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Abstract

Let the torus T^{2n} be equipped with the standard symplectic structure and a periodic Hamiltonian $\mathcal{H} \in C^3(S^1 \times T^{2n}, \mathbb{R})$. We look for periodic orbits of the Hamiltonian flow $\dot{\boldsymbol{u}}(t) = J\nabla\mathcal{H}(t, (t))$. A *subharmonic solution* is a periodic orbit with minimal period an integral multiple m of the period of $\mathcal{H}$, with $m > 1$.

We prove that *if the Hamiltonian flow has only finitely many orbits with the same period as $\mathcal{H}$, then there are subharmonic solutions with arbitrarily high minimal period. Thus there are always infinitely many distinct periodic orbits.*

The results proved here were proved in the nondegenerate case by Conley and Zehnder and in the case $n = 1$ by Le Calvez.

1. Introduction

In [CZ83], Conley and Zehnder published a proof of the Arnold Conjecture on the torus, which says that the Hamiltonian flow from a periodic Hamiltonian $\mathcal{H}$ on T^{2n} has at least as many orbits with period the same as that of $\mathcal{H}$, as a function on T^{2n} has critical points. Here is a rough idea of why it should be true: Periodic orbits of the Hamiltonian flow are the critical points of the action function $\mathcal{A}$ on the free loop space. When $\mathcal{H} \equiv 0$, the orbits of the Hamiltonian flow are the constant curves; thus there is a degenerate manifold $\approx T^{2n}$ of critical points. A perturbation of the Hamiltonian results in a perturbation of the action function; the critical set continues to "carry" the homology of the torus. The difficulty that all critical points have infinite index is resolved using a finite-dimensional approximation.

In [CZ84], Conley and Zehnder, again using a finite-dimensional approximation, proved the nondegenerate case of the theorem proved here: If all orbits with the same period as $\mathcal{H}$ are nondegenerate, then there are subharmonic solutions with arbitrarily high minimal period. The idea is that the topology of the free loop space Λ_m of closed curves on T^{2n} with period m-times the period of $\mathcal{H}$ is independent of m; thus for each m, the set of critical points in Λ_m "carries" the homology of the torus. *A priori* (as happens in the case $\mathcal{H} \equiv 0$) this topology might be carried on iterates of critical points of smaller *minimal* period. But the iterates of a *nondegenerate* critical point can "carry" the top- or bottom-dimensional homology (dimension n or $-n$ with Maslov grading) of the torus in only a finite number of the spaces Λ_m. (This follows from the Bott formula for the index of the iterates, and will be explained below.)

Conley conjectured the result proved here in 1984 [SZ92]. The conjecture cannot be proved by a refinement of the argument used in the nondegenerate case: In the degenerate case it *is* possible for all the top- or bottom-dimensional homology in Λ_m to be carried on critical points that have minimal period $m = 1$. This is the case for example for an *autonomous* (time-independent) Hamiltonian, all of whose maxima and minima are isolated and completely degenerate (so that the second derivative of $\mathcal{H}$ vanishes at these critical points). It is easy to see how to construct such a function, starting with a nondegenerate Morse function; it is also easy to see that all the top- and bottom-dimensional homology of Λ_m for each m can be carried on the constant curves at the maxima and minima.

On the other hand, when $n = 1$ it is easy to see that the existence of these degenerate critical points in Λ_1 forces the existence of subharmonic solutions: For an autonomous Hamiltonian when $n = 1$, the orbits of the Hamiltonian flow are the level sets of $\mathcal{H}$. Clearly any neighborhood of the constant curve at a totally degenerate, strict maximum or minimum contains an infinite number of subharmonic solutions, with minimal period going to infinity. Similarly in our proof, squeezing so much homology onto one (necessarily) degenerate critical point forces the existence of new critical points. It is remarkable however that our proof does *not* ensure the existence of subharmonic solutions in a C^0 neighborhood of the degenerate critical points that carry the homology. In Step 3 of the outline below, there is information about the new critical *values*, but not about the critical points. This is intriguing in that it seems that the existence of a "topologically degenerate" critical point can ensure the existence of infinitely many other critical points, (perhaps) far away!

We will call an isolated critical point of the action *topologically degenerate* if its iterates are homologically visible (i.e., have nontrivial local homology) in dimension $-n$ in infinitely many Λ_m. This local homology forces the critical point to be degenerate by the Bott formula: The Maslov index of the iterates of a nondegenerate critical point would eventually have to drop. The local homology of a topologically degenerate critical point is *unstable*: on a very small scale (Figure 1, left) there is local homology in dimension $-n$, but if you back away (Figure 1, right) it appears that the index has dropped; that is, there is one fewer negative direction. One gets another view of instability by turning both views upside down: The nontrivial local class represented by $\Sigma_m^{\geq}$ in the view at left "slips off" and

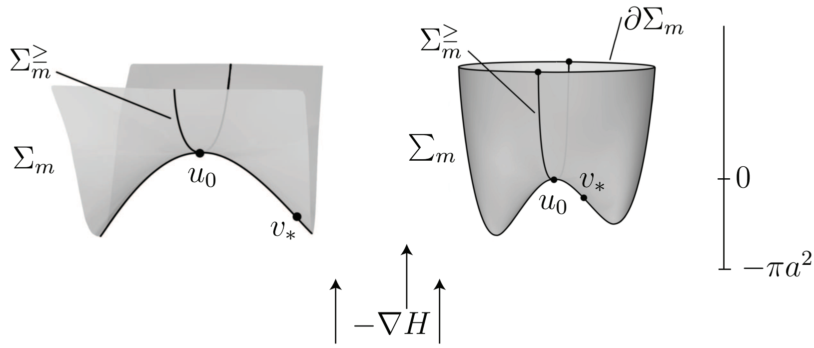

Figure 1. At left, a close-up view of the graph of the action on Σ_m, showing the (weakly) negative direction $\mathbb{R}\boldsymbol{v}_*$. The negative Hamiltonian is like a wind blowing upward. At right, the molar: a larger piece of Σ_m, showing the effects of the wind.

becomes trivial at right. A similar phenomenon appears in the study of closed geodesics; see [Hin93] and [Hin97]. All these arguments are variations on the theme of "Bangert's lemma" [Ban80].

The torus is special because one can use finite-dimensional approximation. For other symplectic manifolds, the best technique seems to be Floer theory, which was used in more general proofs of the Arnold Conjecture. (There is quite a jump from the standard techniques of Fourier series, which are the basis of the finite-dimensional approximation, to Floer theory, whose basic local properties are still mysterious. Still many believe that this is just a technical gap, i.e., that the torus is not really special.) In 1990 Salamon and Zehnder [SZ92] proved, using Floer homology, the existence of infinitely many subharmonic solutions in the nondegenerate case on a symplectic manifold with vanishing second fundamental group. The general case of Conley's conjecture on subharmonic solutions is still open. We hope to extend the techniques of this paper to other symplectic manifolds.

Here is an outline of our proof:

Step 0. Assume the Hamiltonian flow has finitely many orbits with the same period as $\mathcal{H}$. If there is a bound for the minimal period of subharmonic solutions, then the action has a topologically degenerate (as defined above) critical point $\boldsymbol{u}_0$, of the same period as $\mathcal{H}$. This argument comes from Conley and Zehnder [CZ86].

For the rest of the proof we assume the existence of an isolated, topologically degenerate critical point $\boldsymbol{u}_0$. Let $\boldsymbol{u}_0^m$ (also called $\boldsymbol{u}_0$ for simplicity, and also called the *m-th iterate* of $\boldsymbol{u}_0$) be the image $\boldsymbol{u}_0$ of in Λ_m. The rest of the paper is devoted to proving the following, which clearly implies the theorem.

PROPOSITION. *If the action has an isolated, topologically degenerate critical point u_0, then there are subharmonic solutions with arbitrarily high minimal period.*

Step 1. In a suitable local coordinate system, the critical point $\boldsymbol{u}_0$ is the constant loop at the origin, and the Hamiltonian has a strict local maximum at the origin for each fixed t.

Step 2. If m is sufficiently large, there is a *molar* at $\boldsymbol{u}_0$ (see Lemmas 7, 8 and Figure 1): Fix m. Let c_m be the critical value $\boldsymbol{u}_0^m$. A *molar at* u_0 is a pair $(\Sigma_m, \Sigma_m^{\geq})$ of submanifolds such that $\boldsymbol{u}_0 \in \Sigma_m^{\geq}$, $\Sigma_m^{\geq} \subset \Sigma_m$ as a submanifold of codimension 1, and $\partial\Sigma_m^{\geq} \subset \partial\Sigma_m$, together with the following three properties:

(A) $\mathcal{A} \geq c_m$ on $\Sigma_m^{\geq}$, and $\Sigma_m^{\geq}$ represents a nontrivial local homology class at $\boldsymbol{u}_0$.

(B) $\mathcal{A} \geq c_m - \varepsilon$ on Σ_m for some $\varepsilon > 0$.

(C) $\mathcal{A} > c_m$ on $\partial\Sigma_m$.

In fact, the proof will show that we can make ε as small as we like for sufficiently large m.

Step 3. Given $\varepsilon > 0$, if m is sufficiently large, Λ_m has a critical value in the interval $[c_m - \varepsilon, c_m)$.

Question. Is an isolated, topologically degenerate critical point necessarily the C^0 limit of a sequence of subharmonic solutions? Perhaps a more cunning trap could capture the new critical points C^0-locally.

2. Preliminaries

2.1. *Definitions and notation.* See [CZ84] and [MS95]. Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let the torus T^{2n} be equipped with the standard symplectic structure, the standard almost complex structure J, and a Hamiltonian $\mathcal{H} : S^1 \times T^{2n} = \{(t, \boldsymbol{u})\} \to \mathbb{R}$. On $\mathbb{R}^{2n}$, T^{2n}, or the disk D^{2n} we will use coordinates $\boldsymbol{u} = (x, y)$. ∇ will denote differentiation with the x, y variables, with t fixed, and " $\dot{}$ " will denote differentiation with respect to t, with x, y fixed. Let Λ_m be the free loop space $H_0^1(\mathbb{R}/2m\pi\mathbb{Z}, T^{2n})$ of contractible loops of period $2m\pi$. The action $\mathcal{A} = \mathcal{A}_{\mathcal{H}} : \Lambda_m \to \mathbb{R}$ will be "Enclosed area $-\, 2m\pi$(Average value of $\mathcal{H}$)":

$$\mathcal{A}_{\mathcal{H}}(\boldsymbol{u}) = \mathcal{A}_0(\boldsymbol{u}) - \overline{\mathcal{H}}(\boldsymbol{u})$$
$$= \frac{1}{2}\int_0^{2m\pi} \langle \dot{\boldsymbol{u}}(t), J\boldsymbol{u}(t)\rangle dt - \int_0^{2m\pi} \mathcal{H}(t, \boldsymbol{u}(t))dt.$$

The first variation of the action at a point $\boldsymbol{u}$ in the direction of a vector field X along $\boldsymbol{u}$ is

$$\delta\mathcal{A}(\boldsymbol{u})(X) = -\int_0^{2m\pi} \langle \mathcal{D}\boldsymbol{u}, X\rangle dt, \tag{1}$$

where

$$\mathcal{D}\boldsymbol{u}(t) = \mathcal{D}_{\mathcal{H}}\boldsymbol{u}(t) = J\dot{\boldsymbol{u}}(t) + \nabla\mathcal{H}(t, \boldsymbol{u}(t)). \tag{2}$$

The *Hamiltonian flow* $(\mathbb{R} \times T^{2n}) \to T^{2n}$ is a 1-parameter family of symplectomorphisms defined by $\dot{\boldsymbol{u}}(t) = J\nabla\mathcal{H}(t, \boldsymbol{u}(t))$. (In spite of the inelegant local coordinates, the symplectic gradient $J\nabla\mathcal{H}$ depends only on $\mathcal{H}$ and the symplectic structure.) The critical points of $\mathcal{A}$ on Λ_m are precisely the orbits of the Hamiltonian flow of period $2m\pi$. If $\boldsymbol{u}$ is such a critical point, and X and Y are vector fields along $\boldsymbol{u}$, the second variation is given by

$$\delta^2\mathcal{A}(X, Y) = -\int_0^{2m\pi} \langle J\dot{X}(t) + \nabla^2\mathcal{H}(t, \boldsymbol{u}(t)X, Y(t)\rangle dt. \tag{3}$$

It will be convenient to have an expression for the "enclosed area" $\mathcal{A}_0(\boldsymbol{u})$ using Fourier coefficients: If $\boldsymbol{u}(t) = (x(t), y(t))$, where $x + iy = \sum \alpha_k e^{ikt/m}$ with $x, y \in \mathbb{R}^n$ and $\alpha_k \in \mathbb{C}^n$, then

$$\mathcal{A}_0(\boldsymbol{u}) = \pi \sum k|\alpha_k|^2. \tag{4}$$

The *energy* $E(\boldsymbol{u})$ is

$$E(\boldsymbol{u}) := \int_0^{2m\pi} |\dot{\boldsymbol{u}}|^2 dt = \frac{2\pi}{m}(\sum k^2|\alpha_k|^2).$$

2.2. *Generating functions.* See [MS95, p. 287]. Let $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a symplectic map sufficiently C^1-close to the identity map. Then there is a smooth function $S : \mathbb{R}^{2n} \to \mathbb{R}$ such that

$$\Phi(x_0, y_0) = (x_1, y_1) \quad \text{if and only if} \quad (\boldsymbol{x}_1, y_1) - (\boldsymbol{x}_0, y_0) = J\nabla S(\boldsymbol{x}_1, y_0). \tag{5}$$

Conversely, if $S : \mathbb{R}^{2n} \to \mathbb{R}$ is a C^2 function with sufficiently small first and second derivatives, then (5) defines a symplectomorphism of $\mathbb{R}^{2n}$.

