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Exponential growth and an asymptotic formula for the ranks of homotopy groups of a finite 1-connected complex

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Dedicated to J.-L. Koszul for his 87th birthday

Abstract

Let X be an n-dimensional, finite, simply connected CW complex and set $\alpha_X = \limsup_i (\log\ \operatorname{rank} \pi_i(X))/i$. We prove that either $\operatorname{rank} \pi_i(X) = 0$, $i \geq 2n$, or else that $0 < \alpha_X < \infty$ and that for any $\varepsilon > 0$ there is a $K = K(\varepsilon)$ such that

$$e^{(\alpha_X - \varepsilon)k} \leq \sum_{i=k+2}^{k+n} \operatorname{rank} \pi_i(X) \leq e^{(\alpha_X + \varepsilon)k}, \quad \text{for all } k \geq K.$$

In particular, this sum grows exponentially in k.

1. **Introduction**

The homotopy groups of a finite simply connected CW complex, X, have the form $\pi_i(X) = \mathbb{Z}^{\rho_i} \oplus T_i$, where ρ_i is the *rank* of $\pi_i(X)$ and T_i is a finite abelian subgroup. While the properties of the T_i remain by and large a mystery, even for spheres, considerable information is available about the ranks. In fact the ranks of $\pi_i(S^n)$ and of $\pi_i(S^n \vee S^n)$ are simply the dimensions in degree $i-1$ of the free Lie algebra on one (resp. two) generators of degree $n-1$ and, in particular, satisfy

$$(1)\qquad \begin{aligned} &\operatorname{rank} \pi_i(S^n) = 0, \qquad \text{for all } i \geq 2n, \quad \text{while} \\ &\sum_{i=k+2}^{k+n} \operatorname{rank} \pi_i(S^n \vee S^n) \text{ grows exponentially in } k. \end{aligned}$$

(Note that we use the sum $\sum_{i=k+2}^{k+n}$ in (1) because $\operatorname{rank} \pi_i(S^n \vee S^n) = 0$ unless $i \equiv 1 \pmod{n-1}$.)

The principal result here is a remarkable asymptotic property for the ranks, which in particular generalizes (1) to all finite simply connected CW complexes.

Definition. If X is a finite simply connected CW complex, then its *log index*, α_X, is the number given by

$$\alpha_X = \limsup_i \frac{\log \ \operatorname{rank} \pi_i(X)}{i} .$$

THEOREM 1. *If X is an n-dimensional, finite, simply connected* CW *complex, then either $\alpha_X = -\infty$ and*

$$\operatorname{rank} \pi_i(X) = 0 , \qquad i \geq 2n ,$$

or else $0 < \alpha_X < \infty$, and for every $\varepsilon > 0$ there is a $K = K(\varepsilon)$ such that

$$e^{(\alpha_X - \varepsilon)k} \leq \sum_{i=k+2}^{k+n} \operatorname{rank} \pi_i(X) \leq e^{(\alpha_X + \varepsilon)k}, \qquad \text{for all } k \geq K .$$

In particular this sum grows exponentially in k.

In fact a slightly weaker version of Theorem 1 (Theorem 4 in §5) holds more generally for simply connected CW complexes Y of finite type and finite Lusternik-Schnirelmann rational category, whose betti numbers $\dim H_k(Y;\mathbb{Q})$ grow at most exponentially in k. Such spaces are called elliptic if the total rank, $\sum_{i=2}^{\infty} \operatorname{rank} \pi_i(Y)$, is finite, and hyperbolic otherwise.

Now it is known, [12], that if Y is elliptic then Y has the rational homotopy type of a finite complex; in particular $H_i(Y;\mathbb{Q}) = 0$, $i > n$, for some n. Moreover in the elliptic case, it is shown in [12] that, if Y is finite, then $\operatorname{rank} \pi_i(Y) = 0$, $i \geq 2n$ and an explicit algorithm is given that determines all possible sequences $\{\operatorname{rank} \pi_i(Y)\}_{2 \leq i \leq 2n-1}$.

On the other hand if Y is hyperbolic then ([3], [5], [7]) it is shown that for some $\lambda, K > 0$,

$$\sum_{i=2}^{k} \operatorname{rank} \pi_i(Y) \geq e^{\lambda k} , \quad \text{for all } k > K .$$

Replacing this by the much stronger asymptotic formulas of Theorems 1 and 4 is the object of this paper.

Exponential growth of the sum $a_k = \sum_{i=k+2}^{k+n} \operatorname{rank} \pi_i(X)$, was conjectured in the early eighties [7, Prob. 6] and Theorem 1 was first proved for some special families of spaces by Lambrechts [14]. The strongest general result prior to Theorem 1 was given in [8], where it is shown that the a_k grow faster than any polynomial in k. More recently further examples have been given of families for which Theorem 1 holds in [9], [10] and [11].

As for the proof, Theorem 1 is essentially a corollary of Theorem 4, and the key step in the proof of Theorem 4 is a remarkable growth property of certain graded Lie algebras whose proof in Section 3 (Theorem 2) and Section 4 (Theorem 3) is the heart of the paper. It is preceded by preliminary definitions and remarks in Section 2.

Since Theorem 1 may be thought of as establishing a strong 'regularity' property for the ranks of the homotopy groups, it seems reasonable to ask the

Question. Are there 'regularity' properties as $i \to \infty$ of the torsion subgroups T_i of the homotopy groups of a finite simply connected CW complex X?

Finally, the authors would like to thank the referee for his careful reading and many helpful questions and comments.

2. **Growth and depth in graded Lie algebras**

In this section we work over any field $\Bbbk$ of characteristic $\neq 2$. The dual of a graded vector space $V = \{V_k\}$ is denoted by $V^\#$, $V_k^\# = \mathrm{Hom}\,(V_{-k}, \Bbbk)$, and V is *connected* (resp. of *finite type*) if $V_k = 0$, $k \leq 0$ (resp. if each V_k is finite dimensional). Note that our definition of a connected graded vector space differs from that sometimes applied elsewhere, in which $V_0 = \Bbbk$. We denote by $V_{[k,l]}$, $V_{(k,l)}$, $V_{\geq k}$ and $V_{\leq k}$ the graded subspaces $\{\, V_i \mid k \leq i \leq l \,\}$, $\{\, V_i \mid k < i < l \,\}$, $\{\, V_i \mid i \geq k \,\}$ and $\{\, V_i \mid i \leq k \,\}$ respectively.

If V is connected and of finite type we define

$$\log \mathrm{index}\, V = \limsup_k \frac{\log \dim V_k}{k}\,.$$

It is obvious that log index $V < \infty$ if and only if for some $\mu > 0$, $\dim V_k \leq e^{\mu k}$, for all k. In this case we say that V *grows at most exponentially*.

A *graded Lie algebra*, L, is a graded vector space equipped with a bilinear bracket $[\ ,\ ] : L \otimes L \to L$ of degree zero and satisfying

$$[x, y] = -(-1)^{\deg x \cdot \deg y}[y, x]$$

and

$$[[x, [y, z]] = [[x, y], z] + (-1)^{\deg x \cdot \deg y}[y, [x, z]].$$

If $\mathrm{char}\,\Bbbk = 3$ we also require $[x, [x, x]] = 0$ when $\deg x$ is odd. (This last condition is automatic for char $\Bbbk \neq 3$.) A cft *graded Lie algebra* is a graded Lie algebra that, as a graded vector space, is connected and of finite type.

The universal enveloping algebra of a graded Lie algebra L is, as usual, denoted UL and a (graded) left UL-module will be called simply an L-module. In particular the adjoint representation makes L into an L-module, and we denote the UL action by $(a, x) \mapsto a \circ x$, $a \in UL$, $x \in L$.

LEMMA 1. *Given δ and $\beta > 0$, there exists an integer $N(\delta,\beta)$ such that: if F is a* cft *graded Lie algebra satisfying $F_k = 0$, for all $k < N(\delta,\beta)$, and* $\dim(F/[F,F])_k \le e^{k\beta}$, *for all k, then* $\dim(UF)_k \le e^{(\beta+\delta)k}$, *for all $k \ge 0$.*

Proof. First note that the function $1 - x - x^n$ has a single zero $\lambda(n)$ in $(0,1)$ and that $\lim_{n\to\infty}\lambda(n) = 1$. Choose $K = K(\delta)$ so that $\lambda(K) > e^{-\delta/2}$.

