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Abstract

For a smooth manifold M we define the Teichmüller space $\mathcal{T}(M)$ of all Riemannian metrics on M and the Teichmüller space $\mathcal{T}^{\epsilon}(M)$ of ϵ-pinched negatively curved metrics on M, where $0 \leq \epsilon \leq \infty$. We prove that if M is hyperbolic, the natural inclusion $\mathcal{T}^{\epsilon}(M) \hookrightarrow \mathcal{T}(M)$ is, in general, not homotopically trivial. In particular, $\mathcal{T}^{\epsilon}(M)$ is, in general, not contractible.

Introduction

Let M be a closed smooth manifold. We denote by $\mathcal{MET}(M)$ the space of all smooth Riemannian metrics on M with the smooth topology. Note that the space $\mathcal{MET}(M)$ is contractible. We also denote by Diff(M) the group of all smooth self-diffeomorphisms of M. Diff(M) acts on $\mathcal{MET}(M)$ by pulling back metrics: $\phi g = (\phi^{-1})^* g = \phi_* g$ for $g \in \mathcal{MET}(M)$ and $\phi \in \text{Diff}(M)$, that is, ϕg is the metric such that $\phi : (M, g) \to (M, \phi g)$ is an isometry.

Let $\mathbb{R}^+$ be the set of positive real numbers, which we consider as a group with multiplication. Denote by $\mathcal{D}(M)$ the group $\mathbb{R}^+ \times \text{Diff}(M)$. The group $\mathcal{D}(M)$ acts on $\mathcal{MET}(M)$ by scaling and pulling back metrics, that is, $(\lambda, \phi)g = \lambda(\phi^{-1})^* g = \lambda\phi_* g$ for $g \in \mathcal{MET}(M)$ and $(\lambda, \phi) \in \mathcal{D}(M)$. The quotient space $\mathcal{M}(M) = \mathcal{MET}(M)/\mathcal{D}(M)$ is called the moduli space of metrics on M. It is sometimes said that a geometric property is a property that is invariant by isometries, that is, by an action of Diff(M). Hence if two Riemannian metrics represent the same element in $\mathcal{M}(M)$, then they posses the same geometric properties. Clearly, the study of the moduli space of metrics is of fundamental importance not just in geometry but in other areas of mathematics too. See, for instance, [Bes87, Ch. 4].

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It is also interesting to consider subspaces of $\mathcal{M}(M)$ that represent some geometric property. One obvious choice is to consider metrics with constant curvature. For instance, let M_g be an orientable two-dimensional manifold of genus $g > 1$. Consider the moduli space of all hyperbolic metrics on M_g, that is, the subspace of $\mathcal{M}(M_g)$ formed by elements that are represented by Riemannian metrics of constant sectional curvature equal to -1. The moduli space of all hyperbolic metrics is the quotient of another well-known space, the Teichmüller space of M_g. This space is a subspace of the quotient of $\mathcal{MET}(M_g)$ by the subgroup of $\mathrm{Diff}(M_g)$ formed by all smooth self-diffeomorphisms of M_g that are homotopic to the identity; namely, it is the subspace represented by hyperbolic metrics. Then the moduli space is the quotient of the Teichmüller space by the action of $\mathrm{Out}(\pi_1(M_g))$, the group of outer automorphisms of the fundamental group of M_g.

We want to generalize the definition of the Teichmüller space to higher dimensions. The obvious choice for a definition would be the quotient of the space of all hyperbolic metrics by the action of the group of all smooth self-diffeomorphisms that are homotopic to the identity. But Mostow's rigidity theorem implies that, in dimensions no less than 3, this space contains (at most) one point.

Let us go back to dimension two for a moment. Recall that uniformization techniques (see [EE69], or, more recently, Hamilton's Ricci flow [Ham88]) show that every Riemannian metric on M_g for $g > 1$ can be canonically deformed to a hyperbolic metric. Moreover, Hamilton's Ricci flow [Ham88] shows that every negatively curved metric on M_g for $g > 1$ can be canonically deformed (through negatively curved metrics) to a hyperbolic metric. Hence the space of all hyperbolic metrics on M_g is canonically a deformation retract of the space of all negatively curved Riemannian metrics on M_g. This deformation commutes with the action of $\mathrm{Diff}(M_g)$ (this is true at least for the Ricci flow); therefore the Teichmüller space of M_g is canonically a deformation retract of the space that is the quotient of all negatively curved Riemannian metrics on M_g by the action of the group of all smooth self-diffeomorphisms that are homotopic to the identity. Also, instead of considering the space of all negatively curved metrics, we can consider the space of all *pinched* negatively curved metrics, or for that matter, the space of all Riemannian metrics. These are the concepts that we will generalize. Next we make definitions and introduce notation.

As before, let M be a closed smooth manifold. We denote by $\mathrm{Diff}_0(M)$ the subgroup of $\mathrm{Diff}(M)$ of all smooth diffeomorphisms of M that are homotopic to the identity 1_M. Also, denote by $\mathcal{D}_0(M)$ the group $\mathbb{R}^+ \times \mathrm{Diff}_0(M)$. We call the quotient space $\mathcal{T}(M) = \mathcal{MET}(M)/\mathcal{D}_0(M)$ the Teichmüller space of metrics on M.

Given $0 \leq \epsilon \leq \infty$, let $\mathcal{MET}^\epsilon(M)$ denote the space of all ϵ-pinched negatively curved Riemannian metrics on M, that is, $\mathcal{MET}^\epsilon(M)$ is the space of all negatively curved Riemannian metrics g on M for which

$$\frac{\text{sup-sec}\, g}{\text{inf-sec}\, g} \leq 1+\epsilon,$$

where

$$\text{sup-sec}\, g = \sup\{|g\text{-sectional curvature of } P_x|,\ x \in M,\ P_x \text{ a 2-plane in } T_x M\},$$

$$\text{inf-sec}\, g = \inf\{|g\text{-sectional curvature of } P_x|,\ x \in M,\ P_x \text{ a 2-plane in } T_x M\}.$$

Therefore $g \in \mathcal{MET}^{\epsilon}(M)$ if and only if there is a positive real number λ such that λg has all its sectional curvatures in the interval $[-(1+\epsilon), -1]$. Note that a 0-pinched metric is a metric of constant negative sectional curvature and an ∞-pinched metric is just a negatively curved Riemannian metric.

The quotient space $\mathcal{M}^{\epsilon}(M) = \mathcal{MET}^{\epsilon}(M)/\mathcal{D}(M)$ is called the moduli space of ϵ-pinched negatively curved metrics on M, and $\mathcal{T}^{\epsilon}(M) = \mathcal{MET}^{\epsilon}(M)/\mathcal{D}_0(M)$ is called the Teichmüller space of ϵ-pinched negatively curved metrics on M. In particular, $\mathcal{T}^{\infty}(M)$ is the Teichmüller space of all negatively curved metrics on M. Note that the inclusions $\mathcal{MET}^{\epsilon}(M) \hookrightarrow \mathcal{MET}(M)$ induce inclusions $\mathcal{T}^{\epsilon}(M) \hookrightarrow \mathcal{T}(M)$. Also note that, for $\delta \geq \epsilon$, these inclusions factor as

$$\mathcal{MET}^{\epsilon}(M) \hookrightarrow \mathcal{MET}^{\delta}(M) \hookrightarrow \mathcal{MET}(M) \quad \text{and} \quad \mathcal{T}^{\epsilon}(M) \hookrightarrow \mathcal{T}^{\delta}(M) \hookrightarrow \mathcal{T}(M).$$

Remark 1. If M_g is an orientable two-dimensional manifold of genus $g > 1$, then the original Teichmüller space of M_g is denoted (in our notation) by $\mathcal{T}^0(M_g)$, and $\mathcal{T}^0(M_g)$ is homeomorphic to $\mathbb{R}^{6g-6}$; see [EL88]. Hence $\mathcal{T}^0(M_g)$ is contractible. By the uniformization techniques mentioned above [EE69], [Ham88], it follows that $\mathcal{T}^{\epsilon}(M_g)$, $\mathcal{T}^{\infty}(M_g)$, and $\mathcal{T}(M_g)$ are all contractible. (This is also true for nonorientable surfaces of Euler characteristic < 0.)

Remark 2. Let M be a closed hyperbolic manifold. If $\dim M \geq 3$, Mostow's rigidity theorem implies that $\mathcal{T}^0(M) = *$; that is, $\mathcal{T}^0(M)$ contains exactly one point. Therefore $\mathcal{MET}^0(M) = \mathcal{D}_0(M)$. It also follows (see Remark 1 above) that $\mathcal{T}^0(M)$ is contractible when $\dim M \geq 2$.