2.3. *Intersection in the symplectic group and the Maslov index.* See [Bot56], [Cha93], [CZ86], [Hin86], [Lon99], and [Lon02]. If $|\omega| = 1$, let $B(\omega)$ be the set in the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ where ω is an eigenvalue, that is,

$$B(\omega) = \{X \mid \det(X - \omega I) = 0\} \subset \mathrm{Sp}(2n, \mathbb{R}).$$

For example, $\mathrm{Sp}(2, \mathbb{R})$ is topologically an open solid torus, in which $B(1)$ sits as a cone with vertex at the identity matrix I, and with two boundary circles out at infinity. We will assume the reader is familiar with the topology of $\mathrm{Sp}(2, \mathbb{R})$.

The nonsingular part $B^*(\omega)$ of $B(\omega)$ is given by

$$B^*(\omega) = \{Y \mid N_Y(\omega) = 1\}, \quad \text{where } N_Y(\omega) := \dim\ker(Y - \omega I).$$

The cone of *positive directions* (*plus directions*), defined by

$$+\,\mathrm{Cone} = \{A \mid JA \text{ is positive definite}\} \subset \mathfrak{sp}(2n, \mathbb{R})$$

provides an orientation for $B^*(\omega)$. A plus-curve $\gamma(t)$ intersects $B(\omega)$ at isolated values of t. The intersection number of an oriented curve γ in $\mathrm{Sp}(2n, \mathbb{R})$ with $B(\omega)$ can be defined, if the endpoints of γ do not lie on $B(\omega)$. This intersection number is invariant under homotopies of γ leaving the endpoints fixed. The intersection number of a plus-curve intersecting $B(\omega)$ at a single point Y is $N_Y(\omega)$. If $\gamma : [0, T] \to \mathrm{Sp}(2n, \mathbb{R})$, with $\gamma(0) = I$, we define

$$\text{Maslov Index}(\gamma) := \mathbf{\Omega}_\gamma := \#(B(1) \cap \alpha * \gamma * \beta),$$

where $\alpha : [0, 1] \to \mathrm{Sp}(2n, \mathbb{R})$ is a path of real diagonal matrices intersecting $B(1)$ only at $\alpha(1) = I$, and β is a sufficiently short minus-curve beginning at $\gamma(T)$. More generally we define the ω-index

$$\mathbf{\Omega}_\gamma(\omega) := \#(B(\omega) \cap \alpha * \gamma * \beta). \tag{6}$$

The Maslov index is roughly twice the "winding number" of γ around the loop generating $\Pi_1(\mathrm{Sp}(2n, \mathbb{R}) = \mathbb{Z}$ (roughly because γ is not closed, and because of the small corrections α and β. The curve α moves the left endpoint of γ away from $B(1)$, and β (if necessary) moves the right endpoint of γ away from $B(\omega)$.) Let $\tau : [-1, 1] \to \mathrm{Sp}(2n, \mathbb{R})$ be a plus-curve; assume for the moment τ intersects $B(\omega)$ only at $X = \tau(0)$. Let τ^-=$\tau[-1, 0)$. The *splitting numbers* of $X = \tau(0) \in B_\omega$ are

$$S_X^\pm(\omega) := \lim_{\varepsilon \to 0^+} \#(B(\omega e^{\pm i\varepsilon}) \cap \tau^-).$$

They depend only upon the conjugacy class of X in $\mathrm{Sp}(2n, \mathbb{R})$. If X breaks up into invariant symplectic blocks, then $S^\pm$ and N are additive. Moreover,

$$\begin{aligned} S^\pm(\omega) &= S^\mp(\overline{\omega}), \\ 0 \le S_X^\pm(\omega) &\le N_X(\omega), \\ \lim_{\varepsilon \to 0^+} \mathbf{\Omega}_\gamma(\omega e^{\pm i\varepsilon}) &= \mathbf{\Omega}_\gamma(\omega) + S_{\gamma(1)}^\pm(\omega). \end{aligned} \tag{7}$$

Thus for a fixed path $\gamma : [0, T] \to \mathrm{Sp}(2n, \mathbb{R})$ beginning at I, the ω-index is constant except at the eigenvalues of $\gamma(1)$; the jump at an eigenvalue is given by the splitting numbers. If $|\omega| \neq 1$, and $N_X(\omega) = 1$, then

$$|\mathbf{\Omega}_\gamma(\omega e^{i\varepsilon}) - \mathbf{\Omega}_\gamma(\omega e^{-i\varepsilon})| = |S_{\gamma(1)}^+(\omega) - S_{\gamma(1)}^-(\omega)| = 1. \tag{8}$$

If γ is a plus-curve, and if $\gamma(s) = \gamma|_{[0,s]}$, then

$$\mathbf{\Omega}_{\gamma(s)}(\omega) + N_{\gamma(s)}(\omega) \le \mathbf{\Omega}_{\gamma(r)}(\omega) \quad \text{if } s < r. \tag{9}$$

That is, the index increases along plus-curves.

The index transforms under the iteration map $\gamma \to \gamma^m : \Lambda \to \Lambda_m$ by *Bott's formula*

$$\mathbf{\Omega}_{\gamma^m} = \sum_{\omega^m = 1} \mathbf{\Omega}_\gamma(\omega).$$

For fixed γ, the *average index*

$$\overline{\mathbf{\Omega}}_\gamma = \lim_{m\to\infty} \frac{\mathbf{\Omega}_{\gamma^m}}{m} \tag{10}$$

$$= \frac{1}{2\pi}\int \mathbf{\Omega}_\gamma(\omega)\,d\omega \tag{11}$$

exists, and

$$|\mathbf{\Omega}_{\gamma^m}(\omega) - m\overline{\mathbf{\Omega}}_\gamma| \le n. \tag{12}$$

2.4. *Coordinate change.* Consider a domain $\mathcal{D} \subseteq S^1 \times \mathbb{R}^{2n}$. (The typical example will be a periodically traveling disk.) A *coordinate change* will be a map $\boldsymbol{u}(t,\boldsymbol{w}) : \mathcal{D} \to \mathbb{R}^{2n}$ that is a symplectic diffeomorphism for each fixed t. This coordinate change identifies $\mathcal{D}$ with $\mathcal{D}' = \{(t, \boldsymbol{u}(t,\boldsymbol{w})) \mid (t,\boldsymbol{w}) \in \mathcal{D}\}$. When composing coordinate changes in the beginning of Section 5, it will be convenient to write the coordinate change in the form $\psi : \mathcal{D} \to S^1 \times \mathbb{R}^{2n}$ with $\psi(t,\boldsymbol{w}) = (t, \boldsymbol{u}(t,\boldsymbol{w}))$. The flow

$$\boldsymbol{w}_{\mathcal{H}}(t) : \dot{\boldsymbol{w}}_{\mathcal{H}} = J\nabla\mathcal{H}(t, \boldsymbol{w}_{\mathcal{H}}) \tag{13}$$

of a periodic Hamiltonian $\mathcal{H}$ on $\mathcal{D}$ will transform to that of a periodic Hamiltonian $\mathcal{K}$ on $\mathcal{D}'$ in the sense that

$$\boldsymbol{u}_{\mathcal{K}}(t) = \boldsymbol{u}(t, \boldsymbol{w}_{\mathcal{H}}(t)) : \dot{\boldsymbol{u}}_{\mathcal{K}} = J\nabla\mathcal{K}(t, \boldsymbol{u}_{\mathcal{K}}). \tag{14}$$

Thus the Hamiltonian flow and the orbits are preserved under the transformation $\boldsymbol{w} \mapsto \boldsymbol{u}$ and $\mathcal{H} \mapsto \mathcal{K}$. The coordinate change induces a local map $\Lambda_m \to \Lambda_m$ by $\boldsymbol{w} = \boldsymbol{w}(t) \mapsto \boldsymbol{u}(t) := \boldsymbol{u}(t, \boldsymbol{w}(t))$. Though the action may not be preserved, the critical points of $\mathcal{A}_{\mathcal{H}}$ are mapped bijectively to the critical points of $\mathcal{A}_{\mathcal{K}}$. Here are two examples:

Example (Symplectic transformation). A symplectic diffeomorphism $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ induces the map $\boldsymbol{u}(t,\boldsymbol{w}) = \varphi(\boldsymbol{w})$; therefore $\Lambda_m \mapsto \Lambda_m$ by $\boldsymbol{w} \mapsto \varphi \circ \boldsymbol{w}$. This map preserves the symplectic area $\mathcal{A}_0$, that is, $\mathcal{A}_0(\varphi\circ\boldsymbol{w}) = \mathcal{A}_0(\boldsymbol{w})$. See [HZ94, p. 7]. The Hamiltonian $\mathcal{H}$ transforms to $\mathcal{H} \circ (\iota \times \varphi^{-1})$, where ι is the identity map on S^1; see [HZ94, p. 9]. Thus $\overline{\mathcal{H}} \to \overline{H \circ (\iota \times \varphi^{-1})}$ and $\mathcal{A}$ is also preserved.

Example (Hamiltonian coordinate change). Let $\mathcal{D} \subseteq S^1 \times \mathbb{R}^{2n}$. A *Hamiltonian coordinate change* will be a differentiable map $\boldsymbol{u}(t,\boldsymbol{w}) : \mathcal{D} \to \mathbb{R}^{2n}$ that is a symplectic diffeomorphism for each fixed t and has $\boldsymbol{u}(0,\boldsymbol{w}) = \boldsymbol{w}$ for all $\boldsymbol{w}$. Such a coordinate change is generated by a Hamiltonian $h : \mathcal{D}' \to \mathbb{R}$ using

$$\frac{\partial}{\partial t}\boldsymbol{u}(t,\boldsymbol{w}) = J\nabla h(t, \boldsymbol{u}(t,\boldsymbol{w})).$$

See [MS95, p. 287]. Note that by our definition, the flow of h is a *loop* in the semi-group of symplectic maps, i.e., the time-2π map of h's flow is the identity map. A Hamiltonian flow $\mathcal{H}$ in $\mathcal{D}$ transforms to $\mathcal{K}$ in $\mathcal{D}'$ where

$$\mathcal{K} = h \# \mathcal{H} \quad \text{that is, } \mathcal{K}(t, \boldsymbol{u}) = h(t, \boldsymbol{u}) + \mathcal{H}(t, \boldsymbol{w}). \tag{15}$$

See [HZ94, p. 144]. The product # is associative, and h has inverse

$$\overline{h}(t, \boldsymbol{w}) = -h(t, \boldsymbol{u})$$

in the sense that $\overline{h} \# h = h \# \overline{h} = 0$.

Suppose $\mathcal{D} = S^1 \times D^{2n}$, and suppose two Hamiltonians H and K are given with $H(t, \mathbf{0}) = \nabla H(t, \mathbf{0}) = K(t, \mathbf{0}) = \nabla K(t, \mathbf{0}) = 0$ for all t. Suppose also that H and K have the same time-2π map. Then there is a Hamiltonian coordinate change transforming H to $\mathcal{K}$ with time-t map given by

$$\boldsymbol{v}_h^t = \boldsymbol{u}_K^t \circ (\boldsymbol{w}_H^t)^{-1}. \tag{16}$$

The formula for h is $h = K \# \overline{H}$; see [HZ94, p. 144]. Let $\gamma, \delta : \mathbb{R} \to \mathrm{Sp}(2n, \mathbb{R})$ give the linearized flow of H and K at their mutual orbit $\boldsymbol{w} = \mathbf{0} = \boldsymbol{u}$, and suppose further that the closed curve $\gamma^{-1} * \delta$ is contractible.

LEMMA 1. *For $R > 0$, there is an $r > 0$ and a differentiable 1-parameter family of Hamiltonian coordinate changes $\boldsymbol{\eta}_s : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that are supported in the disk of radius R and transform H to K inside a disk of radius r. Specifically,*

(i) $\boldsymbol{\eta}_0(t, \boldsymbol{w}) = \boldsymbol{w}$ *for all* $\boldsymbol{w}$,

(ii) $\boldsymbol{\eta}_s(t, \mathbf{0}) = \mathbf{0}$ *for all* s, t,

(iii) $\boldsymbol{\eta}_s(t, \boldsymbol{w}) = \boldsymbol{w}$ *if* $|w| \geq R$,

(iv) $\boldsymbol{\eta}_1(t, \boldsymbol{w}) = \boldsymbol{v}_h(t, \boldsymbol{w})$ *if* $|w| \leq r$.

