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# Local rigidity of affine actions of higher rank groups and lattices

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### Abstract

Let $J$ be a semisimple Lie group with all simple factors of real rank at least two. Let $\Gamma < J$ be a lattice. We prove a very general local rigidity result about actions of $J$ or $\Gamma$. This shows that almost all so-called "standard actions" are locally rigid. As a special case, we see that any action of $\Gamma$ by toral automorphisms is locally rigid. More generally, given a manifold $M$ on which $\Gamma$ acts isometrically and a torus $\mathbb{T}^n$ on which it acts by automorphisms, we show that the diagonal action on $\mathbb{T}^n \times M$ is locally rigid.

This paper is the culmination of a series of papers and depends heavily on our work in two recent articles. The reader willing to accept the main results of those papers as "black boxes" should be able to read the present paper without referring to them.


## 1. Introduction

Throughout this paper $J$ is a (connected) semisimple Lie group with no compact factors and all simple factors of real rank at least two, and $\Gamma < J$ is a lattice. The purpose of this paper is to prove the following:

THEOREM 1.1. *Let $\rho$ be a quasi-affine action of $J$ or $\Gamma$ on a compact manifold $X$. Then the action is $C^{\infty,\infty}$ and $C^{3,0}$ locally rigid. Furthermore, there exists an integer $k_0$, such that the action is $C^{k,k-n}$ locally rigid for all $k > k_0$, where $n = \frac{1}{2}\dim X + 3$.*

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*Remarks on regularity.* The number $k_0 = \max(k_1, n)$ where $k_1$ is determined by properties of certain foliations associated to the dynamics of $\rho(g_i)$, for a specific finite set of choices of $g_1, \ldots, g_k$ in $J$ or $\Gamma$. If $k$ is even, we can let $n = \frac{1}{2}\dim X + 2$ instead.[1]

We now proceed to define the terms in the theorem. We say $H$ is a *connected real algebraic group* if it is the connected component of the real points $H(\mathbb{R})^0$ of an algebraic group defined over $\mathbb{R}$.

*Definition* 1.2. a) Let $H$ be a connected real algebraic group, $\Lambda < H$ a cocompact lattice. Assume a topological group $G$ acts continuously on $H/\Lambda$. We say that the $G$ action on $H/\Lambda$ is *affine* if every element of $G$ acts via an affine diffeomorphism.

b) More generally, let $M$ be a compact manifold. Assume a group $G$ acts affinely on $H/\Lambda$. Choose a Riemannian metric on $M$ and a cocycle over the $G$ action $\iota : G \times H/\Lambda \to \mathrm{Isom}(M)$. We call the *skew product action* of $G$ on $H/\Lambda \times M$ defined by $d\cdot(x,m) = (d\cdot x, \iota(d,x)\cdot m)$ a *quasi-affine action*.

We always write $X = H/\Lambda \times M$. Recall that an *affine diffeomorphism* $d$ of $H/\Lambda$ is one covered by a diffeomorphism $\tilde{d}$ of $H$ where $\tilde{d} = A \circ T_h$ where $A$ is an automorphism of $H$ such that $A(\Lambda) = \Lambda$ and $T_h$ is left translation by $h \in H$. The full group of affine diffeomorphisms of $H/\Lambda$ is a finite dimensional Lie group which we write as $\mathrm{Aff}(H/\Lambda)$. The definition of acting affinely given above is equivalent to saying the action is given by a homomorphism $\pi : G \to \mathrm{Aff}(H/\Lambda)$. See [FM03, §6.1] for a description of $\mathrm{Aff}(H/\Lambda)$ and a classification of affine actions of $J$ or $\Gamma$ as above. Note also that the case of quasi-affine actions as defined here includes products of affine actions with trivial actions. Another class of examples give the following:

COROLLARY 1.3. *Let $J$ be as above and $\Gamma < J$ a lattice. Then any action of $\Gamma$ by automorphisms of $\mathbb{T}^m$ is $C^{\infty,\infty}$ and $C^{3,0}$ locally rigid. Furthermore there exists a positive integer $k_0 \geq 3$, depending on the action, such that the action is $C^{k,k-\frac{1}{2}m-3}$ locally rigid for all $k \geq \min(\frac{1}{2}m+3, k_0)$.*

We now formally define local rigidity in this context.

*Definition* 1.4. Given a topological group $G$ and a continuous $C^\infty$ action, $\rho : G \times X \to X$, by diffeomorphisms on a manifold $X$, we say that the *action is $C^{k,r}$ locally rigid*, where $r \leq k$, if any continuous action $\rho'$ by $C^k$ diffeomorphisms, that is sufficiently $C^k$ close to $\rho$ is *conjugate* to $\rho$ by a small $C^r$ diffeomorphism. We say

[1] While this paper was under review, we learned that results in [RT05] combined with certain results from [FM05] allow us to achieve remarkably less loss of regularity, yielding $C^{k,k-3}$ local rigidity in place of $C^{k,k-n}$ local rigidity. We explain this briefly at the end of the paper.

that a continuous action $\rho$ is $C^{\infty,\infty}$ *locally rigid* if any continuous $C^{\infty}$ action which is sufficiently $C^{\infty}$ close to $\rho$ is conjugate to $\rho$ by a small $C^{\infty}$ diffeomorphism.

The special case of $C^{k,k}$ local rigidity says exactly that the homomorphism $\rho : G \to \mathrm{Diff}^k(X)$ is locally rigid. In other words that any homomorphism close to $\rho$ is conjugate to $\rho$ by a small element of $\mathrm{Diff}^k(X)$. Since the $C^{\infty}$ topology is defined as the inverse limit of the $C^k$ topologies, two $C^{\infty}$ diffeomorphisms are $C^{\infty}$ close if they are $C^k$ close for some large $k$. Our proof shows explicitly that a $C^{\infty}$ perturbation $\rho'$ of $\rho$ which is $C^k$ close to $\rho$ is conjugate to $\rho$ by a $C^{\infty}$ diffeomorphism which is $C^{k-n}$ close to the identity where $n$ is as in Theorem 1.1. The topology we take on $\mathrm{Hom}(G, \mathrm{Diff}^k(X))$ to define *close* above is the compact-open topology.

Gromov in [Gro88] and Zimmer in [Zim87] suggested that one might be able to "essentially classify" all volume-preserving smooth actions of higher rank semisimple groups and their lattices on compact manifolds. This would be, in a sense, a "nonlinear" analogue of the second author's superrigidity theorems, since one of the consequences of the superrigidity theorems is a classification of all finite dimensional linear representations of higher rank lattices (modulo issues concerning finite image representations). In [Zim87], Zimmer also proposed the study of local rigidity of known actions of higher rank lattices on compact manifolds, as a "nonlinear" analogue of the classical local rigidity theorems of Calabi-Vesentini, Selberg and Weil. These show that any cocompact lattice $\Gamma$ in any simple Lie group $J$ is locally rigid, as long as $J$ is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ [CV60], [Sel], [Wei62]. That is, any embedding of $\Gamma$ in $J$ close to the defining one $i : \Gamma \to J$ is simply a conjugate of $i$ by a small element of $J$. Since $J$ acts transitively on $J/\Gamma$, our theorem can be taken to be a generalization of Weil's result in the case when $J$ is a higher rank simple group. A perturbation $\Gamma'$ of $\Gamma$ in $J$ defines a perturbation of the original $J$ action on $J/\Gamma$ since $J/\Gamma$ and $J/\Gamma'$ are diffeomorphic. The conjugacy between these actions can easily be seen to give a conjugacy between $\Gamma$ and $\Gamma'$.

Many results have been proven concerning local rigidity of affine actions of higher rank lattices and Lie groups, particularly when the action is assumed to satisfy some strong hyperbolicity condition. The first results of this kind are due to Hurder [Hur92]. He proved that the standard action on $\mathbb{T}^n$ of any finite index subgroup in $\mathrm{SL}_n(\mathbb{Z})$ is deformation rigid for $n \geq 3$. (This involves assuming a path of nearby actions and obtaining a path of conjugacies.) The same actions were shown to be locally rigid in [KL91] and [KLZ96]. Many other results along these lines were obtained by many authors; we refer to the introduction of [MQ01] for a more detailed discussion. Here we mention that all standard Anosov actions on tori and nilmanifolds were proven to be locally rigid in [KS97] and all so-called weakly hyperbolic actions were proven to be locally rigid in [MQ01].

For isometric actions, there are also results. In [Ben00], Benveniste shows that any isometric action of any cocompact lattice in a group $J$ as above is $C^{\infty,\infty}$ locally rigid. The interested reader should refer to the introduction to [FM05] for a discussion of earlier, weaker results by Zimmer concerning (certain) perturbations of isometric actions of groups with property (T). In our previous paper [FM05], we have proven:

THEOREM 1.5. *Let $G$ be a locally compact, compactly generated group with property* (*T*)*. Let $(X, g)$ be a compact Riemannian manifold, and let $\rho$ be an action of $G$ on $X$ by isometries. Then the action $C^{\infty,\infty}$ is locally rigid and is $C^{k,k-\kappa}$ locally rigid for any $k \geq 2$ and any $\kappa > 0$.*

We remark that Theorem 1.5 holds for a much broader class of groups than Theorem 1.1.

The proof of Theorem 1.1 uses a foliated generalization of Theorem 1.5 also proven in [FM05]. This result is recalled below in Section 4 where it is applied in the course of our proof.

For actions which are neither weakly hyperbolic nor isometric for all previous results, due to Nitica and Torok, concern affine actions which are products of Anosov actions and trivial actions [NT95], [NT01], [Tör03]. For example, take the standard action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ and let $\rho$ denote the action obtained on $\mathbb{T}^n \times S^1$ by taking $\gamma(t, s) = (\gamma t, s)$. Then Nitica and Torok show that, given $k > 0$, any $C^\infty$ action $\rho'$ that is sufficiently $C^2$ close to $\rho$ is conjugate to $\rho$ by a $C^0$ small, $C^k$ diffeomorphism. (This result does not imply $C^\infty$ local rigidity because the size of the perturbation must be made smaller to obtain more derivatives in the conjugacy.) Their full result is more general, allowing one to replace the standard action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ by any so-called TNS action of a higher rank lattice on a torus. They also prove some more general results for deformation rigidity, but always for products of TNS and trivial actions.

We briefly note two of the more major differences between the cases considered by Nitica and Torok and the general case considered here. First, in the general case, the *central foliation* for the group action is not necessarily by compact leaves. Secondly, in the general case, the action along the central foliation is isometric but not necessarily trivial. (For a definition of the central foliation, see § 2.1 below.)

We note here that nonlocally rigid volume-preserving actions of higher rank semisimple groups and their lattices on compact manifolds have been constructed, first in [KL96] and later and more generally in [Ben96]. Those in [Ben96] are even shown to have smooth volume-preserving deformations. See also [Fis08] for a more general construction and another proof that the deformations are nontrivial. A weaker result is shown in [KL96].

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## 2. **Affine actions, perturbations and quotients**

This section primarily recalls results from [FM03]. Let $\tilde{J}$ be the universal cover of $J$ and $\tilde{\Gamma}$ the pre-image of $\Gamma$ under the covering map $\tilde{J} \to J$. Any action of $J$ or $\Gamma$ can be viewed as an action of $\tilde{J}$ or $\tilde{\Gamma}$ respectively, and so we may assume, without loss of generality, that $J$ is simply connected.

2.1. *Describing affine actions.* In this section we recall from [FM03] another description of the actions we are considering. This description provides an extremely simple description of the derivative cocycle for the action which allows a simple description of the dynamical foliations for elements of the acting group, as well as of the central foliation of the entire group.

Throughout this section $H$ will be a connected real algebraic group and $\Lambda < H$ will be a cocompact lattice. We now recall three technical results from [FM03].

THEOREM 2.1. *Let $\rho$ be an affine action of $J$ on $H/\Lambda$. Then the action $\rho$ is given by $\rho(j)[h] = [\pi_0(j)h]$ where $\pi_0 : J \to H$ is a continuous homomorphism.*

This is a special case of [FM03, Th. 6.4]. As indicated there, the result holds with the weaker assumption that $J$ has no compact simple factors.

The analogous result for $\Gamma$ actions is more complicated and can require that we view $H/\Lambda$ as a homogeneous space for a different Lie group. The following is a rearrangement of [FM03, Prop. 6.3]. Given a Lie group $L$, we denote its automorphism group by $\mathrm{Aut}(L)$. Since $\mathrm{Aut}(L)$ is a closed subgroup of $\mathrm{GL}(\dim(L))$ it is a Lie group.

PROPOSITION 2.2. *Given a real algebraic group $H$ there is a connected cover $p : H' \to H$ and a realization of $H'$ as a connected real algebraic group*, *such that*

(1) *the connected component* $\mathrm{Aut}(H')^0$ *of* $\mathrm{Aut}(H')$ *has the structure of a connected real algebraic group*,
(2) $\mathrm{Aut}(H')^0 < \mathrm{Aut}(H')$ *is a finite index subgroup*,
(3) $\mathrm{Aut}(H')^0$ *acts rationally on* $H'$.

The key point is to choose the algebraic structure on $H'$ so that the connected component of the center of $H'$ is contained in the unipotent radical. It follow that $\mathrm{Aut}(H')^0 \ltimes H'$ is a connected real algebraic group.

Let $\Lambda'$ be $p^{-1}(\Lambda)$. It follows from the construction given in [FM03] that, possibly after passing to a finite index subgroup $\Gamma'$ in $\Gamma$, any affine action $\rho$ of $\Gamma$

on $H/\Lambda$ remains affine when we view $H/\Lambda$ as $H'/\Lambda'$. For the remainder of this paper, we assume that we have replaced our group $H$ with a group $H'$ as described in Proposition 2.2.

Before giving the analogous description of affine $\Gamma$ actions, we need to recall a consequence of the superrigidity theorems [Mar77], [Mar91], where these are only stated for $J$ algebraic. But the extension to $J$ as assumed here is sketched in [Fis02, Th. 7.12]; see [FM03] for detailed proofs. We will use the notation introduced here in the statements below. If $J$ is as above and $\Gamma < J$ is a lattice, and $L$ is an algebraic group, we call a homomorphism $\pi : \Gamma \to L$ *superrigid* if it almost extends to a homomorphism of $J$. This means that there is a continuous homomorphism $\pi^E : J \to L$ and a homomorphism $\pi^K : \Gamma \to L$ with bounded image such that $\pi(\gamma) = \pi^E(\gamma)\pi^K(\gamma)$ and $\pi^E(\Gamma)$ commutes with $\pi^K(\Gamma)$. The superrigidity theorems imply that any continuous homomorphism of $\Gamma$ into an algebraic group is superrigid. This can be deduced easily from Lemma VII.5.1 and Theorems VII.5.15 and VII.6.16 of [Mar91].

THEOREM 2.3. *Let $\rho$ be an affine action of $\Gamma$ on $H/\Lambda$. Then there are a finite index subgroup $\Gamma' < \Gamma$ and a homomorphism $\pi_0 : \Gamma' \to \mathrm{Aff}(H) = \mathrm{Aut}(H) \ltimes H$ such that $\rho(\gamma)[h] = [\pi_0(\gamma)h]$. Furthermore, we can assume that $\pi_0(\Gamma')$ is contained in $\mathrm{Aut}(H)^0 \ltimes H$ and that $\pi_0(\gamma) = \pi_0^E(\gamma)\pi_0^K(\gamma)$ where $\pi_0^E : J \to \mathrm{Aff}(H)$ is a homomorphism and $\pi_0^K : \Gamma' \to \mathrm{Aff}(H)$ is a homomorphism with bounded image, and the images of $\pi_0^E$ and $\pi_0^K$ commute.*

This is a rephrasing of [FM03, Th. 6.5]. The final conclusion concerning the fact that $\pi_0$ is the product of a the restriction of a homomorphism of $J$ and a homomorphism with bounded image follows from the superrigidity theorems discussed above.

We can now describe the *central foliation* for a quasi-affine action $\rho$ of either $J$ or $\Gamma$. We will denote the central foliation $\mathfrak{F}$. If $\rho$ is a $J$ action, and $M$ is trivial, then the central foliation is just the orbit foliation for the left action of $Z = Z_H(\pi_0(J))$ on $H/\Lambda$. If $M$ is nontrivial, we have a projection $H/\Lambda \times M \to H/\Lambda$ and the central foliation is given by the pre-images in $H/\Lambda \times M$ of $Z$ orbits in $H/\Lambda$.

Let $\Gamma'$ be the subgroup of finite index given by Theorem 2.3 and further assume that $\Gamma'$ is normal. Let $A$ be the connected component of $\mathrm{Aut}(H)$, and let $L = A \ltimes H$. Note that $L$ is an algebraic group. In this case, we let $Z = Z_L(\pi_0^E(\Gamma')) \cap H$. If $M$ is trivial, the *central foliation* for the action, which we denote $\mathfrak{F}$, is then defined to be the foliation given by orbits of $Z$ on $H/\Lambda$. If $M$ is nontrivial, we have a projection $H/\Lambda \times M \to H/\Lambda$ and the central foliation is given by the pre-images in $H/\Lambda \times M$ of $Z$ orbits in $H/\Lambda$.

We will refer to the tangent space of the central foliation $\mathfrak{F}$ as *the central distribution for the group action*.

We now want to define a Riemannian metric on $H/\Lambda\times M$ so that the $\rho$ action is isometric along leaves of $\mathfrak{F}$. Since $M$ is assumed in Definition 1.2 to be a Riemannian manifold with Riemannian metric $g_M$, and $\rho$ is defined to be isometric along $M$ fibers, it suffices to define a Riemannian metric on $H/\Lambda$ for which the affine action on $H/\Lambda$ is isometric along $Z$ orbits. Let $\mathfrak{h}$ be the Lie algebra of $H$. An inner product on $\mathfrak{h}$ defines a right invariant Riemannian metric on $H$ and therefore a Riemannian metric on $H/\Lambda$. For the case of $J$ actions, we have that the derivative action on $H/\Lambda$ is given by $D\rho(j)([h],v) = ([\pi_0(j)h, \mathrm{Ad}_H(\pi_0(j))v)$ for $j \in J$. Since $\mathrm{Ad}_H \circ\pi_0|_{\mathfrak{z}}$ is the trivial representation of $J$, it is clear that any inner product on $\mathfrak{h}$ defines a Riemannian metric with the desired property. Let $\mathfrak{m}$ be an $\mathrm{Ad}(\pi_0(J))|_{\mathfrak{h}}$ invariant complement to $\mathfrak{z}$. For simplicity in arguments below, we choose a metric on $\mathfrak{h}$ such that $\mathfrak{m}$ is orthogonal to $\mathfrak{z}$.

For $\Gamma$ actions, we need to be slightly more careful. Let $\mathfrak{l}$ be the Lie algebra of $L$, which contains $\mathfrak{h}$ as an ideal. We denote by $\mathrm{Ad}_{\mathfrak{h}}$ the restriction of the adjoint action of $L$ on $\mathfrak{l}$ to $\mathfrak{h}$. Recall that $\pi^K(\Gamma') < C$ where $C < L$ is compact and take an $\mathrm{Ad}_{\mathfrak{h}}(C)$ invariant metric on $\mathfrak{h}$. This then defines a Riemannian metric on $H/\Lambda\times M$ for which $\rho(\Gamma')$ is isometric along the central foliation defined above.

We perform one further modification to the metric to guarantee that the action of all of $\Gamma$, and not just $\Gamma'$, is isometric along the central foliation. Since the image of $\pi_0$ of $\Gamma'$ is semisimple, we can choose an $\mathrm{Ad}_{\mathfrak{h}}(\pi_0(\Gamma'))$ invariant subspace $\mathfrak{m} < \mathfrak{h}$ orthogonal to $\mathfrak{z}$ such that $\mathfrak{h} = \mathfrak{z}\oplus\mathfrak{m}$, and we can choose an inner product on $\mathfrak{h}$ for which $\mathfrak{z}$ is orthogonal to $\mathfrak{m}$. There are corresponding subbundles of $T(H/\Lambda)$ which we can write as $H/\Lambda\times\mathfrak{m}$ and $H/\Lambda\times\mathfrak{z}$. Note that because these bundles are $\Gamma'$ invariant and $\Gamma'$ can be chosen to be normal in $\Gamma$, they are also $\Gamma$ invariant. Given a vector space $V$, denote by $S^2_+(V)$ the cone of positive definite symmetric two tensors on $V$. The Riemannian metric on $H/\Lambda$ is a section $g_{\mathfrak{h}}$ of $H/\Lambda\times S^2_+(\mathfrak{h})$ which lies in the subbundle given by $H/\Lambda\times S^2_+(\mathfrak{z})\oplus S^2_+(\mathfrak{m})$ or equivalently as the sum in $H/\Lambda\times S^2_+(\mathfrak{h})$ of a section of $g_{\mathfrak{z}} \in H/\Lambda\times S^2_+(\mathfrak{z})$ and a section of $g_{\mathfrak{m}} \in H/\Lambda\times S^2_+(\mathfrak{m})$. Since $g_{\mathfrak{z}}$ is $\Gamma'$ invariant, and $\Gamma' < \Gamma$ is a finite index, we can average $g_{\mathfrak{z}}$ over coset representatives for $\Gamma/\Gamma'$ to obtain a $\Gamma$ invariant section $g'_{\mathfrak{z}}$ in $H/\Lambda\times S^2_+(\mathfrak{z})$. Replacing $g_{\mathfrak{h}}$ by $g'_{\mathfrak{z}}\oplus g_{\mathfrak{m}}$ we have a Riemannian metric on $H/\Lambda$ such that the entire $\Gamma$ action is isometric along the central foliation.

2.2. *Semiconjugacy.* Let $H, \Lambda, \Gamma$ and $J$ be as in the preceding subsection, let $G = J$ or $\Gamma$ and let $\rho$ be a quasi-affine action of $G$ on $H/\Lambda\times M$. Then by Theorems 2.1 and 2.3 there is a finite index subgroup $G' < G$ such that $\rho|_{G'}$ is defined by a continuous homomorphism $\pi : G'\to \mathrm{Aut}(H)\ltimes H$. It follows that the $G'$ action lifts to $H\times M$. As explained in [FM03], following the statement of Theorem 6.7, for any small enough $C^0$ perturbation $\rho'$ of $\rho$, the $G'$ action defined by $\rho'$ also lifts to $H\times M$. (We note that this is trivially true for actions of connected groups.)

By the discussion in Section 2.1 there is a unique subgroup $Z$ in $H$ which is the maximal subgroup of $H$ such that the derivative of $\rho$ on $Z$ cosets is an isometry for an appropriate choice of metric on $H/\Lambda$. The description given there shows that the lift of $\rho(G')$ to $H\times M$ descends to an action $\bar{\rho}$ of $G'$ on $Z\backslash H$. We denote by $p$ the natural projection $H\times M\to Z\backslash H$.

THEOREM 2.4. *Let $H/\Lambda\times M, \rho, G', Z$ and $\bar{\rho}$ be as in the preceding paragraph. Given any action $\rho'$ sufficiently $C^1$ close to $\rho$, there is a continuous $G'\times\Lambda$ equivariant map $f:(H\times M,\rho')\to(Z\backslash H,\bar{\rho})$, and $f$ is $C^0$ close to $p$. Furthermore if $\rho'\to\rho$ in the $C^1$ topology then $f\to p$ in the $C^0$ topology.*

This is [FM03, Th. 1.8]. We note that $f\to p$ in the $C^0$ topology means that $d(f(x),p(x))\to 0$ uniformly on $H\times M$. There is some ambiguity in this, since there is no $\Lambda$ invariant metric on $Z\backslash H$, but it is true that $d(f(x),p(x))\to 0$ in the metric on $Z\backslash H$ which makes $p$ a Riemannian submersion. For the remainder of this subsection, we assume that $G'=G$.

The map $f$ defines a partition $\tilde{\mathfrak{F}}'$ of $H\times M$ into sets of the form $f^{-1}(x)$ where $x$ is in $Z\backslash H$. Since $f$ is $\Lambda$ equivariant, this partition defines a partition $\mathfrak{F}'$ of $H/\Lambda\times M$. We will show that, as a consequence of Theorem 2.4 there is a $\Lambda$ equivariant map $\theta: H\times M\to H\times M$ mapping $\mathfrak{F}'$ to $\mathfrak{F}$ and intertwining the actions of $G$ on $\mathfrak{F}$ and $\mathfrak{F}'$, but first we need some definitions.

If $\mathcal{P}$ is a partition of a topological space $X$ and $\rho$ is an action of a group $D$ on $X$, then we say $\rho$ preserves $\mathcal{P}$, if for any set $V\in\mathcal{P}$, the set $\rho(d)V$ is in $\mathcal{P}$ for any $d$ in $D$.

Given two actions $\rho$ and $\rho'$ of a group $D$ on a topological space $X$ and two partitions $\mathcal{P}$ and $\mathcal{P}'$ of the space $X$ where $\rho$ preserves $\mathcal{P}$ and $\rho'$ preserves $\mathcal{P}'$, we call a map $\phi: X\to X$ a *partition semi-conjugacy from* $(X,\rho,\mathcal{P})$ *to* $(X,\rho'\mathcal{P}')$ if for any subset $V\in\mathcal{P}$ we have

(1) $\phi(V)$ is an element of the partition $\mathcal{P}'$ and,

(2) $\phi(\rho(d)V)=\rho'(d)\phi(V)$ for any $d$ in $D$.

If $\phi$ is a homeomorphism, we call $\phi$ a *partition conjugacy between* $(X,\rho,\mathcal{P})$ *and* $(X,\rho'\mathcal{P}')$. Similarly one can refer to actions as being *partition* (*semi-*)*conjugate*.

To be consistent with the vocabulary of [HPS77], when we are given two actions $\rho$ and $\rho'$ of a group $D$ on a topological space $X$ where $\rho$ (resp. $\rho'$) preserves a foliation $\mathfrak{F}$ of $X$ (resp. a foliation $\mathfrak{F}'$ of $X$), a partition (semi-)conjugacy from $(X,\rho,\mathfrak{F})$ to $(X,\rho',\mathfrak{F}')$ will be called a *leaf* (*semi-*)*conjugacy*. Similarly, when we do not want to make explicit reference to the (semi-)conjugacy, we will say that two actions are *leaf* (*semi-*)*conjugate*.

