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Dedicated to Ross Street on his sixtieth birthday

Abstract

The Deligne groupoid is a functor from nilpotent differential graded Lie algebras concentrated in positive degrees to groupoids; in the special case of Lie algebras over a field of characteristic zero, it gives the associated simply connected Lie group. We generalize the Deligne groupoid to a functor γ from L_∞-algebras concentrated in degree $> -n$ to n-groupoids. (We actually construct the nerve of the n-groupoid, which is an enriched Kan complex.) The construction of gamma is quite explicit (it is based on Dupont's proof of the de Rham theorem) and yields higher dimensional analogues of holonomy and of the Campbell-Hausdorff formula.

In the case of abelian L_∞ algebras (i.e., chain complexes), the functor γ is the Dold-Kan simplicial set.

1. Introduction

Let A be a differential graded (dg) commutative algebra over a field K of characteristic 0. Let $\Omega_\bullet$ be the simplicial dg commutative algebra over K whose n-simplices are the algebraic differential forms on the n-simplex Δ^n. Sullivan [Sul77, § 8] introduced a functor

$$A \mapsto \mathsf{Spec}_\bullet(A) = \mathsf{dAlg}(A, \Omega_\bullet)$$

from dg commutative algebras to simplicial sets; here, $\mathsf{dAlg}(A, B)$ is the set of morphisms of dg algebras from A to B. (Sullivan uses the notation $\langle A \rangle$ for this functor.) This functor generalizes the spectrum, in the sense that if A is a commutative

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algebra, $\mathrm{Spec}_\bullet(A)$ is the discrete simplicial set

$$\mathrm{Spec}(A) = \mathsf{Alg}(A, \mathsf{K}),$$

where $\mathsf{Alg}(A, B)$ is the set of morphisms of algebras from A to B.

If E is a flat vector bundle on a manifold M, the complex of differential forms $(\Omega^*(M, E), d)$ is a dg module for the dg Lie algebra $\Omega^*(M, \mathrm{End}(E))$; denote the action by ρ. To a one-form $\alpha \in \Omega^1(M, \mathrm{End}(E))$ is associated a covariant derivative

$$\nabla = d + \rho(\alpha) : \Omega^*(M, E) \to \Omega^{*+1}(M, E).$$

The equation

$$\nabla^2 = \rho\big(d\alpha + \tfrac{1}{2}[\alpha, \alpha]\big)$$

shows that ∇ is a differential if and only if α satisfies the Maurer-Cartan equation

$$d\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0.$$

This example, and others such as the deformation theory of complex manifolds of Kodaira and Spencer, motivates the introduction of the Maurer-Cartan set of a dg Lie algebra $\mathfrak{g}$ [NR66]:

$$\mathsf{MC}(\mathfrak{g}) = \{\alpha \in \mathfrak{g}^1 \mid \delta\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0\}.$$

There is a close relationship between the Maurer-Cartan set and Sullivan's functor $\mathrm{Spec}_\bullet(A)$, which we now explain. The complex of Chevalley-Eilenberg cochains $C^*(\mathfrak{g})$ of a dg Lie algebra $\mathfrak{g}$ is a dg commutative algebra whose underlying graded commutative algebra is the graded symmetric algebra $S(\mathfrak{g}[1]^\vee)$; here, $\mathfrak{g}[1]$ is the shifted cochain complex $(\mathfrak{g}[1])^i = \mathfrak{g}^{i+1}$, and $\mathfrak{g}[1]^\vee$ is its dual.

If $\mathfrak{g}$ is a dg Lie algebra and Ω is a dg commutative algebra, the tensor product complex $\mathfrak{g} \otimes \Omega$ carries a natural structure of a dg Lie algebra, with bracket

$$[x \otimes a, y \otimes b] = (-1)^{|a||y|}[x, y]\, ab.$$

PROPOSITION 1.1. *Let $\mathfrak{g}$ be a dg Lie algebra whose underlying cochain complex is bounded below and finite-dimensional in each degree. Then there is a natural identification between the n-simplices of* $\mathrm{Spec}_\bullet(C^*(\mathfrak{g}))$ *and the Maurer-Cartan elements of $\mathfrak{g} \otimes \Omega_n$.*

Proof. Under the stated hypotheses on $\mathfrak{g}$, there is a natural identification

$$\mathsf{MC}(\mathfrak{g} \otimes \Omega) \cong \mathsf{dAlg}(C^*(\mathfrak{g}), \Omega)$$

for any dg commutative algebra Ω. Indeed, there is an inclusion

$$\mathsf{dAlg}(C^*(\mathfrak{g}), \Omega) \subset \mathsf{Alg}(C^*(\mathfrak{g}), \Omega) = \mathsf{Alg}(S(\mathfrak{g}[1]^\vee), \Omega) \cong (\mathfrak{g} \otimes \Omega)^1.$$

It is easily seen that a morphism in $\mathsf{Alg}(C^*(\mathfrak{g}),\Omega)$ is compatible with the differentials on $C^*(\mathfrak{g})$ and Ω if and only if the corresponding element of $(\mathfrak{g}\otimes\Omega)^1$ satisfies the Maurer-Cartan equation. □

Motivated by this proposition, we introduce for any dg Lie algebra the simplicial set

$$\mathsf{MC}_\bullet(\mathfrak{g}) = \mathsf{MC}(\mathfrak{g}\otimes\Omega_\bullet).$$

According to rational homotopy theory, the functor $\mathfrak{g}\mapsto\mathsf{MC}_\bullet(\mathfrak{g})$ induces a correspondence between the homotopy theories of nilpotent dg Lie algebras over $\mathbb{Q}$ concentrated in degrees $(-\infty,0]$ and nilpotent rational topological spaces. The simplicial set $\mathsf{MC}_\bullet(\mathfrak{g})$ has been studied in great detail by Hinich [Hin97]; he calls it the nerve of $\mathfrak{g}$ and denotes it by $\Sigma(\mathfrak{g})$.

However, the simplicial set $\mathsf{MC}_\bullet(\mathfrak{g})$ is not the subject of this paper. Suppose that $\mathfrak{g}$ is a nilpotent Lie algebra, and let G be the simply-connected Lie group associated to $\mathfrak{g}$. The nerve $N_\bullet G$ of G is substantially smaller than $\mathsf{MC}_\bullet(\mathfrak{g})$, but they are homotopy equivalent. In this paper, we construct a natural homotopy equivalence

$$N_\bullet G\hookrightarrow\mathsf{MC}_\bullet(\mathfrak{g}), \tag{1-1}$$

as a special case of a construction applicable to any nilpotent dg Lie algebra.

To motivate the construction of the embedding (1-1), we may start by comparing the sets of 1-simplices of $N_\bullet G$ and of $\mathsf{MC}_\bullet(\mathfrak{g})$. The Maurer-Cartan equation on $\mathfrak{g}\otimes\Omega_1$ is tautologically satisfied, since $\mathfrak{g}\otimes\Omega_1$ vanishes in degree 2; thus $\mathsf{MC}_1(\mathfrak{g})\cong\mathfrak{g}[t]dt$. Let $\alpha\in\Omega^1(G,\mathfrak{g})$ be the unique left-invariant one-form whose value $\alpha(e):T_eG\to\mathfrak{g}$ at the identity element $e\in G$ is the natural identification between the tangent space T_eG of G at e and its Lie algebra $\mathfrak{g}$. Consider the path space

$$P_*G=\{\tau\in\mathrm{Mor}(\mathbb{A}^1,G)\mid\tau(0)=e\}$$

of algebraic morphisms from the affine line $\mathbb{A}^1$ to G. There is an isomorphism between P_*G and the set $\mathsf{MC}_1(\mathfrak{g})$, induced by associating to a path $\tau:\mathbb{A}^1\to G$ the one-form $\tau^*\alpha$.

There is a foliation of P_*G, whose leaves are the fibres of the evaluation map $\tau\mapsto\tau(1)$, and whose leaf space is G. Under the isomorphism between P_*G and $\mathsf{MC}_1(\mathfrak{g})$, this foliation is simple to characterize: the tangent space to the leaf containing $\alpha\in\mathsf{MC}_1(\mathfrak{g})$ is the image under the covariant derivative

$$\nabla:\mathfrak{g}\otimes\Omega_1^0\to\mathfrak{g}\otimes\Omega_1^1\cong T_\alpha\,\mathsf{MC}_1(\mathfrak{g})$$

of the subspace

$$\{x\in\mathfrak{g}\otimes\Omega_1^0\mid x(0)=x(1)=0\}.$$

The exponential map $\exp : \mathfrak{g} \to G$ is a bijection for nilpotent Lie algebras; equivalently, each leaf of this foliation of $\mathsf{MC}_1(\mathfrak{g})$ contains a unique constant one-form. The embedding $N_1 G \hookrightarrow \mathsf{MC}_1(\mathfrak{g})$ is the inclusion of the constant one-forms into $\mathsf{MC}_1(\mathfrak{g})$.

What is a correct analogue in higher dimensions for the condition that a one-form on Δ^1 is constant? Dupont's explicit proof of the de Rham theorem [Dup76], [Dup78] relies on a chain homotopy $s_\bullet : \Omega_\bullet^* \to \Omega_\bullet^{*-1}$. This homotopy induces maps $s_n : \mathfrak{g} \otimes \Omega_n^1 \to \mathfrak{g} \otimes \Omega_n^0$, and we impose the gauge condition $s_n \alpha = 0$, which when $n = 1$ is the condition that α is constant. The main theorem of this paper shows that the simplicial set

$$\gamma_\bullet(\mathfrak{g}) = \{\alpha \in \mathsf{MC}_\bullet(\mathfrak{g}) \mid s_\bullet \alpha = 0\} \tag{1-2}$$

is isomorphic to the nerve $N_\bullet G$.

The key to the proof of this isomorphism is the verification that $\gamma_\bullet(\mathfrak{g})$ is a Kan complex, that is, that it satisfies the extension condition in all dimensions. In fact, we give explicit formulas for the required extensions, which yield in particular a new approach to the Campbell-Hausdorff formula.

The definition of $\gamma_\bullet(\mathfrak{g})$ works *mutatis mutandi* for nilpotent dg Lie algebras; we argue that $\gamma_\bullet(\mathfrak{g})$ is a good generalization to the differential graded setting of the Lie group associated to a nilpotent Lie algebra. For example, when $\mathfrak{g}$ is a nilpotent dg Lie algebra concentrated in degrees $[0, \infty)$, the simplicial set $\gamma_\bullet(\mathfrak{g})$ is isomorphic to the nerve of the Deligne groupoid $\mathscr{C}(\mathfrak{g})$.

Recall the definition of this groupoid (cf. [GM88]). Let G be the nilpotent Lie group associated to the nilpotent Lie algebra $\mathfrak{g}^0 \subset \mathfrak{g}$. This Lie group acts on $\mathsf{MC}(\mathfrak{g})$ by the formula

$$e^X \cdot \alpha = \alpha - \sum_{n=0}^{\infty} \frac{\operatorname{ad}(X)^n (\delta_\alpha X)}{(n+1)!}. \tag{1-3}$$

The Deligne groupoid $\mathscr{C}(\mathfrak{g})$ of $\mathfrak{g}$ is the groupoid associated to this group action. There is a natural identification between $\pi_0(\mathsf{MC}_\bullet(\mathfrak{g}))$ and $\pi_0(\mathscr{C}(\mathfrak{g})) = \mathsf{MC}(\mathfrak{g})/G$. Following Kodaira and Spencer, we see that this groupoid may be used to study the formal deformation theory of such geometric structures as complex structures on a manifold, holomorphic structures on a complex vector bundle over a complex manifold, and flat connections on a real vector bundle.

In all of these cases, the dg Lie algebra $\mathfrak{g}$ controlling the deformation theory is concentrated in degrees $[0, \infty)$, and the associated formal moduli space is $\pi_0(\mathsf{MC}_\bullet(\mathfrak{g}))$. On the other hand, in the deformation theory of Poisson structures on a manifold, the associated dg Lie algebra, known as the Schouten Lie algebra, is concentrated in degrees $[-1, \infty)$. Thus, the theory of the Deligne groupoid does not apply, and in fact the formal deformation theory is modeled by a 2-groupoid. (This

2-groupoid was constructed by Deligne [Del94], and, independently, by Getzler [Get02, § 2].) The functor $\gamma_\bullet(\mathfrak{g})$ allows the construction of a candidate Deligne ℓ-groupoid, if the nilpotent dg Lie algebra $\mathfrak{g}$ is concentrated in degrees $(-\ell, \infty)$. We present the theory of ℓ-groupoids in Section 2, following Duskin [Dus79], [Dus01] closely.

It seemed most natural in writing this paper to work from the outset with a generalization of dg Lie algebras called L_∞-algebras. We recall the definition of L_∞-algebras in Section 4; these are similar to dg Lie algebras, except that they have a graded antisymmetric bracket $[x_1, \dots, x_k]$, of degree $2-k$, for each k. In the setting of L_∞-algebras, the definition of a Maurer-Cartan element becomes

$$\delta\alpha + \sum_{k=2}^{\infty} \frac{1}{k!} [\underbrace{\alpha, \dots, \alpha}_{k \text{ times}}] = 0.$$

Given a nilpotent L_∞-algebra $\mathfrak{g}$, we define a simplicial set $\gamma_\bullet(\mathfrak{g})$, whose n-simplices are Maurer-Cartan elements $\alpha \in \mathfrak{g} \otimes \Omega_n$ such that $s_n\alpha = 0$. We prove that $\gamma_\bullet(\mathfrak{g})$ is a Kan complex, and that the inclusion $\gamma_\bullet(\mathfrak{g}) \hookrightarrow \mathsf{MC}_\bullet(\mathfrak{g})$ is a homotopy equivalence, by a method similar to that of [Kur62, § 2].