Proof. We are given a map $\boldsymbol{v}_h(t, w)$ defined on $S^1 \times D^{2n}$, such that $\boldsymbol{v}_h(t, 0) = 0$. (What is relevant to the problem at hand is the loop of germs of the $\boldsymbol{v}_h^t$ at the point 0.) By (16), the loop of derivatives of the $\boldsymbol{v}_h^t$ at the point 0 is homotopic to $\gamma^{-1} * \delta$, and thus contractible. Assume first that $\boldsymbol{v}_h^t$ is C^1-close to the identity map inside the disk of radius $2r$ for some $r < R/2$, and write $\boldsymbol{v}_h(t, x_0, y_0) = (x_t, y_t)$. There is a generating function $S : S^1 \times D^{2n} \to \mathbb{R}$ with

$$(x_t, y_t) - (x_0, y_0) = J \nabla S(t, x_t, y_0) \quad \text{whenever } |(x_0, y_0)| < 2r.$$

Let $0 \leq \rho \leq 1$ with

$$\rho(s, x, y) = \begin{cases} 0 & \text{if } s = 0 \text{ or } |(x, y)| \geq 2r, \\ s & \text{if } |(x, y)| \leq r. \end{cases}$$

We will use the generating functions $\rho S(s, t, x, y)$; let $\eta_s(t, x_0, y_0) = (x_{s,t}, y_{s,t})$ with $(x_{s,t}, y_{s,t}) - (x_0, y_0) = J \nabla (\rho S)(s, t, x_{s,t}, y_0)$. It can be checked that η_s then has properties (i)–(iv). By composing coordinate changes, it is clear that the set of loops of germs $\boldsymbol{v}_h^t$ that can be obtained in this way is the connected component of the identity map. □

Translating the orbit $\boldsymbol{u}_0$ *to the constant loop at the origin.* If $\boldsymbol{u}_0$ is an orbit of the flow $\dot{\boldsymbol{u}}_{\mathcal{H}} = J\nabla\mathcal{H}(t, \boldsymbol{u}_{\mathcal{H}})$ of period 2π, with $\boldsymbol{u}_0(0) = 0$, then we can locally transform $\boldsymbol{u}_0$ to the curve $\boldsymbol{w} \equiv 0$ in D^{2n} and $\mathcal{H}$ to a Hamiltonian H (as in (14) above) on $S^1 \times D^{2n}$ with $\nabla H(t, \mathbf{0}) \equiv 0$ and $H(t, \mathbf{0}) \equiv 0$. To do this, we make a Hamiltonian coordinate change using the function h with $\nabla h(t, \boldsymbol{v}) = \nabla\mathcal{H}(t, \boldsymbol{u}_0(t))$ for all $\boldsymbol{v}$, and $h(t, \boldsymbol{u}_0(t)) \equiv \mathcal{H}(t, \boldsymbol{u}_0(t))$. Then

$$\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_0 \tag{17}$$

is a local coordinate on D^{2n}. Because this transformation is a translation for each fixed t, $\nabla_{\boldsymbol{u}} = \nabla_{\boldsymbol{w}}$, and (15) implies that

$$\nabla_{\boldsymbol{u}}\mathcal{H}(t, \boldsymbol{u}) = \nabla_{\boldsymbol{u}} h(t, \boldsymbol{u}) + \nabla_{\boldsymbol{w}} H(t, \boldsymbol{w}).$$

When $|\boldsymbol{w}|$ is small, by a short computation using (2), (14), and (17), we get

$$\mathcal{D}_H(\boldsymbol{w})(t) = \mathcal{D}_{\mathcal{H}}(\boldsymbol{u})(t) \tag{18}$$

from which it follows, since $-\mathcal{D}$ is the L^2-gradient of $\mathcal{A}$, that

$$\mathcal{A}_H(\boldsymbol{w}) = \mathcal{A}_{\mathcal{H}}(\boldsymbol{u}) + C, \tag{19}$$

where $\mathbb{C} = -\mathcal{A}_{\mathcal{H}}(\boldsymbol{u}_0)$. By adding a constant $g(t)$ to $\mathcal{H}$ (which will not affect the flow) we can assume $\mathbb{C} = 0$. This means that under the coordinate change $\boldsymbol{u} \mapsto \boldsymbol{w}$, the action is preserved, even though the Hamiltonian and the symplectic area may not be.

2.5. *Finite-dimensional approximation*, *local homology*, *and characteristic manifold.* Characteristic manifold [GM69] and finite-dimensional approximation [AZ80], [CZ83] can be understood loosely as two manifestations of the same idea, “saddle point reduction": Let V be a vector space, and let $\mathrm{F} : V \to \mathbb{R}$. If we can find coordinates (u, v, w) on V such that

$$\mathrm{F}(u, v, w) = f(u) + |v|^2 - |w|^2, \tag{20}$$

then we restrict our attention to f, the restriction of F to the smaller space V_{red} : $\{v = w = 0\}$. The critical points of F on V are exactly the critical points of f on V_{red}. Even if F is not explicitly in the form (20), we can find a reduction if we can find a foliation of V with the property that the restriction of F to each leaf has a unique nondegenerate critical point. Given such a foliation, let $V_{\mathrm{red}} \subset V$ be the space of these critical points, one for each leaf, and let f be the restriction of F to V_{red}. There will again be a one-to-one correspondence between critical points of F and critical points of f. If the critical point of the restriction of F to each leaf has finite index λ, then the local homology (see below) of a critical point p in V and the local homology in V_{red} will be related by a shifting theorem:

$$\mathcal{H}_*(\mathrm{F}, p) = \mathcal{H}_{*+\lambda}(f, p). \tag{21}$$

Finite-dimensional approximation. Take a Hamiltonian $\mathcal{H}: S^1 \times T^{2n} = \{(t, \boldsymbol{u})\} \to \mathbb{R}$. Following Amman and Zehnder [AZ80] and Conley and Zehnder [CZ83], we get a finite-dimensional approximation for the free loop space ΛT^{2n} as follows: Let

$$N > 2|\nabla^2 \mathcal{H}| \quad \text{(independent of } m!\text{)}. \tag{22}$$

For fixed N, let F_m be the $2n(2mN+1)$-dimensional space of Fourier series of order mN, mapping $[0, 2m\pi] \to \mathbb{R}^{2n}$. An element of F_m can be written $(x(t), y(t))$, where

$$x + iy = \sum_{|k| \leq mN} \alpha_k e^{ikt/m}, \quad \text{with } x, y \in \mathbb{R}^n \text{ and } \alpha_k \in \mathbb{C}^n.$$

Let $Z_m = T^{2n} \times F_m^+ \times F_m^- \cong T^{2n} \times \mathbb{R}^{2nmN} \times \mathbb{R}^{2nmN}$ be the quotient of F_m by the action of $\mathbb{Z}^{2n}$ (identifying curves whose images on the torus are the same). We identify Z_m with its image in Λ_m.

The finite-dimensional approximation $\mathcal{F}_m$ is

$$\mathcal{F}_m := \{\boldsymbol{u} \in \Lambda_m \mid \mathcal{D}(\boldsymbol{u}) \in F_m\}, \tag{23}$$

where

$$\mathcal{D}(\boldsymbol{u})(t) = J(\dot{\boldsymbol{u}}(t)) + \nabla \mathcal{H}(t, \boldsymbol{u}(t)). \tag{24}$$

Note $\mathcal{F}_m$ depends on $\mathcal{H}$!

This construction fits into the "saddle point reduction" framework described above, as follows: We foliate Λ_m with leaves that are level sets of the first $(2mN+1)$ Fourier integrals. Thus the leaf containing a point $\boldsymbol{u}_0 = (\boldsymbol{x}_0, y_0)$ is

$$\mathcal{L}(\boldsymbol{u}_0) = \left\{(x_0, y_0) + (x, y) \mid x + iy = \textstyle\sum_{|k| > mN} \alpha_k e^{ikt/m}\right\}. \tag{25}$$

Call this the *standard Fourier foliation*. If $\boldsymbol{u} \in \mathcal{F}_m$, Equations (1) and (25) imply that all variations of the action at $\boldsymbol{u}$ in leaf directions vanish; thus the points in $\mathcal{F}_m$ are the critical points of the restriction of the action $\mathcal{A}$ to the leaves. If N is large enough, the restriction of $\mathcal{A}$ to each leaf will have a unique, nondegenerate critical point.

We will assume that N is "large enough" (i.e., (22)) through Section 4. (In Section 5, we will need to recalculate N after a coordinate change.) The reader can check that (22) is the condition that makes the proof in [AZ80] and [CZ83] work by [CZ83, Eq. (8), p. 37] with $A_m := mA$ and $h_m(t, x) := h(mt, x)$, or by [Cha93, p. 188]. In terms of the saddle point reduction framework, the condition

(22) ensures that in the second derivative of the action along a leaf, the (standard quadratic) term coming from the symplectic area dominates the Hamiltonian term. This computation will come out below in (55).

The finite-dimensional approximation $\mathcal{F}_m$ is a smooth submanifold of Λ_m of dimension $2n(2mN+1)$, and the critical points of the action on Λ_m are precisely the critical points of the restriction of the action to $\mathcal{F}_m$. Each leaf of the foliation contains a unique point that can be written as a Fourier series of order $\leq mN$, and a unique critical point of the restricted function; this correspondence gives a smooth bijection

$$\xi : Z_m \xrightarrow{\sim} \mathcal{F}_m. \tag{26}$$

The *nullity* (the dimension of the null space of the second derivative) of a critical point $\boldsymbol{u}_0$ in $\mathcal{F}_m$ is the same as its nullity in Λ_m: By (23), any null direction in Λ_m is automatically tangent to $\mathcal{F}_m$, and the fact that the nullity in $\mathcal{F}_m$ is no greater than the nullity in Λ_m follows from the fact that the isomorphism (26) preserves the first $(2mN+1)$ Fourier integrals. As seen in [CZ83], the relationship between the Morse index in $\mathcal{F}_m$, and the Maslov index of the linearized flow $\gamma : [0, 2\pi] \to \mathrm{Sp}(2n, \mathbb{R})$ (also called the Maslov index of $\boldsymbol{u}_0$) is

$$\text{Maslov Index} = \text{Morse Index} - \tfrac{1}{2}\dim \mathcal{F}_m.$$

For convenience we always shift the grading on H_* by $(2mN+1)n$ so that it coincides with the Maslov index, rather than the Morse index.

Local homology at a critical point. We define the *local homology* at an isolated critical point p of a function f with $f(p) = c$ on a *finite-dimensional* manifold as

$$\mathcal{H}_*(f, p) := H_*(f_c, f_c \setminus \{p\}) = H_*(f_c \cap U, (f_c \cap U) \setminus \{p\}),$$

where $f_c = f^{-1}(-\infty, c]$ and U is any neighborhood of p. This definition, which clearly does not depend upon local coordinates, comes from K. C. Chang [Cha93, p. 32], where it is later shown in Theorem 4.2, page 35, that this is equivalent to the definition used by Gromoll and Meyer. If p is a *nondegenerate* (nullity $= 0$) critical point of index λ, then

$$\mathcal{H}_k(f, p) = \begin{cases} \mathbb{Z} & \text{if } k = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

If the manifold is complete, if c is the only critical value of f in $[c-\varepsilon, c+\varepsilon]$, and if $p_1, \ldots, p_m$ are the critical points of f in $f^{-1}(c)$, then

$$H_*(f_{c+\varepsilon}, f_{c-\varepsilon}) = \sum \mathcal{H}_*(f, p_i).$$

If a and b are regular values with $a < b$, then there are (by Morse theory) long exact sequences building up $H_*(f_b, f_a)$ out of the groups $\mathcal{H}_*(f, p_i)$ with $f(p_i) \in [a, b]$.

Characteristic manifold. The local homology at an isolated critical point (which can be complicated, or trivial, if the critical point is degenerate) is carried by a *characteristic manifold* [GM69]. If $\mathrm{F} : V \to \mathbb{R}$ has an isolated critical point at the origin in a finite-dimensional vector space V, there are local coordinates $(u, v, w) \in \mathbb{R}^\eta \times \mathbb{R}^\theta \times \mathbb{R}^\lambda$ on V such that

$$\mathrm{F}(u, v, w) = f(u) + |v|^2 - |w|^2, \quad \text{with } \nabla f(0) = \nabla^2 f(0) = 0.$$

In this case the manifold $\mathscr{C} : v = w = 0$ is *characteristic*. The dimension of $\mathscr{C}$ is the nullity of the critical point. A *shifting theorem* [GM69] relates the local homology of the critical point of F on V to that of f on $\mathscr{C}$ as

$$\mathscr{H}^V_{*+\lambda}(\mathrm{F}, \{0\}) = \mathscr{H}^{\mathscr{C}}_*(f, \{0\}). \tag{27}$$

In particular, local homology only appears in dimensions between the index and the index plus nullity:

$$\mathscr{H}^V_k(\mathrm{F}, \{0\}) = 0, \quad \text{unless } \lambda \leq k \leq \lambda + \eta. \tag{28}$$

The characteristic manifold can be thought of as a local (and ultimate) finite-dimensional approximation, and can be constructed in the same way: One takes a foliation of V with the leaf through the origin, transverse to the null space of F. Near the origin each leaf will have a unique nondegenerate critical point, and the set of these critical points is a smooth submanifold $\mathscr{C}$ of V with the desired properties.

Invariance of the local homology. Suppose a fixed orbit $\boldsymbol{u}_0$ of the flow $\dot{\boldsymbol{u}}_{\mathscr{H}} = J\nabla\mathscr{H}(t, \boldsymbol{u}_{\mathscr{H}})$ of period 2π is given. Fix m, and suppress the subscripts in $\mathscr{F}_m$, F_m, $\hat{Z}_m$, and Z_m. Once a finite-dimensional approximation $\mathscr{F}$ is chosen for Λ, the local homology of the action at $\boldsymbol{u}_0$ is defined. One would hope that the local homology is independent of the finite-dimensional approximation. The space $\mathscr{F}$ near $\boldsymbol{u}_0$ depends upon the Fourier foliation near $\boldsymbol{u}_0$ (which depends on the coordinates), and upon the action in a neighborhood of $\boldsymbol{u}_0$. Once $\mathscr{F}$ is chosen, the local homology depends only upon the restriction of the action to a neighborhood of $\boldsymbol{u}_0$ in $\mathscr{F}$. The action is not preserved by a general coordinate change. We will show however that, with N fixed, the local homology at $\boldsymbol{u}_0$ is invariant under (i) (time-independent) linear symplectic transformations, (ii) translations of the orbit $\boldsymbol{u}_0$ to the origin, and (iii) Hamiltonian coordinate changes that lie in the path-connected component of the identity transformation in the space of coordinate changes fixing the constant loop at the origin. Finally we show (iv) that the local homology is independent of N if N is sufficiently large.