We now construct a map $\tilde{\theta}: H\times M\to H\times M$ using $f$ and $p$. The space $H\times M$ is a smooth locally trivial fiber bundle over $Z\backslash H$ with fiber $Z\times M$; so given

$x \in Z\backslash H$, we can find a neighborhood $U$ of $x$ such that $p^{-1}(U)$ is diffeomorphic to $U \times Z \times M$. We can therefore introduce coordinates on $p^{-1}(U)$ of the form $(x, y)$ where $x$ is in $U$ and $y \in Z \times M$. In these coordinates, $p(x, y) = x$. We can further write $y = (z, m)$ where $z \in Z$ and $m \in M$.

Moreover, if we let $\mathfrak{m}$ be the $G$ invariant complement of $\mathfrak{z}$ in $\mathfrak{h}$, then the tangent space to $H \times M$ at any point $(x, z, m)$ can be written as $T(H \times M)_{(x,z,m)} = \mathfrak{m} \oplus \mathfrak{z} \oplus TM_m$. We can further choose the local product structure on $p^{-1}(U)$ such that $(U, y) = \exp_y W$ where $W$ is the product of a fixed small ball in $\mathfrak{m}$ with a small ball in $TM_m$. By shrinking $W$ and therefore $U$ slightly, we obtain a trivialization of $p^{-1}(U)$ that extends to a trivialization of $p^{-1}(U')$ for $U'$ an open set strictly containing $U$.

By choosing $\rho'$ close enough to $\rho$, we can arrange for $f$ to be arbitrarily $C^0$ close to $p$, uniformly on $H \times M$. This implies that given any compact set $K$ in $H \times M$, by restricting to sufficiently small $C^1$ perturbations $\rho'$ of $\rho$, we can make the Hausdorff distance between $f^{-1}(x) \cap K$ and $p^{-1}(x) \cap K$ as small as desired for every $x$ in $f(K) \cap p(K)$. Since $f$ and $p$ are $\Lambda$ equivariant and the $\Lambda$ action on $H \times M$ is cocompact, for small enough perturbations we have $f^{-1}(x) \subset p^{-1}(U)$. Then for a point $(x, y)$, we let $\tilde{\theta}(x, y) = (U, y) \cap p^{-1}(f(x, y))$. Therefore $\tilde{\theta}(f^{-1}(x)) = p^{-1}(x)$ for any $x \in Z\backslash H$. Since $\tilde{\theta}$ is $\Lambda$ equivariant by construction, we have a map $\theta : H/\Lambda \times M \to H/\Lambda \times M$ and have established the following:

COROLLARY 2.5. *Let $\rho'$ be an action of $G$ on $H/\Lambda \times M$ sufficiently $C^1$ close to $\rho$. Then there is a $C^0$ small map $\theta : H/\Lambda \times M \to H/\Lambda \times M$ with the following properties.*

(1) *If $\rho' \to \rho$ in the $C^1$ topology then $\theta \to \mathrm{Id}$ in the $C^0$ topology.*

(2) $\theta(\mathfrak{F}') = \mathfrak{F}$.

(3) *the map $\theta$ is a partition semi-conjugacy from*

$$(H/\Lambda \times M, \rho, \mathfrak{F}') \quad \textit{to} (H/\Lambda \times M, \rho, \mathfrak{F}).$$

*Remarks.*

(1) One can deduce Corollary 2.5 directly from the proof of Theorem 2.4.

(2) The argument there proves more. It shows that the set of maps $\{\theta \circ \rho'(g) | g \in G\}$ defines a $G$ action on $H/\Lambda \times M$ that is $C^0$ close to $\rho$.

(3) Remark (2) can be deduced easily from [FM03, Th. 6.7], but to avoid introducing additional notation and definitions, we do not do this here.

(4) The conclusion of Remark (2) will follow once we show, in Section 3.3, that $\theta$ is a $C^0$ small homeomorphism.

## 3. **Hyperbolic dynamics and stability modulo central directions**

In this section, we show that the map $\theta : H/\Lambda\times M\to H/\Lambda\times M$ defined in the last subsection is a homeomorphism. Since $\theta(\mathfrak{F}') = \mathfrak{F}$, this implies that $\mathfrak{F}'$ is a foliation. We further show that $\mathfrak{F}'$ is a foliation by $C^r$ leaves where $r$ depends only on the $C^1$ size of the perturbation. The map $\theta$ is then easily seen to be $C^r$ along leaves of $\mathfrak{F}$. For technical reasons involving the last steps of our proof, once we have shown that $\theta$ is a homeomorphism, we prefer to work with $\phi = \theta^{-1}$.

We now briefly outline the argument of the section. Some of the terminology used here is defined below in Section 3.1. First in Section 3.1 we recall some definitions concerning partially hyperbolic diffeomorphisms and a theorem of Hirsch, Pugh and Shub. In Section 3.2, we prove some basic facts concerning the dynamics of the affine actions of $G$ that we are considering and produce a finite subset $\Phi$ of $G$ such that the intersections of the central foliations of $\rho(g)$ for $g$ in $\Phi$ is the central foliation for $G$ defined above in Section 2.1. In Section 3.3, we show that the map $\theta : H/\Lambda\times M\to H/\Lambda\times M$ defined in Section 2.2 is a homeomorphism and we let $\phi = \theta^{-1}$. It then follows that $\mathfrak{F}' = \phi(\mathfrak{F})$ is $\rho'$ invariant foliation. Finally in Section 3.4, we show that any leaf $\mathfrak{L}$ of $\mathfrak{F}'$ is the transverse intersection of central leaves $\mathcal{W}^c_{\rho'(g)}$ for the diffeomorphisms $\rho'(g)$ where $g$ is in $\Phi$. Since the theorem of Hirsch, Pugh and Shub implies that each foliation $\mathcal{W}^c_{\rho'(g)}$ is by $C^r$ leaves, it follows that $\mathfrak{F}'$ is a foliation by $C^r$ leaves, where $r$ depends only on the $C^1$ size of the perturbation $\rho'$.

3.1. *Hyperbolic dynamics and foliations.* The use of the word *foliation* varies with context. Here a *foliation by $C^k$ leaves* will be a continuous foliation whose leaves are $C^k$ injectively immersed submanifolds that vary continuously in the $C^k$ topology in the transverse direction. To specify transverse regularity we will say that a foliation is transversely $C^r$. A foliation by $C^k$ leaves which is transversely $C^k$ is called simply a $C^k$ foliation. Note our language does not agree with that in the reference [HPS77] where our foliation by $C^k$ leaves is a $C^k$ unbranched lamination and sometimes a $C^k$ injective leaf immersion. Given a foliation $\mathfrak{F}$, we denote the leaf through a point $x$ by $\mathfrak{F}(x)$.

Given a foliation by $C^1$ leaves, $\mathfrak{F}$, of a manifold $X$, a diffeomorphism $f$ is said to be *$r$-normally hyperbolic* to the foliation $\mathfrak{F}$ if there exists a continuous $f$ invariant splitting $TX = E^u_f\oplus T\mathfrak{F}\oplus E^s_f$ such that for every $x\in X$,

(1) $\|Df_x|_{E^u_f}{}^{-1}\|^{-1} > \|Df_x|_{\mathfrak{F}}\|^r$ and,

(2) $\|Df_x|_{E^s_f}\| < \|Df_x|_{\mathfrak{F}}{}^{-1}\|^{-r}$.

For any invariant subbundle $V$ of $TX$ and any fixed Riemannian metric on $X$, the norm above is the operator norm of $Df_x|_{V_x}$. See [HPS77, Chap. 1] for a more detailed discussion of normal hyperbolicity. There, $r$-normally hyperbolic is also called "immediately, relatively $r$-normally hyperbolic." Also the definition given there is slightly different, and applies also to noninvertible $f$. That the definitions are equivalent for $f$ invertible is the content of the remark following Definition 1 in the introduction to [HPS77].

We note that $f$ being $r$-normally hyperbolic to $\mathfrak{F}$ is equivalent to the existence of constants $a, b > 1$ with $a > b^r$ and a continuous $f$ invariant splitting $TX = E^u_f \oplus T\mathfrak{F} \oplus E^s_f$ such that

(1) $\|Df^n(v^u)\| \geq a^n \|v^u\|$ for all $v^u \in E^u_f$,
(2) $\|Df^n(v^s)\| \leq a^{-n} \|v^s\|$ for all $v^s \in E^s_f$ and
(3) $b^{-n}\|v^0\| < \|Df^n(v^0)\| \leq b^n \|v^0\|$ for all $v^0 \in T\mathfrak{F}$ and all integers $n$.

The definition of $r$-normally hyperbolic is motivated by the theory of partially hyperbolic diffeomorphisms. Given an automorphism $f$ of a vector bundle $E \to X$ and constants $a > b \geq 1$, we say $f$ is $(a, b)$-*partially hyperbolic* or simply *partially hyperbolic* if there is a metric on $E$ and a constant and $C \geq 1$ a continuous $f$ invariant nontrivial splitting $E = E^u_f \oplus E^c_f \oplus E^s_f$ such that:

(1) $\|f^n(v^u)\| \geq Ca^n \|v^u\|$ for all $v^u \in E^u_f$,
(2) $\|f^n(v^s)\| \leq C^{-1} a^{-n} \|v^s\|$ for all $v^s \in E^s_f$ and
(3) $C^{-1} b^{-n} \|v^0\| < \|f^n(v^0)\| \leq Cb^n \|v^0\|$ for all $v^0 \in E^c_f$ and all integers $n$.

A $C^1$ diffeomorphism $f$ of a manifold $X$ is $(a, b)$-*partially hyperbolic* if the derivative action $Df$ is $(a, b)$-partially hyperbolic on $TX$. For any partially hyperbolic diffeomorphism, there always exists an *adapted metric* for which $C = 1$. Note that $E^c_f$ is called the *central distribution* of $f$, $E^u_f$ is called the *unstable distribution* of $f$ and $E^s_f$ the *stable distribution* of $f$. We will also refer to the sums $E^{\mathrm{cu}}_f = E^u_f \oplus E^c_f$ and $E^{\mathrm{cs}}_f = E^s_f \oplus E^c_f$ as the weak unstable and weak stable distributions, respectively.

Integrability of various distributions for partially hyperbolic dynamical systems is the subject of much research. The stable and unstable distributions are always tangent to invariant foliations which we call the stable and unstable foliations and denote by $\mathcal{W}^s_f$ and $\mathcal{W}^u_f$. If the central distribution is tangent to an $f$ invariant foliation, we call that foliation a *central foliation* and denote it by $\mathcal{W}^c_f$. If there is a unique foliation tangent to the central distribution we call the central distribution *uniquely integrable*. For smooth distributions unique integrability is a consequence of integrability, but the central distribution is usually not smooth. For general partially hyperbolic diffeomorphisms, unique integrability of central foliations is difficult to establish. If the central distribution of an $(a, b)$-partially hyperbolic diffeomorphism $f$ is tangent to an invariant foliation $\mathcal{W}^c_f$, then $f$ is $r$-normally hyperbolic to $\mathcal{W}^c_f$ for any $r$ such that $a > b^r$.

Given $C^k$ partially hyperbolic diffeomorphism $f$ which is $l$-normally hyperbolic to a central foliation, for $k, l \geq 1$, it follows from [HPS77, Th. 6.8] that there are foliations tangent to the weak (un)stable distribution, which we call the *weak (un)stable foliation* and denote by $\mathcal{W}_f^{\mathrm{cu}}$ and $\mathcal{W}_f^{\mathrm{cs}}$.

In Section 3.4, we need to use the work of Hirsch-Pugh-Shub on perturbations of partially hyperbolic actions of $\mathbb{Z}$. We state a special case of some of their results from [HPS77].

THEOREM 3.1. *Let $f$ be an $(a,b)$-partially hyperbolic $C^k$ diffeomorphism of a compact manifold $M$ which is $k$-normally hyperbolic to a $C^k$ central foliation $\mathcal{W}_f^c$. Then for any $\delta > 0$, if $f'$ is a $C^k$ diffeomorphism of $M$ which is sufficiently $C^1$ close to $f$ we have the following*:

(1) *$f'$ is $(a', b')$-partially hyperbolic, where $|a - a'| < \delta$ and $|b - b'| < \delta$, and the splitting $TM = E^u_{f'} \oplus E^c_{f'} \oplus E^s_{f'}$ for $f'$ is $C^0$ close to the splitting for $f$*;

(2) *There exist $f'$ invariant foliations by $C^k$ leaves $\mathcal{W}^{\mathrm{cs}}_{f'}$ tangent to $E^c_{f'} \oplus E^s_{f'}$, $\mathcal{W}^{\mathrm{cu}}_{f'}$ tangent to $E^c_{f'} \oplus E^u_{f'}$, $\mathcal{W}^s_{f'}$ tangent to $E^s_{f'}$, $\mathcal{W}^u_{f'}$ tangent to $E^u_{f'}$ and $\mathcal{W}^c_{f'}$ tangent to $E^c_{f'}$, and each foliation is close in the natural topology on foliations by $C^k$ leaves to the corresponding foliation for $f$.*

Statement (1) is standard. Statement (2) follows from [HPS77, Th. 6.1, statement f]; see also Theorem 6.8 of that book for more details. (The exact results in [HPS77] are more general.)

3.2. *Dynamics of affine actions.* For the remainder of this section the group $G$ is either a connected, simply connected, semisimple Lie group, $J$, with all simple factors of real rank at least two or a lattice $\Gamma < J$. We fix a manifold $M$, a real algebraic group $H$, and a cocompact lattice $\Lambda < H$ and fix a quasi-affine action $\rho$ of $G$ on $H/\Lambda \times M$. We recall from Section 2.1 that there is a finite index subgroup $G' < G$ such that $\rho$ is defined by a homomorphism $\pi : G' \to < \mathrm{Aut}(H) \ltimes H$ where $\pi = \pi^E \pi^K$, where $\pi^E$ is a continuous homomorphism of $J$, the homomorphism $\pi^K$ has bounded image, and the images of $\pi^E$ and $\pi^K$ commute. For the remainder of this section, we assume that $G' = G$. As above we let $L$ be the product of the Zariski closure of $\pi(G')$ and $H$ and let $Z = Z_L(\pi^E(J)) \cap H$ and $\mathfrak{z}$ be the Lie algebra of $Z$. (Note that when $G = J$, $L$ is always just $H$.) According to the discussion in Section 2.1, we can fix a Riemannian metric on $H/\Lambda \times M$ such that $\rho(g)$ is an isometry of the metric restricted to the tangent space of $\mathfrak{F}$. Given $g \in G$ there is a natural choice of $\rho(g)$ invariant sub-bundles of $T(H/\Lambda)$ with respect to which $\rho(g)$ is partially hyperbolic whenever $\mathrm{Ad}(\pi(g))|_{\mathfrak{h}}$ has an eigenvalue off the unit circle. We first describe the case of affine actions. Writing coordinates on $T(H/\Lambda) \cong H/\Lambda \times \mathfrak{h}$ as $([h], v)$ with $[h]$ in $H/\Lambda$ and $v \in \mathfrak{h}$, we have $D\rho(g) = (\rho(g)([h]), \mathrm{Ad}_{\mathfrak{h}}(\pi(g))v)$. We let $\mathfrak{f}^s_{\rho(g)}$ (resp. $\mathfrak{f}^u_{\rho(g)}$) be the subspace of $\mathfrak{h}$ for which $\mathrm{Ad}(\pi(g))$ has all eigenvalues of modulus less than one (resp. all eigenvalues of modulus greater than one) and $\mathfrak{f}^c_{\rho(g)}$ be the subspace of $\mathfrak{h}$ where $\mathrm{Ad}_{\mathfrak{f}}(\pi(g))$ has all eigenvalues of modulus one. We can then define sub-bundles of $T(H/\Lambda)$ as $E^s_{\rho(g)} = H/\Lambda \times \mathfrak{f}^s_{(\rho(g))}$, $E^u_{\rho(g)} = H/\Lambda \times \mathfrak{f}^u_{(\rho(g))}$ and $E^c_{\rho(g)} = H/\Lambda \times \mathfrak{f}^c_{(\rho(g))}$. It is straightforward to verify that $\rho(g)$ is partially hyperbolic with respect to this splitting whenever this splitting is nontrivial. For the remainder of this paper, whenever we refer to $\rho(g)$ as a partially hyperbolic diffeomorphism, we mean partially hyperbolic with respect to this choice of splitting. We collect here some basic consequences for the dynamics of the action $\rho$.

PROPOSITION 3.2. *For any affine action $\rho$ of $G$ on $H/\Lambda$ and any $g \in G$ there are Lie subgroups $F^s_{\rho(g)}$, $F^u_{\rho(g)}$ and $F^c_{\rho(g)}$ in $H$ such that the foliations $\mathcal{W}^s_{\rho(g)}$, $\mathcal{W}^u_{\rho(g)}$ and $\mathcal{W}^c_{\rho(g)}$ consist of orbits of the corresponding group acting on the left on $H/\Lambda$. Furthermore*

(1) *the groups $F^s_{\rho(g)}$ and $F^u_{\rho(g)}$ are nilpotent,*
(2) *$Z < F^c_{\rho(g)}$ and $Z \cap F^s_{\rho(g)} = Z \cap F^u_{\rho(g)} = 1$,*
(3) *for every point in $H/\Lambda$ the orbit maps for $F^s_{\rho(g)}$ and $F^u_{\rho(g)}$ are injective immersions.*

*Proof.* That $F^s_{\rho(g)}$, $F^u_{\rho(g)}$ and $F^c_{\rho(g)}$ are subgroups, as well as claims 1 and 2 are consequences of the fact $\mathfrak{f}^s_{\rho(g)}$, $\mathfrak{f}^u_{\rho(g)}$ and $\mathfrak{f}^c_{\rho(g)}$ are Lie subalgebras of $\mathfrak{h}$. This is true since if $v$ and $w$ are eigenvectors of $\mathrm{Ad}_L \circ \pi_0|_{\mathfrak{h}}(g)$ with eigenvalues $\lambda$ and $\mu$, then $[v, w]$ is an eigenvector with eigenvalue $\lambda\mu$.

We prove (3) for $\mathcal{W}^s_{\rho(g)}$; the proof is identical for $\mathcal{W}^u_{\rho(g)}$. Assume (3) is false, then there is an element of $f \in \Lambda \cap h^{-1} F^s_{\rho(g)} h$. Since $f$ is in $\Lambda$, $f$ is an element of $\pi_1(H/\Lambda)$, which we can represent by a curve $\bar{f}$ lying entirely in $h^{-1} F^s_{\rho(g)} h$. Since $\rho(g)$ is a contraction on $F^s_{\rho(g)}$ and therefore $h^{-1} F^s_{\rho(g)} h$, for some large $n$, the curve $\rho^n(g)\bar{f}$ is small and therefore contractible, a contradiction, since $\rho^n(g)$ is a diffeomorphism and $\bar{f}$ is not contractible in $H/\Lambda$. □

We now discuss the case of a quasi-affine action $\rho$. We denote by $\hat{\rho}$ the affine action of $G$ from which $\rho$ is defined. We can define a splitting of $T(H/\Lambda \times M)$ as $E^s_{\rho(g)} = H/\Lambda \times \mathfrak{f}^s_{(\hat{\rho}(g))}$, $H/\Lambda \times E^u_{\hat{\rho}(g)} = \mathfrak{f}^u_{(\hat{\rho}(g))}$ and $E^c_{\hat{\rho}(g)} = H/\Lambda \times \mathfrak{f}^c_{(\hat{\rho}(g))} \times TM$. Again it is easy to see that $\rho(g)$ is partially hyperbolic with respect to this splitting whenever this splitting is nontrivial. These sub-bundles are tangent to foliations where $\mathcal{W}^s_{\rho(g)}$ consists of $F^s_{\hat{\rho}(g)}$ orbits, $\mathcal{W}^u_{\rho(g)}$ consists of $F^u_{\hat{\rho}(g)}$ orbits, and $\mathcal{W}^c_{\rho(g)}$ consists of products of $F^c_{\hat{\rho}(g)}$ orbits with $M$. It follows that all dynamical foliations for any quasi-affine action are smooth.

We define $E^0_\rho(G)$ to be the distribution $H/\Lambda \times \mathfrak{z} \times TM$ which is tangent to the foliation $\mathfrak{F}$. We state a lemma here which says that there are finitely many elements in the acting group the sum of whose (un)stable directions is the complement of

$E^0_\rho(G)$ and that, therefore, the intersection of their central distributions is exactly $E^0_\rho(G)$.

LEMMA 3.3. *There exits a finite set $\Phi$ of elements in $G$ such that*

$$T(H/\Lambda\times M) = H/\Lambda\times\Big(\sum_{g\in\Phi} E^s_{\rho(g)}\Big)\times\mathfrak{z}\times TM.$$

*Proof.* The derivative of $\rho$ on $T(H/\Lambda\times M)$ leaves invariant $TM$ and $\mathfrak{h}$. We let Ad be the adjoint representation of $L$. It follows from the description of affine actions in Section 2.1, that $\mathfrak{z}$ is invariant under $\mathrm{Ad}\,|_{\mathfrak{h}}(\pi(G))$, that there is an $\mathrm{Ad}\,|_{\mathfrak{h}}(\pi(G))$ invariant complement $\mathfrak{m}$ to $\mathfrak{z}$ and that $\mathrm{Ad}\,|_{\mathfrak{m}}(\pi(g))$ is $\mathrm{Ad}\,|_{\mathfrak{m}}(\pi^E(g)\pi^K(g))$ where the $J$ representation $\pi^E$ does not contain the trivial representation.

Recall from [FM03, §3.8] that for any element $g$ of $\mathrm{GL}_n(\mathbb{R})$, there is a unique decomposition of $g=us=su$ where $u$ is unipotent and $s$ is semisimple. Further, we have a unique decomposition $s=cp=pc$ where all eigenvalues of $p$ are positive and all eigenvalues of $c$ have modulus one. We refer to $p$ as the *polar part* of $g$ and denote it by $\mathrm{pol}(g)$. As remarked there, one can define the polar part of an element for elements of any real algebraic group and this definition is independent of the realization of the group as an algebraic group.

By [FM03, Lemma 3.21] there is a finite collection $\Psi$ of elements in $G'$ whose polar parts are Zariski dense in $J$. Combined with the fact that the representation $\mathrm{Ad}\circ\pi^E|_{\mathfrak{m}}$ of $J$ does not contain invariant vectors, this implies that $\mathfrak{m}\cap\bigcap\mathfrak{f}^c_{\rho(g)}=0$. Letting $\Phi=\Psi\cap\Psi^{-1}$ completes the proof. □

As above, we let $\bar{\rho}$ be the action on $Z\backslash H$ defined by lifting $\rho$ to an action on $H\times M$ and looking at the action on the leaves of the central foliation there.

To describe some further properties of the dynamics, we recall the local product structure on $H\times M$ as a bundle over $Z\backslash H$ from Section 2.2. Recall that the sub-bundle $H\times M\times\mathfrak{m}$ of $T(H\times M)$ is a $G$ invariant complement to $H\times\mathfrak{z}\times TM$. Letting exp be the exponential map for our fixed metric on $H\times M$ and letting $B_{\mathfrak{m}}(0,\varepsilon)$ be the ball of radius $\varepsilon$ in $\mathfrak{m}$, by choosing $\varepsilon$ small enough, we can guarantee that $\exp_x(B_{\mathfrak{m}}(0,\varepsilon))$ defines a family of manifolds transverse to the fibers of $p$. Furthermore if we write $W_x=\exp_x(B_{\mathfrak{m}}(0,\varepsilon))$ then, for small enough $\varepsilon$, we have a local product structure on $H\times M$ given by

$$p^{-1}(W_x)\cong B_{\mathfrak{m}}(0,\varepsilon)\times p^{-1}(x)\cong B_{\mathfrak{m}}(0,\varepsilon)\times(Z\times M).$$

We define a Riemannian metric on $Z\backslash H$ so that the map $p: H\times M\to Z\backslash H$ is a Riemannian submersion. The next lemma says that for any small enough perturbation $\rho'$ of $\rho$, points on the same transversal to $p^{-1}(x)$ can be moved apart.

LEMMA 3.4. *There exists $c > 0$ depending only on $\rho$ such that for $x$ in $H \times M$ and $y, z \in W_x$, there are $g \in \Phi$ and a nonnegative integer $n$ such that $d(\rho'(g)^n z, \rho'(g)^n y) > c$.*

*Proof.* As this is a special case of a fairly standard fact from (partially) hyperbolic dynamics, we merely sketch the proof. It suffices to consider points $y \in W_x$ with $d(x, y) < c$, since otherwise the lemma is true for any $g \in \Phi$ and $n = 0$. We assume $c$ is small enough that $B(x, c)$ is a convex, normal neighborhood of $x$. Therefore the geodesic between $z$ and $y$ is entirely contained in $B(x, c)$ and we can pull the geodesic back to the $T_x(H/\Lambda \times M)$ where it can be approximated to first order by a segment on a straight line which we denote by $V_{z,y}$.

Since $W_x$ is transverse to $\mathfrak{F}$, by Lemma 3.3, we can choose $g \in \Phi$ such that the angle between $V_{z,y}$ and $\mathcal{W}^{cs}_{\rho'(g)}(x)$ is bounded away from zero for all sufficiently small $\rho'$. The dynamics of $\rho'(g)$ then force the angle between $D\rho'(g)^n V_{z,y}$ and $\mathcal{W}^{cs}_{\rho'(g)}(\rho'(g)^n x)$ to be uniformly bounded away from zero. This implies that $\|D\rho'(g)^n V_{z,y}\|$ grows at an exponential rate controlled by the uniform lower bound on the angle, on the constants $a, b$ for which $\rho(g)$ is $(a, b)$-partially hyperbolic and the $C^1$ size of the perturbation $\rho'$. When $c$ is small enough, the first order behavior of $d(\rho'(g)^n z, \rho'(g)^n y)$ is given by $\|D\rho'(g)^n V_{z,y}\|$, so that, possibly after shrinking $c$ again, we can assume when $\|D\rho'(g)^n V_{z,y}\| > 2c$ that $d(\rho'(g)^n x, \rho'(g)^n y) > c$. □

COROLLARY 3.5. *There is a constant $c$ depending only on $\rho$ such that for any $x, y \in H \times M$ with $p(x) \neq p(y)$ and any $\rho'$ sufficiently $C^1$ close to $\rho$, there exist $g \in \Phi$ and a nonnegative integer $n$ such that*

$$d(p(\rho(g)^n(x)), p(\rho(g)^n(y))) > c.$$

*Proof.* Since we are only concerned with the distance between projections under $p$, to prove the corollary, it suffices to consider $y \in W_x$. This case is immediate from Lemma 3.4. □

3.3. *Fiber structure.* Throughout this subsection, we keep the notation and assumptions of Section 3.2. Here, we show that the map $\theta$ is a homeomorphism. As an immediate consequence, $\mathfrak{F}'$ is a foliation and $\theta$ is a leaf conjugacy between $(H/\Lambda \times M, \rho', \mathfrak{F}')$ and $(H/\Lambda \times M, \rho, \mathfrak{F})$.

