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On the shape of Bruhat intervals

By ANDERS BJÖRNER and TORSTEN EKEDAHL

Abstract

Let (W, S) be a crystallographic Coxeter group (this includes all finite and affine Weyl groups), and let $J \subseteq S$. Let W^J denote the set of minimal coset representatives modulo the parabolic subgroup W_J. For $w \in W^J$, let $f_i^{w,J}$ denote the number of elements of length i below w in Bruhat order on W^J (with notation simplified to f_i^w in the case when $W^J = W$). We show that

$$0 \leq i < j \leq \ell(w) - i \quad \text{implies} \quad f_i^{w,J} \leq f_j^{w,J}.$$

Also, the case of equalities $f_i^w = f_{\ell(w)-i}^w$ for $i = 1, \ldots, k$ is characterized in terms of vanishing of coefficients in the Kazhdan-Lusztig polynomial $P_{e,w}(q)$.

We show that if W is finite then the number sequence $f_0^w, f_1^w, \ldots, f_{\ell(w)}^w$ cannot grow too rapidly. Further, in the finite case, for any given $k \geq 1$ and any $w \in W$ of sufficiently great length (with respect to k), we show

$$f_{\ell(w)-k}^w \geq f_{\ell(w)-k+1}^w \geq \cdots \geq f_{\ell(w)}^w.$$

The proofs rely mostly on properties of the cohomology of Kac-Moody Schubert varieties, such as the following result: if $\overline{X}_w$ is a Schubert variety of dimension $d = \ell(w)$, and $\lambda = c_1(\mathcal{L}) \in H^2(\overline{X}_w)$ is the restriction to $\overline{X}_w$ of the Chern class of an ample line bundle, then

$$(\lambda^k) \cdot : H^{d-k}(\overline{X}_w) \to H^{d+k}(\overline{X}_w)$$

is injective for all $k \geq 0$.

1. Introduction

Let (W, S) be a Coxeter system with S finite. Fix a subset $J \subseteq S$, and let W^J denote the set of minimal coset representatives modulo the parabolic subgroup

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$W_J = \langle J \rangle$. For $w \in W^J$, let

$$f_i^{w,J} := \mathrm{card}\{u \in W^J \mid u \leq w \text{ and } \ell(u) = i\}.$$

In words, $f_i^{w,J}$ is the number of length i elements contained in the Bruhat interval $[e,w]^J = [e,w] \cap W^J$. For terminology and basic facts concerning Coxeter groups, Weyl groups, and Bruhat order, we refer to [BB05], [Hum90].

For a certain class of Coxeter groups we can apply geometric methods to study the numbers $f_i^{w,J}$. The groups to which our methods are applicable are those for which the order of a product of two generators is 2, 3, 4, 6, or ∞; these are usually called the *crystallographic* Coxeter groups. They are precisely the groups that appear as Weyl groups of Kac-Moody algebras (cf. [Kac83, Proposition 3.13]). The main purpose of this paper is to prove the following relations and some of their ramifications.

THEOREM A. *Let* (W, S) *be a crystallographic Coxeter group*, $J \subseteq S$, *and* $w \in W^J$. *Then*

$$0 \leq i < j \leq \ell(w) - i \quad \textit{implies} \quad f_i^{w,J} \leq f_j^{w,J}.$$

This theorem follows immediately from a certain cohomological injectivity condition. Similar injectivity properties have recently appeared in other contexts (cf. [HS02, Theorem 7.4] and [Swa06, §3]). For the definition of Schubert variety see the beginning of Section 4.

THEOREM B. *Let* $\overline{X}_w$ *be a Schubert variety of dimension* $d = \ell(w)$ *of a Kac-Moody group*, *and let* $\lambda := c_1(\mathcal{L})$ *be the Chern class of an ample line bundle on* $\overline{X}_w$. *Then the map*

$$(\lambda^k) \cdot : H^{d-k}(\overline{X}_w) \to H^{d+k}(\overline{X}_w)$$

is injective for all $k \geq 0$.

Combining the cohomological arguments used for proving Theorem A with a linear-algebraic argument of Stanley [Sta80], we can sharpen Theorem A to a combinatorial statement giving structural reasons for these inequalities.

THEOREM C. *Let* (W, S) *be a crystallographic Coxeter group*, *and let* $J \subseteq S$. *Fix* $w \in W^J$ *and* i *such that* $0 \leq i < \ell(w)/2$. *Then*, *in* $[e,w]^J$ *there exist* $f_i^{w,J}$ *pairwise disjoint chains* $u_i < u_{i+1} < \cdots < u_{\ell(w)-i}$ *such that* $\ell(u_j) = j$.

The inequalities of Theorem A are equivalent to the combination of the two sets of inequalities

$$f_i^{w,J} \leq f_{\ell(w)-i}^{w,J} \quad \text{for all } i < \ell(w)/2, \text{ and} \tag{1}$$

$$f_0^{w,J} \leq f_1^{w,J} \leq \cdots \leq f_{\lceil \ell(w)/2 \rceil}^{w,J}. \tag{2}$$

For the rest of this section we treat only the $J = \varnothing$ case. Then $W^J = W$, so for simplicity we drop "J" from the notation. In this case the relations (1), conjectured independently by Carrell [Car95] and by the present authors, sharpen the inequalities $\sum_{i \le k} f_i^w \le \sum_{i \le k} f_{\ell(w)-i}^w$ for all $0 \le k < \ell(w)/2$, due to Brion [Bri00, Corollary 2].

The case of equality in some of the relations (1) is interesting. Fix $w \in W$, and let $m := \lfloor(\ell(w)-1)/2\rfloor$. Let

$$P_{e,w}(q) = 1 + \beta_0 + \beta_1 q + \cdots + \beta_m q^m$$

be the Kazhdan-Lusztig polynomial of the interval $[e, w]$. It is known [Kum02, Theorem 12.2.9] that all Kazhdan-Lusztig polynomials have nonnegative coefficients if W is crystallographic, and that $\beta_0 = 0$.

THEOREM D. *Suppose that (W, S) is crystallographic. Let $w \in W$ and $0 \le k \le m$. Then the following conditions are equivalent*:

(a) $f_i^w = f_{\ell(w)-i}^w$, *for $i = 0, \ldots, k$, and*

(b) $\beta_i = 0$, *for $i = 0, \ldots, k$.*

Furthermore, if $k < m$ then (a) *and* (b) *imply*

(c) $\beta_{k+1} = f_{\ell(w)-k-1}^w - f_{k+1}^w$.

In the case $k = m$, the equivalence of (a) and (b) specializes to a criterion, due to Carrell and Peterson [Car94], for rational smoothness of the Schubert variety X_w.

The next result shows, among other things, that for finite groups the increasing sequence (2) cannot grow too fast. The condition of being an M-sequence is recalled in Section 6.