The Dold-Kan functor $K_\bullet(V)$ [Dol58], [Kan58] is a functor from positively graded chain complexes (or equivalently, negatively graded cochain complexes) to simplicial abelian groups. The set of n-simplices of $K_n(V)$ is the abelian group

$$K_n(V) = \mathsf{Chain}(C_*(\Delta^n), V) \tag{1-4}$$

of morphisms of chain complexes from the complex $C_*(\Delta^n)$ of normalized simplicial chains on the simplicial set Δ^n to V. Eilenberg-Mac Lane spaces are obtained when the chain complex is concentrated in a single degree [EML53].

The functor $\gamma_\bullet(\mathfrak{g})$ is a nonabelian analogue of the Dold-Kan functor $K_\bullet(V)$: if $\mathfrak{g}$ is an abelian dg Lie algebra and concentrated in degrees $(-\infty, 1]$, there is a natural isomorphism between $\gamma_\bullet(\mathfrak{g})$ and $K_\bullet(\mathfrak{g}[1])$, since (1-4) has the equivalent form

$$K_n(V) = Z^0(C^*(\Delta^n) \otimes V, d + \delta),$$

where $C^*(\Delta^n)$ is the complex of normalized simplicial cochains on the simplicial set Δ^n.

The functor $\gamma_\bullet$ has many good features: it carries surjective morphisms of nilpotent L_∞-algebras to fibrations of simplicial sets, and carries a large class of weak equivalences of L_∞-algebras to homotopy equivalences. And of course, it yields generalizations of the Deligne groupoid, and of the Deligne 2-groupoid, for L_∞-algebras. It shares with $\mathsf{MC}_\bullet$ an additional property: there is an action of the symmetric group S_{n+1} on the set of n-simplices $\gamma_n(\mathfrak{g})$ making $\gamma_\bullet$ into a functor from L_∞-algebras to symmetric sets, in the sense of [FL91]. In order to simplify the discussion, we have not emphasized this point, but this perhaps indicates that the correct setting for ℓ-groupoids is the category of symmetric sets.

2. Kan complexes and ℓ-groupoids

Kan complexes are a natural nonabelian analogue of chain complexes: just as the homology groups of chain complexes are defined by imposing an equivalence relation on a subset of the chains, the homotopy groups of Kan complexes are defined by imposing an equivalence relation on a subset of the simplices.

Recall the definition of the category of simplicial sets. Let Δ be the category of finite non-empty totally ordered sets. This category Δ has a skeleton whose objects are the ordinals $[n] = (0 < 1 < \cdots < n)$; this skeleton is generated by the face maps $d_k : [n-1] \to [n]$ for $0 \le k \le n$, which are the injective maps

$$d_k(i) = \begin{cases} i & \text{if } i < k, \\ i+1 & \text{if } i \ge k, \end{cases}$$

and the degeneracy maps $s_k : [n] \to [n-1]$ for $0 \le k \le n-1$, which are the surjective maps

$$s_k(i) = \begin{cases} i & \text{if } i \le k, \\ i-1 & \text{if } i > k. \end{cases}$$

A simplicial set $X_\bullet$ is a contravariant functor from Δ to the category of sets. This amounts to a sequence of sets $X_n = X([n])$ indexed by the natural numbers $n \in \{0, 1, 2, \dots\}$, and maps

$$\begin{aligned} \delta_k &= X(d_k) : X_n \to X_{n-1} \quad \text{for } 0 \le k \le n, \\ \sigma_k &= X(s_k) : X_{n-1} \to X_n \quad \text{for } 0 \le k \le n, \end{aligned}$$

satisfying certain relations. (See [May92] for more details.) A degenerate simplex is one of the form $\sigma_i x$; a nondegenerate simplex is one that is not degenerate. Simplicial sets form a category; we denote by $\mathsf{sSet}(X_\bullet, Y_\bullet)$ the set of morphisms between two simplicial sets $X_\bullet$ and $Y_\bullet$.

The geometric n-simplex $\mathbf{\Delta}^n$ is the convex hull of the unit vectors e_k in $\mathbb{R}^{n+1}$:

$$\mathbf{\Delta}^n = \{(t_0, \dots, t_n) \in [0,1]^{n+1} \mid t_0 + \cdots + t_n = 1\}.$$

Its $\binom{n+1}{k+1}$ faces of dimension k are the convex hulls of the nonempty subsets of $\{e_0, \dots, e_n\}$ of cardinality $k+1$.

The n-simplex Δ^n is the representable simplicial set $\Delta^n = \Delta(\,\cdot\,, [n])$. Thus, the nondegenerate simplices of Δ^n correspond to the faces of the geometric simplex $\mathbf{\Delta}^n$. By the Yoneda lemma, $\mathsf{sSet}(\Delta^n, X_\bullet)$ is naturally isomorphic to X_n.

Let $\Delta[k]$ denote the full subcategory of Δ whose objects are the simplices $\{[i] \mid i \le k\}$, and let sk_k be the restriction of a simplicial set from Δ^{op} to $\Delta[k]^{\mathrm{op}}$.

The functor sk_k has a right adjoint cosk_k, called the k-coskeleton, and we have

$$\mathrm{cosk}_k(\mathrm{sk}_k(X))_n = \mathsf{sSet}(\mathrm{sk}_k(\Delta^n), X_\bullet).$$

For $0 \le i \le n$, let $\Lambda^n_i \subset \Delta^n$ be the union of the faces $d_k[\Delta^{n-1}] \subset \Delta^n$ for $k \neq i$. An n-horn in $X_\bullet$ is a simplicial map from Λ^n_i to $X_\bullet$, or equivalently, a sequence of elements

$$(x_0, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n) \in (X_{n-1})^n$$

such that $\partial_j x_k = \partial_{k-1} x_j$ for $0 \le j < k \le n$.

Definition 2.1. A map $f : X_\bullet \to Y_\bullet$ of simplicial sets is a *fibration* if the maps

$$\xi^n_i : X_n \to \mathsf{sSet}(\Lambda^n_i, X_\bullet) \times_{\mathsf{sSet}(\Lambda^n_i, Y_\bullet)} Y_n$$

defined by

$$\xi^n_i(x) = (\partial_0 x, \dots, \partial_{i-1} x, \cdot, \partial_{i+1} x, \dots, \partial_n x) \times f(x)$$

are surjective for all $n > 0$ and $0 \le i \le n$. A simplicial set $X_\bullet$ is a *Kan complex* if the map from $X_\bullet$ to the terminal object Δ^0 is a fibration.

A Kan complex is *minimal* if the face map $\partial_i : X_n \to X_{n-1}$ factors through ξ^n_i for all $n > 0$ and $0 \le i \le n$.

A groupoid is a small category with invertible morphisms. Denote the sets of objects and morphisms of a groupoid G by G_0 and G_1, the source and target maps by $s : G_1 \to G_0$ and $t : G_0 \to G_1$, and the identity map by $e : G_0 \to G_1$. The nerve $N_\bullet G$ of a groupoid G is the simplicial set whose 0-simplices are the objects G_0 of G, and whose n-simplices for $n > 0$ are the composable chains of n morphisms in G:

$$N_n G = \{[g_1, \dots, g_n] \in (G_1)^n \mid s g_i = t g_{i+1}\}.$$

The face and degeneracy maps are defined using the product and the identity of the groupoid:

$$\partial_k[g_1, \dots, g_n] = \begin{cases} [g_2, \dots, g_n] & \text{if } k = 0, \\ [g_1, \dots, g_k g_{k+1}, \dots, g_n] & \text{if } 0 < k < n, \\ [g_1, \dots, g_{n-1}] & \text{if } k = n, \end{cases}$$

$$\sigma_k[g_1, \dots, g_{n-1}] = \begin{cases} [etg_1, g_1, \dots, g_{n-1}] & \text{if } k = 0, \\ [g_1, \dots, g_{k-1}, etg_k, g_k, \dots, g_{n-1}] & \text{if } 0 < k < n, \\ [g_1, \dots, g_{n-1}, esg_{n-1}] & \text{if } k = n. \end{cases}$$

The following characterization of the nerves of groupoids was discovered by Grothendieck; we sketch the proof.

PROPOSITION 2.2. *A simplicial set $X_\bullet$ is the nerve of a groupoid if and only if the maps $\xi^n_i : X_n \to \mathsf{sSet}(\Lambda^n_i, X_\bullet)$ are bijective for all $n > 1$.*

Proof. The nerve of a groupoid is a Kan complex; in fact, it is a very special kind of Kan complex for which the maps ξ_i^2 are not just surjective, but bijective. The unique filler of the horn $(\cdot, g, h)$ is the 2-simplex $[h, h^{-1}g]$, the unique filler of the horn $(g, \cdot, h)$ is the 2-simplex $[h, g]$, and the unique filler of the horn $(g, h, \cdot)$ is the 2-simplex $[hg^{-1}, g]$. Thus, the uniqueness of fillers in dimension 2 exactly captures the associativity of the groupoid and the existence of inverses.

The nerve of a groupoid is determined by its 2-skeleton, in the sense that

$$N_\bullet G \cong \operatorname{cosk}_2(\operatorname{sk}_2(N_\bullet G)). \tag{2-5}$$

It follows from (2-5) and the bijectivity of the maps ξ_i^2 that the maps ξ_i^n are bijective for all $n > 1$.

Conversely, given a Kan complex $X_\bullet$ such that ξ_i^n is bijective for $n > 1$, we can construct a groupoid G such that $X_\bullet \cong N_\bullet G$: $G_i = X_i$ for $i = 0, 1$, $s = \partial_1 : G_1 \to G_0$, $t = \partial_0 : G_1 \to G_0$, and $e = \sigma_0 : G_0 \to G_1$.

Denote by $\langle x_0, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_n\rangle$ the unique n-simplex that fills the horn

$$(x_0, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n) \in \mathsf{sSet}(\Lambda_i^n, X).$$

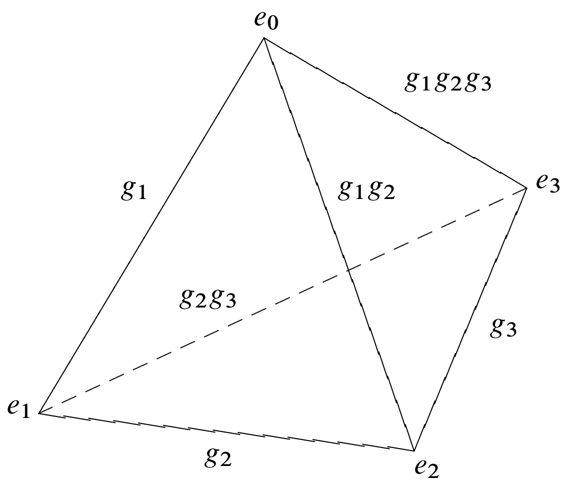
Given a pair of morphisms $g_1, g_2 \in G_1$ such that $sg_1 = tg_2$, define their composition by the formula

$$g_1 g_2 = \partial_1\langle g_2, \cdot, g_1\rangle.$$

Given three morphisms $g_1, g_2, g_3 \in G_1$ such that $sg_1 = tg_2$ and $sg_2 = tg_3$, the 3-simplex $x = [g_1, g_2, g_3] \in X_3$ satisfies

$$g_1(g_2g_3) = \partial_1\partial_2 x = \partial_1\partial_1 x = (g_1g_2)g_3;$$

hence composition in G_1 is associative. Here is a picture of the 3-simplex x:



The inverse of a morphism $g \in G_1$ is defined by the formulas

$$g^{-1} = \partial_0\langle \cdot\,, etg, g\rangle = \partial_2\langle g, esg, \cdot\rangle.$$

To see that these two expressions are equal, call them respectively $g^{-\ell}$ and $g^{-\rho}$, and use associativity:

$$g^{-\ell} = g^{-\ell}(gg^{-\rho}) = (g^{-\ell}g)g^{-\rho} = g^{-\rho}.$$

It follows easily that $(g^{-1})^{-1} = g$, that $g^{-1}g = esg$ and $gg^{-1} = etg$, and that $sg^{-1} = tg$ and $tg^{-1} = sg$.