In two dimensions, Earle and Eells [EE69] proved that $\mathcal{D}_0(M)$ (and hence $\mathcal{MET}^0(M)$) is contractible. The same was proved in three dimensions by Gabai [Gab01]. This is certainly false in dimensions ≥ 6, because $\pi_0(\mathcal{D}_0(M))$ is not finitely generated when $n \geq 11$ (see [FJ89b, Cors. 16 and 10.28])and $\pi_0(\mathcal{D}_0(M))$ is non-trivial when $6 \leq n \leq 10$ (see Theorem 1 and Corollary), and it is reasonable to conjecture that $\mathcal{D}_0(M)$ is also not contractible for $n = 5$.

Remark 3. Let M be a hyperbolic manifold. Then the action of $\mathcal{D}_0(M)$ on $\mathcal{MET}(M)$ is free (see Lemma 1.1). Since $\mathcal{MET}(M)$ is contractible and because of Ebin's slice theorem [Ebi70], we have that $\mathcal{D}_0(M) \to \mathcal{MET}(M) \to \mathcal{T}(M)$ is a principal $\mathcal{D}_0(M)$-bundle and $\mathcal{T}(M)$ is the classifying space $B\mathcal{D}_0(M)$ of $\mathcal{D}_0(M)$.

Therefore, if M is a closed hyperbolic manifold, then $\mathcal{MET}^{\epsilon}(M)$ interpolates between $\mathcal{MET}^0(M)$ (which is homotopy equivalent to $\mathcal{D}_0(M)$) and $\mathcal{MET}(M)$

(which is contractible). Likewise $\mathcal{T}^{\epsilon}(M)$ interpolates between $\mathcal{T}(M)$ (which is equal to $B\mathcal{D}_0(M)$) and $\mathcal{T}^0(M)$ (which is contractible). Schematically, we have the diagram

$$\begin{tikzcd}
\mathcal{MET}^0(M) \arrow[r, hook] \arrow[d] & \mathcal{MET}^{\epsilon}(M) \arrow[r, hook] \arrow[d] & \mathcal{MET}^{\infty}(M) \arrow[r, hook] \arrow[d] & \mathcal{MET}(M) \arrow[d] \\
\mathcal{T}^0(M) \arrow[r, hook] & \mathcal{T}^{\epsilon}(M) \arrow[r, hook] & \mathcal{T}^{\infty}(M) \arrow[r, hook] & \mathcal{T}(M).
\end{tikzcd}$$

All vertical arrows represent quotient maps by the action of the group $\mathcal{D}_0(M)$.

The main result of this paper states that for a hyperbolic manifold the last two horizontal arrows of the lower row of this diagram are not in general homotopic to a constant map. In particular $\mathcal{T}^{\epsilon}$ for $0 \le \epsilon \le \infty$ is in general not contractible. More specifically, we prove that under certain conditions on the dimension n of the hyperbolic manifold M, the manifold M has a finite cover N (which depends on ϵ) such that $\pi_k(\mathcal{T}^{\epsilon}(N)) \to \pi_k(\mathcal{T}(N))$ is nonzero. In particular, $\mathcal{T}^{\epsilon}(N)$ is not contractible. The requirements on the dimension n are implied by one of the following conditions: n is larger than some constant $n_0(4)$, or n is larger than 5, but in this last case we need that $\Theta_{n+1} \neq 0$, where Θ_{ℓ} denotes the group of homotopy spheres of dimension ℓ. Here is a more detailed statement of our main result:

THEOREM 1. *For every integer $k_0 \geq 1$, there is an integer $n_0 = n_0(k_0)$ such that the following holds. Given $\epsilon > 0$ and a closed real hyperbolic n-manifold M with $n \geq n_0$, there is a finite sheeted cover N of M such that, for every $1 \le k \le k_0$ with $n + k \equiv 3 \bmod 4$, the map $\pi_k(\mathcal{T}^{\epsilon}(N)) \to \pi_k(\mathcal{T}(N))$ induced by the inclusion $\mathcal{T}^{\epsilon}(N) \hookrightarrow \mathcal{T}(N)$ is nonzero. Consequently $\pi_k(\mathcal{T}^{\epsilon}(N)) \neq 0$. In particular, $\mathcal{T}^{\delta}(N)$ is not contractible for every δ such that $\epsilon \le \delta \le \infty$ (provided $k_0 \geq 4$).*

Here (and in the corollary below) we consider the given hyperbolic metric as the basepoint for $\mathcal{T}(N)$ and $\mathcal{T}^{\epsilon}(N)$.

For $k_0 = 1$, we will show that we can take $n_0(1) = 6$, and that we can drop the condition $n + k \equiv 2 \bmod 4$. Hence we obtain the following corollary to (the proof of) Theorem 1.

COROLLARY. *Let M be a closed real hyperbolic manifold of dimension $n \geq 6$. Assume that $\Theta_{n+1} \neq 0$. Then for every $\epsilon > 0$, there is a finite sheeted cover N of M such that $\pi_1(\mathcal{T}^{\epsilon}(N)) \neq 0$. Therefore $\mathcal{T}^{\epsilon}(N)$ is not contractible.*

Recall that an n-dimensional π manifold is a manifold that embeds in $\mathbb{R}^{2n+2}$ with trivial normal bundle. Every real hyperbolic manifold has a finite sheeted cover that is a π manifold; see [Sul79, p. 553]. We have the following addition to the statements of Theorem 1 and its corollary.

ADDENDUM. *We can choose $N = M$ in the statements of Theorem* 1 *and its corollary, provided M is a π-manifold and the radius of injectivity of M at some point is sufficiently large.* (*How large depends only on the dimension of M.*)

We now make some comments on Theorem 1 and the diagram above.

Remark 4. Since $\mathcal{MET}(M)$ is contractible, Theorem 1 implies that, for a general hyperbolic manifold M, the map $\pi_k(\mathcal{MET}^\epsilon(M)) \to \pi_k(\mathcal{T}^\epsilon(M))$, induced by the second vertical arrow of the diagram, is not onto for some k.

Remark 5. By Remark 1, the lower row of the diagram above is homotopically trivial in dimension 2. In dimension 3 one could ask the same: is the lower row of the diagram above homotopically trivial in dimension 3? In view of a result of Gabai [Gab01], this is equivalent to asking: Is $\mathcal{T}(M)^\infty$ contractible?

Remark 6. Let M be a hyperbolic manifold. Consider the upper row of the diagram. It follows from a result of [Ye93] on the Ricci flow that, provided the dimension of M is even, there is an $\epsilon_0 = \epsilon_0(M) > 0$ such that for all $\epsilon \leq \epsilon_0$ the inclusion map $\mathcal{MET}^\epsilon \to \mathcal{MET}^\infty$ is $\mathcal{D}_0(M)$-equivariantly homotopic to a retraction $\mathcal{MET}^\epsilon \to \mathcal{MET}^0(M) \subset \mathcal{MET}^\infty$. This has the following consequences. First, the retraction above descends to a retraction $\mathcal{T}^\epsilon(M) \to \mathcal{T}^0(M)$; hence the inclusion map $\mathcal{T}^\epsilon(M) \to \mathcal{T}^\infty(M)$ is homotopic to a constant map (provided $\epsilon \leq \epsilon(M)$), and hence induces the zero homomorphism $\pi_k(\mathcal{T}^\epsilon(M)) \to \pi_k(\mathcal{T}(M))$ for all k. Second, the inclusion map $\mathcal{MET}^0(M) \to \mathcal{MET}^\epsilon(M)$ induces monomorphisms $\pi_k(\mathcal{D}_0(M)) = \pi_k(\mathcal{MET}^0(M)) \to \pi_k(\mathcal{MET}^\epsilon(M))$, provided $\epsilon \leq \epsilon(M)$. Theorem 1 then shows that in many cases $\epsilon_0(M) < \infty$.

Remark 7. We recall an open problem posed by K. Burns and A. Katok [BK85, Quest. 7.1] about hyperbolic manifolds M. Is $\mathcal{MET}^\infty(M)$ path connected? More generally, one could ask if $\mathcal{MET}^\infty(M)$ is contractible. Equivalently, is $\mathcal{T}^\infty(M) \to \mathcal{T}(M) = B\mathcal{D}_0(M)$ a homotopy equivalence? Before Theorem 1, it was conceivable that the opposite extreme in the interpolation between $\mathcal{T}^0(M)$ and $\mathcal{T}(M)$ could be true, that is, $\mathcal{T}^\infty(M)$ is always contractible, or equivalently $* = \mathcal{T}^0(M) \to \mathcal{T}^\infty(M)$ is always a homotopy equivalence. See [FO06], [FO07] for recent results relevant to these questions.