(i) A linear symplectic transformation preserves the leaves of the Fourier foliation and the action function. Thus the set $\mathscr{F}$ is preserved pointwise, and $\mathscr{A}_{\mathscr{H}}|_{\mathscr{F}}$ is preserved.

(ii) Consider the translation $T : \boldsymbol{u} \mapsto \boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_0$ described in Section 2.4, where $\boldsymbol{u}_0$ is a critical point of $\mathscr{A}_{\mathscr{K}}$. The map T takes a neighborhood of $\boldsymbol{u}_0$ to D^{2n}, and $\mathscr{A}_{\mathscr{H}}(\boldsymbol{w}) = \mathscr{A}_{\mathscr{K}}(\boldsymbol{u})$ by (19). The map T also preserves the leaves of the standard Fourier foliation. Because the finite-dimensional approximation depends only on the foliation and the action function in a neighborhood of the critical point, the map $\boldsymbol{u} \mapsto \boldsymbol{w}$ maps the finite-dimensional approximation for $\mathscr{A}_{\mathscr{H}}$ near $\boldsymbol{w} = \boldsymbol{0}$ (by the standard foliation) to the finite-dimensional approximation for $\mathscr{A}_{\mathscr{K}}$ near $\boldsymbol{u}_0$, using the standard foliation. This means that the local homology is unchanged.

(iii) Next assume $\mathscr{H}(\boldsymbol{0}) = \nabla\mathscr{H}(\boldsymbol{0}) = 0$. Let

$$\Psi : [0,1] \times S^1 \times D^{2n} \to D^{2n}$$

be a differentiable family of Hamiltonian coordinate changes, parametrized by $s \in [0,1]$, with $\Psi(s,t,\boldsymbol{0}) = 0$ for all s,t, and $\Psi(0,t,\boldsymbol{w}) = \boldsymbol{w}$ for all $t, \boldsymbol{w}$. For each s, Ψ induces a smooth map $\Psi_s : \Lambda D^{2n} \to \Lambda D^{2n}$ and a transformed Hamiltonian $\mathscr{K}_s = h_s \# \mathscr{H}$. The finite-dimensional approximation (expressed in terms of the coordinate $\boldsymbol{u} = \Psi_s(t,\boldsymbol{w})$ on the right) is

$$\mathscr{F}_s := \{\boldsymbol{u} \in \Lambda : D_{\mathscr{K}_s}(\boldsymbol{u}) \in F\}.$$

This is a smooth family of submanifolds of ΛD^{2n}, each containing the point $\boldsymbol{u} \equiv 0$. We consider the local homology of the function $\mathscr{A}_{\mathscr{K}_s}$ at the point $\boldsymbol{u} \equiv 0$ in $\mathscr{F}_s$. If N is sufficiently large (depending upon s), the leaves of the standard foliation give a smooth map $\varphi_s : F \to \mathscr{F}_s$. Thus the associated local homology can be viewed as the local homology coming from a differentiable family of functions ${}_s = \varphi_s^{-1} \circ \mathscr{A}_{\mathscr{K}_s} \circ \varphi_s$ on F, each with an isolated critical point at the origin. (The critical point is isolated for each s since critical points of the restriction of $\mathscr{A}_{\mathscr{K}_s}$ to $\mathscr{F}_s$ are period 2π orbits of the flow of the Hamiltonian $\mathscr{K}_s$, which are, by construction, the image under Ψ_s of period 2π orbits of the flow of the Hamiltonian $\mathscr{H}$.) In these circumstances Gromoll and Meyer proved [GM69, Lemma 4, p. 366] that the local homology is independent of s.

(iv) The finite-dimensional approximation $\mathscr{F}^N$, with N sufficiently large, can be thought of as a saddle point reduction intermediate between the (unreduced) infinite-dimensional ΛT^{2n} and the (maximally reduced) manifold $\mathscr{C}$. If, as in the case with Fourier foliations $\mathscr{L}^N$, the foliations are *compatible* in the sense that $\mathscr{L}^P(\boldsymbol{u}) \subset \mathscr{L}^N(\boldsymbol{u})$ if $P > N$, then $\mathscr{F}^N \subset \mathscr{F}^P$; the critical points of $\mathscr{A}$ on $\mathscr{F}^N$ are the same as the critical points of $\mathscr{A}$ on $\mathscr{F}^P$, and the local homology of $\mathscr{F}^N$ at each critical point is related to the local homology of $\mathscr{F}^P$ by a shifting theorem. If we use the Maslov grading, there is no shift, and the local homology of $\mathscr{F}^N$ at a critical point $\boldsymbol{u}$ is independent of N, provided only that N is large enough that the leaf $\mathscr{L}^P(\boldsymbol{u})$ does not contain any null directions. A characteristic manifold at an isolated critical point in $\mathscr{F}^N$ is automatically also characteristic in $\mathscr{F}^P$.

We call an isolated critical point of the action *topologically degenerate* if its iterates are homologically visible (i.e., have nontrivial local homology) in dimension $-n$ in infinitely many Λ_m. By the above, the local homology at u_0, and the property of being topologically degenerate are independent of N, and independent of coordinates so long as we consider only coordinate changes in the semigroup generated by (time-independent) linear symplectic transformation, translation of the orbit u_0 to the origin, and Hamiltonian coordinate changes that lie in the path component of the identity transformation in the space of coordinate changes fixing the constant loop at the origin. We will start with the standard coordinates on $S^1 \times T^{2n}$, and use only coordinate changes of the above type throughout the proof.

Remark. One can define a characteristic manifold without first having a finite-dimensional approximation. However if the index is infinite, there can be no shifting theorem. For our purposes it seems simplest to define the local homology and characteristic manifold only on finite-dimensional manifolds. The local homology was defined by Morse in [Mor96].

3. Global to local

We use the homology of the free loop space to infer the existence of a topologically degenerate critical point. This is Step 0 in our outline.

LEMMA 2 (Existence of a topologically degenerate critical point). *Given a periodic Hamiltonian $\mathcal{H}$ on T^{2n}, suppose the Hamiltonian flow has finitely many orbits with the same period as $\mathcal{H}$. If there is a bound for the minimal period of subharmonic solutions, then there is a topologically degenerate critical point $\boldsymbol{u}_0$, of the same period as $\mathcal{H}$.*

Proof. This argument comes from [AZ80] and [CZ83]: Take the standard finite-dimensional approximation using Fourier series. Let $D^{\pm}$ be a ball of radius K in $F_m^{\pm}$. Put

(29) $$B = T^{2n} \times D^{+} \times D^{-} \quad \text{and} \quad B^{-} = T^{2n} \times D^{+} \times \partial D^{-}.$$

If K is sufficiently large, then all critical points of A lie in $\varphi(B)$, and B^{-} is the exit set for the flow of $\nabla(A \circ \varphi)$ on B. Assume all critical points of A are isolated. Since $H_*(B, B^-) = H_{*-n}(T^{2n})$ (here we shift the grading of the homology of B by $\frac{1}{2}\dim \mathcal{F}_m$ so that it coincides with Maslov grading), a standard Morse theory argument ensures the existence of a critical point x_m in each $\mathcal{F}_m$ (and thus, by the basic property of a finite-dimensional approximation, a critical point in each Λ_m) with local homology in dimension $-n$. If there is a bound for the minimal period of subharmonic solutions, and if m is a sufficiently large prime, any critical point in Λ_m must have minimal period 2π and be the iterate of an orbit in Λ_1. Also, if Λ_1 has only finitely many critical points, then one of these critical points must have iterates that are homologically visible in dimension $-n$ in infinitely many Λ_m. That is, there is a topologically degenerate critical point. $\square$

4. **Local results**

We now focus on local properties of an isolated, topologically degenerate critical point $\boldsymbol{u}_0$ of $\mathcal{A}_{\mathcal{H}}$. As described in Section 2.4, we can make a coordinate change locally transforming our Hamiltonian $\mathcal{H}$ to a Hamiltonian K with $\nabla K(t, \mathbf{0}) \equiv 0$ and $K(t, \mathbf{0}) \equiv 0$ on the disk $\boldsymbol{D}^{2n}$, and our critical point $\boldsymbol{u}_0$ to the critical point $\boldsymbol{w} \equiv 0$ of $\mathcal{A}_K$. The local homology of the iterates is invariant under this coordinate change, so the translated critical point is still topologically degenerate. The goal of this section is to complete Step 1 of our outline: After a further coordinate change on $S^1 \times \boldsymbol{D}^{2n}$, supported near $S^1 \times \{\mathbf{0}\}$, the Hamiltonian will have a strict local maximum at the origin for each fixed t. At the beginning of Section 5, we will sum up exactly how the four local lemmas of this section fit together with Lemma 1 to that end.

4.1. *Local minimum in the characteristic manifold.*

LEMMA 3. *Given a periodic Hamiltonian on D^{2n}, suppose there is an isolated, topologically degenerate critical point $\boldsymbol{u}_0$. Let $\gamma : [0, 2\pi] \to \mathrm{Sp}(2n, \mathbb{R})$ be the linearized flow. Then* (*possibly after replacing $\boldsymbol{u}_0$ by one of its powers*),

(A) *the restriction of the action to the characteristic manifold has a strict local minimum at the origin*;

(B) *the linear Poincaré map $\gamma(2\pi)$* (*the derivative of the time-2π Hamiltonian flow at the point $\boldsymbol{u}_0(0)$*) *breaks up into 2×2 symplectic blocks*, *has no eigenvalue but* 1, *and has splitting number n at the eigenvalue* 1;

(C) *if $\tau : [0, 2\pi] \to \mathrm{Sp}(2n, \mathbb{R})$ has the same endpoints as γ, and if $\tau(t)$ has no eigenvalue but 1 for all t, then $\gamma * \tau^{-1}$ is contractible.*

Proof. The nullity η of the critical point $\boldsymbol{u}_0$ is the dimension of the nullspace of (3), i.e., $\eta = N_{\gamma(2\pi)}(1)$; thus $\eta \leq 2n$. If the average index of $\boldsymbol{u}_0$ is not 0, then by (12) the index of large iterates $\boldsymbol{u}_0^m$ will be far away from 0. But then by (28), large iterates cannot have local homology in dimension $-n$, a contradiction. By (10) the average index of $\boldsymbol{u}_0^m$ is 0 for all m. By (12), the index of $\boldsymbol{u}_0^m$ is $\geq -n$. On the other hand, the fact that $\boldsymbol{u}_0$ (possibly replaced by a power) is homologically visible in dimension $-n$ implies that $\mathrm{Index}(\boldsymbol{u}_0) \leq -n$ by (28). Thus $\mathrm{Index}(\boldsymbol{u}_0) = -n$; that is, the critical point is homologically visible in the dimension of its index. By the shifting theorem (27), the characteristic manifold has local homology in dimension 0. That an isolated critical point on a compact manifold with local homology in dimension 0 is a strict local minimum is certainly well known; there is a proof in [Hin93, p. 256]. This proves (A).

Let σ be a very short, generic minus-path starting at $\gamma(2\pi)$ By (9) and continuity,

$$\Omega_{\gamma*\sigma}(1) \leq -n \quad \text{and} \quad \overline{\Omega}_{\gamma*\sigma} \approx 0. \tag{30}$$

By the first part of (30) and by (7) and (8), $\Omega_{\gamma*\sigma}(\omega) \leq 0$ for all ω, which implies by (11) that $\overline{\Omega}_{\gamma*\sigma} \leq 0$. Using the second part of (30) and then (7) and (8) again, we see that the right endpoint of σ will have distinct eigenvalues, all lying on the unit circle and all close to 1, (1 itself will *not* be an eigenvalue). Thus the only eigenvalue of $\gamma(2\pi)$ is 1, and $\gamma(2\pi)$ has splitting number n at the eigenvalue 1. Because the splitting number is additive, and is ≤ 1 on each invariant symplectic block, it also follows that the Poincaré map $\gamma(2\pi)$ splits into 2×2 blocks. Thus (B). (Each block is either the 2×2 identity (with nullity 2) or a shear (with nullity 1). The "sign" of the shear map is determined by the fact that the splitting number is 1.)

To prove (B), suppose τ is such a path with $\tau(0) = I$ and $\tau(2\pi) = \gamma(2\pi)$. Then $\sigma_s = \gamma * (\tau|_{[s,1]})^{-1}$ is a path from I to $\tau(s)$ whose right endpoint has no eigenvalue but 1. By continuity, using (6) and (11), σ_s has the same average index as γ, namely 0. But σ_0 is a closed path; the average index is a nontrivial homomorphism from $\pi_1(\mathrm{Sp}(2n, \mathbb{R})) \cong \mathbb{Z}$ to $\mathbb{Z}$. Thus σ_0 is homotopically trivial, and (C) follows. $\square$

4.2. *Relatively autonomous Hamiltonian.* The next step is to find a coordinate change that will render the Hamiltonian flow in a neighborhood of the translated critical point $\boldsymbol{w} \equiv \mathbf{0}$ tame in a sense we will call "relatively autonomous". This is necessary since a wild coordinate change will transform even the simplest Hamiltonian into something very complicated. By Lemma 1, it is enough to find a relatively autonomous Hamiltonian H with the same time-2π map as our Hamiltonian K (from the beginning of §4), and with a loop of derivatives in the correct homotopy class. We will apply the next lemma to the time-2π map Φ of Hamiltonian K on the disk $\boldsymbol{D}^{2n}$ with the three properties of Lemma 3. From Lemma 4(iv) below it will follow that the linearized flows of H and K at their mutual orbit $\boldsymbol{w} \equiv \mathbf{0}$ are homotopic, and thus that hypotheses of Lemma 1 are satisfied.