It is clear that

$$sh = \partial_1\partial_2\langle g, \cdot\,, h\rangle = \partial_1\partial_1\langle g, \cdot\,, h\rangle = s(gh)$$

and that

$$tg = \partial_0\partial_0\langle g, \cdot\,, h\rangle = \partial_0\partial_1\langle g, \cdot\,, h\rangle = t(gh).$$

We also see that

$$\begin{aligned} g &= \partial_1\sigma_1[g] = \partial_1[g, esg] = g(esg) \\ &= \partial_1\sigma_0[g] = \partial_1[etg, g] = (etg)g. \end{aligned}$$

Thus, G is a groupoid. Since $\mathrm{sk}_2(X_\bullet) \cong \mathrm{sk}_2(N_\bullet G)$, we conclude by (2-5) that $X_\bullet \cong N_\bullet G$. □

Duskin has defined a sequence of functors Π_ℓ from the category of Kan complexes to itself, which give a functorial realization of the Postnikov tower. (See [Dus79] and [Gle82], and for a more extended discussion, [Bek04].) Let $\sim_\ell$ be the equivalence relation of homotopy relative to the boundary on the set X_ℓ of ℓ-simplices. Then $\mathrm{sk}_\ell(X_\bullet)/{\sim_\ell}$ is a well-defined ℓ-truncated simplicial set, and there is a map of truncated simplicial sets

$$\mathrm{sk}_\ell(X_\bullet) \to \mathrm{sk}_\ell(X_\bullet)/{\sim_\ell},$$

and by adjunction, a map of simplicial sets

$$X_\bullet \to \mathrm{cosk}_\ell(\mathrm{sk}_\ell(X_\bullet)/{\sim_\ell}).$$

Define $\Pi_\ell(X_\bullet)$ to be the image of this map. Then the functor Π_ℓ is an idempotent monad on the category of Kan complexes. If $x_0 \in X_0$, we have

$$\pi_i(X_\bullet, x_0) = \begin{cases} \pi_i(\Pi_\ell(X_\bullet), x_0) & \text{if } i \le \ell, \\ 0 & \text{if } i > \ell. \end{cases}$$

Thus $\Pi_\ell(X_\bullet)$ is a realization of the Postnikov ℓ-section of the simplicial set $X_\bullet$. For example, $\Pi_0(X_\bullet)$ is the discrete simplicial set $\pi_0(X_\bullet)$, and $\Pi_1(X_\bullet)$ is the nerve of the fundamental groupoid of $X_\bullet$. It is interesting to compare $\Pi_\ell(X_\bullet)$ to other realizations of the Postnikov tower, such as $\mathrm{cosk}_{\ell+1}(\mathrm{sk}_{\ell+1}(X_\bullet))$: it is a more economic realization of this homotopy type, and has a more geometric character.

We now recall Duskin's notion of higher groupoid: he calls these ℓ-dimensional hypergroupoids, but we simply call them weak ℓ-groupoids.

Definition 2.3. A Kan complex $X_\bullet$ is a *weak ℓ-groupoid* if $\Pi_\ell(X_\bullet) = X_\bullet$ or, equivalently, if the maps ξ^n_i are bijective for $n > \ell$; it is a *weak ℓ-group* if in addition it is reduced (has a single 0-simplex).

The 0-simplices of an ℓ-groupoid are interpreted as its objects and the 1-simplices as its morphisms. The composition gh of a pair of 1-morphisms with $\partial_1 g = \partial_0 h$ equals $\partial_1 z$, where $z \in X_2$ is a filler of the horn

$$(g, \cdot, h) \in \mathsf{sSet}(\Lambda^2_1, X_\bullet).$$

If $\ell > 1$, this composition is not canonical — it depends on the choice of the filler $z \in X_2$ — but it is associative up to a homotopy, by the existence of fillers in dimension 3.

A weak 0-groupoid is a discrete set, while a weak 1-groupoid is the nerve of a groupoid, by Proposition 2.2. Duskin [Dus01] identifies weak 2-groupoids with the nerves of bigroupoids. A bigroupoid G is a bicategory whose 2-morphisms are invertible and whose 1-morphisms are equivalences; the nerve $N_\bullet G$ of G is a simplicial set whose 0-simplices are the objects of G, whose 1-simplices are the morphism of G, and whose 2-simplices x are the 2-morphisms with source $\partial_2 x \circ \partial_0 x$ and target $\partial_1 x$.

The singular complex of a topological space is the simplicial set

$$S_n(X) = \mathrm{Map}(\mathbf{\Delta}^n, X).$$

To see that this is a Kan complex, we observe that there is a continuous retraction from $\mathbf{\Delta}^n = |\Delta^n|$ to $|\Lambda^n_i|$. The fundamental ℓ-groupoid of a topological space X is the weak ℓ-groupoid $\Pi_\ell(S_\bullet(X))$. For $\ell = 0$, this equals $\pi_0(X)$, while for $\ell = 1$, it is the nerve of the fundamental groupoid of X.

Often, weak ℓ-groupoids come with explicit choices for fillers of horns: tentatively, we refer to such weak ℓ-groupoids as ℓ-groupoids. (Often, this term is used for what we call strict ℓ-groupoids, but the latter are of little interest for $\ell > 2$.) We may axiomatize ℓ-groupoids by a weakened form of the axioms for simplicial T-complexes, studied by Dakin [Dak83] and Ashley [Ash88].

Definition 2.4. An *ℓ-groupoid* is a simplicial set $X_\bullet$ together with a set of thin elements $T_n \subset X_n$ for each $n > 0$, satisfying the following conditions:

(i) every degenerate simplex is thin;

(ii) every horn has a unique thin filler;

(iii) every n-simplex is thin if $n > \ell$.

If $\mathfrak{g}$ is an ℓ-groupoid and $n > \ell$, we denote by $\langle x_0, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n \rangle$ the unique thin filler of the horn

$$(x_0, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n) \in \mathsf{sSet}(\Lambda^n_i, X_\bullet).$$

Definition 2.5. An ∞-*groupoid* is a simplicial set $X_\bullet$ together with a set of thin elements $T_n \subset X_n$ for each $n > 0$, satisfying the following conditions:

(i) every degenerate simplex is thin;

(ii) every horn has a unique thin filler.

It is clear that every ℓ-groupoid is a weak ℓ-groupoid and that every ∞-groupoid is a Kan complex. Not every weak ℓ-groupoid underlies an ℓ-groupoid. However, if $X_\bullet$ is a weak ℓ-groupoid, then any minimal simplicial subcomplex $Z_\bullet$ of $X_\bullet$ underlies an ℓ-groupoid; it suffices to take the set of thin n-simplices $T_n \subset Z_n$ to be a section of the map $\xi^n_0 : Z_n \to \mathsf{sSet}(\Lambda^n_0, Z_\bullet)$, taking care to select the (necessarily unique) degenerate simplex in each fiber of ξ^n_0 when there is one.

Note also that while the nerve of a bigroupoid is a weak 2-groupoid in our sense, it is not in general a 2-groupoid unless the identity and inverse 1-morphisms are strict.

The Dold-Kan simplicial set $K_\bullet(V)$ is an ℓ-groupoid if and only if V_i vanishes for $i > \ell$; it is minimal if and only if V has vanishing differential. In Section 5, we will find analogues of these observations for L_∞-algebras.

3. The simplicial de Rham theorem

Let Ω_n be the free graded commutative algebra over K with generators t_i of degree 0 and dt_i of degree 1, and relations $T_n = 0$ and $dT_n = 0$, where $T_n = t_0 + \cdots + t_n - 1$:

$$\Omega_n = \mathsf{K}[t_0, \dots, t_n, dt_0, \dots, dt_n]/(T_n, dT_n).$$

There is a unique differential on Ω_n such that $d(t_i) = dt_i$ and $d(dt_i) = 0$.

The dg commutative algebras Ω_n are the components of a simplicial dg commutative algebra $\Omega_\bullet$: the simplicial map $f : [k] \to [n]$ acts by the formula

$$f^* t_i = \sum_{f(j)=i} t_j \quad \text{for } 0 \le i \le n.$$

Using the simplicial dg commutative algebra $\Omega_\bullet$, we can define the dg commutative algebra of piecewise polynomial differential forms $\Omega(X_\bullet)$ on a simplicial set $X_\bullet$ [Sul77], [BG76], [Dup76], [Dup78].

Definition 3.1. The complex of differential forms $\Omega(X_\bullet)$ on a simplicial set $X_\bullet$ is the space $\Omega(X_\bullet) = \mathsf{sSet}(X_\bullet, \Omega_\bullet)$ of simplicial maps from $X_\bullet$ to $\Omega_\bullet$.

When $\mathsf{K} = \mathbb{R}$ is the field of real numbers, $\Omega(X_\bullet)$ may be identified with the complex of differential forms on the realization $|X_\bullet|$ that are polynomial on each geometric simplex of $|X_\bullet|$.

The following lemma may be found in [BG76]; we learned this short proof from a referee.

LEMMA 3.2. *For each $k \geq 0$, the simplicial abelian group $\Omega^k_\bullet$ is contractible.*

Proof. The homotopy groups of the simplicial set $\Omega^k_\bullet$ equal the homology groups of the complex $C_\bullet = \Omega^k_\bullet$ with differential

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i : C_n \to C_{n-1}.$$

Thus, to prove the lemma, it suffices to construct a contracting chain homotopy for the complex $C_\bullet$.

For $0 \leq i \leq n$, let $\pi_i : \mathbf{\Delta}^{n+1} \to \mathbf{\Delta}^n$ be the affine map

$$\pi_i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_{i-1}, t_i + t_{n+1}, t_{i+1}, \dots, t_n).$$

Define a chain homotopy $\eta : C_n \to C_{n+1}$ by $\eta\omega = (-1)^{n+1} \sum_{i=0}^n t_i\, \pi_i^* \omega$. For $\omega \in \Omega^k_n$, we see that

$$\partial_i \eta\omega = \begin{cases} -\eta\partial_i\omega & \text{if } 0 \leq i \leq n, \\ (-1)^{n+1}\omega & \text{if } i = n+1. \end{cases}$$

It follows that $(\partial\eta + \eta\partial)\omega = \omega$. □

Given a sequence $(i_0, \dots, i_k)$ of elements of the set $\{0, \dots, n\}$, let

$$I_{i_0 \dots i_k} : \Omega_n \to \mathsf{K}$$

be the integral over the k-chain on the n-simplex spanned by the sequence of vertices $(e_{i_0}, \dots, e_{i_k})$; this is defined by the explicit formula

$$I_{i_0 \dots i_k}\big(t_{i_1}^{a_1} \cdots t_{i_k}^{a_k}\, dt_{i_1} \cdots dt_{i_k}\big) = \frac{a_1! \cdots a_k!}{(a_1 + \cdots + a_k + k)!}.$$

Specializing K to the field of real numbers, this becomes the usual Riemann integral.

The space C_n of elementary forms is spanned by the differential forms

$$\omega_{i_0 \dots i_k} = k! \sum_{j=0}^{k} (-1)^j t_{i_j}\, dt_{i_0} \cdots \widehat{dt}_{i_j} \cdots dt_{i_k}.$$

(The coefficient $k!$ normalizes the form so that $I_{i_0 \dots i_k}(\omega_{i_0 \dots i_k}) = 1$.) The spaces C_n are closed under the action of the exterior differential, that is,

$$d\omega_{i_0 \dots i_k} = \sum_{i=0}^{n} \omega_{i\, i_0 \dots i_k},$$

and assemble to a simplicial subcomplex of $\Omega_\bullet$. The complex C_n is isomorphic to the complex of simplicial chains on Δ^n, and this isomorphism is compatible with the simplicial structure. Whitney [Whi57] constructs an explicit projection P_n from Ω_n to C_n:

$$P_n\omega = \sum_{k=0}^{n} \sum_{i_0<\cdots<i_k} \omega_{i_0\ldots i_k}\, I_{i_0\ldots i_k}(\omega). \tag{3-6}$$

The projections P_n assemble to form a morphism of simplicial cochain complexes $P_\bullet : \Omega_\bullet \to C_\bullet$. If $X_\bullet$ is a simplicial set, the complex of elementary forms

$$C(X_\bullet) = \mathsf{sSet}(X_\bullet, C_\bullet) \subset \Omega(X_\bullet)$$

on $X_\bullet$ is naturally isomorphic to the complex of normalized simplicial cochains.

Definition 3.3. A *contraction* is a simplicial endomorphism $s_\bullet : \Omega^*_\bullet \to \Omega^{*-1}_\bullet$ such that

$$\mathrm{id} - P_\bullet = ds_\bullet + s_\bullet d. \tag{3-7}$$

If $X_\bullet$ is a simplicial complex, a contraction $s_\bullet$ induces a chain homotopy $s : \Omega^*(X_\bullet) \to \Omega^{*-1}(X_\bullet)$ between the complex of differential forms on $X_\bullet$ and the complex $C(X_\bullet)$ of simplicial cochains. In other words, a contraction is an explicit form of the de Rham theorem.

Next, we derive some simple properties of a contraction which we will need later. If a and b are operators on a chain complex homogeneous of degree k and ℓ respectively, we denote by $[a, b]$ the graded commutator

$$[a, b] = ab - (-1)^{k\ell} ba.$$

In particular, of a is homogeneous of odd degree, then $\frac{1}{2}[a, a] = a^2$.

LEMMA 3.4. *Let $s_\bullet$ be a contraction. Then*

(i) $P_\bullet s_\bullet = 0$ *and*

(ii) $s_\bullet P_\bullet = [d, (s_\bullet)^2]$.

Proof. To show that $P_\bullet s_\bullet = 0$, we must check that $I_{i_0\ldots i_k} \circ s_n = 0$ for each sequence $(i_0 \ldots i_k)$. By the compatibility of $s_\bullet$ with simplicial maps, this follows from the formula

$$I_{0\ldots k} \circ s_k = 0,$$

which is clear, since $s_k\omega$ is a differential form on $\mathbf{\Delta}^k$ of degree less than k.