THEOREM 3.6. *The map $\theta : H/\Lambda \times M \to H/\Lambda \times M$ defined in Section 2.2 is a homeomorphism.*

Recall that our assumptions, stated in Section 3.2, imply that the $G$ action on $H/\Lambda \times M$ defined by $\rho$ lifts to $H \times M$. As remarked in Section 2.2, if $\rho$ lifts to $H \times M$ then all small enough perturbations of $\rho$ also lift to $H \times M$.

*Proof.* We show that $\theta$ is a homeomorphism by showing it is a homeomorphism on each fiber $f^{-1}(x)$.

We first show $\theta|_{f^{-1}(x)}$ is injective by a dynamical argument. Assume $\theta|_{f^{-1}(x)}$ is not injective, then there are two distinct points, $w, z \in f^{-1}(x)$ such that $\theta(z) = \theta(w)$. This forces $p(z) \neq p(w)$ since otherwise $z = w$. Since, $w, z \in W_x$ by definition, it follows from Lemma 3.3 that for some $g \in G$ the distance between $d(p(\rho'(g)(z)), p(\rho'(g)w))$ is greater than some constant $c$ depending only on $\rho$. Since $f$ can be made arbitrarily $C^0$ close to $p$ by restriction to small enough $C^1$ perturbations $\rho'$ of $\rho$, we know that $d(f(\rho'(g)z), f(\rho'(g)w) \geq \frac{c}{2}$ so that $f(\rho'(g)z)$ and $f(\rho'(g)w)$ are distinct. By equivariance of $f$ we have $\bar{\rho}(g)f(z) \neq \bar{\rho}(g)f(w)$ which implies that $f(z) \neq f(w)$, a contradiction.

We now show that $\theta|_{f^{-1}(x)}$ is surjective. Let $\pi_{U'}$ be composition of the restriction of $\theta$ to $p^{-1}(U)$ composed with projection on the second coordinate. We can then look at the set $V_y = {\pi_{U'}}^{-1}(y)$. Given $(x, y) \in p^{-1}(U)$, we show that the map $f : V_y \to U$ is onto. This has the desired implication, since if $\theta|_{f^{-1}(x)}$ is not surjective then there exists $y$ such that ${\pi_{U'}}^{-1}(y) \cap f^{-1}(x)$ is empty and therefore $x \notin f(V_y)$. Since $f$ is $C^0$ close to $p$, the map $\psi'_U$ is $C^0$ close to projection on the second coordinate. Therefore, after identifying $V_y$ with a subset of $(U', y)$ by a vertical projection, $f : V_y \to U'$ is $C^0$ close to the identity map. Our result now follows from the following general topological lemma.

LEMMA 3.7. *Let $B$ be the ball or radius $r$ about zero in a Euclidean space $E$. Let $F$ be any continuous map from $B$ into $E$ such that $d(F(x), x) < \varepsilon$ for all $x \in B$. Then $F(B)$ contains the ball of radius $r - 2\varepsilon$ about* 0.

*Proof.* This generalizes the key point in the proof of the Browder fixed point theorem: we assume the map is not surjective and use this to construct a deformation retract from a closed ball onto its boundary. Let $B'$ be the ball of radius $r - 2\varepsilon$ and $x$ a point in $B'$. Assume $x \notin F(B)$. Let $B_\varepsilon = B(x, 2\varepsilon)$ and look at $F(B_\varepsilon) \subset B$. Let $S^{k-1}$ be the boundary of $B(x, 2\varepsilon)$ Define a map from $B_\varepsilon$ to $S^{k-1}$ by taking $y \in B_\varepsilon$ to $F(y)$ and then projecting to $S^{k-1}$ along the ray from $x$ to $F(y)$. This gives a continuous map $\bar{F}$ from $B^k$ to $S^{k-1}$ which, when restricted to $S^{k-1}$ is $C^0$ close to the identity. The map $\bar{F}$ is $C^0$ close to the identity on $S^{k-1}$ and so is of degree one and homotopic to the identity. Therefore we can define a map from $B^k$ to $S^{k-1}$ which is the identity on $S^{k-1}$ as follows. Take $B^k$ and embed it in a larger closed ball $B^k_1$. Let $S^{k-1}$ denote the boundary of $B^k$ and $S^{k-1}_1$ denote the boundary of $B^k_1$. Our map is defined by first taking $\bar{F}$ to get a map from $B^k_1$ to $B^k_1 \backslash \mathrm{Int}(B^k)$ and then composing with the deformation retract from $B^k_1 \backslash \mathrm{Int}(B^k)$ to $S^{k-1}_1$ described by the homotopy from $\bar{F}$ to the identity. This then gives a new map $\bar{F}'$ which is a deformation retract from $B^k_1$ to $S^{k-1}_1$. This is impossible since $\pi_{k-1}(S^{k-1}) = \mathbb{Z}$ and $\pi_{k-1}(B) = 0$. Therefore $x \in F(B)$. $\square$

3.4. *The leaves of $\mathfrak{F}'$ are smooth.* This subsection is essentially a proof that $\mathfrak{F}'$ is a foliation by $C^k$ leaves, as defined at the beginning of Section 3.1. The degree of smoothness will depend on the $C^1$ size of the perturbation $\rho'$ or, more precisely, on the largest $r$ such that $\rho'(g)$ is $r$-normally hyperbolic for all $g \in \Phi$ where $\Phi$ is as in Lemma 3.3. For technical reasons both here and in later sections, we let $\psi = \theta^{-1}$ and work with $\psi$ rather than $\theta$.

THEOREM 3.8. *Given $k$, there is a neighborhood $U$ of $\rho$ in the space*

$$\mathrm{Hom}(G, \mathrm{Diff}^1(H/\Lambda \times M))$$

*such that for any $\rho' \in U$ that is a $C^k$ action the homeomorphism*

$$\psi : H/\Lambda \times M \to H/\Lambda \times M$$

*defined above has the following properties.*

(1) *The central foliation $\mathfrak{F}$ of $H/\Lambda \times M$ for the action $\rho$ is mapped by $\psi$ to a foliation $\mathfrak{F}'$ of $H/\Lambda \times M$ that is central for the action $\rho'$.*
(2) *The leaves of $\mathfrak{F}'$ are $C^k$ and $\psi$ is $C^k$ along leaves with $k$-jet depending continuously on $H/\Lambda \times M$.*
(3) *The homeomorphism $\psi$ is a leaf conjugacy between $(H/\Lambda \times M, \mathfrak{F}, \rho)$ and $(H/\Lambda \times M, \mathfrak{F}', \rho')$.*
(4) *The map $\psi$ is $C^0$ close to the identity and also $C^k$ small along leaves.*

The main point is the improvement in the regularity of leaves of $\mathfrak{F}'$, and all other conclusions follow quickly from this one. The key fact is:

LEMMA 3.9. *If $\psi$ is as defined above and $\rho(g)$ is partially hyperbolic, then* $\psi(\mathcal{W}^c_{\rho(g)}) = \mathcal{W}^c_{\rho'(g)}$, $\psi(\mathcal{W}^{\mathrm{cu}}_{\rho(g)}) = \mathcal{W}^{\mathrm{cu}}_{\rho'(g)}$ *and* $\psi(\mathcal{W}^{\mathrm{cs}}_{\rho(g)}) = \mathcal{W}^{\mathrm{cs}}_{\rho'(g)}$.

*Proof.* The proof proceeds in two steps. First we show that any leaf $\mathcal{V}'$ of $\mathcal{W}^{\mathrm{cs}}_{\rho'(g)}$ (resp. $\mathcal{W}^{\mathrm{cu}}_{\rho'(g)}$) is a union of leaves of $\mathfrak{F}'$. Second, we show that for any leaf $\mathcal{V}'$ of $\mathcal{W}^{\mathrm{cs}}_{\rho'(g)}$ (resp. of $\mathcal{W}^{\mathrm{cu}}_{\rho'(g)}$)there is a leaf $\mathcal{V}$ of $\mathcal{W}^{\mathrm{cs}}_{\rho(g)}$ (resp. of $\mathcal{W}^{\mathrm{cu}}_{\rho(g)}$) such that $\mathcal{V}' \subset \psi(\mathcal{V})$. Interchanging the roles of $\rho'$ and $\rho$ in the argument proves the reverse inclusion, forcing $\psi(\mathcal{V}) = \mathcal{V}'$. Since any leaf of $\mathcal{W}^c_{\rho(g)}$ is a transverse intersection of leaves of $\mathcal{W}^{\mathrm{cu}}_{\rho(g)}$ and $\mathcal{W}^{\mathrm{cs}}_{\rho(g)}$ and [HPS77, Th. 6.8] implies that any leaf of $\mathcal{W}^c_{\rho'(g)}$ is a transverse intersection of leaves of $\mathcal{W}^{\mathrm{cu}}_{\rho'(g)}$ and $\mathcal{W}^{\mathrm{cs}}_{\rho'(g)}$, this immediately implies that for any $\mathcal{V}'$ of $\mathcal{W}^c_{\rho'(g)}$, there is a leaf $\mathcal{V}$ of $\mathcal{W}^c_{\rho(g)}$ such that $\mathcal{V}' = \psi(\mathcal{V})$. To prove all of these statements, we will use the construction of leaves of $\mathcal{W}^{\mathrm{cs}}_{\rho'(g)}$ (resp. $\mathcal{W}^{\mathrm{cu}}_{\rho'(g)}$) from [HPS77], which we recall in the following paragraph.

Following [HPS77, §6], we pick a smooth local transversal $\eta$ to the tangent bundle $E^c_{\rho(g)}$ to the foliation $\mathcal{W}^c_{\rho(g)}$. As noted there, this can be chosen to be a smooth approximation to $E^s_{\rho(g)} \oplus E^u_{\rho(g)}$. Since in our setting, $E^s_{\rho(g)} \oplus E^u_{\rho(g)}$ is smooth, we let $\eta = E^s_{\rho(g)} \oplus E^u_{\rho(g)}$. We denote by $V$ the manifold which is the disjoint union of all leaves of $\mathcal{W}^c_{\rho(g)}$. Note that $V$ does not have a countable base and may not be separable, see [HPS77, Exs. 2 and 2′, p. 68] and following for related discussion. Let $i : V \to H/\Lambda \times M$ be the inclusion and pull $\eta$ back to a bundle $i^*\eta$. Note that there is a metric on $\eta$ and therefore $i^*\eta$ defined by our choice of Riemannian metric on $H/\Lambda \times M$ and let $i^*\eta(l)$ be the bundle of discs of radius $l$. Then as described in [HPS77], there are numbers $r > 0$ and $\varepsilon_0 > 0$ such that $\exp \circ i_* : i^*\eta(\varepsilon_0) \to H/\Lambda \times M$ is a diffeomorphism when restricted to $i^*\eta|_{B(x,r)}$ where $x \in V$ and $B(x, r)$ is a ball of radius $r$ in a leaf of $\mathcal{W}^c_{\rho(g)}$. (In [HPS77], the sets $B(x, r)$ are replaced by plaques of a plaquation of $V$. In our context, i.e. when the action of $\rho(g)$ is isometric along $V$, it is easy to see that one can find a plaquation by small enough balls.) Now for any $\varepsilon < \varepsilon_0$, we can pull back the action of $\rho(g)$ (resp. $\rho'(g)$) on $H/\Lambda \times M$ to a (partially defined or overflowing) $i^*\rho(g)$ (resp. $i^*\rho'(g)$) action on $i^*\eta(\varepsilon)$. See [HPS77, pp. 94–95] for details. As in [HPS77], we define a submanifold $\tilde{\mathcal{W}}^{cs}_{\rho'(g)}$ of $i^*\eta(\varepsilon)$ by $\tilde{\mathcal{W}}^{cs}_{\rho'(g)} = \cap_{n \le 0} i^*\rho'(g)^n(i^*\eta(\varepsilon))$. By [HPS77, p. 107] this is a $C^k$ submanifold of $i^*\eta(\varepsilon)$ such that $\exp \circ i_*(\tilde{\mathcal{W}}^{cs}_{\rho'(g)})$ is the foliation $\mathcal{W}^{cs}_{\rho'(g)}$. Replacing $\rho'(g)$ by $\rho'(g^{-1})$, gives $\tilde{\mathcal{W}}^{cu}_{\rho'(g)} = \cap_{n \ge 0} i^*\rho'(g)^n(i^\eta(\varepsilon))$ a $C^k$ submanifold such that $\exp \circ i_*(\tilde{\mathcal{W}}^{cu}_{\rho'(g)})$ is the foliation $\mathcal{W}^{cu}_{\rho'(g)}$. As in [HPS77, Th. 6.8] the intersection $\tilde{\mathcal{W}}^{cu}_{\rho'(g)} \cap \tilde{\mathcal{W}}^{cs}_{\rho'(g)}$ is transverse and the image of a section $\sigma_{\rho'(g)} : V \to i^*\eta(\varepsilon)$ such that $\exp \circ i_* \circ \sigma_{\rho'(g)}(\mathcal{V})$ is a leaf of $\mathcal{W}^c_{\rho'(g)}$.

Note that there is a foliation $i^*\mathfrak{F}$ (resp. $i^*\mathfrak{F}'$) of $i^*\eta(\varepsilon)$ defined on each component of $V$ by pulling back $\mathfrak{F} \cap \exp(\eta(\varepsilon)|_{\mathcal{V}})$ where $\mathcal{V} \in \mathcal{W}^c_{\rho(g)}$. Note that we consider the leaves of these foliations to be connected components of pre-images of leaves rather than entire pre-images of leaves. This foliation is preserved by $i^*\rho(g)$ (respectively $i^*\rho'(g)$) for any $g$ in $G$.

Since $\psi$ is a $C^0$ small homeomorphism, for any leaf $\mathcal{V} \in \mathcal{W}^c_{\rho(g)}$ and any $x \in \mathcal{V}$, we have

$$\psi(\exp(\eta(\frac{\varepsilon}{2})|_{B_{\mathcal{V}}(x,\frac{r}{2})})) \subset \exp(\eta(\varepsilon)|_{B_{\mathcal{V}}(x,r)})$$

and so we can pull back $\psi$ to a map $i^*\psi : i^*\eta(\frac{\varepsilon}{2}) \to i^*\eta(\varepsilon)$.

We now have the following diagram of $\mathbb{Z}$ actions:

$$\begin{array}{ccc}
(i^*\eta(\frac{\varepsilon}{2}), i^*\rho(g)) & \xrightarrow[\exp\circ i_*]{} & (H/\Lambda \times M, \rho(g)) \\
{\scriptstyle i^*\psi}\downarrow & & \downarrow{\scriptstyle \psi} \\
(i^*(\eta)(\varepsilon), i^*\rho'(g)) & \xrightarrow[\exp\circ i_*]{} & (H/\Lambda \times M, \rho'(g))
\end{array}$$

where the horizontal arrows are equivariant and the vertical arrows are leaf conjugacies.

We first show that each leaf of $\mathcal{W}^{cs}_{\rho'(g)}$ is a union of leaves of $\mathfrak{F}'$. Given a leaf $\mathcal{U}$ of $\mathfrak{F}$, we can find a leaf $\mathcal{L}$ of $\mathcal{W}^c_{\rho(g)}$ such that $\mathcal{U} \subset \mathcal{L}$. Note that for the leaf $\mathcal{L}$ of

$\mathcal{W}^c_{\rho(g)}$, we have that $\tilde{\mathcal{W}}^{\mathrm{cs}}_{\rho(g)} \cap i^*\eta(\varepsilon)|_{\mathcal{L}} = (\exp \circ i_*)^{-1}(\mathcal{V}) \cap i^*\eta(\varepsilon)|_{\mathcal{L}}$ where $\mathcal{V}$ is the unique leaf of $\mathcal{W}^{\mathrm{cs}}_{\rho(g)}$ containing $\mathcal{L}$. Furthermore, from the definition of $\tilde{\mathcal{W}}^{\mathrm{cs}}_{\rho(g)}$ it then follows that

$$(i^*\circ\exp)^{-1}(\mathcal{U}) \cap i^*\eta(\varepsilon)|_{\mathcal{L}} \subset i^*\rho(g)^n(i^*\eta(\frac{\varepsilon}{2}))$$

for all $n < 0$. Since $i^*\psi$ is a leaf conjugacy, possibly after shrinking $\varepsilon$, we have

$$i^*\psi((i^*\circ\exp)^{-1}(\mathcal{U}) \cap i^*\eta(\varepsilon)|_{\mathcal{L}} \subset i^*\rho'(g)^n(i^*\eta(\varepsilon))$$

for all $n$. This implies that

$$i^*\psi((i^*\circ\exp)^{-1}(\mathcal{U}) \cap i^*\eta(\varepsilon)|_{\mathcal{L}}) = (i^*\circ\exp)^{-1}(\psi(\mathcal{U})) \cap i^*\eta(\varepsilon)|_{\mathcal{L}}$$

is contained in $\tilde{\mathcal{W}}^{\mathrm{cs}}_{\rho'(g)} \cap i^*\eta(\varepsilon)|_{\mathcal{L}}$. This then implies that $\psi(\mathcal{U})$ is contained in $\mathcal{W}^{\mathrm{cs}}_{\rho'(g)}$ as desired.

Now the fact that $i^*\psi$ is a leaf conjugacy, the definition of $\tilde{\mathcal{W}}^{\mathrm{cs}}_{\rho'(g)}$ and $\tilde{\mathcal{W}}^{\mathrm{cs}}_{\rho(g)}$ and the fact that each leaf of $\mathcal{W}^{\mathrm{cs}}_{\rho'(g)}$ (resp. $\mathcal{W}^{\mathrm{cs}}_{\rho(g)}$) is a union of leaves of $\mathfrak{F}'$ (resp. $\mathfrak{F}$) implies that $i^*\psi(\tilde{\mathcal{W}}^{\mathrm{cs}}_{\rho(g)}) \subset \tilde{\mathcal{W}}^{\mathrm{cs}}_{\rho'(g)}$. This implies that for any $\mathcal{V} \in \mathcal{W}^{\mathrm{cs}}_{\rho}(g)$ there is $\mathcal{V}' \in \mathcal{W}^{\mathrm{cs}}_{\rho'}(g)$ such that $\psi(\mathcal{V}) \subset \mathcal{V}'$. Interchanging the roles of $\rho$ and $\rho'$ and replacing $\psi$ by $\psi^{-1}$, the same argument proves that for any $\mathcal{V}' \in \mathcal{W}^{\mathrm{cs}}_{\rho'}(g)$ there is $\mathcal{V}'' \in \mathcal{W}^{\mathrm{cs}}_{\rho}(g)$ such that $\psi^{-1}(\mathcal{V}') \subset \mathcal{V}''$. These two facts then imply that for any $\mathcal{V} \in \mathcal{W}^{\mathrm{cs}}_{\rho}(g)$ there is $\mathcal{V}' \in \mathcal{W}^{\mathrm{cs}}_{\rho'}(g)$ such that $\psi(\mathcal{V}) = \mathcal{V}'$.

A similar argument using $g^{-1}$ in place of $g$ implies that for any $\mathcal{V} \in \mathcal{W}^{\mathrm{cu}}_{\rho(g)}$ there is $\mathcal{V}' \in \mathcal{W}^{\mathrm{cu}}_{\rho'(g)}$ such that $\psi(\mathcal{V}) = \mathcal{V}'$. As remarked above, it then follows that for any leaf $\mathcal{V}$ of $\mathcal{W}^c_{\rho(g)}$ we have $\psi(\mathcal{V}) = \mathcal{V}'$ where $\mathcal{V}'$ is a leaf of $\mathcal{W}^c_{\rho'(g)}$ . □

*Remark.* The proof of Lemma 3.9 does not depend on all of our assumptions and the statement could be made more axiomatic. All we require is that $\psi$ is a leaf conjugacy and that any leaf of $\mathcal{W}^c_{\rho(g)}$ is a union of leaves of $\mathfrak{F}$ for any $g \in \Phi$.

We recall two definitions and a lemma from [MQ01, pp. 145–146]:

*Definition* 3.10. (1) Let $N$ be a smooth Riemannian manifold and $N_1$, $N_2$ two immersed $C^k$ manifolds. We say that $N_1$ and $N_2$ *intersect s-transversely* if $N_1 \cap N_2$ is a manifold $N'$of dimension

$$\dim(TN_1(x) \cap TN_2(x))$$

for any $x \in N_1 \cap N_2$.

(2) Let $N$ be a smooth Riemannian manifold and $N_1, \ldots, N_l$ a collection of $C^k$ immersed submanifolds. We say that the family $N_1, \ldots, N_l$ *intersects s-transversely* if $\cap_{i=1}^{j-1} N_i$ *intersects s-transversely* with $N_j$ for $j = 2, \ldots l$.

*Definition* 3.11. Let $N$ be a smooth manifold and $N_1, \ldots, N_l$ a collection of $C^k$ manifolds. We call a collection $N'_1, \ldots, N'_l$ of $C^k$ submanifolds of $N$ a *topologically trivial ε-perturbation* of $N_1, \ldots, N_l$ if there is a homeomorphism $h : N \to N$ such that

(1) $N'_i = h(N_i)$

(2) $d(h(x), x) < \varepsilon$ for all $x \in N$, and

(3) for any $x \in N$ and any $i$, let $B_i(x)$ (resp. $B'_i(h(x))$) be the unit balls in the tangent space $TN_i(x)$ (resp. $TN'_i(h(x))$). Then $B_i(x_i)$ and $B'_i(h(x))$ are within an $\varepsilon$ neighborhood of each other in $TN$.

*Remark.* On [MQ01, p. 145], the same notion is called an $\varepsilon$-perturbation. We choose to modify the terminology, since the original terminology is somewhat deceptive.

LEMMA 3.12. *Let $N$ be a compact manifold and $N_1, \ldots, N_l$ be $C^k$ submanifolds intersecting $s$-transversely, then*

(1) $\cap_{i=1}^{l} N_i$ *is a $C^k$ submanifold, and*

(2) *there exists $\varepsilon > 0$ depending only on $N$, such that if $N'_1, \ldots, N'_l$ is any topologically trivial ε-perturbation $N'_1, \ldots, N'_l$ of $N_1, \ldots, N_l$, then $N'_1, \ldots, N'_l$ intersect $s$-transversely.*

Point (1) for $l = 2$ is Lemma 5.5(1) of [MQ01], where $C^k$ replaces the word smooth. The proof is the same. As noted on page 146 of [MQ01], the case $l > 2$ follows by induction. Similarly part (2) follows from [MQ01, Lemma 5.5(2)] and induction. The proof of [MQ01, Lemma 5.5(2)] implicitly uses that if $N_1, N_2$ are $C^k$ submanifolds of $N$ and $N'_1, N'_2$ are a topologically trivial $\varepsilon$-perturbation of $N_1, N_2$ then $\dim(TN'_1(x) \cap TN'_2(x)) = \dim(N'_1 \cap N'_2)$ for every $x$ in $N$. *A priori* $\dim(TN'_1(x) \cap TN'_2(x))$ could drop, but it is in fact bounded below by $\dim(N'_1 \cap N'_2)$. This can be deduced from standard facts about transversality by an argument similar to the proof of [MQ01, Lemma 5.5(1)]. This is not noted explicitly in [MQ01].

Since we will need to know not just that the leaves of $\mathfrak{F}'$ are $C^k$ submanifolds but that $\mathfrak{F}'$ is a foliation by $C^k$ leaves, we require a slight strengthening of Lemma 3.12, also remarked on [MQ01, p. 146]. Let $\mathfrak{F}_1, \ldots, \mathfrak{F}_l$ be foliations by $C^k$ leaves of a compact manifold $N$. We say that the $\mathfrak{F}_i$ intersect $s$-transversely, if for each $x \in N$, leaves $\mathfrak{F}_i(x)$ intersect $s$-transversely and the dimension of the intersections $\cap_{i=1}^{j} \mathfrak{F}_i(x)$ is independent of $x$ for any $j$ from 2 to $l$. We say that a collection of foliations $\mathfrak{F}'_1, \ldots, \mathfrak{F}'_l$ of $N$ is an $\varepsilon$-perturbation of $\mathfrak{F}_1, \ldots, \mathfrak{F}_l$ if:

(1) There exists a homeomorphism $h : N \to N$ with $h(\mathfrak{F}_i) = \mathfrak{F}'_i$ for $i$ from 1 to $l$ and $d(h(x), x) < \varepsilon$ for every $x \in N$;

(2) For any $x \in N$ and any $i$, let $B_i(x)$ (resp. $B'_i(h(x))$) be the unit balls in the tangent space $T\mathfrak{F}_i(x)$ (resp. $T\mathfrak{F}'_i(h(x))$). Then $B_i(x_i)$ and $B'_i(h(x))$ are within an $\varepsilon$ neighborhood of each other in $TN$.

As remarked in [MQ01], a slight modification of the proof of Lemma 3.12 shows that:

(1) If $\mathfrak{F}_1, \ldots, \mathfrak{F}_l$ are $s$-transverse foliations by $C^k$ leaves, then the foliation defined by intersections of leaves of $\mathfrak{F}_1, \ldots, \mathfrak{F}_l$ is a foliation by $C^k$ leaves.

(2) If $N$ is compact, there exists $\varepsilon > 0$ such that any topologicially trivial $\varepsilon$-perturbation of $\mathfrak{F}_1, \ldots, \mathfrak{F}_l$ is $s$-transverse.