THEOREM E. *Let (W, S) be a finite Weyl group and $w \in W$. Then the vectors $(f_0^w, f_1^w, \ldots, f_{\ell(w)}^w)$ and $(f_0^w, f_1^w - f_0^w, f_2^w - f_1^w, \ldots, f_{\lfloor\ell(w)/2\rfloor}^w - f_{\lfloor\ell(w)/2\rfloor-1}^w)$ are M-sequences.*

The increasing inequalities (2) have decreasing counterparts at the upper end of the Bruhat interval, but the information we are able to give about this is much weaker.

THEOREM F. *For all $k \ge 1$, there exists a number N_k such that for every finite Coxeter group (W, S) and every $w \in W$ such that $\ell(w) \ge N_k$, we have*

$$f_{\ell(w)-k}^w \ge f_{\ell(w)-k+1}^w \ge \cdots \ge f_{\ell(w)}^w.$$

The paper is organized as follows. Sections 2 and 3 contain preliminary material on the algebraic geometry underlying the proofs of Theorems A, B and D in Section 4. The proofs of Theorems C, E and F can be found in Sections 5, 6 and 7, respectively. Section 8 expands on some algebraic geometry needed for the proofs.

2. **The pure cohomology**

Let F be an endomorphism of a graded $\mathbf{Q}_\ell$-vector space V that is finite dimensional in each degree. It will be said to be *of weight* $\leq w$ (respectively *of pure weight* w) with respect to a positive integer q if the eigenvalues of F on V^i are algebraic numbers all of whose conjugates have the same absolute value $q^{j/2}$ for some $j \leq w+i$ (respectively $j = w+i$). A theorem of Deligne provides a large number of such vector spaces in the following way.

Consider a proper variety X_0 over a finite field $\mathbb{F}_q$ and étale cohomology $H^*(X, \mathbf{Q}_\ell)$, considered as a graded vector space. (We shall follow the usual convention of using 0 as a subscript to denote objects over a finite field and drop the subscript when we extend scalars to an algebraic closure of that field). The Frobenius map on X_0 induces an endomorphism F of $H^*(X, \mathbf{Q}_\ell)$ and Deligne's theorem [BBD82, 5.1.14] says that the action of F on $H^*(X, \mathbf{Q}_\ell)$ is of weight ≤ 0. We shall only be interested in the *pure part*, $H_p^*(X, \mathbf{Q}_\ell)$, of $H^*(X, \mathbf{Q}_\ell)$, which by definition is obtained from $H^*(X, \mathbf{Q}_\ell)$ by factoring out by the F-generalized eigenspaces of weight < 0. Our first result will identify $H_p^*(X, \mathbf{Q}_\ell)$ with the image of $H^*(X, \mathbf{Q}_\ell)$ in the (middle perversity) *intersection cohomology* $\mathcal{I}H^*(X, \mathbf{Q}_\ell)$ of X.

We start by recalling the construction of [GM83, §5.1] (which, as the authors note, works also in an étale context) of a map of sheaves $\mathbf{Q}_{\ell,X} \to j_{!*}\mathbf{Q}_{\ell,U}$, where $j : U \to X$ is the inclusion of its nonsingular locus of the algebraic variety X (when confusion needs to be avoided, we shall use $\mathbf{Q}_{\ell,Z}$ for the constant sheaf $\mathbf{Q}_\ell$ on the scheme Z). This map induces a map from ordinary cohomology to intersection cohomology. (We use the degree conventions of [BBD82], which means that we have a map $\mathbf{Q}_{\ell,X} \to j_{!*}\mathbf{Q}_{\ell,U}$ rather than $\mathbf{Q}_{\ell,X}[n] \to j_{!*}\mathbf{Q}_{\ell,U}$ as in [GM83].)

Note now that, by definition, if X_0 is an algebraic variety over a finite field $\mathbb{F}_q$, a mixed complex $\mathcal{E}_0$ on X_0 (cf. [Del80, 1.1.2]) is of weight $\leq w$ precisely when the graded vector space $H^*(\mathcal{E}_{\bar{s}})$ is of weight $\leq w$ (with respect to q) for every geometric point $\bar{s}$ of X_0 with image a closed point of X_0 whose residue field has cardinality q.

THEOREM 2.1. *Let X_0 be a proper variety over a finite field. Then the kernel of the map from ordinary cohomology to intersection cohomology $H^*(X, \mathbf{Q}_\ell) \to \mathcal{I}H^*(X, \mathbf{Q}_\ell)$ consists exactly of the part of $H^*(X, \mathbf{Q}_\ell)$ of weight less than* 0. *In particular, $H_p^*(X, \mathbf{Q}_\ell)$ is the image of $H^*(X, \mathbf{Q}_\ell)$ in $\mathcal{I}H^*(X, \mathbf{Q}_\ell)$.*

Proof. Let $j : U_0 \to X_0$ be the inclusion of the smooth locus. We fit the map $\mathbf{Q}_{\ell,X_0} \to j_{!*}\mathbf{Q}_{\ell,U_0}$ into a distinguished triangle

$$\to \mathcal{F} \to \mathbf{Q}_{\ell,X_0} \to j_{!*}\mathbf{Q}_{\ell,X_0} \to .$$

We start by showing that $\mathscr{F}$ is of weight < 0. By [BBD82, Corollary 5.4.3] we know that $j_{!*}\mathbf{Q}_\ell$ is pure of weight 0. Let now $\bar{s}$ be a geometric point of X_0 with image a closed point (whose residue field then is finite). We have an exact sequence

$$H^{i-1}((j_{!*}\mathbf{Q}_{\ell,U_0})_{\bar{s}}) \to H^i(\mathscr{F}_{\bar{s}}) \to H^i((\mathbf{Q}_{\ell,X_0})_{\bar{s}}) \to H^i((j_{!*}\mathbf{Q}_{\ell,U_0})_{\bar{s}}),$$

and, since $H^{i-1}((j_{!*}\mathbf{Q}_{\ell,U_0})_{\bar{s}})$ is of weight $\leq i-1$, it is enough to show that $H^i((\mathbf{Q}_{\ell,X_0})_{\bar{s}}) \to H^i((j_{!*}\mathbf{Q}_{\ell,U_0})_{\bar{s}})$ is injective, which is a nontrivial condition only for $i=0$. In that case it is indeed injective as the composite $\mathbf{Q}_{\ell,X_0} \to j_{!*}\mathbf{Q}_{\ell,U_0} \to Rj_*\mathbf{Q}_{\ell,U_0}$ induces an isomorphism on $H^0(\cdot)$. By Deligne's theorem [BBD82, 5.1.14] we get that $H^*(X,\mathscr{F})$ is of weight < 0, and since the sequence

$$H^*(X,\mathscr{F}) \to H^*(X,\mathbf{Q}_\ell) \to H^*(X, j_{!*}\mathbf{Q}_\ell)$$

is exact, we see that the kernel of $H^*(X,\mathbf{Q}_\ell) \to H^*(X, j_{!*}\mathbf{Q}_\ell)$ is of weight < 0.

