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Abstract

For the Jacobian of a curve, the Riemann singularity theorem gives a geometric interpretation of the singularities of the theta divisor in terms of special linear series on the curve. This paper proves an analogous theorem for Prym varieties. Applications of this theorem to cubic threefolds and Prym varieties of dimension five are also considered.

Introduction

A principally polarized abelian variety (ppav) can be studied through the geometry of its theta divisor. While in general this geometry is not well understood, one can simplify the problem by focusing on ppavs related to curves. This paper will consider such varieties defined over the complex numbers. Among the most studied examples are Jacobians: the Jacobian of a smooth curve C of genus g is the g-dimensional ppav $JC = H^0(C, \omega_C)^*/H_1(C, \mathbb{Z})$. The polarization is given by a theta divisor Θ, whose geometry is closely related to the curve C. The Abel-Jacobi theorem allows us to identify JC with $\operatorname{Pic}^{g-1}(C)$, the space of isomorphism classes of line bundles over C of degree $g-1$, and it is often convenient to make this identification when studying Θ. A fundamental result is Riemann's singularity theorem, which states that $\operatorname{mult}_x \Theta = h^0(C, L_x)$, where L_x is a line bundle of degree $g-1$ in the isomorphism class associated to x.

There is a close connection between the singular points of the theta divisor and the canonical image of C: the tangent cone to a general singular point of Θ contains the canonical image of C, and a theorem of Green [Gre84] implies that for a smooth curve C of genus $g \geq 4$, with no g^1_2, g^1_3 or g^2_5, the canonical image of C is cut out by the tangent cones to double points of Θ. In general, a theorem due to Torelli states that the Jacobian (JC, Θ) uniquely determines the curve C.

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Another commonly studied ppav is the Prym variety of a connected étale double cover of a smooth curve. The study of these varieties goes back to Riemann, and the geometry of the Prym theta divisor is in many ways parallel to that of the Jacobian theta divisor. Recall that associated to $\pi : \widetilde{C} \to C$, a connected étale double cover of a smooth curve C of genus g, is an involution $\tau : \widetilde{C} \to \widetilde{C}$, which induces an involution on $H^0(\widetilde{C}, \omega_{\widetilde{C}})^*$ and $H_1(\widetilde{C}, \mathbb{Z})$. Denoting by $(H^0(\widetilde{C}, \omega_{\widetilde{C}})^*)^-$ and $H_1(\widetilde{C}, \mathbb{Z})^-$ the negative eigenspaces of the involution, the Prym variety P associated to such a cover is the $(g-1)$-dimensional abelian subvariety of $J\widetilde{C}$ defined as $P = (H^0(\widetilde{C}, \omega_{\widetilde{C}})^*)^- / H_1(\widetilde{C}, \mathbb{Z})^-$. As in the case of Jacobians, it is convenient when studying Pryms to identify $J\widetilde{C}$ with $\mathrm{Pic}^{2g-2}(\widetilde{C})$. In this case, P can be described set theoretically as

$$P = \{L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Norm}(L) = \omega_C,\ h^0(L) \equiv 0 (\mathrm{mod}\ 2)\}.$$

There is a principal polarization on P given by a theta divisor Ξ; as a set, $\Xi = \{L \in P \mid h^0(L) \geq 2\}$. In this paper, we will prove an analogue of the Riemann singularity theorem in the case of Pryms; that is, we will relate the multiplicity of a point $x \in \mathrm{Sing}\ \Xi$ to the dimension of special linear systems on $\widetilde{C}$ and C.

The study of this question goes back to Mumford, who proved in [Mum74] that if $x \in \Xi$, then $\mathrm{mult}_x\ \Xi \geq h^0(L_x)/2$, and that if in addition $h^0(L_x) = 2$, then $x \in \mathrm{Sing}\ \Xi$ if and only if $L_x = \pi^*(M) \otimes \mathcal{O}_{\widetilde{C}}(B)$, where M is a line bundle on C such that $h^0(M) = 2$ and B is an effective divisor on $\widetilde{C}$. The proof of the second statement is based on Kempf's generalization of the Riemann singularity theorem [Kem73]. Smith and Varley used a similar method to prove the following theorem extending Mumford's assertions:

THEOREM 1 (Smith and Varley [SV04]). *Let $\pi : \widetilde{C} \to C$ be a connected étale double cover of a smooth curve C of genus g, and let (P, Ξ) be the associated Prym variety. If $x \in \Xi$ corresponds to a line bundle $L \in \mathrm{Pic}^{2g-2}(\widetilde{C})$, and $C_x\widetilde{\Theta}$ is the tangent cone to $\widetilde{\Theta}$ at x, then the following are equivalent*:

(a) $T_x P \subseteq C_x\widetilde{\Theta}$;

(b) $\mathrm{mult}_x\ \Xi > h^0(L)/2$;

(c) $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$, $h^0(C, M) > h^0(\widetilde{C}, L)/2$, $B \geq 0$, *and* $B \cap \tau^* B = \varnothing$.

Furthermore, if (c) *holds, then M and $\mathcal{O}_{\widetilde{C}}(B)$ are unique up to isomorphism, and* $h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(B)) = 1$.

In this paper, we will complete the analogue of the Riemann singularity theorem by determining the exact multiplicity of singular points of this type:

THEOREM 2. *In the above notation, let x be a singular point of Ξ that corresponds to a line bundle $L \in \mathrm{Pic}^{2g-2}(\widetilde{C})$ such that $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$, with $h^0(C, M) > h^0(\widetilde{C}, L)/2$, $B \geq 0$, and $B \cap \tau^* B = \varnothing$. Then* $\mathrm{mult}_x\ \Xi = h^0(M)$.

To prove the theorem, we will consider a deformation of the line bundle L, and then relate $\mathrm{mult}_x\, \Xi$ to the obstruction to lifting sections of L to sections of the deformation. In the course of proving the theorem, we will give a short proof of Smith and Varley's theorem.

More generally [LB92, p. 371], a ppav such that Z is an abelian subvariety of JC with $\Theta \cap Z = e \cdot \Xi$. Necessarily, e is the exponent of Z in JC. Every ppav of dimension g is a Prym-Tjurin variety of exponent $3^{g-1}(g-1)!$, and it is possible that there are situations, other than the exponent 2 case examined in this paper, where these techniques can be used to compute the multiplicity of a singular point of Ξ.

For a ppav (A, Θ), suppose $\mathrm{Sing}_k\, \Theta = \{x \in \mathrm{Sing}\, \Theta \mid \mathrm{mult}_x\, \Theta \geq k\}$. A result of Kollár [Kol95] shows if $\dim(A) = d$, then $\dim(\mathrm{Sing}_k\, \Theta) \leq d-k$. Generalizing a result of Smith and Varley [SV96], Ein and Lazarsfeld [EL97] showed that $\dim(\mathrm{Sing}_k\, \Theta) = d-k$ only if (A, Θ) splits as a k-fold product. In particular, for an irreducible ppav of dimension d and a point $x \in \Theta$, $\mathrm{mult}_x\, \Theta \leq d-1$. For the Jacobian of a smooth curve of genus g, applying the Riemann singularity theorem and Martens's theorem [Mar67], one can see that these bounds are not optimal; in fact $\dim(\mathrm{Sing}_k\, \Theta) \leq g-2k+1$, with equality holding only if C is hyperelliptic. This implies in particular that $\mathrm{mult}_x\, \Theta \leq (g+1)/2$.

For a Prym variety associated to a connected étale double cover of a smooth curve C of genus g, and $x \in \mathrm{Sing}\, \Xi$, the bounds given above yield that $\mathrm{mult}_x\, \Xi \leq g-2 = \dim(P)-1$. Although in [Mum74] Mumford used a strengthened version of Martens's theorem to prove statements about the dimension of the singular locus of the Prym theta divisor, no bound could be given on the multiplicity of these points without the results of Theorem 2. The implications of Theorem 2 for the singular locus of the Prym theta divisor are described in Corollary 5.1.2. In particular, it is shown that for an irreducible Prym variety (P, Ξ) and a point $x \in \Xi$, we have $\mathrm{mult}_x\, \Xi \leq (\dim(P)+1)/2$.

The rich connection between singularities of the Jacobian theta divisor and the canonical image of the curve is reflected in the Prym case by the connection between singularities of the Prym theta divisor and the Prym canonical image of the base curve C. A result of Tjurin [Tju75, Lemma 2.3, translation p. 963] generalized by Smith and Varley in [SV01, Prop. 5.1] shows that the Prym canonical image of C is contained in the tangent cone to Ξ at x for all singular double points x such that $T_x P \not\subseteq C_x \widetilde{\Theta}$. In addition, there are many such points for curves of high genus. Analogous to Green's theorem for Jacobians [Gre84], a primary open question for Prym varieties is to determine when the quadric tangent cones to Ξ cut out the Prym canonical image of C. In [Deb89], Debarre showed that this is true for general curves of genus no less than 8. In other words, the Prym map $\mathcal{P} : \mathcal{R}_g \to \mathcal{A}_{g-1}$, taking a connected étale double cover of a smooth curve to its associated Prym variety, is generically injective. Friedman and Smith have shown in [FS82] that the Prym map is generically injective for $g \geq 7$. On the other hand, unlike the case of Jacobians, there are examples in every dimension of Pryms arising from nonisomorphic double covers [SV02, page 237], and the question remains: Exactly which Pryms arise from a unique double cover of curves?

Towards understanding this question, we will consider the intersection of the k-secant variety of the Prym canonical image of C, that is the variety of k-dimensional secants (see Section 6.2), with the tangent cone to Ξ at singular points. In particular, if $\widetilde{C}$ is not hyperelliptic, Ψ is the Prym canonical morphism of C, and $C_x\Xi$ is the projective tangent cone to Ξ at x, then we will show the following: suppose $x \in \operatorname{Sing}\Xi$ corresponds to the line bundle $L \in \operatorname{Pic}^{2g-2}(\widetilde{C})$ and $h^0(L) = 2n$, then the $(n-1)$-secant variety to $\Psi(C)$ is not contained in $C_x\Xi$, while the $(n-2)$-secant variety is contained in $C_x\Xi$. As a consequence, $\Psi(C) \subseteq C_x\Xi$ if and only if $h^0(L) \geq 4$. Thus, if one hopes to recover the curve C as the base locus of quadric tangent cones to Ξ, one must exclude the tangent cones at points with $h^0(L) = 2$. This was suspected to be true since Smith and Varley [SV02] observed that if (P, Ξ) is the Jacobian of a nonhyperelliptic curve and x is a generic double point such that $h^0(L) = 2$, then $\Psi(C) \not\subseteq C_x\Xi$; see Remark 6.2.6.

Prym varieties also arise in the study of conic bundles, and in particular, in the study of cubic threefolds. Mumford stated in [Mum74] that the intermediate Jacobian of a smooth cubic threefold in $\mathbb{P}^4$ is isomorphic to the Prym variety of a connected étale double cover of a smooth plane quintic. Using this description of the intermediate Jacobian, he stated the following theorem: if X is a smooth cubic threefold in $\mathbb{P}^4$ with intermediate Jacobian (JX, Θ), then $\operatorname{Sing}\Theta = \{x\}$, $\operatorname{mult}_x \Theta = 3$, and $C_x\Theta \cong X$. It follows from this that JX determines X up to isomorphism and X is irrational; both statements were first proved by Clemens and Griffiths in [CG72].

In [CMF05], a converse to Mumford's theorem was proved: if (A, Θ) is a ppav of dimension 5, $\operatorname{Sing}\Theta = \{x\}$ and $\operatorname{mult}_x \Xi = 3$, then (A, Θ) is isomorphic to the intermediate Jacobian of a smooth cubic threefold. If one removes the condition that $\operatorname{Sing}\Theta = \{x\}$ and requires instead the weaker condition that exactly one of the singular points of Θ has multiplicity 3, then it was shown that the only other possibility is that (A, Θ) is isomorphic to JC or $JC \times JC'$ for some hyperelliptic curves C and C' (i.e., curves having a line bundle L such that $\deg(L) = h^0(L) = 2$). One would like to have a complete description of all ppavs of dimension five whose theta divisor has a triple point. The following theorem is a consequence of Theorem 2:

THEOREM 3. *Let* (P, Ξ) *be a* 5*-dimensional Prym variety*; *let* $x \in \operatorname{Sing}\Xi$. *If* $\operatorname{mult}_x \Xi = 3$, *then* (P, Ξ) *is a hyperelliptic Jacobian*, *the product of two hyperelliptic Jacobians*, *or the intermediate Jacobian of a smooth cubic threefold.*

The proof relies on a theorem of Mumford's [Mum74] about Prym varieties of hyperelliptic curves and on a theorem due to Beauville [Bea77b] (see also Donagi and Smith [DS81]) about Prym varieties of plane quintics.

The Prym map $\mathcal{P} : \mathcal{R}_6 \to \mathcal{A}_5$ has dense image, and so to extend Theorem 3 to all ppavs of dimension five, it remains only to check the statement on the boundary of the image. Beauville has shown in [Bea77a] that the ppavs on the boundary correspond to Prym varieties of admissible double covers of stable curves of genus 6. It is reasonable to expect that Theorem 2 can be extended to this case, at least for curves of low genus. Consequently, one should be able to describe all ppavs of dimension five whose theta divisors have triple points. This is work in progress.

The outline of the paper is as follows. Section 1 focuses on how to calculate the multiplicity of points of Sing Ξ. Section 2 concerns some general results that will be useful for computations in later sections. Section 3 gives a short proof of Smith and Varley's theorem. In Section 4 we prove Theorem 2, and Section 5 proves some immediate consequences, including Theorem 3. Section 6 establishes the connection between the computations made in Sections Section 4 and Section 5 and the Prym canonical image of the base curve. In particular, we examine the secant variety to the Prym canonical curve. We also give a brief description of the equation defining the tangent cone to Ξ at certain singular points.

1. **Theta divisors**

In this section we state the key results from [CMF05], which we will need in what follows. The proofs of these facts will be omitted except in the cases where certain generalizations are needed.

1.1. *Preliminaries on theta divisors.* Let S be a scheme, C be a smooth, connected, complete curve, and $\mathcal{L}$ be a line bundle over $C \times S$ whose relative degree is $g-1$. Then a result of Grothendieck [Gro61, 6,7] as formulated by Mumford in [Mum70, Th., p. 46] or [Har77, Lemma III.12.3] gives, locally on S, a complex of locally free $\mathcal{O}_S$-modules of the same rank, given by $d : \mathcal{C}^0 \to \mathcal{C}^1$; its cohomology is $R^0\pi_{2*}\mathcal{L}$ in dimension zero and $R^1\pi_{2*}\mathcal{L}$ in dimension one. If $(\det d)$ is not a zero divisor, then $(\det d)$ is an effective Cartier divisor that is independent of the choice of the complex $\mathcal{C}^\bullet$, and hence defines a global effective divisor Θ_S on S, which satisfies the following:

THEOREM 1.1.1. *In the above notation*, Θ_S *is an effective nonzero Cartier divisor on* S *with the properties that*

(a) *the support of* Θ_S *is equal to the set of* $s \in S$ *such that* $h^0(C;\mathcal{L}_s) \neq 0$;

(b) *if* $S = \mathrm{Pic}^{g-1}(C)$ *and* $\mathcal{L}$ *is a Poincaré line bundle*, *then* Θ_S *is the usual theta divisor*;

(c) *the construction is functorial, that is, if* $f : S' \to S$ *is a morphism and* $\mathscr{L}' = (\mathrm{Id} \times f)^*\mathscr{L}$, *then* $\Theta_{S'} = f^*\Theta_S$;

(d) *if* S *is smooth and* $\dim S = 1$, *then* $\Theta_S = \sum_{s \in S} \ell((R^1\pi_{2*}\mathscr{L})_s) \cdot s$, *where* $(R^1\pi_{2*}\mathscr{L})_s$ *refers to the stalk at* s.

Proof. Parts (a)–(c) are standard. A reference for the proof of (d) is Friedman and Morgan [FM94, Prop. 3.9, p. 384]. □

Let us now restrict to the case that S is a smooth curve with $s_0 \in S$. Let t be a local coordinate for S centered at s_0 that only vanishes there, and set $S_k = \operatorname{Spec} \mathbb{C}[t]/t^{k+1}$. For each k, there is a map $S_k \to S$, so that if we set $C_k = C \times S_k$, then there are induced maps $C_k \to C \times S$. For example, $C_0 = C$, and $C_0 \to C \times S$ is the inclusion of the fiber over s_0. Finally, let $\mathscr{L}_k$ be the restriction of $\mathscr{L}$ to C_k. It follows that $\mathscr{L}_0 = L$ is the restriction of $\mathscr{L}$ to $C \times \{s_0\}$, and $\mathscr{L}_k = \mathscr{L}/t^{k+1}\mathscr{L}$.

LEMMA 1.1.2 [CMF05, 1.5]. *For all* k,

$$\ell(H^0(C_k, \mathscr{L}_k)) \le \ell(H^0(C_{k+1}, \mathscr{L}_{k+1})).$$

Furthermore, there is an $N \in \mathbb{Z}$ *such that for all* $k \ge N$, $\ell(H^0(C_k, \mathscr{L}_k))$ *is independent of* k *and*

$$\ell(H^0(C_k, \mathscr{L}_k)) = \ell((R^1\pi_{2*}\mathscr{L})_{s_0}) = \operatorname{mult}_{s_0} \Theta_S.$$

We can be more explicit about the value of N. There is an exact sequence

$$0 \to t\mathscr{L}_k \longrightarrow \mathscr{L}_k \longrightarrow L \to 0, \tag{1.1.3}$$

where $t\mathscr{L}_k \cong \mathscr{L}_{k-1}$, and the obvious surjection $\mathscr{L}_k \to \mathscr{L}_{k-1}$ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathscr{L}_{k-1}) \arrow[r] \arrow[d] & H^0(\mathscr{L}_k) \arrow[r] \arrow[d] & H^0(L) \arrow[r, "\partial_k"] \arrow[d, equal] & H^1(\mathscr{L}_{k-1}) \cdots \\ 0 \arrow[r] & H^0(\mathscr{L}_{k-2}) \arrow[r] & H^0(\mathscr{L}_{k-1}) \arrow[r] & H^0(L) \arrow[r, "\partial_{k-1}"] & H^1(\mathscr{L}_{k-2}) \cdots. \end{tikzcd} \tag{1.1.4}$$

LEMMA 1.1.5 [CMF05, 1.6]. *Suppose in the above notation that* ∂_{N+1} *is injective for some* N. *Then the natural inclusion* $t^{k-N}\mathscr{L}_k \subseteq \mathscr{L}_k$ *induces an equality* $H^0(t^{k-N}\mathscr{L}_k) = H^0(\mathscr{L}_N)$ *for all* $k \ge N$. *In particular for all* $k \ge N$, *we have* $\ell(H^0(\mathscr{L}_k)) = \ell(H^0(\mathscr{L}_N))$.

