3.4. *The Shapovalov form and real* z. Define an anti-involution

$$\tau:\mathfrak{gl}_{r+1}\to\mathfrak{gl}_{r+1},\quad E_{i,j}\mapsto E_{j,i}\quad \text{for all } i,j.$$

Let W be a highest weight $\mathfrak{gl}_{r+1}$-module with highest weight vector w. The *Shapovalov form* on W is the unique symmetric bilinear form S defined by the conditions

$$S(w,w)=1\quad\text{and}\quad S(gu,v)=S(u,\tau(g)v)$$

for all $u,v\in W$ and $g\in\mathfrak{gl}_{r+1}$; see [Kac90]. The Shapovalov form is nondegenerate on an irreducible module W and is positive definite on the real part of the irreducible module W.

Let $L_{\Lambda_1}\otimes\cdots\otimes L_{\Lambda_n}$ be the tensor product of irreducible highest weight $\mathfrak{gl}_{r+1}$-modules. Let $v_{\Lambda_i}\in L_{\Lambda_i}$ be a highest weight vector and S_i the corresponding Shapovalov form on L_{Λ_i}. Define the symmetric bilinear form on the tensor product by the formula $S=S_1\otimes\cdots\otimes S_n$. The form S is called the *tensor Shapovalov form.*

THEOREM 3.5 [MTV06, Prop. 9.1]. *The Gaudin Hamiltonians* $K_i(x)$ *for* $i=1,\dots,r+1$ *are symmetric with respect to the tensor Shapovalov form* S:

$$S(K_i(x)u,v)=S(u,K_i(x)v)\quad \textit{for all } i,x,u,v.$$

COROLLARY 3.6. *If all of $z_1, \dots, z_n, x$ are real numbers, then the Gaudin Hamiltonians $K_i(x)$ for $i = 1, \dots, r+1$ are real linear operators on the real part of the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$. These operators are symmetric with respect to the positive definite tensor Shapovalov form. Hence they are simultaneously diagonalizable and have real spectrum.*

3.5. *Proof of Theorem* 1.4. If $z_1, \dots, z_n, x$ are real, then all of the Gaudin Hamiltonians on $\operatorname{Sing} L_{\omega_r}^{\otimes n}[\mu]$ have real spectrum, since they are symmetric operators on a Euclidean space. If $\boldsymbol{t}$ is a critical point of $\Phi_{\boldsymbol{d}}(\,\cdot\,;\boldsymbol{z})$, then the eigenvalues $\lambda_1(x), \dots, \lambda_{r+1}(x)$ of the corresponding Bethe vector $\omega(\boldsymbol{t};\boldsymbol{z})$ are real rational functions. Hence the fundamental differential operator $D_{\boldsymbol{t}}$ has real coefficients. Therefore, the fundamental vector space of polynomials $V_{\boldsymbol{t}}$ is real. Thus for generic real $z_1, \dots, z_n$ we have $N(\boldsymbol{d})$ distinct real spaces of polynomials with exponents $\boldsymbol{d}$ at infinity and Wronskian $\prod_{s=1}^{n}(x - z_s)$. Thus Theorem 1.4 is proved. □

Appendix A

A.1. *The differential operator K has polynomial solutions only.* Suppose $z_1, \dots, z_n \in \mathbb{C}$. Let $\Lambda_1, \dots, \Lambda_n, \Lambda_\infty \in \mathfrak{h}^*$ be dominant integral weights. Assume that the irreducible $\mathfrak{sl}_{r+1}$-module L_{Λ_∞} is a submodule of the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.