On the other hand, again by Deligne's theorem and duality, $H^*(X, j_{!*}\mathbf{Q}_\ell)$ is pure of weight 0, and hence everything in $H^*(X,\mathbf{Q}_\ell)$ of weight < 0 lies in the kernel. $\square$

Remark. (i) Over the complex numbers this result is proved in [Web04] (using instead Deligne's Hodge-theoretically defined weight filtration). For the applications of this paper, that result could also be used. In any case, the filtration by weights of ℓ-adic cohomology is defined for a variety over any field, commutes appropriately under specialization of the base field, and coincides with Deligne's Hodge-theoretic weight filtration over the complex numbers. Thus our results are compatible with those of [Web04].

(ii) Nothing changes in the argument if one replaces U_0 by a smaller open dense subset and $\mathbf{Q}_\ell$ by $j_*\mathscr{E}$, where $\mathscr{E}$ is a local system of pure weight 0.

Since it seems reasonable, we shall introduce the *pure Betti numbers* $b_i^p := \dim_{\mathbf{Q}_\ell} H_p^i(X,\mathbf{Q}_\ell)$. Our next result is a weakening of some well-known numeric consequences for the Betti numbers of a smooth and proper variety that arise as a consequence of the hard Lefschetz theorem. The stated restriction to varieties defined over finite fields is easily dispensed with, but we keep it to avoid too many details.

THEOREM 2.2. *Let X_0 be a projective variety over a finite field, of pure dimension n. We have that $b_i^p \leq b_{i+2j}^p$ for all $0 \leq j \leq n-i$. In particular, for $i \leq n$ we have that $b_{n-i}^p \leq b_{n+i}^p$.*

Proof. By Theorem 2.1, it follows that the map $H^*(X,\mathbf{Q}_\ell) \to \mathscr{I}H^*(X,\mathbf{Q}_\ell)$ has $H_p^*(X,\mathbf{Q}_\ell)$ as its image. Since it is induced by a map of $\mathbf{Q}_\ell$-complexes of sheaves, this map is an $H^*(X,\mathbf{Q}_\ell)$-module map. In particular, for $0 \leq j \leq n-i$ it commutes with multiplication by $c_1(\mathscr{L})^j$ (the first Chern class of an ample line bundle), giving a commutative diagram

$$\begin{tikzcd}
H_p^i(X,\mathbf{Q}_\ell) \arrow[r] \arrow[d, "\cap c_1(\mathscr{L})^j"] & \mathscr{I}H^i(X,\mathbf{Q}_\ell) \arrow[d, "\cap c_1(\mathscr{L})^j"] \\
H_p^{i+2j}(X,\mathbf{Q}_\ell) \arrow[r] & \mathscr{I}H^{i+2j}(X,\mathbf{Q}_\ell).
\end{tikzcd}$$

By what we have proved, the horizontal maps are injective, and the right vertical map is an injection by the hard Lefschetz theorem (see [BBD82, Theorem 5.4.10], and note that there they have made a shift by n of the cohomology sheaves, hence the difference in indexing). This implies that the left vertical map is injective, giving

$$b_i^p = \dim_{\mathbf{Q}_\ell} H_p^i(X,\mathbf{Q}_\ell) \le \dim_{\mathbf{Q}_\ell} H_p^{i+2j}(X,\mathbf{Q}_\ell) = b_{i+2j}^p. \qquad \square$$

Using these theorems for motivation and consistency we define the pure cohomology $H_p^*(X,\mathbf{Q}_\ell)$, for any proper variety X over an algebraically closed field, as the image of $H^*(X,\mathbf{Q}_\ell)$ in $\mathscr{I}H^*(X,\mathbf{Q}_\ell)$, and similarly for rational coefficients when the base field is the field of complex numbers.

3. **The number of cells**

We define a *stratification* of a proper variety X as a (necessarily finite) collection $\{V_\alpha\}_{\alpha\in I}$ of subvarieties of X, called *strata*, such that X is the disjoint union of them and the closure of each stratum is a union of strata. We get a partial order on the index set I of the strata by saying that $\alpha \le \beta$ when $V_\alpha \subseteq \overline{V_\beta}$. It then follows that if $J \subseteq I$ is *downwards closed* (i.e., if $\alpha \le \beta$ and $\beta \in J$ then $\alpha \in J$) then $X_J := \bigcup_{\alpha\in J} V_\alpha$ is a closed subset, as it is the union of the closed subsets $\overline{V_\alpha} = \bigcup_{\beta\le\alpha} V_\beta$ for all $\alpha \in J$.

An *algebraic cell decomposition* of X is a stratification for which each stratum (which in this case will also be called a *cell*) is isomorphic to the n-dimensional affine space $\mathbf{A}^n$ for some n.

THEOREM 3.1. *Let X be a proper variety over an algebraically closed field, having an algebraic cell decomposition with f_i cells of dimension i.*

(i) $H^{2i+1}(X,\mathbf{Q}_\ell) = 0$ *for all integers i. In particular, $b_{2i+1} = b_{2i+1}^p = 0$.*

(ii) $H^{2i}(X,\mathbf{Q}_\ell) = H_p^{2i}(X,\mathbf{Q}_\ell)$ *for all i, and this space has a basis in bijection with the set of cells of dimension i. In particular, $b_{2i} = b_{2i}^p = f_i$ for all i.*

(iii) *Assuming that X is projective of pure dimension n we have that $f_i \le f_j$ for all $i \le j \le n-i$.*

Proof. Although parts (i) and (ii) are well known, we give a sketch for the convenience of the reader. In any case, by standard specialization arguments we may assume that X is defined over the algebraic closure of a finite field $\mathbb{F}_q$, and after possibly extending the finite field we may assume that X as well as the strata are defined over $\mathbb{F}_q$. Parts (i) and (ii) are now proved by induction over the number of cells using the long exact sequence of cohomology,

$$\cdots \to H^i_c(U, \mathbf{Q}_\ell) \to H^i(X, \mathbf{Q}_\ell) \to H^i(F, \mathbf{Q}_\ell) \to \cdots,$$

which holds for $U \subset X$ open with complement F, where the cohomology with compact support, H^*_c, is as defined in [AGV$^+$73, exposé XVII, définition 5.1.9]. This is then combined with the fact that $H^i_c(\mathbf{A}^n) = 0$ for $i \neq 2n$ and $H^{2n}_c(\mathbf{A}^n) = \mathbf{Q}_\ell$ is pure of weight $2n$. As X, U, and F are defined over $\mathbb{F}_q$ the Frobenius map acts on all the vector spaces involved compatibly with all the maps. (Note that this long exact sequence is for cohomology with compact supports, but as X and F are compact, cohomology with compact supports is equal to ordinary cohomology.)