One would like to have a way of computing $\ell(H^0(\mathscr{L}_k))$. Define W_k to be the image of the map $H^0(\mathscr{L}_k) \to H^0(L)$ induced by the exact sequence (1.1.3), and let $d_k = \dim(W_k)$. We will say that a section $s \in H^0(L)$ lifts to order k if $s \in W_k$. It is clear from the commutativity of the diagram (1.1.4) that $W_{k+1} \subseteq W_k$ for all k and hence $d_{k+1} \le d_k$.

LEMMA 1.1.6. *In the notation above,* $\ell(H^0(\mathcal{L}_k)) = \sum_{i=0}^{k} d_i$.

Proof. This follows by induction on k using the exact sequence

$$0 \to H^0(\mathcal{L}_i) \longrightarrow H^0(\mathcal{L}_{i+1}) \longrightarrow H^0(L) \xrightarrow{\partial_{i+1}} H^1(\mathcal{L}_i) \longrightarrow \cdots .$$

Indeed, $\ell(H^0(\mathcal{L}_{i+1})) = \ell(H^0(\mathcal{L}_i)) + d_{i+1}$. □

1.2. *Obstructions to lifting.* For an appropriate affine cover $\{U_i\}$ of C, we may assume that L has transition functions λ_{ij} and that the transition functions for $\mathcal{L}$ are of the form $\lambda_{ij}(t) = \lambda_{ij}(1 + \sum_{k=1}^{\infty} \alpha_{ij}^{(k)} t^k)$. By definition, these satisfy the condition $\lambda_{ik}(t) = \lambda_{ij}(t)\lambda_{jk}(t)$, and it follows that the cochain $\xi = \alpha_{ij}^{(1)}$ is a cocycle in $H^1(\mathcal{O}_C)$. Likewise set

$$\lambda_{ij;N}(t) = \lambda_{ij}\big(1 + \textstyle\sum_{k=1}^{N} \alpha_{ij}^{(k)} t^k\big).$$

Assume that $s \in H^0(L)$ and that s_{N-1} is a lifting of s to a section of $\mathcal{L}_{N-1}$. Then using the trivialization over the open cover $\{U_i\}$ we have

$$s_{i;N-1} = \sum_{k=0}^{N-1} \sigma_i^{(k)} t^k \quad \text{for some functions } \sigma_i^{(k)} \in \mathcal{O}_C(U_i),$$

with $s_{i;N-1} = \lambda_{ij;N-1}(t) s_{j;N-1}$ on $(U_i \cap U_j) \times \operatorname{Spec} \mathbb{C}[t]/(t^N)$. The section s_{N-1} lifts to a section s_N if and only if there exists a $\sigma_i^{(N)} \in \mathcal{O}_C(U_i)$ such that, if we set $s_{i;N} = \sum_{k=0}^{N} \sigma_i^{(k)} t^k$ then $s_{i;N} = \lambda_{ij;N}(t) s_{j;N}$ on $(U_i \cap U_j) \times \operatorname{Spec} \mathbb{C}[t]/(t^{N+1})$. Since $s_{i;N-1}$ is already a section of $\mathcal{L}_{N-1}$, this is equivalent to the condition

$$\sigma_i^{(N)} = \lambda_{ij}\sigma_j^{(N)} + \sum_{k=0}^{N-1} \lambda_{ij}\alpha_{ij}^{(N-k)}\sigma_j^{(k)}.$$

Let $\gamma_N(s_{N-1})$ be the 1-cochain defined by $\sum_{k=0}^{N-1} \lambda_{ij}\alpha_{ij}^{(N-k)}\sigma_j^{(k)}$, i.e., the obstruction to lifting s_{N-1} to order N. I claim $\gamma_N(s_{N-1})$ is a 1-cocyle in $H^1(L)$. Indeed, let ∂_k° denote the map $H^0(\mathcal{L}_{k-1}) \to H^1(L)$ induced from the exact sequence

$$0 \to L \longrightarrow \mathcal{L}_k \longrightarrow \mathcal{L}_{k-1} \to 0. \tag{1.2.1}$$

A computation in the Čech complex will then show this:

LEMMA 1.2.2. *Suppose* $s_{N-1} \in H^0(\mathcal{L}_{N-1})$. *Then*

$$\gamma_N(s_{N-1}) = \partial_{N-1}^{\circ}(s_{N-1}) \in H^1(L),$$

and thus s_{N-1} *lifts to a section* $s_N \in H^0(\mathcal{L}_N)$ *if and only if* $\gamma_N(s_{N-1}) = 0$ *in* $H^1(L)$.

Computing these obstructions is the central step in the proofs of the main theorems. In these proofs, we will be restricting our attention to a particular class of deformations described in the next section, and in that case we will write down explicit formulas for the first and second order obstructions. We will also outline a particular technique for determining their class in $H^1(L)$. The basic idea is illustrated by the following lemmas regarding first order lifts.

LEMMA 1.2.3. *Let $\xi \in H^1(\mathcal{O}_C) = \mathrm{Ext}^1(L, L)$ be the extension class corresponding to $\mathcal{L}_1$. Then $\gamma_1(s) = s \cup \xi \in H^1(L)$, where the cup product is $H^0(L) \otimes H^1(\mathcal{O}_C) \to H^1(L)$.*

LEMMA 1.2.4. *Let D be an effective divisor on C, and let ∂ be the coboundary map $H^0(\mathcal{O}_D(D)) \to H^1(\mathcal{O}_C)$ induced by the short exact sequence*

$$0 \to \mathcal{O}_C \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_D(D) \to 0.$$

Suppose that $\xi \in H^1(\mathcal{O}_C)$ is of the form $\partial(t)$ for some $t \in H^0(\mathcal{O}_D(D))$. Then $s \cup \xi = \partial_L(s \cdot t)$, where $s \cdot t$ is the section of $L(D)|_D$ given by taking the cup product of s and t, and ∂_L is the coboundary homomorphism arising from

$$0 \to L \longrightarrow L(D) \longrightarrow L(D) \otimes \mathcal{O}_D \to 0.$$

Now consider the following useful observation. Let p be a point of C, and fix once and for all a local coordinate z at p. More precisely, let $\{U_i\}$ be an open cover of C, and assume that $p \in U_0$, that $p \notin U_i$ for $i \neq 0$, and that $z \in \mathcal{O}_C(U_0)$ is a coordinate centered at p. A calculation then shows this:

LEMMA 1.2.5. *For $a \in \mathbb{C}$, let $\xi \in H^1(\mathcal{O}_C)$ be the image of a/z under the coboundary map induced by the short exact sequence*

$$0 \to \mathcal{O}_C \longrightarrow \mathcal{O}_C(p) \longrightarrow \mathcal{O}_C(p)|_p \to 0.$$

Let $s \in H^0(L)$ be a section such that $s(p) = 0$. Then $s \cup \xi = 0$ in $H^1(L)$, and in fact, defining the 1-cocycle ξ and the 0-cochain $\sigma^{(1)}$ by

$$\xi_{ij} = \begin{cases} a/z & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases} \quad \text{and} \quad \sigma_i^{(1)} = \begin{cases} as/z & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases}$$

we have $s \cup \xi = \delta\sigma^{(1)}$, where δ is the Čech coboundary map. In other words, if $\mathcal{L}_1$ is the first order deformation of L with transition functions $\lambda_{ij}(1 + \xi_{ij} t)$, then $s_i + \sigma_i^{(1)} t$ is a lifting of s to first order.

1.3. *The line bundles $\mathcal{L}^{\pm}_{p;a}$.* For the rest of the paper, we will focus on a particular class of deformations. For a fixed point $p \in C$, consider the line bundles $\pi_1^*\mathcal{O}_C(p) \otimes \mathcal{O}_{C\times C}(-\Delta)$ and $\pi_1^*\mathcal{O}_C(-p) \otimes \mathcal{O}_{C\times C}(\Delta)$ over $C \times C$, where $\Delta \subseteq C \times C$ is the diagonal. We fix the coordinate z centered at p as before, and let t be the coordinate z, viewed as a coordinate on an affine open subset of the second copy of C. Let $S \subseteq \mathbb{C}$ be a small disk, and let $u : S \to C$ be the inverse to t. On $C \times S$, define the line bundles $\Lambda_p^{\pm} = (1 \times u)^*(\pi_1^* \mathcal{O}_C(\pm p) \otimes \mathcal{O}_{C \times C}(\mp \Delta))$. From the definition, it is clear that $\Lambda_p^- = (\Lambda_p^+)^{-1}$. These line bundles induce holomorphic maps $f : S \to JC$, and if $w \in S$, then $(\Lambda_p^{\pm})_w = \mathcal{O}_C(\pm p \mp u(w))$. Using the open cover of C as in Lemma 1.2.5, it follows that the transition functions for Λ_p^+ are equal to 1 if neither i nor j is zero, and (for small t),

$$\lambda_{0j}(t) = \frac{z}{z-t} = \sum_{k=0}^{\infty} \left(\frac{t^k}{z^k}\right).$$

Similarly, the transition functions for Λ_p^- are equal to 1 if neither i nor j is zero, and (for small t),

$$\lambda_{0j}(t) = \frac{z-t}{z} = 1 - \frac{t}{z}.$$

For $L \in \operatorname{Pic}^d(\widetilde{C})$, define $\mathcal{L}_p^{\pm} = \Lambda_p^{\pm} \otimes \pi_1^* L$, a line bundle over $C \times S$. If the transition functions for L are given by λ_{ij}, then it follows that the transition functions for $\mathcal{L}_p^+$ are equal to λ_{ij} if neither i nor j is zero, and (for small t),

$$\lambda_{0j}(t) = \lambda_{0j} \cdot \left(\frac{z}{z-t}\right) = \lambda_{0j} \cdot \sum_{k=0}^{\infty} \left(\frac{t^k}{z^k}\right).$$

Similarly, the transition functions for $\mathcal{L}_p^-$ are equal to λ_{ij} if neither i nor j is zero, and (for small t),

$$\lambda_{0j}(t) = \lambda_{0j} \cdot \left(\frac{z-t}{z}\right) = \lambda_{0j} \cdot \left(1 - \frac{t}{z}\right).$$

For making computations, it will be useful to rescale t. For $a \in \mathbb{C}$, define a local deformation $\mathcal{L}_{p;a}^+$ by setting the transition functions equal to λ_{ij} if neither i nor j is zero, and (for small t),

$$\lambda_{0j}(t) = \lambda_{0j} \cdot \left(\frac{z}{z-at}\right) = \lambda_{0j} \cdot \sum_{k=0}^{\infty} \left(\frac{a^k}{z^k} t^k\right).$$

In short, we are considering a second small disk S' in $\mathbb{C}$, a map $S' \to S$ given by $w \mapsto aw$, and setting $\mathcal{L}_{p;a}^+ = \mathcal{L}_p^+|_{C \times S'}$. There is then an induced holomorphic map $f_a : S' \to JC$ for each a, and if $w \in S'$, then $(\mathcal{L}_{p;a}^+)_w = L \otimes \mathcal{O}_C(p - u(aw))$.

Similarly, define a local deformation $\mathcal{L}_{p;a}^-$ by setting the transition functions equal to λ_{ij} if neither i nor j is zero, and (for small t),

$$\lambda_{0j}(t) = \lambda_{0j} \cdot \left(\frac{z-at}{z}\right) = \lambda_{0j} \cdot \left(1 - \frac{a}{z} t\right).$$

If $w \in S'$, then $(\mathcal{L}_{p;a}^-)_w = L \otimes \mathcal{O}_C(-p + u(aw))$.

A section $s \in H^0(L)$ lifts to first order as a section of $(\mathscr{L}^+_{p;a})_1$ if and only if there exists a $\sigma^{(1)}$ satisfying

$$\sigma_i^{(1)} - \lambda_{ij}\sigma_j^{(1)} = \lambda_{ij}\alpha_{ij}^{(1)} s_j = \begin{cases} 0 & \text{for } i \neq 0, \\ as_0/z & \text{for } i = 0, \end{cases}$$

and a section $(s + \sigma^{(1)}t) \in H^0((\mathscr{L}^+_{p;a})_1)$ lifts to second order if and only if there exists a $\sigma^{(2)}$ satisfying

$$\sigma_i^{(2)} - \lambda_{ij}\sigma_j^{(2)} = \lambda_{ij}\alpha_{ij}^{(2)} s_j + \lambda_{ij}\alpha_{ij}^{(1)}\sigma_j^{(1)} = \begin{cases} 0 & \text{for } i \neq 0, \\ a^2 s_0/z^2 + \lambda_{ij} a\sigma_j^{(1)}/z & \text{for } i = 0. \end{cases}$$

Here the $i = 0$ case can be rewritten as

$$a^2 s_0/z^2 + \lambda_{ij} a\sigma_j^{(1)}/z = a^2 s_0/z^2 + (a\sigma_0^{(1)}/z - a^2 s_0/z^2) = a\sigma_0^{(1)}/z.$$

Likewise, a section $s \in H^0(L)$ lifts to first order as a section of $(\mathscr{L}^-_{p;a})_1$ if and only if there exists a $\sigma^{(1)}$ satisfying

$$\sigma_i^{(1)} - \lambda_{ij}\sigma_j^{(1)} = \lambda_{ij}\alpha_{ij}^{(1)} s_j = \begin{cases} 0 & \text{for } i \neq 0, \\ -as_0/z & \text{for } i = 0, \end{cases}$$

and a section $(s + \sigma^{(1)}t) \in H^0((\mathscr{L}^-_{p;a})_1)$ lifts to second order if and only if there exists a $\sigma^{(2)}$ satisfying

$$\sigma_i^{(2)} - \lambda_{ij}\sigma_j^{(2)} = \lambda_{ij}\alpha_{ij}^{(2)} s_j + \lambda_{ij}\alpha_{ij}^{(1)}\sigma_j^{(1)} = \begin{cases} 0 & \text{for } i \neq 0, \\ \lambda_{ij}(-a/z)\sigma_j^{(1)} & \\ \quad = -a\sigma_0^{(1)}/z - a^2 s_0/z^2 & \text{for } i = 0. \end{cases}$$

A straightforward calculation in the Čech complex will prove the following two lemmas:

LEMMA 1.3.1. *Let $\partial_{L,p}$ be the coboundary map*

$$\partial_{L,p} : H^0(L(p) \otimes \mathbb{O}_p) \to H^1(L)$$

induced from the exact sequence $0 \to L \longrightarrow L(p) \longrightarrow L(p) \otimes \mathbb{O}_p \to 0$, *and let* $A_1(s) \in H^0(L(p) \otimes \mathbb{O}_p)$ *be defined as*

$$A_1(s) = \begin{cases} -as_0/z & \text{for } \mathscr{L}^+_{p;a}, \\ as_0/z & \text{for } \mathscr{L}^-_{p;a}. \end{cases}$$

Then $\gamma_1(s) = \partial_{L,p}(A_1(s))$.

COROLLARY 1.3.2. *If* $s \in H^0(L(-p))$, *then* $\gamma_1(s) = 0$.

Remark 1.3.3. This is a weaker statement than was proved in Lemma 1.2.5, where an explicit first order lifting of s was given.

LEMMA 1.3.4. *Let $\partial_{L,2p}$ be the coboundary map*

$$\partial_{L,2p} : H^0(L(2p) \otimes \mathcal{O}_{2p}) \to H^1(L)$$

induced from the exact sequence $0 \to L \longrightarrow L(2p) \longrightarrow L(2p) \otimes \mathcal{O}_{2p} \to 0$, and let $A_2(s + \sigma^{(1)}t) \in H^0(L(2p) \otimes \mathcal{O}_{2p})$ be defined as

$$A_2(s + \sigma^{(1)}t) = \begin{cases} -a\sigma_0^{(1)}/z & \text{for } \mathcal{L}^+_{p;a}, \\ a\sigma_0^{(1)}/z + a^2 s_0/z^2 & \text{for } \mathcal{L}^-_{p;a}. \end{cases}$$

Then $\gamma_2(s + \sigma^{(1)}t) = \partial_{L,2p}(A_2(s + \sigma^{(1)}t))$.

COROLLARY 1.3.5. *If $s \in H^0(L(-2p))$, then there exists a first order lift of s, say $s + \sigma^{(1)}t$, such that $\gamma_2(s + \sigma^{(1)}t) = 0$.*

Proof. Let $s \in H^0(L(-2p))$. Then since $s \in H^0(L(-p))$, let $s + \sigma^{(1)}t$ be the standard lift of s, as given in Lemma 1.2.5. Recall that we set

$$\sigma_i^{(1)} = \begin{cases} as/z & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Since s vanishes to order 2 at p, we see that $A_2(s + \sigma^{(1)}) = 0$, and hence s lifts to second order. □

For our computations, we will want to consider a more general class of deformations modeled on the $\mathcal{L}^{\pm}_{p;a}$. Define $\Delta_i \subseteq C \times C^k$ as

$$\Delta_i = \{(x_0, \dots, x_k) \in C \times C^k \mid x_0 = x_i\},$$

and for a particular choice of points $p_1, \dots, p_k$, let $D = \sum_{i=1}^k p_i$. On $C \times C^k$, consider the line bundle $\pi_1^* \mathcal{O}_C(D) \otimes \mathcal{O}_{C \times C^k}(-\sum \Delta_i)$. Let u_i be a map from a disk $S \subseteq \mathbb{C}$ to a neighborhood of the point p_i, and let $u : S \to C^k$ be the map given by $w \mapsto (u_1(w), \dots, u_k(w))$. Then set

$$\Lambda^+_D = (1 \times u)^*(\pi_1^* \mathcal{O}_C(D) \otimes \mathcal{O}_{C \times C^k}(-\textstyle\sum \Delta_i)).$$

For $L \in \operatorname{Pic}^d(C)$, let $\mathcal{L}^+_D = \Lambda^+_D \otimes \pi_1^* L$. This has fiber over a point $w \in S$ equal to $L \otimes \mathcal{O}_C((p_1 - u_1(w)) \otimes \cdots \otimes \mathcal{O}_C((p_k - u_k(w))$. As before Λ^+_D induces a holomorphic map $S \to JC$, and it is also clear that $\mathcal{L}^+_D = \pi_1^* L \otimes \Lambda^+_{p_1} \otimes \cdots \otimes \Lambda^+_{p_k}$. We can similarly define $\mathcal{L}^-_D$, and by rescaling the local coordinate, construct the line bundle

$$\mathcal{L} = \pi_1^* L \otimes \Lambda^+_{p_1;a_1} \otimes \cdots \otimes \Lambda^+_{p_{k_1};a_{k_1}} \otimes \Lambda^-_{p_{k_1+1};a_{k_1+1}} \otimes \cdots \otimes \Lambda^-_{p_k;a_k}.$$

The fiber of $\mathcal{L}$ over a point $w \in S$ is given by

$$\mathcal{L}_w = L \otimes \mathcal{O}_C(p_1 - u_1(a_1 w)) \otimes \cdots \otimes \mathcal{O}_C(-p_k + u_k(a_k w)).$$

Remark 1.3.6. The calculations in Lemmas 1.3.1 and 1.3.4 are local, in the sense that the obstructions for $\mathscr{L}$ are sums of the local contributions calculated in those lemmas.