LEMMA 4. *Let Φ be a symplectic diffeomorphism of the disk $\boldsymbol{D}^{2n}$ with $\Phi(0) = 0$. Suppose its derivatives at the origin breaks up into 2×2 blocks and satisfies $(d\Phi - I)^2 = 0$. Let $\varepsilon, \delta > 0$ be given. Then after making a linear symplectic change of coordinates $\varphi : \boldsymbol{D}^{2n} \to \boldsymbol{D}^{2n}$, the restriction of the transformed map $\varphi^{-1} \circ \Phi \circ \varphi$ to a smaller disk is the time-*1 *map of a Hamiltonian H that satisfies $H(t, \mathbf{0}) = \nabla H(t, \mathbf{0}) = 0$ for all t, and these four properties*:

(i) *$H = S + g$, where S is autonomous and*

$$|\nabla g(t, \boldsymbol{w})| + \left| \frac{\partial \nabla g(t, \boldsymbol{w})}{\partial t} \right| < \varepsilon |\nabla S(\boldsymbol{w})| \quad (\textit{relatively autonomous}). \tag{31}$$

(ii) $|\nabla^2 S| < \delta$.

(iii) *For all* t, $\nabla^2 H(t, \mathbf{0}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -\alpha \end{smallmatrix}\right)$, *with* α *constant and diagonal. If the time-*2π *map of* H *has splitting number* n *at the eigenvalue* 1, *then* α *is nonnegative.*

(iv) *The linearized flow* $\tau(t)$ *associated to* H *has no eigenvalue but* 1.

Proof. The Hamiltonian S will be a generating function for Φ, and H will be the Hamiltonian generating the flow whose time-t map has generating function tS.

First we make a linear coordinate change so that $d\Phi(0)$ is in standard form

$$d\Phi = \begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix}, \tag{32}$$

with α diagonal. (That this is possible can be checked in $\mathrm{Sp}(2, \mathbb{R})$, because of the 2×2 blocks.) As Hamiltonian S we take a generating function for Φ. Its leading term at the origin is easily computed as

$$S(x, y) = \sum -\alpha_i y_i^2 + \cdots .$$

Let $\delta > 0$ be given. After a further linear change of coordinates

$$(x, y) \to (\lambda x, \lambda^{-1} y), \tag{33}$$

we can assume that $|\alpha| < \delta/4$. Thus (ii). By restricting our attention to a sufficiently small disk $\boldsymbol{D}$, we can assume that

$$|\nabla S| < \eta, \qquad |\nabla^2 S| < \delta, \qquad |\nabla^3 S| < Q$$

everywhere in $\boldsymbol{D}$ for η as small as we like, and for some $Q \in \mathbb{R}$. Let (x_t, y_t) be the flow whose time-t map has generating function tS. Then

$$(x_t, y_t) - (x_0, y_0) = tJ\nabla S(x_t, y_0).$$

Differentiating, we get

$$(\dot{x_t}, \dot{y_t}) = J\nabla S(x_t, y_0) + t\frac{\partial}{\partial x_t}(J\nabla S(x_t, y_0))\dot{x_t},$$

so that

$$J\nabla H(t, x_t, y_t) = (\dot{x}_t, \dot{y}_t) = (\tau(x_t, y_0))^{-1} J\nabla S(x_t, y_0), \tag{34}$$

where

$$\tau(x_t, y_0) = \begin{pmatrix} 1 + t\,\partial^2 S/\partial x_t \partial y_0 & 0 \\ -t\,\partial^2 S/(\partial x_t)^2 & 1 \end{pmatrix}.$$

Write $\tau = I + t\sigma$, with $\sigma = O(\delta)$ and $|\nabla\sigma| \le Q$.

Let $\phi : [0, 1] \times \boldsymbol{D} \to \boldsymbol{D} \times \boldsymbol{D}$ by $\phi(t, x_0, y_0) = (x_0, y_0, x_t, y_t)$. Then

$$\frac{\partial\phi}{\partial t} = k \circ \phi, \quad \text{where } k(x_0, y_0, x_t, y_t) = (0, 0, (\tau(x_t, y_0))^{-1} J\nabla S(x_t, y_0)). \tag{35}$$

Then

$$|\nabla k| \le \mu, \quad \text{with } \mu = O(Q\eta + \delta), \quad \text{and} \quad \frac{\partial}{\partial t}\nabla\phi = \nabla k \cdot \nabla\phi.$$

Let $C > 0$. (We have in mind $C \approx 2$.) If

$$|\boldsymbol{v} - \boldsymbol{w}| \le C|\nabla S(\boldsymbol{w})|, \tag{36}$$

then

$$|\nabla S(\boldsymbol{v}) - \nabla S(\boldsymbol{w})| \le \delta C|\nabla S(\boldsymbol{w})|. \tag{37}$$

The quantity $|\nabla S(\boldsymbol{w})|$ is used in (36) and (37) as a measuring scale in two different ways. In (36) it is meant to measure the distance between points $\boldsymbol{w}$ and $\boldsymbol{v}$ relative to the displacement by Φ. By (5), $|\nabla S(\boldsymbol{w})|$ represents the displacement of $\boldsymbol{u}$ by Φ if $\boldsymbol{w} = (x_1, y_0)$ and $\boldsymbol{u} = (x_0, y_0)$. This is somewhat awkward, but note that for small δ the distinction between $\boldsymbol{w}$ and $\boldsymbol{u}$ is relatively unimportant, as

$$|\boldsymbol{u} - \boldsymbol{w}| = |(x_1, y_0) - (x_0, y_0)| \le |(x_1, y_1) - (x_0, y_0)| = |\nabla S(\boldsymbol{w})|.$$

So if δC is small, (36) and (37) say that variations in ∇S are relatively small (compared to $|\nabla S(\boldsymbol{w})|$) in a region centered at $\boldsymbol{w}$, roughly the size of displacement of $\boldsymbol{w}$ by Φ. On the other hand, if ∇S varies by less than (say) 40% over the path $\{(x_t, y_0) \mid 0 \le t \le T \le 1\}$, and if δ is small, then (using (35))

$$\frac{\partial}{\partial t}|\phi - I| \le |(\tau(x_t, y_0))^{-1}|\,|\nabla S(\phi(t)|$$

implies

$$|\phi - I| \le \tfrac{3}{2}|\nabla S(\phi(0)| \quad \text{if } t \le 1.$$

In particular,

$$|(x_t, y_t) - (x_0, y_0)| \le \tfrac{3}{2}|\nabla S(\phi(0)|.$$

Putting all this together we see that if δ is sufficiently small, each orbit of the flow stays relatively close to its starting point. In particular, the orbits of all points in some neighborhood of the origin stay inside the disk where Φ is defined for $0 \le t \le 1$, and there

$$|\nabla\phi - I| \le e^{\mu} - 1 = O(\mu). \tag{38}$$

To see (38), let $w(t) = |\nabla\phi - I|$. Then $dw/dt \le \mu(w + 1)$ and $w(0) = 0$ imply $w(t) \le e^{\mu t} - 1$.

We need to bound $|\nabla g(t, \boldsymbol{w})|$ and $|\partial\nabla g/\partial t|$ relative to $|\nabla S(\boldsymbol{w})|$. By (34), $\nabla H(t, x_t, y_t)$ is relatively close to $\nabla S(x_t, y_0)$ (depending upon δ); by Equation (37), $\nabla S(x_t, y_0)$ is relatively close to $\nabla S(x_t, y_t)$ (also depending upon δ). Together these imply that

$$|\nabla g(t, \boldsymbol{w})| \le O(\delta)|\nabla S(\boldsymbol{w})|. \tag{39}$$

To bound $|\partial\nabla g/\partial t|$, we first find $\partial\nabla H/\partial t$ along the orbit, with x_0 and y_0 fixed. Using (34) and (37) we get

(40)
$$\begin{aligned}\left|\frac{\partial\nabla H}{\partial t}\right|_{x_0,y_0} &= \Big|(I+t\sigma)^{-1}\Big(\sigma+t\frac{\partial\sigma}{\partial x_t}\frac{\partial x_t}{\partial t}\Big)(I+t\sigma)^{-1}J\nabla S(x_t,y_0)\\ &\qquad\qquad + (I+t\sigma)^{-1}J\frac{\partial}{\partial x_t}(\nabla S(x_t,y_0))\Big|\left|\frac{\partial x_t}{\partial t}\right|\\ &\le (1+tO(\delta))\big(O(\delta)+tQ(1+tO(\delta))|\nabla S|\big)|\nabla S|\\ &\qquad\qquad + (1+tO(\delta))O(\delta)(1+tO(\delta))|\nabla S|\\ &\le \big(O(\delta)+QO(\eta)\big)|\nabla S|.\end{aligned}$$

The difference between $|\partial(\nabla H)/\partial t|_{x_0,y_0}$ (the derivative we have in (40)) and

$$|\partial(\nabla g)/\partial t|_{x_t,y_t} = |\partial(\nabla H)/\partial t|_{x_t,y_t}$$

(the derivative we want in (31)) is bounded by

$$\begin{aligned}\left|\frac{\partial\nabla H}{\partial(x_t,y_t)}\right||(\dot{x}_t,\dot{y}_t)| &\le \left|\frac{\partial\nabla H}{\partial(x_t,y_0)}\right|\left|\frac{\partial(x_t,y_0)}{\partial(x_t,y_t)}\right||(\dot{x}_t,\dot{y}_t)|\\ &\le (O(\delta)+QO(\eta))(1+O(\mu))|\nabla S|\end{aligned}$$

by a similar computation (using (38)). Given ε, we saw that δ can be made arbitrarily small by the linear coordinate change (33) followed by a shrinking of the disk. These choices determine Q. Since $|\nabla S|=0$ at the origin, by a further shrinking of the disk, we can make $Q\eta$ small; thus (i).

Condition (iii) can be checked directly; that α is nonnegative can be checked in $\mathrm{Sp}(2,\mathbb{R})$. Condition (iv) is a consequence of (iii). □

4.3. *Expanded characteristic manifold.* In the most degenerate case, the Poincaré map $\gamma(2\pi)$ will be the identity $I\in\mathrm{Sp}(2n,\mathbb{R})$, and the characteristic manifold will be of dimension $2n$, tangent to "the constants" in an appropriate coordinate system. The finite-dimensional approximation $\mathcal{F}$ when $N=0$ gives an explicit characteristic manifold in this case. If the nullity is less than $2n$, the same formula defines a useful expansion of the characteristic manifold to dimension $2n$.

LEMMA 5 (Minimum in the expanded characteristic manifold). *Assume we have a Hamiltonian on the $2n$-disk D with $H(t,0)=\nabla H(t,\mathbf{0})=0$ for all t, and that $\nabla^2H(t,\mathbf{0})$ is of the form $\begin{pmatrix}0&0\\0&-\alpha\end{pmatrix}$, where α is constant diagonal matrix with no negative entries. Assume the critical point* (*of the action*) *at the origin is isolated, and that the restriction of the action to the characteristic manifold has a strict local minimum at the critical point. We define the expanded characteristic manifold $\mathcal{E}$ as the space of period 2π solutions to the differential equation*

$$J\dot{\boldsymbol{w}}+\nabla H(t,\boldsymbol{w}(t))=\text{const.} \tag{41}$$

Then we have the following:

(i) *The expanded characteristic manifold* $\mathscr{E}$ *is a smooth* $2n$*-dimensional submanifold of* Λ *in a neighborhood of the origin.*

(ii) *The tangent space to* $\mathscr{E}$ *at the origin is the space of constant vector fields.*

(iii) *In some neighborhood of the origin, for each fixed* t *the evaluation map from* $\mathscr{E}$ *to* D^{2n} *is a diffeomorphism.*

(iv) *If* $\mathscr{N}$ *is the nullspace of* $\nabla^2 H(t, 0)$, *then*

$$\mathscr{C} : J\dot{\boldsymbol{w}} + \nabla H(t, \boldsymbol{w}(t)) \in \mathscr{N}$$

defines locally a manifold that is characteristic in $\mathscr{E}$ *and is also characteristic in the finite-dimensional approximation* $\mathscr{F}$.

(v) *The reductions* $\mathscr{F}$, $\mathscr{E}$, *and* $\mathscr{C}$ *are compatible in the following sense*: *Starting in any finite-dimensional approximation* $\mathscr{F}$ *with* $N \geq 0$, *we can obtain* $\mathscr{E}$ *and* $\mathscr{C}$ *by successive saddle point reduction*; *the reduction from* $\mathscr{F}$ *to* $\mathscr{E}$ *has index* $2nmN$ *(i.e.,* $\lambda = 2nmN$ *in* (21)), *and the reduction from* $\mathscr{E}$ *to* $\mathscr{C}$ *has index* 0.