Part (iii) now follows from Theorem 2.2. □

4. **Proofs of Theorems** A, B **and** D

As was mentioned in the introduction, the crystallographic groups are precisely the ones that appear as Weyl groups of Kac-Moody algebras. We are now going to apply the results of previous sections to such Weyl groups. General references for background to this material are the books by Kac [Kac83] and Kumar [Kum02]; for the algebraic-geometric aspects see [Kum02], [Slo86].

We start by recalling some properties of the Schubert varieties for a Kac-Moody algebra (group). (It seems that no attempt has been made to extend the construction of Kac-Moody groups to positive characteristic, à la Chevalley, so we restrict ourselves to characteristic zero from now on.) Let (W, S) be the Weyl group of the Kac-Moody algebra (which is a Coxeter group on the generating set S), pick $J \subseteq S$, and let W_J be the subgroup of W generated by the elements of J. As is well known, there is a unique element w of minimal length in any W_J-coset $\overline{w}$. The set of such elements will be denoted W^J.

For each $w \in W^J$ there exists [Kum02, Chapter 7; Slo86, §2.2] a complex projective variety $\overline{X}_w$, which we shall call the *Schubert variety* of w, containing locally closed subvarieties X_u for all $u \in [e, w]^J$ whose closures are disjoint unions

$$\overline{X}_u = \biguplus_{z \leq u} X_z,$$

where z is assumed to be in W^J. The partial order $\leq$ is the Bruhat order, and X_u is a subvariety of $\overline{X}_w$ isomorphic to $\mathbf{A}^{\ell(u)}$ [Slo86, Theorem 2.4].

Remark. The variety $\overline{X}_w$ depends, at least *a priori*, on the choice of a dominant weight. However, we are just going to use its existence and not any uniqueness. (It is in any case true that any two choices give varieties that are related by algebraic maps that are homeomorphisms [Slo86], and hence have the same cohomology.)

Going further, the Kac-Moody group has a Borel subgroup B which acts on each $\overline{X}_w$ such that the X_u for $u \leq w$ are the orbits. Note that B is not an algebraic group but only a group scheme (i.e., not of finite type). However, the action on any $\overline{X}_w$ factors through a quotient which is an algebraic group, and we shall therefore allow ourselves to act as if B itself was an algebraic group. Using this action we get the next result.

LEMMA 4.1. *The restriction of a B-complex* (see §8 for the definition) *of $\overline{X}_w$ to some X_u, for $u \leq w$, is constant.*

Proof. Recall (cf. [Slo86, 1.8]) that there is a subvariety U_u of B and a point x on X_u such that the map $g \mapsto gx$ gives an isomorphism $U_u \to X_u$. Now, if C is the B-complex, then by assumption we have an isomorphism between p_2^*C and m^*C on $B \times \overline{X}_{\overline{w}}$. Taking the restriction of this isomorphism to $U_u \times \{x\}$, we obtain an isomorphism between C and the constant extension of C_x to X_u. □

Remark. By Proposition 8.3 this result can be applied to the intersection complex. In the setup of [BM01], rather than assuming a B-action, an equisingularity condition along X_u is assumed (which should follow from the fact that X_u is a B-orbit). This implies that the restriction to X_u is locally constant, and then the fact that X_u is isomorphic to an affine space and hence contractible implies that locally constant complexes are constant.

Using the Kac-Moody Schubert varieties we can now undertake the proofs of the main results.

Proof of Theorems A *and* B. Theorem B follows immediately from Theorem 2.2, and Theorem A follows immediately from Theorem 3.1, both applied to $\overline{X}_w$. □

Remark. In connection with the inequalities of Theorem A it might be tempting to speculate that the f-vectors $(f_0^{w,J}, f_1^{w,J}, \ldots, f_{\ell(w)}^{w,J})$ are *unimodal*, meaning that they increase up to some maximum and then decrease. However, as pointed out by Carrell [Car95], this is false. See Stanton [Sta90] for nonunimodal examples in groups of type A_n modulo maximal parabolic subgroups.

For the proof of Theorem D we need to extend the monotonicity theorem of Braden and MacPherson (see [BM01, Corollary 3.7]) to the case of a general Kac-Moody Schubert variety.

THEOREM 4.2. *Let $x \leq y \leq z$ in a crystallographic Coxeter group. Then $P^i_{x,z} \geq P^i_{y,z}$, where $P^i_{\cdot,z}$ denotes the coefficients of q^i of the respective Kazhdan-Lusztig polynomials.*

Remark. It is no doubt true that the Whitney stratification condition, which is one of the standing hypotheses of [BM01], is indeed fulfilled also in this case. But rather than trying to verify that, we note that the fact that the Schubert cells are the orbits of a group action can be used more directly to prove the necessary conditions. Since the proofs of [BM01] are also sometimes somewhat sketchy, we have therefore chosen to go through the needed steps rather than leaving to the reader the task of checking that the proofs of Braden and MacPherson go through. At the same time, this allows us to give the results in our context of ℓ-adic cohomology rather than in the de Rham-cohomology context of [BM01]. We have deferred to Section 8 the part of the argument that does not directly pertain to Kac-Moody Schubert varieties. We will here take that material for granted.

Proof. Here, as in the more familiar case of finite and affine Weyl groups, these Kazhdan-Lusztig coefficients can be interpreted as the dimension of the fibre of the cohomology of the intersection complex for $\overline{X}_z$ at a point of X_x. Hence the monotonicity Theorem 4.2 follows from our analogue of [BM01, Theorem 3.6], which we now have the appropriate tools for proving. We state it as a separate proposition. $\square$

PROPOSITION 4.3. *For $x \leq y \leq z \in W^J$ we have a surjective map of $\mathbf{Q}_\ell$-vector spaces $\mathcal{IH}^*(\overline{X}_z)_x \to \mathcal{IH}^*(\overline{X}_z)_y$, where the $\mathcal{IH}^*(\overline{X}_z)$ are the cohomology sheaves of the intersection complex of $\overline{X}_z$ and $(\cdot)_t$ denotes a fibre at any point of X_t.*

Proof. This is [BM01, Theorem 3.6] in our context. Its proof, as well as proofs of the supporting [BM01, Lemma 3.1, Lemma 3.3, Proposition 3.4, and Theorem 3.5], can now be carried through:

- The analogue of [BM01, Lemma 3.1] follows from [Spr84, Proposition 1] and Proposition 8.3.
- The analogues of [BM01, Lemma 3.3, Proposition 3.4, and Theorem 3.5] can be proved with the same proofs, using [Del80] instead of [Sai89] for the weight results (note that in sheaf theory the relative cohomology $H^*(X, U, \mathcal{F})$ is defined as the cohomology with support $H^*_Y(X, \mathcal{F})$, where $Y := X \setminus U$).
- In the proof of (the analogue of) [BM01, Theorem 3.6], we use Lemma 4.1 to conclude that the intersection complex is constant on each X_t and hence its cohomology on X_t is equal to its fibre for any point on X_t.
- To get a contracting action on X_z with the same fixed points as for the torus $T \subset B$, we use Corollary 8.2 applied to Z consisting of a single fixed point (note that $\overline{X_z}$ is irreducible). To verify the needed conditions, we use that $\overline{X_z}$ lies in an ind-variety that set-theoretically is equal to G/P_J, where G is the Kac-Moody group and P_J is the parabolic subgroup corresponding to J. The

fixed points of T on G/P_J are the cosets wP_J, where w runs over W^J (where W is identified with the normalizer N of T in G divided by T itself). What we want to show is that the cone spanned by the weights of the cotangent space of $x = xP_J \in \overline{X}_z$ does not contain a line. For this it is enough to show the same thing for the cotangent space of x in G/P_J. By multiplying by $x^{-1} \in N$ we reduce to $x = e$ and by (vector space) duality to the case of the tangent space of $e \in G/P_J$. However, that tangent space is equal to $\mathfrak{g}/\mathfrak{p}_J$, where $\mathfrak{g}$ is the corresponding Kac-Moody Lie algebra and $\mathfrak{p}_J$ is the corresponding parabolic subalgebra. That means that the weights form a subset of the weights of $\mathfrak{g}/\mathfrak{b}$, where $\mathfrak{b}$ is the Lie algebra of B. These weights are exactly the negative roots of the root system of the Lie algebra, and they lie in the cone generated by the negatives of the simple roots, a cone that indeed does not contain a line. $\square$

We are now ready to prove our fourth main theorem.

Proof of Theorem D. Assume condition (b), and let

$$F_w(q) = \sum_{i=0}^{\ell(w)} a_i q^i := \sum_{x \le w} q^{\ell(x)} P_{x,w}(q).$$

Theorem 4.2 implies that the q^i-coefficient of $P_{x,w}(q)$ is 0 for all $i = 1, \dots, k$ and all $x \le w$. Hence, since also $\deg P_{x,w}(q) \le \lfloor (\ell(w) - \ell(x) - 1)/2 \rfloor$, we get

$$\begin{aligned}
a_0 &= 1, & a_{\ell(w)} &= 1,\\
a_1 &= f_1^w, & a_{\ell(w)-1} &= f_{\ell(w)-1}^w,\\
&\ \vdots & &\ \vdots\\
a_k &= f_k^w, & a_{\ell(w)-k} &= f_{\ell(w)-k}^w,\\
a_{k+1} &= f_{k+1}^w + \beta_{k+1}, & a_{\ell(w)-k-1} &= f_{\ell(w)-k-1}^w.
\end{aligned}$$

Here the last row requires that $k < m$.

Now use that $a_i = a_{\ell(w)-i}$ for all i. This is valid in all Coxeter groups by [KL79, Lemma 2.6(v)] (in this case it is also implied by Poincaré duality of middle intersection cohomology of $\overline{X}_w$). From this we conclude condition (a), as well as (c).

Finally, assume that condition (b) fails, say $d \le k$ is minimal such that $\beta_d \ne 0$. Applying that (b) implies (c), we get that $f_{\ell(w)-d}^w - f_d^w = \beta_d \ne 0$, so also condition (a) fails. $\square$

5. **Proof of Theorem** C

As mentioned in the introduction we shall use an argument of Stanley. When adapting it, it is more natural, both from a geometric and linear algebra point of view, to consider homology than cohomology (it will be clear that formally this is not required). Even though there is a sheaf theoretic definition, for our purposes it is enough to define the homology $H_i(X, \mathbf{Q}_\ell)$, for a projective variety X, as the dual vector space to $H^i(X, \mathbf{Q}_\ell)$. Then the homology becomes a covariant instead of a contravariant functor. We denote by f_* the map induced by a map $f : X \to Y$. Furthermore, if X is proper and purely n-dimensional, then there is the *trace map* $H^{2n}(X, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell$, which is surjective and hence gives an element, the *fundamental class*, $[X] \in H_{2n}(X)$. If X is a closed subvariety of Y, then we get an element $[X] := i_*[X]$, the *class* of X, where $i : X \to Y$ is the inclusion. Now, if $U \subseteq X$ is an open n-dimensional subset, then the composite $H_c^{2n}(U) \to H^{2n}(X) \to \mathbf{Q}_\ell$ is surjective and depends only on U. When $U = \mathbf{A}^n$, the map $H_c^*(\mathbf{A}^n) \to \mathbf{Q}_\ell$ is an isomorphism. Assuming now that X has a cell decomposition, combining this with Theorem 3.1 (or rather its proof) gives that $H_{2i}(X)$ has the classes of the closures of the i-dimensional cells as basis.

Remark. This result may seem to contradict [Sta80, Theorem 2.1], as together they would imply that if X is n-dimensional, then the number of i-cells is equal to the number of $(n-i)$-cells, which is not true in general (e.g., for a Schubert variety this is true, by the Carrell-Peterson criterion, if and only if it is rationally smooth). However, when X is smooth, [Sta80, Theorem 2.1] *is* true, and that is the only case considered there.

Now, as $H^*(X)$ is a $\mathbf{Q}_\ell$-algebra, $H_*(X)$ becomes a module over $H^*(X)$. Furthermore, if $f : X \to Y$ is a map, then $H_*(X)$ becomes a $H^*(Y)$-module through $y \cdot x = f^*y \cdot x$ and then, purely formally, we have the *projection formula* $y \cdot f_*x = f_*(f^*y \cdot x)$ for $y \in H^*(Y)$ and $x \in H_*(X)$. We are now ready to prove the analogue of [Sta80, Lemma 2.2] (which as it stands is true only in the smooth case).

LEMMA 5.1. *Let X be a variety with an algebraic cell decomposition, and let $\mathcal{L}$ be a line bundle on X. Then for any cell C, the expansion $c_1(\mathcal{L}) \cdot [\overline{C}] = \sum_D d_{C,D}[\overline{D}]$, where D runs over the cells of X, has the property that $d_{C,D} = 0$ unless $D \subset \overline{C}$.*

Proof. Let $i : C \to X$ be the inclusion, and consider $[\overline{C}] \in H_*(\overline{C})$. Since $\overline{C}$ has a cell decomposition, the cells of which are the D for which $D \subseteq \overline{C}$, we get that

$$i^*c_1(\mathcal{L}) \cdot [\overline{C}] = \sum_D d_{C,D}[\overline{D}] \in H_*(\overline{C}),$$

where D runs over the cells of $\overline{C}$. Applying i_* to this formula gives $c_1(\mathcal{L}) \cdot [\overline{C}] = \sum_{D \subset \overline{C}} d_{C,D} i_*[\overline{D}]$. However, $[\overline{D}] \in H_*(\overline{C})$ is equal to $j_*[\overline{D}]$, where $j : \overline{D} \to \overline{C}$ is the inclusion. Hence $i_*[\overline{D}] = i_* j_*[\overline{D}] = (ij)_*[\overline{D}]$, but the right side is by definition the class of $\overline{D}$ in X. □

We are now almost ready to adapt Stanley's proof. However, in [Sta80] the hard Lefschetz theorem is combined with [Lemma 1.1] for the desired conclusion and in our situation the hard Lefschetz theorem does not quite give a bijective map. Luckily the proof of [Lemma 1.1] needs only a slight modification to be applicable to our situation (where we also, contrary to [Sta80], do not turn the geometric poset "upside down").