Now let E be the free graded Lie algebra generated by a graded vector space, W, with $W_k = 0$, for all $k < K$ and $\dim W_k$ the integral part of $e^{k\beta}$, for all $k \ge K$. Then UE is the tensor algebra, TW. It follows that the respective Hilbert series satisfy ($\ll$ denotes coefficientwise inequality)

$$W(z) \ll (ze^{\beta})^K \sum_{k=0}^{\infty} (ze^{\beta})^k = \frac{(ze^{\beta})^K}{1 - ze^{\beta}}$$

and

$$(UE)(z) = \frac{1}{1 - W(z)} \ll \frac{1 - ze^{\beta}}{1 - ze^{\beta} - (ze^{\beta})^K}\,.$$

But the radius of convergence of $(1-ze^{\beta})/(1-ze^{\beta}-(ze^{\beta})^K)$ is just $e^{-\beta}\lambda(K)$, and so the radius of convergence, ρ, of $UE(z)$ satisfies

$$\rho \ge e^{-\beta}\lambda(K) > e^{-\beta-\delta/2}\,.$$

Thus for some $N(\delta,\beta) \ge K$, $\dim(UE)_k \le e^{(\beta+\delta)k}$, for all $k \ge N(\delta,\beta)$. Let $\mathbb{L}$ be the free graded sub Lie algebra generated by the W_k, $k \ge N(\delta,\beta)$. Then $(U\mathbb{L})_k = 0$, $1 \le k < N(\delta,\beta)$ and for any F satisfying the hypothesis of the lemma we have

$$\dim(UF)_k \le \dim(U\mathbb{L})_k \le \dim(UE)_k \le e^{(\beta+\delta)k}\,, \qquad k \ge N(\delta,\beta)\,.$$

Since $\dim(UF)_k = 0$, $1 \le k < N(\delta,\beta)$, the lemma is proved. □

LEMMA 2. *Suppose $\delta,\gamma > 0$ and N satisfies the condition for $N(\delta,\gamma)$ in Lemma* 1. *Suppose also that N is sufficiently large so that* $\log(x+1)/x \le \delta$, $x \ge N$. *Finally, suppose $E \subset L$ and $V_n \subset L_n$ are respectively a sub Lie algebra and a subspace of a* cft *graded Lie algebra, L.*

If $\dim(UE)_i \le e^{\beta i}$, *for all i and some $\beta \le \gamma$, if* $\dim V_n \le e^{\gamma n}$, *and if $n \ge N$, then the sub Lie algebra, F, generated by E and V_n satisfies* $\dim(UF)_i \le e^{(\gamma+2\delta)i}$, *for all i.*

Proof. Put $W = UE \circ V_n$. Then $W_i = (UE)_{i-n} \circ V_n$ and so $\dim W_i \le e^{\gamma i}$, for all i. By Lemma 1 the sub Lie algebra, $G \subset L$, generated by W satisfies $\dim(UG)_i \le e^{(\gamma+\delta)i}$, all i. But since $[E,G] \subset G$, $E + G$ is a sub Lie algebra; i.e., $E + G = F$. If $i < n$ then $(UF)_i = (UE)_i$, while if $i \ge n$ then $\frac{\log(i+1)}{i} \le \delta$ and $\dim(UF)_i \le \sum_{j=0}^{i} \dim(UE)_j \cdot \dim(UG)_{i-j} \le (i+1)e^{(\gamma+\delta)i} \le e^{(\gamma+2\delta)i}$. □

LEMMA 3. *Suppose for some* $\alpha > 0$ *and some* cft *graded Lie algebra*, L, *that there is an infinite sequence* $0 = q_1 < q_2 < \cdots$ *such that* $\limsup_i \frac{\log \dim L_{q_i}}{q_i} \geq \alpha$. *Then there is an infinite subsequence* $r_\lambda = q_{i_\lambda}$ *and a sub Lie algebra* E *such that* $\lim_{\lambda\to\infty} \frac{\log \dim (E/[E,E])_{r_\lambda}}{r_\lambda} = \alpha$.

Proof. Choose an infinite strictly increasing sequence $0 < \alpha_1 < \alpha_2 < \cdots$ so that $\alpha_\lambda \to \alpha$. Set $\beta_\lambda = \frac{1}{2}(\alpha_\lambda + \alpha_{\lambda+1})$. Now we construct inductively the sequence $r_\lambda = q_{i_\lambda}$, together with an increasing sequence of sub Lie algebras, $E(\lambda)$, to satisfy $r_\lambda > \frac{2\log 2}{\alpha_\lambda - \alpha_{\lambda-1}}$, $\lambda \geq 2$, and $e^{\alpha_\lambda r_\lambda} - 1 \leq \dim E(\lambda)_{r_\lambda} < e^{\alpha_\lambda r_\lambda}$ and $\dim(UE(\lambda))_i \leq e^{\beta_\lambda i}$, for all λ, i.

Indeed set $r_1 = q_1 = 0$ and $E(1) = 0$; these conditions are then satisfied for $\lambda = 1$. Suppose the r_λ and $E(\lambda)$ are constructed for $\lambda < m$. By hypothesis there are arbitrarily large q_i with $\dim L_{q_i} > e^{\alpha_m q_i}$. Since for each i, $\dim E(m-1)_i \leq e^{\beta_{m-1} i} < e^{\alpha_m i}$, we may choose r_m arbitrarily large but such that $E(m-1)_{r_m}$ extends to a subspace $V_{r_m} \subset L_{r_m}$ satisfying $e^{\alpha_m r_m} - 1 \leq \dim V_{r_m} < e^{\alpha_m r_m}$. But for r_m sufficiently large Lemma 2 asserts that the Lie algebra $E(m)$ generated by $E(m-1)$ and V_{r_m} satisfies $\dim(UE(m))_i \leq e^{\beta_m i}$, for all i. This closes the induction.

Set $E = \cup_\lambda E(\lambda)$. By construction, $(E/[E,E])_{r_\lambda} = V_{r_\lambda}/W_{r_\lambda}$, where $W_{r_\lambda} = V_{r_\lambda} \cap [E(\lambda-1), E(\lambda-1)]_{r_\lambda}$. Thus

$$e^{\alpha_\lambda r_\lambda} - 1 - e^{\beta_{\lambda-1} r_\lambda} \leq \dim(E/[E,E])_{r_\lambda} \leq e^{\alpha_\lambda r_\lambda} - 1\,.$$

But $\beta_{\lambda-1}\, r_\lambda = \frac{\alpha_\lambda + \alpha_{\lambda-1}}{2}\, r_\lambda = \alpha_\lambda r_\lambda + \frac{\alpha_{\lambda-1} - \alpha_\lambda}{2} r_\lambda < \alpha_\lambda r_\lambda - \log 2$. This gives $\frac{1}{2}(e^{\alpha_\lambda r_\lambda} - 2) \leq \dim(E/[E,E])_{r_\lambda} \leq e^{\alpha_\lambda r_\lambda} - 1$, and completes the proof. □

The invariants $\operatorname{Ext}^*_A(M, N)$ and $\operatorname{Tor}^A_*(M, N)$ for graded modules over a graded algebra, A, will play an important role in this paper. We recall that each $\operatorname{Ext}^p_A(M, N)$ converts direct sums in the first factor and direct products in the second factor to direct products. However, if N has finite type as a graded vector space then a direct sum decomposition is a direct product decomposition. In particular we have the classical and useful

Remark. If N is A-free and N has finite type as a graded vector space then $\operatorname{Ext}^p_A(M, N) \neq 0$ if and only if $\operatorname{Ext}^p_A(M, A) \neq 0$.

Next let $V = \{V_i\}_{i \geq 0}$ be a graded vector space of finite type and denote by $\bigwedge V^\#$ the free graded commutative algebra on $V^\#$. Then $\bigwedge^q V^\#$ is the linear span of the products $f_1 \ldots f_q$, $f_i \in V^\#$. The dual $\Gamma V = (\bigwedge V^\#)^\#$ is just the free divided powers algebra on V with multiplication dual to the comultiplication in $\bigwedge V^\#$ defined by $f \mapsto f \otimes 1 + 1 \otimes f$, $f \in V^\#$. Clearly $\Gamma V = \bigoplus_q \Gamma^q V$ with $\Gamma^q V = (\bigwedge^q V^\#)^\#$.

