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The Quasi-Additivity Law in conformal geometry

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Abstract

On a Riemann surface S of finite type containing a family of N disjoint disks D_i ("islands"), we consider several natural conformal invariants measuring the distance from the islands to ∂S and the separation between different islands. In a near degenerate situation we establish a relation between them called the Quasi-Additivity Law. We then generalize it to a Quasi-Invariance Law providing us with a transformation rule of the moduli in question under covering maps. This rule (and in particular, its special case called the Covering Lemma) has important applications in holomorphic dynamics.

1. Introduction

Several central problems in holomorphic dynamics depend on the so-called *a priori* bounds, that is, uniform lower bounds on the conformal moduli of certain dynamically defined annuli. So far, the only analytic tools suitable to this end (for unreal maps) were the basic properties of the moduli of annuli (transformation rules and the Grötzcsh Inequality). In this paper we design a new analytic tool, the *Covering Lemma*, that provides us, in a near degenerate situation, with a much stronger version of the transformation rule for conformal moduli under covering maps. In the following papers, it is used to generalize the Yoccoz Theorem (on local connectivity of non-renormalizable Julia sets) to higher degree unicritical maps [KL1] and to prove *a priori* bounds (and hence MLC) for some classes of infinitely renormalizable quadratic maps [K], [KL2], [KL3]. Further applications of this method (to multicritical maps) are under way, see [KS], [QY], [RY].

We will derive the Covering Lemma from a "Quasi-Additivity Law" relating three natural conformal moduli for a Riemann surface with several Jordan disks marked. Let us formulate it precisely.

Let S stand for a compact Riemann surface with boundary. We denote the *extremal length* of a family $\mathcal{G}$ of curves by $\mathcal{L}(\mathcal{G})$, and we let $\mathcal{W}(\mathcal{G}) = \mathcal{L}(\mathcal{G})^{-1}$ be the corresponding *extremal width* (see the Appendix). Given a compact subset

$K \subset \operatorname{int} S$, we let $\mathcal{L}(S,K)$ and $\mathcal{W}(S,K)$ be respectively the extremal length and width of the family of curves in $S \smallsetminus K$ connecting ∂S to K.

An open subset $A \Subset \operatorname{int} S$ is called an (*open*) *archipelago* if its closure is a Riemann surface of finite type (not necessarily connected) with smooth boundary. Its connected components are called *islands*.

Let A_j $(j = 1, \dots, N)$ be a finite family of archipelagos in S with disjoint closures. We call the number

$$\operatorname{Top} = \operatorname{Top}_S\{A_j\} = -\chi(S) + \sum_j \# \operatorname{Comp} \partial A_j$$

the *topological complexity* of the family of archipelagos.

Let us introduce three conformal moduli of this family of archipelagos:

$$X = X_S\{A_j\} = \mathcal{W}(S, \bigcup_{j=1}^{N} A_j); \tag{1.1}$$

$$Y = Y_S\{A_j\} = \sum_{j=1}^{N} \mathcal{W}(S, A_j),$$

$$Z = Z_S\{A_j\} = \sum_{j=1}^{N} \mathcal{W}(S \smallsetminus \bigcup_{k \neq j} A_k,\ A_j).$$

The first modulus measures the (inverse) extremal distance from the union of the archipelagos to the boundary of S, the second one is the sum of the inverse extremal distances from the individual archipelagos to the boundary of S, while the last one measures the (inverse) separation between the archipelagos.

There are some obvious relations between these moduli: $X \le Y \le Z$ and $Y \le NX$. The goal of this paper is to establish one non-obvious relation in a near degenerate situation, (i.e., when Y is big), namely, to bound Y by the geometric mean of X and Z with an absolute constant. The number N of the archipelagos does not appear in the estimate: it only influences how degenerate the situation should be:

Quasi-Additivity Law. *There exists K depending only on the topological complexity of the family of archipelagos such that*:

$$Y \ge K \Rightarrow Y^2 \le 2XZ\,.[1]$$

The proof of this law will occupy most of the paper.

[1] In fact, our proof shows that "2" can be replaced with any constant $C > 4/3$. On the other hand, one can show that $C < 32/27$ would not work.

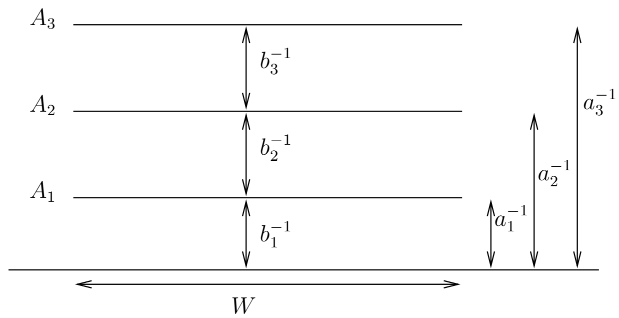

Figure 1.1: Example: every island is a horizonal line segment.

A simple example. Figure 1.1 presents a simple configuration of archipelagos (consisting of a single island each) for which the asymptotics for the X-, Y- and Z-moduli can be calculated explicitly, so that the QA Law can be verified directly. At the same time, this configuration is nearly optimal as the constant in the QA Law is concerned.