1.4. *The Prym case.* For the rest of the paper, we will be considering the following situation: C will be a smooth curve of genus g, $\pi : \widetilde{C} \to C$ will be a connected étale double cover, τ will be the associated involution on $\widetilde{C}$, $\eta \in \mathrm{Pic}^0(C)$ will be the associated semiperiod, and $P \subseteq J\widetilde{C}$ will be the Prym variety. If $\widetilde{\Theta}$ is the canonical theta divisor of $J\widetilde{C}$, then Mumford [Mum74] has shown that $\widetilde{\Theta} \cap P = 2 \cdot \Xi$, where Ξ is the class of a principal polarization on P. Recall, if we identify JC with $\mathrm{Pic}^{2g-2}(\widetilde{C})$, then as a set P can be described as

$$P = \{L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Norm}(L) = \omega_C,\ h^0(L) \equiv 0 \pmod 2\}.$$

It follows that $\Xi = \{L \in P \mid h^0(L) \geq 2\}$.

The following straightforward lemma is fundamental for what follows.

LEMMA 1.4.1. *Let H be a hypersurface, not necessarily reduced, defined in an open neighborhood of* 0 *in* $\mathbb{C}^n$ *and containing* 0. *Let S be a disk in* $\mathbb{C}$ *containing* 0, *and let $f : S \to \mathbb{C}^n$ be a holomorphic map with $f(0) = 0$. Then* $\mathrm{mult}_0 H \leq \mathrm{mult}_0 f^*H$, *and equality holds if and only if $f_*(T_0 S)$ is not contained in the tangent cone to H at* 0.

As an application, suppose S is a smooth curve with $s_0 \in S$ and $\mathscr{L}$ is a line bundle over $\widetilde{C} \times S$ of relative degree $2g-2$. Let $f : S \to J\widetilde{C}$ be the induced morphism, and let $L \in \mathrm{Pic}^{2g-2}(\widetilde{C})$ be the line bundle associated to the point $x = f(s_0) \in J\widetilde{C}$.

LEMMA 1.4.2 [CMF05, 1.4]. *If $f(S) \subseteq P$, then*

$$\tfrac{1}{2} h^0(L) \leq \mathrm{mult}_x \Xi \leq \tfrac{1}{2} \deg_{s_0} \Theta_S = \tfrac{1}{2} \ell((R^1 \pi_{2*} \mathscr{L})_{s_0}).$$

Moreover, there exists a choice of S and a line bundle $\mathscr{L}$ as above such that $\mathrm{mult}_x \Xi = \frac{1}{2} \ell((R^1 \pi_{2*} \mathscr{L})_{s_0})$.

Recall that for t a local coordinate on S centered at s_0 and only vanishing there, $\widetilde{C}_k = \widetilde{C} \times \mathrm{Spec}\, \mathbb{C}[t]/(t^{k+1})$, W_k is the image of the map $H^0(\mathscr{L}_k) \to H^0(L)$ induced by the exact sequence (1.1.3), and $d_k = \dim(W_k)$.

PROPOSITION 1.4.3. *In the notation above, we have these facts*:

(a) $\ell((R^1 \pi_{2*} \mathscr{L})_{s_0}) \geq \ell(H^0(\mathscr{L}_k)) = \sum_{i=0}^k d_i$ *for every k, and if $d_N = 0$, then equality holds for all $k \geq N$.*

(b) *if $f(S) \subseteq P$ and $L = \pi^*(M) \otimes \mathscr{O}_{\widetilde{C}}(B)$, with $h^0(M) > h^0(L)/2$, $B \geq 0$, and $B \cap \tau^* B = \varnothing$, then $d_1 \geq 2h^0(M) - h^0(L)$.*

(c) *If* $x \in \Xi$ *and* $L = \pi^*(M) \otimes \mathcal{O}_{\widetilde{C}}(B)$, *with* $h^0(M) > h^0(L)/2$, $B \geq 0$, *and* $B \cap \tau^* B = \varnothing$, *then* $\operatorname{mult}_x \Xi \geq h^0(M)$. *Furthermore*, $\operatorname{mult}_x \Xi = h^0(M)$ *if and only if there exists a choice of* S *and a line bundle* $\mathcal{L}$ *as above such that* $f(S) \subseteq P$, $d_1 = 2h^0(M) - h^0(L)$, *and* $d_2 = 0$.

Proof. (a) is a restatement of the lemmas in Section 1.1, and the proof of (b) is contained in the proof of [CMF05, Th. 2.3].

(c) It follows from part (a) and Lemma 1.4.2 that for any deformation $f : S \to P$ with $f(s_0) = x$,

$$\operatorname{mult}_x \Xi \leq \tfrac{1}{2} \operatorname{mult}_{s_0} \Theta_S = \tfrac{1}{2} \textstyle\sum_{k \geq 0} d_k.$$

Furthermore, there exist deformations for which equality holds, so that

$$\operatorname{mult}_x \Xi = \inf_{\substack{f:S\to P\\ f(s_0)=x}} \{\tfrac{1}{2} \textstyle\sum_{k \geq 0} d_k\}.$$

By definition $d_0 = h^0(L)$, and by (b) we know that $d_1 \geq 2h^0(M) - h^0(L)$ for all such deformations. Since $d_k \geq d_{k+1} \geq 0$ for all k, it follows that

$$\operatorname{mult}_x \Xi = \inf\{\tfrac{1}{2} \textstyle\sum_{k \geq 0} d_k\} \geq \tfrac{1}{2}(h^0(L) + 2h^0(M) - h^0(L)) = h^0(M),$$

and the inequality becomes an equality if and only if there is a deformation such that $d_1 = 2h^0(M) - h^0(L)$ and $d_2 = 0$. □

Let $\{U_i\}_{i \in I}$ be an open affine cover for $\widetilde{C}$, where $p_i \in U_j$ for $i \in \{1, \ldots, n\}$ if and only if $i = j$, and $\tau(p_i) \in U_j$ if and only if $j = i + n$. For $1 \leq i \leq n$ we will define the index $\tau(i) = i + n$. On each open set U_i define z_i to be a local coordinate, which is centered at p_i for $i \in \{1, \ldots, n\}$ and is centered at $\tau(p_i)$ for $i \in \{\tau(1), \ldots, \tau(n)\}$. We also choose local coordinates so that $\tau^* z_i = z_{\tau(i)}$.

Let $q_1, \ldots, q_n$ be general points of C, let $\pi^{-1}(q_i) = \{p_i, \tau(p_i)\}$, and let $D = \sum_{i=1}^n (p_i + \tau(p_i))$. Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$.

Definition 1.4.4. Let $S \subseteq \mathbb{C}$ be a disk containing the origin, and let $L \in \operatorname{Pic}^{2g-2}(\widetilde{C})$ be the line bundle associated to a point $x \in P$. With D and a as above, define the *deformation of* L *associated to* D *and* a, denoted by $\mathcal{L}_{D;a} = \mathcal{L}$, to be the line bundle over $\widetilde{C} \times S$ given by

$$\mathcal{L} = \pi_1^* L \otimes \Lambda^+_{p_1;a_1} \otimes \cdots \otimes \Lambda^+_{p_n;a_n} \otimes \Lambda^-_{\tau(p_1);a_1} \otimes \cdots \otimes \Lambda^-_{\tau(p_n);a_n}.$$

The fiber over a point $w \in S$ is given by

$$\mathcal{L}_w = L \otimes \mathcal{O}_{\widetilde{C}}(p_1 - u_1(a_1 w) - \tau(p_1) + \tau(u_1(a_1 w))) \otimes \cdots \otimes \mathcal{O}_{\widetilde{C}}(p_n - u_n(a_n w) - \tau(p_n) + \tau(u_n(a_n w))).$$

LEMMA 1.4.5. *Given* $\mathcal{L}_{D,a}$, *let* $f : S \to J\widetilde{C}$ *be the associated morphism. Then* f *is holomorphic*, *and* $f(S) \subseteq P$.

Proof. We have seen in the previous section that f is holomorphic. Let $s_0 \in S$ be such that $f(s_0) = x$. Then $f(S) \subseteq P$, since $\mathrm{Norm}((\mathscr{L}_{D;a})_s) = \omega_C$ for all $s \in S$, and $(\mathscr{L}_{D;a})_{s_0} = L \in P$. Indeed, $\mathrm{Norm}^{-1}(\omega_C)$ has two connected components distinguished by the parity of h^0. Since $f(S)$ includes a point in the Prym variety, namely L, $f(S)$ is contained in P. □

We will now reinterpret Lemmas 1.3.1 and 1.3.4 in the case of a deformation $\mathscr{L}_{D;a}$. To begin, we fix a trivialization of L and $\mathscr{L}_1$ at the points p_1, $\tau(p_1)$, $\dots$, p_n, $\tau(p_n)$. We then choose a fixed basis for $H^0(L(D) \otimes \mathcal{O}_D)$ given by $\{1/z_1, 1/z_{\tau(1)}, \dots, 1/z_n, 1/z_{\tau(n)}\}$, and a basis for $H^0(L(2D) \otimes \mathcal{O}_{2D})$ given by $\{1/z_1^2, 1/z_1, 1/z_{\tau(1)}^2, 1/z_{\tau(1)}, \dots, 1/z_n^2, 1/z_n, 1/z_{\tau(n)}^2, 1/z_{\tau(n)}\}$. With respect to these bases and trivializations, the lemmas can then be restated as follows:

LEMMA 1.4.6. *Let $\partial_{L,D}$ be the coboundary map*

$$\partial_{L,D} : H^0(L(D) \otimes \mathcal{O}_D) \to H^1(L)$$

induced from the exact sequence $0 \to L \longrightarrow L(D) \longrightarrow L(D) \otimes \mathcal{O}_D \to 0$, *and let* $A_1(s) \in H^0(L(D) \otimes \mathcal{O}_D)$ *be defined as*

$$A_1(s) = (-a_1 s(p_1), a_1 s(\tau(p_1)), \dots, -a_n s(p_n), a_n s(\tau(p_n))).$$

Then $s \in H^0(L)$ lifts to first order as a section of $\mathscr{L}_{D;a}$ if and only if $s \in$ $\ker(\partial_{L,D} \circ A_1)$.

COROLLARY 1.4.7. $H^0(L(-D)) \subseteq \ker(A_1)$, *and if $a_i \neq 0$ for all i, then* $H^0(L(-D)) = \ker(A_1)$.

LEMMA 1.4.8. *Let $\partial_{L,2D}$ be the coboundary map*

$$\partial_{L,2D} : H^0(L(2D) \otimes \mathcal{O}_{2D}) \longrightarrow H^1(L)$$

induced from the exact sequence $0 \to L \longrightarrow L(2D) \longrightarrow L(2D) \otimes \mathcal{O}_{2D} \to 0$, *and let* $A_2(s + \sigma^{(1)} t) \in H^0(L(2D) \otimes \mathcal{O}_{2D})$ *be defined as*

$$A_2(s+\sigma^{(1)}t) = \Big(0, -a_1\sigma^{(1)}(p_1),\ a_1^2 s(\tau(p_1)),\ a_1\sigma^{(1)}(\tau(p_1)) + a_1^2\frac{ds}{dz}(\tau(p_1)), \dots\Big).$$

Then $s + \sigma^{(1)} t \in H^0(\mathscr{L}_1)$ lifts to second order if and only if $s \in$ $\ker(\partial_{L,2D} \circ A_2)$.

COROLLARY 1.4.9. $H^0(L(-2D)) \subseteq W_2$.

Proof. The section $s \in W_2$ if and only if there is a first order lift $s + \sigma^{(1)} t$ such that $\partial_{L,2D} \circ A_2(s + \sigma^{(1)} t) = 0$. Using the standard lift of s given in Lemma 1.2.5, and using the same analysis as in Corollary 1.3.5, one can easily show that if $s \in H^0(L(-2D))$, then $A_2(s + \sigma^{(1)} t) = 0$. □

2. **Linear systems and linear algebra**

Here we collect some general results that will be useful for computations in subsequent sections.

2.1. *Linear systems on a double cover.* Given a line bundle L on $\widetilde{C}$, and a point $q \in C$, we will want to know when the points of $\pi^{-1}(q)$ impose independent conditions on the linear system $|L|$.

LEMMA 2.1.1. *Suppose that $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$, where M is a line bundle on C such that $h^0(\widetilde{C}, L) > h^0(C, M) > 0$, and suppose $B \geq 0$ is an effective divisor on $\widetilde{C}$ such that $B \cap \tau^* B = \varnothing$. If $p \in \widetilde{C}$ is a general point, then $h^0(L(-p - \tau(p))) = h^0(L) - 2$.*

Proof. Let $b \in H^0(\mathcal{O}_{\widetilde{C}}(B))$ with $(b)_0 = B$. Let $v \in H^0(L) - \pi^* H^0(M) \cdot b$. We can consider (v/b) as a rational section of $\pi^* M$, and I claim that (v/b) is not τ-invariant. Indeed, if it were, then it could not have poles along B since it has none along $\tau(B)$, and so (v/b) would be a regular section; i.e. $v/b \in H^0(\pi^* M)^+ = \pi^* H^0(M)$. This would be a contradiction, as we would then have $v \in \pi^* H^0(M) \cdot b$.

It follows that $(v/b)(p) \neq (v/b)(\tau(p))$ for a general point $p \in \widetilde{C}$. Now let $s \in \pi^* H^0(M)$ be a nonzero section, and choose $\lambda \in \mathbb{C}$ so that $v(p) - \lambda s(p) b(p)$ is equal to zero. It is immediate to check that $v(\tau(p)) - \lambda s(\tau(p)) b(\tau(p)) \neq 0$. This completes the proof, since $h^0(L(-p)) = h^0(L) - 1$ for a general p, and we have found a section $v - \lambda s b \in H^0(L(-p))$ that does not vanish at $\tau(p)$. $\square$

COROLLARY 2.1.2. *Suppose that $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$, where M is a line bundle on C such that $h^0(\widetilde{C}, L) \geq h^0(C, M) > 0$, and $B \geq 0$ is an effective divisor on $\widetilde{C}$ such that $B \cap \tau^* B = \varnothing$. Let $b \in H^0(B)$ be a section vanishing on B. Then $H^0(L) = \pi^* H^0(M) \cdot b$ if and only if $h^0(L(-p - \tau(p))) = h^0(L) - 1$ for a general point $p \in \widetilde{C}$.*

Proof. Let p be any point of $\widetilde{C}$, and let $q = \pi(p)$. If $H^0(L) = \pi^* H^0(M) \cdot b$, then it follows that $H^0(L(-p - \tau(p))) = \pi^* H^0(M(-q)) \cdot b$. Consequently, $h^0(L(-p - \tau(p))) = h^0(L) - 1$. Conversely, if $h^0(L(-p - \tau(p))) = h^0(L) - 1$ for a general point p, then Lemma 2.1.1 implies that $h^0(L) = h^0(M)$. $\square$

For the duration of the paper we will use the following notation. Given a collection $p_1, \tau(p_1), \ldots, p_k, \tau(p_k)$ of distinct points of $\widetilde{C}$, we will set $D_k = \sum_{i=1}^{k}(p_i + \tau(p_i))$.

COROLLARY 2.1.3. *Suppose that $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$, where M is a line bundle on C such that $h^0(\widetilde{C}, L) \geq h^0(C, M) > 0$, and $B \geq 0$ is an effective divisor on $\widetilde{C}$ such that $B \cap \tau^* B = \varnothing$. Let $h^0(M) = n_1$, $h^0(L) - h^0(M) = n_2$, and $p_1, \tau(p_1), \ldots, p_k, \tau(p_k)$ be $2k$ points of $\widetilde{C}$, where $p_1, \ldots, p_k$ are general.*

(a) *Suppose $h^0(M) > h^0(L)/2$ and $k \leq n_2$. Then $h^0(L(-D_k)) = h^0(L) - 2k$.*

(b) *Suppose* $h^0(M) > h^0(L)/2$ *and* $n_2 \le k \le n_1$. *Then* $h^0(L(-D_k)) = h^0(L) - n_2 - k$. *Furthermore, in this case*

$$H^0(L(-D_k)) = \pi^* H^0(M(-\textstyle\sum_{i=1}^k q_i)) \cdot b,$$

where $q_i = \pi(p_i)$ *and* $b \in H^0(\mathbb{O}_{\tilde{C}}(B))$ *vanishes on* B.

(c) *Suppose* $h^0(M) > h^0(L)/2$, $n_2 \le k \le n_1$, *and* $1 \le k_1 \le k$. *Then*

$$\begin{aligned} H^0(L(-D_k - D_{k_1})) &= H^0\Big(L\Big(-D_k - \sum_{i=1}^{k_1} p_i\Big)\Big) \\ &= H^0\Big(L\Big(-D_k - \sum_{i=1}^{k_1} \tau(p_i)\Big)\Big), \\ h^0(L(-D_k - D_{k_1})) &= \max(h^0(L) - n_2 - k - k_1, 0) \end{aligned}$$

Furthermore, in this case

$$H^0(L(-D_k - D_{k_1})) = \pi^* H^0(M(-\textstyle\sum_{i=1}^k q_i - \sum_{j=1}^{k_1} q_j)) \cdot b.$$

(d) *Suppose* $h^0(M) \le h^0(L)/2$ *and* $k \le n_1$. *Then* $h^0(L(-D_k)) = h^0(L) - 2k$.

Proof. (a) In the case $h^0(L) = h^0(M)$, there is nothing to prove since $k \le n_2 = 0$. So assume that $h^0(L) > h^0(M)$. We will now use induction on k. For $k = 1$, we are done by Lemma 2.1.1. So assume we have proved (a) for all $k \le m$, where m is some integer less than n_2. Let $q_i = \pi(p_i)$, $L' = L(-D_m)$, and $M' = M(-\sum_{i=1}^m q_i)$. Then

$$L' = \pi^* M' \otimes \mathbb{O}_{\tilde{C}}(B) \quad \text{and} \quad h^0(L') = h^0(L) - 2m$$

by induction, and $h^0(M') = h^0(M) - m$ since the points q_i are general. Now $h^0(L') = h^0(L) - 2m > h^0(M) - m = h^0(M') > 0$, since $m < n_2 < h^0(M)$ and $h^0(L) - h^0(M) = n_2 > m$. It follows that L' and M' satisfy the condition of the lemma, and hence for general points p_{m+1} and $\tau(p_{m+1})$ we have $h^0(L(-D_{m+1})) = h^0(L) - 2m - 2$.

(b) In the case $k = n_2$, we have seen that $h^0(L(-D_{n_2})) = h^0(L) - 2n_2 = n_1 - n_2 = h^0(M(-\sum_{i=1}^{n_2} q_i))$. Hence the natural inclusion

$$\pi^* H^0(M(-\textstyle\sum_{i=1}^{n_2} q_i))) \cdot b \subseteq H^0(L(-D_{n_2}))$$

is an equality. For $k > n_2$ we use induction and the previous corollary.