In particular, if L is a cft graded Lie algebra and M and N are L-modules then $\mathrm{Tor}_*^{UL}(M,N)$ and $\mathrm{Ext}_{UL}^*(M,N)$ may be computed as the homology of complexes respectively of the form $\Gamma^*(sL)\otimes_{\Bbbk} M\otimes_{\Bbbk} N$ and $\mathrm{Hom}_{\Bbbk}(\Gamma^*(sL)\otimes_{\Bbbk} M, N)$ with twisted differentials [15]. (Here sL is the suspension of L; $(sL)_k = L_{k-1}$.) Now suppose $E\subset L$ is a sub Lie algebra and write $L = E\oplus S$. This defines a multiplicative isomorphism $\Gamma(sE)\otimes\Gamma(sS)\xrightarrow{\cong}\Gamma(sL)$. Moreover, the filtration $\mathscr{F}_p$ of $\Gamma(sL)\otimes M\otimes N$ corresponding under this isomorphism to $\bigoplus_{i\le p}\Gamma(sE)\otimes\Gamma^i(sS)\otimes M\otimes N$ is independent of the choice of S and is preserved by the differential. The corresponding first quadrant spectral sequence converges from

$$E^1_{p,q} = \mathrm{Tor}_q^{UE}(\Gamma^p s(L/E)\otimes M, N) \qquad \text{to} \qquad \mathrm{Tor}^{UL}_{p+q}(M,N)\,.$$

This spectral sequence was first introduced by Koszul in his thesis [13] for the case of the Lie algebra of a connected closed subgroup, H, of a connected compact Lie group, G. The generalization by Hochschild and Serre to general Lie algebras and Lie modules includes the result that when E is an ideal then

$$E^2_{p,q} = \mathrm{Tor}_p^{UL/E}(\Bbbk, \mathrm{Tor}_q^{UE}(M,N))$$

and the spectral sequence is generally known as the Hochschild-Serre spectral sequence. In general the E^2-term is mysterious; however in the original case considered by Koszul, $E^2_{p,q} = H_p(G/H;\mathbb{R})\otimes H_q(H;\mathbb{R})$. Note there is a "dual" Hochschild-Serre spectral sequence converging from $\mathrm{Ext}^q_{UE}(\Gamma^p s(L/E)\otimes M, N)$ to $\mathrm{Ext}^{p+q}_{UL}(M,N)$.

LEMMA 4. *Suppose M and N are L-modules where L is a* cft *graded Lie algebra and each N_i is finite dimensional. If $\mathrm{Ext}^m_{UL}(M,N)\neq 0$ then for some finitely generated sub Lie algebra $E\subset L$ and for some finitely generated L-submodule $P\subset M$ the restrictions $\mathrm{Ext}^m_{UL}(M,N)\to\mathrm{Ext}^m_{UE}(M,N)$ and $\mathrm{Ext}^m_{UL}(M,N)\to\mathrm{Ext}^m_{UL}(P,N)$ are nonzero.*

Proof. It suffices to observe that $\mathrm{Ext}^*_{UL}(M,N) = [\mathrm{Tor}_*^{UL}(M,N^\#)]^\#$ and that Tor commutes with direct limits in L and in M. □

PROPOSITION 1. *Suppose L is an abelian* cft *graded Lie algebra and M is an L-module. If $\mathrm{Ext}^m_{UL}(M,UL)\neq 0$, some m, then for some $x\in M$ and some n the map $U(L_{\ge n})\to M$, $a\mapsto a\cdot x$, is injective.*

Proof. If $m=0$ then, because $\mathrm{Ext}^0_{UL} = \mathrm{Hom}_{UL}$, there is an $x\in M$ and an $f\in\mathrm{Hom}_{UL}(M,UL)$ such that $f(x)\neq 0$. Choose n such that $f(x)\in U(L_{<n})$. Since $UL = U(L_{<n})\otimes U(L_{\ge n})$ the proposition follows in this case.

We now proceed by induction and assume the proposition holds for $p<m$. It is clearly sufficient to prove the conclusion for some subquotient of M. By Lemma 4 we may suppose M finitely generated. Then a simple exact sequence argument reduces to the case where M has the form $M = UL\cdot z$.

Denote the ideals (all subspaces of L are ideals) of elements of even (resp. odd) degrees in L by E (resp. F). Write $E(k)$ for $E_{<k}$. Next, let $N \subset M$ be the L-submodule of elements y such that $UE \to UE \cdot y$ is not injective. Either $\mathrm{Ext}^m_{UL}(N, UL) \neq 0$ or $\mathrm{Ext}^m_{UL}(M/N, UL) \neq 0$ and we consider these two cases separately.

In the first case there is a finitely generated L-submodule $P \subset N$ such that $\mathrm{Ext}^m_{UL}(P, UL) \neq 0$. Suppose $y_1, \dots, y_r$ is a set of generators for P and let $0 \neq a_i \in UE$ satisfy $a_i y_i = 0$. Since UE has no zero divisors, $0 \neq a = \prod_i a_i$; clearly $aP = 0$. Let k be sufficiently large that $a \in UE(k)$ and write $L = E(k) \oplus I$.

Now denote $\mathrm{Tor}^{UE(k)}_q(P, (UE(k))^\#)$ simply by T_q. Since $a \cdot P = 0$ it follows that $\mathrm{Hom}_{UE(k)}(P, UE(k)) = 0$, as $UE(k)$ has no zero divisors. But this space is the dual of T_0 and so $T_0 = 0$. On the other hand, the Hochschild-Serre spectral sequence converges from $\mathrm{Ext}^p_{UI}(T_{m-p}, UI)$ to $\mathrm{Ext}^m_{UL}(P, UL)$ and so $\mathrm{Ext}^p_{UI}(T_{m-p}, UI) \neq 0$ some $p < m$. Thus by induction for some $\alpha \in T_{m-p}$ and some $\ell > k$, $U(L_{\geq \ell}) \to U(L_{\geq \ell}) \cdot \alpha$ is injective (note that $L_{\geq \ell} = I_{\geq \ell}$).

Suppose that the proposition fails for P and so there are integers $l \leq \sigma_0 < \sigma_1 < \cdots < \sigma_r$ and nonzero elements $b_i \in U(L_{(\sigma_{i-1}, \sigma_i)})$ such that $b_i y_i = 0$. Set $b = \prod_i b_i$. Since $\otimes_i UL_{(\sigma_{i-1}, \sigma_i)} \subset UL$ it follows that $b \neq 0$. But $bP = 0$, which implies that $b \cdot T_{m-p} = 0$. This contradiction establishes the proposition when $\mathrm{Ext}^m_{UL}(N, UL) \neq 0$.

The second case is when $\mathrm{Ext}^m_{UL}(M/N, UL) \neq 0$ and it is sufficient to establish the conclusion of the proposition for M/N. Thus we may restrict consideration to modules M satisfying $ay \neq 0$ if $0 \neq a \in UE$ and $0 \neq y \in M$. If the proposition fails here we may as above find an infinite sequence $0 < \sigma_0 < \sigma_1 < \cdots$ and nonzero elements $a_i \in U(L_{(\sigma_{i-1}, \sigma_i)})$ such that $a_i z = 0$. Now it is easy to find $b_i \in U(L_{(\sigma_{i-1}, \sigma_i)})$ such that $0 \neq b_i a_i = v_i w_i$ with $v_i \in U(F_{(\sigma_{i-1}, \sigma_i)})$ and $w_i \in U(E_{(\sigma_{i-1}, \sigma_i)})$. Then $w_i v_i z = 0$ and so by construction, each $v_i z = 0$. Since $M = UL \cdot z$ it follows that each $v_i M = 0$. In particular $\mathrm{Hom}_{UF}(M, UF) = 0$, and so it follows from the sublemma below that $\mathrm{Ext}^*_{UF}(M, UF) = 0$. Since UL is UF-free this implies (see the remark above) that $\mathrm{Ext}^*_{UF}(M, UL) = 0$.

Thus the Hochschild-Serre spectral sequence

$$\mathrm{Ext}^q_{UL/F}(\Bbbk, \mathrm{Ext}^p_{UF}(M; UL)) \Longrightarrow \mathrm{Ext}^{p+q}_{UL}(M, UL)$$

implies that $\mathrm{Ext}^*_{UL}(M, UL) = 0$, a contradiction. □

SUBLEMMA. *If F is any* cft *graded Lie algebra concentrated in odd degrees and if Q is any F-module then* $\mathrm{Ext}^p_{UF}(Q, UF) = 0$, $p \geq 1$.

Proof. Since F is concentrated in odd degrees it is abelian. In view of Lemma 4 and the Remark before it, it is sufficient to prove the sublemma when F is finitely generated (and therefore finite dimensional). Let $v_1, \dots, v_r$ be a basis of F. Since

$0 \to v_1 Q \to Q \to Q/v_1 Q \to 0$ is a short exact sequence it is sufficient to prove the sublemma for $v_1 Q$ and $Q/v_1 Q$. But $v_1^2 = \frac{1}{2}[v_1, v_1] = 0$ in UF and so $v_1 \cdot v_1 Q = 0$. Thus we are reduced to the case of F-modules Q for which $v_1 Q = 0$. Iterating this argument for $v_2, \dots, v_r$ reduces us to the case $F \cdot Q = 0$. Thus Q is the direct sum of one dimensional F-modules. But $\mathrm{Ext}^*_{UF}(-, UF)$ converts a direct sum in the first factor to a direct product, which reduces us to proving the sublemma for $Q = \Bbbk$. In this case, however, the assertion $\mathrm{Ext}^*_{UF}(\Bbbk, UF) = \mathrm{Ext}^0_{UF}(\Bbbk, UF)$ is proved in [4]. This completes the proof of the sublemma and of Proposition 1. □

The invariants $\mathrm{Ext}_{UL}(M, UL)$ will play a key role in the proof of the main theorem, where UL acts on itself by left multiplication. We recall that the *depth* of L is the least integer m (or ∞) for which $\mathrm{Ext}^m_{UL}(\Bbbk, UL) \neq 0$. We shall also need a generalization, to be baptized *weak depth*.