For any $s = 1, \dots, n, \infty$, and $i = 1, \dots, r$, set $m_{s,i} = (\Lambda_s, \sum_{j=1}^{i} \alpha_j)$ and

$$l = \frac{1}{r+1} \sum_{i=1}^{r} \Big(\sum_{s=1}^{n} m_{s,i} - m_{\infty,i} \Big).$$

For any $s = 1, \dots, n$, we will consider the $\mathfrak{sl}_{r+1}$-module L_{Λ_s} as the $\mathfrak{gl}_{r+1}$-module of highest weight $(0, -m_{s,1}, -m_{s,2}, \dots, -m_{s,r})$. Considered as a submodule of the $\mathfrak{gl}_{r+1}$-module $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$, the $\mathfrak{sl}_{r+1}$-module L_{Λ_∞} has the $\mathfrak{gl}_{r+1}$-highest weight

$$(-l, -l - m_{\infty,1}, -l - m_{\infty,2}, \dots, -l - m_{\infty,r}).$$

THEOREM A.1. *Consider the operator $\boldsymbol{K}$ as a differential operator acting on $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$-valued functions in x.*

(i) *Then all singular points of the operator $\boldsymbol{K}$ are regular and lie in the set $\{z_1, \dots, z_n, \infty\}$.*

(ii) *Let $u(x)$ be any germ of an $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$-valued function such that $\boldsymbol{K}u = 0$. Then u is the germ of an $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$-valued polynomial in x.*

(iii) *Let $w \in \operatorname{Sing}(L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n})[\Lambda_\infty]$ be an eigenvector of the operators $K_1(x), \dots, K_{r+1}(x)$ with the eigenvalues $\lambda_1(x), \dots, \lambda_{r+1}(x)$, respectively.*

Consider the scalar differential operator

$$D_w = \frac{d^{r+1}}{dx^{r+1}} + \lambda_1(x)\frac{d^r}{dx^r} + \cdots + \lambda_{r+1}(x).$$

Then the exponents of the differential operator D_w at ∞ are

$$-l,\ -m_{\infty,1} - 1 - l, \ldots,\ -m_{\infty,r} - r - l.$$

(iv) *If $z_1, \ldots, z_n$ are distinct, then for any $s = 1, \ldots, n$, the exponents of the differential operator D_w at z_s are* $0, m_{s,1} + 1, \ldots, m_{s,r} + r$.

(v) *The kernel of the differential operator D_w is an $(r+1)$-dimensional space of polynomials.*

Proof. Part (i) is a direct corollary of the definition of the operator $\boldsymbol{K}$.

We first prove part (ii) in the special case of $\Lambda_1 = \cdots = \Lambda_n = \omega_r$ and generic $z_1, \ldots, z_n$. By construction, the operator $\boldsymbol{K}$ commutes with the $\mathfrak{gl}_{r+1}$-action on $L_{\omega_r}^{\otimes n}$. This fact and Theorem 3.1 imply that $\boldsymbol{K}$ has an eigenbasis consisting of the Bethe vectors and their images under the $\mathfrak{gl}_{r+1}$-action. Then by Theorems 3.3 and 2.1, all solutions of the differential equation $\boldsymbol{K}u = 0$ are polynomials.

The proof of part (ii) for arbitrary $\Lambda_1, \ldots, \Lambda_n$ and $z_1, \ldots, z_n$ clearly follows from the special case and the following remarks:

- The operator $\boldsymbol{K}$ is well defined for any $z_1, \ldots, z_n$, not necessarily distinct, and rationally depends on $z_1, \ldots, z_n$.
- If for generic $z_1, \ldots, z_n$, all solutions of the differential equation $\boldsymbol{K}u = 0$ are polynomial, then for any $z_1, \ldots, z_n$, all solutions of the differential equation $\boldsymbol{K}u = 0$ are polynomial.
- Assume that some of $z_1, \ldots, z_n$ coincide. Partition the set $\{z_1, \ldots, z_n\}$ into several groups of coinciding points of sizes $n_1, \ldots, n_k$, whose sum is n. Denote the representatives in the groups by $u_1, \ldots, u_k \in \mathbb{C}$, where $u_1, \ldots, u_k$ are distinct. Denote $W_s = L_{\omega_r}^{\otimes n_s}$ for $s = 1, \ldots, k$. Choose an irreducible module $L_{\nu_s} \subset W_s$ for every s. Then the operator $\boldsymbol{K}$ defined for those $z_1, \ldots, z_n$ on $W_1 \otimes \cdots \otimes W_k$ preserves the space of functions with values in the submodule $L_{\nu_1} \otimes \cdots \otimes L_{\nu_k}$. If we restrict $\boldsymbol{K}$ to the space of functions with values in $L_{\nu_1} \otimes \cdots \otimes L_{\nu_k}$, then this restriction coincides with the operator $\boldsymbol{K}$ defined for the tensor product $L_{\nu_1} \otimes \cdots \otimes L_{\nu_k}$ and $u_1, \ldots, u_k$.
- Any highest weight irreducible finite dimensional $\mathfrak{gl}_{r+1}$-module with highest weight $(m_0, \ldots, m_r)$, where $m_0 \in \mathbb{Z}_{\leqslant 0}$, is a submodule of a suitable tensor power of L_{ω_r} (considered as the $\mathfrak{gl}_{r+1}$-module with highest weight $(0, \ldots, 0, -1)$).

Part (ii) is proved.

To calculate the exponents of the operator D_w at singular points, we calculate the exponents of its formal adjoint operator. Namely, we consider the operator

$$D_w^* = \frac{d^{r+1}}{dx^{r+1}} - \frac{d^r}{dx^r}\lambda_1(x) + \cdots + (-1)^{r+1}\lambda_{r+1}(x)$$
$$= \frac{d^{r+1}}{dx^{r+1}} + \mu_1(x)\frac{d^r}{dx^r} + \cdots + \mu_{r+1}(x).$$

The vector w is an eigenvector of the operators $M_1(x), \ldots, M_{r+1}(x)$, with eigenvalues $\mu_1(x), \ldots, \mu_{r+1}(x)$, respectively.

LEMMA A.2. *Let the exponents of D_w^* at a point $x = z$ be $p_1, \ldots, p_{r+1}$. Then the exponents of D_w at the point $x = z$ are $r - p_{r+1}, \ldots, r - p_1$.* □

Consider the $U(\mathfrak{gl}_{r+1})$-valued polynomial

$$A(x) = \sum_{\sigma \in \Sigma_{r+1}} (-1)^\sigma \big((x-r)\delta_{1,\sigma(1)} - E_{\sigma(1),1}\big) \cdots \big((x-1)\delta_{r,\sigma(r)} - E_{\sigma(r),r}\big)\big(x\delta_{r+1,\sigma(r+1)} - E_{\sigma(r+1),r+1}\big). \tag{A.1}$$

It is known that the coefficients of this polynomial are central elements in $U(\mathfrak{gl}_{r+1})$; see for example [MNO96, Remark 2.11]. If v is a singular vector of a $\mathfrak{gl}_{r+1}$-weight $(p_1, \ldots, p_{r+1})$, then formula (A.1) yields

$$A(x)v = \prod_{i=1}^{r+1}(x - r - 1 + i - p_i)v.$$

Hence, the operator $A(x)$ acts on L_{Λ_s} as the identity operator multiplied by

$$\psi_s(x) = \prod_{i=0}^{r}(x - r + i + m_{s,i}) .$$

Let $s = 1, \ldots, n$. It follows from (3.2) that the indicial polynomial of D_w^* at the singular point z_s is the eigenvalue of the operator $1^{\otimes(s-1)} \otimes A(x) \otimes 1^{\otimes(n-s)}$ acting on the vector w, that is, $\psi_s(x)$. Similarly, the indicial polynomial of D_w^* at infinity is the eigenvalue of $A(-x)$ acting on the vector w that belongs to the submodule L_{Λ_∞} of the $\mathfrak{gl}_{r+1}$-module $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$, that is,

$$\psi_\infty(x) = \prod_{i=0}^{r}(-x - r + i + l + m_{\infty,i}) .$$

Hence, by Lemma A.2, the exponents of the operator D_w are as required. This proves parts (iii) and (iv). Part (v) follows from parts (i)–(iv). □