The second part of the lemma is a simple calculation. □

Dupont [Dup76], [Dup78] found an explicit contraction: we now recall his formula. Given $0 \le i \le n$, define the dilation map $\varphi_i : [0, 1] \times \mathbf{\Delta}^n \to \mathbf{\Delta}^n$ by the formula

$$\varphi_i(u,\mathbf{t}) = u\mathbf{t} + (1-u)e_i .$$

Let $\pi_* : \Omega^*([0,1]\times\mathbf{\Delta}^n) \to \Omega^{*-1}(\mathbf{\Delta}^n)$ be integration along the fibers of the projection $\pi : [0,1]\times\mathbf{\Delta}^n \to \mathbf{\Delta}^n$. Define the operator $h^i_n : \Omega^*_n \to \Omega^{*-1}_n$ by the formula

$$h^i_n\omega = \pi_*\varphi_i^*\omega, \tag{3-8}$$

Let $\varepsilon^i_n : \Omega_n \to \mathsf{K}$ be evaluation at the vertex e_i. Stokes's theorem implies the Poincaré lemma, that h^i_n is a chain homotopy between the identity and ε^i_n:

$$dh^i_n + h^i_n d = \mathrm{id}_n - \varepsilon^i_n . \tag{3-9}$$

The flow $\varphi_i(u)$ is generated by the vector field $E_i = \sum_{j=0}^n (t_j - \delta_{ij})\partial_j$. Let ι_i be the contraction $\iota(E_i)$: we have

$$\iota_j\varphi_i(u) = \varphi_i(u)(u\iota_j + (1-u)\iota_i) \tag{3-10}$$

and also

$$\iota_i\omega_{i_0\dots i_k} = k\sum_{p=0}^k (-1)^{p-1}\delta_{ii_p}\,\omega_{i_0\dots\widehat{\imath_p}\dots i_k} . \tag{3-11}$$

The formula (3-8) for h^i_n may be written more explicitly as

$$h^i_n = \int_0^1 u^{-1}\varphi_i(u)\iota_i\,du.$$

LEMMA 3.5. $h^ih^j + h^jh^i = 0$.

Proof. Let $\varphi_{ij} : [0,1]\times[0,1]\times\mathbf{\Delta}^n \to \mathbf{\Delta}^n$ be the map

$$\varphi_{ij}(u,v,\mathbf{t}) = uvt_k + (1-u)e_i + u(1-v)e_j .$$

Then we have $h^ih^j\omega = \pi_*\varphi^*_{ij}\omega$. We have $\varphi_{ji}(u,v) = \varphi_{ij}(\tilde v,\tilde u)$, where $\tilde u$ and $\tilde v$ are determined implicitly by the equations

$$(1-u)v = 1-\tilde u \quad\text{and}\quad 1-v = (1-\tilde v)\tilde u .$$

Since this change of variables is a diffeomorphism of the interior of the square $[0,1]\times[0,1]$, the lemma follows. □

LEMMA 3.6. $I_{i_0\dots i_k}(\omega) = (-1)^k\,\varepsilon^{i_k}_n h^{i_{k-1}}_n \dots h^{i_0}_n\omega$.

Proof. For $k=0$, this holds by definition. We argue by induction on k. We may assume that ω has positive degree and hence that $\omega = d\nu$ is exact. By Stokes's theorem,

$$I_{i_0\dots i_k}(d\nu) = \sum_{j=0}^k (-1)^{j-1} I_{i_0\dots\widehat{\imath_j}\dots i_k}(\nu).$$

On the other hand, by (3-9), we have

$$\begin{aligned}\varepsilon_n^{i_k} h_n^{i_{k-1}} \cdots h_n^{i_0} d\nu &= \sum_{j=0}^{k-1} (-1)^j \varepsilon_n^{i_k} h_n^{i_{k-1}} \cdots [d, h_n^{i_j}] \cdots h_n^{i_0} \nu \\ &= \sum_{j=0}^{k-1} (-1)^j \varepsilon_n^{i_k} h_n^{i_{k-1}} \cdots \widehat{h_n^{i_j}} \cdots h_n^{i_0} \nu + (-1)^k \varepsilon_n^{i_k} \varepsilon_n^{i_{k-1}} h_n^{i_{k-2}} \cdots h_n^{i_0} \nu.\end{aligned}$$

But $\varepsilon_n^{i_k} \varepsilon_n^{i_{k-1}} = \varepsilon_n^{i_{k-1}}$. □

THEOREM 3.7 (Dupont). *The operators*

$$s_n = \sum_{k=0}^{n-1} \sum_{i_0 < \cdots < i_k} \omega_{i_0 \dots i_k} h_n^{i_k} \cdots h_n^{i_0} \quad \text{for } n \geq 0 \tag{3-12}$$

form a contraction.

Proof. It is straightforward to check that $s_\bullet$ is simplicial. In the proof of (3-7), we abbreviate h_n^i to h^i. In the definition of s_n, we may take the upper limit of the sum over k to be n. We now have

$$\begin{aligned}[d, s_n] = \sum_{k=0}^{n-1} \sum_{i_0 < \cdots < i_k} \sum_{i \notin \{i_0, \dots, i_k\}} \omega_{i\, i_0 \dots i_k} h^{i_k} \cdots h^{i_0} \\ + \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \sum_{i_0 < \cdots < i_k} \omega_{i_0 \dots i_k} h^{i_k} \cdots [d, h^{i_j}] \cdots h^{i_0}.\end{aligned} \tag{3-13}$$

By (3-9), we have

$$\begin{aligned}\sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \sum_{i_0 < \cdots < i_k} \omega_{i_0 \dots i_k} h^{i_k} \cdots [d, h^{i_j}] \cdots h^{i_0} \\ = \mathrm{id} + \sum_{k=1}^{n} \sum_{j=0}^{k} (-1)^j \sum_{i_0 < \cdots < i_k} \omega_{i_0 \dots i_k} h^{i_k} \cdots \widehat{h}^{i_j} \cdots h^{i_0} \\ - \sum_{k=0}^{n} (-1)^k \sum_{i_0 < \cdots < i_k} \omega_{i_0 \dots i_k} \varepsilon^{i_k} h^{i_{k-1}} \cdots h^{i_0}.\end{aligned}$$

The first term on the right side equals the identity operator, the second cancels the first sum of (3-13), while by Lemma 3.6, the third sum equals P_n. □

We will need special class of contractions, which we call gauges.

Definition 3.8. A *gauge* is a contraction such that $(s_\bullet)^2 = 0$.

In fact, Dupont's operator $s_\bullet$ is a gauge. But by a trick of Lambe and Stasheff [LS87], any contraction gives rise to a gauge.

PROPOSITION 3.9. *If $s_\bullet$ is a contraction, then the operator*

$$\tilde{s}_\bullet = s_\bullet d s_\bullet\,(\mathrm{id} - P_\bullet)$$

is a gauge. If $s_\bullet$ is a gauge, then $\tilde{s}_\bullet = s_\bullet$.

Proof. Let $\bar{s}_\bullet$ be the contraction $\bar{s}_\bullet = s_\bullet(\mathrm{id} - P_\bullet)$. By construction, we have $\bar{s}_\bullet P_\bullet = 0$; hence by Lemma 3.4, $[d, (\bar{s}_\bullet)^2] = 0$. Then $\tilde{s}_\bullet = \bar{s}_\bullet d\bar{s}_\bullet$ is a contraction:

$$\begin{aligned}[d, \tilde{s}_\bullet] = [d, \bar{s}_\bullet d\bar{s}_\bullet] &= [d, \bar{s}_\bullet] d\bar{s}_\bullet + \bar{s}_\bullet d[d, \bar{s}_\bullet] \\ &= (\mathrm{id} - P_\bullet) d\bar{s}_\bullet + \bar{s}_\bullet d(\mathrm{id} - P_\bullet) \\ &= d(\mathrm{id} - P_\bullet)\bar{s}_\bullet + \bar{s}_\bullet(\mathrm{id} - P_\bullet)d = [d, \bar{s}_\bullet] = \mathrm{id} - P_\bullet.\end{aligned}$$

Since $d(\bar{s}_\bullet)^2 d = (\bar{s}_\bullet)^2 d^2 = 0$, the operator $\tilde{s}_\bullet$ is a gauge:

$$(\tilde{s}_\bullet)^2 = (\bar{s}_\bullet d\bar{s}_\bullet)(\bar{s}_\bullet d\bar{s}_\bullet) = \bar{s}_\bullet d(\bar{s}_\bullet)^2 d\bar{s}_\bullet = 0.$$

If $s_\bullet$ happens to be a gauge, then $s_\bullet P_\bullet = 0$ by Lemma 3.4. It follows that

$$\begin{aligned}\tilde{s}_\bullet - s_\bullet &= s_\bullet(ds_\bullet(\mathrm{id} - P_\bullet) - \mathrm{id}) \\ &= s_\bullet(ds_\bullet - \mathrm{id}) \\ &= -s_\bullet(s_\bullet d + P_\bullet) = -(s_\bullet)^2 d + s_\bullet P_\bullet = 0.\end{aligned}$$ □

We now turn to the proof that Dupont's operator $s_\bullet$ is a gauge. Denote by $\varepsilon(\alpha)$ the operation of multiplication by a differential form α on Ω_n.

LEMMA 3.10. *If $i \notin \{i_0, \dots, i_k\}$, then*

$$\varepsilon(\omega_{i_0\dots i_k})h^i = (-1)^k h^i\big(\varepsilon(\omega_{i_0\dots i_k}) + \varepsilon(\omega_{i_0\dots i_k i})h^i\big).$$

Proof. We have

$$\begin{aligned}(-1)^k h^i \varepsilon(\omega_{i_0\dots i_k}) &= (-1)^k \int_0^1 w^{-1}\varphi_i(w)\iota_i\,\varepsilon(\omega_{i_0\dots i_k})\,dw \\ &= \varepsilon(\omega_{i_0\dots i_k}) \int_0^1 w^k\,\varphi_i(w)\,\iota_i\,dw.\end{aligned}$$

On the other hand, by (3-11),

$$\begin{aligned}(-1)^k h^i \varepsilon(\omega_{i_0\dots i_k i})h^i &= (-1)^k \int_0^1\int_0^1 (uv)^{-1}\varphi_i(u)\,\iota_i\,\varepsilon(\omega_{i_0\dots i_k i})\,\varphi_i(v)\,\iota_i\,dv\,du \\ &= (k+1)\int_0^1\int_0^1 (uv)^{-1}\varphi_i(u)\,\varepsilon(\omega_{i_0\dots i_k})\,\varphi_i(v)\,\iota_i\,dv\,du \\ &= (k+1)\,\varepsilon(\omega_{i_0\dots i_k})\int_0^1\int_0^1 u^k v^{-1}\varphi_i(uv)\,\iota_i\,dv\,du.\end{aligned}$$

Changing variables from u to $w = uv$, we see that

$$\int_0^1 \int_0^1 u^k v^{-1} \varphi_i(uv)\, dv\, du = \int_0^1 \left(\int_w^1 v^{-k-2}\, dv \right) w^k \varphi_i(w)\, dw$$
$$= (k+1)^{-1} \int_0^1 (w^{-1} - w^k) \varphi_i(w)\, dw,$$

establishing the lemma. □

THEOREM 3.11. *The operator $s_\bullet$ is a gauge.*

Proof. By induction on k, the above lemma shows that

$$h^{i_k} \cdots h^{i_0} s = \sum_{\ell=0}^{n-1} (-1)^{k\ell+\ell} \sum_{\substack{j_0 < \cdots < j_\ell \\ \{i_0,\dots,i_k\} \bigcap \{j_0,\dots,j_\ell\} = \varnothing}} \omega_{j_0 \dots j_\ell} h^{i_k} \cdots h^{i_0} h^{j_\ell} \cdots h^{j_0}.$$

It follows that s^2 is equal to

$$\sum_{k,\ell=0}^{\infty} (-1)^{k\ell+\ell} \sum_{\substack{i_0 < \cdots < i_k;\ j_0 < \cdots < j_\ell \\ \{i_0,\dots,i_k\} \bigcap \{j_0,\dots,j_\ell\} = \varnothing}} \omega_{i_0 \dots i_k} \omega_{j_0 \dots j_\ell} h^{i_k} \cdots h^{i_0} h^{j_\ell} \cdots h^{j_0}. \tag{3-14}$$

We have

$$\omega_{i_0 \dots i_k} \omega_{j_0 \dots j_\ell} h^{i_k} \cdots h^{i_0} h^{j_\ell} \cdots h^{j_0}$$
$$= (-1)^{k\ell + (k+1)(\ell+1)} \omega_{j_0 \dots j_\ell} \omega_{i_0 \dots i_k} h^{j_\ell} \cdots h^{j_0} h^{i_k} \cdots h^{i_0}.$$

The expression (3-14) changes sign on exchange of $(i_0, \dots, i_k)$ and $(j_0, \dots, j_\ell)$, and thus vanishes. □

4. The Maurer-Cartan set of an L_∞-algebra

L_∞-algebras are a generalization of dg Lie algebras in which the Jacobi rule is only satisfied up to a hierarchy of higher homotopies. In this section, we start by recalling the definition of L_∞-algebras. Following [Sul77] and [Hin97], we represent the homotopy type of an L_∞-algebra $\mathfrak{g}$ by the simplicial set $\mathsf{MC}_\bullet(\mathfrak{g}) = \mathsf{MC}(\mathfrak{g} \otimes \Omega_\bullet)$. We prove that this is a Kan complex, and that under certain additional hypotheses, it is a homotopy invariant of the L_∞-algebra $\mathfrak{g}$.