Definition. Let L be a cft graded Lie algebra.

(i) An L-module, M, is *weakly locally finite* if M is the increasing union of finite dimensional subspaces $M(1) \subset M(2) \subset \dots$ such that $M(k)$ is preserved by $L_{\leq k}$.

(ii) A sub-Lie algebra $E \subset L$ is *admissible* if the quotient L/E is a weakly locally finite E-module for the adjoint representation.

Note that if M and N are weakly locally finite L-modules, so are $M \otimes N$, each $\bigwedge^q M$, and any sub-quotient module of M. If M has finite type then $\Gamma^q(M) := (\bigwedge^q M^\#)^\#$ is also weakly locally finite.

Definition. The *weak depth* of L, w-depth L, is the least m (or ∞) such that $\mathrm{Ext}^m_{UL}(M, UL) \neq 0$ for some weakly locally finite L-module, M.

LEMMA 5. (i) *If E is admissible in a* cft *graded Lie algebra*, L, *then*

$$\text{w-depth}\, E \leq \text{w-depth}\, L\,;$$

(ii) *w-depth $L \leq$ depth L, and equality holds if L is finitely generated.*

Proof. (i) Choose a weakly locally finite L-module M such that $\mathrm{Ext}^m_{UL}(M, UL)$ is nonzero, where $m =$ w-depth L. Then the Hochschild-Serre spectral sequence implies that

$$\mathrm{Ext}^q_{UE}(\Gamma^p s(L/E) \otimes M, UL) \neq 0\,, \qquad \text{for some } p + q = m\,.$$

But because E is admissible in L, L/E is a weakly locally finite E-module; hence so is $\Gamma^p s(L/E) \otimes M$, and w-depth $E \leq q \leq m$.

(ii) The inequality is obvious. Moreover if w-depth $L = m$ then choose a weakly locally finite L-module M for which $\mathrm{Ext}^m_{UL}(M, UL) \neq 0$. Since L is finitely generated, M is the union of an increasing sequence of finite dimensional modules $M(r)$. Thus (Lemma 4), for some r, $\mathrm{Ext}^m_{UL}(M(r), UL) \neq 0$. Now, since $M(r)$ is finite dimensional, a simple long exact sequence argument shows that $\mathrm{Ext}^m_{UL}(\Bbbk, UL) \neq 0$, i.e., depth $L \leq m$. □

LEMMA 6. *Let $E \subset L$ be a sub Lie algebra of a* cft *graded Lie algebra L. Suppose for some weakly locally finite L-module, M, and some m, that the restriction map $\mathrm{Ext}^m_{UL}(M, UL) \to \mathrm{Ext}^m_{UE}(M, UL)$ is nonzero. Let Z be the centralizer of E in L. Then for some q, $Z_{\geq q}$ is concentrated in odd degrees. If $M = \Bbbk$ then Z is finite dimensional.*

Proof. In view of Lemma 4 we may and do assume E is a finitely generated Lie algebra. Then by Lemma 5, depth $E < \infty$. Since $Z \cap E$ is contained in the center of E it is finite dimensional [4]. Let $F = Z_{\geq q}$ where q is chosen so that $F \cap E = 0$.

Again by Lemma 4 we may write F as the increasing union of finitely generated sub Lie algebras $F(k)$, and M as the increasing union of finite dimensional $(E \oplus F(k))$-modules, $M(k)$, such that $\rho(k) : \mathrm{Ext}^m_{U(E\oplus F(k))}(M(k), UL) \to \mathrm{Ext}^m_{UE}(M(k), UL)$ is nonzero. Because $M(k)$ is finite dimensional,

$$\mathrm{Ext}^*_{UL}(M(k), UL) = \mathrm{Ext}^*_{UL}(\Bbbk, M(k)^\# \otimes UL),$$

where $U(E \oplus F(k))$ acts diagonally in $M(k)^\# \otimes UL$. Because UL is $U(E \oplus F(k))$-free and $M(k)^\#$ is finite dimensional it follows that $M(k)^\# \otimes UL$ is $U(E \oplus F(k))$-free. We therefore deduce that

$$\mathrm{Ext}^m_{U(E\oplus F(k))}(\Bbbk, U(E \oplus F(k))) \to \mathrm{Ext}^m_{UE}(\Bbbk, U(E \oplus F(k)))$$

is nonzero. But this is dual to the map

$$\begin{aligned}\mathrm{Tor}^{UE}_m(\Bbbk, (UE)^\#) \otimes (UF(k))^\# &\to \mathrm{Tor}^{U(E\oplus F(k))}_m(\Bbbk, (UE)^\# \otimes (UF(k))^\#) \\ &= \bigoplus_{p+q=m} \mathrm{Tor}^{UE}_p(\Bbbk, (UE)^\#) \otimes \mathrm{Tor}^{UF(k)}_q(\Bbbk, (UF(k))^\#),\end{aligned}$$

whose image is in $\mathrm{Tor}^{UE}_m \otimes \mathrm{Tor}^{UF(k)}_0$. Thus $\mathrm{Ext}^{UF(k)}_0(\Bbbk, (UF(k))) \neq 0$ and this, by [4] implies that $F(k)$ is concentrated in odd degrees. Hence so is $F = \cup F(k)$. Finally, if $M = \Bbbk$ then we may replace $M(k)$ by $\Bbbk$ and $F(k)$ by F in the argument above to conclude that depth $F = 0$. This implies that F is finite dimensional [4]. □

LEMMA 7. *Let L be a* cft *graded Lie algebra of finite depth. Then for some r the sub Lie algebra $Z = \{ x \in L \mid [x, L_{\leq r}] = 0 \}$ is finite dimensional.*

Proof. Choose r so that for some m, $\mathrm{Ext}^m_{UL}(\Bbbk, UL) \to \mathrm{Ext}^m_{UE}(\Bbbk, UL)$ is nonzero, where E is the sub Lie algebra generated by $L_{\leq r}$ (Lemma 4). Then Z is the centralizer of E and Lemma 7 is a special case of Lemma 6. □

LEMMA 8. *Let $E \subset L$ be a sub Lie algebra of a graded Lie algebra L such that L/E is finite dimensional and concentrated in odd degrees. Then for any*

L-module M, the restriction morphism

$$\mathrm{Ext}^*_{UL}(M, UL) \to \mathrm{Ext}^*_{UE}(M, UL)$$

is injective.

Proof. Choose r so that for some It is clearly sufficient to consider the case $L = E \oplus \Bbbk x$ with x of odd degree. In this case E is an ideal in L and the quotient Lie algebra L/E has the exterior algebra $\bigwedge x$ as its universal enveloping algebra. It follows that the isomorphism $UL \cong UE \otimes \bigwedge x$ induces an isomorphism $\mathrm{Ext}^*_{UE}(M, UL) \cong \mathrm{Ext}^*_{UE}(M, UE) \otimes \bigwedge x$, because $\bigwedge x$ is finite dimensional. Moreover, the action of x in $\mathrm{Ext}^*_{UE}(M, UL)$ satisfies $x \cdot (a \otimes 1) = x \cdot a \otimes x \pm a \otimes x$, $a \in \mathrm{Ext}^*_{UE}(M, UE) \otimes 1$, and so each $\mathrm{Ext}^q_{UE}(M, UL)$ is a free $\bigwedge x$-module. Thus in the Hochschild-Serre spectral sequence, converging to $\mathrm{Ext}^*_{UL}(M, UL)$ we have

$$E_2^{p,q} = \mathrm{Ext}^p_{\bigwedge x}(\Bbbk, \mathrm{Ext}^q_{UE}(M, UL)) = 0\,, \qquad p \geq 1\,.$$

Hence

$$\mathrm{Ext}^q_{UL}(M, UL) = E_2^{0,q} = \mathrm{Hom}_{\bigwedge x}(\Bbbk, \mathrm{Ext}^q_{UE}(M, UL))\,.$$

Since the restriction morphism $\mathrm{Ext}^q_{UL}(M, UL) \to \mathrm{Ext}^q_{UE}(M, UL)$ is the composition

$$\mathrm{Ext}^q_{UL}(M, UL) \stackrel{\cong}{\to} E_2^{0,q} \hookrightarrow \mathrm{Ext}^q_{UE}(M, UL)\,,$$

the injectivity of the restriction morphism follows. □

3. **The first Lie algebra growth theorem**

We consider the following conditions on a cft graded Lie algebra L:

(T_1) For some $\beta > 0$ there is an infinite sequence $q_1 < q_2 < \cdots$ of even integers such that $\frac{q_{j+1}}{q_j} \to 1$, and

$$\liminf_j \frac{\log \dim L_{q_j}}{q_j} \geq \beta.$$

(T_2) w-depth $L < \infty$.