COROLLARY A.3. *Assume that the operators* $K_i(x), \dots, K_{r+1}(x)$ *acting on the subspace of weight singular vectors* $\mathrm{Sing}(L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n})[\Lambda_\infty]$ *are diagonalizable and have simple joint spectrum. Then there exist*

$$\dim \mathrm{Sing}(L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n})[\Lambda_\infty]$$

distinct polynomial $(r+1)$*-dimensional spaces* V *with the following properties. If* D *is the fundamental differential operator of such a space, then* D *has singular points at* $z_1, \dots, z_n, \infty$ *only, with the exponents*

$$\begin{aligned} &0, m_{s,1}+1, \dots, m_{s,r}+r && \text{at } z_s \text{ for any } s, \\ -&l,\ -m_{\infty,1}-1-l, \dots,\ -m_{\infty,r}-r-l && \text{at } \infty. \end{aligned}$$

Consider all $(r+1)$-dimensional polynomial spaces V, whose fundamental operator has exponents at $z_1, \dots, z_n, \infty$ as indicated in Corollary A.3. Schubert calculus says that the number of such spaces is not greater than the dimension of $\mathrm{Sing}(L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n})[\Lambda_\infty]$; see for example [MV04]. Thus, according to Corollary A.3, the simplicity of the spectrum of the Gaudin Hamiltonians on $\mathrm{Sing}(L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n})[\Lambda_\infty]$ implies the transversality of the Schubert cycles corresponding to these exponents at $z_1, \dots, z_n, \infty$; cf. [MV04] and [EH83].

The operators $K_1(x), \dots, K_{r+1}(x)$ acting on $\mathrm{Sing}(L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n})[\Lambda_\infty]$ are diagonalizable if $z_1, \dots, z_n$ are real; see Section 3.4.

Remark A.4. It was conjectured in [CT04] that the monodromy of the differential operator $\boldsymbol{M}$, acting on $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$-valued functions in x, is trivial. However, the proof of this statement in [CT04] is not satisfactory. On the other hand, Theorem A.1 implies that the monodromy of the differential operator $\boldsymbol{K}$, acting on $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$-valued functions in x, is trivial. Together with Theorem 3.5, this implies that the monodromy of the operator $\boldsymbol{M}$ is trivial as well.

A.2. *Bethe vectors in* $\mathrm{Sing}(L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n})[\Lambda_\infty]$. Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be a point in $\mathbb{C}^n$ with distinct coordinates. Let $\Lambda_1, \dots, \Lambda_n, \Lambda_\infty \in \mathfrak{h}^*$ be dominant integral weights. Assume that the irreducible $\mathfrak{sl}_{r+1}$-module L_{Λ_∞} is a submodule of the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.

Introduce $\boldsymbol{l} = (l_1, \dots, l_r)$ by the formula $\Lambda_\infty = \sum_{s=1}^n \Lambda_s - \sum_{i+1}^r l_i \alpha_i$. Set $l = l_1 + \cdots + l_r$. Consider the associated master function

$$\Phi(\boldsymbol{t}; \boldsymbol{z}) = \prod_{i=1}^{r} \prod_{j=1}^{l_i} \prod_{s=1}^{n} (t_j^{(i)} - z_s)^{-(\Lambda_s, \alpha_i)} \prod_{i=1}^{r} \prod_{1 \leqslant j < s \leqslant l_i} (t_j^{(i)} - t_s^{(i)})^2 \times \prod_{i=1}^{r-1} \prod_{j=1}^{l_i} \prod_{k=1}^{l_{i+1}} (t_j^{(i)} - t_k^{(j+1)})^{-1}.$$