An operation $[x_1, \dots, x_k]$ on a graded vector space $\mathfrak{g}$ is called graded antisymmetric if

$$[x_1, \dots, x_i, x_{i+1}, \dots, x_k] + (-1)^{|x_i||x_{i+1}|} [x_1, \dots, x_{i+1}, x_i, \dots, x_k] = 0$$

for all $1 \le i \le k-1$. Equivalently, $[x_1, \dots, x_k]$ is a linear map from $\bigwedge^k \mathfrak{g}$ to $\mathfrak{g}$, where $\bigwedge^k \mathfrak{g}$ is the k-th exterior power of the graded vector space $\mathfrak{g}$, that is, the k-th symmetric power of $s^{-1}\mathfrak{g}$.

Definition 4.1. An L_∞-*algebra* is a graded vector space $\mathfrak{g}$ with a sequence $[x_1,\dots,x_k]$ for $k>0$ of graded antisymmetric operations of degree $2-k$, or equivalently, homogeneous linear maps $\bigwedge^k \mathfrak{g} \to \mathfrak{g}$ of degree 2, such that for each $n>0$, the n-Jacobi rule holds:

$$\sum_{k=1}^n (-1)^k \sum_{\substack{i_1<\dots<i_k;\ j_1<\dots<j_{n-k} \\ \{i_1,\dots,i_k\}\bigcup\{j_1,\dots,j_{n-k}\}=\{1,\dots,n\}}} (-1)^\varepsilon [[x_{i_1},\dots,x_{i_k}],x_{j_1},\dots,x_{j_{n-k}}]=0.$$

Here, the sign $(-1)^\varepsilon$ equals the product of the sign $(-1)^\pi$ associated to the permutation

$$\pi = \begin{pmatrix} 1 & \dots & k & k+1 & \dots & n \\ i_1 & \dots & i_k & j_1 & \dots & j_{n-k} \end{pmatrix}$$

with the sign associated by the Koszul sign convention to the action of π on the elements $(x_1,\dots,x_n)$ of $\mathfrak{g}$.

In terms of the graded symmetric operations

$$\ell_k(y_1,\dots,y_k) = (-1)^{\sum_{i=1}^k (k-i+1)|y_i|} s^{-1}[sy_1,\dots,sy_k]$$

of degree 1 on the graded vector space $s^{-1}\mathfrak{g}$, the Jacobi rule simplifies to

$$\sum_{k=1}^n \sum_{\substack{i_1<\dots<i_k;\ j_1<\dots<j_{n-k} \\ \{i_1,\dots,i_k\}\bigcup\{j_1,\dots,j_{n-k}\}=\{1,\dots,n\}}} (-1)^{\tilde\varepsilon} \{\{y_{i_1},\dots,y_{i_k}\},y_{j_1},\dots,y_{j_{n-k}}\}=0,$$

where $(-1)^{\tilde\varepsilon}$ is the sign associated by the Koszul sign convention to the action of π on the elements $(y_1,\dots,y_n)$ of $s^{-1}\mathfrak{g}$. This is a small modification of the conventions of Lada and Markl [LM95]: their operations l_k are related to ours by a sign

$$l_k(x_1,\dots,x_k) = (-1)^{\binom{k+1}{2}}[x_1,\dots,x_k].$$

The operation $x \mapsto [x]$ makes the graded vector space $\mathfrak{g}$ into a cochain complex, by the 1-Jacobi rule $[[x]]=0$. Because of the special role played by the operation $[x]$, we denote it by δ. An L_∞-algebra with $[x_1,\dots,x_k]=0$ for $k>2$ is the same thing as a dg Lie algebra. A quasi-isomorphism of L_∞-algebras is a quasi-isomorphism of the underlying cochain complexes.

The lower central filtration on an L_∞-algebra $\mathfrak{g}$ is the canonical decreasing filtration defined inductively by $F^1\mathfrak{g}=\mathfrak{g}$ and, for $i>1$,

$$F^i\mathfrak{g} = \sum_{i_1+\dots+i_k=i} [F^{i_1}\mathfrak{g},\dots,F^{i_k}\mathfrak{g}].$$

Definition 4.2. An L_∞-algebra $\mathfrak{g}$ is *nilpotent* if the lower central series terminates, that is, if $F^i\mathfrak{g}=0$ for $i \gg 0$.

If $\mathfrak{g}$ is a nilpotent L_∞-algebra, the curvature

$$\mathscr{F}(\alpha) = \delta\alpha + \sum_{\ell=2}^{\infty} \frac{1}{\ell!}[\alpha^{\wedge\ell}] \in \mathfrak{g}^2$$

is defined and polynomial in α. If $\mathfrak{g}$ is a dg Lie algebra, the curvature equals

$$\mathscr{F}(\alpha) = \delta\alpha + \tfrac{1}{2}[\alpha, \alpha];$$

this expression is familiar from the theory of connections on principal bundles.

Definition 4.3. The *Maurer-Cartan set* $\mathsf{MC}(\mathfrak{g})$ of a nilpotent L_∞-algebra $\mathfrak{g}$ is the set of those $\alpha \in \mathfrak{g}^1$ satisfying the Maurer-Cartan equation

$$\mathscr{F}(\alpha) = 0. \tag{4-15}$$

An L_∞-algebra is abelian if the bracket $[x_1, \dots, x_k]$ vanishes for $k > 1$. In this case, the Maurer-Cartan set is the set of 1-cocycles $Z^1(\mathfrak{g})$ of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a nilpotent L_∞-algebra. For any element $\alpha \in \mathfrak{g}^1$, the formula

$$[x_1, \dots, x_k]_\alpha = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}[\alpha^{\wedge\ell}, x_1, \dots, x_k]$$

defines a new sequence of brackets on $\mathfrak{g}$, where $[\alpha^{\wedge\ell}, x_1, \dots, x_k]$ is an abbreviation for $[\alpha, \dots, \alpha, x_1, \dots, x_k]$, in which α occurs l times.

PROPOSITION 4.4. *If* $\alpha \in \mathsf{MC}(\mathfrak{g})$, *then the brackets* $[x_1, \dots, x_k]_\alpha$ *make* $\mathfrak{g}$ *into an* L_∞*-algebra.*

Proof. Applying the $(m+n)$-Jacobi relation to the sequence

$$(\alpha^{\wedge m}, x_1, \dots, x_n)$$

and summing over m, we obtain the n-Jacobi relation for the brackets $[x_1, \dots, x_k]_\alpha$. □

LEMMA 4.5. *The curvature satisfies the Bianchi identity*

$$\delta\mathscr{F}(\alpha) + \sum_{\ell=1}^{\infty} \frac{1}{\ell!}[\alpha^{\wedge\ell}, \mathscr{F}(\alpha)] = 0. \tag{4-16}$$

Proof. The n-Jacobi relation for $(\alpha^{\wedge n})$ shows that

$$\sum_{\ell=0}^{n} \frac{1}{\ell!(n-\ell)!}[\alpha^{\wedge\ell}, [\alpha^{\wedge(n-\ell)}]] = 0.$$

Summing over $n > 0$, we obtain the lemma. □

If $\mathfrak{g}$ is an L_∞-algebra and Ω is a dg commutative algebra, then the tensor product $\mathfrak{g} \otimes \Omega$ is an L_∞-algebra, with brackets

$$[x \otimes a] = [x] \otimes a + (-1)^{|x|} x \otimes da,$$
$$[x_1 \otimes a_1, \dots, x_k \otimes a_k] = (-1)^{\sum_{i<j} |x_i||a_j|} [x_1, \dots, x_k] \otimes a_1 \cdots a_k \quad \text{for } k \neq 1.$$

The functor $\mathsf{MC}(\mathfrak{g})$ extends to a covariant functor $\mathsf{MC}(\mathfrak{g}, \Omega) = \mathsf{MC}(\mathfrak{g} \otimes \Omega)$ from dg commutative algebras to sets, that is, a presheaf on the category of dg affine schemes over K. If $X_\bullet$ is a simplicial set, we have

$$\mathsf{MC}(\mathfrak{g}, \Omega(X_\bullet)) \cong \mathsf{sSet}(X_\bullet, \mathsf{MC}_\bullet(\mathfrak{g})).$$

If $\mathfrak{g}$ is a nilpotent L_∞-algebra, let $\mathsf{MC}_\bullet(\mathfrak{g})$ be the simplicial set $\mathsf{MC}_\bullet(\mathfrak{g}) = \mathsf{MC}(\mathfrak{g}, \Omega_\bullet)$. In other words, the n-simplices of $\mathsf{MC}_\bullet(\mathfrak{g})$ are differential forms α on the n-simplex $\mathbf{\Delta}^n$, of the form $\alpha = \sum_{i=0}^n \alpha_i$, where $\alpha_i \in \mathfrak{g}^{1-i} \otimes \Omega^i(\mathbf{\Delta}^n)$, such that

$$(d + \delta)\alpha + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} [\alpha^{\wedge \ell}] = 0. \tag{4-17}$$

Before developing the properties of this functor, we recall how it emerges naturally from Sullivan's approach [Sul77] to rational homotopy theory.

If $\mathfrak{g}$ is an L_∞-algebra that is finite-dimensional in each degree and bounded below, we may associate to it the dg commutative algebra $C^*(\mathfrak{g})$ of cochains. The underlying graded commutative algebra of $C^*(\mathfrak{g})$ is $\bigwedge \mathfrak{g}^\vee = S(\mathfrak{g}[1]^\vee)$, the free graded commutative algebra on the graded vector space $\mathfrak{g}[1]^\vee$ that equals $(\mathfrak{g}^{1-i})^\vee$ in degree i. The differential δ of $C^*(\mathfrak{g})$ is determined by its restriction to the space of generators $\mathfrak{g}[1]^\vee \subset C^*(\mathfrak{g})$, on which it equals the sum over k of the adjoints of the operations ℓ_k. The resulting graded derivation satisfies the equation $\delta^2 = 0$ if and only if $\mathfrak{g}$ is an L_∞-algebra.

As explained in Section 1, the simplicial set $\mathrm{Spec}_\bullet(\mathscr{A}) = \mathsf{dAlg}(\mathscr{A}, \Omega_\bullet)$ may be viewed as an analogue in homotopical algebra of the spectrum of a commutative algebra. Applied to $C^*(\mathfrak{g})$, we obtain a simplicial set $\mathrm{Spec}_\bullet(C^*(\mathfrak{g}))$, which has a natural identification with the simplicial set $\mathsf{MC}_\bullet(\mathfrak{g})$.

The homotopy groups of a nilpotent L_∞-algebra $\mathfrak{g}$ are defined as $\pi_i(\mathfrak{g}) = \pi_i(\mathsf{MC}_\bullet(\mathfrak{g}))$. In particular, the set of components $\pi_0(\mathfrak{g})$ of $\mathfrak{g}$ is the quotient of $\mathsf{MC}(\mathfrak{g})$ by the nilpotent group associated to the nilpotent Lie algebra $\mathfrak{g}^0$. This plays a prominent role in deformation theory: it is the moduli set of deformations of $\mathfrak{g}$.

In order to establish that $\mathsf{MC}_\bullet(\mathfrak{g})$ is a Kan complex, we use the Poincaré lemma. Let $0 \leq i \leq n$. By (3-9), we see that

$$\mathrm{id}_n = \varepsilon_n^i + (d + \delta) h_n^i + h_n^i (d + \delta).$$

If $\alpha \in \mathsf{MC}_n(\mathfrak{g})$, we see that

$$\begin{aligned}\alpha &= \varepsilon_n^i\alpha + (d+\delta)h_n^i\alpha + h_n^i(d+\delta)\alpha \\ &= \varepsilon_n^i\alpha + R_n^i\alpha - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}h_n^i[\alpha^{\wedge\ell}],\end{aligned}$$

where $R_n^i = (d+\delta)h_n^i$. Introduce the space $\mathsf{mc}_n(\mathfrak{g}) = \{(d+\delta)\alpha \mid \alpha \in (\mathfrak{g}\otimes\Omega)^0\}$.

LEMMA 4.6. *Let $\mathfrak{g}$ be a nilpotent L_∞-algebra. The map $\alpha \mapsto (\varepsilon_n^i\alpha, R_n^i\alpha)$ induces an isomorphism between $\mathsf{MC}_n(\mathfrak{g})$ and $\mathsf{MC}(\mathfrak{g})\times\mathsf{mc}_n(\mathfrak{g})$.*

Proof. Given $\mu \in \mathsf{MC}(\mathfrak{g})$ and $\nu \in \mathsf{mc}_n(\mathfrak{g})$, let $\alpha_0 = \mu+\nu$, and define differential forms $(\alpha_k)_{k>0}$ inductively by the formula

$$\alpha_{k+1} = \alpha_0 - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}h_n^i[\alpha_k^{\wedge\ell}]. \tag{4-18}$$

Then for all k, we have $\varepsilon_n^i\alpha_k = \mu$ and $R_n^i\alpha_k = \nu$. The sequence is eventually constant, since by induction, we see that

$$\alpha_{k+1}-\alpha_k = \sum_{\ell=2}^{\infty}\frac{1}{\ell!}\sum_{j=1}^{\ell}h_n^i\big[\alpha_{k-1}^{\wedge j-1},\alpha_{k-1}-\alpha_k,\alpha_k^{\wedge\ell-j}\big] \in F^{k+1}\mathfrak{g}\otimes\Omega_n.$$

The limit $\alpha = \lim_{k\to\infty}\alpha_k$ satisfies

$$\alpha = \alpha_0 - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}h_n^i[\alpha^{\wedge\ell}].$$

Applying the operator $d+\delta$, we see that

$$(d+\delta)\alpha = \delta\mu - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}(d+\delta)h_n^i[\alpha^{\wedge\ell}]$$

and hence that

$$\begin{aligned}\mathcal{F}(\alpha) &= \delta\mu + \sum_{\ell=2}^{\infty}\frac{1}{\ell!}[\alpha^{\wedge\ell}] - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}(d+\delta)h_n^i[\alpha^{\wedge\ell}] \\ &= \mathcal{F}(\mu) + \sum_{\ell=2}^{\infty}\frac{1}{\ell!}h_n^i(d+\delta)[\alpha^{\wedge\ell}] = \mathcal{F}(\mu) + h_n^i(d+\delta)\mathcal{F}(\alpha).\end{aligned}$$

The Bianchi identity (4-16) implies that

$$\mathcal{F}(\alpha) = \mathcal{F}(\mu) - \sum_{\ell=1}^{\infty}\frac{1}{\ell!}h_n^i[\alpha^{\wedge\ell},\mathcal{F}(\alpha)] = \sum_{\ell=1}^{\infty}\frac{1}{\ell!}h_n^i[\alpha^{\wedge\ell},\mathcal{F}(\alpha)].$$

The nilpotence of $\mathfrak{g}$ implies that $\mathscr{F}(\alpha)=0$; it follows that α is an element of $\mathsf{MC}_n(\mathfrak{g})$ with $\varepsilon^i_n\alpha=\mu$ and $R^i_n\alpha=\nu$.