Let S be the closure of the upper half-plane in the Riemann sphere. Given a sequence $a_1 > a_2 > \cdots > a_n > 0$, let us consider archipelagos $A_i = [0, W] \times \{a_i^{-1}\}$, where W is large in terms of the a_i^{-1}. (Here our archipelagos are closed rather than open; see §2.10.1 for a discussion.) Then

$$X \sim W a_1, \quad Y \sim W \sum_{j=1}^{n} a_j,$$

and

$$Z \sim W \sum_{i=1}^{n} (b_i + b_{i+1}),$$

where $b_i^{-1} = a_{i-1}^{-1} - a_i^{-1}$ (and $b_1 = a_1$, $b_{n+1} = 0$). Then the QA Law in this case follows immediately from the arithmetic inequality

$$\left(\sum_{j=1}^{n} a_j \right)^2 \leq \frac{4}{3} b_1 \sum_{j=1}^{n} b_j,$$

which is proved in Section 2.8.

Given $\xi \geq 1$, we say that the archipelagos are *ξ-separated* if $Z \leq \xi Y$. The following immediate corollary shows that in a near degenerate situation, under the separation assumption, the moduli X and Y are comparable:

QA LAW WITH SEPARATION. *Assume that the archipelagos $A_j \Subset \operatorname{int} S$ are ξ-separated. Then there exists K depending only on ξ and the topological complexity of the family of archipelagos such that*:

$$Y \geq K \Rightarrow Y \leq 2\xi X.$$

In Section 2.10 we give several variations of the QA Law adapted to the needs of holomorphic dynamics.

We then generalize the QA Law to a Quasi-Invariance Law providing us with a transformation rule of the moduli in question under covering maps in a near degenerate situation. Keeping in mind further applications, we formulate in Section 3.1 a number of variations and special cases of this law. Let us formulate here one of them.

If we have a branched covering $f : U \to V$ of degree D between two disks that restricts to a branched covering $f : \Lambda \to B$ of degree d between smaller disks, then a simple general estimate shows that $\operatorname{mod}(V \setminus B) \leq D \operatorname{mod}(U \setminus \Lambda)$. It turns out that given d, in a near degenerate situation the above moduli are, in fact, comparable (under a "collar assumption"):

Covering lemma. *Fix some $\eta \in (0, 1]$. Let $U \supset \Lambda' \supset \Lambda$ and $V \supset B' \supset B$ be two nests of Jordan disks. Let $f : (U, \Lambda', \Lambda) \to (V, B', B)$ be a branched covering between the respective disks, and let $D = \deg(U \to V)$, $d = \deg(\Lambda' \to B')$. Under the following Collar Assumption:*

$$\operatorname{mod}(B' \setminus B) > \eta \operatorname{mod}(U \setminus \Lambda),$$

there exists an $\varepsilon > 0$ (depending on η and D) such that if

$$0 < \operatorname{mod}(U \setminus \Lambda) < \varepsilon$$

then

$$\operatorname{mod}(V \setminus B) < 2\eta^{-1} d^2 \operatorname{mod}(U \setminus \Lambda).$$

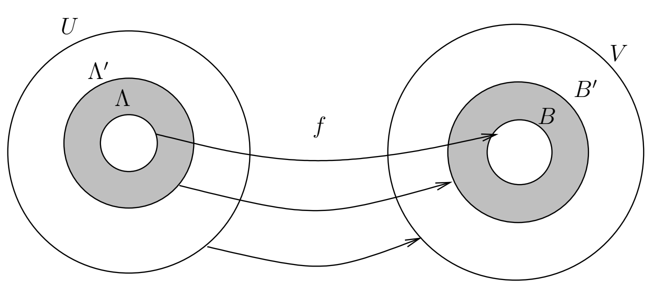

Figure 1.2: Covering between two nests of three disks

We derive the QI Law (and, in particular, this Covering Lemma) from the QA Law by passing to an appropriate Galois covering of U.

The needed background in the extremal length techniques is summarized in the Appendix.

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2. Quasi-Additivity Law

2.1. *Outline of the proof.* Let us assume for simplicity (in this outline only) that S and all the archipelagos A_j are disks. Then each $S \smallsetminus A_j$ is an annulus. Let us endow it with the vertical foliation $\mathcal{F}_j$ (the one that becomes genuinely vertical after uniformization of $S \smallsetminus A_j$ by a standard Euclidean cylinder). Then $Y = \sum \mathcal{W}(\mathcal{F}_j)$.

We begin with analyzing topology of these foliations relative to our family of archipelagos (§§2.2–2.3). Namely, we associate to each leaf γ of each $\mathcal{F}_j$ a combinatorial invariant called its *route*. This invariant records the archipelagos visited by γ (in order of first appearance) and some extra homotopy data about γ. These data are selected in the minimal way to ensure that if two disjoint paths are parallel (i.e., have the same route), then together with appropriate arcs of the boundary of $S \cup \bigcup A_j$ they bound a rectangle. Moreover, any vertical path in this rectangle has the same route. Thus, the vertical paths with a given route vertically foliate a rectangle.