(vi) *The restriction of the action to* $\mathscr{E}$ *has a strict local minimum at the origin.*

Remark. The expanded characteristic manifold $\mathscr{E}$ is intermediate between the characteristic manifold construction of Gromoll and Meyer and the finite-dimensional approximation of Amann and Zehnder. It follows from (v) that the respective local homologies of the critical point at the origin in $\mathscr{F}$, $\mathscr{E}$, or $\mathscr{C}$ are related by a shifting theorem. The size of the shift is equal to the index. We foliate Λ_m near the critical point with leaves that are level sets of the average value of $\boldsymbol{u}$. The solutions to (41) are the critical points of the restriction of the action to the leaves. In the case of maximal degeneracy, $\mathscr{E}$ is a characteristic manifold in the sense of Gromoll and Meyer.

Proof. Statement (i) follows from the implicit function theorem: Let $\Phi : \Lambda \times \mathbb{R}^{2n} \to L^2(S^1, \mathbb{R}^{2n})$ by $\Phi(\boldsymbol{w}, \boldsymbol{v}_0) = J\dot{\boldsymbol{w}} + \nabla H(t, \boldsymbol{w}(t)) - \boldsymbol{v}_0$. The derivative is

$$d\Phi(\mathbf{0}, \mathbf{0})(X, Z_0) = J\dot{X} + \nabla^2 H(t, \mathbf{0})X - Z_0.$$

Because of the simple form of $\nabla^2 H(t, \mathbf{0})$, it is easy to solve for the kernel of $d\Phi(\mathbf{0}, \mathbf{0})$ and the nullspace of $D^2\mathscr{A}$ (see (3)) at the critical point, using Fourier series. The nullspace of $D^2\mathscr{A}$ is the space of constant vector fields with values in $\mathscr{N}$. The kernel of $d\Phi(\mathbf{0}, \mathbf{0})$ is the space of (X, Z_0) with X constant and $Z_0 = \nabla^2 H(t, \mathbf{0})X$, which has dimension $2n$. The map $d\Phi(\mathbf{0}, \mathbf{0})$ is surjective: if $Y \in L^2(S^1, \mathbb{R}^{2n})$ is orthogonal to the image, i.e., if

$$\langle d\Phi(\mathbf{0}, \mathbf{0})(X, Z_0), Y\rangle = \int \langle J\dot{X} + \nabla^2 h(t, 0)X - Z_0, Y\rangle dt = 0 \quad \text{for all } X, Z_0,$$

then the average value of Y is 0 (using Z_0), and Y is in the nullspace of $D^2\mathcal{A}$ (using X). Thus $Y = 0$, and $\Phi^{-1}(0)$ is a smooth $2n$-dimensional manifold of Λ. The tangent space to $\mathcal{E}$ at the origin is the space of constant vector fields; thus for each fixed t the evaluation map from $\mathcal{E}$ to T^{2n} is a diffeomorphism.

The second derivative of the action at the origin in $\mathcal{E}$ is nonnegative on constant vector fields, and thus on the tangent space to $\mathcal{E}$. This means the index of the critical point in $\mathcal{E}$ is 0. The construction of $\mathcal{C}$ involves a foliation with leaves that are level sets of the projection of the average value of $\boldsymbol{w}$ onto $\mathcal{N}$. Note the tangent space to the leaf containing the critical point is transverse to the nullspace. The reader can check that in this context, the proof of Gromoll and Meyer of the existence of a characteristic manifold begins (starting in $\mathcal{F}$ or $\mathcal{E}$) by constructing the manifold $\mathcal{C}$. The argument of Gromoll and Meyer (a Morse lemma with parameters) shows that $\mathcal{C}$ is characteristic. Thus we have (iv). Since $\mathcal{E}$ is a saddle point reduction, its local relative homology is the local relative homology of the characteristic manifold, shifted by the index of the critical point. It follows that the critical point in $\mathcal{E}$ is homologically visible in dimension 0, and thus (using again the argument from [Hin93, p. 256]) that the action has a strict local minimum at the origin. □

4.4. *Negative Hamiltonian.*

LEMMA 6. *Assume we have a Hamiltonian on the* $2n$*-disk* $\boldsymbol{D}$ *such that* $H(t,0) = \nabla H(t,0) = 0$ *for all* t. *Assume that* H *is relatively autonomous* (31) *with sufficiently small constant* ε, *that* $|\nabla^2 S| < \delta$ (*Lemma* 4(ii)) *for sufficiently small* δ, *and that the restriction of the action to* $\mathcal{E}$ *as in* (41) *has a strict local minimum at the origin. Then* $H(t,\boldsymbol{w}) < 0$ *for all* t *if* $|\boldsymbol{w}|$ *is sufficiently small and* $\boldsymbol{w} \neq \boldsymbol{0}$.

Remark (of importance). If H is autonomous, then the solutions to (41) are constant. In this case Lemma 6 follows immediately from Lemma 5(vi). We will show that “relatively autonomous" is enough.

Proof. If $\boldsymbol{w} \in \mathcal{E}$, then

$$\dot{\boldsymbol{w}} = J(\nabla S + \nabla g - \boldsymbol{W}) \tag{42}$$

with ∇S constant in space, $\boldsymbol{W}$ constant along orbits, and $|\nabla g|$ small compared to $|\nabla S|$. Our first goal is to establish that $|\dot{\boldsymbol{w}}|$ is also small compared to $|\nabla S|$. On a fixed orbit, $\nabla S = \boldsymbol{K} + \boldsymbol{V}(t)$ with $\boldsymbol{K}$ constant and $|\boldsymbol{V}(t)| \leq \delta|\dot{\boldsymbol{w}}|_{\max}$, where $|\nabla^2 S| < \delta$ everywhere in D, and $|\dot{\boldsymbol{w}}|_{\max}$ is the maximum value of $|\dot{\boldsymbol{w}}|$ on the orbit. Integrating and using the fact that $\boldsymbol{w}$ is a closed curve, we have $|\boldsymbol{K} - \boldsymbol{W}| \leq |\nabla g|_{\max} + \delta|\dot{\boldsymbol{w}}|_{\max}$. At a point on the orbit where $|\dot{\boldsymbol{w}}|$ is maximal, we have

$$\begin{aligned}|\dot{\boldsymbol{w}}|_{\max} = |\boldsymbol{K} - \boldsymbol{W} + \boldsymbol{V}(t) + \nabla g| &\leq 2|\nabla g|_{\max} + 2\delta|\dot{\boldsymbol{w}}|_{\max} \\ &\leq 2\varepsilon|\nabla S|_{\max} + 2\delta|\dot{\boldsymbol{w}}|_{\max},\end{aligned}$$

which implies

$$|\dot{\boldsymbol{w}}|_{\max} \leq \frac{2\varepsilon|\nabla S|_{\max}}{1-2\delta}.$$

If $\delta < \frac{1}{4}$, and ε is sufficiently small, this implies the orbit is small on the scale $|\nabla S|$; using the fact (37) that $|\nabla S|$ is relatively constant, we can conclude that $|\dot{\boldsymbol{w}}| \leq 3\varepsilon|\nabla S|/(1-2\delta)$ at each point. From (42) and (31) we now conclude that $|\nabla S - \boldsymbol{W}| < O(\varepsilon)|\nabla S|$ and thus

$$|\nabla H - W| < O(\varepsilon)|\nabla H|. \tag{43}$$

∇H and $\boldsymbol{W}$ are bounded away from 0 away from the origin since ∇S is.

Fix T, and parametrize $\mathscr{E}$ by evaluation at time T. Let $\boldsymbol{U}(t,\boldsymbol{w})$ be the vector field tangent to $\mathscr{E}$ with $\boldsymbol{U}(T,\boldsymbol{w}) = \nabla H(T,\boldsymbol{w})$. If $\boldsymbol{w}(t)$ is an orbit of the $\mathscr{E}$-flow, with $|\boldsymbol{w}(0)|$ sufficiently small, we claim

$$\int_{\text{orbit}} \langle \boldsymbol{U}(t,\boldsymbol{w}(t)), \boldsymbol{W}(t,\boldsymbol{w}(t))\rangle dt > 0. \tag{44}$$

The restriction of $\boldsymbol{U}$ to an orbit of the $\mathscr{E}$-flow lies in the tangent space to $\mathscr{E}$ and thus is almost constant, since the tangent space to $\mathscr{E}$ at the origin is the space of constant vector fields. (Here we use the fact that $\mathscr{E}$ is smooth, i.e., that it has a continuous tangent space.) Since $\boldsymbol{W}$ is constant on orbits, and $\boldsymbol{W}$ is relatively close to ∇H according to (43), $\boldsymbol{U}$ and $\boldsymbol{W}$ are relatively equal along each orbit near the origin; thus the inner product (44) is positive. It follows, since $\boldsymbol{W}$ is minus the L^2-gradient of the action on Λ, that the action decreases along the integral curves of (the restricted) vector field ∇H on the (time-T-parametrized) $\mathscr{E}$, at a rate bounded away from 0 away from the origin. But since the action has a strict local minimum at the origin in $\mathscr{E}$, it follows that the integral curves of ∇H go to the origin, and thus that $H(T,\boldsymbol{w}) < 0$ if $\boldsymbol{w} \neq 0$. □

5. **Global results**

5.1. *Global coordinate change.* We will compile the local results from Section 4 and paste them onto the torus: Let a periodic Hamiltonian $\mathscr{H}$ be given on the torus T^{2n}, with three continuous derivatives. Assume $\mathscr{A}_{\mathscr{H}}$ has a topologically degenerate critical point $\boldsymbol{u}_0$. By Lemmas 1, 3, 4, 5, and 6, we can make a local coordinate change $S^1 \times D_a^{2n} \to S^1 \times T^{2n}$, with $D_a^{2n} = \{\boldsymbol{w} \mid |\boldsymbol{w}| < a\}$ for some $a > 0$, that takes the constant loop at the origin to $\boldsymbol{u}_0$, and transforms a Hamiltonian H to $\mathscr{H}$, where $H : S^1 \times D^{2n} \to \mathbb{R}$ has

(45) $$H(t,0) = \nabla H(t,0) = 0 \quad \text{for all } t,$$

(46) $$H(t,x,y) = -\sum \alpha_i y_i^2 + \cdots ,$$

(47) H is "relatively autonomous" as in (31) for $\varepsilon = 1$,

(48) $$H(t,\boldsymbol{w}) < 0 \quad for \boldsymbol{w} \neq 0.$$

What follows is an explicit description of this local coordinate change: Define a translation $\psi : S^1 \times D^{2n} \to S^1 \times T^{2n}$, $(t,\boldsymbol{w}) \mapsto (t,\boldsymbol{u})$, where $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$. This translation takes the orbits of $K(t,\boldsymbol{w})$ to those of $\mathcal{K}(t,\boldsymbol{u})$, where $\mathcal{K} = h \,\#\, K$, using the function h satisfies $\nabla h(t,\boldsymbol{v}) = \nabla\mathcal{K}(t,\boldsymbol{u}_0(t))$ for all $\boldsymbol{v}$ and $h(t,\boldsymbol{u}_0(t)) \equiv \mathcal{K}(t,\boldsymbol{u}_0(t))$. Let Φ be the time-2π map of the Hamiltonian K. By Lemma 3, the linearized flow of K satisfies the three parts of Lemma 3, and thus Φ satisfies the hypotheses of Lemma 4. Let $\varphi : D^{2n} \to D^{2n}$ be the linear symplectic map mentioned in Lemma 4. The symplectic transformation $S^1 \times D^{2n} \to S^1 \times D^{2n}$, $(t,\boldsymbol{w}) \mapsto (t, \varphi \circ \boldsymbol{w})$ takes the orbits of $K \circ (\iota \times \varphi)$ to those of K. The time-2π map of $K \circ (\iota \times \varphi)$ is $\varphi^{-1} \circ \Phi \circ \varphi$. Let H be the relatively autonomous Hamiltonian guaranteed by Lemma 4. From Lemma 3(C) and Lemma 4(iv), it will follow that the linearized flows of H and K at their mutual orbit $\boldsymbol{w} \equiv \boldsymbol{0}$ are homotopic, and the hypotheses of Lemma 1 are satisfied. By Lemma 1, there is a 1-parameter family of coordinate changes $\eta_s : S^1 \times D_r^{2n} \to S^1 \times D_r^{2n}$ taking the orbits of H to those of $K \circ (\iota \times \varphi)$ near the origin. Because the local homology at $\boldsymbol{u}_0$ is invariant under these coordinate changes, the transformed critical point is still topologically degenerate, and the conclusions of Lemma 3 still hold. Thus the hypotheses of Lemma 5 and then of Lemma 6 hold for H, and H has a strict local maximum at the origin for each fixed t. Thus (45)–(48) hold for some $a > 0$. In order to complete the proof, we will need to extend this local coordinate change to a global coordinate change $\Xi : S^1 \times T^{2n} \to S^1 \times T^{2n}$ that has the effect that a translation of $\boldsymbol{u}_0$ back to the origin by h now results in a coordinate system in which the Hamiltonian $H : S^1 \times D_a^{2n} \to \mathbb{R}$ is as above in some neighborhood D_a^{2n} of the origin. Let $\varphi : D^{2n} \to D^{2n}$ be a symplectic diffeomorphism that is the linear symplectic map mentioned in Lemma 4 near the origin, and is the identity outside a small disk D_R^{2n}. (That such a map exists follows from an argument similar to that used in the proof of Lemma 1.) The 1-parameter family of coordinate changes from Lemma 1 can also be assumed to have support on D_R^{2n}. The local coordinate change $\psi^{-1} \circ \eta_1 \circ (\times) \circ \psi$ is the identity near the boundary of $\psi(S^1 \times D_R^{2n})$ and thus can be extended by the identity map to a global coordinate change $\Xi : S^1 \times T^{2n} \to S^1 \times T^{2n}$ with the desired properties. The transformed Hamiltonian will be

$$\mathcal{H}(t,\boldsymbol{u}) = \begin{cases} h \,\#\, H(t,\boldsymbol{u}) & \text{if } \boldsymbol{u}(t) - \boldsymbol{u}_0(t) \in D_R^{2n}, \\ \mathcal{K}(t,\boldsymbol{u}) & \text{otherwise.} \end{cases}$$

Ξ maps the orbits of the Hamiltonian flow of $\mathscr{H}$ to the orbits of the Hamiltonian flow of $\mathscr{K}$ in the sense of (13) and (14). Thus if the flow of $\mathscr{H}$ has subharmonic solutions of arbitrarily high minimal period, the same is true for $\mathscr{K}$.