THEOREM 2. *Suppose L is a* cft *graded Lie algebra satisfying* (T_1) *and* (T_2). *Then*

$$\lim_{d\to\infty} \liminf_k \frac{\log \sum_{i=k}^{k+d} \dim L_i}{k} \geq \beta\,.$$

Proof. Since $\sum_{i=k}^{k+d} \dim L_i$ increases with d so does

$$\liminf_k \frac{\log \sum_{i=k}^{k+d} \dim L_i}{k}\,.$$

Thus the limit in the theorem exists. Denote this limit by γ:

$$\gamma = \lim_{d\to\infty} \liminf_{k} \frac{\log \sum_{i=k}^{k+d} \dim L_i}{k},$$

so that the theorem simply asserts the inequality $\beta \leq \gamma$. We assume this inequality fails, and deduce a contradiction.

Our assumption allows us to choose $\delta > 0$ so that

$$(T_3) \qquad 2\delta < \beta - \gamma .$$

Then, by the definition of γ, for any d there are arbitrarily large integers k such that

$$(T_4) \qquad \sum_{i=k}^{k+2d} \dim L_i < e^{(\gamma+\delta)k} .$$

Next, let $N(\delta, \gamma + 2\delta)$ be as in Lemma 1, and choose N so that

$$(T_5) \qquad \begin{cases} N > N(\delta, \gamma + 2\delta) , \\ 1 + e^{(\gamma+\delta)i} < e^{(\gamma+2\delta)i} , \ i \geq N , \text{ and} \\ \dim L_{q_j} > e^{(\beta-\delta)q_j} , \qquad q_j \geq N . \end{cases}$$

Then, let $F \subset L$ be the sub Lie algebra generated by the subspaces L_i for which $i \geq N$ and $\dim L_i < e^{(\gamma+\delta)i}$.

The main step in the proof of the theorem is the inductive construction of an infinite sequence $z_0, z_1, \dots$ of nonzero elements in L of strictly increasing even degrees, and satisfying property (T_6) below. Namely let $E(r)$ denote the sub Lie algebra generated by $F_{\leq r}$ and $z_0, \dots, z_{r-1}$, and let $UE(r)$ act on L by the adjoint representation. Then we shall construct the z_r so that

(T_6) $\quad UE(r)_+ \circ z_r$ is finite dimensional and concentrated in odd degrees.

We show first that the existence of this sequence leads to the desired contradiction, and then complete the proof of the theorem by carrying out the construction.

The relation (T_6) clearly implies that for any r, $i \geq 0$, $[E(r), z_{r+i}]$ is finite dimensional and concentrated in odd degrees. Now let $E = \cup_r E(r)$ be the union of the increasing sequence of sub Lie algebras $E(r)$. Since $F_{\leq r} \subset E(r)$ it follows that $F \subset E$ and so $L_i \subset E$ if $i \geq N$ and $\dim L_i < e^{(\gamma+\delta)i}$. Thus (T_4) implies that $(L/E)_i = 0$ for $i \in [k, k+d]$ where k and d can be arbitrarily large. In particular E is admissible in L and so, by Lemma 5, w-depth $E \leq$ w-depth $L < \infty$.

Let w-depth $E = m$, and let $Z(r) \subset E(r)$ be the sub Lie algebra of elements that commute with each z_{r+i}, $0 \leq i \leq m$. Since $[E(r), z_{r+i}]$ is finite dimensional and concentrated in odd degrees it follows that $E(r)/Z(r)$ is also finite dimensional and concentrated in odd degrees. By Lemma 8, the restriction morphism

$$\mathrm{Ext}_{UE(r)}(M, UE) \to \mathrm{Ext}_{UZ(r)}(M, UE)$$

is injective for any E-module M.

On the other hand, since w-depth $E = m$ there is a weakly locally finite E-module M such that $\mathrm{Ext}^m_{UE}(M, UE) \neq 0$. Moreover, since $E = \cup_r E(r)$, for some r the restriction morphism $\mathrm{Ext}^m_{UE}(M, UE) \to \mathrm{Ext}^m_{UE(r)}(M, UE)$ is nonzero. Hence also the composite

$$(T_7) \qquad \mathrm{Ext}^m_{UE}(M, UE) \to \mathrm{Ext}^m_{UZ(r)}(M, UE) \text{ is nonzero.}$$

Since each $[E(i), z_i]$ is concentrated in odd degrees, it follows that $[z_i, z_j] = 0$, $0 \le j < i < \infty$. Thus $Z = \bigoplus_{i=0}^m \Bbbk z_{r+i}$ is an abelian Lie algebra commuting with $Z(r)$. Because $Z(r) \to E$ factors as $Z(r) \to Z(r) + Z \to E$, the restriction map $\mathrm{Ext}^m_{UE}(M, UE) \to \mathrm{Ext}^m_{UZ(r)}(M, UE)$ factors through $\mathrm{Ext}^m_{U(Z(r)+Z)}(M, UE)$. Since UE is a free $U(Z(r) + Z)$-module, it follows from (T_7) that

$$\text{w-depth}(Z(r) + Z) \le m.$$

Since Z is an ideal in $Z(r) + Z$, there is a Hochschild-Serre spectral sequence converging from

$$\mathrm{Ext}^p_{U((Z(r)+Z)/Z)}(\Bbbk, \mathrm{Ext}^q_{U(Z)}(-,-))$$

to $\mathrm{Ext}^{p+q}_{U(Z(r)+Z)}(-,-)$. It follows that w-depth $Z \le m$. But Z is finitely generated and so Lemma 5 asserts that depth $Z \le m$. On the other hand, since Z is abelian and concentrated in even degrees, [4] asserts that depth $Z = m + 1$, and we have the desired contradiction.

It remains to construct the sequence (z_r). To begin note that $E(0) = 0$ and let z_0 be any nonzero element of L of even degree $\ge N$. Then assume by induction that $z_0, \dots, z_{r-1}$ are constructed. Since the degrees of the z_i are strictly increasing with $\deg z_0 > N$ and since F is generated by the spaces L_i with $i > N$ and $\dim L_i < e^{(\gamma+\delta)i}$, it follows from (T_5) that

$$\dim \left(\frac{E(r)}{[E(r), E(r)]} \right)_i < 1 + e^{(\gamma+\delta)i} < e^{(\gamma+2\delta)i}, \qquad \text{for all } i \ge 0.$$

Moreover $E(r)_i = 0$, $i < N$, and so Lemma 1 asserts that

$$(T_8) \qquad \dim (UE(r))_j < e^{(\gamma+3\delta)j}, \qquad \text{for all } j \ge 0.$$

Choose d so that

$$(T_9) \qquad d > r + \deg z_{r-1}.$$

Then choose s so that

$$(T_{10}) \qquad q_s > d\left(\frac{\gamma}{\delta} + 1\right),$$

and

$$(T_{11}) \qquad \frac{q_{j+1}}{q_j} < 1+\sigma\,, \qquad \text{for all } j \geq s\,,$$

where $\sigma = \dfrac{\beta-\gamma-2\delta}{2\gamma+5\delta}$. Finally, by (T_4) we may choose k so that

$$(T_{12}) \qquad k > q_s \qquad \text{and} \qquad \sum_{i=k}^{k+2d} \dim\, L_i < e^{(\gamma+\delta)k}\,.$$

Then $q_s > d > \deg z_{r-1} \geq \deg z_0 > N$ and so by the choice of N (cf. T_5),

$$(T_{13}) \qquad \dim L_{q_j} > e^{(\beta-\delta)q_j}\,, \qquad \text{for all } j \geq s\,.$$

Since (cf. T_3) $\beta-\gamma > 2\delta$, it follows from (T_{12}) and (T_{13}) that no q_j is in the interval $[k, k+2d]$. Thus for some ℓ,

$$(T_{14}) \qquad q_s \leq q_\ell < k < k+2d < q_{\ell+1}\,.$$

The adjoint representation of $UE(r)_j$ in L 'dualizes' to linear maps

$$\theta_i : L_{q_\ell} \to L_i \otimes (UE(r))^{\#}_{i-q_\ell}\,.$$

Then

$$\begin{aligned}
\sum_{i=k+d}^{k+2d} &\dim\bigl(L_i \otimes (UE(r))^{\#}_{i-q_\ell}\bigr) \\
&< \Bigl(\sum_{i=k+d}^{k+2d} \dim L_i\Bigr) e^{(\gamma+4\delta)(k+2d-q_\ell)} && (T_8)\\
&< e^{\gamma(2k+2d-q_\ell)+\delta(5k+8d-4q_\ell)} && (T_{12})\\
&< e^{\gamma(2q_{\ell+1}-q_\ell)+\delta(5q_{\ell+1}-4q_\ell)} && (T_{14})\\
&< e^{(\gamma+\delta)q_\ell+(2\gamma+5\delta)\sigma q_\ell} && (T_{11})\\
&< e^{(\beta-\delta)q_\ell} && (T_3) \text{ and } (T_{11})\\
&< \dim L_{q_\ell} && (T_{13}).
\end{aligned}$$

The inequalities above imply that some nonzero element $x \in L_{q_l}$ is in the intersection of the $\operatorname{Ker}\theta_i$, $k+d \leq i \leq k+2d$. Therefore

$$UE(r)_p \circ x = 0\,, \qquad p \in [k+d-q_l, k+2d-q_l]\,.$$

Since $E(r)$ is generated by $F_{\leq r}$ and $z_0, \dots, z_{r-1}$, and since d was chosen (cf. T_9) so that $d > r$ and $d > \deg z_i$, $0 \leq i \leq r-1$, we have $(UE(r))_{\geq k+2d-q_l} \circ x = 0$, and so $UE(r) \circ x$ is finite dimensional.