Let us consider one such rectangle, P, and let $(A_1, \dots, A_l)$ be the list of the archipelagos visited by P. This rectangle comes together with a sequence of associated "big" and "little" rectangles,

$$P_k \subset S \smallsetminus \bigcup_{j=k}^{N} A_j, \quad Q_k \subset S \smallsetminus \bigcup_{j=1}^{N} A_j, \quad k = 1, \dots, l.$$

The big rectangles correspond to the pieces of its vertical boundary $\partial^v P$ until its first entry to the archipelago A_k, while the little ones correspond to the last piece of $\partial^v P$ in $S \cup \bigcup A_j$. The first of these rectangles, Q_1, is called "initial".

Cutting off from P two buffers of width four each, we obtain a truncated rectangle $\tilde{P}$ coming together with the associated truncated rectangles $\tilde{P}_j$ and $\tilde{Q}_j$.

At this point, we make use of a *Small Overlapping Principle* asserting that families of curves with large extremal width have a relatively small intersection (see §2.5). This implies that if two truncated little rectangles overlap (with matching vertical orientation) then the corresponding big rectangles have comparable routes (i.e., one route is an extension of the other), see §2.6.

This allows us to relate the moduli X and Z to the widths of the truncated small rectangles (§2.7). Namely, the total width of the truncated little rectangles is bounded by Z, while the total width of the truncated initial little rectangles is bounded by X. On the other hand, the total width of the truncated big rectangles is bounded from below by $(1-\delta)Y$, as long as $Y > 16s/\delta$, where s is the total number of these rectangles, which can be bounded in terms of the topological complexity.

Moreover, by the Series Law for the extremal length, the width of each truncated big rectangle is bounded by the harmonic sum of the widths of the associated little ones. By an "arithmetic inequality" of Section 2.8, this yields the desired quadratic relation between the moduli X, Y and Z.

2.2. *Paths and rectangles.* Let S be a Riemann surface with boundary. All the curves $\gamma : [0, 1] \to S$ below will be considered naturally oriented. A curve $\gamma : [0, 1] \to S$ is called *proper* if $\gamma\{0, 1\} \subset \partial S$. Two proper curves are called *properly homotopic* in S if they are homotopic through a family of proper curves. A proper curve is called *trivial* if it is properly homotopic to a curve $[0, 1] \to \partial S$. A *path* in S is a curve without self-intersections, i.e., an embedded (oriented) interval $[0, 1] \to S$.

In this paper, a *standard* (*Euclidean*) *rectangle* E will mean $I \times [0, h]$ where I is an interval of arbitrary type (closed, semi-closed, or open), and $h > 0$. Its *horizontal boundary* $I \times \{0, h\}$ comprises the *base* $I \times \{0\}$ and the *roof* $I \times \{h\}$. A *vertical path* in E is a path connecting its horizontal sides. Every vertical path is naturally oriented (from the base to the roof) which endows E with *vertical orientation.* The intervals $\{x\} \times [0, h]$ will be referred to as *genuine vertical paths* in E; together, they form the *genuine vertical foliation.*

A (*topological*) *rectangle* P on a surface S will mean an embedded Euclidean rectangle, coming together with all the previously described affiliated structure: the horizontal boundary $\partial^h P$ comprising the base and the roof, and the vertical orientation. In what follows we will often deal with *properly embedded* rectangles, i.e., such that $\partial^h P \subset \partial S$. Any topological rectangle can be conformally uniformized by a standard rectangle, supplying the former with the genuine vertical foliation.

Similarly, a standard cylinder will mean $C = \mathbb{T} \times [0, h]$, where $\mathbb{T}$ is a round circle, coming together with the base and the roof, and the vertical orientation (and the genuine vertical foliation, too). A (*topological*) *annulus* R on S is an embedded cylinder supplied with all the affiliated structure.

If we cut the annulus along two disjoint vertical paths, we obtain two rectangles. This situation is special, as only one rectangle would be cut off from any other Riemann surface:

LEMMA 2.1. *Assume S is connected and not an annulus. Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two disjoint properly homotopic non-trivial paths in S such that* $\operatorname{int} \mathcal{C}^i \subset \operatorname{int} S$.

(i) *Then there exist two unique arcs α and ω on the boundary ∂S which together with the paths $\mathcal{C}^i$ bound a closed rectangle P with base α and roof ω.*

(ii) *Let $(\mathcal{C}^t)$, $1 \le t \le 2$, be a proper homotopy between the above paths, and let $(e^t) \subset \partial S$ be the corresponding motion of the endpoint e^t of $\mathcal{C}^t$. Then* *the curve* $(e^t)_{1\le t\le 2}$ *is homotopic in* ∂S *rel its endpoints to the arc* ω *oriented from* e^1 *to* e^2.