If α and β are a pair of elements of $\mathsf{MC}_n(\mathfrak{g})$ such that $\varepsilon^i_n\alpha=\varepsilon^i_n\beta$ and $R^i_n\alpha=R^i_n\beta$, then

$$\alpha-\beta=-\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\sum_{j=1}^{\ell}h^i_n\big[\alpha^{\wedge j-1},\alpha-\beta,\beta^{\wedge\ell-j}\big].$$

This shows, by induction, that $\alpha-\beta\in F^i\mathfrak{g}$ for all $i>0$ and hence, by the nilpotence of $\mathfrak{g}$, that $\alpha=\beta$. □

The following result applies when $\mathfrak{g}$ is a dg Lie algebra.

PROPOSITION 4.7 [Hin97]. *If $f:\mathfrak{g}\to\mathfrak{h}$ is a surjective morphism of nilpotent L_∞-algebras, the induced morphism $\mathsf{MC}_\bullet(f):\mathsf{MC}_\bullet(\mathfrak{g})\to\mathsf{MC}_\bullet(\mathfrak{h})$ is a fibration of simplicial sets.*

Proof. Let $0\le i\le n$. Given a horn $\beta\in\mathsf{sSet}(\Lambda^n_i,\mathsf{MC}_\bullet(\mathfrak{g}))$ and an n-simplex $\gamma\in\mathsf{MC}_n(\mathfrak{h})$ such that $\partial_j\gamma=f(\partial_j\beta)$ for $j\ne i$, we wish to construct an element $\alpha\in f^{-1}(\gamma)\subset\mathsf{MC}_n(\mathfrak{g})$ such that $\partial_j\alpha=\partial_j\beta$ for $j\ne i$.

Since $f:\mathfrak{g}\otimes\Omega_\bullet\to\mathfrak{h}\otimes\Omega_\bullet$ is a Kan fibration, there exists an extension $\rho\in\mathfrak{g}\otimes\Omega_n$ of β of total degree 1 such that $f(\rho)=\alpha$. Let α be the unique element of $\mathsf{MC}_n(\mathfrak{g})$ such that $\varepsilon^i_n\alpha=\varepsilon^i_n\rho$ and $R^i_n\alpha=R^i_n\rho$. If $j\ne i$, we have $\varepsilon^i_n\partial_j\alpha=\varepsilon^i_n\partial_j\beta$ and $R^i_n\partial_j\alpha=R^i_n\partial_j\beta$ and hence, by Lemma 4.6, $\partial_j\alpha=\partial_j\beta$. Thus, α fills the horn β. Also $f(\varepsilon^i_n\alpha)=f(\varepsilon^i_n\rho)=\varepsilon^i_n\gamma$ and $f(R^i_n\alpha)=f(R^i_n\rho)=R^i_n\gamma$; hence $f(\alpha)=\gamma$. □

The category of nilpotent L_∞-algebras concentrated in degrees $(-\infty,0]$ is a variant of Quillen's model [Qui69] for rational homotopy of nilpotent spaces. By the following theorem, the functor $\mathsf{MC}_\bullet(\mathfrak{g})$ carries quasi-isomorphisms of such L_∞-algebras to homotopy equivalences of simplicial sets.

THEOREM 4.8. *If $\mathfrak{g}$ and $\mathfrak{h}$ are both L_∞-algebras concentrated in degrees $(-\infty,0]$ and if $f:\mathfrak{g}\to\mathfrak{h}$ is a quasi-isomorphism, then*

$$\mathsf{MC}_\bullet(f):\mathsf{MC}_\bullet(\mathfrak{g})\to\mathsf{MC}_\bullet(\mathfrak{h})$$

is a homotopy equivalence.

Proof. Filter $\mathfrak{g}$ by L_∞-algebras $F^j\mathfrak{g}$, where

$$(F^{2j}\mathfrak{g})^i=\begin{cases}0 & \text{if } i+j>0,\\ Z^{-j}(\mathfrak{g}) & \text{if } i+j=0,\\ \mathfrak{g}^i & \text{if } i+j<0,\end{cases}\qquad (F^{2j+1}\mathfrak{g})^i=\begin{cases}0 & \text{if } i+j>0,\\ B^{-j}(\mathfrak{g}) & \text{if } i+j=0,\\ \mathfrak{g}^i & \text{if } i+j<0,\end{cases}$$

and similarly for $\mathfrak{h}$. If $j > k$, there is a morphism of fibrations of simplicial sets

$$\begin{tikzcd} \mathsf{MC}_\bullet(F^j\mathfrak{g}) \arrow[r] \arrow[d] & \mathsf{MC}_\bullet(F^k\mathfrak{g}) \arrow[r] \arrow[d] & \mathsf{MC}_\bullet(F^k\mathfrak{g}/F^j\mathfrak{g}) \arrow[d] \\ \mathsf{MC}_\bullet(F^j\mathfrak{h}) \arrow[r] & \mathsf{MC}_\bullet(F^k\mathfrak{h}) \arrow[r] & \mathsf{MC}_\bullet(F^k\mathfrak{h}/F^j\mathfrak{h}). \end{tikzcd}$$

We have

$$\mathsf{MC}_\bullet(F^{2j}\mathfrak{g}/F^{2j+1}\mathfrak{g}) \cong \mathsf{MC}_\bullet(H^{-j}(\mathfrak{g})) \cong \mathsf{MC}_\bullet(H^{-j}(\mathfrak{h})) \cong \mathsf{MC}_\bullet(F^{2j}\mathfrak{h}/F^{2j+1}\mathfrak{h}).$$

The simplicial sets

$$\begin{aligned} &\mathsf{MC}_\bullet(F^{2j+1}\mathfrak{g}/F^{2j+2}\mathfrak{g}) \cong B^{-j}(\mathfrak{g}) \otimes \Omega_\bullet^{j+1} \quad \text{and} \\ &\mathsf{MC}_\bullet(F^{2j+1}\mathfrak{h}/F^{2j+2}\mathfrak{h}) \cong B^{-j}(\mathfrak{h}) \otimes \Omega_\bullet^{j+1} \end{aligned}$$

are contractible by Lemma 3.2. The proposition follows. □

Let $\mathbf{m}$ be a nilpotent commutative ring; that is, $\mathbf{m}^{\ell+1} = 0$ for some ℓ. If $\mathfrak{g}$ is an L_∞-algebra, then $\mathfrak{g} \otimes \mathbf{m}$ is nilpotent; this is the setting of formal deformation theory. In this context too, the functor $\mathsf{MC}_\bullet(\mathfrak{g}, \mathbf{m}) = \mathsf{MC}_\bullet(\mathfrak{g} \otimes \mathbf{m})$ takes quasi-isomorphisms of L_∞-algebras to homotopy equivalences of simplicial sets.

PROPOSITION 4.9. *If $f : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism of L_∞-algebras and* $\mathbf{m}$ *is a nilpotent commutative ring*, *then*

$$\mathsf{MC}_\bullet(f, \mathbf{m}) : \mathsf{MC}_\bullet(\mathfrak{g}, \mathbf{m}) \to \mathsf{MC}_\bullet(\mathfrak{h}, \mathbf{m})$$

is a homotopy equivalence.

Proof. We argue by induction on the nilpotence length ℓ of $\mathbf{m}$. There is a morphism of fibrations of simplicial sets

$$\begin{tikzcd} \mathsf{MC}_\bullet(\mathfrak{g}, \mathbf{m}^2) \arrow[r] \arrow[d] & \mathsf{MC}_\bullet(\mathfrak{g}, \mathbf{m}) \arrow[r] \arrow[d] & \mathsf{MC}_\bullet(\mathfrak{g} \otimes \mathbf{m}/\mathbf{m}^2) \arrow[d] \\ \mathsf{MC}_\bullet(\mathfrak{h}, \mathbf{m}^2) \arrow[r] & \mathsf{MC}_\bullet(\mathfrak{h}, \mathbf{m}) \arrow[r] & \mathsf{MC}_\bullet(\mathfrak{h} \otimes \mathbf{m}/\mathbf{m}^2). \end{tikzcd}$$

The abelian L_∞-algebras $\mathfrak{g} \otimes \mathbf{m}/\mathbf{m}^2$ and $\mathfrak{h} \otimes \mathbf{m}/\mathbf{m}^2$ are quasi-isomorphic; hence the morphism $\mathsf{MC}_\bullet(\mathfrak{g} \otimes \mathbf{m}/\mathbf{m}^2) \to \mathsf{MC}_\bullet(\mathfrak{h} \otimes \mathbf{m}/\mathbf{m}^2)$ is a homotopy equivalence. The result follows by induction on ℓ. □

5. The functor $\gamma_\bullet(\mathfrak{g})$

In this section, we study the functor $\gamma_\bullet(\mathfrak{g})$; we prove that it is homotopy equivalent to $\mathsf{MC}_\bullet(\mathfrak{g})$ and show that it specializes to the Deligne groupoid when $\mathfrak{g}$

is concentrated in degrees $[0,\infty)$. Fix a gauge $s_\bullet$, for example Dupont's operator (3-12).

The simplicial set $\gamma_\bullet(\mathfrak{g})$ associated to a nilpotent L_∞-algebra is the simplicial subset of $\mathsf{MC}_\bullet(\mathfrak{g})$ consisting of those Maurer-Cartan forms annihilated by $s_\bullet$:

$$\gamma_\bullet(\mathfrak{g}) = \{\alpha \in \mathsf{MC}_\bullet(\mathfrak{g}) \mid s_\bullet\alpha = 0\}. \tag{5-19}$$

For any simplicial set $X_\bullet$, the set of simplicial maps $\mathsf{sSet}(X_\bullet, \gamma_\bullet(\mathfrak{g}))$ equals the set of Maurer-Cartan elements $\alpha \in \mathsf{MC}(\mathfrak{g}, X_\bullet)$ such that $s_\bullet\alpha = 0$. This is reminiscent of gauge conditions, such as the Coulomb gauge, in gauge theory.

PROPOSITION 5.1. *If $\mathfrak{g}$ is abelian, then there is a natural isomorphism $\gamma_\bullet(\mathfrak{g}) \cong K_\bullet(\mathfrak{g}[1])$.*

Proof. If $\alpha \in \gamma_n(\mathfrak{g})$, then $(d+\delta)\alpha = s_n\alpha = 0$. Hence by (3-7),

$$\alpha = P_n\alpha + s_n(d+\delta)\alpha + (d+\delta)s_n\alpha = P_n\alpha.$$

Thus $\gamma_n(\mathfrak{g}) \subset K_n(\mathfrak{g}[1])$. Conversely, if $\alpha \in K_n(\mathfrak{g}[1])$, then $P_n\alpha = \alpha$, and hence $s_n\alpha = 0$. Thus $K_n(\mathfrak{g}[1]) \subset \gamma_n(\mathfrak{g})$. $\square$

We show that $\gamma_\bullet(\mathfrak{g})$ is an ∞-groupoid and, in particular, a Kan complex: the heart of the proof is an iteration, similar to the iteration (4-18), that solves the Maurer-Cartan equation on the n-simplex Δ^n in the gauge $s_n\alpha = 0$.

Definition 5.2. An n-simplex $\alpha \in \gamma_n(\mathfrak{g})$ is *thin* if $I_{0\dots n}(\alpha) = 0$.