Remark 8. Let M be a hyperbolic manifold. Since $\mathrm{Diff}(M)/\mathrm{Diff}_0(M) \cong \mathrm{Out}(\pi_1(M))$ we have $\mathcal{M}(M) \cong \mathcal{T}(M)/\mathrm{Out}(\pi_1(M))$ or, in general, $\mathcal{M}^\epsilon(M) \cong \mathcal{T}^\epsilon(M)/\mathrm{Out}(\pi_1(M))$. Note that $\mathrm{Out}(\pi_1(M))$ is a finite group, provided that $\dim M \geq 3$. We do not know whether our results descend to the moduli spaces. See [FO08] for recent results relevant to these questions.

Remark 9. Let M be a hyperbolic manifold. We can consider the quotients of $\mathcal{MET}(M)$ and $\mathcal{MET}^\epsilon(M)$ by $\mathrm{Diff}^0(M)$, the connected component of the identity 1_M in $\mathrm{Diff}(M)$, instead of by the larger group $\mathrm{Diff}_0(M)$. Since the quotient group

$\mathrm{Diff}_0(M)/\,\mathrm{Diff}^0(M)$ is discrete, it can be easily checked from the proof of our results that the statement of Theorem 1 also holds for the inclusion of the quotients: $\mathcal{MET}^\epsilon(M)/\,\mathrm{Diff}^0(M) \to \mathcal{MET}(M)/\,\mathrm{Diff}^0(M)$, with the strengthened restriction "$2 \leq k \leq k_0$" and proviso "(provided $k_0 \geq 5$)".

Theorem 1 follows from the more technical Theorems 2 and 3 below. To state these results we need some notation. Write $\mathcal{G} \subset \mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1,2], \partial)$ for the group of all smooth isotopies φ of the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ that are the identity near 1 and 2 and are homotopic to the identity by a homotopy that is constant near 1 and 2. That is, $\varphi : \mathbb{S}^{n-1} \times [1,2] \to \mathbb{S}^{n-1} \times [1,2]$, and $\varphi(x,s) = (x,s)$ for s near 1 and 2, and φ is homotopic to the identity by a homotopy H such that $H_t(x,s) = (x,s)$ for s near 1 and 2 and all $t \in [0,1]$. Note that $\mathcal{G}$ depends only on the dimension $(n-1)$ of the sphere. If we need to express this dependency explicitly, we shall write $\mathcal{G}_n$.

Let N be a real hyperbolic manifold of dimension n, and let B be a closed geodesic ball of radius $2r$ centered at some point $p \in N$ (hence, the radius of injectivity of N at p is larger that $2r$). Identify $B \setminus \{p\}$ with $\mathbb{S}^{n-1} \times (0,2r]$, where the lines $t \mapsto (x,t)$ are the speed-one geodesics emanating from p. Now, every element in $\mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1,2], \partial)$ gives rise to an element in $\mathrm{Diff}_0(N)$ by identifying $\mathbb{S}^{n-1} \times [1,2]$ with $\mathbb{S}^{n-1} \times [r,2r]$. That is, we have a map $\Lambda = \Lambda(N,p,r)$: $\mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1,2], \partial) \to \mathrm{Diff}_0(N)$, defined by

$$\Lambda\varphi(p) = \begin{cases} p & \text{if } p \notin (\mathbb{S}^{n-1} \times [r,2r]) \subset N, \\ (\varphi_{t/r}(x), t) & \text{if } p = (x,t) \in (\mathbb{S}^{n-1} \times [r,2r]) \subset N, \end{cases}$$

where $\varphi \in \mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1,2], \partial)$ and $\varphi(x,s) = (\varphi_s(x), s)$. We will denote the restriction $\Lambda|_{\mathcal{G}}$ by the same symbol Λ.

Remark (a technical point). Rigorously, for Λ to be well defined (that is, for $\Lambda\varphi$ to be smooth), we will assume every element in $\mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1,2], \partial)$ is the identity near $\mathbb{S}^{n-1} \times \{1\}$ and $\mathbb{S}^{n-1} \times \{2\}$. This does not cause problems since standard extension methods (along collars) show that the inclusion of the space of all elements in $\mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1,2], \partial)$, with the properties above, into $\mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1,2], \partial)$ is a homotopy equivalence.

Remark (another technical point). The map Λ depends also on the identification between $B \setminus \{p\}$ and $\mathbb{S}^{n-1} \times (0,2r]$ used above. This identification is uniquely determined if an orthonormal basis $\mathcal{B}$ of T_pM is specified. Hence we should write $\Lambda = \Lambda(N,p,r,\mathcal{B})$. To alleviate the notation we still write $\Lambda = \Lambda(N,p,r)$ since the choice of $\mathcal{B}$ is not essential. Note also that such a map Λ can be defined whenever the radius of injectivity of N at p is larger than $2r$.

THEOREM 2. *Given $\epsilon > 0$ and a compact subset $K \subset \mathcal{G}$, there is a real number $r > 0$ such that the following holds. Let (N, g^0) be a closed real hyperbolic* *manifold, and let $p \in N$ with radius of injectivity at p larger than $3r$. Then the map $K \to \mathcal{MET}^{\epsilon}(N)$ given by $\phi \mapsto (\Lambda\phi)g^0$ is contractible; i.e. homotopic to a constant map. Here $\Lambda = \Lambda(N, p, r)$.*

THEOREM 3. *For every integer $k \geq 0$, there is an integer $n_1 = n_1(k)$ and elements $\alpha_{k,n} \in \pi_k(\mathcal{G}_n)$ with $n \geq n_1$ such that the following holds. If N is a closed real hyperbolic n-manifold, with $n \geq n_1$ and $n + k \equiv 2 \bmod 4$, which is a π-manifold, then $\Lambda_{\#}(\alpha_{k,n}) \neq 0 \in \pi_k(\mathrm{Diff}_0(N))$. Here $\Lambda = \Lambda(N, p, r)$, where $p \in N$ is any point and $2r$ is less than the injectivity radius of N at p.*

To prove the corollary we will need this addendum to Theorem 3:

ADDENDUM. *For $k = 0$, we can choose $n_1 = n_1(1) = 6$ and we can drop the condition $n + k \equiv 2 \bmod 4$, provided $\Theta_{n+1} \neq 0$.*

Note that, while Theorem 2 is a geometric result, Theorem 3 is purely topological: the point p and the number r are arbitrary, and the only restriction is that $2r$ is less than the injectivity radius of N at p.

In Section 1, we deduce Theorem 1 and its corollary from Theorems 2 and 3. In Section 2, we prove Theorem 2, and in Section 3, we prove Theorem 3 together with its addendum.

We are grateful for the referee's useful comments and suggestions.

1. Proof of Theorem 1 and its corollary

Here we prove Theorem 1 and its corollary assuming Theorems 2 and 3. First we give a lemma and some remarks.

Recall that $\mathrm{Diff}_0(P)$ and $\mathcal{D}_0(P)$ act on $\mathcal{MET}(P)$ for any closed smooth manifold P.

LEMMA 1.1. *If P is aspherical and the center of $\pi_1 P$ is trivial, then the action of $\mathrm{Diff}_0(P)$ and $\mathcal{D}_0(P)$ on $\mathcal{MET}(P)$ is free.*

Proof. Let $g \in \mathcal{MET}(P)$. The isotropy group $H = \{\phi \in \mathrm{Diff}_0(P) : \phi g = g\}$ of the action of $\mathrm{Diff}_0(P)$ at g is $\mathrm{Iso}_0(M, g)$, the group of all isometries of the Riemannian manifold (M, g) that are homotopic to the identity. Hence this isotropy group H is compact. Let $\gamma : \mathrm{Diff}(P) \to \mathrm{Out}(\pi_1 P)$ be the homomorphism induced by $\phi \mapsto \phi_*$. Borel, Conner and Raymond showed [CR77, p. 43] that under the assumptions above, γ restricted to compact subgroups is monic. But $\gamma(H)$ is trivial, since every element in $\mathrm{Diff}_0(P)$ is, by definition, homotopic to the identity. It follows that H is trivial. Hence the action of $\mathrm{Diff}_0(P)$ is free. Therefore the action of $\mathcal{D}_0(P)$ is also free. □

The lemma implies that there is a fibration $\mathcal{D}_0(P) \to \mathcal{MET}(P) \to \mathcal{T}(P)$, and, since $\mathcal{MET}(P)$ is contractible, we have

$$\pi_{k-1}(\mathrm{Diff}_0(P)) \cong \pi_{k-1}(\mathcal{D}_0(P)) \cong \pi_k(\mathcal{T}(P)).$$

We can give an explicit isomorphism between $\pi_{k-1}(\mathrm{Diff}_0(P))$ and $\pi_k(\mathcal{T}(P))$. Let g^0 be any metric on P. Let $\beta : \mathbb{S}^{k-1} \to \mathrm{Diff}_0(P)$ be an element in $\pi_{k-1}(\mathrm{Diff}_0(P))$. Define $\beta' : \mathbb{S}^{k-1} \to \mathcal{MET}(P),\ u \mapsto \beta(u)g^0$. Since $\mathcal{MET}(P)$ is contractible, we can extend β' to a map $\bar{\beta}$ defined on the whole disc $\mathbb{D}^k$. Then the isomorphism is given by $\beta \mapsto p\bar{\beta}$, where $p : \mathcal{MET}(P) \to \mathcal{T}(P)$ is the quotient map that assigns to each metric its $\mathcal{D}_0(P)$ orbit.