(iii) *Let* $\mathcal{C}^3$ *be a third path which is disjoint and properly homotopic to the above two. Let* P_j, $j = 1, 2, 3$, *be the rectangles bounded by the pairs of these three paths. Then one of these rectangles is tiled by the other two.*

Proof. (i) Let us consider the universal covering $\pi : \hat{S} \to S$ of S. It is conformally equivalent to $\bar{\mathbb{D}} \smallsetminus K$, where $\bar{\mathbb{D}}$ is the closed unit disk and $K \subset \mathbb{T}$ is a nowhere dense compact subset of the unit circle (the limit set of the Fuchsian group of deck transformations). Since the paths $\mathcal{C}^i$ are properly homotopic, they lift to (disjoint) properly homotopic paths $\hat{\mathcal{C}}^i$ in $\hat{S}$. Let these lifts begin at points $b^i \in \mathbb{T}$ and end at points $e^i \in \mathbb{T}$. Then b^1 and b^2 (resp., e^1 and e^2) bound an arc $\hat{\alpha} \subset \partial\hat{S}$ (resp. $\hat{\omega} \subset \partial\hat{S}$). These two arcs are disjoint since the paths $\mathcal{C}^i$ are non-trivial. They are also disjoint from the $\operatorname{int}\mathcal{C}^i \subset \operatorname{int}\hat{S}$. Hence the four paths, $\mathcal{C}^1$, $\mathcal{C}^2$, $\hat{\alpha}$ and $\hat{\omega}$, bound a closed rectangle $\hat{P}$ in $\hat{S}$.

Let us consider all the lifts $\hat{\mathcal{C}}^i_j$ of $\mathcal{C}^i$ that cross $\hat{P}$, where $\hat{\mathcal{C}}^i_0 \equiv \hat{\mathcal{C}}^i$. For each $i = 1, 2$, the lifts $\hat{C}^i_j$ are pairwise disjoint since the paths $\mathcal{C}^i$ do not have self-intersections. Any two paths $\hat{\mathcal{C}}^1_j$ and $\hat{\mathcal{C}}^2_k$ are disjoint as well since $\mathcal{C}^1$ and $\mathcal{C}^2$ do not cross each other. Hence each $\hat{\mathcal{C}}^i_j$ is completely contained in P and moreover, $\partial\hat{\mathcal{C}}^i_j \subset \hat{\alpha} \cup \hat{\omega}$. But $\partial\hat{\mathcal{C}}^i_j$ cannot belong to one horizontal side, α or ω, since the paths $\mathcal{C}^i$ are non-trivial. Thus, we obtain a family of disjoint vertical paths $\hat{\mathcal{C}}^i_j$ in $\hat{P}$.

If one of the above curves, say $\mathcal{C}^1$, has more than one lift in $\hat{P}$, then we consider the lift $\hat{\mathcal{C}}^1_1$ such that there are no other lifts in between $\hat{\mathcal{C}}^1_0$ and $\hat{\mathcal{C}}^1_1$. Then $\hat{\mathcal{C}}^1_0$ and $\hat{\mathcal{C}}^1_1$, together with two subarcs of $\hat{\alpha}$, and $\hat{\omega}$ bound a rectangle $\hat{\Pi}$. The projection of this rectangle to S is a clopen annulus R in S. Since S is connected, $S = R$ contradicting our assumption.

Thus, each curve $\mathcal{C}^i$ has only one lift to $\hat{P}$, so that $\hat{P} \cap \pi^{-1}(\mathcal{C}^i) = \hat{\mathcal{C}}^i$. It follows that the paths $\mathcal{C}^i$ lie on the boundary of $P \equiv \pi(\hat{P})$. Hence $\pi(\partial\hat{P}) \subset \partial P$, and the map $\pi : \hat{P} \to P$ is proper. Moreover, it is injective over $\mathcal{C}^i$ and hence has degree 1. Thus, the map $\pi : \hat{P} \to P$ is a homeomorphism.

If there were two rectangles P^1 and P^2 as above then they would be glued along the paths $\mathcal{C}^i$ to form an annulus.

(ii) The homotopy $(\mathcal{C}^t)$ lifts to a proper homotopy $\hat{\mathcal{C}}^t$ on $\hat{S}$ between the lifts $\hat{\mathcal{C}}^i$ considered in (i). The endpoint $\hat{e}^t$ of this lift moves along a component $\hat{\xi}$ of $\partial\hat{S}$. Since $\hat{\xi}$ is an interval, the curve $(\hat{e}^t)$ is homotopic to the arc $\hat{\omega}$ on $\hat{\xi}$ rel the endpoints. Hence (e^t) is homotopic to ω on ∂S rel the endpoints.

(iii) The paths $\mathcal{C}^i$ lift to proper paths $\hat{\mathcal{C}}^i$ in $\hat{S}$ that begin and end on the same component of $\partial\hat{S}$. Then one of the lifted rectangles $\hat{P}_j$ is tiled by the other two. Since $\pi : \hat{P}_j \to P_j$ is a homeomorphism, the same is true for the P_j's. □

Somewhat loosely, we will say that the above rectangle P is *bounded* by the curves $\mathcal{C}^1$ and $\mathcal{C}^2$.

To avoid the ambiguity in the choice of the rectangle P, *in what follows we assume that the Riemann surface S under consideration is not an annulus.* A simple trick shows that this assumption does not reduce generality (see §2.4).

Let us consider an archipelago A in S. Given a proper path $\mathcal{C}$ in S that crosses $\bar{A}$, let a be the last point of intersection of $\mathcal{C}$ with $\bar{A}$, and let $\delta \subset S \smallsetminus A$ be the *terminal* closed segment of $\mathcal{C}$ which connects a to ∂S. Note that $\operatorname{int} \delta \subset \operatorname{int}(S \smallsetminus A)$. If we have several paths $\mathcal{C}^i$ as above, we naturally label the corresponding objects as a^i and δ^i, etc.