LEMMA 3.13. *Given $k$, if $\rho'$ is a sufficiently $C^1$ small, $C^k$ perturbation of $\rho$, the leaves of the foliation $\mathfrak{F}'$ are the $s$-transverse intersections of leaves of $\mathcal{W}^c_{\rho'(g)}$ for $g \in \Phi$. Therefore $\mathfrak{F}'$ is a foliation by $C^k$ leaves. Furthermore, the foliation $\mathfrak{F}'$ is close to $\mathfrak{F}$ in the natural topology on foliations by $C^k$ leaves.*

*Proof.* We fix a neighborhood $U$ of $\rho$ in $\mathrm{Hom}(D, \mathrm{Diff}^1(M))$ such that

(1) for $g \in \Phi$, $\rho(g')$ is close enough to $\rho(g)$ to satisfy the hypotheses of Theorem 3.1,

(2) the map $\psi = \theta^{-1}$ constructed in Section 2.2 satisfies $d(\psi(x), x) < \varepsilon$ (or equivalently $d(\theta(x), x) < \varepsilon$) for $\varepsilon$ as in Lemma 3.12.

Let $\mathfrak{L}'$ be an arbitrary leaf of $\mathfrak{F}'$ and let $\mathfrak{L} = \psi^{-1}(\mathfrak{L}')$. Note that the leaf $\mathfrak{L}$ of $\mathfrak{F}$ is the $s$-transverse intersection of leaves $\mathcal{V}^c_{\rho(g)}$ of $\mathcal{W}^c_{\rho(g)}$ for $g \in \Phi$. By Lemma 3.9, for every $g \in \Phi$, we know that $\psi(\mathcal{V}^c_{\rho(g)})$ is a leaf $\mathcal{V}^c_{\rho'(g)}$ of $\mathcal{W}^c_{\rho'(g)}$ and therefore by Theorem 3.1 a $C^k$ submanifold of $H/\Lambda \times M$ which is $C^k$ close to some leaf of $\mathcal{W}^c_{\rho(g)}$. Since $\psi$ is a homeomorphism and can be made arbitrarily small by choosing $\rho'$ close enough to $\rho$, Lemma 3.12(2) implies that $\psi(\mathcal{V}^c_{\rho(g)}) = \mathcal{V}^c_{\rho'(g)}$ intersects $s$-transversely in a $C^k$ manifold. Since

$$\cap_{d \in \Phi} \psi(\mathcal{V}^c_{\rho(g)}) = \psi(\cap_{d \in \Phi} \mathcal{V}^c_{\rho(g)}) = \psi(\mathfrak{L}) = \mathfrak{L}'$$

it follows that every leaf $\mathfrak{L}'$ of $\mathfrak{F}'$ is a $C^k$ submanifold of $H/\Lambda \times M$. The remarks following Lemma 3.12 then imply that $\mathfrak{F}'$ is a foliation by $C^k$ leaves. □

*Proof of Theorem* 3.8. The homeomorphism $\psi^{-1} = \theta$ is constructed in Section 2.2 and shown to be a homeomorphism in Theorem 3.6. Since $\psi^{-1}$ is given by projecting from leaves of $\mathfrak{F}'$ to leaves of $\mathfrak{F}$ via a smooth transversal, and leaves of $\mathfrak{F}'$ are $C^k$ by Lemma 3.13, the map $\psi$ is $C^k$ and $C^k$ small along fibers.

The remaining conclusions follow from Corollary 2.5 and Theorem 3.6. □

We will eventually need one additional fact concerning $\psi$ which is now straightforward. To state this fact about $\psi$, we need to define some additional dynamical foliations. Let $E^0_{\rho'}(G)$ be the distribution tangent to $\mathfrak{F}'$. Given $g \in \Phi$, we take the distributions $E^0_{\rho}(G) \oplus E^s_{\rho(g)}$ and $E^0_{\rho'}(G) \oplus E^s_{\rho'(g)}$. Recall that $\mathfrak{F}$ is tangent to $E^0_{\rho}(G)$. For the $\rho$ action there is a smooth foliation tangent to $E^0_{\rho}(G) \oplus E^s_{\rho(g)}$, which we denote by $\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)}$. To see this, one notes that the group $Z$ normalizes the group $F^s_{\rho(g)}$ and so the product $ZF^s_{\rho(g)}$ is a subgroup of $H$. For the case of affine actions, the foliation $\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)}$ is just the orbit foliation for the left action of $ZF^s_{\rho(g)}$ on $H/\Lambda$. For quasi-affine actions, we recall that there is a natural projection $H/\Lambda \times M \to M$ and the foliation $\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)}$ is given by pre-images in $H/\Lambda \times M$ of the $ZF^s_{\rho(g)}$ orbits in $H/\Lambda$. We note that the lift of any leaf of $\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)}$ to $H \times M$ is of the form $p^{-1}(\mathcal{V})$ where $\mathcal{V}$ is a leaf of $\mathcal{W}^s_{\bar{\rho}(g)}$.

PROPOSITION 3.14. *For any $g \in \Phi$, there is a $\rho'$ invariant foliation $\mathfrak{F}' \oplus \mathcal{W}^s_{\rho'(g)}$ of $H/\Lambda \times M$ tangent to $E^0_{\rho'}(G) \oplus E^s_{\rho'(g)}$ such that $\psi(\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)}) = \mathfrak{F}' \oplus \mathcal{W}^s_{\rho'(g)}$.*

*Proof.* We can define the foliation $\mathfrak{F}' \oplus \mathcal{W}^s_{\rho'(g)}$ to be $\psi(\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)})$. A leaf $\mathcal{V}$ of the foliation $\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)}$ is given by sets of points sharply forward asymptotic to a leaf $\mathcal{L}$ of $\mathfrak{F}$. Here, as in [HPS77], $x$ is sharply forward asymptotic to $\mathfrak{L}$ if $d(\rho^n(g)x, \rho^n(\mathcal{L}))$ goes to zero at least as fast as $\exp(-\lambda n)$ for some $\lambda > 0$ depending on the dynamics. Since $\psi$ is a leaf conjugacy it follows that a leaf $\mathcal{V}'$ of $\mathfrak{F}' \oplus \mathcal{W}^s_{\rho'(g)}$ is given by sets of points sharply forward asymptotic to a leaf $\mathcal{L}'$ of $\mathfrak{F}'$. By [HPS77, Th. 6.8 (e) and (f)], the leaves of $\mathcal{W}^s_{\rho'(g)}$ are exactly sets of points sharply forward asymptotic to the orbit of a point on a leaf of $\mathcal{W}^c_{\rho'(g)}$. Therefore, a leaf $\mathcal{V}'$ of $\mathfrak{F}' \oplus \mathcal{W}^s_{\rho'(g)}$ is the union of all leaves of $\mathcal{W}^s_{\rho'(g)}$ through a leaf $\mathfrak{L}'$ of $\mathfrak{F}'$. This immediately implies that $\mathfrak{F}' \oplus \mathcal{W}^s_{\rho'(g)}$ is tangent to $E^0_{\rho'}(G) \oplus E^s_{\rho'(g)}$ and completes the proof. $\square$

## 4. **Property T and conjugacy**

In this section, we modify the leaf conjugacy obtained at the end of the last section to obtain a semiconjugacy. The *a priori* regularity of this semiconjugacy will be somewhat bad. In Section 5 we show it is a homeomorphism, in Section 6 we show it is differentiable along many foliations and in Section 7 we show it is differentiable and even $C^\infty$ when $\rho'$ is $C^\infty$. The key ingredient in the arguments of this section is [FM05, Th. 2.16], so we begin by recalling some notation and definitions from subsection 2.3 of that paper. For most of this section $G$ will be a compactly generated topological group, though for our applications, $G$ will be $J$ or $\Gamma$ as above.

Throughout this section $X$ will be a second countable, compact, Hausdorff manifold and $\mathfrak{F}$ will be a foliation of $X$ by $C^k$ leaves. For background on foliated spaces, their tangent bundles, and transverse invariant measures, the reader is referred to [CC00] or [MS88].

We let $\mathrm{Diff}^k(X,\mathfrak{F})$ be the set of homeomorphisms of $X$ which preserve $\mathfrak{F}$ and restrict to $C^k$ diffeomorphisms on each leaf with derivatives depending continuously on $x$ in $X$. For $1 \le k \le \infty$, there is a natural $C^k$ topology on $\mathrm{Diff}^k(X,\mathfrak{F})$. The definition of this topology is straightforward and is recalled in [FM05, §2.3].

We now define a special class of perturbations of actions.

*Definition* 4.1. Let $G$ be a compactly generated, topological group and $\rho$ an action of $G$ on $X$ defined by a homomorphism from $G$ to $\mathrm{Diff}^\infty(X,\mathfrak{F})$. Let $\rho'$ be another action of $G$ on $X$ defined by a homomorphism from $G$ to $\mathrm{Diff}^k(X,\mathfrak{F})$. Let $U$ be a (small) neighborhood of the identity in $\mathrm{Diff}^k(X,\mathfrak{F})$ and $K$ be a compact generating set for $G$. We call $\rho'$ a $(U,C^k)$*-foliated perturbation* of $\rho$ if:

(1) for every leaf $\mathfrak{L}$ of $\mathfrak{F}$ and every $g \in G$, we have $\rho(g)\mathfrak{L} = \rho'(g)\mathfrak{L}$ and,

(2) $\rho'(g)\rho(g)^{-1}$ is in $U$ for every $g$ in $K$.

We fix a continuous, leafwise smooth Riemannian metric $g_{\mathfrak{F}}$ on $T\mathfrak{F}$, the tangent bundle to the foliation, and note that $g_{\mathfrak{F}}$ defines a volume form and corresponding measure on each leaf $\mathfrak{L}$ of $\mathfrak{F}$, both of which we denote by $\nu_{\mathfrak{F}}$. (Metrics $g_{\mathfrak{F}}$ exist by a standard partition of unity argument.) Let $G$ be a group and $\rho$ an action of $G$ on $X$ defined by a homomorphism from $G$ to $\mathrm{Diff}^k(X,\mathfrak{F})$. We say the action is *leafwise isometric* if $g_{\mathfrak{F}}$ is invariant under the action. When $G = \mathbb{Z}$ and $\mathbb{Z} = \langle f \rangle$, we will call $f$ a *leafwise isometry*.

For the remainder of this section, we will assume that the foliation has a transverse invariant measure $\nu$. By integrating the transverse invariant measure $\nu$ against the Riemannian measure on the leaves of $\mathfrak{F}$, we obtain a measure $\mu$ on $X$ which is finite when $X$ is compact. In this case, we normalize $g_{\mathfrak{F}}$ so that $\mu(X) = 1$. We will write $(X,\mathfrak{F},g_{\mathfrak{F}},\mu)$ for our space equipped with the above data, sometime leaving one or more of $\mathfrak{F}, g_{\mathfrak{F}}$ and $\mu$ implicit. We will refer to the subgroup of $\mathrm{Diff}^k(X,\mathfrak{F})$ which preserves $\nu$ as $\mathrm{Diff}^k_\nu(X,\mathfrak{F})$. Note that if $\rho$ is an action of $G$ on $X$ defined by a homomorphism into $\mathrm{Diff}^k_\nu(X,\mathfrak{F})$ and $\rho$ is leafwise isometric, then $\rho$ preserves $\mu$. Furthermore if $\rho$ is an action of $\Gamma$ on $X$ defined by a homomorphism into $\mathrm{Diff}^k_\nu(X,\mathfrak{F})$ and $\rho'$ is a $(U,C^k)$-leafwise perturbation of $\rho$, then it follows easily from the definition that $\rho'$ is defined by a homomorphism into $\mathrm{Diff}^k_\nu(X,\mathfrak{F})$ since the induced map on transversals is the same.

Before stating one of the main results of [FM05], we will need a coarse quantitative measure of the $C^k$ size of the $C^k$ map. We denote by $B_{\mathfrak{F}}(x,r)$ the ball in $\mathfrak{L}_x$ about $x$ of radius $r$. For a sufficiently small value of $r > 0$, we can canonically identify each $B_{\mathfrak{F}}(x,2r)$ with the ball of radius $2r$ in Euclidean space via the exponential map from $T\mathfrak{F}_x$ to $\mathfrak{L}_x$. We first consider the case when $k$ is an integer, where we can give a pointwise measure of size. Recall that a $C^k$ self map of a manifold $Z$ acts on $k$-jets of $C^k$ functions on $Z$. Any metric on $TZ$ defines a pointwise norm on each fiber of the bundle of $J^k(Z)$ of $k$-jets of functions on $Z$.

For any $C^k$ diffeomorphism $f$ we can define $\|j^k(f)(z)\|$ as the operator norm of the map induced by $f$ from $J^k(Z)_z$ to $J^k(Z)_{f(z)}$. For a more detailed discussion on jets and an explicit construction of the norm on $J^k(Z)_z$, see [FM05, §4]. We say that a map $f$ has $C^k$ size less than $\delta$ on a set $U$ if $\|j^k(f)(z)\| < \delta$ for all $z$ in $U$. If $k$ is not an integer, we say that $f$ has $C^k$ size less than $\delta$ on $U$ if $f$ has $C^{k'}$ size less than $\delta$ on $U$ where $k'$ is the greatest integer less than $k$ and $j^{k'}(f)$ satisfies a (local) Hölder estimate on $U$. See [FM05, §4] for a more detailed discussion of Hölder estimates.

*Remark.* This notion of $C^k$ size is not very sharp. The size of the identity map will be 1, as will be the size of any isometry of the metric. We only use this notion of size to control estimates on a map at points where the map is known to be "fairly large" and where we only want bounds to show it is "not too large".

For the following theorem, we assume that the holonomy groupoid of $(X, \mathfrak{F})$ is Hausdorff. This is a standard technical assumption that allows us to define certain function spaces on "pairs of points on the same leaf of $(X, \mathfrak{F})$". See [FM05, §6.1], [CC00] and [MS88] for further discussion. All the foliations considered in this paper for the proof of Theorem 1.1 are covered by fiber bundles, and in that case the holonomy groupoid is Hausdorff. We now recall [FM05, Th. 2.16].

THEOREM 4.2. *Let $G$ be a locally compact, $\sigma$-compact group with property (T). Let $\rho$ be a continuous leafwise isometric action of $G$ on $X$ defined by a homomorphism from $G$ to $\mathrm{Diff}^{\infty}_{\nu}(X, \mathfrak{F})$. Then for any $k \geq 3, \kappa > 0$ and any $\varsigma > 0$ there exists a neighborhood $U$ of the identity in $\mathrm{Diff}^k(X, \mathfrak{F})$ such that for any continuous $(U, C^k)$-foliated perturbation $\rho'$ of $\rho$ there exists a measurable map $\phi : X \to X$ such that*:

1. *$\phi \circ \rho(g) = \rho'(g) \circ \phi$ for all $g \in G$,*
2. *$\phi$ maps each leaf of $\mathfrak{F}$ into itself.*
3. *There is a subset $S \subset X$ with $\mu(S) = 1 - \varsigma$ and $\Gamma \cdot S$ has full measure in $X$, and a constant $r \in \mathbb{R}^+$, depending only on $X, \mathfrak{F}$ and $g_{\mathfrak{F}}$, such that, for every $x \in S$, the map $\phi : B_{\mathfrak{F}}(x, r) \to \mathfrak{L}_x$ is $C^{k-1-\kappa}$-close to the identity; more precisely, with our chosen identification of $B_{\mathfrak{F}}(x, 2r)$ with the ball of radius $2r$ in Euclidean space, $\phi - \mathrm{Id} : B_{\mathfrak{F}}(x, r) \to B_{\mathfrak{F}}(x, 2r)$ has $C^{k-1-\kappa}$ norm less than $\varsigma$ for every $x \in S$.*
4. *There exists $0 < t < 1$ depending only on $G$ and $K$ such that the set of $x \in X$ where the $C^{k-1-\kappa}$ size of $\phi$ on $B_{\mathfrak{F}}(x, r)$ is not less than $(1 + \varsigma)^{l+1}$ has measure less than $t^l \varsigma$.*

*Furthermore, for any $l \geq k$, if $\rho'$ is a $C^{2l-k+1}$ action, then by choosing $U$ small enough, we can choose $\phi$ to be $C^l$ on $B_{\mathfrak{F}}(x, r)$ for almost every $x$ in $X$. In* *particular, if $\rho'$ is $C^\infty$ then for any $l \geq k$, by choosing $U$ small enough, we can choose $\phi$ to be $C^l$ on $B_{\mathfrak{F}}(x,r)$ for almost every $x$ in $X$.*

*Remarks.*

(1) Since $\rho'$ is a foliated perturbation of $\rho$, the transverse measure $\nu$ is $\rho'$ invariant. This is because $\rho'$ defines the same action on transversals as $\rho$.

(2) The map $\phi$ constructed in the theorem is not even $C^0$ close to the identity on $X$. However, the proof of the theorem shows that for every $1 \leq q < \infty$, possibly after changing $U$ depending on $q$, we have $\int_X (d(x,\phi(x))^q d\mu \leq \varsigma$.

We now proceed to show how Theorem 4.2 can be applied in the proof of Theorem 1.1. As before, we fix a semisimple Lie group $J$ with all simple factors of real rank at least two and a lattice $\Gamma$ in $J$ and let $G$ be one of $J$ and $\Gamma$. We also fix an algebraic group $H$, a cocompact lattice $\Lambda < H$, a compact manifold $M$ and a quasi-affine action $\rho$ of $G$ on $H/\Lambda \times M$. Once again, we assume that $\rho$ lifts to $H \times M$. We fix the foliation $\mathfrak{F}$ of $H/\Lambda \times M$ by central leaves for $\rho(G)$ as in Section 2.1. We further assume that the $G$ action defined by $\rho$ lifts to an action on $H \times M$. We note that there is a transverse invariant measure $\nu$ to $\mathfrak{F}$ defined by lifting to $H \times M$ and identifying local transversals with their projections to $Z \backslash H$.

PROPOSITION 4.3. *When $U \subset \mathrm{Hom}(G, \mathrm{Diff}^k_\nu(H/\Lambda \times M, \mathfrak{F}))$ is a neighborhood of $\rho$, there is a neighborhood $V$ of $\rho$ in $\mathrm{Hom}(G, \mathrm{Diff}^1(H/\Lambda \times M))$ such that if $\rho' \in V$ is a $C^k$ action and $\psi$ is the homeomorphism from Theorem 3.8, then $\psi^{-1} \circ \rho \circ \psi$ is in $U$. Furthermore, given $m \geq k$, by assuming $\rho'$ is $C^m$ and possibly after shrinking $V$, we can also guarantee that $\psi^{-1} \circ \rho \circ \psi$ is in*

$$\mathrm{Hom}(G, \mathrm{Diff}^m_\nu(H/\Lambda \times M, \mathfrak{F})).$$

*Proof.* This is immediate from the definitions and Theorem 3.8. □

THEOREM 4.4. *For every $k \geq 3$, $\kappa > 0$ and $\varsigma > 0$, there is a neighborhood $V$ of $\rho$ in $\mathrm{Hom}(G, \mathrm{Diff}^k(H/\Lambda \times M))$ such that if $\rho' \in V$ then there exists a measurable map $\varphi : H/\Lambda \times M \to H/\Lambda \times M$ such that:*

(1) *$\varphi \circ \rho(g) = \rho'(g) \circ \varphi$ for all $g \in G$.*

(2) *$\varphi$ maps each leaf of $\mathfrak{F}$ into a leaf of $\mathfrak{F}'$.*

(3) *$\varphi(\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)}) = \mathfrak{F} \oplus \mathcal{W}^s_{\rho'(g)}$ on a set of full measure in $H/\Lambda \times M$ for any $g \in G$.*

(4) *There is a subset $S \subset X$ with $\mu(S) = 1 - \varsigma$ and $\Gamma \cdot S$ is of full measure in $H/\Lambda \times M$, and a constant $r \in \mathbb{R}^+$, depending only on $X$, $\mathfrak{F}$ and $g_{\mathfrak{F}}$, such that, for every $x \in S$, the map $\varphi : B_{\mathfrak{F}}(x,r) \to \mathfrak{F}(x)$ is $C^{k-1-\kappa}$-close to the identity; more precisely, with our chosen identification of $B_{\mathfrak{F}}(x,2r)$ with the ball of radius $2r$ in Euclidean space, $\varphi - \mathrm{Id} : B_{\mathfrak{F}}(x,r) \to B_{\mathfrak{F}}(x,2r)$ has $C^{k-1-\kappa}$ norm less than $\varsigma$ for every $x \in S$.*

(5) *There exists* $0 < t < 1$ *depending only on* $\Gamma$ *and* $K$ *such that the set of* $x \in X$ *where the* $C^{k-1-\kappa}$ *size of* $\varphi$ *on* $B_{\mathfrak{F}}(x, r)$ *is not less than* $(1 + \varsigma)^{l+1}$ *has measure less than* $t^l \varsigma$ *and*,

*Furthermore*, *for any* $l \geq k$, *if* $\rho'$ *is a* $C^{2l-k+1}$ *action*, *then by choosing* $U$ *small enough*, *we can choose* $\varphi$ *to be* $C^l$ *on* $B_{\mathfrak{F}}(x, r)$ *for almost every* $x$ *in* $X$. *In particular*, *if* $\rho'$ *is* $C^\infty$ *then for any* $l \geq k$, *by choosing* $U$ *small enough*, *we can choose* $\phi$ *to be* $C^l$ *on* $B_{\mathfrak{F}}(x, r)$ *for almost every* $x$ *in* $X$.

*Remark.* Conclusion (4) combined with equivariance of $\varphi$ and the fact that the central foliation is the quotient of the fibers of a bundle, imply that for almost every $x$, the map $\varphi$ is $C^{k-1-\kappa}$ along $\mathfrak{F}(x)$ and that the derivative $D\varphi : T\mathfrak{F}(x) \to T\mathfrak{F}(x)$ is an isomorphism at all points of $\mathfrak{F}(x)$.

*Proof.* By Proposition 4.3, we can apply Theorem 4.2 to the actions $\rho$ and $\psi^{-1} \circ \rho' \circ \psi$. This produces a map $\phi$ satisfying the conclusions of Theorem 4.2. We let $\varphi = \psi \circ \phi$ which satisfies (1) and (2) by construction. Since $\psi$ is uniformly $C^k$ small when restricted to any leaf of $\mathfrak{F}$, the estimates in (4) and (5) follow from the estimates in Theorem 4.2 (3) and (4). Point (3) follows from Proposition 3.14, the fact that $\phi$ maps almost every leaf of $\mathfrak{F}$ to itself, and the fact that leaves of $\mathfrak{F} \oplus \mathcal{W}^s_{\rho(g)}$ are unions of leaves of $\mathfrak{F}$. ☐

The majority of the remainder of this paper is devoted to a proof that the map $\varphi$ constructed above is a small diffeomorphism with regularity depending on the regularity of $\rho'$. This suffices to prove Theorem 1.1 in the case when the $\rho$ action lifts to $H \times M$. An additional argument in Section 5.3 completes the proof. With this in mind, we fix:

4.1. *Notation for the remainder of this paper.* As above $J$ will be a semisimple Lie group with all simple factors of real rank at least two and $\Gamma < J$ will be a lattice. We will fix $G$ to be one of $J$ and $\Gamma$ and also fix a quasi-affine action $\rho$ of $G$ on $H/\Lambda \times M$. Until Section 5.3, we will assume that the $G$ action $\rho$ lifts to $H \times M$. In Section 5.3, we explain how to remove this assumption. In addition, we will fix an integer $k$, and $\rho'$ will always denote a perturbation of $\rho$ which is sufficiently $C^k$ small so as to be able to apply Theorem 4.4, and $\varphi$ will be the resulting semi-conjugacy. We allow the possibility that $\rho'$ is $C^l$ for some $l > k$, including $l = \infty$, so as to be able to prove the $C^\infty$ case of Theorem 1.1. We also fix the maps $\psi$ and $\phi$ from Theorems 3.8 and 4.2, the projection $p : H \times M \to Z \backslash H$ and the map $f : H \times M \to Z \backslash H$ from Theorem 2.4.

## 5. **Continuity along dynamical foliations**

In this section, we show that $\varphi$ is a homeomorphism by showing that it is a homeomorphism when restricted to certain dynamical foliations. To do this we show that for any $g \in \Phi$, $\varphi$ maps contracting leaves for the action $\rho(g)$ on $H/\Lambda \times M$ to contracting leaves for $\rho'(g)$ and deduce from this that $\varphi$ is a homeomorphism along those foliations. Throughout this section, all notation is as fixed at the end of Section 4 or as in Section 3.1. Once we have shown that $\varphi$ is a homeomorphism, we show, in Section 5.3, how to remove the assumption that the action $\rho$ lifts to a $G$ action on $H \times M$.