Proof of Theorem 1, *assuming Theorems* 2 *and* 3. Let $\epsilon > 0$ and $k_0 > 0$. Let $n_0 = n_0(k_0) = \max\{n_1(k-1),\ 1 \le k \le k_0\}$, where $n_1(k-1)$ is as in Theorem 3. Let n be such that $n \ge n_0$. Define the compact subset K of $\mathcal{G}$ by

$$K = \{\alpha_{k-1}(u) : u \in \mathbb{S}^{k-1},\ 1 \le k \le k_0,\ n + (k-1) \equiv 2 \bmod 4\},$$

where the $\alpha_{k-1} = \alpha_{(k-1),n}$ are explicit representatives of the elements given in Theorem 3. Note that

$$K = \bigcup_k \mathrm{image}(\alpha_{k-1}), \quad \text{where } 1 \le k \le k_0 \text{ and } n + (k-1) \equiv 2 \bmod 4.$$

Let r be as in Theorem 2 for ϵ and K as above.

Let M be a closed hyperbolic manifold of dimension n. By taking successive finite sheeted covers, we find a finite sheeted cover N of M such that

- N is a π manifold (see [Sul79, p. 553]), and
- N has a point p with injectivity radius larger than $3r$. (Recall that $\pi_1(M)$ is residually finite; see [Mag69].)

Let g^0 be the hyperbolic metric of N pulled back from M. Write $\Lambda = \Lambda(N, p, r)$. Define $\beta_{k-1} = \Lambda\alpha_{k-1} \in \pi_{k-1}(\mathrm{Diff}_0(N))$. By Theorem 3, all β_{k-1} are nonzero. Define also $\beta'_{k-1} : \mathbb{S}^{k-1} \to \mathcal{MET}^\epsilon(N)$ by $\beta'_{k-1}(u) = \beta_{k-1}(u)g^0$ for $u \in \mathbb{S}^{k-1}$. By Theorem 2, we can extend each $\beta' = \beta'_{k-1}$ to the whole disc $\mathbb{D}^k$, obtaining maps $\bar{\beta} : \mathbb{D}^k \to \mathcal{MET}^\epsilon(N)$.

Recall that $p : \mathcal{MET}(P) \to \mathcal{T}(P)$ is the quotient map that assigns to each metric its $\mathcal{D}_0(P)$ orbit (see the comments following the proof of Lemma 1.1). Since $p\bar{\beta}(\mathbb{S}^{k-1})$ contains exactly one point (this point is $p(g^0)$), we have that $p\bar{\beta}$ determines an element in $\pi_k(\mathcal{T}(N))$. Also, since $\beta_{k-1} \ne 0$, we have that $p\bar{\beta} \ne 0 \in \pi_k(\mathcal{T}(N))$. But image$(\bar{\beta}) \subset \mathcal{MET}^\epsilon(N)$; hence $p\bar{\beta}$ is in the image of the map $\pi_k(\mathcal{T}^\epsilon(N)) \to \pi_k(\mathcal{T}(N))$ induced by the inclusion $\mathcal{T}^\epsilon(M) \hookrightarrow \mathcal{T}(M)$. This proves Theorem 1 assuming Theorems 2 and 3. □

The proof of the corollary is similar: just use the addendum to Theorem 3. The proof of the addendum to Theorem 1 and its corollary is also similar.

2. **Proof of Theorem** 2

First we introduce some notation and give a lemma. We denote by $\mathcal{G}' \subset \mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1,2])$ the group of all smooth isotopies φ of the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ that are the identity near 1 and constant near 2. That is, if $\varphi : \mathbb{S}^{n-1} \times [1,2] \to \mathbb{S}^{n-1} \times [1,2]$ takes (x,s) to (y,s), then y does not depend on s for s near 2, and $\varphi(x,s) = (x,s)$ for s near 1. Note that $\mathcal{G}'$ depends only on the dimension $n-1$ of the sphere. We have an inclusion $\mathcal{G} \hookrightarrow \mathcal{G}'$.

LEMMA 2.1. *$\mathcal{G}'$ is contractible.*

Proof. Recall that the space of all isotopies is homeomorphic to the space of smooth paths of vector fields V_s for $s \in [1,2]$ on the sphere (or any closed manifold). This correspondence is given explicitly in the following way. An isotopy φ corresponds to the smooth path of vector fields V_s, where $V_s(x) = (d/ds)\varphi_s(x)|_s$. Here $\varphi(x,s) = (\varphi_s(x), s)$. Conversely, given a smooth path of vector fields V_s for $s \in [1,2]$, we can integrate it and obtain the flow φ_s of V_s. Then V_s corresponds to the isotopy $\varphi(x,s) = (\varphi_s(x), s)$. But every vector field (or path of vector fields) can be deformed to the zero vector field by homotheties: $(\mu, V) \mapsto (1-\mu)V$ for $\mu \in [0,1]$ is a homotopy of V to the zero vector field. Integrating this homotopy (for each μ) we obtain a homotopy from the identity $\mathrm{id}_{\mathcal{G}'}$ to the constant map $\mathcal{G}' \to \{\mathrm{id}_{\mathbb{S}^{n-1}\times[1,2]}\} \subset \mathcal{G}'$. □

Note that the homotopy given in the proof of the lemma fixes the identity $\mathrm{id}_{\mathbb{S}^{n-1}\times[1,2]}$. Note also that the homotopy does not necessarily leave $\mathcal{G}$ invariant.

Proof of Theorem 2. We first prove the theorem for the case in which K has exactly one element. Fix $\epsilon > 0$ and $\varphi \in \mathcal{G}$. Let (N, g^0) be a real hyperbolic manifold, and let $p \in N$ with injectivity radius (at p) larger than $3r$. We will construct a deformation of $(\Lambda\varphi)g^0 = (\Lambda\varphi)_* g^0$ to g^0, through metrics in $\mathcal{MET}^\epsilon(N)$, assuming that r is large enough. Here $\Lambda = \Lambda(N, p, r)$.

Denote by $B \subset N$ the closed geodesic ball centered at p of radius $3r$. As before, we identify $B \setminus \{p\}$ with $\mathbb{S}^{n-1} \times (0, 3r]$. In fact, this identification can be done isometrically: $B \setminus \{p\}$ with metric g^0 is isometric to $\mathbb{S}^{n-1} \times (0,3r]$ with metric $\sinh^2(t)h + dt^2$, where h is the Riemannian metric on the sphere $\mathbb{S}^{n-1}$ with constant curvature equal to 1. In view of this identification, we write then

$$g^0(x,t) = \sinh^2(t)h(x) + dt^2.$$

Write $\phi = \Lambda\varphi$. Also write $g^1 = \phi g^0$. The metric g^1 on $B \setminus \{p\}$ is given by

$$g^1(x,t) = \begin{cases} g^0(x,t) & \text{if } t \notin [r, 2r], \\ \varphi_* g^0(x,t) & \text{if } t \in [r, 2r]. \end{cases}$$

By the above Lemma 2.1 we have a path of isotopies $\varphi^\mu \in \mathcal{G}'$ for $\mu \in [0,1]$, with $\varphi^0 = \varphi$ and $\varphi^1 = \mathrm{id}_{\mathbb{S}^{n-1}\times[1,2]}$. Write $\theta^\mu = \varphi_2^\mu$ for the final map of the isotopy

φ^μ, that is, $\varphi^\mu(x,2) = (\theta^\mu(x),2)$. Then $\theta^0 = \theta^1 = \mathrm{id}_{\mathbb{S}^{n-1}}$. Define

$$\phi^\mu : \mathbb{S}^{n-1} \times [r,2r] \to \mathbb{S}^{n-1} \times [r,2r]$$

by rescaling φ to the interval $[r,2r]$, that is, $\phi^\mu(x,t) = (\varphi^\mu_{t/r}(x),t)$.