(c) By part (b), $H^0(L(-D_k)) = \pi^* H^0(M(-\sum_{i=1}^k q_i)) \cdot b$. Thus the vanishing locus of a section of $H^0(L(-D_k))$ is invariant away from the support of B. It follows that

$$\begin{aligned} H^0(L(-D_k - p_1 - \tau(p_1))) &= H^0(L(-D_k - p_1)) \\ &= H^0(L(-D_k - \tau(p_1))) = \pi^* H^0(M(-\textstyle\sum_{i=1}^k q_i - q_1)) \cdot b, \end{aligned}$$

since the points q_i are general. We also have that

$$h^0(M(-\sum_{i=1}^k q_i - q_1)) = \max(h^0(M(-\sum_{i=1}^k q_i)) - 1, 0),$$

since if q_1 is a base point of $H^0(M(-\sum_{i=1}^k q_i))$, then it is a ramification point of the map associated to $|M|$. (The ramification locus is finite, and the q_i are general, so we can assume q_1 is not a base point of $H^0(M(-\sum_{i=1}^k q_i))$.) One can then proceed by induction on k_1.

(d) An induction argument similar to that in part (a) will prove this. □

LEMMA 2.1.4. *Suppose $h^0(L) = 2n > 0$, and that $h^0(L(-D_n)) > 0$ for every choice of distinct points $p_1, \tau(p_1), \dots, p_n, \tau(p_n)$ of $\widetilde{C}$. Then $L = \pi^*M \otimes \mathcal{O}_{\widetilde{C}}(B)$, where M is a line bundle on C such that $h^0(C, M) > h^0(\widetilde{C}, L)/2$, $B \geq 0$ is an effective divisor on $\widetilde{C}$ such that $B \cap \tau^*B = \varnothing$, and $h^0(\widetilde{C}, B) = 1$.*

Proof. For every $s \in H^0(L)$, the divisor $(s)_0$ can be decomposed into an invariant part, say $N = \pi^*N'$, and the residual part, say B, which by definition must have the property $B \cap \tau^*B = \varnothing$. Hence, setting $M = \mathcal{O}_C(N')$ we can always write $L = \pi^*M \otimes \mathcal{O}_{\widetilde{C}}(B)$, with $h^0(L) \geq h^0(M) > 0$, $B \cap \tau^*B = \varnothing$, and $B \geq 0$.

We first prove the lemma in the case that there do not exist 2 points p and $\tau(p)$ such that $h^0(L(-p-\tau(p))) = h^0(L) - 2$. Then Corollary 2.1.2 implies that $h^0(M) = h^0(L)$. Since B is effective, $h^0(B) \geq 1$, and the inequality

$$\dim|M| = \dim|L| \geq \dim|\pi^*M| + \dim|B| \geq \dim|M| + \dim|B|$$

implies $h^0(B) = 1$. Hence $L = \pi^*M \otimes \mathcal{O}_{\widetilde{C}}(B)$, $h^0(L) = h^0(M)$, $B \cap \tau^*B = \varnothing$, and $h^0(B) = 1$.

We will now prove the lemma by induction on $h^0(L)$. The case $h^0(L) = 2$ is a consequence of the case above. So suppose we have proved the result for all line bundles L' for which $h^0(L') \leq 2n-2$, and then consider a line bundle L with $h^0(L) = 2n > 2$. By the case above, we may assume there are points p and $\tau(p)$ that impose independent conditions on $H^0(L)$. Let $L' = L(-p-\tau(p))$, so that $h^0(L') = 2n-2$. There do not exist $2n-2$ distinct points $p_1, \tau(p_1), \dots, p_{n-1}, \tau(p_{n-1})$ on $\widetilde{C}$ imposing independent conditions on $H^0(L')$: otherwise, after possibly replacing $p_1, \tau(p_1), \dots, p_{n-1}, \tau(p_{n-1})$ with a more general choice of points, $p, \tau(p), p_1, \tau(p_1), \dots, p_{n-1}, \tau(p_{n-1})$ would be distinct points imposing independent conditions on $H^0(L)$, contradicting our assumptions.

Thus, by induction, $L' = \pi^*M' \otimes \mathcal{O}_{\widetilde{C}}(B)$, with

$$h^0(M') > n-1, \quad B \cap \tau^*B = \varnothing, \quad B \geq 0, \quad h^0(B) = 1.$$

Setting $q = \pi(p)$, it follows that $L = \pi^*(M'(q)) \otimes \mathcal{O}_{\widetilde{C}}(B)$. If we let $M = M'(q)$, then $h^0(M) \geq h^0(M') \geq n$. In the case $h^0(M) = n$, we would arrive at a contradiction, since Corollary 2.1.3(c) with $k = n_1 = n$ would imply that there were $2n$ distinct points $p_1, \tau(p_1), \ldots, p_n, \tau(p_n)$ imposing independent conditions on $H^0(L)$. Thus, $h^0(M) > n$. $\square$

LEMMA 2.1.5 (Smith and Varley [SV04]). *Suppose $L = \pi^*M \otimes \mathcal{O}_{\widetilde{C}}(B)$, where M is a line bundle on C such that $h^0(C, M) > h^0(\widetilde{C}, L)/2 > 0$, and $B \geq 0$ is an effective divisor on $\widetilde{C}$ such that $B \cap \tau^*B = \varnothing$. Then M and $\mathcal{O}_{\widetilde{C}}(B)$ are unique up to isomorphism.*

Proof. Suppose M' and B' satisfy the respective properties of M and B. Then $\pi^*H^0(M)\cdot b$ and $\pi^*H^0(M')\cdot b'$ are both linear subspaces of $H^0(L)$, and $\dim(\pi^*H^0(M)\cdot b)+\dim(\pi^*H^0(M')\cdot b') \geq h^0(L)+2$. Hence they have nontrivial intersection.

Let $s \in \pi^*H^0(M)$ and $s' \in \pi^*H^0(M')$ be sections such that $s\cdot b = s'\cdot b'$. Then $(s)_0 + (b)_0 = (s')_0 + (b')_0$. The invariant parts of these divisors must agree, and so we see $(s)_0 = (s')_0$ and $(b)_0 = (b')_0$. Thus $M \cong \mathcal{O}_C((s)_0) \cong \mathcal{O}_C((s')_0) \cong M'$, and $\mathcal{O}_{\widetilde{C}}(B) \cong \mathcal{O}_{\widetilde{C}}(B')$.

Remark 2.1.6. Suppose L can be written in the form $\pi^*M \otimes \mathcal{O}_{\widetilde{C}}(B)$, where M is a line bundle on C such that $h^0(C, M) > h^0(\widetilde{C}, L)/2 > 0$, and $B \geq 0$ is an effective divisor on $\widetilde{C}$ such that $B \cap \tau^*B = \varnothing$. Then the proof of Corollary 2.1.3 actually shows that if k is the maximum number such that there exist points $p_1, \tau(p_1), \ldots, p_k, \tau(p_k)$ imposing $2k$ conditions on $H^0(L)$, then $h^0(M) = h^0(L) - k$.

2.2. *Subspaces of complementary dimension.* In computations made in subsequent sections we will have to examine the intersection of linear subspaces of a given vector space. Specifically, we will be given a vector space $V = \mathbb{C}^d$, two fixed subspaces V_1 and V_2 such that $\dim(V_1) + \dim(V_2) = d$, and a family F of linear subspaces of V parametrized by a second copy of $\mathbb{C}^d$ and defined as follows: for $a = (a_1, \ldots, a_d) \in \mathbb{C}^d$

$$F_a = \{v \in V \mid \text{there exists a } v^1 \in V_1 \\ \text{such that } \pi_i(v) = a_i\pi_i(v^1) \text{ for } i = 1, \ldots, d\},$$

where π_i is projection onto the i-th factor. Our goal will be to determine whether or not there exists an $a \in \mathbb{C}^d$ such that $F_a \cap V_2 = 0$.

To begin, let $\dim(V_1) = d_1$ and $\dim(V_2) = d_2$. Define a coordinate m-plane to be the linear subspace of V defined by the vanishing of $d - m$ of the π_i. We will now prove the following proposition.

PROPOSITION 2.2.1. *If V_2 intersects each coordinate d_1-plane trivially, then there is a Zariski open subset $U \subseteq \mathbb{C}^d$ such that $F_a \cap V_2 = 0$ for all $a \in U$.*

Proof. Let V_1 and V_2 have respective bases

$$\{(v^1_{11},\dots,v^1_{1d}),\dots,(v^1_{d_1 1},\dots,v^1_{d_1 d})\},\quad \{(v^2_{11},\dots,v^2_{1d}),\dots,(v^2_{d_2 1},\dots,v^2_{d_2 d})\}.$$

A basis for F_a is then given by $\{(a_1v^1_{11},\dots,a_dv^1_{1d}),\dots,(a_1v^1_{d_1 1},\dots,a_dv^1_{d_1 d})\}$.

Let M be the matrix

$$M=\begin{pmatrix} a_1v^1_{11} & \dots & a_dv^1_{1d}\\ \vdots & & \vdots\\ a_1v^1_{d_1 1} & \dots & a_dv^1_{d_1 d}\\ v^2_{11} & \dots & v^2_{1d}\\ \vdots & & \vdots\\ v^2_{d_2 1} & \dots & v^2_{d_2 d}\end{pmatrix}.$$

It follows that $F_a\cap V_2=0$ if and only if $\det(M)\neq 0$. We now appeal to the following lemma, where we will use the notation $\mathbf{M}(m,n)$ for the space of $m\times n$ matrices over $\mathbb{C}$.

LEMMA 2.2.2. *Let $d'<d\in\mathbb{N}$. Let $A\in\mathbf{M}(d',d)$ and $B\in\mathbf{M}(d-d',d)$ have columns A_i and B_i, respectively, and let C be the matrix*

$$C=\begin{pmatrix} A_1 & \dots & A_d\\ B_1 & \dots & B_d\end{pmatrix}.$$

Then

$$\det(C)=\sum_{i_1<\dots<i_{d'}}(-1)^{\epsilon+i_1+\dots+i_{d'}}\det(A_{i_1}\cdots A_{i_{d'}})\det(B_{k_1}\cdots B_{k_{d-d'}}),$$

where $\{i_1,\dots,i_{d'}\}\cup\{k_1,\dots,k_{d-d'}\}=\{1,\dots,d\}$, $k_1<\dots<k_{d-d'}$, and ϵ is an integer satisfying $\epsilon+d'(d'+1)/2\equiv 0 \pmod 2$.

Proof. Let $D:\mathbf{M}(d,d)\to\mathbb{C}$ be given by the above formula. $D(I)=1$, D is alternating, and D is multilinear in the columns. Thus $D=\det$. □

If we let $V^i_j=(v^i_{1j},\dots,v^i_{d_i j})^T$, then as an immediate consequence of the lemma

$$\det(M)=\sum_{i_1<\dots<i_{d_1}}(-1)^{\epsilon+\Sigma i_j}a_{i_1}\dots a_{i_{d_1}}\det(V^1_{i_1}\cdots V^1_{i_{d_1}})\det(V^2_{k_1}\cdots V^2_{k_{d_2}}).$$

The monomials in the a_i that appear in the formula above are distinct. Also, since the dimension of V_1 is d_1, there must be some choice of $i_1<\dots<i_{d_1}$ such that $\det(V^1_{i_1}\cdots V^1_{i_{d_1}})\neq 0$. I claim that for any choice of $k_1<\dots<k_{d_2}$, $\det(V^2_{k_1}\dots V^2_{k_{d_2}})\neq 0$. It follows that the determinant of M is not identically zero as a polynomial in the a_i, and thus we can take $U=\{\det(M)\neq 0\}$.

We now proceed to prove the claim.

LEMMA 2.2.3. *Suppose* $\Lambda = \{\lambda_{ij}\} \in \mathbf{M}(d', d)$ *has rank* d', *and let* H *be the* d' *dimensional linear subspace of* $\mathbb{C}^d$ *spanned by the rows of* Λ. *Suppose further that one of the following equivalent conditions hold*:

(a) *the intersection of* H *with any coordinate* $(d-d')$*-plane is trivial*;

(b) *a linear combination of the rows of* Λ *with* d' *or more entries equal to zero is identically zero*;

(c) *if* $\sum_{i=1}^{d'} \alpha_i \lambda_{ij} = 0$ *for some* $\alpha_1, \dots, \alpha_{d'} \in \mathbb{C}$ *and for all* $j \in S \subseteq \{1, \dots, d\}$ *with* $|S| = d'$, *then* $\alpha_i = 0$ *for all* i.

Then every choice of d' *columns of* Λ *is linearly independent.*

Proof. If there exist d' columns of Λ that are dependent, then there would be a dependence among the rows of those columns. This would imply that some nontrivial linear combination of the rows of Λ had at least d' entries that were zero, which contradicts our assumption on Λ. □

Clearly applying this lemma to the space V_2 finishes the proof of the claim and hence of Proposition 2.2.1. □

Due to Lemma 2.2.3, there is a useful restatement of the proposition.

COROLLARY 2.2.4. *In the notation above*, *suppose* V_2 *satisfies the condition that if* $v_2 \in V_2$ *and* $\pi_i(v_2) = 0$ *for* $i \in I \subseteq \{1, \dots, d\}$ *with* $|I| = d_2$, *then* $v_2 = 0$. *Then there exists a Zariski open set* $U \in \mathbb{C}^d$ *such that* $F_a \cap V_2 = 0$ *for all* $a \in U$.

3. **Proof of Theorem** 1

3.1. *The preliminary lemma.*

LEMMA 3.1.1 [CMF05, Lemma 2.1]. *Suppose that* x *is a singular point of* Ξ, *corresponding to a line bundle* $L \in \mathrm{Pic}^{2g-2}(\widetilde{C})$, *and there exist* $2n$ *points* $p_1, \tau(p_1), \dots, p_n, \tau(p_n)$ *of* $\widetilde{C}$ *imposing independent conditions on* $H^0(L)$; *i.e. if* $D = \sum_{i=1}^{n}(p_i + \tau(p_i))$, *then* $h^0(\widetilde{C}, L(-D)) = 0$. *Then* $\mathrm{mult}_x\, \Xi = h^0(\widetilde{C}, L)/2$.

Proof. With D as above, let $\mathscr{L}_{D;(1,\dots,1)}$ be the deformation of L defined in *Definition* 1.4.4. By Riemann-Roch, $h^0(L(D)) = h^0(\omega_{\widetilde{C}} \otimes L^{-1}(-D)) + 2n$. The fact that $L \in P$ implies that $\omega_{\widetilde{C}} \otimes L^{-1} \cong \tau^* L$, and since D is τ-invariant, there is an isomorphism $H^0(L(-D)) \cong H^0(\tau^* L(-D))$ given by $s \mapsto \tau^* s$. Hence, $h^0(L(D)) = 2n$.

According to Lemma 1.4.6, there is a long exact sequence

$$0 \to H^0(L) \longrightarrow H^0(L(D)) \xrightarrow{E} H^0(L(D) \otimes \mathbb{O}_D) \xrightarrow{\partial_{L;D}} H^1(L) \cdots$$

and a map $A_1 : H^0(L) \to H^0(L(D) \otimes \mathbb{O}_D)$ such that $W_1 = \ker(\partial_{L;D} \circ A_1)$. Recall that $A_1(s) = (-s(p_1), s(\tau(p_1)), \dots, -s(p_n), s(\tau(p_n)))$. As observed in Corollary 1.4.7, it follows from this formula that $\ker(A_1) = H^0(L(-D)) = 0$.

On the other hand, since $h^0(L) = h^0(L(D))$, it follows that $\ker(\partial_{L;D}) = 0$. Hence $W_1 = \ker(\partial_{L;D} \circ A_1) = 0$, and so $d_k = 0$ for all $k \geq 1$. By Lemma 1.4.2 and Proposition 1.4.3(a),

$$h^0(L)/2 \leq \operatorname{mult}_x \Xi \leq \ell((R^1\pi_*\mathcal{L})_{s_0})/2 = d_0/2 = h^0(L)/2.$$

Thus $\operatorname{mult}_x \Xi = h^0(L)/2$. □

3.2. *Proof of Theorem* 1. We first prove (a) if and only if (b). By the Riemann singularity theorem, $\operatorname{mult}_x \widetilde{\Theta} = h^0(L)$. Since $\widetilde{\Theta} \cap P = 2 \cdot \Xi$, it is clear that $\operatorname{mult}_x \Xi = (\operatorname{mult}_x \widetilde{\Theta})/2$ if and only if $T_x P \not\subseteq C_x \widetilde{\Theta}$.

We next prove that (b) implies (c). Suppose $\operatorname{mult}_x \Xi > h^0(L)/2$. Then by Lemma 3.1.1, every choice of $2n$ points $p_1, \tau(p_1), \ldots, p_n, \tau(p_n)$ of $\widetilde{C}$ do not impose independent conditions on $H^0(L)$. That is, if $D = \sum_{i=1}^n (p_i + \tau(p_i))$, then $h^0(\widetilde{C}, L(-D)) > 0$. Then (c) follows from Lemma 2.1.4.

Next we show (c) implies (a). So suppose (c) holds. Then by Proposition 1.4.3(c), we have $\operatorname{mult}_x \Xi \geq h^0(M) > h^0(L)/2$.

To prove the last statement suppose again that (c) holds. Then M and $\mathcal{O}_{\widetilde{C}}(B)$ are unique up to isomorphism by Lemma 2.1.5, and $h^0(\mathcal{O}_{\widetilde{C}}(B)) = 1$ by Lemma 2.1.4. □

4. **Proof of Theorem** 2

The basic aim of the proof is to find a deformation of L lying in the Prym variety for which $d_1 = 2h^0(C, M) - h^0(\widetilde{C}, L)$ and $d_2 = 0$. One then concludes using Proposition 1.4.3(c). The computations needed to prove the theorem are quite lengthy, and consequently the proof will be broken down into five parts as follows.

In Section 4.1 we will fix the class of deformations to be used in the proof, and establish some preliminary results on linear systems associated to L. In Section 4.2 we will give a description of the space of sections of L lifting to first order — a necessary computation for the subsequent sections. In Section 4.3 we will consider sections lifting to second order, and we will show that a section lifting to second order must vanish along a chosen divisor D. In Section 4.4, we will show that any section lifting to second order that vanishes along D must be the zero section, and after this we will complete the proof of Theorem 2.

4.1. *Preliminaries.* We will use the following notation. Let

$$h^0(\widetilde{C}, L) = 2n, \qquad h^0(C, M) = n_1, \qquad h^0(\widetilde{C}, L) - h^0(C, M) = n_2.$$

With regard to Proposition 1.4.3, we note that $n_1 - n_2 = 2h^0(M) - h^0(L)$. Let $b \in H^0(\mathcal{O}_{\widetilde{C}}(B))$ be a section such that $(b)_0 = B$. Let $D' = \sum_{i=1}^n q_i$, where the q_i are general points of C, and let $\pi^* D' = D = \sum_{i=1}^n (p_i + \tau(p_i))$, where $\pi^{-1}(q_i) = \{p_i, \tau(p_i)\}$. With this notation fixed, let $\mathcal{L}$ be a family of deformations of L parametrized by $\mathbb{C}^n$, whose fiber $\mathscr{L}_a$ over a point $a \in \mathbb{C}^n$ is the deformation $\mathscr{L}_{D;a}$.