Since $x \in L_{q_\ell}$ and q_ℓ is even we may choose z_r to be a nonzero element of maximum even degree in $UE(r) \circ x$. Then $\deg z_r \geq q_\ell \geq q_s > d > \deg z_{r-1}$ (cf. (T_9) and (T_{10})), and by choice $UE(r)_+ \circ z_r$ is finite dimensional and concentrated in odd degrees. This completes the inductive step of the construction and, with it, the proof of Theorem 2. □

4. **The second Lie algebra growth theorem**

Consider the following conditions on a cft graded Lie algebra, L:

(X_1) For some integer $m \geq 2$ and some fixed $\alpha \in (0, \infty)$, there is an infinite sequence $p_1 < p_2 < \cdots$ of integers such that $p_{j+1} \leq m p_j$, all j, and

$$\liminf_j \frac{\log \dim L_{p_j}}{p_j} \geq \alpha .$$

(X_2) L has finite depth.

THEOREM 3. *Suppose L is a* cft *graded Lie algebra satisfying (X_1) and (X_2). Then there is an integer d such that*

$$\liminf_k \frac{\log \sum_{i=k}^{k+d} \dim L_i}{k} \geq \alpha .$$

THEOREM 3′. *Suppose L is a* cft *graded Lie algebra satisfying (X_1) and (X_2), with $\alpha =$* logindex L. *Then there is an integer d for which, given any $\varepsilon > 0$ there is a $K = K(\varepsilon)$ such that*

$$e^{(\alpha-\varepsilon)k} \leq \sum_{i=k}^{k+d} \dim L_i \leq e^{(\alpha+\varepsilon)k} , \qquad \textit{for all } k \geq K .$$

Proof of Theorem 3. The main step of the proof is to show that

(X_3) For any $\beta < \alpha$ there is an infinite sequence $q_1 < q_2 < \cdots$ of even integers such that $\frac{q_{j+1}}{q_j} \to 1$, and $\liminf_j \frac{\log \dim L_{q_j}}{q_j} \geq \beta$.

Indeed, given (X_3) we may apply Theorem 2 to find

$$\lim_{s\to\infty} \liminf_k \frac{\log \sum_{i=k}^{k+s} \dim L_i}{k} \geq \beta$$

if $\beta < \alpha$. It follows that

$$\lim_{s\to\infty} \liminf_k \frac{\log \sum_{i=k}^{k+s} \dim L_i}{k} \geq \alpha .$$

Choose d (Lemma 7) so that $Z = \{ u \mid [u, L_{\leq d}] = 0 \}$ is finite dimensional. Choose D so that $Z_{\geq D} = 0$.

Next, for any $k > s > 0$, write

$$\sum_{i=k}^{k+s} \dim L_i = e^{\gamma(k,s)k} .$$

Then for some $j \in [k-s, k]$, $\dim L_j \geq \frac{1}{s+1} e^{\gamma(k-s,s)(k-s)}$. Let $u_1, \dots, u_p$ be a basis for $L_{\leq d}$ and note that if $j \geq D$ then for some λ we have $\dim[u_\lambda, L_j] \geq \frac{1}{p} \dim L_j$. Proceeding in this way yields an infinite sequence (u_{λ_ν}) such that

$$\dim[u_{\lambda_q}, [u_{\lambda_{q-1}}, [\dots [u_{\lambda_1}, L_j] \dots] \geq \left(\frac{1}{p}\right)^q \dim L_j \, , \qquad \text{for all } q \, .$$

But for some $q \leq s$ we have $\sum_{\nu=1}^{q} \deg u_{\lambda_\nu} + j \in [k, k+d]$. It follows that

$$\gamma(k,d) \geq (1 - s/k)\gamma(k-s,s) - \frac{Q(s)}{k} \, ,$$

for some $Q(s)$ independent of k. Fixing $s \geq d$ and letting $k \to \infty$ we see that $\liminf_k \gamma(k,d) = \liminf_k \gamma(k,s)$. Thus

$$\liminf_k \frac{\log \sum_{i=k}^{k+d} \dim L_i}{k} = \lim_{s\to\infty} \liminf_k \frac{\log \sum_{i=k}^{k+s} \dim L_i}{k} \geq \alpha \, .$$

It remains to establish (X_3). First we prove a weaker statement:

(X_4) For any $\beta < \alpha$ there is an infinite sequence $\ell_1 < \ell_2 < \cdots$ such that $\frac{\ell_{j+1}}{\ell_j} \to 1$, and $\liminf_j (\log \dim L_{\ell_j})/\ell_j \geq \beta$.

For this we suppose (X_4) false, and deduce a contradiction. By hypothesis, then, there are numbers $\beta < \alpha$ and $\sigma > 1$ such that there is no sequence $\ell_1 < \ell_2 < \cdots$ with $\ell_{j+1} < \sigma \ell_j$, all j, and $\liminf_j \frac{\log \dim L_{\ell_j}}{\ell_j} \geq \beta$. In particular there exists k_1 with $\dim L_j < e^{\beta j}$, $j \in [k_1, \sigma k_1]$. Since there is no sequence beginning with σk_1, there is $k_2 > \sigma k_1$ such that $\dim L_j < e^{\beta j}$, $j \in [k_2, \sigma k_2]$. This process gives an infinite sequence $k_1 < k_2 < \cdots$ such that $\dim L_j < e^{\beta j}$, $j \in [k_s, \sigma k_s]$, all s. By starting with a sufficiently large k_1 we may also assume that for each s there is a largest i_s such that $p_{i_s} < \frac{\sigma - 1}{2} k_s$. Denote p_{i_s} by r_s. Since $p_{i_s+1} \leq m p_{i_s}$ we obtain

$$(X_5) \qquad \frac{\sigma-1}{2m} k_s \leq r_s \leq \frac{\sigma-1}{2} k_s.$$

Now apply Lemma 3 to see that if we replace (k_s) and (r_s) by infinite subsequences we may find a sub Lie algebra $E \subset L$ such that $\frac{\log \dim (E/[E,E])_{r_s}}{r_s} \to \alpha$. Denote $[E, E]$ simply by I.

Now the Hochschild-Serre spectral sequence

$$\mathrm{Ext}_{UE}^{q}(\Gamma^p s(L/E), UL) \Rightarrow \mathrm{Ext}_{UL}^{p+q}(\Bbbk, UL)$$

implies that for some p, $\mathrm{Ext}^*_{UE}(\Gamma^p s(L/E), UL) \neq 0$, since L has finite depth. As UL is UE-free it follows that $\mathrm{Ext}^*_{UE}(\Gamma^p s(L/E), UE) \neq 0$. The proof of Lemma 4.2 in [6] now applies verbatim to show that

$$\mathrm{Ext}^*_{UE}(\Gamma^p s(L/E), U(E/I)) \neq 0 .$$

But there is also a Hochschild-Serre spectral sequence converging to

$$\mathrm{Ext}^*_{UE}(\Gamma^p s(L/E), U(E/I))$$

from $\mathrm{Ext}^*_{U(E/I)}(\mathrm{Tor}^{UI}(\Bbbk, \Gamma^p s(L/E)), U(E/I))$. The latter Ext^* is therefore nonzero. Thus by Proposition 1 there are an n and a $\gamma \in \mathrm{Tor}^{UI}_{p'}(\Bbbk, \Gamma^p s(L/E))$, some p', such that

(X_6) $U((E/I)_{\geq n}) \to U((E/I)_{\geq n}) \cdot \gamma$ is injective.

Now $\mathrm{Tor}^{UI}_*(\Bbbk, \Gamma^p s(L/E))$ is the homology of a complex of E-modules of the form $\Gamma(sI) \otimes \Gamma^p s(L/E)$, where the representation of E is simply the adjoint representation. In particular we may represent γ by a cycle $z \in \Gamma^{p'} sI \otimes \Gamma^p s(L/E)$.