Let $\delta : [2,3] \to [0,1]$ be smooth with $\delta(2)=1$, $\delta(3)=0$, and δ constant near 2 and 3. We now define a path of metrics g^μ on $B \setminus \{p\} = \mathbb{S}^{n-1} \times (0,3r]$:

$$g^\mu(x,t) = \begin{cases} g^0(x,t) & \text{if } t \in (0,r], \\ (\phi^\mu)_* g^0(x,t) & \text{if } t \in [r,2r], \\ \sinh^2(t) g\big(\delta(\frac{t}{r})(\theta^\mu)_* h(x) + (1-\delta(\frac{t}{r}))h(x)g\big) + dt^2 & \text{if } t \in [2r,3r]. \end{cases}$$

Since δ and all isotopies we used are constant near the endpoints of their intervals of definitions, it is straightforward to show that g^μ is a smooth metric on $B \setminus \{p\}$ and that g^μ joins g^1 to g^0. Moreover, $g^\mu(x,t) = g^0(x,t)$ for t near 0 and 3. Hence we can extend g^μ to the whole manifold N by defining $g^\mu(q) = g^0(q)$ for $q = p$ or $q \notin B$.

CLAIM 2.2. *We have $g^\mu \in \mathcal{MET}^\epsilon(N)$, provided r is large enough, and the necessary size of r depends only on φ^μ and δ.*

Proof. The metric $g^\mu(x,t)$ is equal to $g^0(x,t)$ for $t \in (0,r]$; hence $g^\mu(x,t)$ is hyperbolic for $t \in (0,r]$. Also, $g^\mu(x,t)$ is the push-forward (by ϕ^μ) of the hyperbolic metric g^0 for $t \in [r,2r]$; hence $g^\mu(x,t)$ is hyperbolic for $t \in [r,2r]$. For $t \in [2r,3r]$, the metric $g^\mu(x,t)$ is similar to the ones constructed in [FJ89a, §3] or [Ont94, Th. 3.1]. It can be checked from those references that the sectional curvatures of g^μ are ϵ close to -1, provided r is large enough. How large we need r to be depends only on the partial derivatives (up to order two) of φ^μ and δ. This proves the claim and Theorem 2 for the case in which the compact set K has exactly one element. □

For the general case, just note that since K is compact, so is the set $\overline{K} = \{\varphi^\mu : \varphi \in K\}$, where φ^μ denotes the canonical deformation of an element $\varphi \in K$ to the identity (given by Lemma 2.1 above). Then all partial derivatives (up to order two) of all elements in $\overline{K}$ are bounded. Therefore there is a real number r for which the argument used in the claim above works for all $\varphi \in K$. This proves Theorem 2. □

3. **Proof of Theorem** 3

In this section we will always assume that the manifold N is a closed real hyperbolic manifold which is a π-manifold. To prove Theorem 3 we will first reduce the problem to another problem.

3.1. *First reduction.* Recall that for any manifold L, $\mathrm{Diff}(L)$ is the space of all self-diffeomorphisms of L, with the smooth topology, and if $\partial L \neq \varnothing$, then

$\mathrm{Diff}(L, \partial)$ denotes the space of all self-diffeomorphisms of L that are the identity on the boundary.

Let $\Omega\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)$ be the space of all (continuous) loops in $\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)$ based at the identity $1_{\mathbb{D}^{n-1}}$. A loop $t \mapsto f_t$ for $t \in [0, 1]$ is smooth if the map $(x, t) \mapsto f_t(x)$ is smooth. Classical approximation methods (for example, convolution) show that the inclusion of the space of all smooth loops into $\Omega\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)$ is a homotopy equivalence. Hence we will assume, when necessary, that loops are smooth. We will also assume, if necessary that the loops are constant near 0 and 1. This does not cause any problems either.

We define a map

$$\alpha : \Omega\mathrm{Diff}(\mathbb{D}^{n-1}, \partial) \to \mathrm{Diff}(\mathbb{D}^n, \partial)$$

by the formula $\alpha(f_t)(x, t) = (f_t(x), t)$ for $(x, t) \in \mathbb{D}^{n-1} \times [0, 1] = \mathbb{D}^n$. Here $t \mapsto f_t$ denotes a loop in $\mathrm{Diff}(\mathbb{D}^{n-1})$, and we are identifying $\mathbb{D}^{n-1} \times [0, 1]$ with $\mathbb{D}^n$. (Certainly here we must assume that the loops are smooth. We also must have smooth corners.)

Remark. This map α and the standard constructions of it used here have appeared (much earlier) in Gromoll's fundamental work [Gro66] on positive curvature questions.

Identify $\mathbb{D}^{n-1}$ with, say, the northern hemisphere of the sphere $\mathbb{S}^{n-1}$. Then we have inclusions $\mathbb{D}^n = \mathbb{D}^{n-1} \times [1, 2] \hookrightarrow \mathbb{S}^{n-1} \times [1, 2] \hookrightarrow N$. The composition induces a map $\mathrm{Diff}(\mathbb{D}^n, \partial) \hookrightarrow \mathrm{Diff}_0(N)$, and this map factors through $\mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1, 2], \partial)$:

$$\mathrm{Diff}(\mathbb{D}^n, \partial) \hookrightarrow \mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1, 2], \partial) \xhookrightarrow{\Lambda} \mathrm{Diff}_0(N)\ ,$$

and we denote this composition also by Λ.

Remark. As in the remark before the statement of Theorem 3, we will assume that the elements in $\mathrm{Diff}(\mathbb{D}^n, \partial) = \mathrm{Diff}(\mathbb{D}^{n-1} \times [0, 1], \partial)$ are constant near $\partial(\mathbb{D}^{n-1} \times [0, 1]) = \mathbb{D}^{n-1} \times \{0, 1\} \cup \mathbb{S}^{n-2} \times [0, 1]$. We make this assumption so that the map $\mathrm{Diff}(\mathbb{D}^n, \partial) \hookrightarrow \mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1, 2], \partial)$ is well defined. Again, as before, this does not cause problems since standard extension methods (along collars) show that the inclusion of the space of all elements in $\mathrm{Diff}(\mathbb{D}^n, \partial)$ with the properties above into $\mathrm{Diff}(\mathbb{D}^n, \partial)$ is a homotopy equivalence.

Now, we note a simple but important fact: an element in $\mathrm{Diff}(\mathbb{D}^n, \partial)$ is mapped to $\mathcal{G}$ by the map $\mathrm{Diff}(\mathbb{D}^n, \partial) \to \mathrm{Diff}_0(\mathbb{S}^{n-1} \times [1, 2], \partial)$ if and only if it is in the image of α. Therefore we have reduced the proof of Theorem 3 to this:

THEOREM 3′. *Fix $k \geq 0$. For sufficiently large n with $n + k \equiv 2 \bmod 4$, the composition map*

$$\pi_k(\Omega\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)) \xrightarrow{\alpha_\#} \pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial)) \xrightarrow{\Lambda_\#} \pi_k(\mathrm{Diff}_0(N^n))$$

is nonzero. Also, if $k = 0$, $n \geq 10$ and $\Theta_{n+1} \neq 0$, the composition map is also nonzero.

Here, as we mentioned at the beginning of this section, N^n is any closed real hyperbolic manifold of dimension n which is a π-manifold. Note that the last statement of Theorem 3′ corresponds to the addendum to Theorem 3. We now further reduce the statement of Theorem 3′ to another statement.

3.2. *Second reduction.* First we recall some definitions. For a compact smooth manifold L, denote by $\widetilde{\mathrm{Diff}}(L)$ the semisimplicial group whose l-simplices are self-diffeomorphisms of $\Delta^l \times L$ that send faces $\sigma \times L$ to themselves. Here Δ^l is the l-simplex, and σ is any subsimplex of Δ^l; see [Wal70, §17A]. We can consider $\mathrm{Diff}(L)$ as contained in $\widetilde{\mathrm{Diff}}(L)$ in two (homotopy equivalent) ways: as the set of vertices of $\widetilde{\mathrm{Diff}}(L)$ or as the semisimplicial subgroup whose i-simplices are self-diffeomorphisms of $\Delta^l \times L$ that commute with the projection to Δ^l. Also, define $\widetilde{\mathrm{Diff}}(L, \partial)$ as before, but with the extra requirement that the self-diffeomorphisms of $\Delta^l \times L$ be the identity on $\Delta^l \times \partial L$.