Recall that we are setting $\widetilde{C}_k = \widetilde{C} \times \operatorname{Spec} \mathbb{C}[t]/(t^{k+1})$ and denoting by $\mathscr{L}_k$ the restriction of $\mathscr{L}$ to $\widetilde{C}_k$. We will denote by $W_i(a)$ the image of the map $H^0(\mathscr{L}_{a;i}) \to H^0(L)$ induced from the exact sequence

$$0 \to \mathscr{L}_{a;i-1} \longrightarrow \mathscr{L}_{a;i} \longrightarrow L \to 0,$$

and by $d_i(a)$ the dimension of this space. In other words, for a given a, $W_i(a)$ is the space of sections lifting to order i.

Using Lemma 2.1.1 and its corollaries we can compute the dimensions of some pertinent linear systems. We note that for a line bundle $L \in \operatorname{Pic}^{2g-2}(\widetilde{C})$ corresponding to a point $x \in P$, and for a divisor E on $\widetilde{C}$, the Riemann-Roch theorem takes the form

$$h^0(L(E)) - h^0(L(-\tau^* E)) = \deg(E),$$

since $\omega_{\widetilde{C}} \otimes L^{-1} \cong \tau^* L$, and the map $\tau^* : H^0(\tau^* L(-E)) \to H^0(L(-\tau^* E))$ is an isomorphism.

LEMMA 4.1.1. *In the notation above*

(a) $h^0(L) = 2n$;

(b) $h^0(L(-D)) = n_1 - n$, *and* $H^0(L(-D)) = \pi^* H^0(M(-D')) \cdot b$;

(c) $h^0(L(D)) = n_1 + n$;

(d) $h^0(M(-2D')) = h^0(L(-2D)) = h^0(L(-D - \sum_{i=1}^n p_i)) = 0$;

(e) $h^0(L(2D)) = 4n$;

(f) $h^0(L(2D - \sum_{i=1}^n p_i)) = 3n$;

(g) $h^0(L(D + \sum_{i=1}^{n_1-n} \tau(p_i))) = h^0(L(D))$, *so that the natural inclusion induces an isomorphism* $H^0(L(D)) \cong H^0(L(D + \sum_{i=1}^{n_1-n} \tau(p_i)))$;

(h) $h^0(L(-\sum_{i=1}^n \tau(p_i))) = n$;

(i) $h^0(L(\sum_{i=1}^n p_i)) = h^0(L)$, *so that the natural inclusion induces an isomorphism* $H^0(L) \cong H^0(L(\sum_{i=1}^n p_i))$;

(j) $h^0(L(-\sum_{i=1}^n \tau(p_i) - \sum_{i=1}^{n_2} p_i) = h^0(L(-D))$, *so that the natural inclusion induces an isomorphism* $H^0(L(-D)) \cong H^0(L(-\sum_{i=1}^n \tau(p_i) - \sum_{i=1}^{n_2} p_i))$.

Proof. Part (b) follows from Corollary 2.1.3(b) with $k = n$. For part (c), observe by Riemann-Roch that $h^0(L(D)) - h^0(L(-D)) = 2n$. Therefore $h^0(L(D)) = n_1 - n + 2n = n_1 + n$. Part (d) follows from Corollary 2.1.3(c) with $k = k_1 = n$. Part (e) follows from (d) by Riemann-Roch. For part (f), use Corollary 2.1.3(c) with $k = k_1 = n$ to get $H^0(L(-D - \sum_{i=1}^n p_i)) = 0$. Part (f) then follows by

Riemann-Roch. For (g), Corollary 2.1.3(c) with $k = n$ and $k_1 = n_1 - n$ gives $H^0(L(-D - \sum_{i=1}^{n_1-n} p_i)) = 0$. Then by Riemann-Roch,

$$h^0(L(D + \textstyle\sum_{i=1}^{n_1-n} \tau(p_i))) = 2n + (n_1 - n) = n + n_1 = h^0(L(D)).$$

For (h), observe that points $\tau(p_1), \dots, \tau(p_n)$ are general. Then (i) follows from (h) by Riemann-Roch. For (j), apply Corollary 2.1.3(b) with $k = n_2$ to get $H^0(L(-\sum_{i=1}^{n_2}(p_i + \tau(p_i)))) = \pi^* H^0(M(-\sum_{i=1}^{n_2} q_i)) \cdot b$, which has dimension $2n - 2n_2$. The same argument as in Corollary 2.1.3(c) will show that the remaining points impose $n - n_2$ conditions. Hence

$$\begin{aligned} h^0(L(-\textstyle\sum_{i=1}^{n} \tau(p_i) - \sum_{i=1}^{n_2} p_i)) &= 2n - 2n_2 - (n - n_2) \\ &= n - n_2 = n_1 - n = h^0(L(-D)). \qquad \square \end{aligned}$$

4.2. *Sections lifting to first order.* We are now ready to study the sections of L that lift to first order. We would like to find some Zariski open subset $\Omega_1 \subseteq \mathbb{C}^n$ such that $d_1(a) = n_1 - n_2$ for all $a \in \Omega_1$. In addition, in order to make the second order computations easier, we will want to understand the relationship between $W_1(a)$ and $H^0(L(-D))$.

PROPOSITION 4.2.1. *Let* $\Omega_1 = \bigcap_{i=1}^{n}\{a_i \neq 0\} \subseteq \mathbb{C}^n$. *Then for all* $a \in \Omega_1$

(a) $H^0(L(-D)) \subseteq W_1(a)$;

(b) $d_1(a) = n_1 - n_2$;

(c) $H^0(L(-\sum_{i=1}^{n} \tau(p_i))) \cap W_1(a) = H^0(L(-D))$.

Proof. By Lemma 1.4.6, there is a long exact sequence

$$0 \to H^0(L) \longrightarrow H^0(L(D)) \xrightarrow{E_1} H^0(L(D) \otimes \mathcal{O}_D) \xrightarrow{\partial_{L;D}} H^1(L) \cdots$$

and a map $A_1 : H^0(L) \to H^0(L(D) \otimes \mathcal{O}_D)$ such that $W_1(a) = \ker(\partial_{L;D} \circ A_1)$. Recall that

$$A_1(s) = (-a_1 s(p_1), a_1 s(\tau(p_1)), \dots, -a_n s(p_n), a_n s(\tau(p_n))).$$

Thus $\ker(A_1) = H^0(L(-D))$ on Ω_1, proving (a). To prove (b), consider that

$$\begin{aligned} n_1 - n_2 &\leq \dim(W_1(a)) \\ &= \dim(\mathrm{im}(A_1) \cap \ker(\partial_{L,D})) + \dim(\ker(A_1)) \\ &\leq \dim(\ker(\partial_{L,D})) + \dim(\ker(A_1)) \\ &= (h^0(L(D)) - h^0(L)) + h^0(L(-D)) = n_1 - n_2. \end{aligned} \tag{4.2.2}$$

Finally, to prove (c), consider a section $s \in H^0(L(-\sum_{i=1}^{n} \tau(p_i))) \cap W_1(a)$. That $s \in W_1(a)$ is equivalent to $A_1(s) \in \mathrm{im}(E_1)$, and it is clear that if s lies in

$H^0(L(-\sum_{i=1}^n \tau(p_i)))$, then $A_1(s)$ lies in $E_1(H^0(L(D-\sum_{i=1}^n \tau(p_i))))$. But

$$H^0(L(D-\textstyle\sum_{i=1}^n \tau(p_i))) = H^0(L(\sum_{i=1}^n p_i)),$$

and we have seen in Lemma 4.1.1(i) that $H^0(L(\sum_{i=1}^n p_i)) = H^0(L)$. Therefore $A_1(s) \in E_1(H^0(L)) = 0$, and it follows that $s \in H^0(L(-D))$. $\square$

Remark 4.2.3. Because the inequalities in (4.2.2) are actually all equalities, $\ker(\partial_{L,D}) \subseteq \operatorname{im}(A_1)$. In fact, $\ker(\partial_{L,D}) = A_1(W_1(a))$, so

$$\ker(\partial_{L,D}) = \{(-a_1 s(p_1), a_1 s(\tau(p_1)), \dots, -a_n s(p_n), a_n s(\tau(p_n))) \mid s \in W_1(a)\}.$$

This will be important in later computations where we will exploit the fact that $\ker(\partial_{L,D})$ does not depend on the a_i, whereas $W_1(a)$ does.

4.3. *Second order lifts*: *a necessary condition.* We are now in a position to consider W_2, the space of sections lifting to second order. Our eventual goal will be to show $W_2 = 0$. We begin with the following proposition.

PROPOSITION 4.3.1. *There is a nonempty Zariski open subset $\Omega_2 \subseteq \Omega_1$ such that $W_2(a) \subseteq H^0(L(-D))$ for all $a \in \Omega_2$. Therefore*

$$W_2(a) \subseteq H^0(L(-D)) \subseteq W_1(a).$$

Proof. By Lemma 1.4.8, there is a long exact sequence

$$0 \to H^0(L) \longrightarrow H^0(L(2D)) \xrightarrow{E_2} H^0(L(2D) \otimes \mathcal{O}_{2D}) \xrightarrow{\partial_{L;2D}} H^1(L) \cdots$$

and a map $A_2 : H^0(\mathcal{L}_1) \to H^0(L(2D) \otimes \mathcal{O}_{2D})$ such that

$$\operatorname{im}(H^0(\mathcal{L}_2) \to H^0(\mathcal{L}_1)) = \ker(\partial_{L;2D} \circ A_2).$$

In other words, a section $s \in H^0(L)$ lifts to second order if and only if there exists some first order lift $s + \sigma^{(1)} t \in H^0(\mathcal{L}_1)$ such that $A_2(s + \sigma^{(1)} t) \in \operatorname{im}(E_2)$. Recall that

$$A_2(s+\sigma^{(1)}t) = \Big(0, -a_1\sigma^{(1)}(p_1),\ a_1^2 s(\tau(p_1)),\ a_1\sigma^{(1)}(\tau(p_1)) + a_1^2 \frac{ds}{dz}(\tau(p_1)), \dots\Big)$$

and, for $\varphi \in H^0(L(2D))$,

$$E_2(\varphi) = \Big(\varphi(p_1), \frac{d\varphi}{dz}(p_1), \varphi(\tau(p_1)), \frac{d\varphi}{dz}(\tau(p_1)), \dots\Big),$$

so that if a section s lifts to second order, then there must be some section φ in $H^0(L(2D))$ such that $\varphi(p_i) = 0$ and $\varphi(\tau(p_i)) = a_i^2 s(\tau(p_i))$ for all i.

Now let us examine this condition. Let F be the family of linear subspaces of $\mathbb{C}^n$ defined by $F_a = \{(a_1^2 s(\tau(p_1)), \dots, a_n^2 s(\tau(p_n))) \in \mathbb{C}^n \mid s \in W_1\}$, and let $V_2 = \{(\varphi(\tau(p_1)), \dots, \varphi(\tau(p_n))) \in \mathbb{C}^n \mid \varphi \in H^0(L(2D - \sum_{i=i}^n p_i))\}$. If a section s lifts to second order, then $(a_1^2 s(\tau(p_1)), \dots, a_n^2 s(\tau(p_n))) \in F_a \cap V_2$.

I claim there is a nonempty Zariski open set $\Omega_2 \subseteq \Omega_1$ such that $F_a \cap V_2 = 0$ for all $a \in \Omega_2$. It follows that if a section lifts to second order, it must vanish at $\tau(p_i)$ for all i. Since $H^0(L(-\sum_{i=1}^n \tau(p_i))) \cap W_1(a) = H^0(L(-D))$, this means that a section lifting to second order must be in $H^0(L(-D))$, which completes the proof of the proposition. □

Now we must address the unproved claim.

LEMMA 4.3.2. *There is a nonempty Zariski open set $\Omega_2 \subseteq \Omega_1$ such that if $a \in \Omega_2$, then $F_a \cap V_2 = 0$.*

Proof. This will be an application of Proposition 2.2.1. We begin by introducing some notation: with respect to the basis we have been using for $H^0(L(D) \otimes \mathbb{O}_D)$, let $\mathrm{pr}_2 : H^0(L(D) \otimes \mathbb{O}_D) \to \mathbb{C}^n$ be the projection onto the even factors. Let $V_1 = \mathrm{pr}_2(\ker(\partial_{L,D}))$.

CLAIM 4.3.3. *In the above notation*, $\dim(F_a) = n - n_2$ *for* $a \in \Omega_1$. *Also*, $F_a = \{(a_1 v_1, \dots, a_n v_n) \in \mathbb{C}^n \mid (v_1, \dots, v_n) \in V_1\}$.

Proof. Since $H^0(L(-\sum_{i=1}^n \tau(p_i))) \cap W_1(a) = H^0(L(-D))$, it follows that $F_a \cong W_1(a)/H^0(L(-D))$. Hence $\dim(F_a) = (n_1 - n_2) - (n_1 - n) = n - n_2$. The second statement is a direct consequence of Remark 4.2.3, which implies that $V_1 = \mathrm{pr}_2(\ker(\partial_{L,D})) = \{(a_1 s(\tau(p_1)), \dots, a_n s(\tau(p_n))) \mid s \in W_1(a)\}$. □

CLAIM 4.3.4. *In the above notation*, $\dim(V_2) = n_2$. *Furthermore*, *if* $v = (v_1, \dots, v_n) \in V_2$ *and* $v_i = 0$ *for* $i \in I \subseteq \{1, \dots, n\}$ *with* $|I| = n_2$, *then* $v = 0$.

Proof. First, $V_2 \cong H^0(L(2D - \sum_{i=1}^n p_i))/H^0(L(D))$. By Lemma 4.1.1(c) and (f),

$$h^0(L(2D - \textstyle\sum_{i=1}^n p_i)) - h^0(L(D)) = 3n - (n + n_1) = 2n - n_1 = n_2,$$

and hence $\dim(V_2) = n_2$. By Lemma 4.1.1(g),

$$H^0(L(2D - \textstyle\sum_{i=1}^n p_i - \sum_{i \in I} \tau(p_i))) = H^0(L(D + \sum_{i \in I^c} \tau(p_i))) = H^0(L(D)),$$

since $|I^c| = n - n_2 = n_1 - n$. Hence if $v = (\varphi(\tau(p_1)), \dots, \varphi(\tau(p_n))) \in V_2$ and $\varphi(\tau(p_i)) = 0$ for $i \in I$, then $v = 0$. □

The proof of the lemma is now just an application of Proposition 2.2.1. To see this, set $V = \mathbb{C}^n$. Then $\dim V_1 + \dim V_2 = \dim V$, and $F_a = \{(a_1 v_1, \dots, a_n v_n) \in \mathbb{C}^n \mid (v_1, \dots, v_n) \in V_1\}$, so that V, V_1, V_2, and F are as in Section 2.2. Furthermore, Claim 4.3.4 shows that V_2 satisfies the conditions of Corollary 2.2.4. Hence there exists a nonempty Zariski open subset $\Omega_2 \subseteq \Omega_1$ such that if $a \in \Omega_2$, then $F_a \cap V_2 = 0$. □

4.4. *Second order lifts*: *a sufficient condition.* Now that we have shown that $W_2(a) \subseteq H^0(L(-D)) \subseteq W_1(a)$, we can focus our attention on sections in $H^0(L(-D))$. This is a great advantage, as we know the exact form of the first order lifts of such sections. In order to take full advantage of this information, we prove the following lemma, which addresses an important special case. First, let us define the following notation. Let $\{e_1^*, \dots, e_{4n}^*\}$ be the basis dual to the basis we have been using for $H^0(L(2D) \otimes \mathbb{O}_{2D})$. Let $H = \{v \in H^0(L(2D) \otimes \mathbb{O}_{2D}) \mid e_{i+4j}^*(v) = 0 \text{ for } i = 1, 3, 4 \text{ and } 0 \le j \le n-1\}$; i.e., $H = \{(0, *, 0, 0, \dots, 0, *, 0, 0) \in H^0(L(2D) \otimes \mathbb{O}_{2D})\}$.

LEMMA 4.4.1. *Suppose that $s + \sigma^{(1)}t \in \operatorname{im}(H^0(\mathscr{L}_2) \to H^0(\mathscr{L}_1))$ and that $A_2(s + \sigma^{(1)}t) \in H$. Then $A_2(s + \sigma^{(1)}t) = 0$.*

Proof. If $s + \sigma^{(1)}t$ lifts to second order, then $A_2(s + \sigma^{(1)}t) = E_2(\varphi)$ for some $\varphi \in H^0(L(2D))$. Due to the form of H, we can see that

$$\varphi \in H^0(L(2D - D - \textstyle\sum_{i=1}^n \tau(p_i))) = H^0(L(\textstyle\sum_{i=1}^n p_i)).$$

By Lemma 4.1.1(i), $H^0(L(\sum_{i=1}^n p_i)) = H^0(L)$, so that $\varphi \in H^0(L)$ and thus $E_2(\varphi) = 0$. Therefore $A_2(s + \sigma^{(1)}t) = 0$. $\square$

With this we will prove the next proposition.

PROPOSITION 4.4.2. *There is a Zariski open subset $\Omega_3 \subseteq \Omega_2$ such that $W_2(a) = 0$ for all $a \in \Omega_3$.*

Proof. Let $s \in W_2(a)$. By Proposition 4.3.1, $s \in H^0(L(-D))$. Let $s + \sigma^{(1)}t$ be the standard lifting given in Lemma 1.2.5, and recall that on an open set $U_i \subseteq \widetilde{C}$ in our cover,

$$\sigma_i^{(1)} = \begin{cases} a_i s/z & \text{if } p_i \in U_i, \\ -a_i s/z & \text{if } \tau(p_i) \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

A general lifting of s will be given by $s + (\sigma^{(1)} + \varphi)t$ for some $\varphi \in H^0(L)$. Observe that

$$A_2(s + (\sigma^{(1)} + \varphi)t) = (0, -a_1^2(s/z)(p_1) - a_1\varphi(p_1), 0, a_1\varphi(\tau(p_1)), \dots), \tag{4.4.3}$$

since $(s/z)(\tau(p_i)) = (ds/dz)(\tau(p_i))$ when $s(\tau(p_i)) = 0$.

I claim there is a $\psi \in W_1$ such that $\psi(\tau(p_i)) = \varphi(\tau(p_i))$ for all i. Indeed if $A_2(s + (\sigma^{(1)} + \varphi)t) \in \operatorname{im}(E_2)$, then it must be in $E_2(H^0(L(D)))$, because $s \in H^0(L(-D))$. Consider the diagram

$$\begin{array}{ccc} H^0(L(2D)) & \xrightarrow{E_2} & H^0(L(2D)/L) \\ \bigcup & & \bigcup \\ H^0(L(D)) & \xrightarrow{E_1} & H^0(L(D)/L), \end{array}$$

where the inclusion on the right, in our chosen bases, is given by

$$(x_1, x_{\tau(1)}, \ldots, x_n, x_{\tau(n)}) \mapsto (0, x_1, 0, x_{\tau(1)}, \ldots, 0, x_n, 0, x_{\tau(n)}).$$

It follows that $A_2(s+\sigma^{(1)}t) \in H^0(L(D)/L) \subseteq H^0(L(2D)/L)$, so that

$$(-a_1^2(s/z)(p_1) - a_1\varphi(p_1), a_1\varphi(\tau(p_1)), \ldots) \in E_1(H^0(L(D))).$$

Recall from Remark 4.2.3 that $\mathrm{im}(E_1) = A_1(W_1)$, so that there must be a $\psi \in W_1$ such that

$$\begin{aligned} A_1(\psi) &= (-a_1\psi(p_1), a_1\psi(\tau(p_1)), \ldots) \\ &= (-a_1^2(s/z)(p_1) - a_1\varphi(p_1), a_1\varphi(\tau(p_1)), \ldots). \end{aligned}$$

Hence $\psi(\tau(p_i)) = \varphi(\tau(p_i))$ for all i, establishing the claim.