LEMMA 5.2. *Let P be a finite graded poset of rank n. Let P_j denote the set of its elements of rank j, and let V_j be the vector space with basis P_j over some given field. For each $i \leq j < n-i$, assume there exists a linear transformation $\varphi_j : V_{j+1} \to V_j$ such that the following two conditions are satisfied*:

(a) *The composite map $\varphi_i \circ \varphi_{i+1} \circ \cdots \circ \varphi_{n-i-2} \circ \varphi_{n-i-1}$ is surjective.*

(b) *If $x \in P_{j+1}$ and $\varphi_j(x) = \sum_{y \in P_j} c^j_{x,y} y$, then $c^j_{x,y} = 0$ unless $y < x$.*

Then, in P there exist $\operatorname{card}(P_i)$ *pairwise disjoint chains $x_i < x_{i+1} < \cdots < x_{n-i}$ such that* $\operatorname{rank}(x_j) = j$ *for all j.*

Proof. The proof of [Sta80, Lemma 1.1] goes through in this situation with only slight modification. For the reader's convenience we repeat the argument.

Let $m := \operatorname{card}(P_i)$, and put $\varphi := \varphi_i \circ \cdots \circ \varphi_{n-i-1}$. Since φ is surjective, so is $\Lambda^m \varphi : \Lambda^m V_{n-i} \to \Lambda^m V_i$. By the definition of φ, we get that $\Lambda^m \varphi = \Lambda^m(\varphi_i) \circ \cdots \circ \Lambda^m(\varphi_{n-i-1})$. Using the bases P_j we get bases for $\Lambda^m V_j$ and hence a matrix for each $\Lambda^m(\varphi_j)$. An entry of the product matrix of the composite $\Lambda^m(\varphi_i) \circ \cdots \circ \Lambda^m(\varphi_{n-i-1})$ has the form

$$\sum \det \varphi_i[Q_{i+1}, Q_i] \det \varphi_{i+1}[Q_{i+2}, Q_{i+1}] \cdots \det \varphi_{n-i-1}[Q_{n-i}, Q_{n-i-1}],$$

where $Q_j \subseteq P_j$ with $\operatorname{card}(Q_j) = m = \operatorname{card}(P_i)$, Q_{n-i} specifies the entry of the matrix of the composite, $\varphi_j[Q_{j+1}, Q_j]$ is the submatrix of φ_j corresponding to the sets of basis elements Q_j and Q_{j+1}, and the sum runs over all the choices of Q_j for $i \leq j < n-i$. By assumption there is a Q_{n-i} such that this sum is nonzero, and hence there is a summand that is nonzero. This gives us a set of Q_j such that all the $\det \varphi_j[Q_{j+1}, Q_j]$ are nonzero. In particular, one term of the expansion of this matrix must be nonzero, which gives us a bijection $\sigma_j : Q_{j+1} \to Q_j$ such that $c^j_{x,\sigma_j(x)} \neq 0$ for all $x \in Q_{j+1}$. By assumption (b), this implies that $\sigma_j(x) < x$ for all $x \in Q_{j+1}$. □

Proof of Theorem C. We apply Lemma 5.2 to the interval $[e, w]^J$, a graded poset of rank $\ell(w)$. We identify the vector space V_j with $H_{2j}(X)$, where X is the Schubert variety of w, and we let $\varphi_j : H_{2j+2}(X) \to H_{2j}(X)$ be multiplication by $c_1(\mathscr{L})$. Condition (a) of the lemma follows from the proof of Theorem 2.2, which shows that multiplication by $c_1(\mathscr{L})^{n-2i}$ gives an injective map from $H^{2i}(X)$ to

$H^{2n-2i}(X)$ and hence by duality it gives a surjective map from $H_{2n-2i}(X)$ to $H_{2i}(X)$. Condition (b) follows from Lemma 5.1. □

6. **Proof of Theorem** E

We begin by recalling the definition of an M-sequence. For $n, k \geq 1$ there is a unique expansion

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i},$$

with $a_k > a_{k-1} > \cdots > a_i \geq i \geq 1$. This given, let

$$\partial^k(n) := \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-1} \quad \text{and} \quad \partial^k(0) := 0.$$

THEOREM (Macaulay and Stanley [Sta78, Theorem 2.2]). *For an integer sequence* $(1, m_1, m_2, \ldots)$ *the following conditions are equivalent* (*and this defines an M-sequence*):

(1) $\partial^k(m_k) \leq m_{k-1}$ *for all* $k \geq 1$;

(2) $\dim(A_k) = m_k$ *for some graded commutative algebra* $A = \oplus_{k \geq 0} A_k$ (*over some field*) *such that A is generated by A_1*.

PROPOSITION 6.1. *Let $H^*(\overline{X}_w)$ be the cohomology algebra of a Schubert variety $\overline{X}_w$.*

(a) *If W is finite then $H^*(\overline{X}_w)$ is generated in degree one for all $w \in W$.*

(b) *There exist elements u in the affine Weyl group $\widetilde{C}_2$ for which $H^*(\overline{X}_u)$ is not generated in degree one.*

Proof. Suppose that W is finite. We are to show that $H^*(\overline{X}_w)$ is generated in degree one (or, equivalently, in dim $= 2$). For $w = w_0$ this is classical — it can be seen either from the description of $H^*(\overline{X}_{w_0})$ in terms of special Schubert classes or from the isomorphism of $H^*(\overline{X}_{w_0})$ with the coinvariant algebra of W.

For $w \neq w_0$ we use that the inclusion $\overline{X}_w \hookrightarrow \overline{X}_{w_0}$ induces an injective map on homology $H_*(\overline{X}_w) \to H_*(\overline{X}_{w_0})$, as is apparent from the cell decomposition. Hence, dually there is an algebra surjection $H^*(\overline{X}_{w_0}) \to H^*(\overline{X}_w)$. Since $H^*(\overline{X}_{w_0})$ is generated in degree one, so is $H^*(\overline{X}_w)$. This proves part (a).