If we replace "diffeomorphism" above by "homeomorphism" or "simple homotopy equivalence", then we obtain spaces $\widetilde{\mathrm{Top}}(L)$ and $\widetilde{G}(L)$ (and also $\widetilde{\mathrm{Top}}(L, \partial)$). Here $G(L)$ is the H-space of all simple homotopy equivalences of L. Since a self-homotopy equivalence does not have to be one-to-one, we have that $G(L)$ and $\widetilde{G}(L)$ are homotopy equivalent. We have fibrations (see [Wal70, §17A])

$$\widetilde{\mathrm{Diff}}(L) \to \widetilde{G}(L) \to \widetilde{G}/\widetilde{\mathrm{Diff}}(L),$$
$$\widetilde{\mathrm{Top}}/\widetilde{\mathrm{Diff}}(L) \to \widetilde{G}/\widetilde{\mathrm{Diff}}(L) \to \widetilde{G}/\widetilde{\mathrm{Top}}(L).$$

It is known that $\pi_i(\widetilde{\mathrm{Top}}/\widetilde{\mathrm{Diff}}(L)) \cong [L \times \mathbb{D}^i, \partial\,; \mathrm{Top}/O]$, where $[\cdot\,, \cdot]$ denotes "homotopy classes of maps". Since Top/O is an infinite loop space, it defines a (nonreduced) generalized cohomology theory such that

$$h^{-i}(L) = [L \times \mathbb{D}^i, \partial\,; \mathrm{Top}/O].$$

We now come back to the proof. The map $\Lambda : \mathrm{Diff}(\mathbb{D}^n, \partial) \to \mathrm{Diff}_0(N)$ clearly induces a semisimplicial map $\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial) \to \widetilde{\mathrm{Diff}}(N)$.

LEMMA 3.3. *$\pi_i(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial)) \to \pi_i(\widetilde{\mathrm{Diff}}(N))$ is a monomorphism for every i, provided $n \geq 5$.*

(Recall that we are assuming that N is any closed real hyperbolic manifold of dimension n that is also a π-manifold.)

Proof. We claim the following.

(i) $\pi_i(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial)) \cong \pi_{i+1}(\widetilde{G}/\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial))$.

(ii) $\pi_i(\widetilde{\mathrm{Diff}}(N)) \cong \pi_{i+1}(\widetilde{G}/\widetilde{\mathrm{Diff}}(N))$ when $i \geq 1$, and $\pi_1(\widetilde{G}/\widetilde{\mathrm{Diff}}(N))$ naturally injects into $\pi_0(\widetilde{\mathrm{Diff}}(N))$ (when $i = 0$).

(iii) $\pi_i(\widetilde{\mathrm{Top}}/\widetilde{\mathrm{Diff}}(\mathbb{D}^n,\partial)) \cong \pi_i(\widetilde{G}/\widetilde{\mathrm{Diff}}(\mathbb{D}^n,\partial))$.

(iv) $\pi_i(\widetilde{\mathrm{Top}}/\widetilde{\mathrm{Diff}}(N)) \cong \pi_i(\widetilde{G}/\widetilde{\mathrm{Diff}}(N))$, provided $n \geq 5$.

To prove (i) and (ii) use the first fibration above and just note that (a) every homotopy equivalence on the disc $\mathbb{D}^n$, modulo boundary, can be canonically deformed (by Alexander's trick) to the identity, and hence $\widetilde{G}(\mathbb{D}^n,\partial)$ is contractible; (b) since N is negatively curved, $\widetilde{G}(N) \cong \mathrm{Out}(\pi_1(N))$, which is a discrete set. To prove (iii) and (iv), use the second fibration above and just note that (c) $\widetilde{G}/\widetilde{\mathrm{Top}}(\mathbb{D}^n,\partial)$ is contractible because the canonical deformation mentioned above preserves homeomorphisms and (d) $\widetilde{G}/\widetilde{\mathrm{Top}}(N)$ is contractible by Farrell-Jones rigidity results [FJ91].

It follows from (i)–(iv) above that

$$\begin{aligned}\pi_i(\widetilde{\mathrm{Diff}}(\mathbb{D}^n,\partial)) &\cong \pi_{i+1}(\widetilde{\mathrm{Top}}/\widetilde{\mathrm{Diff}}(\mathbb{D}^n,\partial))\\ &\cong [\mathbb{D}^{n+i+1},\partial\,;\mathrm{Top}/O] = \tilde{h}^{-(i+1)}(\mathbb{S}^n),\\ \pi_i(\widetilde{\mathrm{Diff}}(N)) &\cong \pi_{i+1}(\widetilde{\mathrm{Top}}/\widetilde{\mathrm{Diff}}(N))\\ &\cong [N\times\mathbb{D}^{i+1},\partial\,;\mathrm{Top}/O] = h^{-(i+1)}(N),\end{aligned}$$

provided $n \geq 5$ and $i \geq 1$.

We have then the commutative diagram

$$\begin{array}{ccc} \tilde{h}^{-(i+1)}(\mathbb{S}^n) = [\mathbb{D}^{n+i+1},\partial;\mathrm{Top}/O] & \longrightarrow & [N\times\mathbb{D}^{i+1},\partial;\mathrm{Top}/O] = h^{-(i+1)}(N) \\ \downarrow & & \downarrow \\ \pi_i(\widetilde{\mathrm{Diff}}(\mathbb{D}^n,\partial)) & \longrightarrow & \pi_i(\widetilde{\mathrm{Diff}}(N)). \end{array}$$

Here the vertical arrows are the canonical isomorphisms mentioned above, and, when $i = 0$, the first vertical arrow is also an isomorphism while the second vertical is still injective (by (ii) above). Now, since every diffeomorphism in the image of the map $\mathrm{Diff}(\mathbb{D}^n,\partial) \to \mathrm{Diff}(N)$ is the identity outside $\mathbb{D}^n \subset N$, the image of an element in $[\mathbb{D}^{n+i+1},\partial\,;\mathrm{Top}/O]$ by the map $[\mathbb{D}^{n+i+1},\partial\,;\mathrm{Top}/O] \to [N\times\mathbb{D}^{i+1},\partial\,;\mathrm{Top}/O]$ in the diagram above has the property that it is constant outside $\mathbb{D}^n \subset N$. Hence the map

$$[\mathbb{D}^{n+i+1},\partial\,;\mathrm{Top}/O] \to [N\times\mathbb{D}^{i+1},\partial\,;\mathrm{Top}/O]$$

is induced by the map

$$c\times 1_{\mathbb{D}^{i+1}} : N\times\mathbb{D}^{i+1} \to \mathbb{S}^n\times\mathbb{D}^{i+1},$$

where $c : N \to N/\text{closure}\,(N\setminus\mathbb{D}^n) = \mathbb{D}^n/\partial = \mathbb{S}^n$ is the collapsing map. Therefore, at the cohomology level, the map $h^{-(i+1)}(\mathbb{S}^n) \to h^{-(i+1)}(N)$ induced by the degree-one collapsing map c composed with the canonical monomorphism $\tilde{h}^{-(i+1)}(\mathbb{S}^n) \to h^{-(i+1)}(\mathbb{S}^n)$ is the homomorphism of Lemma 3.3. The lemma now follows from the following result:

If N^n is a closed stably parallelizable manifold and $f : N \to \mathbb{S}^n$ is a degree-one map, then $f^* : h^*(\mathbb{S}^n) \to h^*(N)$ is a monomorphism for any representable generalized cohomology theory h.

For the proof of this result, see [FJ89a, Cl. 2.4, p. 902]. (Replace Top/O in the proof of that claim by the infinite loop space corresponding to h.) Also, the referee points out that this result has long been known to be an easy corollary to results of G. W. Whitehead [Whi62] and W. Browder [Bro72], since π-manifolds are orientable for such theories h^*. This completes the proof of the lemma. □

Note that $\pi_k(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial)) \cong [\mathbb{D}^{n+k+1}, \partial\,; \mathrm{Top}/O] \cong \Theta_{n+k+1}$, the group of homotopy spheres of dimension $n + k + 1$.

Consider now the commutative diagram

$$\begin{tikzcd} \pi_k(\Omega\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)) \arrow[r, "\alpha_\#"] & \pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial)) \arrow[r, "\Lambda_\#"] \arrow[d] & \pi_k(\mathrm{Diff}(N)) \arrow[d] \\ & \pi_k(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial)) \arrow[r] & \pi_k(\widetilde{\mathrm{Diff}}(N)) \end{tikzcd}$$

and, in particular, just the left part of this diagram:

$$\pi_k(\Omega\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)) \xrightarrow{\alpha_\#} \pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial)) \longrightarrow \pi_k(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial)). \tag{1}$$

From Lemma 3.3 above, we see that Theorem $3'$ is implied by the following statement:

THEOREM $3''$. *Fix $k \geq 0$. For sufficiently large n with $n + k \equiv 2 \bmod 4$, the composition map in diagram* (1) *above is nonzero. Also, if $k = 0$, $n \geq 6$ and $\Theta_{n+1} \neq 0$, the composition map is also nonzero.*

Note we have succeeded in eliminating the manifold N from the problem.