Remark. The map $\Lambda_m \to \Lambda_m$ induced by the coordinates change Ξ fixes $\boldsymbol{u}_0$. The loop $\boldsymbol{u}_0$ is an isolated, topologically degenerate critical point for both $\mathscr{A}_{\mathscr{K}}$ and $\mathscr{A}_{\mathscr{H}}$. The coordinate change stretches (33) a neighborhood of $\boldsymbol{u}_0(0)$ in $\{0\} \times T^{2n}$ in order to make the shear term α in the linear time-2π map (32) small, and straightens the flow in a neighborhood of $\boldsymbol{u}_0$, using η_s.

We pause to demonstrate that we have nothing up our sleeve. For the rest of the proof we will use the Hamiltonian $\mathscr{H}$ and the coordinates

$$(t, \boldsymbol{u}) \circ \Xi \quad \text{on } S^1 \times T^{2n} \quad \text{and} \quad (t, \boldsymbol{w}) \circ \Xi \quad \text{on } S^1 \times D^{2n},$$

where $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_0$. The constants $(\delta, \varepsilon, Q, N)$ in the proof so far have served their purpose in producing a negative Hamiltonian, and will not appear again (though we may use these letters again in a new context). The only hypothesis for the remainder of the proof is the existence of the Hamiltonian $\mathscr{H}$ with a topologically degenerate critical point $\boldsymbol{u}_0$ and the property that translation of $\boldsymbol{u}_0$ back to the origin results in a coordinate system on $S^1 \times D_a^{2n}$ for some $a > 0$ in which the Hamiltonian $H : S^1 \times D_a^{2n} \to \mathbb{R}$ satisfies (45)–(48). Let $a_0 > 0$ be such that H is as above if $a < a_0$.

The final step is to show that $H(t, \boldsymbol{w}) < 0$ (locally) implies the existence of a "molar". The existence of subharmonic solutions of arbitrarily high minimal period will follow immediately by a standard argument of finite-dimensional Morse theory.

5.2. *Shifted Fourier series.* In our new global coordinates, we begin by taking a finite-dimensional approximation (see (23) and (24)) again. (After our coordinate change we will need to recalculate N; we need $N > 2|\nabla^2 \mathscr{H}|$. We will also need later that δ/N is small, where δ is a (new!) bound for $|\nabla^2 H|$ in D^{2n}.) Using the standard Fourier foliation, we produce $\mathscr{F}_m$ just as before.

The loop space Λ_m contains two submanifolds of dimension $2n(2mN + 1)$: $Z_m = F_m/\mathbb{Z}^{2n}$(the image in Λ_m of Fourier series of order $\leq N$) and $\mathscr{F}_m$, the finite-dimensional approximation space. Each is transverse to the leaves of the standard Fourier foliation; moving along a leaf gives the map (26). In what follows, we will use the standard foliation and the standard $\mathscr{F}_m$. However, we will replace Z_m by the shifted space

$$\hat{Z}_m = \{\boldsymbol{u} \mid \boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_0, \text{ with } \boldsymbol{w} \in Z_m\} \subset \boldsymbol{\Lambda}_m.$$

That is, we will expand the loops in Λ_m in Fourier series centered at $\boldsymbol{u}_0$. Just as before, if $N > 2|\nabla^2 \mathscr{H}|$ independent of m, then each leaf will contain a unique point in $\mathscr{F}_m$ (the unique critical point of the restriction of the action to the leaf), and a unique point in $\hat{Z}_m$; let $\hat{\xi} : \hat{Z}_m \to \mathcal{F}_m$ denote this correspondence. The critical points of the action on Λ_m are of course still precisely the critical points of the restriction of the action to $\mathcal{F}_m$.

Fix $a < a_0$. Let $B_m = B_m(a)$ be the ball of loops in F_m that lie inside the $2n$-disk of radius a, and let Δ_m be the boundary of B_m. We will identify B_m with its image in $\hat{Z}_m$:

$$B_m \hookrightarrow \hat{Z}_m : \boldsymbol{w} \mapsto \boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_0.$$

When $|\boldsymbol{w}|$ is small, we have by (18)

$$\mathcal{D}_{\mathcal{H}}(\boldsymbol{u})(t) = \mathcal{D}_H(\boldsymbol{w})(t) = J(\dot{\boldsymbol{w}}(t)) + \nabla H(t, \boldsymbol{w}(t)).$$

The action on $\hat{Z}_m \subset \Lambda_m$ in these local coordinates is, by (4) and (19),

$$\mathcal{A}_{\mathcal{H}}(\boldsymbol{u}) = \mathcal{A}_H(\boldsymbol{w}) = \mathcal{A}_0(\boldsymbol{w}) - \overline{H}(\boldsymbol{w}), \qquad \text{where } \mathcal{A}_0(\boldsymbol{w}) = \pi \sum k|\alpha_k|^2 \tag{49}$$
$$\text{and } \overline{H}(\boldsymbol{w}) = \textstyle\int H(t, \boldsymbol{w}(t))dt$$

with $\{\alpha_k\}$ the Fourier coefficients of $\boldsymbol{w}$.

Because of the shift, and because of our choice of coordinates, the submanifolds $\hat{Z}_m$ and $\mathcal{F}_m$ are tangent at $\boldsymbol{u}_0$. It will come out below that the local homology of $\mathcal{A}_{\mathcal{H}}$ at $\boldsymbol{u}_0$ in $\hat{Z}_m$ is the same as the local homology of $\mathcal{A}_{\mathcal{H}}$ at $\boldsymbol{u}_0$ in $\mathcal{F}_m$. Thus $\hat{Z}_m$ is locally an “approximation" for $\mathcal{F}_m$. Alternatively, $\mathcal{A}_{\mathcal{H}} \circ \hat{\xi}^{-1}$ is locally an approximation for $\mathcal{A}_{\mathcal{H}}$ on $\mathcal{F}_m$. We will produce the molar in two steps: First, in Lemma 7 we show for large m that $\mathcal{A}_{\mathcal{H}}$ has a molar $(\Sigma_m, \Sigma_m^{\geq})$ in $\hat{Z}_m$ at $\boldsymbol{u}_0$. Next, in Lemma 8 we will show that $(\hat{\xi}(\Sigma_m), \hat{\xi}(\Sigma_m^{\geq}))$ is a molar for $\mathcal{A}_{\mathcal{H}}$ in $\mathcal{F}_m$ at $\boldsymbol{u}_0$.

5.3. *Local molar in $\hat{Z}_m$.* Here is one more local result. With (45)–(48), let $H : S^1 \times D^{2n} \to \mathbb{R}$ We start with a splitting

$$F_m = F_m^{\geq} + F_m^{<},$$

where the second derivative of the action (49) is ≥ 0 on $F_m^{\geq}$ and < 0 on $F_m^{<}$, and a vector subspace V_m of F_m containing $F_m^{\geq}$ as a subspace of codimension 1: The vector space $F_m^{\geq}$ consists of (x, y) with

$$F_m^{\geq} : (x_j + iy_j)(t) = \sum_{0 \leq k \leq mN} \rho_k e^{ikt/m} \quad \text{for } 1 \leq j \leq n,$$

and $F_m^{<}$ consists of (x, y) with

$$F_m^{<} : (\lambda x_j + i\lambda^{-1} y_j)(t) = \sum_{-mN \leq k < 0} \rho_k e^{ikt/m} \quad \text{for } 1 \leq j \leq n,$$

with $\lambda < 1$ chosen (depending on m) so that the second derivative of the action is < 0 on $F_m^{<}$. (This is possible since the enclosed area $\mathcal{A}_0$ is independent of λ, but $\overline{H} \to 0$ as $\lambda \to 0$. Note λ is introduced only to demonstrate that $F_m^{\geq}$ is a maximal nonnegative subspace in the proof of Lemma 6(A) below, and will not appear again.) V_m is the subspace of F_m spanned over $F_m^{\geq}$ by $\boldsymbol{v}_* = (x, y)$ with $(x_1 + iy_1) = e^{-it/m}$ and $(x_j + iy_j)(t) = 0$ for $j > 1$. Let

$$\Sigma_m = V_m \cap B_m \subset \hat{Z}_m.$$

Then the boundary $\partial\Sigma_m$ of Σ_m lies in Δ_m. Let $\Sigma_m^{\geq} = \Sigma_m \cap F_m^{\geq}$.

LEMMA 7. *If m is sufficiently large*, $(\Sigma_m, \Sigma_m^{\geq})$ *is a* (C^0*-local*) molar *in* $\hat{Z}_m$; *that is* (*see Figure* 1):

(A) the surface $\Sigma_m^{\geq}$ represents a nontrivial class in the upside-down local homology $H_n(\mathscr{A}_H^0, \mathscr{A}_H^0 \setminus \{0\})$, where $\mathscr{A}_H^0 = \mathscr{A}_H^{-1}[0,\infty) \subset F_m$;

(B) we have $\mathscr{A}_H \geq -\pi a^2$ on Σ_m;

(C) if m is sufficiently large, $\mathscr{A}_H > 0$ on $\partial\Sigma_m$.

Apology. The author sincerely apologizes to the reader for using upside-down homology. It does avoid several other very awkward signs.

Proof. From the explicit expression (49), note that $-\overline{H}(\boldsymbol{w}) > 0$ unless $\boldsymbol{w} \equiv 0$. Also $\mathscr{A}_0(\boldsymbol{w}) \geq 0$ on $F_m^{\geq}$. Thus the action will have a strict local minimum at the origin in $F_m^{\geq}$; (A) follows from the stated properties of the splitting $F_m = F_m^{\geq} + F_m^{<}$.

Let $\boldsymbol{w}(t) = \alpha\boldsymbol{v}_*(t) + (x(t), y(t))$ be in $\partial\Sigma_m$, where

$$x_j + iy_j = \sum_{0 \leq k \leq mN} \alpha_k e^{ikt/m} \quad \text{for } \alpha \in \mathbb{R} \text{ and } \alpha_k \in C. \tag{50}$$

Note $\boldsymbol{w} \in \partial\Sigma_m$ implies that $\|\boldsymbol{w}\|_{\sup} = a$; thus $\int |\boldsymbol{w}(t)|^2 dt \leq \int a^2 dt = 2m\pi a^2$. Since (from (50))

$$\int_0^{2m\pi} |\boldsymbol{w}(t)|^2 dt = 2m\pi(\alpha^2 + \sum |\alpha_k|^2),$$

we have $\alpha^2 \leq a^2$ and thus (B).

To prove (C), we consider three cases: (i) good second derivative, (ii) small energy, and (iii) large L^2 norm. In cases (ii) and (iii), we need to show that the Hamiltonian term $-\overline{H}(\boldsymbol{v})$ is positive enough to make up for the deficit caused by the one negative term $-\alpha^2$ coming from $\alpha\boldsymbol{v}_*$.

(i) If $\alpha^2 < \sum k|\alpha_k|^2$, then $\mathscr{A}_0(\boldsymbol{w}) > 0$, and $-\overline{H}(\boldsymbol{w}) \geq 0$, so $\mathscr{A} > 0$.

(ii) Let $\varepsilon > 0$, and suppose $\alpha^2 < \varepsilon^2$ and $\sum k|\alpha_k|^2 < 2\varepsilon^2$. The maximum possible value of the energy is then (using e.g. Lagrange multipliers)

$$E := \int_0^{2m\pi} |\dot{\boldsymbol{w}}|^2 dt = \frac{2\pi}{m}(\alpha^2 + \sum k^2|\alpha_k|^2) \leq 4\pi\varepsilon^2(N + 1/m).$$

This, together with $\|\boldsymbol{w}\|_{\sup} = a$, implies that $|\boldsymbol{w}| \geq a/2$ on a set of measure at least $L = a^2/4\pi\varepsilon^2 N$. (See Figure 2; it is well known that the path covering a

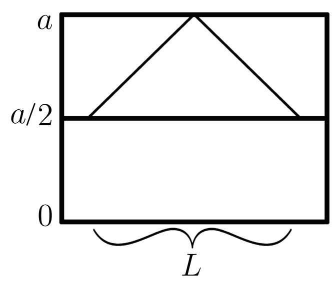

Figure 2. The small energy case: $|\dot{\boldsymbol{v}}| = \text{speed} = a/L$ implies Energy $= \int |\dot{\boldsymbol{v}}|^2 = (a/L)^2 L$. Then Energy $\leq 4\pi\varepsilon^2 N$ implies $L \geq a^2/(4\pi\varepsilon^2 N)$.