Set $sI \oplus sL/E = W$. Then, clearly, for some N and with $q = p + p'$, $z \in \Gamma^q(W_{\leq N})$. Choose S so large that $r_S > N + n$, and so that for some $\varepsilon < \frac{1}{2q}(\alpha - \beta)$, we have $e^{(\alpha-\varepsilon)r_t} < \dim(E/I)_{r_t} < e^{(\alpha+\varepsilon)r_t}$, for all $t \geq S$.

Next, recall that ΓW is an algebra, and that adjoint representations are denoted by "$\circ$". A straightforward calculation shows that

(X_7) $E \circ \Gamma^r W \subset (E \circ W) \cdot \Gamma^{r-1} W$, $r \geq 1$.

Now fix $s \geq S$ and let $\{u_i\} \subset E_{r_s}$ represent a basis of $(E/I)_{r_s}$ and denote by $A^\lambda \subset (UE)_{\lambda r_s}$ the linear span of the elements $u_{i_1} \dots u_{i_\lambda}$, $i_1 < i_2 < \cdots < i_\lambda$. The map $UE \to UE/I$ maps each A^λ injectively and so it follows from (X_6) that

(X_8) $\dim(A^\lambda \circ z) = \dim A^\lambda$.

Moreover given the inequalities above for $\dim(E/I)_{r_s}$ we have

(X_9) $\frac{1}{\lambda!} e^{(\alpha-\varepsilon)\lambda r_s} \leq \dim A^\lambda \leq e^{(\alpha+\varepsilon)\lambda r_s}$.

Then, from the definition of A^λ, the fact that ΓW is (graded) commutative, and from (X_7) we obtain:

(X_{10}) $$A^\lambda \circ z \subset \sum_{\substack{\lambda_1 + \cdots + \lambda_t = \lambda \\ \lambda_1 \leq \cdots \leq \lambda_t}} (A^{\lambda_1} \circ W_{\leq N}) \dots (A^{\lambda_t} \circ W_{\leq N}) \Gamma^{q-t}(W_{\leq N}) .$$

Let $K = \dim \Gamma^{\leq q}(W_{\leq N})$. Then for any λ_i,

$$\dim A^{\lambda_i} \circ W_{\leq N} \leq K e^{(\alpha+\varepsilon)\lambda_i r_s} .$$

Moreover, since $N < r_s$ it follows from (X_5) that for some integer ℓ, $[\ell r_s, \ell r_s + N] \subset [k_s, \sigma k_s]$. Thus $A^\ell \circ W_{\leq N}$ is contained in a graded vector space isomorphic to a subspace of $L_{[k_s, \sigma k_s]}$. Thus $\dim A^\ell \circ W_{\leq N} \leq K e^{\beta(l r_s + N)}$, by the choice of the sequence k_s and of σ.

Now set $\lambda = \ell q$ in (X_{10}). Then in each summand of (X_{10}) some $\lambda_i \geq \ell$, and so $\dim A^{\lambda_i} \circ W_{\leq N} \leq \dim A^{\lambda_i - \ell} \dim A^\ell \circ W_{\leq N}$. Let ρ be the number of partitions of ℓq. Then (X_{10}) yields

$$\dim A^{\ell q} \circ z \leq \rho e^{(\alpha+\varepsilon)(\ell q - \ell) r_s} e^{\beta(\ell r_s + N)} K^{\ell q + 1} .$$

Apply (X_8) and (X_9) to find

$$\frac{1}{(\ell q)!} e^{(\alpha-\varepsilon)\ell q r_s} \leq \rho \, e^{(\alpha+\varepsilon)\ell q r_s} e^{(\beta-\alpha)\ell r_s} e^{\beta N} K^{\ell q + 1} .$$

This last formula holds for q, N, K, ρ fixed and for arbitrarily large r_s. Moreover, it follows from (X_5) that $\sigma k_s < \frac{(\sigma+1)2m}{\sigma-1} r_s$, and so $\ell = \ell(s) < \frac{(\sigma+1)2m}{\sigma-1}$. Take logs, divide by ℓr_s and let $k_s \to \infty$. Then by (X_5), $r_s \to \infty$ and hence

$$(\alpha - \varepsilon) q \leq (\alpha + \varepsilon) q + (\beta - \alpha) ,$$

whence $(\alpha - \beta) \leq 2q\varepsilon$. But ε was chosen so that $\varepsilon < \frac{1}{2q}(\alpha - \beta)$ and this contradiction establishes (X_4).

It remains to deduce (X_3) from (X_4). Denote by J the sub Lie algebra of L generated by elements of odd degree. Since J is (obviously) an ideal it follows from the Hochschild-Serre spectral sequence,

$$\mathrm{Ext}^p_{U(L/J)}(\Bbbk, \mathrm{Ext}^q_{UJ}(\Bbbk, UL)) \Rightarrow \mathrm{Ext}^{p+q}_{UL}(\Bbbk, UL),$$

that $\operatorname{depth} J \leq \operatorname{depth} L$. Choose a finite set $y_1, \ldots, y_r$ of elements of odd degree such that the sub Lie algebra F generated by the y_i satisfies

$$\mathrm{Ext}_{UJ}(\Bbbk, UJ) \to \mathrm{Ext}_{UF}(\Bbbk, UJ) \text{ is a nonzero map};$$

cf. Lemma 4. Then by Lemma 6, the centralizer of F in J is finite dimensional.

It follows that for ℓ large enough there is some $\lambda(\ell)$ such that $\dim[y_{\lambda(l)}, J_\ell] \geq \frac{1}{r} \dim J_\ell$. Since $J_\ell = L_\ell$ for ℓ odd, it follows that

$$\dim L_{\deg y_{\lambda(\ell)} + \ell} \geq \frac{1}{r} \dim L_\ell, \qquad \text{for } \ell \text{ odd and sufficiently large.}$$

Now let $\ell_1 < \ell_2 < \cdots$ be the infinite sequence provided by (X_4) for β,

$$\frac{\ell_j + 1}{\ell_j} \to 1 \quad \text{and} \quad \liminf_j \frac{\log \dim L_{\ell_j}}{\ell_j} \geq \beta .$$

Deducing X_3 from X_4 means showing the existence of a sequence of even integers with the same property. Let $R = \max\{\deg y_\lambda \mid 1 \leq \lambda \leq r\}$. By choosing a subsequence if necessary we may assume the subsequence (ℓ_j) also satisfies $\ell_{j+1} > R + \ell_j$. Define a new infinite sequence $q_1 < q_2 < \cdots$ of even integers by

$$q_j = \begin{cases} \ell_j & \text{if } \ell_j \text{ is even,} \\ \deg y_{\lambda(\ell_j)} + \ell_j & \text{if } \ell_j \text{ is odd.} \end{cases}$$

Then $\frac{q_{j+1}}{q_j} \leq \frac{\ell_{j+1}+R}{\ell_j} \to 1$, and

$$\frac{\log \dim L_{q_j}}{q_j} \geq \frac{\log(1/r)}{q_j} + \frac{\log \dim L_{\ell_j}}{q_j} \geq \frac{\log(1/r)}{q_j} + \frac{\log \dim L_{\ell_j}}{\ell_j} \cdot \frac{\ell_j}{\ell_j + R}.$$

It follows that $\liminf_j \frac{\log \dim L_{q_j}}{q_j} \geq \beta$.

This completes the proof of (X_3) and of Theorem 3. □

Proof of Theorem 3′. Let d be as in Theorem 3. Clearly

$$\log \operatorname{index} L = \limsup_k \frac{\log \sum_{i=k}^{k+d} \dim L_i}{k}.$$

Thus by Theorem 3, this is less than or equal to $\liminf_k \frac{\log \sum_{i=k}^{k+d} \dim L_i}{k}$ and so lim sup and lim inf coincide, whence the formula in the theorem. □

5. **Growth of the ranks of homotopy groups**

It is a classical result of Serre that the following conditions on a simply connected CW complex X are equivalent: (i) Each $H_i(X;\mathbb{Q})$ is finite dimensional, (ii) Each $\pi_i(X)\otimes\mathbb{Q}$ is finite dimensional, and (iii) X has the rational homotopy type of a CW complex with finitely many cells in each degree. When these conditions hold we say X is *rationally of finite type*.