3.4. *Proof of Theorem* $3''$. First we recall some definitions and introduce some notation. For a manifold L, the space of smooth pseudoisotopies of L is denoted by $P(L)$, that is, $P(L)$ consists of all self-diffeomorphisms of $L \times I$ that are the identity on $L \times \{0\} \cup \partial L \times I$. Here $I = [0, 1]$. (The condition of the pseudoisotopies being the identity on $\partial L \times I$ is useful but superfluous: the space of pseudoisotopies of L that are the identity just on $L \times \{0\}$ can be identified with $P(L)$ by "bending around corners"; see [Hat78].) Note that $P(L)$ is a group with the composition. We have stabilization maps $\Sigma : P(L) \to P(L \times I)$. The direct limit of the sequence

$$P(L) \to P(L \times I) \to P(L \times I^2) \to \cdots$$

is called the space of stable pseudoisotopies of L, and it is denoted by $\mathcal{P}(L)$. We mention two important facts: First, $\mathcal{P}(\cdot)$ is a homotopy functor. Therefore we get, for example, that $\mathcal{P}(\mathbb{D}^n) = \mathcal{P}(*)$, where $*$ denotes a point. Second, the map $\pi_k(P(L)) \to \pi_k(\mathcal{P}(L))$ is an isomorphism for $k \ll \dim L$; see [Igu88].

Let $\mathscr{A}_{n-1}$ denote the subgroup of $\mathrm{Diff}(\mathbb{D}^{n-1}, \partial)$ consisting of all self-diffeomorphisms of $(\mathbb{D}^{n-1}, \partial)$ that are pseudoisotopic to the identity. Hence we have a surjective homomorphism $\tau : P(\mathbb{D}^{n-1}) \to \mathscr{A}_{n-1}$, where we just take the "top" of a pseudoisotopy f, that is, $\tau(f) = f|_{\mathbb{D}^{n-1}\times\{1\}}$. The kernel of this map is the space of all pseudoisotopies that are the identity on $\partial(\mathbb{D}^{n-1} \times I)$, and we can identify this space with $\mathrm{Diff}(\mathbb{D}^n, \partial)$. Consequently, we get the sequence

$$\mathrm{Diff}(\mathbb{D}^n, \partial) \xrightarrow{\eta} P(\mathbb{D}^{n-1}) \xrightarrow{\tau} \mathscr{A}_{n-1}. \tag{2}$$

CLAIM 3.5. *Sequence* (2) *is a Hurewicz fibration.*

Proof. Let $t \mapsto f_t$ be a (smooth) path in $\mathscr{A}_{n-1}$ beginning at f_0 and ending at f_1. (Assume also that the path is constant near 0 and 1.) Let F_0 be a lifting of f_0, so $\tau(F_0) = f_0$. To define a lifting $t \mapsto F_t$ of the whole path, beginning at F_0, just take the concatenation of F_0 with the map $(x, t') \mapsto (f_{t'}(x), t')$ for $0 \le t' \le t$. This defines F_t on $\mathbb{D}^{n-1} \times [0, 1 + t]$; now rescale back to the interval [0,1]. □

CLAIM 3.6. *The connecting map* $\beta : \Omega(\mathscr{A}_{n-1}) \to \mathrm{Diff}(\mathbb{D}^n, \partial)$ *of fibration* (2) *is homotopic to* α.

Proof. Let $t \mapsto f_t$ be a (smooth) path in $\mathscr{A}_{n-1}$ beginning at $f_0 = 1_{\mathbb{D}^{n-1}}$ and ending at $f_1 = 1_{\mathbb{D}^{n-1}}$. Take $F_0 = 1_{\mathbb{D}^n}$ as the lifting of f_0. Using the lifting given in the proof of Claim 3.5, we see that the path $t \mapsto f_t$ maps, by the connecting map, to F_1, where F_1 is such that $F_1(x, t) = (x, t)$ for $0 \le t \le 1/2$ and $F_1(x, t) = (f_{2t-1}(x), t)$ for $1/2 \le t \le 1$. By squeezing the interval $[0, 1/2]$ to 0, we see that the connecting map β is homotopic to α. □

Now, to prove Theorem 3$''$ it is enough to prove the following:

THEOREM 3$'''$. *Fix* $k \ge 0$. *For sufficiently large* n *with* $n + k \equiv 2 \bmod 4$, *the composition map*

$$\pi_k(\Omega\mathscr{A}_{n-1}) \xrightarrow{\beta_\#} \pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial)) \longrightarrow \pi_k(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial))$$

is nonzero. Also, if $k = 0$, $n \ge 10$ *and* $\Theta_{n+1} \ne 0$, *the composition map is also nonzero.*

Using the fibration $\mathrm{Diff}(\mathbb{D}^n, \partial) \longrightarrow \widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial) \longrightarrow \widetilde{\mathrm{Diff}}/\mathrm{Diff}(\mathbb{D}^n, \partial)$ (see [Wal70, §17A]) and fibration (2), we can embed the sequence in Theorem 3$'''$ into the larger diagram

$$\begin{tikzcd}
\pi_k(\Omega\mathscr{A}_{n-1}) \arrow[r, "\beta_\#"] & \pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial)) \arrow[r] \arrow[d] & \pi_k(P(\mathbb{D}^{n-1})) \\
 & \pi_k(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial)) = \Theta_{n+k+1} \arrow[d] & \\
 & \pi_k(\widetilde{\mathrm{Diff}}/\mathrm{Diff}(\mathbb{D}^n, \partial)), &
\end{tikzcd} \tag{3}$$

where the upper row and the central column are exact. Note that for $n \gg k$, $\pi_k(P(\mathbb{D}^{n-1})) \cong \pi_k(\mathcal{P}(\mathbb{D}^{n-1})) \cong \pi_k(\mathcal{P}(*))$, which does not depend on n; see [Igu88]. To prove Theorem $3'''$, we have three cases, with increasing degrees of difficulty: the case $k = 0$ (which corresponds to the addendum to Theorem 3), the case $k \not\equiv 3 \bmod 4$, and the case $k \equiv 3 \bmod 4$.

First case: $k = 0$, $n \geq 6$ *and* $\Theta_{n+1} \neq 0$. In this case, since $n - 1 \geq 5$ we have that $\pi_0(P(\mathbb{D}^{n-1})) \cong 0$ by Cerf's foundational work [Cer70]. Also, it follows immediately from the definitions that $\pi_0(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial)) \cong 0$. Since we are assuming that $\Theta_{n+1} \neq 0$, we can (using diagram (3)) pull back a nonzero element $a \neq 0 \in \Theta_{n+1}$ all the way back to $\pi_0(\Omega\mathcal{A}_{n-1})$. This proves Theorem $3'''$ for this case. □

Second case: $k \not\equiv 3 \bmod 4$, $n \gg k$ *and* $n + k \equiv 2 \bmod 4$. We will use the following facts:

(i) For $k \not\equiv 3 \bmod 4$, $\pi_k(\mathcal{P}(*))$ is a finite group. For $k \equiv 3 \bmod 4$, we have $\pi_k(\mathcal{P}(*)) \cong \mathbb{Z} \oplus$(finite group) (see [Dwy80], [FH78], [Wal78]). Denote by a_k the order of the torsion part of $\pi_k(\mathcal{P}(*))$.

(ii) Using Hatcher's spectral sequence (see [Hat78, Props. 2.1 and 2.2]) we have, for $n \gg k$, that the group $\pi_k(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial))$ has a filtration

$$0 = G_0 < \cdots < G_k = \pi_k(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial))$$

such that G_i/G_{i-1} is a subquotient of $H_{k-i}(\mathbb{Z}_2, \pi_{i-1}\mathcal{P}(*))$. Since all the homology groups $H_{k-i}(\mathbb{Z}_2, \pi_{i-1}\mathcal{P}(*))$ are 2-torsion groups for $i < k$ and $H_0(\mathbb{Z}_2, \pi_{k-1}\mathcal{P}(*)) \cong \pi_{k-1}\mathcal{P}(*)/\mathbb{Z}_2$ (the quotient by the action of $\mathbb{Z}_2$), we see that the torsion part of $\pi_k(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial))$ has $2^k a_{k-1}$ for an exponent.

(iii) The group Θ_{4m-1} has a cyclic subgroup of order

$$2^{2m-2}(2^{2m-1} - 1) \cdot \text{numerator}\{4B_m/m\}$$

(see [MS74, p. 285]). Note that this order increases exponentially with m.

The important observation here is that, for $n \gg k$, the group $\pi_k(P(\mathbb{D}^{n-1}))$ and an exponent of the torsion part of $\pi_k(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial))$ do not depend on n.

Remark 10. Here and in the next case (that is, in the proofs of the second and third cases) item (ii) above can be replaced by the following fact that can be deduced using [BL82] or [HJ82, Lem. 2.2]: *the exponent of the odd order torsion part of* $\pi_k(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial))$, *when* $n \gg k$, *does not depend on* n. (Indeed this exponent is a_{k-1}, where a_{k-1} is as in item (i).) To use this fact instead of item (ii), just note that by item (iii), Θ_{4m-1} has elements of order $2^{2m} - 1$, which is large and odd.