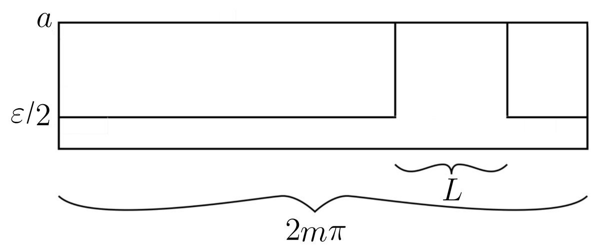

Figure 3. The large L^2 norm case: $\int_0^{2m\pi} |v|^2 dt = La^2 + (2m\pi - L)\varepsilon^2/4 \geq 2m\pi\varepsilon^2$ implies $L \geq 3m\pi\varepsilon^2/(2(a^2 - \varepsilon^2/4)$.

fixed distance in a fixed time at minimal energy has constant velocity; we leave it to the reader to check this implies that, for fixed energy, the minimal elapsed time occurs for constant velocity.) If ε is sufficiently small, this will imply (since $H < 0$) that $-\overline{H}(\boldsymbol{w}) > 2\pi a^2$; together with (A), this implies $\mathcal{A}(\boldsymbol{w}) > 0$. We now so fix ε; assume also $\varepsilon < a$.

(iii) Given $\varepsilon > 0$ (chosen in (ii)), there exists a $\mu > 0$ such that

$$-\overline{H}(\boldsymbol{w}) \geq \mu m \quad \text{if } \alpha^2 + \sum |\alpha_k|^2 \geq \varepsilon^2. \tag{51}$$

The reason is that if $\alpha^2 + \sum |\alpha_k|^2 \geq \varepsilon^2$ and $\|\boldsymbol{w}\|_{\text{sup}} = a$, then $|\boldsymbol{w}| \geq \varepsilon/2$ on a set of measure at least

$$L = \frac{3m\pi}{2} \frac{\varepsilon^2}{a^2 - \varepsilon^2/4}$$

(see Figure 3), which is proportional to m. Now use $H < 0$. Thus (using (51)), $\mathcal{A}(\boldsymbol{w}) > 0$ if m is sufficiently large (depending upon a) and if $\alpha^2 + \sum |\alpha_k|^2 \geq \varepsilon^2$.

Together, (i), (ii), and (iii) imply (C).

In fact looking a little more closely, we see that, given ε positive but sufficiently small, we can make

$$-\overline{H}(\boldsymbol{w}) \geq 2\pi a^2 \tag{52}$$

for m sufficiently large unless

$$\sum k|\alpha_k|^2 \geq 2\varepsilon^2 \text{ and } \alpha^2 + \sum |\alpha_k|^2 < \varepsilon^2. \qquad \square$$

5.4. *The molar*: *Global version.* Lemma 7 implies the existence of a critical point of the restriction of the action to $\hat{Z}_m$ with critical value in $(-\pi a^2, 0)$. But we have a correspondence between critical points on $\mathcal{F}_m$ and those on Λ_m, not between critical points on $\hat{Z}_m$ and those on Λ_m. The next lemma shows that there is also a "molar" in the finite-dimensional approximation space $\mathcal{F}_m$.

LEMMA 8. *Let a periodic Hamiltonian $\mathcal{H}$ be given on the torus T^{2n}, with three continuous derivatives. Assume that there is a topologically degenerate critical point u_0 with the property that translation of u_0 back to the origin results in a coordinate system on $S^1 \times D_a^{2n}$ for some $a > 0$ in which the Hamiltonian $H : S^1 \times D_a^{2n} \to \mathbb{R}$ satisfies* (45)–(48). *If m is sufficiently large*, $(\hat{\xi}(\Sigma_m), \hat{\xi}(\Sigma_m^{\geq}))$ *is a* molar *in $\mathcal{F}_m$*; *that is*,

(A′) *$\mathcal{A}_{\mathcal{H}} \geq 0$ on $\hat{\xi}(\Sigma_m^{\geq})$, and $\hat{\xi}(\Sigma_m^{\geq})$ represents a nontrivial class in the upside-down local homology*

$$H_n(\mathcal{A}_{\mathcal{H}}^0, \mathcal{A}_{\mathcal{H}}^0 \setminus \{0\}), \qquad \textit{where } \mathcal{A}_{\mathcal{H}}^0 = \mathcal{A}_{\mathcal{H}}^{-1}[0,\infty) \subset \mathcal{F}_m;$$

(B′) *$\mathcal{A}_{\mathcal{H}} > -2\pi a^2$ on $\widehat{\xi}(\Sigma_m)$*;

(C′) *if m is sufficiently large*, *$\mathcal{A}_{\mathcal{H}} > 0$ on $\hat{\xi}(\partial\Sigma_m)$*.

COROLLARY 9. *If a is sufficiently small, and m is sufficiently large* (*depending upon a*), *then $\mathcal{F}_m$* (*and thus Λ_m*) *has a critical point of $\mathcal{A}_{\mathcal{H}}$ with critical value in* $(-2\pi a^2, 0)$.

Proof. We will use the coordinate $\boldsymbol{u}$ and the Hamiltonian $\mathcal{H}$ on $\mathcal{F}_m$, since we can no longer work locally. (Remember though that if $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$, with $|\boldsymbol{w}|$ small, then $\mathcal{H}(\boldsymbol{u}) = H(\boldsymbol{w})$.)

(A′): That $\mathcal{A} \geq 0$ on $\widehat{\xi}(\Sigma_m^{\geq})$ will come out below. Together with the facts that $\hat{\xi}(\Sigma_m^{\geq})$ is embedded at $\boldsymbol{u}_0$ and that $\hat{\xi}(\Sigma_m^{\geq})$ has dimension equal to the index plus nullity of the upside-down critical point, this implies that the class $\hat{\xi}(\Sigma_m^{\geq})$ is nontrivial.

Proof of (B′) and (C′): Let $\boldsymbol{u} \in \Lambda_m$. The leaf containing $\boldsymbol{u}$ is the set of all points of the form

$$\boldsymbol{v}(t) = \boldsymbol{u}(t) + \sum_{k > mN} \gamma_k e^{ikt/m} + \sum_{k < -mN} \sigma_k e^{ikt/m}$$

(using complex notation). We are interested in the difference between the action $\mathcal{A}(\boldsymbol{u})$ and the action at the critical point of the restriction of the action to the leaf. The latter action, which we will call $\mathcal{A}_C(\boldsymbol{u})$, is given by

$$\mathcal{A}_C(\boldsymbol{u}) = \sup_{\{\sigma_k\}} \inf_{\{\gamma_k\}} \mathcal{A}(\boldsymbol{v}).$$

Quite clearly,

$$\mathcal{A}_C(\boldsymbol{u}) \geq \mathcal{A}_C^+(\boldsymbol{u}) = \inf_{\{\gamma_k\}} \mathcal{A}(\boldsymbol{v}).$$

Thus in order to find a lower bound for $\mathcal{A}_C(\boldsymbol{u})$, it will be sufficient to find a lower bound for $\mathcal{A}_C^+(\boldsymbol{u})$, the critical point of the restriction of the action to the leaf of the "positive" foliation with leaf consisting of points of the form

$$v(t) = \boldsymbol{u}(t) + \sum_{k>mN} \gamma_k e^{ikt/m}.$$

Consider a path

$$\boldsymbol{u}(s,t) = \boldsymbol{u}(t) + s \sum_{k>mN} \gamma_k e^{ikt/m} \tag{53}$$

along the leaf of the positive foliation. Along this path,

$$\frac{d^2\mathcal{A}}{ds^2} = 2\pi \sum k_{k>mN} |\gamma_k|^2 + \int \nabla^2 H\left(\sum \gamma_k e^{ikt/m}, \sum \gamma_k e^{ikt/m}\right) dt. \tag{54}$$

If $N > 2|\nabla^2 H|$, then

$$\frac{d^2\mathcal{A}}{ds^2} \geq \sum k_{k>mN} |\gamma_k|^2. \tag{55}$$

(This explains the requirement (22); if $N > 2\pi|\nabla^2 H|$, then each leaf has a unique critical point.) This means that we can estimate the minimum value of $\mathcal{A}$ along the path, using the value of $d\mathcal{A}/ds$ at the initial point. This value we will compute in local coordinates. The path begins at a point $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$. Assume that $\boldsymbol{w} \in \Sigma_m$. In local coordinates, the path (53) is given by

$$\boldsymbol{w}(s,t) = \boldsymbol{w}(t) + s \sum_{k>mN} \gamma_k e^{ikt/m}. \tag{56}$$

Using the form for $\mathcal{A}$ (49) in local coordinates, it follows that for $s \geq 0$,

$$\mathcal{A}(s) \geq \mathcal{A}(0) + \tfrac{1}{2} s^2 \sum k_{k>mN} |\gamma_k|^2 - 2\pi s m \sum \mu_k \gamma_k, \tag{57}$$

where $\mu_k \in \mathbb{C}^n$ are the Fourier coefficients of $\nabla H(\boldsymbol{w})$ for $k > mN$. (Note the left and right sides of (57) have the same derivative with respect to s at $s = 0$; we have used $\boldsymbol{w} \in \Sigma_m$, which simplifies the derivative.) It follows (solving to find the sequence $\{\gamma_k\}$ that gives the least value for the minimum value of $\mathcal{A}(s)$ along the path (56)) that

$$\mathcal{A}(s) \geq \mathcal{A}(0) - \frac{2\pi^2}{N^2} \sum_{k>mN} k|\mu_k|^2.$$

But, using $|\nabla^2 H| < \delta$,

$$
\begin{aligned}
(58)\quad \sum_{k>mN} k|\mu_k|^2 &\leq \frac{1}{mN} \sum_{k>mN} k^2|\mu_k|^2 \\
&\leq \frac{1}{2\pi N} \int \left|\frac{d}{dt}\nabla H(t, \boldsymbol{w}(0,t))\right|^2 dt \\
&\leq \frac{2}{2\pi N} \int \left(|\nabla^2 H|^2 \left|\frac{\partial \boldsymbol{w}}{\partial t}\right|^2 + \left|\frac{\partial}{\partial t}\nabla H(t, \boldsymbol{w}(0,t))\right|^2 \right) dt \\
&\leq \frac{2\delta^2}{2\pi N}\frac{2\pi}{m}\Big(\alpha^2 + \sum_{1\leq k\leq mN} k^2|\alpha_k|^2\Big) + \frac{2}{2\pi N}\int |\nabla H(t, \boldsymbol{w}(0,t))|^2)dt \\
&\leq 2\delta^2\Big(\frac{\alpha^2}{mN} + \sum_{1\leq k\leq mN} k|\alpha_k|^2\Big) + \frac{2}{2\pi N}\int 2\delta|H(t, \boldsymbol{w}(0,t))|dt.
\end{aligned}
$$

In step (58) we used the fact that H is relatively autonomous, with $\varepsilon = 1$. In the last step we get $|\nabla H(t, \boldsymbol{w}(0,t))|^2 \leq 2\delta|H(t, \boldsymbol{w}(0,t))|$ by looking, along an integral curve of $\nabla|H|$ starting near the origin, at the comparison equation $d^2w/ds^2 \leq \delta$, $w(0) \approx 0$, $dw/ds(0) \approx 0$, which has $dw/ds \leq \sqrt{2\delta w}$. (Q: If your maximum acceleration is δ, what do you do to attain a given speed in the least distance? A: Floor it.) Note the last term is equal to $\delta(-\overline{H}(0))/\pi N$.

Putting everything together, we have

$$
\begin{aligned}
(59)\quad \mathcal{A}(s) \geq -\pi\alpha^2\Big(1 + \frac{4\pi\delta^2}{mN^3}\Big) - \overline{H}(0)\Big(1 - \frac{4\pi\delta}{N^3}\Big) \\
+ \Big(\pi \sum_{1\leq k\leq mN} k|\alpha_k|^2\Big)\Big(1 - \frac{4\pi}{N^2}\delta^2\Big).
\end{aligned}
$$

(Note each of the three terms has the form (Term from $\mathcal{A}(0)$)$(1 - O(\delta/N)$.) Assume that N is large enough so that each of the terms $O(\delta/N)$ is $< 1/4$. Then (A′) and (B′) are clear from (59). Now assume that $\boldsymbol{w} \in \partial\Sigma$. In the case $\sum k|\alpha_k|^2 \geq 2\varepsilon^2$ and $\alpha^2 + \sum|\alpha_k|^2 < \varepsilon^2$, $\mathcal{A}(s)$ is clearly positive. Otherwise use (52) to conclude that $-\overline{H}(0) \geq 2\pi a^2$, so that $\mathcal{A}(s) > 0$. □

Proof of Corollary 9. $\Sigma_m^{\geq}$ can be deformed to the boundary in B_m. Thus the nontrivial class of $\hat{\xi}(\Sigma_m^{\geq})$ in $H_n(\mathcal{A}^0, \mathcal{A}^0 \setminus \{0\})$ has trivial image in $H_n(\mathcal{A}^{-\pi a^2}, \mathcal{A}^0 \setminus \{0\})$. Under the assumption that Λ_m contains only isolated critical points, a maximin argument in (B, B^+) ((29), [HZ94]) shows that the action has a critical value in $(-\pi a^2, 0)$. □

Proof of the proposition (and thus of the theorem). Let $\boldsymbol{u}_0 \in \Lambda_1$ be an isolated, topologically degenerate critical point. Assume $\mathscr{A}(\boldsymbol{u}_0) = 0$. If a is such that the action on Λ_1 has no critical value in $(-\pi a^2, 0)$, then for every sufficiently large prime p, there will be a critical point in Λ_p with critical value in $(-\pi a^2, 0)$, which must clearly have *minimal* period $2p\pi$. □

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