Two disjoint proper paths $\mathcal{C}^1$ and $\mathcal{C}^2$ in S that cross $\bar{A}$ are called *roof parallel* (rel A) if:

- $\mathcal{C}^1$ and $\mathcal{C}^2$ are properly homotopic in S, and hence they bound a "big rectangle" P;
- The paths δ^i are properly homotopic in $S \smallsetminus A$, and hence they bound a "terminal little rectangle" $Q \subset S \smallsetminus A$;
- The rectangles P and Q share the roof (Figure 2.1 illustrates that this is not automatic).

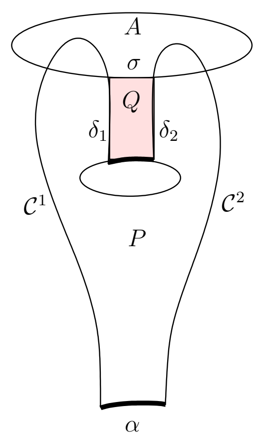

Figure 2.1: Strange configuration of rectangles

Two paths are called *base parallel* (rel A) if after reversing orientation they become roof parallel. Initial segments of these paths bound an initial little rectangle $Q_1 \subset S \smallsetminus A$ which shares the base with P. Two paths are called *parallel* if they are roof and base parallel.

We will now formulate several statements about roof parallel paths. The corresponding statements about base parallel paths are obtained by reversing orientation, and the corresponding statements about parallel paths immediately follow.

LEMMA 2.2. *Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two roof parallel (rel A) proper paths in S, and P and Q be the corresponding big and little rectangles. Let $\mathcal{C}$ be a positively oriented vertical path in P which is disjoint from the $\mathcal{C}^i$. Then it is roof parallel (rel A) to each $\mathcal{C}^i$. Moreover, its terminal segment δ is a vertical path in Q.*

Proof. Any vertical path in P is properly homotopic to the sides $\mathcal{C}^i$. Let P^i be the big rectangles bounded by the paths $\mathcal{C}$ and $\mathcal{C}^i$, and let ω^i be their roofs, $i = 1, 2$. Of course, they tile the roof ω, overlapping at the endpoint e of $\mathcal{C}$.

Let $\mathcal{C}'$ be the path $\mathcal{C}$ with reverse orientation. Since P and Q share the roof, some initial segment of $\mathcal{C}'$ is contained in Q. Since $\mathcal{C}'$ is proper, it must exit Q. Since $\operatorname{int}\mathcal{C}'$ is disjoint from the vertical sides and the roof of Q, it can exit Q only through its base, σ. Let a be the first point of intersection between $\mathcal{C}'$ and σ. Then the terminal segment δ of $\mathcal{C}$ that begins at a is a positively oriented vertical path in Q. Hence it is properly homotopic in $S \smallsetminus A$ to the paths δ^i.

Let $Q^i \subset S \smallsetminus \bar{A}$ be the little rectangles bounded by the paths δ and δ^i, $i = 1, 2$. Since δ is a vertical path in Q ending at e, the arcs ω_i are the roofs of the little rectangles Q^i. Thus, the Q_i respectively share the roofs with the P_i. □

The following lemma will be used for counting the number of parallel classes (see Lemmas 2.7 and 2.8):

LEMMA 2.3. *Let $\mathcal{C}^i$ be three disjoint properly homotopic paths in S crossing the archipelago $\bar{A}$ in such a way that their terminal segments δ^i are properly homotopic in $S \smallsetminus A$. Then at least two of these paths are roof parallel rel $\bar{A}$.*

Proof. For $i = 1, 2, 3$, let P_i be the big rectangle bounded by the paths $\mathcal{C}^k$ and $\mathcal{C}^l$ with $\{i, k, l\} = \{1, 2, 3\}$, and let Q_i be the corresponding little rectangles. Let ω_i be the roofs of the P_i, and let λ_i be the roofs of the Q_i. We need to show that one of the roofs ω_i coincides with the corresponding λ_i.

Since by Lemma 2.1 (iii) one of the big rectangles, say P_1, is tiled by the other two, the roof ω_1 is tiled by ω_2 and ω_3. Denote the complements of the roofs ω_i by ω_i'. If $\omega_i \neq \lambda_i$ for $i = 2, 3$, then $\lambda_2 = \omega_2' = \omega_1' \cup \omega_3$ and similarly $\lambda_3 = \omega_1' \cup \omega_2$. Hence $\lambda_2 \cup \lambda_3 = \omega_1' \cup \omega_2 \cup \omega_3 = \eta$, where η is the whole component of ∂S containing the endpoints of the paths $\mathcal{C}^i$. But it is impossible since one of the roofs λ_i is tiled by the other two (as one of the little rectangles Q_i is tiled by the other two). □

Let us now enlarge the notion of parallel to an equivalence relation on the class $\mathcal{A}$ of all proper curves $\mathcal{C}$ in S crossing the archipelago $\bar{A}$. We say that two curves $\mathcal{C}^1$ and $\mathcal{C}^2$ of class $\mathcal{A}$ are *roof equivalent* if

- They are properly homotopic in $\mathcal{C}$;

- The terminal segments δ^1 and δ^2 are properly homotopic in $S \smallsetminus A$;
- The motions of the endpoints, (e^t) and (q^t), of the above homotopies are homotopic (rel endpoints) curves on ∂S.