Remark 11. Antonelli, Burghelea and Kahn [ABK72] showed that the image of $\pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial)) \to \pi_k(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial))$ is isomorphic to Gromoll's group Γ^{n+k+1}_{k+1}; see [Gro66]. Furthermore they obtained many strong nonvanishing results in [ABK72] about these groups of Gromoll. Their results perhaps combined with recent knowledge about a_k (see [Rog02] and [Rog03]) should yield extremely substantial quantitative improvements to Theorem 3 and hence also to Theorem 1. We are very grateful to the referee for pointing out to us this direction for future investigation.

We continue with the proof of the second case. Fix k with $k \not\equiv 3 \bmod 4$. Now, since we are assuming $n + k \equiv 2 \bmod 4$, we have that $n + k + 1 = 4m - 1$ for some m. Hence, using diagram (3) and the three facts above, we get that, by choosing n large, we can find a nonzero element $x \in \pi_k(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial)) = \Theta_{n+k+1} = \Theta_{4m-1}$ with large order that maps to $0 \in \pi_k(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial))$. Hence x is the image of an element $y \in \pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial))$ with large order. Now, since we are assuming $k \not\equiv 3 \bmod 4$, by fact (i) above we have that $\pi_k(P(\mathbb{D}^{n-1})) \cong \pi_k(\mathscr{P}(*))$ is a finite group of order a_k. Then $a_k y$ maps to $0 \in \pi_k(P(\mathbb{D}^{n-1}))$; hence $a_k y$ pulls back to an element z in $\pi_k(\Omega\mathscr{A}_{n-1})$. Since we can take the order of x and y as large as we want, we can choose x and y such that $a_k x \neq 0$. Hence $a_k y \neq 0$ and it follows that $z \neq 0$. This concludes the proof of Theorem $3'''$ in the second case. □

Third case: $k \equiv 3 \bmod 4$*,* $n \gg k$ *and* $n + k \equiv 2 \bmod 4$. The problem in this case is that now the group $\pi_k(P(\mathbb{D}^{n-1})) \cong \mathbb{Z} \oplus$(finite group); hence it is not finite, and we cannot use the argument above because the element y can map to an infinite order element.

To begin with, we embed diagram (3) in a larger diagram

$$\begin{tikzcd}
\pi_k(P(\mathbb{D}^{n-1})) \arrow[d, "\Sigma_\#"'] & & \\
\pi_k(P(\mathbb{D}^n)) \arrow[r, "\phi"] \arrow[dr, "\tau_\#"'] & \pi_{k+1}(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial)) \arrow[d, "\mu"'] & \\
\pi_k(\Omega\mathscr{A}_{n-1}) \arrow[r] & \pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial)) \arrow[r, "\eta_\#"] \arrow[d, "\nu"'] & \pi_k(P(\mathbb{D}^{n-1})) \\
& \pi_k(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial)) = \Theta_{n+k+1} \arrow[d] & \\
& \pi_k(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial)). &
\end{tikzcd} \tag{4}$$

We explain the new terms. The central column contains one more term than the central column of (3) and, as before, it is a piece of the exact sequence of the homotopy groups of the fibration

$$\mathrm{Diff}(\mathbb{D}^n, \partial) \longrightarrow \widetilde{\mathrm{Diff}}(D^n, \partial) \longrightarrow \widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial).$$

The map $\Sigma_\#$ is induced by the suspension map

$$\Sigma : P(\mathbb{D}^{n-1}) \to P(\mathbb{D}^{n-1} \times I) = P(\mathbb{D}^n).$$

The diagonal arrow is induced the map $\tau : P(D^n) \to \mathrm{Diff}(\mathbb{D}^n)$, which, as defined before, consists of taking the "top" of the pseudoisotopy. An element f in $\pi_{k+1}(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial)) = \pi_{k+1}(\widetilde{\mathrm{Diff}}(\mathbb{D}^n, \partial), \mathrm{Diff}(\mathbb{D}^n, \partial))$ is represented by a self-diffeomorphism of $\mathbb{D}^n \times \mathbb{D}^{k+1}$ which preserves the projection to $\mathbb{D}^{k+1}$ over $\partial\mathbb{D}^{k+1}$ and is the identity when restricted to $(\partial\mathbb{D}^n \times \mathbb{D}^{k+1}) \cup (\mathbb{D}^n \times \{0\} \times \mathbb{D}^k)$, where $\mathbb{D}^{k+1} = I \times \mathbb{D}^k$. Hence $\mu(f)$ is represented by the restriction of the above self-diffeomorphism to $\mathbb{D}^n \times \partial\mathbb{D}^{k+1}$. An element in $\pi_k(P(\mathbb{D}^n))$ is represented by a self-diffeomorphism of $(\mathbb{D}^n \times I) \times \mathbb{D}^k$ which is the identity over

$$\partial((\mathbb{D}^n \times I) \times \mathbb{D}^k) \setminus (\mathbb{D}^n \times \{1\} \times \mathbb{D}^k),$$

and preserves the projection to $\mathbb{D}^k$. Identifying $I \times \mathbb{D}^k$ with $\mathbb{D}^{k+1}$, we obtain a map $\pi_k(P(\mathbb{D}^n)) \to \pi_{k+1}(\widetilde{\mathrm{Diff}}/\,\mathrm{Diff}(\mathbb{D}^n, \partial))$. This map is ϕ. It is easy to verify that $\mu\phi = \tau_\#$. Hence diagram (4) is commutative with the central row and central column exact.

Recall that we are assuming that $k \equiv 3 \bmod 4$, and that $n + k \equiv 2 \bmod 4$. It follows that $n \equiv 3 \bmod 4$. In particular n is odd. Then, assuming also that $n \gg k$, we have the following facts:

(a) $\Sigma_\#$ is an isomorphism (see [Igu88]) and

$$\pi_k(P(\mathbb{D}^n)) \cong \pi_k(P(\mathbb{D}^{n-1})) \cong \pi_k(\mathcal{P}(*)) \cong \mathbb{Z} \oplus \text{(finite group)};$$

see [Dwy80], [FH78], [Wal78]. An element in this group can then be written in the form $j + t$, where $j \in \mathbb{Z}$ and t is in the torsion part of the group.

(b) $\eta_\#\tau_\#\Sigma_\#(x) = x \pm \bar{x}$ for $x \in \pi_k(P(\mathbb{D}^{n-1}))$. Here the map $x \mapsto \bar{x}$ is an involution on $\pi_k(P(\mathbb{D}^{n-1}))$; see [Hat78].

(c) $\pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial)) \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\tau_\# \otimes \mathbb{Q}$ is an isomorphism; see [FH78, Th. 2.1]. Also $\eta_\# \otimes \mathbb{Q}$ is an isomorphism. (For this use also [FH78, Th. 2.1] and combine it with the homotopy exact sequence induced by the fibration (2). One detail: note that $\pi_i(\mathcal{A}_{n-1}) \cong \pi_i(\mathrm{Diff}(\mathbb{D}^n, \partial))$ for $i > 0$, because $\mathrm{Diff}^0(\mathbb{D}^n, \partial) \subset \mathcal{A}_{n-1} \subset \mathrm{Diff}(\mathbb{D}^n, \partial)$, where $\mathrm{Diff}^0(\mathbb{D}^n, \partial)$ is the connected component of the identity in $\mathrm{Diff}(\mathbb{D}^n, \partial)$.)

Define $y_0 = \tau_\#\Sigma_\#(1) \in \pi_k(\mathrm{Diff}(\mathbb{D}^n, \partial))$. Then, by (a) and (c) above, y_0 has infinite order, and, since diagram (4) is commutative, $\nu(y_0) = 0$. We have $\eta_\#(y_0) = \eta_\#\tau_\#\Sigma_\#(1)$, and by (b), $\eta_\#(y_0) = 1 \pm \bar{1}$. But an involution sends 1 to

$\pm 1+t$, where t is a torsion element. Hence $\eta_\#(y_0)$ is either of the form $2+t$ or t, and, since y_0 has infinite order, (c) shows that $\eta_\#(y_0)=2+t$. Write $y_1=a_k y_0$. Then $\eta_\#(y_1)=2a_k$. Note that y_1 also has infinite order and $\nu(y_1)=0$. As in the second case (that is, for $k\not\equiv 3 \bmod 4$), we can find an element $y'\in\pi_k(\mathrm{Diff}(\mathbb{D}^n,\partial))$ with that $\nu(y')\neq 0$ and $\eta_\#(y')=2a_k j$ (here $y'=2a_k y$, where y is as in the proof of the second case). Now take $y''=y'-jy_1$ and we see that $\eta_\#(y'')=0$ and $\nu(y'')=\nu(y')\neq 0$, and we are done. This completes the proof of Theorem 3. $\square$

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