5.1. *Equivariance of contracting foliations.* We use the equivariance of $\varphi$ to show that:

PROPOSITION 5.1. *For every $g \in \Phi$ and almost every $x \in X$, the map $\varphi$ defined in Theorem* 4.4 *maps a set of full measure in the leaf $\mathcal{W}^s_{\rho(g)}(x)$ into the leaf $\mathcal{W}^s_{\rho'(g)}(\varphi(x))$.*

The proof of this proposition takes up the rest of this subsection. If $\varphi$ were continuous as well as equivariant, this would follow easily from standard dynamical arguments. We begin by introducing some terminology and notation. Fix a finite set $\Phi$ of elements in $G$ as in Lemma 3.3 for the remainder of this section. We introduce a function which measures the extent to which $\varphi$ does not take stable leaves to stable leaves for a fixed element $g \in \Phi$. We define this function on a $H/\Lambda \times M \times F^s_{\rho(g)}$, where $F^s_{\rho(g)} < H$ is as in Proposition 3.2. We denote the identity in $F^s_{\rho(g)}$ by $e_F$. We note that $\rho(g)$ induces a contracting automorphism of $F^s_{\rho(g)}$ which we denote by $\varrho(g)$. The diffeomorphism $(\rho(g), \varrho(g))$ of $H/\Lambda \times M \times F^s_{\rho(g)}$ will be written $\tilde{\rho}(g)$. The projection $\pi : H/\Lambda \times M \times F^s_{\rho(g)} \to H/\Lambda \times M$ is equivariant for the $\mathbb{Z}$ action generated by $\tilde{\rho}(g)$ on $H/\Lambda \times M \times F^s_{\rho(g)}$ and the $\mathbb{Z}$ action generated by $\rho(g)$ on $H/\Lambda \times M$. First note that if we take the leaf of $\mathfrak{F}' \oplus \mathcal{W}^s_{\rho'(g)}$ through a point $x$ in $X$, this is foliated by stable leaves, each of which intersects the leaf of $\mathfrak{F}'$ through $x$ in exactly one point. Given a point $y$ on the leaf of $\mathfrak{F}' \oplus \mathcal{W}^s_{\rho'(g)}$ through $x$ we will look at its projection to the leaf $\mathfrak{F}'(x)$ through $x$ defined by this unique intersection point, call this point $p_{\mathfrak{F}'}(y)$. We denote the restriction of the fixed Riemannian metric on $H/\Lambda \times M$ to the foliation $\mathfrak{F}$ by $g_{\mathfrak{F}}$. Note that the bounds on the derivatives of $\varphi$ along $\mathfrak{F}$ from Theorem 4.4(4) and (5) imply that for almost every $x$ in $H/\Lambda \times M$, there is a small ball $B_{\mathfrak{F}}(x, \varepsilon(x))$ such that $\phi$ is a $C^{k-1-\kappa}$ diffeomorphism when restricted to $B_{\mathfrak{F}}(x, \varepsilon(x))$, where $\kappa$ depends only on the size of the perturbation. For $x$ in $S$ as defined in Theorem 4.4 point (4), the number $\varepsilon$ is very close to the number $r$ specified in that theorem. For general $x$, the number $\varepsilon$ depends on the bound from (5) of Theorem 4.4. Regardless, whenever $p_{\mathfrak{F}'}(\varphi(fx))$ is in $\varphi(B_{\mathfrak{F}}(x, \varepsilon(x))$ we define:

$$\delta(x, f) = d_{\mathfrak{F}'}((\varphi|_{B_{\mathfrak{F}}(x,\varepsilon(x))})^{-1}(p_{\mathfrak{F}'}(\varphi(fx))), x),$$

and let $\delta(x, f) = \infty$ otherwise. While the fact that $\varepsilon(x)$ is not $\rho(g)$ invariant prevents us from concluding that $\delta(x, f)$ is $\rho(g)$ invariant, we do have the following weaker condition on $\delta(f, x)$. Since $\varphi$ and $\pi$ are $G$ equivariant, $\rho(g)$ is isometric along $\mathfrak{F}$ and $p_{\mathfrak{F}'}$ is $\rho(g)$ equivariant, it follows that if $\delta(x, f) < \infty$ and $\delta(\rho(g)x, \varrho(g)f) < \infty$, then $\delta(x, f) = \delta(\rho(g)x, \varrho(g)f)$. It is clear that $\delta(x, e_F) = 0$ for almost every $x \in H/\Lambda \times M$.

We recall some basic facts concerning density points. For more discussion of the density points, including a proof of the density point theorem, see e.g. [Mar91, IV.1]. First, we need to specify a $b$-metric on $H/\Lambda \times M \times F^s_{\rho(g)}$. Recall that given a number $b > 1$, a $b$-metric on a topological space $Y$ is a map $d : Y \times Y \to \mathbb{R}^{\geq 0}$ satisfying the usual axioms of a metric, except that the triangle inequality is replaced by $d(x, y) \leq b(d(x, z) + d(z, y))$. Our $b$-metric will be the sum of the metric induced by our choice of Riemannian metric along $H/\Lambda \times M$ with a metric on the fiber analogous to the one introduced in the proof of [Mar91, Cor. IV.1.6]. Given a ball $B$ in $F^s_{\rho(g)}$, we define a left invariant 2-distance function on $E^s_{\rho(g)}$, by letting:

$$n_B(f_1, f_2) = \max\{n \in \mathbb{Z} | (f_1{}^{-1} f_2) \in \varrho(g)^n (B)\},$$

$$d_B(f_1, f_2) = 2^{-n_B(f_1, f_2)}.$$

This is a left invariant 2-distance function by the proof of [Mar91, Cor. IV.1.6]. Given $d_B$ and the distance $d_{H/\Lambda \times M}$ induced by adapted metric on $H/\Lambda \times M$, we define a 2-distance function on $H/\Lambda \times M \times E^s_{\rho(g)}$ by letting

$$d((x_1, f_1), (x_2, f_2)) = d_{H/\Lambda \times M}(x_1, x_2) + d_B(f_1, f_2).$$

Whenever discussing density points in $H/\Lambda \times M \times F^s_{\rho(g)}$ (resp. $F^s_{\rho(g)}$ or $H/\Lambda \times M$) we mean density points with respect to balls in the metric $d$ (resp. $d_B$ or $d_X$). For $d_B, d_X$ and $d$, we will denote by $B_d(x, \varepsilon)$ (resp. $B_{d_B}(x, \varepsilon)$ or $B_{d_{\text{base}}}(x, \varepsilon)$) the $d$ ball about $x$ of radius $\varepsilon$.

Given a topological space $Y$, a $b$-distance function $d$ on $Y$, a measure $\mu$ on $Y$, and a measurable set $C \subset Y$, we call a point $y \in Y$ a density point of $C$ if

$$\lim_{\varepsilon \to 0} \frac{\mu(C \cap B_d(x, \varepsilon))}{\mu(B_d(x, \varepsilon))} = 1.$$

Note that this conclusion is most reasonable in the case where $\mu$ is a regular Borel measure which is $d$-finite dimensional in the sense of [Mar91, IV.1]. The generalization of the classical density point theorem as stated in [Mar91, Th. IV.1.5] says that if $Y, d$ are as above and if $\mu$ is $d$-finite dimensional, then the subset of $C$ consisting of density points of $C$ is of full measure in $C$. We do not give a more detailed discussion here, since we will use the density point theorem only through the following consequence, which is a special case of [Mar91, Cor. IV.1.6].

PROPOSITION 5.2. *Let $F$ be a locally compact, compactly generated topological group, $\varrho : F \to F$ a contracting automorphism of $F$ and $C \subset V$ a (Haar)* *measurable subset. Then if $e_F$ is a density point of $C$, the sequence of sets*

$$\{\varrho(g)^{-n}(C)\}_{n\in N}$$

*converges in measure to $F$.*

*Proof.* In the proof of Corollary IV.1.6 of [Mar91], it is shown that the sets $\{\varrho(g)^{-n}(f^{-1}C)\}_{n\in N}$ converge in measure to $F$ whenever $f$ is a density point of $C$. This implies the desired conclusion. □

Let $V_\beta$ be the set of points in $H/\Lambda\times M\times F^s_{\rho(d)}$ such that $\delta(x,f)\leq\beta$ and let $U_\beta$ be the set of points $x$ in $H/\Lambda\times M$ such that $\varepsilon(x)<\beta$. We now show that for any $\beta>0$, the set $V_\beta$ is of full measure in $U_\beta\times F^s_{\rho(d)}$. Note that by the conclusions of Theorem 4.4, the set $U_\beta\to H/\Lambda\times M$ in measure as $\beta\to 0$. The proof of the lemma below is complicated by the fact that we need to work with points $x$ that are density points both in $H/\Lambda\times M$ and along the leaf of $\mathcal{W}^s_{\rho(g)}(x)$.

LEMMA 5.3. *For every $\beta>0$, the set $V_\beta$ is a set of full measure in $U_\beta\times F^s_{\rho(g)}$.*

*Proof.* We fix $\beta$ and $U_\beta$ and a constant $\eta>0$. Then by Theorem 4.4(5) there are a number $C_1$ and a set $U_1\subset U_\beta$ with $\mu(U_1)\geq(1-\eta)\mu(U_\beta)$ where for any $x\in U_1$, $\varphi|_{U_1}$ is differentiable and $\|D\phi(x)|_{\mathfrak{F}}\|\leq C_1$.

By Luzin's theorem, we can choose a set $U_2\subset H/\Lambda\times M$ with $\mu(U_2)\geq 1-\eta$ and a continuous map $\theta: H/\Lambda\times M\to H/\Lambda\times M$ such that $\varphi=\theta$ on $U_2$. Let $U_3=U_1\cap U_2\cap U_\beta$ and note that $\mu(U_3)\geq(1-2\eta)\mu(U_\beta)$.

Define the map $\Pi: H/\Lambda\times M\times F^s_{\rho(g)}\to H/\Lambda\times M$ by $\Pi(x,f)=fx$. Now, $\tilde{\varphi}=\varphi\circ\Pi$ and $\tilde{\theta}=\theta\circ\Pi$. As a result, $\tilde{\varphi}=\tilde{\theta}$ on $\Pi^{-1}(U_2)$.

The manifold $H/\Lambda\times M\times F^s_{\rho(g)}$ is equipped with a product measure $\mu\times\nu$ where $\nu$ is Haar measure on $F^s_{\rho(g)}$. Note that there is no difficulty in applying Fubini's theorem to this product measure. For all $f\in F^s_{\rho(g)}$ we have

$$\Pi(H/\Lambda\times M, f)=H/\Lambda\times M,$$

and $\Pi(\cdot,f)_*\mu=\mu$ and therefore $\mu(\Pi(\cdot,f)^{-1}U_i)=\mu(U_i)$.

By Fubini's theorem and the density point theorem, the set $U_4$ of points which are density points for $U_3\cap(x\times F^s_{\rho(g)})$ are of full $\nu$ measure in $U_3\cap(x\times F^s_{\rho(g)})$ for almost all $x\in H/\Lambda\times M$. Applying Fubini's theorem again implies that

$$U_4\cap(H/\Lambda\times M\times\{f\})$$

is of full $\mu$ measure in $U_3\cap(H/\Lambda\times M\times\{f\})$ for almost every $f$ such that $(x,f)$ lies in $\Pi^{-1}(U_3)$ for some $x\in H/\Lambda\times M$. By changing basepoint by translating by $f$, we can assume that $U_5=U_4\cap(H/\Lambda\times M\times\{e_F\})$ is of full measure in $U_3\cap(H/\Lambda\times M\times\{e_F\})$ and that $\mu(U_5)\geq(1-2\eta)\mu(U_\beta)$.

Let $N_R(x)=\{i\,|\,\rho(g)^{-i}x\in U_5\}$. The set $N_R(x)$ is infinite for almost every $x\in U_5$ by the Poincaré Recurrence Theorem. Given $f\in F^s_{\rho(g)}$ let $N_R(x,f)=$

$N_R(x) \cap \{j \,|\, \varrho(g)^{-j} f \in (U_3 \cap (\{\rho(g)^{-j}(x)\} \times F^s_{\rho(g)})\}$. Then for $\nu$ almost every $f \in F^s_{\rho(g)}$, Proposition 5.2 implies that $N_R(x, f)$ is infinite for almost every $x$ in $U_5$.

For $x \in U_5$, $f \in F^s_{\rho(g)}$, $y = fx$ and $n \in N_R(x, f)$, it follows that

$$d(\varphi(\rho(g)^{-n}x), \varphi(\rho(g)^{-n}y)) = d(\theta(\rho(g)^{-n}x), \theta(\rho(g)^{-n}y))$$

since the definition of $N_R(x, f)$ implies that $\rho(g)^{-n}x$ and $\rho(g)^{-n}y$ are in $U_2$.

The definition of $U_1$, the fact that $\rho(g)^{-n}(x)$ is in $U_1$ and compactness of $H/\Lambda \times M$ imply that there exists a constant $C$ depending only on the geometry of $H/\Lambda \times M$ such that

$$\begin{aligned} &d_{\mathfrak{F}}((\varphi|_{B_{\mathfrak{F}}(x,\varepsilon(x))})^{-1} p_{\mathfrak{F}'}(\varphi(y)), x) \\ &\quad = d_{\mathfrak{F}}((\varphi|_{B_{\mathfrak{F}}(x,\varepsilon(x))})^{-1}(p_{\mathfrak{F}'}(\varphi(\rho(g)^{-n}y))), \rho(g)^{-n}x) \\ &\quad \leq CC_1 d(\varphi(\rho(g)^{-n}x), \varphi(\rho(g)^{-n}y) = CC_1 d(\theta(\rho(g)^{-n}x), \theta(\rho(g)^{-n}y)) \end{aligned}$$

whenever $x \in U_5$ and $n \in N_R(x, y)$. Since $\theta$ is uniformly continuous and we have $d(\rho(g)^{-n}x, \rho(g)^{-n}y) \to 0$ as $n \to \infty$, by choosing $n \in N_R(x, f)$ large enough, we can guarantee that

$$CC_1 d(\theta(\rho(g)^{-n}x), \theta(\rho(g)^{-n}y)) < \eta.$$

Since the choice of $\eta$ is free, this proves the lemma. □

*Proof of Propostion* 5.1. . Take the sequence $V_{\frac{1}{n}}$. Then $V = \bigcap_{n=1}^{\infty} V_{\frac{1}{n}}$ is a set of full measure in $H/\Lambda \times M \times F^s_{\rho(g)}$ and is also a set of full measure in almost every fiber. By definition of $V_\beta$, for any $x \in U_\beta$ such that $V$ is of full measure in the $F^s_{\rho(g)}$ fiber over $x$, $\varphi$ takes a set of points of full measure in $\mathcal{W}^s_{\rho(g)}(x)$ to points in $\mathcal{W}^s_{\rho'(g)}(\varphi(x))$. □

5.2. *$\varphi$ is a homeomorphism.* In our setting, $\varphi = \phi \circ \psi$ is not *a priori* a homeomorphism since $\phi$ is not even *a priori* continuous. However, we will show that $\varphi$ is agrees almost everywhere with a homeomorphism when restricted to the leaves of any of the foliations $\mathcal{W}^s_{\rho(g)}$ for $g \in \Phi$. We will then use this fact to prove that $\varphi$ is in fact a homeomorphism. We begin with some definitions. Recall that if $X$ is a Riemannian manifold with a foliation $\mathfrak{F}$, there is a natural volume on the leaves of $\mathfrak{F}$ defined by the restriction of the Riemannian metric to $T\mathfrak{F}$. We will call a map of a noncompact space *uniformly small* if it is uniformly close to the identity on all compact sets. Similarly, we say that two homeomorphisms $h, g$ are uniformly close if $hg^{-1}$ is uniformly small and we say that a sequence of homeomorphisms $h_n$ on a noncompact *converge uniformly* to a homeomorphism $h$ if the map $h_n h^{-1}$ is uniformly small.

*Definition* 5.4. Given a Riemannian manifold $X$ equipped with a foliation $\mathfrak{F}$ by smooth manifolds and a map $h : X \to X$, we say that $f$ is:

(1) *essentially continuous* along $\mathfrak{F}$ if for almost every $x \in X$ the restriction of $h$ to $\mathfrak{F}(x)$ agrees almost everywhere with a continuous map,

(2) *essentially a homeomorphism* along $\mathfrak{F}$ if for almost every $x \in X$ the restriction of $h$ to $\mathfrak{F}(x)$ agrees almost everywhere with a uniformly small homeomorphism and,

(3) *essentially uniformly continuous* along $\mathfrak{F}$ if it is essentially a homeomorphism along $\mathfrak{F}$ and for every sequence $x_n \to x$ with $h(x_n) \to h(x)$, the maps $h|_{\mathfrak{F}(x_n)}$ agree almost everywhere with maps which converge uniformly to a homeomorphism $\bar{h} : \mathfrak{F}(x) \to \mathfrak{F}(h(x))$.

The first step in proving continuity of $\varphi$ is proving:

PROPOSITION 5.5. *For any $g \in \Phi$ the map $\varphi$ is essentially uniformly continuous along $\mathcal{W}^s_{\rho(g)}$.*

Before proving the proposition, we require a lemma that follows immediately from the definition of $\psi$. Recall that $\psi$ is continuous and is covered by a map $\tilde{\psi}$ such that:

$$\begin{tikzcd}
(H \times M, \rho) \arrow[r, "\tilde{\psi}"'] \arrow[d, "p"'] & (H \times M, \rho') \arrow[dl, "f"] \\
(Z \backslash H, \bar{\rho}) &
\end{tikzcd}$$

where all maps are right $\Lambda$ equivariant and $p$ and $f$ are left $G$ equivariant. As an immediate consequence of this and the fact that $f$ and $p$ are uniformly $C^0$ close and $\Lambda$ equivariant we have:

LEMMA 5.6. *Let $\mathcal{V}'$ be a leaf of $\mathcal{W}^s_{\rho'(g)}$ and $\tilde{\mathcal{V}}'$ the lift of $\mathcal{V}'$ to $H \times M$. Then $f : \tilde{\mathcal{V}}' \to f(\tilde{\mathcal{V}}')$ is a homeomorphism onto a leaf of $\mathcal{W}^s_{\bar{\rho}(g)}$. Furthermore if $\mathcal{V}$ is a leaf of $\mathcal{W}^s_{\rho(g)}$ with lift $\tilde{\mathcal{V}}$ to $H \times M$, and*

(1) *$\tilde{\mathcal{V}}$ is close to $\tilde{\mathcal{V}}'$ and,*

(2) *$f(\tilde{\mathcal{V}}') = p(\tilde{\mathcal{V}})$*

*then $f : \tilde{\mathcal{V}}' \to f(\tilde{\mathcal{V}}')$ is uniformly close to $p : \tilde{\mathcal{V}} \to p(\tilde{\mathcal{V}})$.*

Since $\phi$ preserves the foliation of $H/\Lambda \times M$ which is covered by fibers of $p$, we have that $\varphi$ commutes locally with the projections $f$ and $p$, i.e. that for any

$U \subset H/\Lambda \times M$, we have

$$\begin{tikzcd} U \arrow[r, "\varphi"'] \arrow[d, "p"'] & \varphi(U) \arrow[dl, "f"] \\ p(U) & \end{tikzcd}$$

where the arrows $p$ and $f$ are defined by viewing $U$ and $\varphi(U)$ as subsets of $H \times M$. With this in mind we can now proceed to prove Proposition 5.5.

*Proof of Proposition* 5.5. By Proposition 5.1, for almost every $x$, we have that

$$\varphi(\mathcal{W}^s_{\rho(g)}(x)) \subset \mathcal{W}^s_{\rho'(g)}(\varphi(x)).$$

By the commutative diagram above, for almost every $x$ there is a neighborhood $U$ of $x$ such that $f(\varphi(\mathcal{W}^s_{\rho(g)}(x) \cap U)) = p(\mathcal{W}^s_{\rho(g)}(\varphi(x)) \cap \varphi(U))$ and therefore $\varphi(\mathcal{W}^s_{\rho(g)}(x) \cap U) = \mathcal{W}^s_{\rho'(g)}(\varphi(x)) \cap \varphi(U)$. Furthermore, since by Lemma 5.6 $f|_{\mathcal{W}^s_{\rho'(g)}(\varphi(x)) \cap U}$ projects $\mathcal{W}^s_{\rho'(g)}(\varphi(x)) \cap U$ homeomorphically onto

$$f(\mathcal{W}^s_{\rho'(g)}(\varphi(x)) \cap \varphi(U)),$$

we can write $\varphi|_{\tilde{\mathcal{W}}^s_{\rho(g)}(x)}$ as

$$p|_{(\mathcal{W}^s_{\rho(g)}(x) \cap \varphi(U))} \circ f|_{\mathcal{W}^s_{\rho'(g)}(\varphi(x)) \cap \varphi(U)}{}^{-1}$$

which is clearly a homeomorphism. The fact that $f|_{(\mathcal{W}^s_{\rho'(g)}(\varphi(x)) \cap \varphi(U))}$ is $C^0$ close to $p|_{\mathcal{W}^s_{\rho(g)}(x)}{}^{-1} \cap U$ by Lemma 5.6 implies that $\varphi$ is essentially a homeomorphism along $\mathcal{W}^s_{\rho(g)}$.

That $\varphi$ is essentially uniformly continuous along $\mathcal{W}^s_{\rho(g)}$ follows from the fact that $f$ and $p$ are $C^0$ and uniformly $C^0$ close on all of $H \times M$, are homeomorphisms when restricted to leaves of $\mathcal{W}^s_{\rho(g)}$ and $\mathcal{W}^s_{\rho'(g)}$ respectively and the fact that the foliations $\mathcal{W}^s_{\rho(g)}$ and $\mathcal{W}^s_{\rho'(g)}$ are continuous. $\square$

At this point we want to conclude that since $\varphi$ is essentially a homeomorphism and essentially uniformly continuous along foliations whose tangent spaces span $T(H/\Lambda \times M)$ at each point, $\varphi$ is a homeomorphism. However it is unclear that $\varphi$ should agree with a single well-defined global homeomorphism. Our proof of this uses the fact that the foliations involved are smooth, or at least absolutely continuous, in order to use Fubini's theorem repeatedly. We first give two general lemmas from which we will deduce continuity of $\varphi$. To avoid technicalities concerning integrability, we will prefer to work with 1 dimensional foliations. For our application we need only the second statement in the following lemma, but we state and prove the first statement since it makes the ideas involved clearer.

LEMMA 5.7. (1) *Let $X$ be an $n$ dimensional compact Riemannian manifold and $V_1, V_2, \ldots V_n$ smooth nowhere vanishing vector fields such that $TX_x =$*

*$\bigoplus_{i=1}^n V_i(x)$ for every $x \in X$. Let $\mathfrak{F}_i$ be the foliation tangent to $V_i$, and let $h : X \to Y$ be a measurable map which is essentially uniformly continuous along each $\mathfrak{F}_i$. Then h is a homeomorphism.*

(2) *Let $X$ be as above, let $V_1 \ldots V_k$ be nowhere vanishing vector fields, and let $\mathfrak{F}^c$ be a smooth foliation of $X$ by manifolds of dimension $n-k$ such that $TX_x = \bigoplus_{i=1}^k V_i(x) \oplus T\mathfrak{F}^c(x)$. Let $h : X \to Y$ be a measurable map that is essentially uniformly continuous along each $\mathfrak{F}_i$. Further assume that we can cover $X$ by foliation charts $U_i$ for $\mathfrak{F}^c$ such that h is a small homeomorphism along most leaves of $\mathfrak{F}^c$ in $U_i$. Then h is a homeomorphism.*

*Remark.* An examination of the proof indicates that we could make slightly more general assumptions on the vector fields $V_i$ provided we choose a collection of vector fields which span the tangent space of $X$ at every point and such that the foliations $\mathfrak{F}_i$ are absolutely continuous. We only state and prove the version needed for our applications to avoid unnecessary technicalities.

*Proof.* We first prove (1) and then explain how to modify the proof to prove (2). We work in a chart $U$ that is a foliation chart for each $\mathfrak{F}_i$ and can in fact assume that $U = \mathbb{R}^n$ and that the foliation $\mathfrak{F}_i$ is given by lines parallel to the line $l_i$ where $x_j = 0$ for $j \neq i$. We denote the line parallel to $l_i$ passing through the point $y$ by $\tilde{l}_i(y)$. Let $W_j = l_1 \times l_2 \times \ldots \times l_j$ and let $\tilde{W}_j$ be a $j$ plane parallel to $W_j$ specified by coordinates $(x_{j+1}, \ldots, x_n)$. We prove by induction that $h$ agrees almost everywhere with a homeomorphism $\tilde{h}_j(x_j + 1, \ldots, x_n)$ along almost every $j$ plane $\tilde{W}_j$. By assumption $h$ agrees almost everywhere with a small homeomorphism along almost every line parallel to $l_1$. Assume $h$ agrees almost everywhere with a continuous function $\tilde{h}_j$ along almost every $j$ plane $\tilde{W}_j$ parallel to $W_j$. Then by Fubini, for almost every such $j$ plane $\tilde{W}_j$, we have that for almost every $y \in \tilde{W}_j$, the map $h$ agrees almost everywhere on $\tilde{l}_{j+1}(y)$ with a small homeomorphism $\tilde{h}_y$. We define a map $h_{j+1}(x_{j+2}, \ldots, x_n)$ on $\tilde{W}_{j+1}$ by letting $h_j = \tilde{h}_y$ where that map agrees almost everywhere with $h$. This map extends continuously to a homeomorphism on $\tilde{W}_{j+1}$ since $h$ is essentially uniformly continuous and therefore the maps $\tilde{h}_y$ are uniformly continuous. The map $h_n$ is a small homeomorphism that agrees almost everywhere on $U$ with $h$. That $h_n$ is independent of the chart chosen follows easily from the definitions.

For (2) we use induction to prove a slightly weaker statement which still suffices. We re-index our vector fields as $V_{k+1}, \ldots, V_n$ and re-index the resulting foliations similarly. We work in a foliation chart $U$ for $\mathfrak{F}^c$ and $\mathfrak{F}_i$ such that $\mathfrak{F}^c$ is given by planes $\tilde{W}_k$ of the form $\mathbb{R}^k \times (x_{k+1}, \ldots, x_n)$ and for $k+1 \leq i \leq n$, each foliation $\mathfrak{F}_i$ is given by lines $l_i$ as above. To begin our induction, we use the fact that $h$ agrees with a small homeomorphism along most of the planes $\tilde{W}_k$. The induction follows as before, with the change that at each step we only assume that $h$ agrees almost everywhere with a homeomorphism on most planes $\tilde{W}_j$ and $\tilde{W}_{j+1}$. $\square$

We now need a lemma to show that in the setting of our applications we can apply Lemma 5.7.

LEMMA 5.8. *Let $X$ be a compact Riemannian manifold, $\mathfrak{F}$ a smooth foliation of $X$ and $V$ a nowhere vanishing smooth vector field on $X$ such that $V(x) \in T\mathfrak{F}_x$ for every $x \in X$. Let $\mathfrak{F}_V$ be the foliation tangent to $V$. If $h : X \to Y$ is essentially uniformly continuous along $\mathfrak{F}$, then $h$ is essentially uniformly continuous along $\mathfrak{F}_V$.*

*Proof.* It suffices to work in a chart $U$ which is a foliation chart for both $\mathfrak{F}$ and $\mathfrak{F}_V$. We can choose coordinates on such a chart so that $U = \mathbb{R}^n$, and the foliation $\mathfrak{F}$ is given by $\mathbb{R}^k \times y$ where $y \in \mathbb{R}^{n-1}$ and the foliation $\mathfrak{F}_V$ is given by $\mathbb{R} \times z$ where $z \in \mathbb{R}^{n-1}$. We are assuming that $h$ agrees with a homeomorphism $\tilde{h}$ of $\mathbb{R}^k \times y$ for almost every $y$, and by Fubini's theorem this implies that $\tilde{h}$ and $h$ agree almost everywhere on $\mathbb{R} \times z$ for almost every $z$. $\square$

We are now prepared to prove continuity of $\varphi$.