The definitions of *base equivalent* and *equivalent* paths are straightforward. Again, we restrict ourselves to a statement concerning roof equivalence only:

LEMMA 2.4. *Two disjoint curves $\mathcal{C}^1$ and $\mathcal{C}^2$ of class $\mathcal{A}$ are roof parallel if and only if they are roof equivalent.*

Proof. If $\mathcal{C}^1$ and $\mathcal{C}^2$ are roof parallel then they are homotopic within the big rectangle P in such a way that the endpoint e^t parametrizes the roof ω. Similarly, the curves δ^1 and δ^2 are homotopic in Q in such a way that q^t parametrizes the same roof ω. So, the motions of the endpoints are homotopic.

Vice versa, by Lemma 2.1 (ii), the homotopy class of the endpoint motion determines the roof of the rectangle. □

In what follows, (*roof*/*base*) *equivalent curves* (*not necessarily disjoint*) *will also be called* (*roof*/*base*) *parallel.* Also, "parallel in S (rel $\emptyset$)" just means "properly homotopic" in S.

COROLLARY 2.5. *Let $\mathcal{F}$ be a family of disjoint properly homotopic paths of class $\mathcal{A}$ such that their terminal and initial segments are* (*respectively*) *properly homotopic in $S \smallsetminus \bar{A}$. Then $\mathcal{F}$ comprises at most four parallel classes.*

We close with two combinatorial lemmas.

LEMMA 2.6. *Suppose that S is a Riemann surface of finite topological type such that each connected component of S has negative Euler characteristic. Then there can be at most $-3\chi(S)$ disjoint non-parallel (rel $\emptyset$) proper paths in S.*

Proof. Removing the boundary from S, we obtain a Riemann surface homeomorphic to a compact Riemann surface $\mathbf{S}$ with finitely many punctures v_k, $k = 1, \dots, n$, where say, the first l of them correspond to the removed boundary. Proper paths in S correspond to paths in $\mathbf{S} \smallsetminus \{v_k\}$ connecting two of the first l punctures. Of course, if we allow ourselves to connect other vertices as well, we obtain only more paths. So, we can assume in the first place that $S = \mathbf{S} \smallsetminus \{v_k\}_{k=1}^n$ and $l = n$ (and of course, we can assume that $n \geq 1$). Since the Euler characteristic is additive, we can also assume that S is connected.

Let us call the punctures "vertices," and non-trivial paths in $\mathbf{S} \smallsetminus \{v_k\}$, connecting them, "edges". It is well-known that any finite family $\mathcal{F}$ of disjoint non-parallel edges can be completed to a triangulation of the surface $\mathbf{S}$ with the same vertices v_k (provided $\chi(S) < 0$). (To see this, let us first complete

$\mathcal{F}$ to a connected graph containing all the vertices v_k. We then consider any "face" D of it, i.e., a component of the complement of the edges. If D has positive genus, we can add to $\mathcal{F}$ a closed non-dividing edge connecting some vertex to itself. Cutting along this edge, we reduce the genus of D. Proceeding in this way, we will eventually obtain a graph whose faces are polygons. None of these faces can be a bigon, since the edges are not parallel. It cannot be a one-gon either since $\chi(S) < 0$. Thus, all the polygons are at least m-gons with $m \geq 3$, and we can further triangulate them.)

Let us apply the Euler formula to this triangulation:

$$F - E + V = \chi(\mathbf{S}),$$

where E is the number of proper paths, $V = n$, and $3F = 2E$. Therefore $-E/3 = \chi(\mathbf{S}) - n = \chi(S)$, and we are done. □

LEMMA 2.7. *Suppose that A is an archipelago on S, and let $\mathcal{F}$ be a set of disjoint proper paths on S. Then there are at most*

$$-108\,\chi(S)\,\chi(S \smallsetminus A)^2$$

distinct parallel classes (*rel A*) *in $\mathcal{F}$.*

Proof. There are at most $-3\chi(S)$ distinct homotopy classes of curves γ in $\mathcal{F}$, and at most $-3\chi(S \smallsetminus A)$ distinct homotopy class for the initial and final segments of γ. By Corollary 2.5, there are at most four distinct parallel classes, given the homotopy classes for γ and its initial and terminal segments. □

2.3. *Routes and associated rectangles.* Let us now consider a finite family $\mathcal{A}$ of archipelagos A_j ($j = 1, \dots, N$) in S with disjoint closures. We consider a path $\mathcal{C}$ in S that begins at $b \subset \partial S$ and ends at a point e on some archipelago $\bar{A}$. Such a path is called *good* if $\operatorname{int} \mathcal{C}$ does not intersect $\partial S \cup \bar{A}$.

Given a good path $\mathcal{C}$ in S, we relabel (if needed) our archipelagos so that $(A_1, \dots, A_l \equiv A)$ is the sequence of distinct archipelagos whose closures are crossed by $\mathcal{C}$ ordered according to their first appearance, while $A_{l+1}, \dots, A_N$ are the archipelagos that are not crossed by $\mathcal{C}$ ordered in an arbitrary way. Thus, for any $1 \leq i < j \leq l$, the path $\mathcal{C}$ enters A_i for the first time before it enters A_j. Note that though $\mathcal{C}$ can enter each archipelagos A_i ($1 \leq i \leq l$) many times, it is recorded only once.