Since ψ lifts to first order, ψt lifts to second order, so $s + (\sigma^{(1)} + \varphi)t - \psi t$ also lifts to second order. But then $A_2(s + (\sigma^{(1)} + \varphi)t - \psi t) \in H$, so that the above lemma implies $s + (\sigma^{(1)} + \varphi)t - \psi t \in \ker(A_2)$. Setting $\rho = \psi - \varphi$, we have $A_2(s + (\sigma^{(1)} + \rho)t) = 0$. In other words, using (4.4.3), if s lifts to second order, then there is a section $\rho \in H^0(L)$ such that $\rho(\tau(p_i)) = 0$ and $\rho(p_i) = a_i(s/z)(p_i)$ for all i.

Now let us examine this condition. Let F be the family of linear subspaces of $\mathbb{C}^n$ defined by

$$F_a = \{(a_1(s/z)(p_1), \ldots, a_n(s/z)(p_n)) \in \mathbb{C}^n \mid s \in H^0(L(-D))\},$$

and let $V_2 = \{(\rho(p_1), \ldots, \rho(p_n)) \in \mathbb{C}^n \mid \rho \in H^0(L(-\sum_{i=1}^n \tau(p_i)))\}$. If a section s lifts to second order, then $(a_1(s/z)(p_1), \ldots, a_n(s/z)(p_n)) \in F_a \cap V_2$.

I claim there is a nonempty Zariski open subset $\Omega_3 \subseteq \Omega_2$ such that for all $a \in \Omega_3$, $F_a \cap V_2 = 0$. It follows that if a section lifts to second order, it must vanish to second order at p_i for all i, so that $s \in H^0(L(-D - \sum_{i=1}^n p_i))$. By Lemma 4.1.1(d), $H^0(L(-D - \sum_{i=1}^n p_i)) = 0$, and hence $s = 0$. □

Now we must address the unproved claim.

LEMMA 4.4.4. *There is a nonempty Zariski open subset $\Omega_3 \subseteq \Omega_2$ such that if $a \in \Omega_3$, then $F_a \cap V_2 = 0$.*

Proof. This will be an application of Proposition 2.2.1. To begin, let

$$V_1 = \{((s/z)(p_1), \ldots, (s/z)(p_n)) \in \mathbb{C}^n \mid s \in H^0(L(-D))\},$$

so that $V_1 = F_{(1,\ldots,1)}$. It is clear that

$$V_1 \cong H^0(L(-D))/H^0(L(-D - \textstyle\sum_{i=1}^n p_i)) = H^0(L(-D)),$$

since $H^0(L(-D - \sum_{i=1}^n p_i)) = 0$.

CLAIM 4.4.5. *In the above notation*, $\dim(F_a) = n_1 - n = n - n_2$ *for* $a \in \Omega_1$.

Proof. On Ω_1, $F_a \cong V_1 \cong H^0(L(-D))$. Thus, $\dim(F_a) = h^0(L(-D)) = n_1 - n$. ☐

CLAIM 4.4.6. *In the above notation*, $\dim(V_2) = n_2$. *Furthermore*, *if* $v = (v_1, \dots, v_n) \in V_2$ *and* $v_i = 0$ *for* $i \in I \subseteq \{1, \dots, n\}$ *with* $|I| = n_2$, *then* $v = 0$.

Proof. $V_2 \cong H^0(L(-\sum_{i=1}^n \tau(p_i))/H^0(L(-D))$. By Lemma 4.1.1(h) and (b), $h^0(L(-\sum_{i=1}^n \tau(p_i))) = n$ and $h^0(L(-D)) = n_1 - n$, so that $\dim(V_2) = n - (n_1 - n) = 2n - n_1 = n_2$. By Lemma 4.1.1(j),

$$H^0(L(-\textstyle\sum_{i=1}^n \tau(p_i) - \sum_{i\in I} p_i)) = H^0(L(-D)).$$

Hence if $v = (\varphi(p_1), \dots, \varphi(p_n)) \in V_2$ and $\varphi(p_i) = 0$ for $i \in I$, then $v = 0$. ☐

The proof of the lemma concludes just as the proof of Lemma 4.3.2. In its final paragraph on page 187, simply replace Claim 4.3.4 with Claim 4.4.6. ☐

Proof of Theorem 2. Let $a \in \Omega_3$, and consider the deformation $\mathscr{L}_{D;a}$. It follows from Proposition 4.2.1 that $d_1 = n_1 - n_2 = 2h^0(M) - h^0(L)$. Then Proposition 4.4.2 implies that $W_2 = 0$ and hence that $d_2 = 0$. Finally, by Proposition 1.4.3(c), $\operatorname{mult}_x \Xi = h^0(M)$. ☐

5. Consequences of Theorem 2

5.1. *Upper bounds on multiplicity of singularities.* For a ppav (A, Θ), let $\operatorname{Sing}_k \Theta = \{x \in \operatorname{Sing}\Theta \mid \operatorname{mult}_x \Theta \geq k\}$. A result of Kollár [Kol95] shows that if $\dim(A) = d$, then $\dim(\operatorname{Sing}_k \Theta) \leq d - k$. Generalizing a result of Smith and Varley [SV96], Ein and Lazarsfeld [EL97] showed that $\dim(\operatorname{Sing}_k \Theta) = d - k$ only if (A, Θ) splits as a k-fold product. Thus $\operatorname{mult}_x \Xi \leq g - 2 = \dim(P) - 1$ for an irreducible Prym variety associated to a connected étale double cover of a smooth curve C of genus g and $x \in \operatorname{Sing}\Xi$. Using Theorem 2, we will improve these estimates for Prym varieties. To begin, we will prove the following lemma on double covers of hyperelliptic curves:

LEMMA 5.1.1 [CMF05, Lemma 3.5]. *Let* $\pi : \widetilde{C} \to C$ *be a connected étale double cover of a smooth curve* C. *If* $\widetilde{C}$ *is hyperelliptic*, *then* C *is hyperelliptic. Furthermore*, *if* $\widetilde{G}$ *is the line bundle corresponding to the* g^1_2 *on* $\widetilde{C}$, *and* G *is the line bundle corresponding to the* g^1_2 *on* C, *then* $\operatorname{Norm}(\widetilde{G}) \cong G$ *and* $\pi^* G \cong \widetilde{G}^{\otimes 2}$.

Proof. Suppose $p_1 + p_2$ and $p_3 + p_4$ are general in $|\widetilde{G}|$. Since Norm preserves linear equivalence, $\pi(p_1) + \pi(p_2) \sim \pi(p_3) + \pi(p_4)$. Thus C is hyperelliptic, and $\operatorname{Norm}(\widetilde{G}) \cong G$ since there is a unique g^1_2 on C. Now let $f : \widetilde{C} \to \mathbb{P}^1$ be the morphism corresponding to the g^1_2 on $\widetilde{C}$. Then $f \circ \tau$ is also a finite degree 2 morphism of $\widetilde{C}$ to $\mathbb{P}^1$, and since there is a unique g^1_2 on $\widetilde{C}$, this implies that $\tau(p_1) + \tau(p_2) \sim p_1 + p_2$. Thus

$$\pi^* G \cong \pi^* \operatorname{Norm}(\widetilde{G}) \cong \mathbb{O}_{\widetilde{C}}(p_1 + p_2 + \tau(p_1) + \tau(p_2)) \cong \widetilde{G}^{\otimes 2}. \qquad \square$$

Recall the definition of the Clifford index:

$$\mathrm{Cliff}(C) = \min\{\deg(D) - 2\dim|D| : h^0(D) \geq 2,\ h^1(D) \geq 2\}.$$

If $x \in \mathrm{Sing}\,\Xi$, then

$$h^0(L) \leq \deg(L)/2 - \mathrm{Cliff}(\widetilde{C})/2 + 1,$$
$$h^0(M) \leq \deg(M)/2 - \mathrm{Cliff}(C)/2 + 1.$$

Letting $W^r_d(C)$ denote the variety of line bundles L on C such that $\deg(L) = d$ and $h^0(L) > r$, Martens's theorem [Mar67] states that if $2 \leq d \leq g-1$ and $0 < 2r \leq d$, then $\dim(W^r_d(C)) \leq d - 2r$, with equality holding only if C is hyperelliptic. Since $\mathrm{Cliff}(C) \leq d - 2r$, these inequalities yield essentially the same information.

COROLLARY 5.1.2. *If* $x \in \mathrm{Sing}\,\Xi$, *then* $\mathrm{mult}_x\,\Xi \leq (g+1)/2$. *If* P *is irreducible*, *then* $\mathrm{mult}_x\,\Xi \leq g/2 = (\dim(P)+1)/2$. *More precisely*, *suppose* $g \geq 5$, *and let* Z *be an irreducible component of* $\mathrm{Sing}_k\,\Xi$. *For* $x \in Z$, *let* L_x *be the corresponding line bundle.*

(a) *Suppose for a general* $x \in Z$, $\mathrm{mult}_x\,\Xi = h^0(L_x)/2$, *i.e.*, $T_xP \not\subseteq C_x\widetilde{\Theta}$. *Then* $k \leq g/2 - \mathrm{Cliff}(\widetilde{C})/4$, *and* $\dim(Z) \leq 2g - 4k$. *If* $\dim(Z) = 2g - 4k$, *then* C *is hyperelliptic.*

(b) *Suppose for a general* $x \in Z$, $\mathrm{mult}_x\,\Xi > h^0(L_x)/2$, *i.e.*, $T_xP \subseteq C_x\widetilde{\Theta}$. *Then* $k \leq (g+1)/2 - \mathrm{Cliff}(C)/2$, *and* $\dim(Z) \leq g - 2k + 1$. *If* $\dim(Z) = g - 2k + 1$, *then* C *is hyperelliptic. If we suppose moreover that* C *is not hyperelliptic*, *and* $\dim(Z) > 0$, *then* $\dim(Z) \leq g - 2k - 1$, *and if* $\dim(Z) = g - 2k - 1$, *then* C *is either trigonal*, *bielliptic*, *or a smooth plane quintic.*

Proof. The first statement of the corollary follows immediately from (a) and (b). The statement for irreducible Prym varieties then follows from Mumford's result [Mum74] that if C is hyperelliptic, then the Prym variety associated to the double cover is a hyperelliptic Jacobian or the product of two such Jacobians.

(a) We have that $\mathrm{mult}_x\,\Xi = h^0(L)/2$. Since $\deg(L) = 2g-2 = g(\widetilde{C}) - 1$, by Riemann-Roch, $h^1(L) = h^0(L) \geq 2$. Thus by Clifford's theorem

$$\mathrm{mult}_x\,\Xi \leq (2g-2)/4 - \mathrm{Cliff}(\widetilde{C})/4 + 1/2 = g/2 - \mathrm{Cliff}(\widetilde{C})/4.$$

In addition, we must have that $Z \subseteq W^{2k-1}_{2g-2}(\widetilde{C})$, and so it follows immediately from Martens's theorem that $\dim(Z) \leq 2g - 2 - 2(2k-1) = 2g - 4k$, with equality holding only if $\widetilde{C}$, and hence C, is hyperelliptic.

(b) Now suppose $T_xP \subseteq C_x\widetilde{\Theta}$, so that

$$L = \pi^*M \otimes \mathcal{O}_{\widetilde{C}}(B), \quad h^0(C, M) > h^0(\widetilde{C}, L)/2, \quad B \geq 0, \quad B \cap \tau^*B = \varnothing,$$

and $\mathrm{mult}_x\,\Xi = h^0(C, M)$. Since M is special, Clifford's theorem implies

$$h^0(C, M) \leq \deg(M)/2 - \mathrm{Cliff}(C)/2 + 1 \leq (g+1)/2 - \mathrm{Cliff}(C)/2.$$

Applying Martens's theorem, we see that

$$\dim(Z) \leq (g-1) - 2(k-1) = g - 2k + 1,$$

with equality holding only if C is hyperelliptic. If we suppose that $\dim(Z) > 0$, then we can assume that $\deg(M) < g-1$, since there are only a finite number of theta characteristics. If we assume further that C is not hyperelliptic, then it follows from Mumford's refinement [Mum74] of Martens's theorem that $\dim(Z) \leq g - 2k - 1$, with equality holding only if C is either trigonal, bielliptic, or a smooth plane quintic. $\square$

Remark 5.1.3. The statements regarding the dimension of $\mathrm{Sing}_k\,\Xi$ in the corollary above were pointed out by the referee. It should also be noted that in [Mum74], Mumford studied a skew symmetric bilinear pairing

$$\beta : H^0(L) \wedge H^0(L) \to H^0(\omega_C \otimes \eta)$$

and showed by a dimension count that if $g \geq 5$ and $\dim(\mathrm{Sing}\,\Xi) \geq g-5$, then β has a two dimensional isotropic subspace $V \subseteq H^0(L)$. Such an isotropic subspace gives rise to an isomorphism $L \cong \pi^*M \otimes \mathcal{O}_{\tilde{C}}(B)$, where $h^0(M) \geq 2$, $B \geq 0$, and $B \cap \tau^*B = \varnothing$. As in the proof of (b) above, Mumford concluded that if C is not hyperelliptic, then $\dim(\mathrm{Sing}\,\Xi) \leq g-5$. Generalizing Mumford's work, Smith and Varley [SV04] have shown that there exists an isotropic subspace of dimension k for β if and only if there exists such a decomposition of L with $h^0(M) \geq k$. Thus it seems possible that through further analysis of the pairing β, one may be able to improve the bound on $\dim(\mathrm{Sing}_k\,\Xi)$.

Remark 5.1.4. The referee has raised the question of whether Ein and Lazarsfeld's bound on the dimension of $\mathrm{Sing}_k\,\Theta$ for irreducible ppavs is sharp. That is, do there exist irreducible ppavs with $\dim(\mathrm{Sing}_k\,\Theta) = d-k-1$? As an example, it would be interesting to know if there exist irreducible ppavs of dimension five with a point of order four on their theta divisor. It appears that the techniques of this paper may extend to Prym varieties associated to double covers of stable curves, and hence in the case of an irreducible ppav of dimension less than or equal to five, it may be possible to answer this question and give a sharp bound on $\dim(\mathrm{Sing}_k\,\Theta)$. This is work in progress.

Remark 5.1.5. For the Jacobian of a curve, Martens's theorem implies that $\mathrm{codim}(\mathrm{Sing}_k\,\Theta) = 2k-1$ only if the curve is hyperelliptic. It is a result of Beauville [Bea77a] that if (A,Θ) is an irreducible generalized Prym variety and $\dim(\mathrm{Sing}\,\Xi) \geq g-4 = \dim(A)-3$, then (A,Θ) is a hyperelliptic Jacobian; see Mumford [Mum74]. Thus at least in dimension less than or equal to five, any irreducible ppav whose theta divisor has double points in codimension three is a hyperelliptic Jacobian. In regards to these results, and Corollary 5.1.2(b), the referee has asked to what extent k-fold points in codimension $2k-1$ characterize hyperelliptic Jacobians among all irreducible ppavs. Despite the examples cited, in general for $k>2$ it appears there may be other components in this locus, since at least in dimension five, the theta divisor of the intermediate Jacobian of a smooth cubic threefold has a triple point, but such a ppav is not a Jacobian; see Clemens and Griffiths [CG72].

Recall that given distinct points $p_1, \tau(p_1), \dots, p_k, \tau(p_k)$ of $\widetilde{C}$, we define $D_k = \sum_{i=1}^{k}(p_i + \tau(p_i))$. Due to Remark 2.1.6, we have the following upper bound on the multiplicity of a point on the Prym theta divisor:

COROLLARY 5.1.6. *Suppose* $x \in \operatorname{Sing}\Xi$ *corresponds to the line bundle* $L \in \operatorname{Pic}^{2g-2}(\widetilde{C})$. *If there exist* $2k$ *distinct points* $p_1, \tau(p_1), \dots, p_k, \tau(p_k)$ *of* $\widetilde{C}$ *such that* $h^0(L(-D_k)) = h^0(L) - 2k$, *then*

$$\operatorname{mult}_x \Xi \le h^0(L) - k,$$

with equality holding if and only if k is the largest number with this property.

5.2. *Prym varieties of dimension five.* A Prym variety of dimension five is associated to a double cover of a genus six curve. For a point $x \in \operatorname{Sing}\Xi$, Corollary 5.1.2 implies that $\operatorname{mult}_x \Xi \le 3$; in this section we will examine exactly which Prym varieties of dimension five have singular theta divisors with triple points. Theorem 3 is a direct consequence of the following theorem.

THEOREM 5.2.1. *Suppose* $\dim P = 5$. *If* $\operatorname{Sing}_3 \Xi \neq \varnothing$, *then one of the following must hold*:

(a) *C is a plane quintic and* $h^0(\mathcal{O}_{\mathbb{P}^2}(1)|_C \otimes \eta) = 1$. *In this case* $\operatorname{Sing}_3 \Xi = \operatorname{Sing}\Xi = \{x\}$ *consists of a unique point corresponding to the line bundle* $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)|_C)$. *Moreover,* $X = C_x\Xi$ *is a smooth cubic threefold, and* $(P, \Xi) \cong (JX, \Theta)$.

(b) *C is hyperelliptic, and either*
   - (a) *$\widetilde{C}$ is hyperelliptic and $(P, \Xi) \cong JC'$ for some hyperelliptic curve C', in which case* $\dim(\operatorname{Sing}\Xi) = 2$ *and* $\operatorname{Sing}_3 \Xi = \{x\}$ *consists of a unique point corresponding to the line bundle* $5g^1_2$ *on* $\widetilde{C}$; *or*
   - (b) *$\widetilde{C}$ is not hyperelliptic and $(P, \Xi) \cong JC' \times JC''$ for some hyperelliptic curves C' and C'', in which case* $\dim(\operatorname{Sing}\Xi) = 3$, *and* $\dim(\operatorname{Sing}_3 \Xi) = 1$.

Proof. (a) Suppose C is not hyperelliptic and L is a line bundle corresponding to a singular point x of multiplicity 3. Since $\deg(L) = 10$, by Clifford's theorem, $h^0(L) \le 6$, with equality holding only if $\widetilde{C}$, and hence C, is hyperelliptic. By Theorems 1 and 2, if $h^0(L) = 2$, then $\operatorname{mult}_x \Xi \le 2$. Thus we may assume $h^0(L) = 4$, and by Theorem 1, $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$, with $h^0(M) \ge 3$, $B \ge 0$ and $B \cap \tau^* B = \varnothing$.