PROPOSITION 2. *If X is a simply connected* CW *complex that is rationally of finite type then the sequence* $\dim \pi_k(X) \otimes \mathbb{Q}$ *grows at most exponentially if and only if the sequence* $\dim H_k(X;\mathbb{Q})$ *grows at most exponentially.*

Proof. Suppose $\dim \pi_k(X) \otimes \mathbb{Q} \leq e^{\mu k}$ for all k. By Sullivan's theory of minimal models ([17], [7]), $H^*(X;\mathbb{Q}) \cong H(\bigwedge V, d)$, where $V^k \cong \pi_k(X) \otimes \mathbb{Q}$. Now $\bigwedge V$ is a quotient of $\bigoplus_{i=2}^{\infty}(\otimes V^i)$ and so

$$\dim(\textstyle\bigwedge V)^k \leq \sum_{\substack{q_1 \leq \cdots \leq q_m \\ q_1 + \cdots + q_m = k}} \dim V^{q_1} \ldots \dim V^{q_m} \leq \rho(k) e^{\mu k},$$

where $\rho(k)$ is the number of partitions of k. Since $\rho(k) < e^{\pi\sqrt{k}}$ [18, Th. 15.7], it follows that the sequence $\dim H_k(X;\mathbb{Q})$ grows at most exponentially.

Conversely, suppose $\dim H_k(X;\mathbb{Q}) \leq e^{\mu k}$, for all k. Then it follows from the classic result of Adams-Hilton [1] that $H_*(\Omega X;\mathbb{Q}) \cong H(TW, d)$ where $W_k \cong$

$H_{k-1}(X;\mathbb{Q})$. The Hilbert series for W and TW satisfy

$$W(z) \ll \frac{ze^{\mu}}{1-ze^{\mu}} \qquad \text{and} \qquad TW(z) \ll \frac{1-ze^{\mu}}{1-2ze^{\mu}}\,.$$

It follows that $\dim(TW)_k$ grows at most exponentially in k. Since

$$\dim \pi_k(X)\otimes\mathbb{Q} = \dim \pi_{k-1}(\Omega X)\otimes\mathbb{Q} \le \dim H_{k-1}(\Omega X;\mathbb{Q}) \le \dim(TW)_{k-1}$$

it follows that $\dim \pi_k(X)\otimes\mathbb{Q}$ grows at most exponentially in k. □

Recall that the Lusternik-Schnirelmann category of a topological space X, $\operatorname{cat} X$, is the least integer m (or ∞) such that X can be covered by $m+1$ open sets, each contractible in X. The rational LS category of X, $\operatorname{cat}_0 X$, is the LS category of the rationalization $X_{\mathbb{Q}}$ and satisfies $\operatorname{cat}_0 X \le \operatorname{cat} X$.

Recall also that the classical result of Milnor-Moore-Cartan-Serre [16] asserts that $H_*(\Omega X;\mathbb{Q})$ is the universal enveloping algebra of a graded Lie algebra $L_X \cong \pi_*(\Omega X)\otimes\mathbb{Q}$: L_X is called the rational homotopy Lie algebra of X.

THEOREM 4. *Let X be a simply connected* CW *complex satisfying the following three conditions*: (i) *The sequence* $\dim H_k(X;\mathbb{Q})$ *grows at most exponentially*; (ii) $\pi_k(X)\otimes\mathbb{Q}$ *is nonzero for infinitely many k, and* (iii) $\operatorname{cat}_0 X = m-1 < \infty$. *Let α_X denote log index L_X. Then* $0 < \alpha_X < \infty$ *and, for some fixed d, and for any* $\varepsilon > 0$, *there is a $K = K(\varepsilon)$ such that*

$$e^{(\alpha_X-\varepsilon)k} \le \sum_{i=k}^{k+d} \dim \pi_i(X)\otimes\mathbb{Q} \le e^{(\alpha_X+\varepsilon)k} \quad \textit{for all } k \ge K\,.$$

Proof. This is immediate from Theorem 3′ once we have verified that the rational homotopy Lie algebra L_X satisfies hypotheses (X_1) and (X_2) with $\alpha = \alpha_X$.

First note that $\alpha_X < \infty$, by Proposition 2. Then recall from [5, Th. 4.1] that there are a sequence p_j and a constant $C > 1$ such that $\dim(L_X)_{p_j} \ge C^{p_j}$. It follows that $\alpha_X > 0$. Now for simplicity denote L_X simply by L.

Choose a sequence

$$u_1 < u_2 < \cdots$$

such that $(\dim L_{u_i})^{1/u_i} \to e^{\alpha_X}$. Then put $a = \left(\dfrac{1}{2m}\right)^m$. The formula in [5, p. 189] gives a sequence

$$u_i = v_0 < v_1 < \cdots < v_k = u_{i+1}$$

such that for $j \le k$, $2v_{j-1}+2 \le v_j + 2 \le m(v_{j-1}+1)$, and

$$\dim L_{v_j} \ge a(\dim L_{v_{j-1}})^{\frac{v_j+2}{v_{j-1}+1}}\,, \qquad j < k\,.$$

Since $v_j \geq 2v_{j-1}$ we have $v_j \geq 2^j v_0$ and $a^{\frac{1}{v_j+2}} \geq a^{\frac{1}{v_j}} \geq a^{\frac{1}{2^j v_0}}$. Thus

$$\begin{aligned}(\dim L_{v_j})^{\frac{1}{v_j+1}} &\geq (\dim L_{v_j})^{\frac{1}{v_j+2}} \\ &\geq a^{\frac{1}{v_j}} (\dim L_{v_{j-1}})^{\frac{1}{v_{j-1}+1}} \geq a^{\frac{1}{2^j v_0}} (\dim L_{v_{j-1}})^{\frac{1}{v_{j-1}+1}} .\end{aligned}$$

It follows that

$$\left(\dim L_{v_j}\right)^{\frac{1}{v_j+1}} \geq \left(a^2 \, \dim L_{v_0}\right)^{\frac{1}{v_0+1}} , \qquad j < k .$$

Interpolating the sequence u_i with the sequences v_j defines a sequence r_j satisfying $r_{j+1} \leq (m+1) r_j$ and

$$\lim_j \frac{\log \dim L_{r_j}}{r_j} = \alpha_X .$$

Thus (X_1) holds. Finally recall from [4] that depth $L \leq \mathrm{cat}_0 X$, so that (X_2) holds also. □

Theorem 1. Since the subject of Theorem 1 is a simply connected finite CW complex X of dimension n, its rational LS category satisfies $\mathrm{cat}_0 X \leq n/2$. If $\mathrm{rank}\, \pi_k(X) = 0$ for all but finitely many k then it is shown in [12] that $\mathrm{rank}\, \pi_k(X) = 0$, $k \geq 2n$. Otherwise we may apply Theorem 4 to obtain for some d and any $\varepsilon > 0$ there is a $K = K(\varepsilon)$ such that

$$e^{(\alpha_X - \varepsilon)k} \leq \sum_{i=k}^{k+d} \mathrm{rank}\, \pi_i(X) \leq e^{(\alpha_X + \varepsilon)k} , \quad \text{for all } k \geq K .$$

We complete the proof by showing that for $d \geq n$,

$$\liminf_k \frac{\log \left(\sum_{i=k}^{k+d} \mathrm{rank} \;\; \pi_i(X) \right)}{k} = \liminf_k \frac{\log \left(\sum_{i=k+2}^{k+n} \mathrm{rank}\, \pi_i(X) \right)}{k} .$$

Put $h = \dim H_*(X; \mathbb{Q})$. In Corollary 7 of [14] Lambrechts shows that for some ℓ_0 and any $p \geq \ell_0$, there is a $p_1 \in (p, p+n)$ such that $\mathrm{rank}\, \pi_{p_1}(X) \geq \frac{1}{h} \mathrm{rank}\, \pi_p(X)$. The desired inequality follows. Indeed let $\varepsilon > 0$ and choose $K \geq K(\varepsilon/2)$ such that $e^{\alpha_X d}(d+1)h^d \leq e^{\varepsilon K/2}$. Then for any $k \geq K + d - 2$,

$$\sum_{i=k-d+2}^{k+2} \mathrm{rank}\, \pi_i(X) \geq e^{(\alpha_X - \frac{\varepsilon}{2})(k-d+2)} .$$

Thus there is $p \in [k-d+2, k+2]$ such that $\mathrm{rank}\, \pi_p(X) \geq e^{(\alpha_X - \frac{\varepsilon}{2})(k-d+2)}/(d+1)$. It follows from Lambrechts' result that we can extend p to a sequence $p = p_0 < p_1 < \cdots$ such that $p_{i+1} < p_i + n$ and $\mathrm{rank}\, \pi_{p_i}(X) \geq e^{(\alpha_X - \varepsilon/2)(k-d+2)}/((d+1)h^i)$.

In particular for some $i \leq d$, $p_i \in [k+2, k+n]$. Therefore,

$$\sum_{i=k+2}^{k+n} \operatorname{rank} \pi_i(X) \geq \frac{e^{(\alpha_X - \frac{\varepsilon}{2})(k-d+2)}}{(d+1)h^d} \geq e^{(\alpha_X - \varepsilon)k}. \qquad \square$$

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