LEMMA 5.3. *If $\mathfrak{g}$ is a nilpotent L_∞-algebra, the map $\alpha \mapsto (\varepsilon_n^i\alpha, P_nR_n^i\alpha)$ induces an isomorphism between $\gamma_n(\mathfrak{g})$ and $\mathsf{MC}(\mathfrak{g}) \times P_n[\mathsf{mc}_n(\mathfrak{g})]$.*

Proof. Let $0 \le i \le n$. By (3-7), we see that

$$\begin{aligned} \mathrm{id}_n &= P_n + (d+\delta)s_n + s_n(d+\delta) \\ &= \varepsilon_n^i + (d+\delta)(P_nh_n^i + s_n) + (P_nh_n^i + s_n)(d+\delta). \end{aligned}$$

It follows that if $\alpha \in \gamma_n(\mathfrak{g})$,

$$\alpha = \varepsilon_n^i\alpha + P_nR_n^i\alpha - \sum_{\ell=2}^{\infty} \frac{1}{\ell!}(P_nh_n^i + s_n)[\alpha^{\wedge\ell}]. \tag{5-20}$$

Given $\mu \in \mathsf{MC}(\mathfrak{g})$ and $\nu \in P_n[\mathsf{mc}_n(\mathfrak{g})]$, let $\alpha_0 = \mu + \nu$, and define differential forms $(\alpha_k)_{k>0}$ inductively by the formula

$$\alpha_k = \alpha_0 - \sum_{\ell=2}^{\infty} \frac{1}{\ell!}(P_nh_n^i + s_n)[\alpha_{k-1}^{\wedge\ell}].$$

Then for all k, we have $s_n\alpha_k = 0$, $\varepsilon^i_n\alpha_k = \mu$ and $P_n R^i_n\alpha_k = \nu$. The sequence (α_k) is eventually constant since, by induction, we see that

$$\begin{aligned}\alpha_k - \alpha_{k-1} &= \sum_{\ell=2}^{\infty}\frac{1}{\ell!}\sum_{j=1}^{\ell}(P_n h^i_n + s_n)\big[\alpha_{k-2}^{\wedge j-1}, \alpha_{k-2}-\alpha_{k-1}, \alpha_{k-1}^{\wedge \ell-j}\big]\\ &\in F^k\mathfrak{g}\otimes\Omega_n.\end{aligned}$$

The limit $\alpha = \lim_{k\to\infty}\alpha_k$ satisfies

$$\alpha = \alpha_0 - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}(Ph^i_n + s_n)[\alpha^{\wedge\ell}].$$

By the same argument as in the proof of Lemma 4.6, it follows that

$$\begin{aligned}\mathscr{F}(\alpha) &= \mathscr{F}(\mu) - \sum_{\ell=1}^{\infty}\frac{1}{\ell!}(Ph^i_n + s_n)[\alpha^{\wedge\ell}, \mathscr{F}(\alpha)]\\ &= \sum_{\ell=1}^{\infty}\frac{1}{\ell!}(Ph^i_n + s_n)[\alpha^{\wedge\ell}, \mathscr{F}(\alpha)].\end{aligned}$$

The nilpotence of $\mathfrak{g}$ implies that $\mathscr{F}(\alpha) = 0$; it follows that α is an element of $\gamma_n(\mathfrak{g})$ with $\varepsilon^i_n\alpha = \mu$ and $PR^i_n\alpha = \nu$.

If α and β are a pair of elements of $\gamma_n(\mathfrak{g})$ such that $\varepsilon^i_n\alpha = \varepsilon^i_n\beta$ and $P_n R^i_n\alpha = P_n R^i_n\beta$, then

$$\alpha - \beta = -\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\sum_{j=1}^{\ell}(P_n h^i_n + s_n)\big[\alpha^{\wedge j-1}, \alpha-\beta, \beta^{\wedge\ell-j}\big].$$

This shows, by induction, that $\alpha-\beta \in F^i\mathfrak{g}$ for all $i > 0$, and hence, by the nilpotence of $\mathfrak{g}$, that $\alpha = \beta$. ☐

THEOREM 5.4. *If $\mathfrak{g}$ is a nilpotent L_∞-algebra, $\gamma_\bullet(\mathfrak{g})$ is an ∞-groupoid. If $\mathfrak{g}$ is concentrated in degrees $(-\ell,\infty)$, respectively $(-\ell,0]$, then $\gamma_\bullet(\mathfrak{g})$ is an ℓ-groupoid, respectively an ℓ-group.*

Proof. Let $\beta \in \mathsf{sSet}(\Lambda^n_i, \gamma_\bullet(\mathfrak{g}))$ be a horn in $\gamma_\bullet(\mathfrak{g})$. The differential form

$$\alpha_0 = \varepsilon^i_n\beta + (d+\delta)\sum_{k=1}^{n-1}\sum_{\substack{i_1<\cdots<i_k\\ i\notin\{i_1,\dots,i_k\}}}\omega_{i_1\dots i_k}\otimes I_{i i_1\dots i_k}(\beta) \in \mathsf{MC}(\mathfrak{g})\times P_n[\mathsf{mc}_n(\mathfrak{g})]$$

satisfies $I_{0\dots n}(\alpha_0) = 0$. The solution $\alpha \in \gamma_n(\mathfrak{g})$ of the equation

$$\alpha = \alpha_0 - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}(P_n h^i_n + s_n)[\alpha^{\wedge\ell}]$$

constructed in Lemma 5.3 is thin and $\xi^n_i(\alpha) = \beta$. Thus $\gamma_\bullet(\mathfrak{g})$ is an ∞-groupoid.

If $\mathfrak{g}^{1-n} = 0$, it is clear that every n-simplex $\alpha \in \gamma_n(\mathfrak{g})$ is thin, while if $\mathfrak{g}^1 = 0$, then $\gamma_\bullet(\mathfrak{g})$ is reduced. $\square$

Given $\mu \in \mathsf{MC}(\mathfrak{g})$ and $x_{i_1\dots i_k} \in \mathfrak{g}^{1-k}$ for $1 \le i_1 < \dots < i_k \le n$, let

$$\alpha^\mu_n(x_{i_1\dots i_k}) \in \gamma_n(\mathfrak{g})$$

be the solution of (5-20) with $\varepsilon^0_n \alpha^\mu_n(x_{i_1\dots i_k}) = \mu$ and

$$R^0_n \alpha^\mu_n(x_{i_1\dots i_k}) = \sum_{k=1}^n \sum_{1\le i_1<\dots<i_k\le n} \omega_{i_1\dots i_k} \otimes x_{i_1\dots i_k}.$$

Definition 5.5. The n-th generalized Campbell-Hausdorff series associated to the gauge $s_\bullet$ is the function of $\mu \in \mathsf{MC}(\mathfrak{g})$ and $x_{i_1\dots i_k} \in \mathfrak{g}^{1-k}$ for $1 \le i_1 < \dots < i_k \le n$ given by the formula

$$\rho^\mu_n(x_{i_1\dots i_k}) = I_{1\dots n}(\alpha^\mu_n(x_{i_1\dots i_k})) \in \mathfrak{g}^{2-n}.$$

If $\mathfrak{g}$ is concentrated in degrees $(-\infty, 0]$, then the Maurer-Cartan element μ equals 0 and may be omitted from the notation for $\alpha_n(x_{i_1\dots i_k})$ and $\rho_n(x_{i_1\dots i_k})$.

Since $\alpha^\mu_2(x_1, x_2, x_{12})$ is a flat connection 1-form on the 2-simplex, its monodromy around the boundary must be trivial. (The 2-simplex is simply connected.) In terms of the generalized Campbell-Hausdorff series $\rho^\mu_2(x_1, x_2, x_{12})$, this gives the equation $e^{x_1} = e^{\rho^\mu_2(x_1,x_2,x_{12})} e^{x_2}$ in the Lie group associated to the nilpotent Lie algebra $\mathfrak{g}^0$. Thus, the simplicial set $\gamma_\bullet(\mathfrak{g})$ (indeed, its 2-skeleton) determines $\rho^\mu_2(x_1, x_2, x_{12})$ as a function of x_1, x_2 and x_{12}. In the Dupont gauge, modulo terms involving more than two brackets, it equals

$$\begin{aligned}\rho^\mu_2(x_1, x_2, x_{12}) = x_1 - x_2 &+ \tfrac12[x_1, x_2]_\mu + \tfrac12[x_{12}]_\mu \\ &+ \tfrac{1}{12}[x_1 + x_2, [x_1, x_2]_\mu]_\mu + \tfrac16[[x_1 + x_2]_\mu, x_1, x_2]_\mu \\ &+ \tfrac16[[x_1 + x_2]_\mu, x_{12}]_\mu - \tfrac{1}{12}[x_1 + x_2, [x_{12}]_\mu]_\mu + \cdots .\end{aligned}$$

Definition 5.6. A nilpotent L_∞-algebra $\mathfrak{g}$ is **minimal** if the following two conditions hold:

(i) $\mathfrak{g}$ is concentrated in degrees $(-\infty, 0]$ and

(ii) the differential δ of $\mathfrak{g}$ vanishes.

An L_∞-algebra $\mathfrak{g}$ is minimal if and only if the dg commutative algebra $C^*(\mathfrak{g})$ is minimal in the sense of [Sul77]. The following result was suggested to the author by P. Ševera.

PROPOSITION 5.7. *If L is minimal, $\gamma_\bullet(L)$ is a minimal Kan complex.*

Proof. If L is minimal, $\mathscr{F}\big(\alpha + \omega_{0\dots n} \otimes x\big)$ is independent of $x \in L^{1-n}$. It follows that

$$\alpha_n(x_{i_1\dots i_k}) = \alpha_n(x_{i_1\dots i_k})_{k<n} + \omega_{0\dots n} \otimes x_{1\dots n}$$

and hence that $\rho_n(x_{i_1\dots i_k}) = \rho_n(x_{i_1\dots i_k})_{k<n}$. This shows that $\partial_0\alpha_n(x_{i_1\dots i_k})$ is independent of $x_{1\dots n}$. The same holds with ∂_i replacing ∂_0, by action of the symmetric group S_n on the n-simplices of $\gamma_\bullet(L)$. □

If $\mathfrak{g}$ is a dg Lie algebra, the thin 2-simplices define a composition on the 1-simplices of $\gamma_\bullet(\mathfrak{g})$ which is strictly associative; this parallels a recent result of Paoli [Pao07].

PROPOSITION 5.8. *If $\mathfrak{g}$ is a dg Lie algebra, the composition $\rho_2^\mu(x_1, x_2)$: $\mathfrak{g}^0 \otimes \mathfrak{g}^0 \to \mathfrak{g}^0$ is associative.*

Proof. It suffices to show that $\rho_3^\mu(x_1, x_2, x_3, x_{ij} = 0) = 0$; in other words, if three faces of a thin 3-simplex are thin, then the fourth is. The iteration leading to the solution α of (5-20) with initial conditions

$$\alpha_0 = \mu + (d + \delta)(t_1x_1 + t_2x_2 + t_3x_3)$$

lies in the space $\mathfrak{g}^0 \otimes \Omega_3^1 \oplus \mathfrak{g}^1 \otimes \Omega_3^0$; hence $I_{123}(\alpha) = 0 \in \mathfrak{g}^{-1}$. □

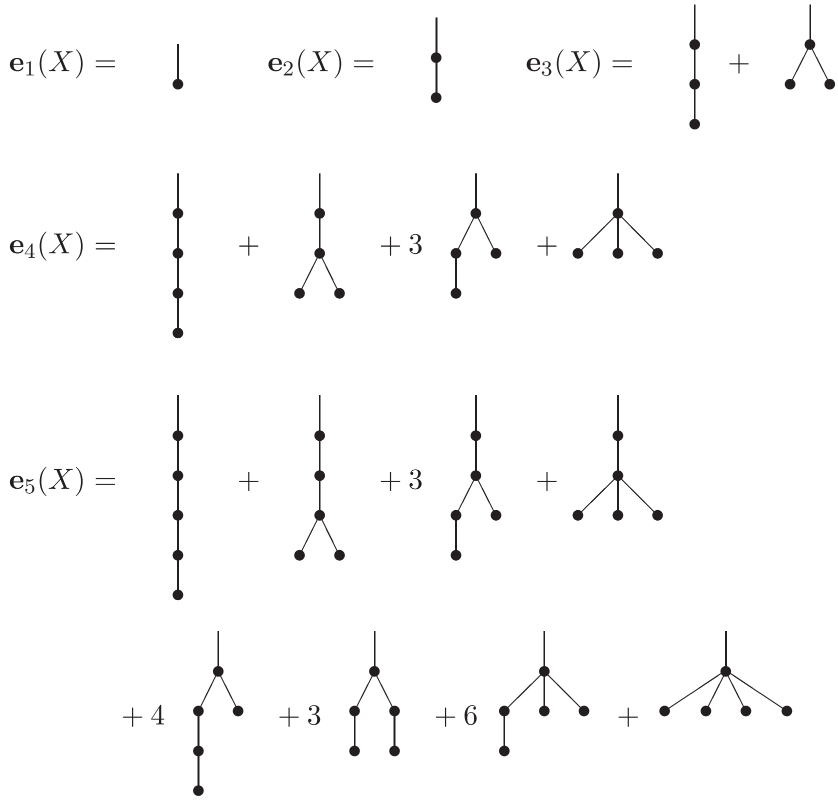
In particular, if $\mathfrak{g}$ is a dg Lie algebra concentrated in degrees $(-2, \infty)$, then $\gamma_\bullet(\mathfrak{g})$ is the nerve of a strict 2-groupoid, that is, a groupoid enriched in groupoids; in this way, we see that $\gamma_\bullet(\mathfrak{g})$ generalizes the Deligne 2-groupoid [Del94], [Get02].

Although it is not hard to derive explicit formulas for the generalized Campbell-Hausdorff series up to any order, we do not know any closed formulas for them except when $n = 1$, in which case it is independent of the gauge. We now derive a closed formula for $\rho_1^\mu(x)$, which resembles Cayley's famous formula for the series solution of the ordinary differential equation $x'(t) = f(x(t))$.