By Clifford's theorem applied to M, either C is hyperelliptic, or $\deg(M) = 5$, $h^0(M) = 3$, and $L = \pi^* M$. We must have that $|M|$ is base point free, or else there would be a g^2_4 on C. Thus $|M|$ defines a morphism to $\mathbb{P}^2$, which is birational since M has prime degree, and is an embedding since the genus of a smooth plane quintic is 6. Hence, $M \cong \mathbb{O}_{\mathbb{P}^2}(1)|_C$, and $h^0(\mathbb{O}_{\mathbb{P}^2}(1)|_C \otimes \eta) = 1$, since $h^0(L) = h^0(M) + h^0(M \otimes \eta)$. A plane quintic has a unique g^2_5 [ACGH85, p. 209], and so L can be the only triple point on Ξ.

Given a smooth plane quintic such that $h^0(\mathbb{O}_{\mathbb{P}^2}(1)|_C \otimes \eta) = 1$, an elementary argument (see Beauville [Bea82]) will show that Ξ has a unique singular point. Finally, Beauville [Bea77b] (see also Donagi and Smith [DS81] as well as [CMF05]) has shown that $X = C_x \Xi$ is a smooth cubic threefold, and $(P, \Xi) \cong (JX, \Theta)$ as principally polarized abelian varieties.

(b) part (a). If $\widetilde{C}$ is hyperelliptic, then the proof of Mumford's theorem [Mum74, p. 344] implies that $(P, \Xi) \cong (JC', \Theta')$, for some hyperelliptic curve C' of genus 5. Hence, $\dim(\operatorname{Sing} \Theta') = 2$ and $\operatorname{Sing}_3 \Theta' = \{x\}$, where x corresponds to the unique $2g^1_2$ on C'. On the other hand, $h^0(\widetilde{C}, 5g^1_2) = 6$, and $\pi_*(5g^1_2) = 5g^1_2 = \omega_C$, so that $5g^1_2$ corresponds to a triple point on Ξ.

(b) part (b). If $\widetilde{C}$ is not hyperelliptic, then the proof of Mumford's theorem implies that $(P, \Xi) \cong (JC' \times JC'', JC' \times \Theta'' + \Theta' \times JC'')$ for some hyperelliptic curves C' and C''. The possible genera for C' and C'' are 1 and 4, or 2 and 3, respectively.

In the former case, $\operatorname{Sing} \Xi = (\Theta' \times \Theta'') \cup (JC' \times \operatorname{Sing} \Theta'')$, and it follows that $\dim(\operatorname{Sing} \Xi) = \dim(\Theta'') = 3$. $\operatorname{Sing}_3 \Xi = \Theta' \times \operatorname{Sing}_2 \Theta''$, and $\operatorname{Sing}_2 \Theta'' = \{g^1_2 + p\}$, which has dimension one, so $\dim(\operatorname{Sing}_3 \Xi) = 1$.

In the latter case, $\operatorname{Sing} \Xi = (\Theta' \times \Theta'') \cup (JC' \times \operatorname{Sing} \Theta'')$, and it follows that $\dim(\operatorname{Sing} \Xi) = \dim(\Theta') + \dim(\Theta'') = 3$. $\operatorname{Sing}_3 \Xi = \Theta' \times \operatorname{Sing}_2 \Theta''$, and thus $\dim(\operatorname{Sing}_3 \Xi) = \dim(\Theta') + \dim(\operatorname{Sing}_2 \Theta'') = 1$. □

Remark 5.2.2. The proof above includes a simplification suggested by the referee, who also observed that this theorem is deducible from the results of Friedman and the author in [CMF05]. To be precise, the proof of Theorem 5.2.1 only uses the special case of Theorem 2 that $h^0(L) = 2$. This special case follows from [CMF05, Th. 2.5, p. 306].

Remark 5.2.3. Some of the statements in part (b) can be proved without using the fact that the Prym of a hyperelliptic curve is a hyperelliptic Jacobian. Namely, one can show that if $\widetilde{C}$ is hyperelliptic, then there is a unique triple point of Ξ, and if $\widetilde{C}$ is not hyperelliptic, then $\dim(\operatorname{Sing}_3 \Xi) \leq 1$. Indeed, as observed in the proof above, if $\widetilde{C}$ is hyperelliptic, then there is a unique g^5_{10} on $\widetilde{C}$, namely $5g^1_2$. Furthermore, $\pi_* 5g^1_2 = 5g^1_2 = \omega_C$, so that in fact $5g^1_2 \in \operatorname{Sing}_3 \Xi$. I claim that there are no triple points with $h^0(L) = 4$. In fact, since $h^0(M) = 3$, we must have

$M \geq 2g^1_2$. But then $\pi^* M \geq 4\tilde{g}^1_2$, so that $h^0(\pi^* M) \geq 5 > 4 = h^0(L)$, a contradiction. On the other hand, if C is hyperelliptic, then $h^0(L) = 4$. Since $h^0(M) = 3$, $M \geq 2g^1_2$. Let $M' = 2g^1_2$, so that $L = \pi^* M' \otimes \mathbb{O}_{\widetilde{C}}(B')$, where now we only require that $B' > 0$ and $\deg(B') = 2$. It follows that $\pi_* L = \pi_* \pi^* g^1_2 \otimes \pi_* B' = 4g^1_2 \otimes \pi_* B'$, so that if $\pi_* L = \omega_C$, then B' must lie above the g^1_2 on C. Since the g^1_2 has dimension one, and there are four choices of B' above each pair of points in the g^1_2, the dimension of triple points can be at most one.

6. **The Prym canonical map**

Let $\Psi_\eta : C \to \mathbb{P}^{g-2}$ be the Prym canonical map, the map induced by the linear system $|\omega_C \otimes \eta|$. One can easily check that $|\omega_C \otimes \eta|$ has a base point if and only if $\widetilde{C}$ is hyperelliptic, and consequently in this section we will restrict our attention to the case that $\widetilde{C}$ is not hyperelliptic. Under this assumption, we will establish a connection between the tangent cone to a singular point of Ξ and the Prym canonical image of C. Our eventual goal will be to determine whether a k-dimensional secant variety to C is contained in the tangent cone to a singular point of Ξ.

6.1. *Preliminaries on Prym images.* Let $q \in C$; set $D = p + \tau(p) = \pi^*(q)$. Consider the exact sequence

$$0 \to \mathbb{O}_{\widetilde{C}} \longrightarrow \mathbb{O}_{\widetilde{C}}(D) \longrightarrow \mathbb{O}_D \to 0,$$

and let ∂_D be the boundary map of the associated long exact sequence. Since $\widetilde{C}$ is not hyperelliptic, $h^0(\mathbb{O}_{\widetilde{C}}) = h^0(\mathbb{O}_{\widetilde{C}}(p + \tau(p))) = 1$, and hence ∂_D is injective.

LEMMA 6.1.1. $\Psi_\eta(q) = \{\partial_D(-a, a) \mid a \in \mathbb{C}\} \in \mathbb{P}(H^1(\mathbb{O}_C)^-)$.

Proof. Serre duality gives an isomorphism $H^1(\mathbb{O}_{\widetilde{C}}) \to (H^0(\omega_{\widetilde{C}}))^*$ given by $\alpha \mapsto (\omega \mapsto \sum_{p' \in \widetilde{C}} \operatorname{Res}_{p'}(\alpha\omega))$. From this description, it is easy to see that this isomorphism induces an isomorphism $H^1(\mathbb{O}_{\widetilde{C}})^- \to (H^0(\omega_{\widetilde{C}})^-)^*$ and hence an isomorphism $\mathbb{P}(H^1(\mathbb{O}_{\widetilde{C}})^-) \to \mathbb{P}((H^0(\omega_{\widetilde{C}})^-)^*)$. The map $C \to \mathbb{P}(H^1(\mathbb{O}_{\widetilde{C}})^-)$ given by $q \mapsto \langle\xi\rangle = \{\partial_D(-a, a) \mid a \in \mathbb{C}\}$ is well defined since ∂_D is injective. Composing with the duality map gives a map $\psi : C \to \mathbb{P}(H^0(\omega_{\widetilde{C}})^-)^*)$, which by definition is given by $q \mapsto \langle \omega \mapsto \sum_{p' \in \widetilde{C}} \operatorname{Res}_{p'}(\xi\omega)\rangle$.

Letting $\xi = \partial_{\pi^*(q)}(-a, a)$, a computation in the Čech complex shows that in the open cover we have been using, ξ is equal to a/z on U_{0i}, to $-a/z$ on $U_{\tau(0)j}$, and to 0 otherwise. Hence $\sum_{p' \in \widetilde{C}} \operatorname{Res}_{p'}(\xi\omega) = 2a\omega(p)$. It follows that $\psi(q) = \langle \omega \mapsto \omega(p)\rangle$, which is the definition of Ψ_η. □

Now consider a point $x \in \operatorname{Sing} \Xi$ that corresponds to the line bundle $L \in \operatorname{Pic}^{2g-2}(\widetilde{C})$, and consider the deformation $\mathscr{L}_{D,a}$ for some $a \in \mathbb{C}$. As before, let the transition functions of $\mathscr{L}_1$ be denoted by $\lambda_{ij}(t)=\lambda_{ij}(1+\alpha_{ij}^{(1)}t)$, and let $f:S\to P$ be the associated morphism. Then

$$f_*:T_{s_0}S\to T_xP=H^1(\mathbb{O}_{\tilde{C}})^-$$

has image equal to the linear span of $\alpha^{(1)}\in H^1(\mathbb{O}_{\tilde{C}})^-$. From our computation of $\alpha^{(1)}$ in Section 1 and the description of $\partial_D(-a,a)$ given in the proof above, we have the following:

LEMMA 6.1.2. *In the above notation*,

$$f_*(T_{s_0}S)=\langle\alpha^{(1)}\rangle=\langle\partial_D(-a,a)\rangle=\Psi_\eta(q)\in\mathbb{P}(H^1(\mathbb{O}_C)^-).$$

More generally, we can ask for the relation between the Prym canonical map and a deformation $\mathscr{L}_{D;a}$, where D has higher degree. Let $q_1,\dots q_k\in C$, and set $D=\pi^*(\sum_{i=1}^k q_i)$. Let $\mathscr{L}$ be a family of line bundles over C, parametrized by $\mathbb{C}^k$, such that $\mathscr{L}_a$ is a line bundle associated to $\mathscr{L}_{D,a}$, and consider the induced family of maps $f_a:S_a\to P$. For all a, let $s_0\in S_a$ be such that $f_a(s_0)=x$. Finally, let $\langle\Psi_\eta(q_1),\dots,\Psi_\eta(q_k)\rangle\subseteq\mathbb{P}^{g-2}$ be the span of the points $\Psi_\eta(q_1),\dots,\Psi_\eta(q_k)$. Extending the proofs of the first two lemmas by linearity, we immediately have the following:

LEMMA 6.1.3. *In the above notation*,

$$\langle\Psi_\eta(q_1),\dots,\Psi_\eta(q_k)\rangle=\{\partial_D(-a_1,a_1,\dots,-a_k,a_k)\mid a\in\mathbb{C}^k\}\subseteq\mathbb{P}(H^1(\mathbb{O}_{\tilde{C}})^-),$$

and for each $a\in\mathbb{C}^k$,

$$(f_a)_*(T_{s_0}S_a)=\langle\alpha_a^{(1)}\rangle=\langle\partial_D(-a_1,a_1,\dots,-a_k,a_k)\rangle\in\mathbb{P}(H^1(\mathbb{O}_C)^-).$$

As a consequence, we have the following proposition:

PROPOSITION 6.1.4. *Suppose* $\Psi_\eta(q_1),\dots,\Psi_\eta(q_k)$ *are all contained in a unique* $(k-1)$*-plane. Then* ∂_D *induces a linear inclusion* $\partial_D:\mathbb{P}^{k-1}\to\mathbb{P}^{g-1}$. *Moreover*, *if* $\operatorname{mult}_x\Xi=\mu$, *then* ∂_D *gives a bijection of sets*

$$\{a\in\mathbb{P}^{k-1}\mid(\operatorname{mult}_{s_0}\Theta_{S_a})/2>\mu\}\leftrightarrow\langle\Psi_\eta(q_1),\dots,\Psi_\eta(q_k)\rangle\cap C_x\Xi.$$

Proof. $\operatorname{mult}_{s_0}\Theta_{S_a}/2>\mu$ if and only if $\langle f_*(T_{s_0}S_a)\rangle\in C_x\Xi$, that is, if and only if $\langle\partial_D(-a_1,a_1,\dots,-a_k,a_k)\rangle\in C_x\Xi$. □

Due to Proposition 1.4.3, we can restate Proposition 6.1.4 as follows:

COROLLARY 6.1.5. *With the same hypothesis as the proposition*,

(a) *if* $\operatorname{mult}_x\Xi=h^0(L)/2$, *then there is a bijection of sets*

$$\{a\in\mathbb{P}^{k-1}\mid d_1(a)>0\}\leftrightarrow\langle\Psi_\eta(q_1),\dots,\Psi_\eta(q_k)\rangle\cap C_x\Xi;$$

(b) *if* $\operatorname{mult}_x \Xi > h^0(L)/2$ *and thus* $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$ *with* $h^0(M) > h^0(L)/2$, $B \geq 0$, *and* $B \cap \tau^* B = \varnothing$, *then there is a bijection of sets*

$$\{a \in \mathbb{P}^{k-1} \mid d_1(a) > 2h^0(M) - h^0(L) \text{ or } d_2(a) > 0\} \\ \leftrightarrow \langle \Psi_\eta(q_1), \dots, \Psi_\eta(q_k)\rangle \cap C_x \Xi.$$

We have the following elementary consequence.

COROLLARY 6.1.6. *With the same hypothesis as the proposition, if*

$$\deg(D) = 2k \leq 2g-2 \quad \textit{and} \quad h^0(L(-2D)) \neq 0,$$

then $C_x \Xi$ *contains a* $(k-1)$*-dimensional hyperplane.*

Remark 6.1.7. In the above analysis in the case that both $d_1 > 0$ and $d_2 > 0$, we did not show that $f(S) \nsubseteq \widetilde{\Theta}$, since we did not rule out the possibility that sections lift to arbitrary order. Nevertheless, in the case that sections lift to arbitrary order, and hence $f(S) \subseteq \widetilde{\Theta}$, it is clear that $f_*(T_{s_0} S) \subseteq C_x \widetilde{\Theta}$, and hence the conclusions of the corollaries hold in these cases as well.

We will now do a computation to prove the following proposition. This will illustrate the basic technique to be used in the next section.

PROPOSITION 6.1.8. *Suppose that* x *is a singular point of* Ξ, *corresponding to a line bundle* $L \in \operatorname{Pic}^{2g-2}(\widetilde{C})$, *such that* $\operatorname{mult}_x \Xi = h^0(L) = 2$. *For a point* $q \in C$, *let* $\pi^{-1}(q) = \{p, \tau(p)\}$. *Then* $\Psi_\eta(q) \in C_x \Xi$ *if and only if* $h^0(L(-\tau(p))) \neq 1$ *or* $h^0(L(-2p - \tau(p))) \neq 0$.

Remark 6.1.9. Since the condition $\Psi_\eta(q) \in C_x \Xi$ is independent of the choice of p versus $\tau(p)$, one can conclude from the proposition that for any point $p \in C$, $h^0(L(-\tau(p))) = 1$ and $h^0(L(-2p - \tau(p))) = 0$ if and only if $h^0(L(-p)) = 1$ and $h^0(L(-p - 2\tau(p))) = 0$.

Proof. By Theorem 1, $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$, with $h^0(C, M) = h^0(\widetilde{C}, L) = 2$, $B \geq 0$, and $B \cap \tau^* B = \varnothing$. Let $D = \pi^*(q)$, consider the deformation $\mathcal{L}_{D;1}$, and let $f : S \to P$ be the associated morphism with $f(s_0) = x$. By Proposition 1.4.3(b), all sections lift to first order, and I claim that if $h^0(L(-\tau(p))) \neq 1$ or $h^0(L(-p - \tau(p) - p)) \neq 0$, then a nontrivial section must lift to second order, so that by Corollary 6.1.5(b), $\Psi(q) \in C_x \Xi$.

Indeed, assume $h^0(L(-\tau(p))) \neq 1$. Then either $h^0(L(-p - \tau(p))) = 2$, in which case $h^0(L(-2D)) \neq 0$ and a section lifts to second order due to Corollary 1.4.9, or $h^0(L(-p - \tau(p))) = 1$. In this case, consider a nonzero section s in $H^0(L(-p - \tau(p)))$. Let $s + \sigma^{(1)} t$ be the standard lifting of s, so that the general lifting of s will be of the form $s + (\sigma^{(1)} + \varphi)t$ for some $\varphi \in H^0(L)$. Then $A_2(s + (\sigma^{(1)} + \varphi)t) = (0, -(s/z)(p) - \varphi(p), 0, \varphi(\tau(p)))$. Now considering the fact that $h^0(L(-\tau(p))) = 2$ and $h^0(L(-p - \tau(p))) = 1$, it follows that the map

$H^0(L(-\tau(p))) \to \mathbb{C}$ given by $\psi \mapsto \psi(p)$ is surjective, so that there is a section $\varphi \in H^0(L)$ such that $\varphi(\tau(p)) = 0$ and $\varphi(p) = -(s/z)(p)$. Thus $A_2(s + (\sigma^{(1)} + \varphi)t) = 0$, and so s lifts to second order.

On the other hand, suppose $h^0(L(-p-\tau(p)-p))$ is nonzero, and let $s \in H^0(L(-p-\tau(p)-p))$ be a nonzero section. Then s lifts to first order since $s \in H^0(L(-D))$, and consequently we set $s + \sigma^{(1)}t$ to be the standard first order lift. $A_2(s + \sigma^{(1)}t) = (0, -(s/z)(p), 0, 0)$, and since $s \in H^0(L(-p-\tau(p)-p))$ it follows that $A_2(s + \sigma^{(1)}t) = 0$.

Conversely, suppose $h^0(L(-2p-\tau(p))) = 0$ and $h^0(L(-\tau(p))) = 1$. In this case I claim that only the trivial section lifts to second order, and hence, by Corollary 6.1.5(b), $\Psi_\eta(q) \notin C_x\,\Xi$. In fact, this follows from the proof of Theorem 2; the key observation is that the proof of the theorem depends only on the numerology of Lemma 4.1.1 and not on the assumption that the chosen points were general. Thus we must check that the conditions

$$h^0(L(-2p-\tau(p))) = 0 \quad \text{and} \quad h^0(L(-\tau(p))) = 1$$

are sufficient to establish the results of Lemma 4.1.1. Using Riemann-Roch, we need only check (b), (d), (f), (g), (h), and (j):

(b) $h^0(L(-D)) = 1;\ 1 \le h^0(M(-q)) \le h^0(L(-D)) \le h^0(L(-\tau(p))) = 1$.