Let e_j be the first point of intersection of $\mathcal{C}$ with $\bar{A}_j$, and let $\mathcal{C}_j$ be the segment of $\mathcal{C}$ bounded by $b \equiv e_0$ and e_j. In this way we obtain the *associated sequence*

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \dots \mathcal{C}_l \equiv \mathcal{C}$$

of good paths in S. We let $|\mathcal{C}| = l$ and call it the *height* of $\mathcal{C}$.

Let

$$\Lambda_j = \bigcup_{i=j}^{N} A_i, \quad \Omega_j = \bigcup_{i=1}^{j-1} A_i.$$

(Note that $\Omega_1 = \emptyset$. Also, we let $\Lambda \equiv \Lambda_1$ be the union of all archipelagos.) Then $\mathcal{C}_j$ is a proper path in $S \smallsetminus \Lambda_j$, and Ω_j is an archipelago in $S \smallsetminus \Lambda_j$. Let $\boldsymbol{\alpha}_j$ be the class of proper paths in $S \smallsetminus \Lambda_j$ parallel to $\mathcal{C}_j$ rel Ω_j. We say that these paths and classes are *associated* to $\mathcal{C}$. The sequence of the associated parallel classes,

$$\mathcal{R}(\mathcal{C}) = (\boldsymbol{\alpha}_j)_{j=1}^{l},$$

is called the *route* of $\mathcal{C}$. Note that the route determines the base component of ∂S where $\mathcal{C}$ begins, and the components of ∂A_j where the curves $\mathcal{C}_j$ end. Two good paths are called *parallel* rel the family $\mathcal{A}$ of archipelagos if they have the same route. Note that parallel paths can cross some particular archipelagos A different number of times (see Figure 2.2).

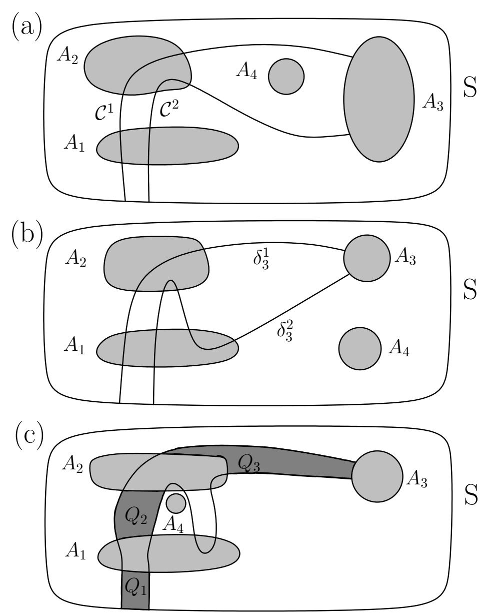

Figure 2.2: This picture illustrates the notion of parallelism. Here the family $\mathcal{A}$ comprises four archipelagos A_i each consisting of a single island. The routes of the paths $\mathcal{C}^1$ and $\mathcal{C}^2$ have height $l = 3$, and $\Lambda_3 = A_3 \cup A_4$, $\Omega_3 = A_1 \cup A_2$. The paths on figure (a) are not parallel since they are not properly homotopic in $S \smallsetminus \Lambda_3$. The paths on figure (b) are not parallel since their terminal arcs, δ_3^1 and δ_3^2, are not properly homotopic in $S \smallsetminus \Lambda$. On the other hand, the paths on (c) are parallel, notwithstanding $\mathcal{C}^2$ visits the island A_1 twice, while $\mathcal{C}^1$ visits it only once. In all three cases, the initial segments of the paths (of height two), $\mathcal{C}_2^1$ and $\mathcal{C}_2^2$, are obviously parallel.

We will now derive a bound on the number of routes:

LEMMA 2.8. *Let $A_1, \ldots, A_N$ be distinct archipelagoes in S, and let $\mathcal{T}$ be a set of disjoint good paths in S. Then among the elements of $\mathcal{T}$ there are at most $s(Top, N) = N!\,(108\mathrm{Top}^3)^{N+1}$ distinct routes rel $\{A_j\}$ (where $\mathrm{Top} = \mathrm{Top}_S\{A_j\}$ is the topological complexity defined in the Introduction).*

Proof. Let us bound the number of routes $\mathcal{R}(\mathcal{C})$ (for $\mathcal{C} \in \mathcal{T}$) for which $A_1, \ldots, A_k$ are visited in sequence (so that the terminal point of $\mathcal{C}_j$ lies in $\bar{A}_j$). By the previous lemma, there are at most

$$-108\,\chi(S \smallsetminus \Lambda_j)\,\chi(S \smallsetminus \Lambda)^2 \le 108\mathrm{Top}^3$$

distinct parallel classes for C_j, so there are at most $(108\,\mathrm{Top}^3)^k$ distinct routes which visit $A_1, \ldots, A_k$ in sequence. There are $\frac{N!}{(N-k)!}$ injective functions $\sigma : \{1, \ldots, k\} \to \{1, \ldots, N\}$, so there are at most

$$\frac{N!}{(N-k)!}(108\mathrm{Top}^3)^k$$

distinct routes of length k. The total number of these routes is bounded as desired. □