THEOREM 5.9. *The map $\varphi$ constructed in the proof of Theorem 4.4 agrees almost everywhere with a $C^0$ small homeomorphism.*

*Proof.* We will apply Lemma 5.7 to $\varphi$. In doing so, we let $\mathfrak{F}^c$ be $\mathfrak{F}$. It follows from Theorem 4.4(4) that by restricting to small enough perturbations, we can cover $H/\Lambda \times M$ by foliation charts where $\varphi$ is $C^{k-1-\kappa}$ small on most leaves of $\mathfrak{F}$ for some $\kappa \leq 1$, and therefore that $\mathfrak{F}$ satisfies the hypotheses on $\mathfrak{F}^c$ in Lemma 5.7. We choose elements $V_i \in \mathfrak{h}$ such that each $V_i \in \mathfrak{f}^s_{\rho(g)}$ for some $g \in \Phi$ and such that $\oplus V_i \oplus \mathfrak{z} = \mathfrak{h}$. Then each $V_i$ defines a smooth nonvanishing vector field $\tilde{V}_i$ on $H/\Lambda \times M$, and $T(H/\Lambda \times M)_x = \oplus V_i(x) \oplus T\mathfrak{F}$. By Lemma 5.8 and Proposition 5.5, we have that $\varphi$ is essentially uniformly continuous along the foliation $\mathfrak{F}_i$ tangent to $V_i$ for each $i$. Therefore, we can apply Lemma 5.7 to $\varphi$ which implies that $\varphi$ is a small homeomorphism. $\square$

5.3. *Additional arguments in the case of discrete groups.* In the case of $\Gamma$ actions, we have been assuming that the unperturbed $\Gamma$ action lifts to the cover $H \times M$. As remarked above, this is always true on a finite index subgroup $\Gamma'$ of $\Gamma$ which depends only on $\rho$. We have constructed a continuous $C^0$ small conjugacy for the $\Gamma'$ actions with additional regularity along $\mathfrak{F}$ and we now explain how to replace this with a $C^0$ small conjugacy for the $\Gamma$ actions with the same additional regularity along $\mathfrak{F}$. The passage to $\Gamma'$ is required in the proof of Theorem 4.2. In that proof, when we conjugate the $\rho'$ action by $\psi$ we only know that the $\Gamma'$ action defined by $\psi^{-1} \circ \rho' \circ \psi$ preserves $\mathfrak{F}$ and therefore is in a small neighborhood of $\rho$ in $\mathrm{Hom}(\Gamma', \mathrm{Diff}^k_\nu(H/\Lambda \times M, \mathfrak{F}))$. Neither of these facts is clear for the full $\Gamma$ action.

In fact it suffices to show that $\psi^{-1} \circ \rho' \circ \psi(\Gamma)$ preserves $\mathfrak{F}$, since closeness to $\rho$ in $\mathrm{Hom}(\Gamma, \mathrm{Diff}^k_\nu(H/\Lambda \times M, \mathfrak{F})$ then follows from the definition of $\psi$. Therefore the remainder of this subsection is dedicated to a proof that $\psi^{-1} \circ \rho' \circ \psi(\Gamma)$ preserves $\mathfrak{F}$. Without loss of generality, we may assume $\Gamma'$ is normal in $\Gamma$.

Given two (closed) subsets $A, B$ of a metric space $(X, d)$, we let $d_S(A, B) = \inf_{a \in A, b \in B} d(a, b)$.

*Definition* 5.10. Let a group $D$ act on a manifold $X$ preserving a foliation $\mathfrak{F}$. We call the action *c-leafwise expansive* if there exists a constant $c$, such that $\mathfrak{L}$ and $\mathfrak{L}'$ are distinct leaves of $\mathfrak{F}$ and there is $f \in F$ such that $d_S(\rho(g)\mathfrak{L}, \rho(g)(\mathfrak{L}')) > c$.

Note that many foliations, e.g. any foliation with a dense leaf, do not admit leafwise expansive actions. We will be applying Definition 5.10 to the lift of $\rho$ to $H \times M$, which is $c$-leafwise expansive by Corollary 3.5.

LEMMA 5.11. *Let $c > 0$ and $\rho$ be a c-leafwise expansive action of a group $D$ on a foliated metric space $(X, d, \mathfrak{F})$. Let $h$ be a homeomorphism of $X$ such that*:

(1) $d(h(x), x) < c$ *for all* $x \in X$,

(2) $h(\rho(g)\mathfrak{L}) = \rho(g)(h(\mathfrak{L}))$ *for any leaf* $\mathfrak{L}$ *of* $\mathfrak{F}$ *and any* $d \in D$, *i.e. $h$ and $\rho$ commute as actions on leafs of* $\mathfrak{F}$,

*then* $h(\mathfrak{L}) = \mathfrak{L}$ *for every leaf* $\mathfrak{L}$ *of* $\mathfrak{F}$.

*Proof.* Assume $h(\mathfrak{L}) \neq \mathfrak{L}$. Then there is a point $x \in \mathfrak{L}$ with $h(x) \notin \mathfrak{L}$. By our assumptions, there exists $g \in D$ such that $d_S(\rho(g)(\mathfrak{F}_{h(x)}), \rho(g)\mathfrak{F}_x) > c$. But then $d_S(h(\rho(g)(\mathfrak{F}_x)), (\rho(g)(\mathfrak{F}_x))) > c$ which contradicts (1) above. $\square$

We define a subgroup $\mathrm{Homeo}(H/\Lambda \times M, \mathfrak{F})$ of $\mathrm{Homeo}(H/\Lambda \times M)$ which consists of all homeomorphisms which map each leaf of $\mathfrak{F}$ to itself.

PROPOSITION 5.12. *Given a quasi-affine action $\rho$ of $\Gamma'$ on $H/\Lambda \times M$ which lifts to $H \times M$, any small enough homeomorphism in the centralizer of $\rho(\Gamma')$ in* $\mathrm{Homeo}(H/\Lambda \times M)$ *is an element of* $\mathrm{Homeo}(H/\Lambda \times M, \mathfrak{F})$.

*Proof.* If $f$ is a small homeomorphism commuting with $\rho(\Gamma')$, there is a unique lift $\tilde{f}$ of $f$ to $H \times M$ such that $\tilde{f}$ is small as a homeomorphism of $H \times M$. Since $\tilde{f}$ is small, for any small enough $\gamma \in \Gamma'$ we have that $[\tilde{f}, \tilde{\rho}(\gamma)]$ is a small homeomorphism of $H \times M$ covering the identity on $H/\Lambda \times M$ and so $\tilde{f}$ and $\tilde{\rho}(\gamma)$ commute. Since $\Gamma'$ is finitely generated this implies that $\tilde{f}$ commutes with $\Gamma'$ on $H \times M$. Let $\tilde{\mathfrak{F}}$ be the lift to $H \times M$ of the foliation $\mathfrak{F}$. Since Corollary 3.5 implies that the $\tilde{\rho}'(\Gamma)$ action on $H \times M$ is leafwise expansive, Lemma 5.11 implies that $\tilde{f}$ maps each leaf of $\tilde{\mathfrak{F}}$ to itself. This then implies that $f$ maps each leaf of $\mathfrak{F}$ to itself. $\square$

We need one more purely algebraic lemma. For simplicity, we will write $\rho''(\gamma)$ for $\varphi^{-1}\circ\rho'(\gamma)\circ\varphi$ and state the lemma only in the form needed for our applications. The lemma is true for any pair of homomorphisms from a group $D$ to a group $H$ which agree on a normal subgroup in $D$.

LEMMA 5.13. *If $\gamma_0$ is in $\Gamma$, then the diffeomorphism $\rho(\gamma_0)\circ\rho''(\gamma_0)^{-1}$ commutes with $\rho(\gamma)$ for all $\gamma\in\Gamma'$.*

*Proof.* For any $g\in\Gamma'$ and $\gamma\notin\Gamma$, we have $\gamma_0\gamma{\gamma_0}^{-1}\in\Gamma'$ which implies that $\rho''(\gamma_0\gamma{\gamma_0}^{-1})=\rho(\gamma_0\gamma{\gamma_0}^{-1})$. Expanding gives:

$$\rho''(\gamma_0)\rho(\gamma)\rho''({\gamma_0}^{-1})=\rho(\gamma_0)\rho(\gamma)\rho({\gamma_0}^{-1})$$

which can be rearranged as

$$\big(\rho(\gamma_0)^{-1}\rho''(\gamma_0)\big)\rho(\gamma)=\rho(\gamma)\big(\rho({\gamma_0}^{-1})\rho''(\gamma_0)\big)$$

proving the lemma. □

We choose a set of coset representatives $\gamma_1,\dots,\gamma_j$ for $\Gamma/\Gamma'$ and assume that $\rho'$ is close enough to $\rho$ so that $\rho(\gamma_i)\rho''(\gamma_i)^{-1}$ is sufficiently $C^0$ small so that Lemma 5.13 and Proposition 5.12 imply that $\rho(\gamma_i)\rho''(\gamma_i)^{-1}$ is close to the identity in $\mathrm{Homeo}(H/\Lambda\times M,\mathfrak{F})$. This implies that $\rho''(\gamma_i)\mathfrak{L}=\rho(\gamma_i)\mathfrak{L}$ or that $\rho'(\gamma)\circ\varphi(\mathfrak{L})=\varphi\circ\rho(g_i)(\mathfrak{L})$ for all $\gamma\in\Gamma$.

We know that $\varphi=\phi\circ\psi$ where $\psi$ is the homeomorphism constructed in Theorem 3.8. This implies that $\phi$ is also a homeomorphism, which, by construction is in $\mathrm{Homeo}(H/\Lambda\times M,\mathfrak{F})$. Combined with the conclusion of the last paragraph, this implies that $\rho'(\gamma)\circ\psi(\mathfrak{L})=\psi\circ\rho(\gamma_i)(\mathfrak{L})$ for all $\gamma\in\Gamma$. This suffices to allow us to apply Theorem 4.2 to the entire $\Gamma$ action in the proof of Theorem 4.4, rather than just to the $\Gamma'$ action. This constructs a map $\phi$ such that $\varphi=\phi\circ\psi$ is $\Gamma$ equivariant and $\phi$ satisfies all the conclusions stated in Theorem 4.4.

*Remarks.* (1) We can now re-apply the arguments of subsections 5.1 and 5.2 to show that $\varphi$ is a $C^0$ small homeomorphism.

(2) It is not clear that the $\varphi$ constructed from the $\Gamma'$ action is actually equivariant for $\Gamma$. In applying Theorem 4.2 to the $\Gamma$ action, we may be finding a different conjugacy.

(3) Due to the arguments of this subsection, for the remainder of this paper, we no longer assume that $\rho$ lifts to an action of $G$ on $H\times M$.

## 6. Smoothness along dynamical foliations

In this section, we adapt the method of Katok-Spatzier to show that $\varphi$ is differentiable along certain special expanding and contracting foliations by constructing transitive $C^k$ group actions along those foliations. All notation are as in the previous section.

6.1. *Some other important dynamical foliations.* In this subsection we define some additional important foliations related to the group actions $\rho$ and $\rho'$. These foliations are the ones to which we will apply the method of Katok-Spatzier, building transitive smooth actions of Lie groups, along the leaves, that are intertwined by $\varphi$. First we define the relevant foliations in the unperturbed setting. The exposition here is similar to the exposition in section 5.1 of [MQ01].

Recall that $G = J$ or $\Gamma$. Let $S$ be a maximal $\mathbb{R}$ split torus in $J$ and $T$ be a maximal torus containing $S$. The foliations defined here depend on the choice of $T$ and in the case of $\Gamma$ actions, we will make restrictions on that choice below.

Recall from Definition 1.2 that $\rho$ is a skew product action on $H/\Lambda\times M$. More precisely, the action on $H/\Lambda\times M$ is defined by an action on $H/\Lambda$ and a cocycle $\iota : G\times H/\Lambda\to K$ over that action where $K < \mathrm{Isom}(M)$. Recall from Theorems 2.1 and 2.3 that, possibly after passing to a subgroup of finite index when $G = \Gamma$, the action on $H/\Lambda$ is defined by a homomorphism $\pi : G\to L$ where $L = \mathrm{Aut}(H)^0\ltimes H$ is an algebraic group. Note that $H$ is normal in $L$, so that $\mathfrak{h}$ is invariant under $\mathrm{Ad}\, L$. We have an invariant splitting of the tangent bundle $T(H/\Lambda\times M) = (H/\Lambda\times\mathfrak{h})\times TM$ and all elements of $G$ are isometries along $TM$. The derivative cocycle leaves $\mathfrak{h}$ and $TM$ invariant, and, again after passing to a subgroup of finite index, if $G = \Gamma$ the restriction to $\mathfrak{h}$ is given by the representation $\sigma = \mathrm{Ad}_L|_{\mathfrak{h}}\circ\pi_0$ of $G$ on $\mathfrak{h}$. From now on when describing the action and the derivative cocycle, we assume that if the acting group is $\Gamma$ we have passed to a finite index subgroup for which this description holds. We recall that $\pi_0 = \pi_0^E\pi_0^K$ where $\pi_0^E$ is (the restriction of) a representation of $J$, $\pi_0^K$ has bounded image, and the images of $\pi_0^K$ and $\pi_0^E$ commute. Therefore we can write $\sigma = \sigma^E\sigma^K$ where $\sigma^E$ is (the restriction of) a representation of $J$, $\sigma^K$ has bounded image and the images of $\sigma^K$ and $\sigma^E$ commute.

For $g \in T\cap G$, define the *Lyapunov exponents* of $\sigma(g)$ as the logs of the absolute values of the eigenvalues of $\sigma(g)$. We obtain homomorphisms $\chi : T\cap G\to\mathbb{R}$ which extend to homomorphisms $\chi : T\to\mathbb{R}$. The $\chi$ are exactly the absolute values of the weights of the representation $\sigma^E$ for the torus $T$, and we will refer to them as *generalized weights*. There is a decomposition of $\mathfrak{h}$ into generalized weight spaces $E_\chi$, $\mathfrak{h} = \bigoplus_\chi E_\chi$. Corresponding to this there is a decomposition of the tangent bundle to $H/\Lambda\times M$ into invariant subbundles for the derivative action, $T(H/\Lambda\times M) = \big((H/\Lambda\times(\bigoplus_\chi E_\chi)\big)\times TM$. We call $H/\Lambda\times M\times E_\chi$ a *Lyapunov distribution* for the $G\cap T$ action defined by $\rho$.

The set $\Omega$ of all generalized weights for $(\sigma, T)$ can be decomposed into disjoint subsets $[\chi]$ such that $\chi' \in [\chi]$ if and only if $\chi' = t\chi$ for some positive real number $t$. We fix a set $\bar{\Omega} \subset \Omega$ of representatives for the subsets $[\chi]$. If $\chi = 0$ identically, we call $H/\Lambda\times M\times E_{0,T}$ the *central distribution* for the action $G\cap T$. It is integrable, and we denote by $\mathcal{W}_{0,T}$ the corresponding foliation. It is clear that

$\cap_T E_{0,T} = E_0$ defines an integrable distribution and that $H/\Lambda \times E_0 \oplus TM = E^c_{\rho(G)}$. Other Lyapunov distributions may or may not be integrable, but $H/\Lambda \times M \times E_{[\chi]} = \bigoplus_{\lambda \in [\chi]} H/\Lambda \times M \times E_\lambda$ is always integrable. We denote the integral foliation on $H/\Lambda \times M$ by $\mathcal{W}_{[\chi]}$, suppressing the dependence on $\rho$ and $T$ in the notation.

We now describe the choice of $T$ for the case $G = \Gamma$. The reader should note that if we choose $T$ so that $T \cap \Gamma = e$, then all of $\mathfrak{h}$ is in $E_0$. To obtain a more useful set of Lyapunov distributions, we need the following theorem which we derive from results of Prasad and Rapinchuk from [PR01]. Without loss of generality, we assume that $\Gamma$ is the direct product of a finite number of irreducible lattices $\Gamma_i$. We write $J = \prod_I J_i$ where $\Gamma_i < J_i$ is irreducible, and for any maximal torus $T < J$, we can write $T = \prod_I T_i$ where $T_i < J_i$ is a maximal torus.

THEOREM 6.1. *Let $J$ and $\Gamma$ be as above. Then there is a maximal torus $T$ in $J$ such that*:

(1) *$T$ contains a maximal $\mathbb{R}$-split torus $S$.*

(2) *$\Sigma = \Gamma \cap T$ is cocompact in $T$.*

(3) *For any $T_i$ there is no proper algebraic torus $T_i' < T_i$ such that $T_i' \cap \Gamma$ is a lattice in $T_i'$.*

*Furthermore if $\mu : J \to \mathrm{GL}_n(\mathbb{R})$ is any linear representation of $J$ and $\sigma \in \Sigma$ projects to an infinite order element in each $\Gamma_i$, then $\xi(\sigma)$ is not a root of unity for any nontrivial weight $\xi$ of $\mu$.*

*Proof.* It suffices to prove the theorem for $\Gamma_i < J_i$. The first two assertions follow from [PR01, Th. 1] and the assertion immediately preceding the proof of that theorem. The third assertion is an immediate consequence of two facts. First $\Gamma_i$ is arithmetic, and therefore $\Gamma_i < \mathbb{G}(k)$ for some field $k$. Combined with [PR01, Prop. 1(ii))] this implies that any infinite order element of $\Gamma_i$ generates a Zariski dense subgroup of $T_i$. The last statement follows from [PR01, Prop. 1(iii)] and the fact that $\mu$ necessarily agrees with a homomorphism of $\Gamma_i$ on subgroup of finite index and so must be defined over a finite extension of $k$. □

From now on we assume that we have picked $T$ satisfying the conclusion of Theorem 6.1. We do not use all the properties of $T$ here, but will need them in Section 6.4.

The following lemma is analogous to [MQ01, Lemma 5.2].

LEMMA 6.2. *Fix a maximal torus $T$ as above. Let $E(T)$ be the sum of $E_{[\chi]}$ for all nontrivial weights $\chi$ for $(\sigma, T)$. Then there exists a finite subset $\Psi \subset G$ such that*:

$$T(H/\Lambda \times M) = TM \times \Big(\sum_{g \in \Psi} D\rho(g)(H/\Lambda \times E(T))\Big) \times E_0.$$

*Proof.* It suffices to prove that

$$T(H/\Lambda) = \Big(\sum\nolimits_{g\,\in\,\Psi} D\rho(g)(H/\Lambda\times E(T))\Big)\oplus E_0).$$

We know that the derivative of $\rho$ is given by linear representation $\sigma: G \to \mathrm{Ad}(L)$. The structure of $\sigma$ implies that $\mathfrak{h} = \mathfrak{h}^E \oplus \mathfrak{h}^K$ where $\sigma^E$ (resp. $\sigma^K$) is trivial on $\mathfrak{h}^K$ (resp. $\mathfrak{h}^E$). From the definitions, it suffices to see that

$$\mathfrak{h}^E = \sum_{g\,\in\,\Psi} D\rho(g)(H/\Lambda\times E(T)).$$

The inclusion of the right-hand in the left-hand side is clear. If the left-hand side contains a subspace $V$ not contained in the right-hand side, then we have that $V$ is in the zero weight space for $\sigma^E|_{G\,\cap\,T}$ with respect to our choice of maximal torus $T$ for $J$. For $G = J$, this is only possible if the representation of $\sigma^E|V$ is trivial, contradicting our assumptions. For $G = \Gamma$, the contradiction follows since we have chosen a $T$ as in Theorem 6.1. □

Fix a nontrivial generalized weight $\chi_0$ for $(\sigma^E, T)$. Then there exists $g_0$ in $T \cap D$ such that $\chi_0(g_0) < 0$. It follows that for all $\chi' \in [\chi_0]$, $\chi'(g_0) < 0$. For every $g \in T \cap G$ with $\chi_0(g) < 0$, note that $\bigoplus_{\chi(g)<0} E_\chi$ is the stable distribution $E^s_{\rho(a)}$. This distribution is tangent to the foliation $\mathcal{W}^s_{\rho(a)}$ . It is clear that $E^s_{\rho(a)}$ contains $E_{[\chi_0]}$. We call an element of $T$ *regular* if for all nontrivial weights $\chi$ for $(\sigma^E, T)$, $\chi(a)\neq 0$. Combined with the usual descending chain arguments, this yields the following lemma.

LEMMA 6.3. *Let $\chi_0$ be a nontrivial weight for $(\sigma^E, T)$. Then*

$$E_{[\chi_0]} = \bigcap E^s_{\rho(a)}$$

*where the intersection is taken over all regular $a$ with $\chi_0(a) < 0$. Furthermore, there exist regular elements $a_1, \ldots a_q \in T \cap G$ with $\chi_0(a_i) < 0$ such that we can take the intersection just over $E^s_{\rho(a_i)}$.*

We now define a finite collection of foliations and distributions which we will use below. Fix a maximal torus $T$ and a set $\bar{\Omega} \subset \Omega$ as above. Also fix a collection of elements $a_1, \ldots, a_q \in T$ as in Lemma 6.3. Given $g \in G$ let $E^g_{[\chi]} = \rho(g)E_{[\chi]}$, $\mathcal{W}^g_{[\chi]} = \rho(g)\mathcal{W}_{[\chi]}$, $E^{g,s}_{\rho(a_i)} = \rho(g)E^s_{\rho(a_i)}$ and $\mathcal{W}^{g,s}_{\rho(a_i)} = \rho(g)\mathcal{W}^s_{\rho(a_i)}$. We can also define $E^{g,s}_{\rho'(a_i)} = \rho'(g)E^s_{\rho'(a_i)}$ and $\mathcal{W}^{g,s}_{\rho'(a_i)} = \rho'(g)\mathcal{W}^s_{\rho'(a_i)}$. We will show that $\varphi$ is smooth along each $\mathcal{W}^g_{[\chi]}$ for $g \in \Psi$ and $\chi \in \bar{\Delta}$. To do this, we first need to identify the image of $\mathcal{W}^g_{[\chi]}$ under $\varphi$.

PROPOSITION 6.4. *For every $m > 0$, if $\rho'$ is sufficiently $C^1$ close to $\rho$, for every $x$ the intersection $\tilde{\mathcal{W}}^g_{[\chi]} = \bigcap \mathcal{W}^{g,s}_{\rho'(a_i)}$ is a $C^k$ submanifold tangent to the distribution $\tilde{E}_{[\chi_0]} = \bigcap \tilde{E}^{s,g}_{a_i}$. Furthermore, $\varphi(\mathcal{W}^g_{[\chi]}(x)) = \tilde{\mathcal{W}}^g_{[\chi]}(\varphi(x))$ for every $x \in X$.*

*Proof.* It suffices to consider the case where $g$ is the identity, since other cases follow by translation. We will show that the intersection is $s$-transverse and then apply Lemma 3.12(1). Since dimension of intersection of the distributions $E^s_{\rho'(a_i)}$ can only be smaller than for the corresponding intersection of $E^s_{\rho(a_i)}$, it suffices to show that dimensions of the intersections of the foliations do not decrease. This is immediate from the fact that $f$ projects $\mathcal{W}^s_{\rho'(a_i)}$ homeomorphically onto $\mathcal{W}^s_{\bar{\rho}(a_i)}$ by Lemma 5.6 and $p$ projects $\mathcal{W}^s_{\rho(a_i)}$ homeomorphically onto $\mathcal{W}^s_{\bar{\rho}(a_i)}$ by definition. So dimensions of all intersections of foliations are equal in the perturbed and unperturbed cases and therefore the intersection $\bigcap \mathcal{W}^{g,s}_{\rho'(a_i)}$ is $s$-transverse.

The final claim follows from the fact that $\varphi(\mathcal{W}^{g,s}_{\rho(a_i)}(x)) = \mathcal{W}^{g,s}_{\rho'(a_i)}(\varphi(x))$ which is true for each $a_i$ for almost every $x$ by Proposition 5.1, and therefore also true for all $x$ by Theorem 5.9 and Proposition 5.5. It is also possible to prove that $\varphi(\mathcal{W}^{g,s}_{\rho(a_i)}(x)) = \mathcal{W}^{g,s}_{\rho'(a_i)}(\varphi(x))$ for all $x$ directly by a dynamical argument. □

The rest of this section describes a variant of the method of Katok- Spatzier which we use to show smoothness of $\varphi$ along the foliations $\mathcal{W}^g_{[\chi]}$ for $g \in \Psi$ and $\chi \in \bar{\Delta}$. The outline here is close to that of [KS97] or [MQ01] , but there are two additional difficulties. First, we need to have estimates on the $C^k$ size of various maps in both the normal form theory of Guysinsky-Katok [GK98], [Guy02] and in the work of Montgomery-Zippin [MZ55]. In both cases, these estimates follow from examination of the existing proofs, as is explained below. Secondly, we will need to show that ergodic components consist of entire leaves of $\mathcal{W}^g_{[\chi]}$ for a more general class of actions than those considered in [MQ01].

The precise statement we prove is:

THEOREM 6.5. *We use the notation introduced before. Let $n$ be the dimension of $\mathcal{W}^g_{[\chi]}$. Let $\theta : D^n \times D^{m-n} \to X$ be a smooth foliation chart for $\mathcal{W}^g_{[\chi]}$. Then there is a number $k_1$ depending only on $\rho$ such that for all $k \geq k_1$ and all $\rho'$ sufficiently $C^k$ close to $\rho$*:

(1) *The map $\varphi : \mathcal{W}_{[\chi]}(x) \to \tilde{\mathcal{W}}_{[\chi]}(x)$ is a $C^k$ injective immersion.*

(2) *the map $Q : D^{m-n} \to \mathrm{Emb}^k(D^n, X)$ given by $\phi \circ \theta(\cdot, y)$ is continuous and $C^0$ close to the map $Q_0$ induced by the identity on $H/\Lambda \times M$.*

*Furthermore if $\rho'$ is $C^l$ for some $l \geq k$ then*:

(1) *The map $\varphi : \mathcal{W}_{[\chi]}(x) \to \tilde{\mathcal{W}}_{[\chi]}(x)$ is a $C^l$ injective immersion*

(2) *The map $Q : D^{m-n} \to \mathrm{Emb}^l(D^n, X)$ given by $\phi \circ \theta(\cdot, y)$ is continuous.*

6.2. *Theory of nonstationary normal forms.* Before giving the construction of the groups acting transitively on foliations, we outline the theory of nonstationary normal forms that will be used to show smoothness of the group actions on the leaves for the perturbed action. The theorems we use are due to Guysinsky and

Katok and the references are [Guy02], [GK98]. Some of our definitions are slightly different from theirs.