Consider the universal weight function $\omega : \mathbb{C}^l \times \mathbb{C}^n \to (L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n})[\Lambda_\infty]$ defined by the formulas of Section 3.2. The value $\omega(\boldsymbol{t};\boldsymbol{z})$ of the universal weight function at a critical point $\boldsymbol{t}$ of the master function $\Phi(\,\cdot\,;\boldsymbol{z})$ is called *a Bethe vector* (see [RV95] and [MV04]), and belongs to $\mathrm{Sing}(L_{\Lambda_1}\otimes\cdots\otimes L_{\Lambda_n})[\Lambda_\infty]$; see [RV95].

For a critical point $\boldsymbol{t}$, define the tuple $\boldsymbol{y}^{\boldsymbol{t}} = (y_1, \dots, y_r)$ of polynomials in variable x by formulas of Section 2.1. Define polynomials $T_1, \dots, T_r$ in x by the formula

$$T_i(x) = \prod_{s=1}^{n} (x - z_s)^{(\Lambda_s, \alpha_i)}.$$

Consider the linear differential operator of order $r + 1$ given by

$$D_{\boldsymbol{t}} = \Big(\frac{d}{dx} - \ln'\Big(\frac{T_1 \cdots T_r}{y_r}\Big)\Big)\Big(\frac{d}{dx} - \ln'\Big(\frac{y_r T_1 \cdots T_{r-1}}{y_{r-1}}\Big)\Big) \cdots \Big(\frac{d}{dx} - \ln'\Big(\frac{y_2 T_1}{y_1}\Big)\Big)\Big(\frac{d}{dx} - \ln'(y_1)\Big).$$

All singular points of $D_{\boldsymbol{t}}$ are regular and lie in $\{z_1, \dots, z_n, \infty\}$. The exponents of $D_{\boldsymbol{t}}$ at z_s are $0, m_{s,1} + 1, \dots, m_{s,r} + r$ for any s, and the exponents of $D_{\boldsymbol{t}}$ at ∞ are $-l,\ -m_{\infty,1} - 1 - l, \dots,\ -m_{\infty,r} - r - l$. The kernel $V_{\boldsymbol{t}}$ of $D_{\boldsymbol{t}}$ is an $(r + 1)$-dimensional space of polynomials; see [MV04].

The tuple $\boldsymbol{y}^{\boldsymbol{t}}$ can be recovered from $V_{\boldsymbol{t}}$ as follows. Let $f_1, \dots, f_{r+1}$ be a basis of $V_{\boldsymbol{t}}$ consisting of monic polynomials of strictly increasing degree. Then $y_1, \dots, y_r$ are respective scalar multiples of the polynomials

$$f_1, \quad \frac{\mathrm{Wr}(f_1, f_2)}{T_1}, \quad \frac{\mathrm{Wr}(f_1, f_2, f_3)}{T_2 T_1^2}, \quad \dots \quad , \frac{\mathrm{Wr}(f_1, \dots, f_r)}{T_{r-1} T_{r-2}^2 \dots T_1^{r-1}};$$

see [MV04].

THEOREM A.5 [MTV06, Th. 9.2]. *For any critical point $\boldsymbol{t}$ of the master function $\Phi(\,\cdot\,;\boldsymbol{z})$, the Bethe vector $\omega(\boldsymbol{t};\boldsymbol{z})$ is an eigenvector of $K_i(x)$ for $i = 1, \dots, r+1$ and the corresponding eigenvalues $\lambda_1(x), \dots, \lambda_{r+1}(x)$ are given by the formula*

$$\frac{d^{r+1}}{du^{r+1}} + \lambda_1(x)\frac{d^r}{dx^r} + \cdots + \lambda_{r+1}(x) = D_{\boldsymbol{t}}.$$

COROLLARY A.6. *Any two distinct nonzero Bethe vectors cannot have the same eigenvalues for all Gaudin Hamiltonians.*

The proof of the corollary is similar to the proof of the second statement of Corollary 3.4.