For part (b) we observe that the Poincaré series of $\widetilde{C}_2$ begins

$$\sum q^{\ell(w)} = 1 + 3q + 5q^2 + 8q^3 + \cdots.$$

Let $u \in \widetilde{C}_2$ be an upper bound (in Bruhat order) to all eight elements of length 3. That such elements exist follows from [BB05, Proposition 2.2.9]. Then $f^u = (1, 3, 5, 8, \ldots)$, which is not an M-sequence since $\partial^3(8) = 6 \not\leq 5$. So, by the Macaulay-Stanley theorem, $H^*(\overline{X}_u)$ is not generated in degree one. □

Proof of Theorem E. The vector $f^w = \{f_0, f_1, \ldots, f_{\ell(w)}\}$ satisfies $f_k = \dim H^{2k}(\overline{X}_w)$. Hence, to prove that f^w is an M-sequence it suffices to show that $H^*(\overline{X}_w)$ is generated in degree one. This was done in Proposition 6.1.

To prove that

$$(f_0^w, f_1^w - f_0^w, f_2^w - f_1^w, \ldots, f_{\lfloor \ell(w)/2 \rfloor}^w - f_{\lfloor \ell(w)/2 \rfloor - 1}^w)$$

is an M-sequence, we apply the Macaulay-Stanley theorem to the algebra

$$H^*(X, \mathbf{Q}_\ell)/c_1(\mathscr{L})H^*(X, \mathbf{Q}_\ell) + H^{>\ell(w)}(X, \mathbf{Q}_\ell),$$

which by Theorem 2.2 corresponds to the desired vector. □

Remark. The M-sequence property fails for general intervals $[x, w]$ in finite groups. For instance, for a particular $x \in C_4$,

$$\sum_{x \leq y \leq w_0} q^{\ell(y)-\ell(x)} = 1 + 4q + 11q^2 + \cdots .$$

This information can be read off from Goresky's tables [Gor81] by letting $x = zw_0$, where z is element number 377 of C_4 (i.e., $x = \overline{1}\,2\,4\,3$ and $z = 1\,\overline{2}\,\overline{4}\,\overline{3}$ in signed one-line notation). Since $\partial^2(11) = 5 \not\leq 4$, this is not an M-sequence.

7. **Proof of Theorem** F

Let $[u, v]$ be a Bruhat interval. The elements of $[u, v]$ of length $\ell(u) + 1$ are its *atoms*. We let $f_{\ell(u)+1}^{u,v}$ denote the number of atoms in $[u, v]$.

It has been shown by Dyer [Dye91] that (up to isomorphism) only finitely many posets of each given length r occur as intervals in the Bruhat order on finite Coxeter groups. Therefore, the function

$$M(r) := \max_{[u,v]}\{\, f_{\ell(u)+1}^{u,v} \mid \ell(v) - \ell(u) \leq r\}$$

is well defined. That is, $M(r)$ denotes the maximum number of atoms of a Bruhat interval of length at most r occurring in any finite Coxeter group.

As usual, denote by w_0 the element of maximal length in any given finite group W. It follows from the classification of irreducible finite Coxeter groups that

$$Q(s) := \max_{(W,S)}\{\, \ell(w_0) \mid \mathrm{card}(W) < \infty,\ \mathrm{card}(S) = s\}$$

is a well-defined number-theoretic function for $s \neq 2$, where the maximum is taken over all irreducible finite Coxeter groups of rank s, of which there are only finitely many. The classification shows that $Q(s) = s^2$ for $s \geq 9$ (the maximum occurring in type B), whereas there are irregularities occurring for $s \leq 8$ due to the exceptional groups.

LEMMA 7.1. *Let (W, S) be a finite irreducible Coxeter group, and let $w \in W$. We have that*

$$\ell(w) > Q(j) \quad \textit{implies} \quad f^w_{\ell(w)-1} > j.$$

Proof. Suppose that $f^w_{\ell(w)-1} \le j$. By Theorem A we have $f^w_1 \le f^w_{\ell(w)-1}$. (*Remark*: Theorem A does not cover the groups H_3 and H_4. They can easily be dealt with directly, since $f^w_1 \le 3$ and $f^w_1 \le 4$ for all w in H_3 and H_4, respectively. Or, one can give a separate argument based on the relation $\beta_1 = f^w_{\ell(w)-1} - f^w_1$ (cf. Theorem D(c)) together with the nonnegativity of β_1, the coefficient of q in Kazhdan-Lusztig polynomials. See for example [BB05, Exercise 5.38] and the reference there to Dyer and Tagawa.)

Hence $f^w_1 \le j$, which means that there is a set $J \subseteq S$ of cardinality $|J| \le j$ (the set of atoms of $[e, w]$) such that every reduced expression for w uses letters only from the set J (by the subword property of Bruhat order [Hum90, Theorem 5.10]). In particular, we have $w \in W_J$, and so $\ell(w) \le \ell(w_0(J)) \le Q(j)$, where $w_0(J)$ denotes the element of maximal length in the parabolic subgroup W_J. □

Proof of Theorem F. Assume that W is irreducible. The easy extension of the proof to the general case is left to the reader. For given $k \ge 1$, put

$$N_k := Q(\,M(k) - 1) + k,$$

and let $n := \ell(w) \ge N_k$. We are going to consider the bipartite graphs induced between adjacent rank levels in Bruhat order near the top of the interval $[e, w]$.

For r such that $1 \le r \le k$, let $V_{n-r} := \{x \in [e, w] \mid \ell(x) = n - r\}$. Now consider the bipartite graph with vertices $V_{n-r} \cup V_{n-r+1}$ and edges $E_r = \{(x, y) \in V_{n-r} \times V_{n-r+1} \mid x < y\}$. If $(x, y) \in E_r$ then

$$\deg(x) \le M(r) \le M(k),$$

where $\deg(x)$ denotes the number of edges adjacent to x in E_r. Similarly, by Lemma 7.1 and the choice of N_k, we have $\deg(y) \ge M(k)$. Thus,

$$|V_{n-r}| \cdot M(k) \ge |E_r| \ge |V_{n-r+1}| \cdot M(k),$$

and hence

$$f^w_{\ell(w)-r} = |V_{n-r}| \ge |V_{n-r+1}| = f^w_{\ell(w)-r+1}. \qquad \square$$

In closing we would like to raise a question: Does there exist $\alpha < 1$ such that

$$f^w_{\lfloor \alpha \cdot \ell(w) \rfloor} \ge \cdots \ge f^w_{\ell(w)}$$

for all $w \in W$?

8. Appendix: Contracting $\mathbf{G}_m$-actions

As was mentioned in Section 4, we have deferred to here some general algebraic-geometric material needed for the proof of Theorem 4.2. The first result is a criterion for when a $\mathbf{G}_m$-action is contracting (which certainly is standard but for which we have not found an appropriate reference).