(d) $h^0(M(-2q)) = h^0(L(-2D)) \le h^0(L(-2p-\tau(p))) = 0$.

(f) $h^0(L(D+\tau(p))) = 3$; this follows from Riemann-Roch.

(g) This is the same as (f) in this case.

(h) $h^0(L(-\tau(p))) = 1$ is given.

(j) This is vacuous. □

6.2. *Secant varieties.* We now direct our attention to secant varieties of the Prym canonical image. For $0 \le k \le r$, the k-secant variety of a curve Γ embedded in $\mathbb{P}^r$ is defined to be the closure of the union of the linear subspaces in $\mathbb{P}^r$ spanned by a $(k+1)$-tuple of distinct points of Γ; that is, the 0-secant variety is Γ, and the 1-secant variety is the usual secant variety.

THEOREM 6.2.1. *Suppose that x is a singular point of Ξ corresponding to a line bundle $L \in \mathrm{Pic}^{2g-2}(\widetilde{C})$ such that $h^0(L) = 2n$.*

(a) *The $(n-1)$-secant variety of $\Psi_\eta(C)$ is not contained in $C_x\,\Xi$. More precisely, if $q_1, \dots, q_n$ are general points of C, then*

$$\langle \Psi_\eta(q_1), \dots, \Psi_\eta(q_n) \rangle \not\subseteq C_x\,\Xi.$$

(b) *The $(n-2)$-secant variety of $\Psi_\eta(C)$ is contained in $C_x\,\Xi$. Hence the k-secant variety of $\Psi_\eta(C)$ is contained in $C_x\,\Xi$ for all $0 \le k \le n-2$.*

Proof. (a) Let $q_1, \dots, q_n$ be n general points of C, let $D' = \sum_{i=1}^n q_i$, and let $D = \pi^* D'$. For $a \in \mathbb{C}^n$, consider the deformation $\mathcal{L}_{D;a}$, and let $f : S \to P$ be the associated morphism, with $f(s_0) = x$. In the case $\operatorname{mult}_x \Xi = n$, the proof of Theorem 1 implies that for a general $a \in \mathbb{C}^n$, $(\operatorname{mult}_{s_0} \Theta_S)/2 = \operatorname{mult}_x \Xi$. In the case $\operatorname{mult}_x \Xi > n$, the proof of Theorem 2 implies the same result. Proposition 6.1.4 then implies that $\langle \Psi_\eta(q_1), \dots, \Psi_\eta(q_n)\rangle \not\subseteq C_x \Xi$.

(b) In the case $\operatorname{mult}_x \Xi = n$, for general points $q_1, \dots, q_{n-1} \in C$, let $D' = \sum_{i=1}^{n-1} q_i$, let $D = \pi^* D'$, and consider the deformation $\mathcal{L}_{D;a}$. For all $a \in \mathbb{C}^{n-1}$, $d_1(a) \geq h^0(L(-D)) > 0$, and hence by Corollary 6.1.5(a), the $(n-2)$-secant variety of $\Psi(C)$ is contained in $C_x \Xi$.

In the case $\operatorname{mult}_x \Xi > n$, if

$$L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B), \quad h^0(C, M) > h^0(\widetilde{C}, L)/2, \quad B \geq 0, \quad B \cap \tau^* B = \varnothing,$$

let $n_1 = h^0(M)$ and $n_2 = h^0(L) - h^0(M)$. For general points $q_1, \dots, q_{n-1} \in C$, let $D' = \sum_{i=1}^{n-1} q_i$, $D = \pi^* D'$, and consider the deformation $\mathcal{L}_{D;a}$. We will find that for general $a \in \mathbb{C}^{n-1}$, there is a nontrivial section that lifts to second order and hence, by Corollary 6.1.5(b), the $(n-2)$-secant variety of $\Psi(C)$ is contained in $C_x \Xi$.

Let $a \in \mathbb{C}$ be such that $a_i \neq 0$ for all i. Let $s \in H^0(L(-D))$, and let $s + \sigma^{(1)} t$ be the standard lift of s, as in Lemma 1.2.5. Then the general lift of s will be of the form $s + (\sigma^{(1)} + \varphi)t$ for some $\varphi \in H^0(L)$. We have seen that

$$A_2(s + (\sigma^{(1)} + \varphi)t) = (0, -a_1^2 (s/z)(p_1) - a_1 \varphi(p_1), 0, a_1 \varphi(\tau(p_1)), \dots).$$

I claim that there is some $s \in H^0(L(-D))$ and some $\varphi \in H^0(L)$ such that $A_2(s + (\sigma^{(1)} + \varphi)t) = 0$. From the equation above, this is equivalent to the claim that there exists some $s \in H^0(L(-D))$ and some $\varphi \in H^0(L)$ such that $\varphi(p_i) = a_i (s/z)(p_i)$ and $\varphi(\tau(p_i)) = 0$ for $1 \leq i \leq n-1$. Let

$$V_1 = \{(a_1(s/z)(p_1), \dots, a_{n-1}(s/z)(p_{n-1})) \in \mathbb{C}^{n-1} \mid s \in H^0(L(-D))\},$$
$$V_2 = \{(\varphi(p_1), \dots, \varphi(p_{n-1})) \in \mathbb{C}^{n-1} \mid \varphi \in H^0(L(-\textstyle\sum_{i=1}^{n-1} p_i))\}.$$

To prove our claim, we need only show that $\dim(V_1) + \dim(V_2) > n-1$. Now

$$\dim(V_1) = h^0(L(-D)) - h^0(L(-D - \textstyle\sum_{i=1}^{n-1} p_i)),$$
$$\dim(V_2) = h^0(L(-\textstyle\sum_{i=1}^{n-1} \tau(p_i))) - h^0(L(-D)).$$

Hence $\dim(V_1) + \dim(V_2) = h^0(L(-\sum_{i=1}^{n-1} \tau(p_i))) - h^0(L(-D - \sum_{i=1}^{n-1} p_i))$. Since the points p_i are general, we have that

$$h^0(L(-\textstyle\sum_{i=1}^{n-1} \tau(p_i))) = 2n - (n-1) = n+1.$$

Furthermore, it follows from Corollary 2.1.3 that $h^0(L(-D - \sum_{i=1}^{n-1} p_i)) = \max(0, n_1 - n + 1 - (n-1)) = \max(0, n_1 - 2n + 2) \leq 2$, with equality holding if and only if $n_1 = 2n$. Thus $\dim(V_1) + \dim(V_2) \geq n + 1 - 2$, with equality holding if and only if $n_1 = 2n$. Therefore, we have established the claim in the case $n_1 \neq 2n$, and it follows that in this case there is a nontrivial section that lifts to second order.

On the other hand, the case $n_1 = 2n$ is much easier. Indeed,

$$h^0(L(-2D)) = 2n - (n-1) - (n-1) = 2 \neq 0,$$

and thus, as observed in the corollary to Lemma 1.4.8, a nontrivial section must lift to second order. $\square$

Remark 6.2.2. This theorem generalizes [SV01, Prop. 5.1], which does not address the issue of the secant variety and which makes the additional assumption that either $\mathrm{mult}_x\, \Xi = (1/2)h^0(L)$ or L is base point free and $(1/2)h^0(L) \leq \mathrm{mult}_x\, \Xi \leq h^0(L) - 1$. Also, as a consequence of Theorem 6.2.1, we see that we could not have proved Theorem 2 using a divisor D of degree less than $h^0(L)$.

Remark 6.2.3. Using similar techniques, one can easily prove the Riemann singularity theorem for Jacobians, as well as the fact that for a Jacobian (JC, Θ) and a point $x \in \mathrm{Sing}\, \Theta$ corresponding to a line bundle $L \in \mathrm{Pic}^{g-1}(C)$ with $h^0(L) = n$, the k-secant variety of the canonical image of the curve is contained in $C_x\Theta$ if and only if $k \leq n - 2$. See [CMF05, Th. 1.9] and [ACGH85, Th. 1.6, p. 232].

COROLLARY 6.2.4. $\Psi_\eta(C) \subseteq C_x\Xi$ *if and only if* $h^0(L_x) \geq 4$.

COROLLARY 6.2.5 (Tjurin [Tju75], Smith and Varley [SV01]). *If* $\mathrm{mult}_x\, \Xi = 2$, *then one of the following must hold*:

(a) $h^0(L) = 4$, *and* L *can not be written in the form* $L = \pi^*(M) \otimes \mathcal{O}_{\tilde{C}}(B)$ *where* $h^0(M) > 2$, $B > 0$ *and* $B \cap \tau^* B = \varnothing$. *In this case*, $\Psi_\eta(C) \subseteq C_x\Xi$; *as a result*, $C_x\Xi$ *is nondegenerate*, *and* $\mathrm{rank}(C_x\Xi) \geq 3$. *In addition*, *the secant variety of* $\Psi_\eta(C)$ *is not contained in* $C_x\Xi$.

(b) $h^0(L) = 2$, *and* $L = \pi^*(M) \otimes \mathcal{O}_{\tilde{C}}(B)$ *where* $h^0(M) = 2$, $B \geq 0$, *and* $B \cap \tau^* B = \varnothing$. *In this case* $\Psi_\eta(C) \not\subseteq C_x\Xi$.

Remark 6.2.6. The fact in (a) that $\Psi_\eta(C) \subseteq C_x\Xi$ was first shown by Tjurin [Tju75, Lem. 2.3, p. 963]. The fact in (a) that the secant variety of $\Psi_\eta(C)$ is not contained in $C_x\Xi$ and fact in (b) that $\Psi_\eta(C) \not\subseteq C_x\Xi$ are consequences of Proposition 6.1.4, and were not previously known in general. In the special case that (P, Ξ) is the Jacobian of a nonhyperelliptic curve, Smith and Varley have observed that at a generic exceptional double point $\Psi_\eta(C) \not\subseteq C_x\Xi$ [SV02, p. 241, l. 8]. Their argument in the case that the curve has no g^1_2, g^1_3, or a g^2_5 is that the Prym canonical curve is contained in every stable quadric, while by Green's theorem [Gre84] the base locus of the quadrics is a canonically embedded curve. This cannot also be a Prym canonically embedded curve, and therefore not all of the exceptional quadrics contain $\Psi_\eta(C)$.

6.3. *Equations for tangent cones.* Kempf's theorem from [Kem73] gives an equation defining the tangent cone to $\widetilde{\Theta}$ at a point x as a subscheme of $H^0(\widetilde{C}, \omega_{\widetilde{C}}) = T_x J\widetilde{C}$. In the case $T_x P \not\subseteq C_x\widetilde{\Theta}$, this equation restricts to $H^0(\widetilde{C}, \omega_{\widetilde{C}})^- = T_x P$ to give the square of an equation defining the tangent cone to Ξ at x as a subscheme of $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$.

The aim of this section will be to give another description of the equation for $C_x\Xi$ in the case $T_x P \not\subseteq C_x\widetilde{\Theta}$, and to give a description of the equation of the tangent cone in the case that $T_x P \subseteq C_x\widetilde{\Theta}$ and $h^0(L) = h^0(M)$. In the former case, we will need only to look at first order liftings. In the latter case, we will need to look at second order liftings, but we will have the advantage of knowing that the space of sections lifting to first order is fixed. The analysis that follows will apply to any situation where this is true.

Given $g-1$ points $q_1, \dots, q_{g-1}$ of C, which are linearly independent as points of $\Psi(C)$, let $D = \sum_{i=1}^{g-1} \pi^{-1}(q_i)$, and consider the deformation $\mathscr{L}_{D;a}$. Let $E_1 : H^0(L(D)) \to H^0(L(D) \otimes \mathbb{O}_D)$ be the map induced from the short exact sequence, and similarly, let $E_2 : H^0(L(2D)) \to H^0(L(2D) \otimes \mathbb{O}_{2D})$. Let M_{E_1} and M_{E_2} be the matrices whose rows span the respective images of these maps. Define $B_1 : H^0(L) \to H^0(L(D) \otimes \mathbb{O}_D)$ to be A_1. By definition, a section $s \in H^0(L)$ lifts to first order if and only if $\partial_{L;D} \circ B_1(s) = 0 \in H^1(L)$.

We will now define a map $B_2 : H^0(L) \to H^0(L(2D) \otimes \mathbb{O}_{2D})$ that will have the property that there is a one-to-one correspondence between sections of L that lift to second order and sections of L in the kernel of $\partial_{L;2D} \circ B_2$. To do this, choose a basis $\{s_1, \dots, s_{d_1}\}$ for W_1 and a set of sections $\{s_{d_1+1}, \dots, s_{2n}\}$ whose images form a basis for $H^0(L)/W_1$. Let $\{s_1 + \sigma_1^{(1)} t, \dots, s_{d_1} + \sigma_{d_1}^{(1)} t\} \subseteq H^0(\mathscr{L}_1)$ be a set of liftings of the basis for W_1, and define the map B_2 by

$$\textstyle\sum_{i=1}^{2n} \alpha_i s_i \mapsto A_2\big(\sum_{i=1}^{d_1} \alpha_i (s_i + \sigma_i^{(1)} t) + \sum_{j=d_1+1}^{2n} \alpha_j s_j t\big),$$

where the $\alpha_i \in \mathbb{C}$. One can easily check that there is a one-to-one correspondence between sections of L that lift to second order and sections of L in the kernel of $\partial_{L;2D} \circ B_2$.

Let M_{B_1} and M_{B_2} be matrices whose rows span the respective images of these maps. Define matrices

$$M_1 = \begin{pmatrix} M_{E_1} \\ M_{B_1} \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} M_{E_2} \\ M_{B_2} \end{pmatrix}.$$

THEOREM 6.3.1. *Suppose that $x \in \operatorname{Sing} \Xi$ corresponds to the line bundle $L \in \operatorname{Pic}^{2g-2}(\widetilde{C})$.*

(a) *If* $\operatorname{mult}_x \Xi = h^0(L)/2$, *then* $\det(M_1)$ *is a homogeneous polynomial of degree $h^0(L)$, which defines $C_x\Xi$ as a subset of $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$.*

(b) *If* $\operatorname{mult}_x \Xi = h^0(L)$, *and* $L = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$ *where* $h^0(M) = h^0(L)$, $B \geq 0$ *and* $B \cap \tau^* B = \varnothing$, *then* $\det(M_2)$ *is a homogeneous polynomial of degree* $2 \cdot h^0(L)$, *which defines* $C_x \Xi$ *as a subset of* $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$.

Proof. (a) Let $\mathcal{L}$ be a family of deformations parametrized by $a \in \mathbb{C}^{g-1}$, with fiber $\mathcal{L}_a = \mathcal{L}_{D;a}$. By Riemann-Roch,

$$h^0(L(D)) = h^0(\omega_{\widetilde{C}} \otimes L^{-1}(-D)) + \deg(D).$$

But $\deg((\omega_{\widetilde{C}} \otimes L^{-1}(-D))) = -g-1$, and so $h^0(L(D)) = 2g-2$. Letting $\{s_1, \ldots, s_n\}$ be a basis for $H^0(L)$, we can take

$$(M_{B_1})_{i;2j-1} = -a_j s_i(p_j) \quad \text{and} \quad (M_{B_1})_{i;2j} = a_j s_i(\tau(p_j))$$

for $1 \leq i \leq n$ and $1 \leq j \leq g-1$. Letting $\{r_1, \ldots, r_{2g-2-n}\}$ be a basis for $H^0(L(D))/H^0(L)$, then we can take

$$(M_{E_1})_{i;2j-1} = r_i(p_j) \quad \text{and} \quad (M_{E_1})_{i;2j} = r_i(\tau(p_j))$$

for $1 \leq i \leq 2g-2-n$ and $1 \leq j \leq g-1$. With this notation, nontrivial sections lift to first order if and only if $\det(M_1) = 0$. From the form of the matrix M_1, it is clear that the determinant is a homogeneous polynomial of degree $h^0(L)$ in the a_i. Now (a) now follows from Corollary 6.1.5(a).

(b) Let $\mathcal{L}$ be a family of deformations parametrized by $a \in \mathbb{C}^{g-1}$, with fiber $\mathcal{L}_a = \mathcal{L}_{D;a}$. In this case we have seen that all sections lift to first order in all directions, and it is easy to check that if $s_i + \sigma_{i;j}$ is a lift of s_i in the direction of the j-th basis vector of $\mathbb{C}^{g-1}$, then $s_i + \sum_j a_j \sigma_{i;j}$ is a lift of s_i in the direction of a. Using the basis described in the definition of B_2, we get that

$$(M_{B_2})_{i;4j-3} = 0, \qquad (M_{B_2})_{i;4j-2} = -a_j \textstyle\sum_k a_k \sigma_{i;k}(p_j),$$
$$(M_{B_2})_{i;4j-1} = a_j^2 s_i(\tau(p_j)),$$

and $(M_{B_2})_{i;4j} = a_j \sum_k a_k \sigma_{i;k}(\tau(p_j)) + a_j^2 (ds_i/dz)(\tau(p_j))$. Again, M_{E_2} is independent of the a_i, and so we see that $\det(M_2)$ is a homogeneous polynomial of degree $2h^0(L)$ in the a_i. Nontrivial sections lift to second order if and only if $\det(M_2) = 0$, and so (b) now follows from Corollary 6.1.5(b). □

Remark 6.3.2. This analysis will go through in any case where the space of sections lifting to first order is fixed. In the case that it is not fixed, the dependence of the entries of M_{B_2} on the a_i is more difficult to ascertain.

Remark 6.3.3. Theorem 1 implies that $T_x P \not\subseteq C_x \widetilde{\Theta}$ in case (b), in which case Kempf's theorem gives an equation for the tangent cone as a scheme.

COROLLARY 6.3.4 (Quadric tangent cones). *Suppose* $x \in \operatorname{Sing} \Xi$ *corresponds to the line bundle* $L \in \operatorname{Pic}^{2g-2}(\widetilde{C})$. *If* $\operatorname{mult}_x \Xi = 2$, *then one of the following must hold*:

(a) $h^0(L)=4$. *Then* $\det(M_1)=q(a_1,\dots,a_g)^2$ *for some irreducible homogeneous quadratic polynomial* $q\in\mathbb{C}[a_1,\dots,a_g]$. *Hence*, $q=\sqrt{\det(M_1)}$ *defines* $C_x\Xi$ *as a subscheme of* $\mathbb{P}^{g-1}$.

(b) $h^0(L)=2$. *In this case*, $\det(M_2)=q^2$ *or* $\ell_1\ell_2^3$, *where* q *and* ℓ_1, ℓ_2 *are homogeneous polynomials of degree two and one respectively. Hence*, *either* $q=\sqrt{\det(M_2)}$ *or* $\ell_1\ell_2$ *defines* $C_x\Xi$ *as a subscheme of* $\mathbb{P}^{g-1}$.

Proof. We have seen in Corollary 6.2.5 that in case (a) the tangent cone is nondegenerate. □

Remark 6.3.5. Smith and Varley in [SV02] have used Kempf's theorem to analyze the rank of quadric tangent cones in case (a). It is reasonable to expect that the description of the tangent cone given in the corollary above will yield new information. This is work in progress.

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