Appendix B

Let $\mathfrak{g}$ be a simple Lie algebra, let $\mathfrak{h}$ its Cartan subalgebra, let $\alpha_i \in \mathfrak{h}^*$ for $i = 1, \dots, r$ be simple roots, and let $(\,\cdot\,,\cdot\,)$ be the standard invariant scalar product on $\mathfrak{g}$. Let $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ be integral dominant weights of $\mathfrak{g}$. Let $\boldsymbol{l} = (l_1, \dots, l_r)$ be nonnegative integers such that the weight

$$\Lambda_\infty = \textstyle\sum_{s=1}^n \Lambda_s - \sum_{i=1}^r l_i \alpha_i$$

is dominant integral. Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be distinct complex numbers. Introduce the associated *master function* depending on variables

$$\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}, \dots, t_1^{(r)}, \dots, t_{l_r}^{(r)})$$

by the formula

$$\Phi_{\mathfrak{g},\boldsymbol{\Lambda},\boldsymbol{l}}(\boldsymbol{t};\boldsymbol{z}) = \prod_{i=1}^{r}\prod_{j=1}^{l_i}\prod_{s=1}^{n}(t_j^{(i)} - z_s)^{-(\Lambda_s,\alpha_i)} \prod_{i=1}^{r}\prod_{1\leqslant j<s\leqslant l_i}(t_j^{(i)} - t_s^{(i)})^{(\alpha_i,\alpha_i)} \times \prod_{1\leqslant i<j\leqslant r}\prod_{s=1}^{l_i}\prod_{k=1}^{l_j}(t_s^{(i)} - t_k^{(j)})^{(\alpha_i,\alpha_j)}.$$

The function Φ is a rational function of $\boldsymbol{t}$ and depends on parameters $\boldsymbol{z}$. The master function is $\Sigma_{\boldsymbol{l}}$-invariant with respect to permutations of variables with the same upper index. The critical set of the master function with respect to variables $\boldsymbol{t}$ is $\Sigma_{\boldsymbol{l}}$-invariant. If $\boldsymbol{z}$ consists of real numbers, then the critical set is invariant with respect to complex conjugation.

CONJECTURE B.1. *If $\boldsymbol{z}$ consists of real numbers, then every orbit of critical points is invariant with respect to complex conjugation.*

For a critical point $\boldsymbol{t}$, define the tuple $\boldsymbol{y}^{\boldsymbol{t}} = (y_1, \dots, y_r)$ of polynomials in variable x by formulas of Section 2.1. Conjecture B.1 can be reformulated as follows. If $\boldsymbol{z}$ consists of real numbers and $\boldsymbol{t}$ is a critical point, then the tuple $\boldsymbol{y}^{\boldsymbol{t}}$ consists of real polynomials.

Theorems 1.1 and 2.1 imply this conjecture for $\mathfrak{g} = \mathfrak{sl}_{r+1}$. In the same way, Theorems 1.1 and 2.1 imply Conjecture B.1 for $\mathfrak{g}$ of types B_r or C_r; see [MV04, §7].

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E-mail address: mukhin@math.iupui.edu
DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY–PURDUE UNIVERSITY, 402 NORTH BLACKFORD ST., INDIANAPOLIS, IN 46202-3216, UNITED STATES

E-mail address: vt@pdmi.ras.ru
E-mail address: vt@math.iupui.edu
ST. PETERSBURG BRANCH OF STEKLOV MATHEMATICAL INSTITUTE, FONTANKA 27, ST. PETERSBURG, 191023, RUSSIA

and

DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY–PURDUE UNIVERSITY, 402 NORTH BLACKFORD ST., INDIANAPOLIS, IN 46202-3216, UNITED STATES

E-mail address: anv@email.unc.edu
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL, CHAPEL HILL, NC 27599-3250, UNITED STATES