Consider a continuous extension $\mathcal{F}$ of a homeomorphism $f$ of a compact connected metric space $X$ to a vector bundle $V$ over $X$ which is smooth along the fibers and preserves the zero section. Let $F = D\mathcal{F}_0$ where the derivative is taken at the zero section in the fiber direction. Fix a continuous family of Riemannian metrics on the fiber of $V$. Consider the induced operator $\mathcal{F}^*$ on the Banach space of continuous sections of $V$ endowed with the uniform norm, given by $\mathcal{F}^*v(x) = F(v(f^{-1}(x)))$. For $i = 1, \ldots l$, let $\Delta_i = [\lambda_i, \mu_i]$ be a finite set of disjoint intervals on the negative half line with $\lambda_{i+1} > \mu_i$. Assume that $V$ splits as a sum of subbundles $V = V_1 \oplus \ldots \oplus V_k$ such that the spectrum of $\mathcal{F}^*$ on the space of sections of $V_i$ is contained in the annulus with inner radius $\exp(\lambda_i)$ and outer radius $\exp(\mu_i)$. If $\mu_l < 0$, then the map $\mathcal{F}$ is a contraction with respect to the continuous family of Riemannian metrics chosen above.

*Remark.* When $\mathcal{F}$ is a contraction it also makes sense to consider $\mathcal{F}$ which is only defined in a neighborhood of the zero section in $V$. Theorem 6.6 below holds in this generality, and with some care a version of Theorem 6.7 can be stated in this context as well.

We say that $\mathcal{F}$ has *narrow band spectrum* if $\mu_i + \mu_l < \lambda_i$ for all $i = 1, \ldots, l$.

We call two extensions $C^k$ conjugate if there exists a continuous family of $C^k$ diffeomorphisms of the fibers $V(x)$, preserving the origin which transforms one extension into the other. The following two theorems on normal forms and centralizers are from [Guy02], [GK98]. We remark that to avoid unnecessary definitions we did not state the theorems in their full generality, but these are sufficient for our applications.

THEOREM 6.6. *Let $f$ be a homeomorphism of a metric space $X$ and suppose that $\mathcal{F}$ is a $C^l$ extension of $f$ which is a contraction, that the linear extension $D\mathcal{F}_0$ has narrow band spectrum determined by the vectors $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_l)$, and that $\mathcal{F}$ is $C^k$ close to $D\mathcal{F}_0$ in a neighborhood of the zero section. There exists a constant $k_1 = k_1(\lambda, \mu)$ such that if $k \geq k_1$ there exist*

(1) *a finite dimensional Lie group $G_{\lambda,\mu}$ which is a subset of all polynomial maps from $\mathbb{R}^m$ to $\mathbb{R}^m$ of degree less than or equal to $d$ for some $d < \infty$,*

(2) *an extension $\tilde{\mathcal{F}}$ such that for every $x \in X$, the map*

$$\tilde{\mathcal{F}}|_{V(x)} : V(x) \to V(f(x))$$

*is an element of $G_{\lambda,\mu}$;*

(3) *a $C^l$ conjugacy $H$ between $\tilde{\mathcal{F}}$ and $\mathcal{F}$ which is $C^k$ small.*

THEOREM 6.7. *Suppose $g$ is a homeomorphism of the space $X$ commuting with $f$ and $\mathcal{G}$ is an extension of $g$ by $C^l$ diffeomorphisms of the fibers commuting with the extensions $\mathcal{F}$ and that $\mathcal{F}$ satisfies all of the hypotheses of Theorem* 6.6 *and* $k \geq k_1$. *Then $H$ conjugates $\mathcal{G}$ to a map of the same form, i.e. one where*:

$$\tilde{\mathcal{G}}|_{V(x)} : V(x) \to V(f(x))$$

*is a polynomial of degree at most $d$ and is in fact in the group $G_{\lambda,\mu}$ from Theorem* 6.6.

*Proof.* The only statement which is not justified explicitly in the proofs of [GK98], [Guy02] is the bound on the size of $H$. Recall that $H$ is constructed in two steps. First one constructs a conjugacy between $\mathcal{F}$ and an extension $\mathcal{F}'$ of $f$ such that $\mathcal{F}'$ is $C^k$ tangent to $\tilde{\mathcal{F}}$ at the zero section. In this step one proceeds by solving an iterative equation for the conjugacy, see [Guy02, Proof of Th. 1, Step 1, p. 851]. It is clear from the formula that if $\mathcal{F}$ and $D\mathcal{F}_0$ are close, then this conjugacy is small and $\mathcal{F}'$ is also $C^k$ close to $D\mathcal{F}_0$. (We note that in [GK98], this step is broken into two steps, first finding the Taylor series of the conjugacy at the zero section, and then proving that one can find a conjugacy with this Taylor series.) In the second step, one constructs an action $\bar{F}$ on a set of local changes of coordinates, and applies a contraction mapping argument to find the conjugacy $H$ between $\mathcal{F}'$ and $\tilde{\mathcal{F}}$. If $\mathcal{F}' = D\mathcal{F}_0$ it is clear from the construction that this contraction $\bar{F}$ has as a unique fixed point the identity map, and that if $\mathcal{F}'$ is $C^k$ close to $D\mathcal{F}_0$ then this unique fixed point of $\bar{F}$ will be $C^k$ close to the identity. □

*Remark.* The number $k_1$ is explicitly computable in terms of the spectrum of the contraction $\mathcal{F}$; see [Guy02], [GK98] for details. The computation yields that Theorem 6.6 and 6.7 are true for

$$k_1 \geq \left|\frac{\lambda_1\lambda_2 \ldots \lambda_{l-1}}{\mu_2\mu_3 \ldots \mu_l}\right|.$$

For some special choices of $\Delta_i$ it is possible to achieve much lower values of $k_1$.

6.3. *Smoothness along contracting foliations.* In this subsection we retain the number $k_1$ as in the last subsection and note that $\varphi$ is the map constructed in Theorem 4.4, which we know to be a $C^0$ small homeomorphism and a conjugacy between the unperturbed and perturbed actions. We first show that to prove Theorem 6.5, it suffices to verify the following lemma.

LEMMA 6.8. *For $k \geq k_1$ there is a connected Lie group $\Delta$ and, for each $x \in H \times M$, there is an open set $U_0 \subset H \times M$ which contains $x$ and is the union of leaves of $\mathcal{W}_{[\chi]}$, such that*:

S1. *There is a locally free $C^\infty$ action $d : \Delta \times U_0 \to U_0$ such that $\delta\mathcal{W}_{[\chi]}(y) = \mathcal{W}_{[\chi]}(y)$ and $\Delta$ acts transitively on $\mathcal{W}_{[\chi]}(y)$ for all $y \in U_0$.*

S2. *The set* $\varphi(U_0) = U_0{}'$ *is the union of leaves of* $\tilde{\mathcal{W}}_{[\chi]}$ *and there is a locally free* $C^0$ *action* $d' : \Delta \times U_0{}' \to U_0{}'$ *such that* $\delta\tilde{\mathcal{W}}_{[\chi]}(z) = \tilde{\mathcal{W}}_{[\chi]}(z)$ *for all* $z \in U_0{}'$.

S3. *For all* $\delta \in \Delta$, *the map* $d'(\delta) : U_0{}' \to U_0{}'$ *is* $C^l$ *when restricted to every leaf of* $\tilde{\mathcal{W}}_{[\chi]}$, *and all partial derivatives along the leaf are globally continuous. Furthermore, the* $k$*-jet of* $d'(\delta)$ *along leaves of* $\tilde{\mathcal{W}}_{[\chi]}$ *tends to the* $k$*-jet of* $d(\delta)$ *along* $\mathcal{W}_{[\chi]}$ *as* $\rho' \to \rho$.

S4. $\varphi$ *is a* $\Delta$ *equivariant map from* $U_0$ *to* $U_0{}'$.

We temporarily defer the proof of Lemma 6.8 and first show how it implies Theorem 6.5. We state a variant of the results of Bochner and Montgomery which we will use in the proof.

THEOREM 6.9 (Bochner and Montgomery). *Let* $\kappa$ *be a continuous action of a Lie group* $Q$ *on a manifold* $N_1 \times N_2$ *such that the action is trivial in the second factor. If for each* $q \in Q$,*and* $n_2 \in N_2$, *the map* $\kappa(q) : N_1 \times n_2 \to N_1 \times n_2$ *is* $C^l$ *with all derivatives continuous in* $N_1 \times N_2$, *then the map* $\kappa : Q \times N_1 \times \{n_2\} \to N_1 \times \{n_2\}$ *is* $C^l$ *for each* $n_2$ *in* $N_2$ *and depends continuously on* $n_2$ *in the* $C^l$ *topology. Furthermore if* $\kappa$ *and* $\kappa'$ *are two such actions which are* $C^0$ *close, such that* $\kappa(q)$ *and* $\kappa'(q)$ *are* $C^k$ *close as maps of* $N_1$ *for all* $g \in Q$, *then the actions* $\kappa$ *and* $\kappa'$ *are* $C^k$ *close as actions on* $N_1 \times \{n_2\}$ *for any* $n_2 \in N_2$.

*Proof.* All statements follow from the proof of the results of Bochner and Montgomery given in [MZ55, Chap. V, §§1 and 2]. The possibility of adding the $N_2$ factor along which the action is trivial is already noted in [MQ01, Proof of Lemma 5.12]. That the actions are actually $C^k$ close follows from the explicit formulas for derivatives of $\kappa$ along $Q$ given in [MZ55, V.2.1]. □

*Proof of Theorem* 6.5 *from Lemma* 6.8. Possibly after shrinking $U_0$, we can assume that $U_0$ is a product of a leaf $\tilde{\mathcal{V}}_{[\chi]}$ and a small transverse neighborhood $V_0$. (We will in fact construct the $\Delta$ action on such a neighborhood.) The hypotheses $S1 - S4$ imply that the map $\varphi$ intertwines two actions $d$ and $d$ of the group $\Delta$ such that orbits of $d(\Delta)$ (resp. orbits of $d'(\Delta)$) are leaves of $\mathcal{W}_{[\chi]}$ (resp. $\tilde{\mathcal{W}}_{[\chi]}$) and such that, for each $\delta \in \Delta$ and $v \in \varphi(V_0)$, the map $\delta'(d) : \tilde{\mathcal{W}}_{[\chi]} \times \{v\} \to \tilde{\mathcal{W}}_{[\chi]} \times \{v\}$ is $C^k$ with all derivatives depending continuously on $v$. Combined with Theorem 6.9, this implies $d' : \Delta \times \tilde{\mathcal{W}}_{[\chi]} \times \{v\} \to \tilde{\mathcal{W}}_{[\chi]} \times \{v\}$ is a $C^l$ action depending continuously on $v \in \varphi(V_0)$, which suffices to prove Theorem 6.5(1). Since $S3$ and Theorem 6.9 also imply that $d$ and $d'$ are close as actions which are $C^k$ along orbits with all derivatives transversely continuous and that $d'$ tends to $d$ in the natural topology on such actions as $\rho' \to \rho$, Theorem 6.5(2) follows as well. □

The remainder of this section is devoted to a proof of Lemma 6.8. We begin by constructing the group $\Delta$ and its actions on $U_0$ and $U_0{}'$. Recall that we have identified the tangent bundle to $H/\Lambda \times M$ with $H/\Lambda \times \mathfrak{h} \times TM$. We note that $E_{[\chi]}$ is a nilpotent Lie subalgebra of $\mathfrak{h}$. Let $F_{[\chi]}$ denote the corresponding Lie subgroup of $H$. The smooth foliation of $H \times M$ by left cosets for $F_{[\chi]}$ is tangent to $H \times M \times E_{[\chi]}$ and the projection of this foliation to $H/\Lambda \times M$ is exactly $\mathcal{W}_{[\chi]}$.

Fix $x \in H \times M$ and an orthogonal complement $E_{[\chi]}^{\perp}$ to $E_{[\chi]}$ in $TX_x$. Note that we can pick $E_{[\chi]}^{\perp}$ to be a direct sum of a subspace of $E_{\mathfrak{h}}^{\perp} \subset \mathfrak{h}$ and $TM$. Let $O$ be a small open disc in $E_{\mathfrak{h}}^{\perp}$. Let $U_1 = \exp_x(O) \subset H$. For any point $x \in H \times M$ we let $x = (x_1, x_2)$ be coordinates for the product structure and choose a small open neighborhood $U_2$ of $x_2$ in $M$. Then $U_0 = F_{[\chi]} U_1 \times U_2$ is an open subset in $H \times M$ containing $x$. If $O$ is small enough then each leaf of $\mathcal{W}_{[\chi]}$ contained in $U_0$ has a unique expression as $(F_{[\chi]} u x_1, x_2')$ where $u \in U_1$ and $x_2' \in U_2$. We then let $\Delta = F_{[\chi]}$ and let $\Delta$ act on $U_0$ via $(\delta, f_0 u x) \to (f_0 \delta^{-1} u x)$. This clearly defines a $\Delta$ action on $U_0$ which is $C^\infty$, free and transitive along the leaves of $\mathcal{W}_{[\chi]}$ and establishes $S1$.

We need to understand the derivative of the $\Delta$ action. Note that

$$T(H/\Lambda \times M)|_{U_0} = F_{[\chi]} U_1 \times \mathfrak{h} \times TM|_{U_2}.$$

Also note that the $\Delta$ action is trivial on the second factor. The facts that we identify the tangent space to $H$ with right invariant vectors and that $d(\delta)$ acts on the right on $F_{[\chi]}$ orbits, imply the following:

LEMMA 6.10. *For all $f_0 \in F_{[\chi]}, u \in U_1, v \in \mathfrak{h}, m \in M$, and $w \in TM_m$ we have $Dd(\delta)(f_0 u, v, m, w) = (f_0 \delta^{-1} u, v, m, w)$.*

The following lemma records the fact that leaves of $\mathcal{W}_{[\chi]}$ are injectively immersed in $H/\Lambda \times M$ and remain injectively when lifted to $H \times M$ and projected to $Z \backslash H$.

LEMMA 6.11. *The projection of $F_{[\chi]} x$ to $H/\Lambda \times M$, $H/\Lambda$ or $Z \backslash H$ is an injective immersion.*

*Proof.* This is immediate from the fact that $F_{[\chi]} < F^s_{\rho(a_i)}$ and that the leaves of $\mathcal{W}^s_{\rho(a_i)}$ are injectively immersed in $H/\Lambda \times M$, $H/\Lambda$ or $Z \backslash H$ by Proposition 3.2. □

We choose a lift $\tilde{\varphi}$ of $\varphi$ to a map from $H \times M$ to $H \times M$ and let $U_0' = \tilde{\varphi}(U_0)$. Let $\Delta$ act on $U_0{}'$ by letting $d'(\delta)(x) = \phi(d(\delta)(\phi^{-1}(x)))$ for every $x \in U_0{}'$ and every $\delta \in \Delta$. The properties $S2$ and $S4$ are immediate from this definition.

We now show how to realize the action $\Delta$ differently, in a way that will allow us to use Theorem 6.7 to prove $S3$.

We first explain why it suffices to consider the case of $\Gamma$ actions. In the case when $\rho$ is a $J$ action, we fix a lattice $\Gamma < J$. As a consequence of Theorem 6.1 and Zariski density of $\Gamma$ in $J$ we can choose the elements $\Psi$ from Lemma 6.2 and the elements $a_1, \ldots, a_q$ in Lemma 6.3 to be in $\Gamma$ even when $\rho$ is a $J$ action. For the remainder of this section, we can therefore restrict our attention to the case of $G = \Gamma$.

Let $T$ be a torus as given by Theorem 6.1 and restrict $\rho$ and $\rho'$ to $\Sigma$ actions. We then form the induced $T$ actions $\rho_{\text{in}}$ and $\rho_{\text{in}}{}'$ on $(T \times H/\Lambda \times M)/\Sigma$. The map $\varphi$ can be extended to an equivariant map which we denote $\varphi_{\text{in}}$ from $X_{\text{in}} = ((T \times H/\Lambda \times M)/\Sigma, \rho_{\text{in}})$ to $X_{\text{in}}{}' = ((T \times H/\Lambda \times M)/\Sigma, \rho_{\text{in}}{}')$. Let $\mathfrak{t}$ be the Lie algebra of $T$ and identify $T((T \times H/\Lambda \times M)/\Sigma)$ with $((T \times H \times TM)/\Lambda)/\Sigma \times \mathfrak{h} \times \mathfrak{t}$. There is a $\rho_{\text{in}}(T)$ invariant smooth foliation of $X_{\text{in}}$ given by $\mathcal{V}_{[\chi]}[t, x] = [t, \mathcal{W}_{[\chi]}(x)]$ tangent to a $\rho_{\text{in}}(T)$ invariant distribution $V_{[\chi]}$ which is $E_{[\chi]}$ viewed as a subbundle of $T(T \times H/\Lambda \times M)/\Sigma)$. Furthermore there are analogously defined foliations and distributions $\tilde{\mathcal{V}}_{[\chi]}$ and $\tilde{V}$ on $X_{\text{in}}{}'$ and $\psi_{\text{in}}$ maps every leaf $\mathcal{V}_{[\chi]}[t, x]$ to the leaf $\tilde{\mathcal{V}}_{[\chi]}(\varphi_{\text{in}}(t, x))$.

As in [KS97] and [MQ01], we will verify $S3$ by verifying it for these induced actions. It is clear that the actions of $\Delta$ on $U_0$ and $U_0{}'$ defined above can be extended to neighborhoods in the universal cover of $X_{\text{in}}$ and $X_{\text{in}}{}'$ simply by taking the trivial action on the first factor. It is also clear that individual leaves of $\mathcal{V}_{[\chi]}$ and $\tilde{\mathcal{V}}_{[\chi]}$ are still injectively immersed in $X_{\text{in}}$ and $X_{\text{in}}{}'$.

LEMMA 6.12. *Let $c_t$ be any nontrivial one-parameter $\mathbb{R}$-split subgroup in $S$ that is in the kernel of $\chi$ and which has noncompact image when projected to any simple quotient of $J$. To prove $S3$ it is sufficient to prove that any leaf of the foliation $\mathcal{V}_{[\chi]}$ is contained in the support of an ergodic component of $c_t$ acting on $X_{\text{in}}$.*

*Proof.* We proceed by giving a different description of the group $\Delta$ acting on $X_{\text{in}}$. Since $c_t$ is in the kernel of $\chi$, it follows that the maps $\rho(c_t) : \mathcal{V}_{[\chi]}(x) \to \mathcal{V}_{[\chi]}(c_t(x))$ are isometries with respect to the metric on the leaves. Since every ergodic component consists of entire leaves of $\mathcal{V}_{[\chi]}$, for any point $y \in \mathcal{V}_{[\chi]}(x)$ there exists a sequence $t_i$ such that $\lim_{i \to \infty} \rho(c_{t_i})x = y$. By passing to a subsequence, we may assume that $\rho(c_t) : \mathcal{V}_{[\chi]}(x) \to \mathcal{V}_{[\chi]}(\rho(c_{t_i})x)$ converges to an isometry $\tilde{\delta} : \mathcal{V}_{[\chi]}(x) \to \mathcal{V}_{[\chi]}(x)$ which takes $x$ to $y$. The group $\Delta_x$ generated by such limits is clearly transitive on $\mathcal{V}_{[\chi]}(x)$. Note also that the tangent map $D\rho(c_{t_i})(x, v) = (\rho(c_{t_i})x, v)$ and so the tangent map $D\delta(x, v) = (\delta x, v)$. Since $\Delta$ and $\Delta_x$ both act by isometries with trivial derivative on $\mathcal{V}_{[\chi]}(x)$ it is clear that they are equal. (For further discussion of this construction see [MQ01] or [KS97].)

We note that, by equivariance,

$$d'(\delta)x = \lim_{i \to \infty} \varphi_{\text{in}} \circ \rho_{\text{in}}(c_{t_i}) \circ \varphi_{\text{in}}{}^{-1}(x) = \lim_{i \to \infty} \rho'_{\text{in}}(c_{t_i})x.$$

Therefore, $d'(\delta) = \lim_{t_i \to \infty} \rho'_{\text{in}}(c_{t_i})$. We let $s \in T$ be an element with $\chi(s) < 0$, so the natural extension of $\rho'(s)$ to the tangent bundle of $\tilde{\mathcal{V}}_{[\chi]}$ is a contraction with narrow band spectrum which is close to its linear part. This follows because this contraction is close to the one defined by $\rho(s)$ which is linear. By Theorem 6.6 as long as $k \geq k_1$ there is a number $d$ depending only on the dynamics of $\rho(t_i)$, and continuous, $C^k$ small along fibers, conjugacy $H$ between $\rho'(s)$ and a polynomial of order less than $d$. Furthermore, since $\rho'(c_t)$ commutes with $\rho'(s)$, by Theorem 6.7 the conjugacy $H$ conjugates each $\rho'(c_t)$ to a polynomial map of order less than $d$. Since $d'(\delta) = \lim_{t_i \to \infty} \rho'_{\text{in}}(c_{t_i})$, it follows that in the coordinates along a fiber given by $H$, the map $d'(\delta)$ is given by a polynomial of order less than $d$. Identifying leaves with fibers we see that $d'(\delta)$ is $C^k$ along each leaf, that the $k$-jet of the $d'(\delta)$ depends continuously on the leaf, and that the $k$-jet is close to the one for $d(\delta)$, since it is given by composing a map close to $d(\delta)$ with a change in coordinates which is $C^k$ small.

We also see that if $\rho$ is $C^l$ for some $l \geq k$, then $d'(\delta)$ is $C^l$ since it is a uniform limit of maps $C^l$ conjugate to polynomials. □

6.4. *Ergodic components and dynamical foliations.* We retain all notation from the previous subsection. To prove Theorem 6.5 it now suffices to prove the following:

PROPOSITION 6.13. *Let $T$ be the torus described in the last section and $r$ in $T$ a regular element for the representation $\sigma$. For any one-parameter subgroup $c_t$ of $S$ which is in the kernel of $\chi$ and which projects to a noncompact subgroup of each simple factor of $J$, the ergodic components of $\rho_{\text{in}}(g_t)$ consist of entire leaves of $\mathcal{W}^s_{\rho_{\text{in}}(r)}$ and therefore of entire leaves of $\mathcal{V}_{[\chi]}$.*

We first note an alternate description of $\rho_{\text{in}}$. Throughout this subsection, we assume that we have passed to a finite index torsion-free subgroup of $\Gamma$. We will also need to pass to further finite index subgroups of $\Gamma$, but will abuse notation by retaining the notation $\Gamma$ for each of these successive subgroups. We recall some facts from [FM03]. First by [FM03, Th. 6.5], the homomorphism $\pi : \Gamma \to \text{Aut}(H) \ltimes H$ defining the action $\rho$ on a subgroup of finite index is a product of two homomorphisms $\pi_A : \Gamma \to \text{Aut}(H)$ and $\pi_H : \Gamma \to H$ whose images commute. It follows from the proof of [FM03, Th. 6.5] that after changing the algebraic structure on $H$ as in Proposition 2.2 and passing to a further subgroup of finite index, that $\pi(\Gamma)$ is actually contained in $\text{Aut}(U)$ where $U$ is the unipotent radical of $H$. Fixing a Levi complement $L$ for $U$ in $H$ and letting $L = ZM$ where $Z$ is a central torus and $M$ is semisimple, the superrigidity theorems imply that (again after passing to a subgroup of finite index) $\pi_A(\Gamma) < M$. After passing to another finite index subgroup, the restriction of $\pi_A$ to $\Sigma = \Gamma \cap T$ extends to a homomorphism $\pi_A^T : T \to \text{Aut}(U)$ and the restriction of $\pi_H$ to $\Sigma$ extends to a homomorphism $\pi_H^T : T \to M$. These homomorphisms are not quite canonical, but suffice for our purposes. It is clear that the images of these homomorphisms commute and so we can define a homomorphism $\pi^T(t) = \pi_T^A \pi_T^H(t)$.

We can now give a simple description of a finite cover of the induced action. In the description in the last paragraph, we passed to a finite index subgroup of $\Gamma$, which also causes us to pass to a finite index subgroup $\Sigma' < \Sigma$. The map $(t,h) \to (t, \pi^T(t)h)$ descends to a map from $(T \times H/\Lambda)/\Sigma'$ to $(T \ltimes H)/(\Sigma' \ltimes \Lambda)$ where the semidirect product is defined by $\pi_T^A$ and $\pi_T^A(\Sigma)$ normalizes $\Lambda$ by definition. This map conjugates the induced action to an action defined by $\rho_T(t_0)[t,h] = [t_0 t, \pi_H^T(t_0)h]$. We summarize this discussion with the following proposition.

PROPOSITION 6.14. (1) *If $\rho$ is an affine action, then there is a finite index subgroup $\Sigma'$ such that the lift of the action $\rho_{\text{in}}$ to $(T \times H/\Lambda)/\Sigma'$ is smoothly conjugate to a left translation action $\rho_T$ of $T$ on $(T \ltimes H)/(\Sigma' \ltimes \Lambda)$ as described above.*

(2) *If $\rho$ is a quasi-affine action, then there is a finite index subgroup $\Sigma'$ in $\Sigma$ and a left translation action $\rho_T$ of $T$ on $(T \ltimes H)/(\Sigma' \ltimes \Lambda)$ as above and a cocycle $\iota : T \times (T \ltimes H)/(\Sigma \ltimes \Lambda) \to \operatorname{Isom}(M)$, such that the lift of $\rho_{\text{in}}$ to $(T \times H/\Lambda \times M)/\Sigma'$ is smoothly conjugate to the skew product action over $\rho_T$ defined by $\iota$.*

We begin by showing that, even for quasi-affine actions, it suffices to consider the action on $(T \ltimes H)/(\Sigma' \ltimes \Lambda)$.