To each rooted tree, associate the word obtained by associating to a vertex with i branches the operation $[x, a_1, \dots, a_i]_\mu$. Multiply the resulting word by the number of total orders on the vertices of the tree such that each vertex precedes its parent. Let $\mathbf{e}_\mu^k(x)$ be the sum of these terms over all rooted trees with k vertices. For example, $\mathbf{e}_\mu^1(x) = [x]_\mu$, $\mathbf{e}_\mu^2(x) = [x, [x]_\mu]_\mu$, and

$$\mathbf{e}_\mu^3(x) = [x, [x, [x]_\mu]_\mu]_\mu + [x, [x]_\mu, [x]_\mu]_\mu.$$

The coefficient of a tree T in $\mathbf{e}_\mu^k(x)$ equals the number of monotone orderings of its vertices, that is, total orderings such that each vertex is greater than its parent. The pictures below show the trees contributing to $\mathbf{e}_\mu^k(X)$ for $k < 5$.



PROPOSITION 5.9. *The* 1*-simplex* $\alpha_1^\mu(x) \in \gamma_1(\mathfrak{g})$ *that is determined by* $\mu \in \mathsf{MC}(\mathfrak{g})$ *and* $x \in \mathfrak{g}^0$ *is given by the formula*

$$\alpha_1^\mu(x) = \mu - \sum_{k=1}^{\infty} \frac{t^k}{k!} \mathbf{e}_\mu^k(x) + x\,dt.$$

Proof. To show that $\alpha_1^\mu(x) \in \gamma_1(\mathfrak{g})$, we must show that it satisfies the Maurer-Cartan equation. Let $\alpha(t) = \mu - \sum_{k=1}^{\infty}(t^k/k!)\mathbf{e}_\mu^k(x)$. It must be shown that $\alpha'(t) + \sum_{n=0}^{\infty}(1/n!)[\alpha(t)^{\wedge n}, x] = 0$ or, in other words, that

$$\begin{aligned} \mathbf{e}_\mu^{k+1}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1!\cdots k_n!} [\mathbf{e}_\mu^{k_1}(x), \dots, \mathbf{e}_\mu^{k_n}(x), x]_\alpha \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1!\cdots k_n!} [x, \mathbf{e}_\mu^{k_1}(x), \dots, \mathbf{e}_\mu^{k_n}(x)]_\alpha. \end{aligned}$$

This is easily proved by induction on k. $\square$

Proposition 5.9 implies a formula for the generalized Campbell-Hausdorff series $\rho_1^\alpha(x)$:

$$\rho_1^\mu(x) = \mu - \sum_{k=1}^{\infty} \frac{1}{k!}\mathbf{e}_\mu^k(x).$$

If $\mathfrak{g}$ is a dg Lie algebra, only trees with vertices of valence 0 or 1 contribute to $\mathbf{e}_\alpha^k(x)$, and we recover the formula (1-3) figuring in the definition of the Deligne groupoid for dg Lie algebras.

There is a relative version of Theorem 5.4, analogous to Proposition 4.7:

THEOREM 5.10. *If $f : \mathfrak{g} \to \mathfrak{h}$ is a surjective morphism of nilpotent L_∞-algebras, the induced morphism $\gamma_\bullet(f) : \gamma_\bullet(\mathfrak{g}) \to \gamma_\bullet(\mathfrak{h})$ is a fibration of simplicial sets.*

Proof. Let $0 \le i \le n$. Given a horn $\beta \in \mathsf{sSet}(\Lambda_i^n, \gamma_\bullet(\mathfrak{g}))$ and an n-simplex $\gamma \in \gamma_n(\mathfrak{h})$ such that $f(\partial_j\beta) = \partial_j\gamma$ for $j \neq i$, our task is to construct an element $\alpha \in f^{-1}(\gamma) \subset \gamma_n(\mathfrak{g})$ such that $\partial_j\alpha = \partial_j\beta$ if $j \neq i$.

Choose a solution $x \in \mathfrak{g}^{1-n}$ of the equation $f(x) = I_{0\dots n}(\gamma) \in \mathfrak{h}^{1-n}$. Let α be the unique element of $\gamma_n(\mathfrak{g})$ such that $\varepsilon_n^i\alpha = \varepsilon_n^i\beta$ and

$$P_nR_n^i\alpha = (d+\delta)\Bigg(\sum_{k=1}^{n-1}\sum_{\substack{i_1<\dots<i_k\\ i\notin\{i_1,\dots,i_k\}}} \omega_{i_1\dots i_k}\otimes I_{i i_1\dots i_k}(\beta) + (-1)^i\,\omega_{0\dots\widehat{i}\dots n}\otimes x\Bigg).$$

If $j \neq i$, we have $\varepsilon_n^i\partial_j\alpha = \varepsilon_n^i\partial_j\beta$ and $P_nR_n^i\partial_j\alpha = P_nR_n^i\partial_j\beta$ and hence, by Lemma 5.3, $\partial_j\alpha = \partial_j\beta$; thus, α fills the horn β. Also $f(\varepsilon_n^i\alpha) = f(\varepsilon_n^i\beta) = \varepsilon_n^i\gamma$ and $f(P_nR_n^i\alpha) = P_nR_n^i\gamma$; hence $f(\alpha) = \gamma$. ☐

COROLLARY 5.11. *If $\mathfrak{g}$ is a nilpotent L_∞-algebra, the inclusion of simplicial sets $\gamma_\bullet(\mathfrak{g}) \hookrightarrow \mathsf{MC}_\bullet(\mathfrak{g})$ is a homotopy equivalence; in other words, $\pi_0(\gamma_\bullet(\mathfrak{g})) \cong \pi_0(\mathfrak{g})$, and for all 0-simplices $\alpha_0 \in \mathsf{MC}_0(\mathfrak{g}) = \mathsf{MC}(\mathfrak{g})$,*

$$\pi_i(\gamma_\bullet(\mathfrak{g}), \alpha_0) \cong \pi_i(\mathfrak{g}, \alpha_0) \quad \textit{for } i > 0.$$

Proof. This is proved by induction on the nilpotence length ℓ of $\mathfrak{g}$. When $\mathfrak{g}$ is abelian, $\mathsf{MC}_\bullet(\mathfrak{g})$ and $\gamma_\bullet(\mathfrak{g})$ are simplicial abelian groups, and their quotient is the simplicial abelian group

$$\mathsf{MC}_n(\mathfrak{g})/\gamma_n(\mathfrak{g}) \cong (d+\delta)s_n(\mathfrak{g}\otimes\Omega_n)^1.$$

This simplicial abelian group is a retract of the contractible simplicial abelian group $\mathfrak{g}\otimes\Omega_\bullet$ and hence is itself contractible.

Let $F^i\mathfrak{g}$ be the lower central series of $\mathfrak{g}$. Given $i > 0$, we have a morphism of principal fibrations of simplicial sets given by

$$\begin{tikzcd}
\gamma_\bullet(F^{i+1}\mathfrak{g}) \arrow[r] \arrow[d] & \gamma_\bullet(F^i\mathfrak{g}) \arrow[r] \arrow[d] & \gamma_\bullet(F^i\mathfrak{g}/F^{i+1}\mathfrak{g}) \arrow[d] \\
\mathsf{MC}_\bullet(F^{i+1}\mathfrak{g}) \arrow[r] & \mathsf{MC}_\bullet(F^i\mathfrak{g}) \arrow[r] & \mathsf{MC}_\bullet(F^i\mathfrak{g}/F^{i+1}\mathfrak{g}).
\end{tikzcd}$$

Since $F^i\mathfrak{g}/F^{i+1}\mathfrak{g}$ is abelian, we see that $\gamma_\bullet(F^i\mathfrak{g}/F^{i+1}\mathfrak{g}) \simeq \mathsf{MC}_\bullet(F^i\mathfrak{g}/F^{i+1}\mathfrak{g})$. The result follows by induction on ℓ. □

When $\mathfrak{g}$ is a nilpotent Lie algebra, the isomorphism

$$\pi_0(\gamma_\bullet(\mathfrak{g})) \cong \pi_0(\mathsf{MC}_\bullet(\mathfrak{g}))$$

is equivalent to the surjectivity of the exponential map. The above corollary may be viewed as a generalization of this fact.

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References

[Ash88] N. Ashley, Simplicial T-complexes and crossed complexes: a nonabelian version of a theorem of Dold and Kan, *Dissertationes Math.* (*Rozprawy Mat.*) **265** (1988), 61. MR 89k:55028 Zbl 1003.55500

[Bek04] T. Beke, Higher Čech theory, *K-Theory* **32** (2004), 293–322. MR 2005i:18007 Zbl 1070.18008

[BG76] A. K. Bousfield and V. K. A. M. Gugenheim, *On* PL *de Rham Theory and Rational Homotopy Type*, *Mem. Amer. Math. Soc.* **8:179**, 1976. MR 54 #13906 Zbl 0338.55008

[Dak83] M. K. Dakin, *Kan complexes and multiple groupoid structures*, *Esquisses Math.* **32**, Univ. Amiens, 1983. MR 766238 Zbl 0566.55010

[Del94] P. Deligne, letter to L. Breen, Feb. 28, 1994. Available at http://math.northwestern.edu/~getzler/Papers/deligne.pdf

[Dol58] A. Dold, Homology of symmetric products and other functors of complexes, *Ann. of Math.* **68** (1958), 54–80. MR 20 #3537 Zbl 0082.37701

[Dup76] J. L. Dupont, Simplicial de Rham cohomology and characteristic classes of flat bundles, *Topology* **15** (1976), 233–245. MR 54 #1243 Zbl 0331.55012

[Dup78] ———, *Curvature and Characteristic Classes*, *Lecture Notes in Math.* **640**, Springer-Verlag, New York, 1978. MR 58 #18477 Zbl 0373.57009

[Dus79] J. Duskin, Higher-dimensional torsors and the cohomology of topoi: the abelian theory, in *Applications of Sheaves* (Univ. Durham, 1977) (M. P. Fourman, C. J. Mulvey, and D. S. Scott, eds.), *Lecture Notes in Math.* **753**, Springer-Verlag, New York, 1979, pp. 255–279. MR 82c:18006 Zbl 0444.18014

[Dus01] J. W. DUSKIN, Simplicial matrices and the nerves of weak n-categories. I: Nerves of bicategories, *Theory Appl. Categ.* **9** (2001), 198–308. MR 2003f:18005 Zbl 1046.18009

[EML53] S. EILENBERG and S. MAC LANE, On the groups of $H(\Pi, n)$. I, *Ann. of Math.* **58** (1953), 55–106. MR 15,54b Zbl 0050.39304

[FL91] Z. FIEDOROWICZ and J.-L. LODAY, Crossed simplicial groups and their associated homology, *Transactions Amer. Math. Soc.* **326** (1991), 57–87. MR 91j:18018 Zbl 0755.18005

[Get02] E. GETZLER, A Darboux theorem for Hamiltonian operators in the formal calculus of variations, *Duke Math. J.* **111** (2002), 535–560. MR 2003e:32026 Zbl 1100.32008

[Gle82] P. G. GLENN, Realization of cohomology classes in arbitrary exact categories, *J. Pure Appl. Algebra* **25** (1982), 33–105. MR 83j:18016 Zbl 0487.18015

[GM88] W. M. GOLDMAN and J. J. MILLSON, The deformation theory of representations of fundamental groups of compact Kähler manifolds, *Inst. Hautes Études Sci. Publ. Math.* **67** (1988), 43–96. MR 90b:32041 Zbl 0678.53059

[Hin97] V. HINICH, Descent of Deligne groupoids, *Internat. Math. Res. Notices* **5** (1997), 223–239. MR 98h:22019 Zbl 0948.22016

[Kan58] D. M. KAN, Functors involving c.s.s. complexes, *Trans. Amer. Math. Soc.* **87** (1958), 330–346. MR 24 #A1720 Zbl 0090.39001

[Kur62] M. KURANISHI, On the locally complete families of complex analytic structures, *Ann. of Math.* (2) **75** (1962), 536–577. MR 25 #4550 Zbl 0106.15303

[LM95] T. LADA and M. MARKL, Strongly homotopy Lie algebras, *Comm. Algebra* **23** (1995), 2147–2161. MR 96d:16039 Zbl 0999.17019

[LS87] L. LAMBE and J. STASHEFF, Applications of perturbation theory to iterated fibrations, *Manuscripta Math.* **58** (1987), 363–376. MR 89d:55041 Zbl 0632.55011

[May92] J. P. MAY, *Simplicial Objects in Algebraic Topology*, *Chicago Lectures in Math.*, University of Chicago Press, Chicago, IL, 1992. MR 93m:55025 Zbl 0769.55001

[NR66] A. NIJENHUIS and R. W. RICHARDSON, JR., Cohomology and deformations in graded Lie algebras, *Bull. Amer. Math. Soc.* **72** (1966), 1–29. MR 33 #4190 Zbl 0136.30502

[Pao07] S. PAOLI, Semistrict Tamsamani n-groupoids and connected n-types, preprint, 2007. arXiv math/0701655v2

[Qui69] D. QUILLEN, Rational homotopy theory, *Ann. of Math.* **90** (1969), 205–295. MR 41 #2678 Zbl 0191.53702

[Sul77] D. SULLIVAN, Infinitesimal computations in topology, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 269–331. MR 58 #31119 Zbl 0374.57002

[Whi57] H. WHITNEY, *Geometric Integration Theory*, Princeton Univ. Press, Princeton, NJ, 1957. MR 19,309c Zbl 0083.28204

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