Suppose that $X = \mathbf{Spec}\, R$ is an affine variety with a $\mathbf{G}_m$-action. The action immediately translates to a grading $R = \bigoplus_{i \in \mathbb{Z}} R_i$ so that $\lambda \in \mathbf{k}^* = \mathbf{G}_m(\mathbf{k})$ acts as $\lambda \cdot r = \lambda^i r$ for $r \in R_i$. The condition that the action be contracting is then equivalent to $R_i = 0$ for $i < 0$. This implies that if $x \in X(\mathbf{k})$ is a $\mathbf{k}$-point, then the linear action of $\mathbf{G}_m$ on the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ has the property that only nonnegative weights occur. (Recall that a linear representation of $\mathbf{G}_m$ is a direct sum of 1-dimensional representations of the form $\lambda \mapsto \lambda^n$, and that n is the weight of that subrepresentation.) It turns out that there is a converse to this result (where for simplicity and because it is the only case we shall use, we consider only the irreducible case).

PROPOSITION 8.1. *Let X be an irreducible variety with a $\mathbf{G}_m$-action, and let Z be a closed subvariety of fixed points such that for each closed point of Z the action of $\mathbf{G}_m$ on its cotangent space has only nonnegative weights. Then the action of $\mathbf{G}_m$ is contracting on the union of the open $\mathbf{G}_m$-invariant affine subsets that meet Z.*

Proof. An extension of the $\mathbf{G}_m$-action to an $\mathbf{A}^1$-action is unique if it exists, as $\mathbf{G}_m$ is dense in $\mathbf{A}^1$. Hence it is enough to show that the action extends to any open $\mathbf{G}_m$-invariant affine subset that meets Z. We may therefore assume that $X = \mathbf{Spec}\, R$ and the $\mathbf{G}_m$-action then corresponds to a grading $R = \bigoplus_i R_i$. We may further assume that Z contains the closed point z. What we want to show is that $R_i = 0$ for $i < 0$.

Now, as Z is irreducible we have that R embeds in the local ring $R_{\mathfrak{m}_z}$, and by a theorem of Krull we have that $\bigcap_n \mathfrak{m}_z^n = 0$ in $R_{\mathfrak{m}_z}$. Hence it will be enough to show that $R_i \subseteq \mathfrak{m}_z^n$ for $i < 0$ and every n. We do this by induction on n, where the case $n = 1$ is true as z is a fixed point so that $\mathbf{G}_m$ acts trivially on $R/\mathfrak{m}_z$. The case $n = 2$ is then true as the map $R_i \to \mathfrak{m}_z/\mathfrak{m}_z^2$ is a map from a $\mathbf{G}_m$-representation, consisting only of negative weight representations, to a representation that by assumption contains only nonnegative weight representations. Multiplication in R now gives a surjective $\mathbf{G}_m$-equivariant map $S^n(\mathfrak{m}_z/\mathfrak{m}_z^2) \to \mathfrak{m}_z^n/\mathfrak{m}_z^{n+1}$, which implies that $\mathfrak{m}_z^n/\mathfrak{m}_z^{n+1}$ consists only of nonnegative weight representations. By the induction assumption we have an induced map $R_i \to \mathfrak{m}_z^n/\mathfrak{m}_z^{n+1}$, which therefore also is the zero map. □

Remark. The case of the action of $\mathbf{G}_m$ on $\mathbf{P}^2$ given by $(x\colon y\colon z) \mapsto (x\colon \lambda y\colon \lambda^2 z)$ is instructive. The point $(1\colon 0\colon 0)$ is then a fixed point whose cotangent space has weights 1 and 2. The only invariant affine open subset that contains $(1{:}\,0{:}\,0)$ is $\{x \neq 0\}$, and on it the action visibly extends to an $\mathbf{A}^1$-action; $(1{:}\,y{:}\,z) \mapsto (1{:}\,\lambda y{:}\,\lambda^2 z)$ makes sense also for $\lambda = 0$.

This proposition has the following corollary, which is the only consequence that we shall actually use.

COROLLARY 8.2. *Assume that T is an algebraic torus* (*i.e.*, *isomorphic to $\mathbf{G}_m^n$ for some $n > 0$*), *and let Γ be its group of characters* (*i.e.*, *the group of algebraic homomorphisms $T \to \mathbf{G}_m$*). *Let X be a variety with an action of T such that there exists an ample line bundle on X with a compatible T-action. Suppose $z \in X$ is a T-fixed point such that the cone generated by the characters that appear in the cotangent space of z does not contain a line* (*or*, *equivalently*, *no nonzero element and its inverse*). *Then there is an algebraic group homomorphism $\mathbf{G}_m \to T$ such that $X^T = X^{\mathbf{G}_m}$ and such that there exists an affine T-invariant neighborhood of z for which the action is contracting.*

Proof. Let Γ^* be the group of cocharacters of T (i.e., algebraic group homomorphisms $\mathbf{G}_m \to T$), which is identified with the dual of Γ by pairing $\phi : \mathbf{G}_m \to T$ and $\varphi : T \to \mathbf{G}_m$ using the integer n for which the composite $\varphi \circ \phi$ is of the form $\lambda \mapsto \lambda^n$. We want to apply Proposition 8.1, so what we are looking for is an element $\phi \in \Gamma^*$ such that its fixed point set is equal to that of T and such that its weights on the cotangent space of z are all nonnegative. For the first condition it is well known that there is a finite set of hyperplanes in Γ^* such that if ϕ is not contained in any of them, then it has the same fixed point set as T. For the second condition it is clear that we are looking for a ϕ with nonnegative values on each character of T that appears in the cotangent space of z. By the assumption on the cone generated by them (and the fact that Γ^* is the dual of Γ) there is such a ϕ outside of any finite number of hyperplanes, so that we may simultaneously fulfill both conditions. Now, by the assumption that X admits a T-linearized ample line bundle, we may find a T-invariant affine neighborhood of z, and then Proposition 8.1 is applicable. □

We are now left with establishing that the intersection complex is a $\mathbf{G}_m$-complex. However, we shall need the corresponding results for other actions, so we put ourselves in a somewhat more general situation. It would be possible to refer to the general results of [BL94]. Unfortunately the discussion there is formulated in a topological rather than an étale context. It is true that enough of their results are valid (with essentially the same proof) for the étale case, but rather than leaving the task of checking that to the reader, we make the following *ad hoc* construction. Thus, if G is an algebraic group acting on a variety X, we say that a complex C of ℓ-adic sheaves is a *G-complex* if the two complexes p_2^*C and m^*C are isomorphic in the derived category, where $p_2, m : G \times X \to X$ is the projection and G-action map, respectively.

PROPOSITION 8.3. *Let X be a variety on which the algebraic group G acts. Then the intersection complex of X is a G-complex.*

Proof. Notice that m is the composite of p_2 and the automorphism

$$t : G \times X \to G \times X, \qquad (g, x) \mapsto (g, gx),$$

so that it will be enough to show that t^*C is isomorphic to C, where C is the pullback by p_2 of the intersection complex on X. We shall prove this by showing that C is the intersection complex of $G \times X$ and then that the intersection complex is invariant under automorphisms. The last property follows directly from the characterization [GM83, Theorem 4.1], and that characterization is also invariant (with the degree conventions of [BBD82]) under smooth pullbacks, which implies that C is indeed the intersection complex on $G \times X$. □

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