LEMMA 6.15. *For any one-parameter subgroup $g_t$ of $T$, and any regular element $r$ in $T$, if the ergodic components of the left translation action $g_t$ on $(T \ltimes H)/(\Sigma' \ltimes \Lambda)$ consist of entire leaves of $\mathcal{W}^s_{\rho_T(r)}$ then the ergodic components of $\rho_{\text{in}}(g_t)$ consist of entire leaves of $\mathcal{W}^s_{\rho_T(r)}$.*

*Proof.* This follows two facts. The first is one of the main results of Zimmer's thesis [Zim76]. This says that if $K$ is a compact group and $K$ acts on a standard probability measure space $(Y, \nu)$, and $\rho$ is an action of locally compact group $G$ by measure-preserving transformations on a standard measure space $(X, \mu)$ and $\iota : G \times X \to K$ is a cocycle, then the ergodic components of the skew-product action of $G$ on $(X \times Y, \mu \times \nu)$ are of the form $E \times L \cdot y$ where $E$ is an ergodic component of $X$, $y$ is a point in $Y$ and $L$ is a subgroup of $K$ such that $\iota$, restricted to $E$, is cohomologous to a cocycle taking values in $L$.

The second fact describes dynamical foliations for skew product extensions. Again, let $K$ be a compact group. Let $X$ be a smooth compact manifold and $Y$ be an associated bundle to a principal $K$ bundle over $X$. Assume $G$ is a locally compact group and that $\rho : G \times X \to X$ and $\tilde{\rho} : G \times Y \to Y$ are two actions which commute with the bundle projection $\pi : Y \to X$. Then the $G$ action on $Y$ is measurably isomorphic to a skew product extension as described in the previous paragraph and for any $g \in G$ which is partially hyperbolic and normally hyperbolic to a central foliation on both $X$ and $Y$, the map $\pi$ is a diffeomorphism from each leaf of $\mathcal{W}^s_{\tilde{\rho}(g)}$ onto a leaf of $\mathcal{W}^s_{\rho(g)}$. This follows from the dynamical characterization of $\mathcal{W}^s_{\rho(g)}$ in [HPS77, Th. 6.8e]

To prove the lemma, we apply these facts twice, first to the $g_t$ action on $(T\times H/\Lambda\times M)/\Sigma'$ covering the $\rho_{\text{in}}(g_t)$ action on $(T\times H/\Lambda\times M)/\Sigma$ and second to the action $\rho_{\text{in}}$ on $(T\ltimes H)/(\Sigma'\ltimes\Lambda)\times M$ which is a skew product action over the action $\bar{\rho}_{\text{in}}$ on $(T\ltimes H)/(\Sigma'\ltimes\Lambda)$. □

We are now reduced to identifying ergodic components for left translation actions on homogeneous spaces. To do this, we will use work of Brezin and Moore [BM81]. Following that paper, we note that for any Lie group $L$, any finite volume homogeneous space $L/\Delta$ has two special quotients, a *maximal toral quotient* and a *maximal semisimple quotient*. An affine quotient of the space $L/\Delta$ is one of the form $P/\phi(\Delta)$ where $\phi : L\to P$ is a surjective homomorphism. The maximal toral quotient is the maximal affine quotient of $L/\Delta$ which is a torus and the maximal semisimple quotient is the maximal affine quotient of $L/\Delta$ where $P$ is semisimple. Given a one-parameter subgroup $l_t$ in $L$, we can project $l_t$ to either the torus or to $M$ and this defines a quotient of the left translation flow of $l_t$ on $L/\Delta$. Let $\phi_1(l_t)$ be the quotient action on the maximal toral quotient and let $\phi_2(l_t)$ be the quotient action on the maximal semisimple quotient. The following is a restatement of [BM81, Th. 6.1].

THEOREM 6.16. *Let $l_t$ be a one-parameter subgroup of $L$ acting by left translation on a finite volume homogeneous space $L/\Delta$ for a Lie group $L$. Then the action of $l_t$ is ergodic if and only if both $\phi_1(l_t)$ and $\phi_2(l_t)$ are ergodic.*

To prove Proposition 6.13, we require an additional lemma which is an immediate consequence of Theorem 6.1(3).

LEMMA 6.17. *Let $c_t$ be a one-parameter subgroup of $T$ which is in the kernel of $\chi$ and projects to a noncompact subgroup in each $J_i$. Then the action of $c_t$ on $T/\Sigma'$ is ergodic.*

*Proof.* This is immediate since an ergodic component of the action is necessarily of the form $\prod_I T_i'/(\Sigma'\cap T_i')$ where $T_i' < T_i$ is a subtorus and $\Sigma'\cap T_i'$ is a lattice in $T_i'$. This forces $T_i'$ to be the Zariski closure of a subgroup of $\Sigma'$ and therefore to be algebraic. Theorem 6.1(3) then implies that $T_i' = T_i$. □

*Proof of Proposition* 6.13. By Proposition 6.14 and Lemma 6.15 we are reduced to showing that ergodic components of the $c_t$ action on $(T\ltimes H)/(\Sigma'\ltimes\Lambda)$ consist of entire leaves of $\mathcal{W}^s_{\rho_L(r)}$. We do this by explicitly identifying ergodic components, or rather explicitly identifying ergodic components modulo finite extensions.

Note that arguments as in Lemma 6.15 show that there is no loss of generality in passing to finite covers, so for simplicity we pass to a finite cover of $H$ such that:

(1) $\Lambda$ is torsion free.

(2) The Levi complement of $L$ is the direct product of $Z$ and $M$.
(3) $M$ is the direct product of its simple factors and $\Lambda$ does not intersect the center of $M$.
(4) $Z$ is a direct product of copies of $S^1$ and copies of $\mathbb{R}^*$.

Let $\sigma_M$ be restriction to $M$ of the map from $L$ to $\mathrm{Aut}(U)$ defining the semi-direct product structure of $H$ as $L\ltimes U$. Let $M^U$ be the kernel of $\sigma_M$ and let $M^U_K$ be the maximal normal connected compact subgroup of $M^U$. Note that our assumptions imply that $M^U_K$ is normal in $H$ and that $H = H\times M^U_K$ where $H'$ is isomorphic to $H/M^U_K$.

We write $M'$ as $M''\times C$ where $C$ is the maximal connected normal compact subgroup of $M'$. It follows from the proof of [Aus63, Ths. 1 and 2] that in $H$,

(1) $\Lambda \cap M^U_K\times U = \Lambda_U$ is a lattice in $M^U_K\times U$ and projects to a lattice in $U$,
(2) the projection of $\Lambda$ to $M''$ is a lattice in $M''$.

Since $\pi_H$ is a homomorphism of $\Gamma$ the superrigidity theorems imply that there are a homomorphism $\pi^E_H : J\to H$ and a homomorphism $\pi^K_H : \Gamma\to H$ with bounded image such that the images commute and $\pi_H(\gamma) = \pi^E_H(\gamma)\pi^K_H(\gamma)$. Note that, after passing to a further finite index subgroup, $\pi_H$ necessarily takes values in $M'$ and $\pi^E_H$ necessarily takes values in $M''$. Using that $\mathrm{Aut}(U)$ is an algebraic group, we can also write $\pi_A$ as a product of $\pi^E_A : J\to \mathrm{Aut}(U)$ and $\pi^K_A : \Gamma\to \mathrm{Aut}(U)$. We write $M'$ as a direct product $M_1M_2$ where $M_1$ is the minimal product of simple factors of $M'$ such that $\pi^E_H$ takes values in $M_1$ and the projection of $\Lambda$ to $M_1$ is a lattice. This implies that $M_1$ is a direct product of semisimple groups $M^i_1$ where the projection of $\Lambda$ to $M_1$ is commensurable to a product of irreducible lattices $\Lambda^i_{M_1} < M^i_1$ and such that the projection of $\pi^T_H(c_t)$ is noncompact in each $M^i_1$. This implies that the left translation action of $c_t$ on $M_1/\Lambda_{M_1}$ defined by $\pi^T_H$ is ergodic. Since $\pi^E_{H'}$ has nontrivial image in each $M^i_1$, it follows that each $M^i_1$ has real rank at least 2 and so there is a compact connected normal subgroup $M^K_1 < M'$ such that $(M_1\times M^K_1)\cap\Lambda = \Lambda_{M_1}$ at least after we replace $\Lambda$ by a subgroup of finite index. We write $M'_1$ for $M_1\times M^k_1$. It is also easy to see that the product $T\cdot M_1$ is a subgroup of $L$ and that $T\cdot M_1\cap(\Sigma'\ltimes\Lambda) = \Sigma'\times\Lambda_{M_1}$.

We now construct a subgroup of $U$. The fact that $\Lambda_U$ projects to a lattice in $U$ defines a rational structure on $U$ and $\mathfrak{u}$. We let $\mu_1$ be the composition of $\pi^E_H$ with the restriction of $\mathrm{Ad}_{T\times H}$ to $\mathfrak{u}$ and let $\mathfrak{u}_H$ be the minimal Lie subalgebra containing all nontrivial root subspaces of $\mu_1$ and invariant under $\pi^T_H(T)$. We let $\mu_2$ be the composition of $\pi^E_A$ with the representation of $\mathrm{Aut}(U)$ on $\mathfrak{u}$ and let $\mathfrak{u}_A$ be the minimal Lie algebra containing all nontrivial root subspaces of $\mu_2$ and invariant under $\pi^T_A$. Finally we let $\mathfrak{u}_0$ be the minimal rational Lie subalgebra of $\mathfrak{u}$ containing both $\mathfrak{u}_A$ and $\mathfrak{u}_H$ and invariant under $T\cdot M_1$. Let $U_0 < U$ be the Lie subgroup with

Lie algebra $\mathfrak{u}_0$. We let $K_0$ be the closure of the projection of $\Lambda_U$ to $M_K^U$ and form the semidirect product $N = (T \times K_0 \times M_1') \ltimes U_0$. By construction it is clear that:

(1) $N \cap \Lambda = \Lambda_N$ is a lattice in $N$.

(2) For any regular elements $r$ in $T$, the space $E^s_{\rho^{\mathrm{in}}}(r)$ is a subspace of $\mathfrak{n}$.

(3) The maximal semisimple quotient of $N/\Lambda_N$ is $M_1/\Lambda_{M_1}$.

(4) The maximal toral quotient of $N/\Lambda_N$ is $T/\Sigma'$.

Together with Lemma 6.17 and the definition of $M_1$ this implies that any ergodic component of the action of $c_t$ on $(T \ltimes H)/(\Sigma' \ltimes \Lambda)$ contains a translate of $N/\Lambda_N$ in $(T \ltimes H)/(\Sigma' \ltimes \Lambda')$ which suffices to prove the proposition. □

## 7. **Final arguments**

7.1. *Elliptic operators and global regularity.* In this section, we prove that $\varphi$ is a diffeomorphism. Here $\rho, G$ and $H/\Lambda \times M$ are as in the remarks at the end of Section 4. The number $k_0$ is the smallest number that allows us to apply the techniques of Section 6.2 to show that $\varphi$ is a diffeomorphism along foliations of the type $\mathcal{W}^g_{[\chi]}$. In keeping with the statement of Theorem 1.1 we let $n = \frac{\dim(H/\Lambda \times M)}{2} + 3$. We now prove:

THEOREM 7.1. *There is a neighborhood $V$ of $\rho$ in* $\mathrm{Hom}(G, \mathrm{Diff}^k(X))$ *such that if $\rho' \in V$, the map $\varphi$ constructed above is a $C^{k-n}$ small $C^{k-n}$ diffeomorphism which is conjugacy between $\rho$ and $\rho'$. Furthermore*

(1) $\varphi \to \mathrm{Id}$ *as $\rho' \to \rho$ and,*

(2) *given $l \geq k$, we can choose $V$ so that if $\rho'$ is $C^\infty$ and $\rho' \in V$, then, the map $\varphi$ is $C^l$.*

*Remark.* The proof below uses only standard facts concerning elliptic operators and is straightforward. The result stated also follows from the main theorem in [KS94], but as that article relies on much deeper and harder results concerning hypo-elliptic operators, we give the proof below.

*Proof.* We choose a finite cover of $H/\Lambda \times M$ by open sets $U_m$ such that:

(1) Each $U_m$ is contained in a neighborhood $W_m$ which is coordinate chart on $H/\Lambda \times M$.

(2) For each $U_m$ we have $\varphi(U_m) \subset W_m$.

(3) Each $U_m$ is a foliation chart for $\mathfrak{F}$ and $\mathcal{W}^g_{[\chi]}$ for all $g \in \Phi$ and $\chi$ in $\bar{\Sigma}$ as defined in Section 6.1.

(4) Each $U_m$ is of the form $U_{1m} \times U_{2m}$ where $U_{1m}$ is an open set in $H/\Lambda$ and $U_{2m}$ is an open set in $M$.

For convenience, we denote $\mathfrak{F}$ by $\mathfrak{F}_0$ and fix an order on the $\mathcal{W}^g_{[\chi]}$ and relabel them $\mathfrak{F}_1,\dots,\mathfrak{F}_q$. We choose a basis $X_{ij}$ of $\mathfrak{h}$ where $X_{0j}$ for $1\leq j\leq \dim(Z)$ is a basis for $\mathfrak{z}$ and $X_{ij}$ is a basis for $\mathfrak{F}_i$ with $1\leq i\leq q$. For $U_{2j}$ we choose an explicit identification with an open ball in $\mathbb{R}^n$ and choose a basis of constant vector fields $X_{0j}$ where $\dim(Z)+1\leq j\leq \dim(Z)+\dim(M)$.

Identifying each $W_m$ with a subset of $\mathbb{R}^n$, we can write $\varphi_m=\varphi|_{U_m}=\mathrm{Id}+h_m$ where $h_m : U_m\to\mathbb{R}^n$ is $C^0$ small.

For any $\eta>0$ and any $l'>k$, by choosing $V$ small enough, and applying Theorem 6.5 and Theorem 4.4, we have that each $h_m$ is

(1) $C^k$ along each leaf of $\mathfrak{F}_l$ for $1\leq l\leq q$ with $\mathrm{supp}_{U_m} X^n_{lj}(h_m)<\eta$ for any $0\leq n\leq k$ and any $1\leq j\leq \dim(\mathfrak{F}_l)$,
(2) $C^{k-2}$ along almost every leaf of $\mathfrak{F}$ with $\int_{U_m}\|X^n_{0j}(h_m)(x)\|^2 d\mu<\eta$ for any $0\leq n\leq k-2$ and any $0\leq j\leq \dim(Z)+\dim(M)$.
(3) If $\rho'$ is $C^\infty$ then $h_m$ is $C^{l'}$ along $\mathfrak{F}_0$ and $C^\infty$ along each $\mathfrak{F}_i$ for $1\leq i\leq q$.

We construct an elliptic operator as follows. Let $c$ be the least even integer less than or equal to $k-2$, then the operator

$$\Delta=\sum_{i=1}^{q}\sum_{j=1}^{\dim(\mathfrak{F}_i)} X^c_{ij}$$

is elliptic with smooth coefficients on each $U_l$. Standard estimates, see e.g. [Zim90, §6.3], imply that:

$$\|u\|_{2,k-2}<C(\|\Delta(u)\|_2+\|u\|_2)$$

for any $u$ in the Sobolev space $W^{2,k-2}(U_l)$, where $W^{2,k-2}(U_l)$ is the Sobolev space of functions with $k$ weak derivatives in $L^2$, $\|\cdot\|_{2,k-2}$ is the Sobolev norm and $\|\cdot\|_2$ is the $L^2$ norm. (For $k-2$ odd, the standard inequality involves the $\|\cdot\|_{2,1}$ for both terms on the right-hand side, but we will not need this.)

We want to apply this estimate to $h_m$, but $h_m$ is not *a priori* in $W^{2,k-2}(U_m)$. We let $U^\varepsilon_m$ be the set of points $x$ in $U_m$ such that $B(x,\varepsilon)\subset U_m$. To complete the argument, we use mollifiers $J_\varepsilon$ such that

(1) $X_{ij}J_\varepsilon=J_\varepsilon X_{ij}$ for $X_{ij}$ above and,
(2) $J_\varepsilon u$ is defined on $U^\varepsilon_m$ and $J_\varepsilon$ maps $L^{1,loc}(U_m)$ to $C^\infty(U^\varepsilon_m)$.
(3) $J_\varepsilon$ is uniformly bounded on $W^{2,k-2}(U_m)\subset L^1_{loc}(U_m)$.
(4) $J_\varepsilon$ converges uniformly to the identity on $L^{1,loc}$ as $\varepsilon\to 0$.

We briefly describe the operators $J_\varepsilon$ which are convolution operators for a family of functions $f_\varepsilon$. We write the function $f_\varepsilon=f_{1\varepsilon}f_{2\varepsilon}$ where $f_{i\varepsilon}$ is a function on $U_{im}$ for $i=1,2$. The function $f_{1\varepsilon}$ is a standard mollifier and we define $J_{1\varepsilon}$ by standard convolution. Also, we define $f_2$ by taking a standard mollifier on a small neighborhood of zero in $\mathfrak{h}$ and pulling back to $H$ via the inverse of the exponential map. We identify $U_{1m}$ with a small neighborhood in $H$ and define $J_{2\varepsilon}(u) = \int_H f_{2\varepsilon}(h)u(xh, m)d\mu$. The fact that we act on the right on $x$ in the formula is necessary to guarantee condition (1) above. We then let $J_\varepsilon = J_{1\varepsilon}J_{2\varepsilon}$. It is easy to see that $J_{1\varepsilon}$ and $J_{2\varepsilon}$ commute, and that $J_\varepsilon$ satisfies (1)–(4) above.

Letting $\varepsilon_n = \frac{1}{n}$, we have that

$$\|(J_{\varepsilon_n} - J_{\varepsilon_{n+1}})h_m\|_{2,k-2} < C(\|(J_{\varepsilon_n} - J_{\varepsilon_{n+1}})\Delta(h_m)\|_2 + \|(J_{\varepsilon_n} - J_{\varepsilon_{n+1}})h_m\|_2).$$

The right-hand side converges to zero, which implies that $\{J_{\varepsilon_n}h_m\}_n$ is a Cauchy sequence in $W^{2,k-2}(U_i)$. Since $\{J_{\varepsilon_n}h_m\}_n$ converges in $L^{1,loc}$ to $h_m$, this implies that $h_m \in W^{2,k-2}$ and

$$\|h_m\|_{2,k-2} < C(\|\Delta(h_m)\|_2 + \|h_m\|_2)$$

which by the properties (1) and (2) of $h_m$ described above, imply that $\|h_m\|_{2,k-2} < C\eta$ for a constant $C$ not depending on $\rho'$. By the Sobolev embedding theorems, this implies that $h_m$ is $C^{k-n}$ small where $n$ is $\frac{\dim(H/\Lambda\times M)}{2}$.

This then implies that $\varphi$ is $C^{k-n}$ close to the identity, which implies that $\varphi$ is a diffeomorphism, since there is a neighborhood of the identity in the space of $C^{k-n}$ maps which consists of diffeomorphisms.

To show that $h_m$ is $C^l$ when $\rho'$ is $C^\infty$ follows a similar outline. We choose $V$ such that $h_m$ is $C^\infty$ along each $\mathfrak{F}_j$ for $1 \leq j \leq q$ and $C^{l'}$ along $\mathfrak{F}_0$ where $l' \geq l + \frac{\dim(X)}{2} + 3$ is even. The same argument with $c = l'$ in the construction of the elliptic operator shows that $h_m$ is in $W^{2,l'}$ and therefore is $C^l$. Note that since we do not have a good bound on the $W^{2,l'}$ norm of $\varphi$ along $\mathfrak{F}$ in Theorem 4.4 or of the $C^{l'}$ norm of $\varphi$ along $\mathcal{W}^g_{[\chi]}$ in Theorem 6.5, we do not obtain a bound on the $C^l$ size of $\varphi$. $\square$

7.2. *Smooth perturbations, smooth conjugacy, and iterations.* We keep all the notation from the previous subsection. For notational convenience in the proof of the $C^\infty$ case of Theorem 1.1, it is convenient to fix right invariant metrics $d_l$ on the connected components of $\mathrm{Diff}^l(X)$ with the additional property that if $\varphi$ is in the connected component of $\mathrm{Diff}^\infty(X)$, then $d_l(\varphi, \mathrm{Id}) \leq d_{l+1}(\varphi, \mathrm{Id})$. To fix $d_l$, it suffices to define inner products $<,>_l$ on $\mathrm{Vect}^l(X)$ which satisfy $< V, V >_l \leq < V, V >_{l+1}$ for $V \in \mathrm{Vect}^\infty(X)$. As remarked in [FM05, §6], after fixing a Riemannian metric $g$ on $X$, it is straightforward to introduce such metrics using the methods of [FM05, §4].

Once we have fixed the family of metrics $d_l$ and fix a generating set $K$ for $G$, it is possible to rephrase parts of Theorem 7.1 more quantitatively as follows:

COROLLARY 7.2. *In the setting of Theorem* 7.1, *given* $k \geq k_0$ *and* $l \geq k$, *for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $\rho'$ *is an action of* $G$ *on* $H/\Lambda\times M$ *with*

*$d_k(\rho'(g)\rho(g)^{-1}, \mathrm{Id}) < \delta$ for all $g \in K$ then there exists a $C^l$ conjugacy $\varphi$ between $\rho$ and $\rho'$ such that $d_{k-n}(\varphi, \mathrm{Id}) < \epsilon$.*

*Remark.* The algorithm presented below for proving smoothness is essentially contained in the proof of the $C^{\infty,\infty}$ case of [FM05, Th. 1.1].

*Proof of $C^{\infty,\infty}$ local rigidity in Theorem* 1.1. . If $\rho'$ is a $C^\infty$ perturbation of $\rho$, then there exists some $k > 1$, such that $\rho'$ is $C^k$ close to $\rho$ and we can assume that $k \geq k_0$. We fix a sequence of positive integers $k = l_0 < l_1 < l_2 < \cdots < l_i < \ldots$ with $l_{i+1} - l_i > n+3$ for each $i$. We construct a sequence of $C^\infty$ diffeomorphisms $\phi_i$ such that the sequence $\{\phi_n \circ \ldots \circ \phi_1\}_{n \in \mathbb{N}}$ converges in the $C^\infty$ topology to a conjugacy between $\rho$ and $\rho'$.

We let $\phi^i = \phi_i \circ \ldots \circ \phi_1$ and $\rho_i = \phi^i \circ \rho' \circ (\phi^i)^{-1}$ and construct $\phi_i$ inductively such that

(1) $\rho_i$ is sufficiently $C^{l_i-n-3}$ close to $\rho$ to apply Corollary 7.2 to $\rho_i$ and $\rho$ with $l = l_{i+1}$ and $\epsilon = \frac{1}{2^{i+2}}$,

(2) $d_{l_i-n-3}(\phi_i, \mathrm{Id}) < \frac{1}{2^i}$ and,

(3) $d_{l_i-n-3}(\rho_i(\gamma) \circ \rho(\gamma)^{-1}, \mathrm{Id}) < \frac{1}{2^i}$ for every $\gamma \in K$.

To construct $\phi_{i+1}$, we assume that $\rho_i$ is close enough to $\rho$ in the $C^{l_i}$ topology to apply Corollary 7.2 with $l = l_{i+1}$ and $\varepsilon = \frac{1}{2^{i+1}}$. Then we have a $C^{l_{i+1}-n-3}$ diffeomorphism $\psi_{i+1}$ with $\psi_{i+1} \circ \rho_i \circ {\psi_{i+1}}^{-1} = \rho$ and

$$d_{l_{i+1}-n-3}(\psi_{i+1}, \mathrm{Id}) < \frac{1}{2^{i+2}}.$$

Using standard approximation theorems, we can choose a $C^\infty$ diffeomorphism $\phi_{i+1}$ with $d_{l_{i+1}-n-3}(\phi_{i+1}, \mathrm{Id}) < \frac{1}{2^{i+1}}$ and $\rho_i = \phi_{i+1} \circ \rho_i \circ {\phi_{i+1}}^{-1}$ close enough to $\rho$ in the $C^{l_{i+1}-n-3}$ topology to apply Corollary 7.2 with $l = l_{i+2}$ and $\varepsilon = \frac{1}{2^{i+3}}$ and so that (3) above is satisfied.

To start the induction it suffices that $\rho'$ is sufficiently $C^k$ close to $\rho$ to apply Theorem 7.1(2) with $l = l_1$ and $\epsilon = \frac{1}{2}$.

It remains to show that the sequence $\{\phi_n \circ \ldots \circ \phi_1\}_{n \in \mathbb{N}}$ converges in the $C^\infty$ topology to a conjugacy between $\rho$ and $\rho'$. Combining condition (2) with the fact that $d_{l_i}(\phi_i, \mathrm{Id}) \leq d_j(\phi_i, \mathrm{Id})$ for all $j \geq l_i$, and the fact that $d_{l_i}$ is right invariant implies that $d_{l_i-n-3}(\phi_j, \mathrm{Id}) = d_{l_i-n-3}(\phi^j, \phi^{j-1}) \leq \frac{1}{2^j}$ for all $j \geq i$. This implies that $\{\phi^j\}$ is a Cauchy sequence in $\mathrm{Diff}^{l_i-n-3}(X)$ for all $i$, so that $\{\phi^j\}$ converges in $\mathrm{Diff}^\infty(X)$. Similarly, condition (3) implies $\rho_i$ converges to $\rho$ in the $C^\infty$ topology. □

*Remark added in proof*, January 10, 2009. As noted in the footnote on page 68, the results of Rauch and Taylor from [RT05] allow us to avoid almost all loss of regularity at this step in the argument. The main point is that we can work in Sobolev spaces of type $W^{p,k-2}$ for a large value of $p$ rather than for $p=2$. The results of Rauch and Taylor show that if the map $h_m$ is in $W^{p,k-2}$ and $W^{p,k-2}$ small along each of $\mathfrak{F}_0,\dots,\mathfrak{F}_q$ then it is in $W^{p,k-2}$ and $W^{p,k-2}$ small on $W_m$. The standard Sobolev embedding theorems then imply that $h_m$ is $C^{k-2-\frac{\dim(M)}{p}}$, so that, in particular, $C^{k-3}$ for large enough $p$.

Since $h_m$ is $C^k$ and uniformly $C^k$ small along each $\mathfrak{F}_i$ for $1\leq i\leq q$, it is clearly in $W^{p,k}$ and $W^{p,k}$ small along these foliations. To see that by choosing $\rho'$ close enough to $\rho$ we can also force $h_m$ small in $W^{p,k-2}$ along $\mathfrak{F}_0$ is immediate from the proof of Theorem 4.2 in [FM05]. It can also be deduced from the statement of Theorem 4.2 and Remark (2) following that theorem.

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