Let us consider two disjoint parallel good paths $\mathcal{C}^1$ and $\mathcal{C}^2$ with route of height l. By Lemma 2.1, these two paths, together with a base α and a roof ω, bound a *good big rectangle* P. Moreover, for each $j = 1, \ldots, l$, the associated good paths $\mathcal{C}^1_j$ and $\mathcal{C}^2_j$, together with a base α_j and a roof ω_j, bound an *associated good big rectangle* $P_j \subset S \smallsetminus \Lambda_j$, where $P_l \equiv P$. In fact, the P_j share the same base, i.e. $\alpha = \alpha_j$, since they share the base with the same *associated initial little rectangle* $Q_1 \equiv P_1$. Furthermore, each rectangle P_j shares the roof with *associated* (*terminal*) *little rectangle* Q_j, $j = 2, \ldots, l$, bounded by the terminal paths δ^1_j and δ^2_j, a base σ_j, and the roof ω_j. Note that the little rectangles Q_j are not necessarily contained in the big rectangle P (see Figure 2.3). All the above rectangles are vertically orientated.

We say that a path $\mathcal{C}$ (*positively*) *vertically overflows* a little rectangle Q_j if $\mathcal{C}$ contains a segment δ which is a (positively oriented) vertical path in Q_j.

The notion of "parallel curves" was designed to ensure the following property:

LEMMA 2.9. *Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two disjoint parallel* (*rel $\mathcal{A}$*) *good paths of height l, and let $P \equiv P_l$ be the corresponding good big rectangle. Let $\mathcal{C}$ be a positively oriented vertical path in P. Then it is parallel to $\mathcal{C}^1$ and $\mathcal{C}^2$* (*rel $\mathcal{A}$*) *and, in particular, it has height l. Moreover, $\mathcal{C}$ positively vertically overflows all associated little rectangles Q_j, $j = 1, \ldots, l$.*

Proof. Let us begin with the last assertion. For $j = l$ and $j = 1$ it immediately follows from Lemma 2.2 (by reversing orientation for $j = 1$). Let

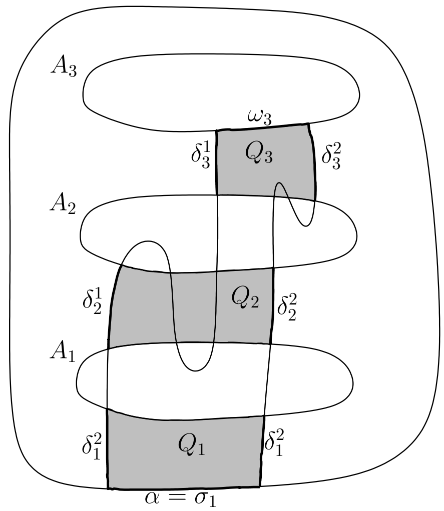

Figure 2.3: This picture illustrates that the little rectangles Q_i (shaded) are not necessarily contained in the big ones.

$1 < j < l$. Since P_j has the same base $\alpha \subset \partial S$ as P, a little initial segment of $\mathcal{C}$ is contained in P_j. On the other hand, the endpoint of $\mathcal{C}$ belongs to the archipelago $\bar{A}_l$ which is disjoint from P_j since

$$P_j \subset (S \setminus \bar{\Lambda}_j) \cup \partial A_j \subset S \setminus \bar{\Lambda}_l.$$

Hence the curve $\mathcal{C}$ must exit the rectangle P_j. But since $\mathcal{C}$ is a vertical curve in P, it can exit P_j only through the roof ω_j. Let e_j be the first intersection point of $\mathcal{C}$ with this roof. Then the initial segment $\mathcal{C}_j$ of $\mathcal{C}$ with endpoint e_j is a vertical path of P_j. By Lemma 2.2, it positively vertically overflows the little rectangle Q_j. All the more, $\mathcal{C}$ does also.

Since each P_j is a good big rectangle as well, we can apply to it the previous result and conclude that for any $i \leq j$, $\mathcal{C}_j$ vertically overflows Q_i. In particular it crosses the roof $\omega_i \subset \partial A_i$, and hence $\mathcal{C}_i \subset \mathcal{C}_j$.

Let us show that $\mathcal{C}_1 \subset \cdots \subset \mathcal{C}_l$ is the associated sequence of good paths. Since all the paths $\mathcal{C}_j$ are good initial segments of $\mathcal{C}$, it is part of the associated sequence. Moreover, $\mathcal{C}$ does not contain any other good initial segment since all other archipelagos A_k, $k = l+1, \dots, N$, are disjoint from P.

In particular, $\mathcal{C}$ has the same height l as $\mathcal{C}^1$. Moreover, by Lemma 2.2, the paths $\mathcal{C}_j$ are parallel to $\mathcal{C}_j^1$ and $\mathcal{C}_j^2$ rel Ω_j. Hence $\mathcal{C}$ is parallel to $\mathcal{C}^1$ and $\mathcal{C}^2$ rel $\mathcal{A}$. □

The previous lemma can be sharpened as follows:

LEMMA 2.10. *Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two disjoint parallel (rel $\mathcal{A}$) good paths of height l, and let $P \equiv P_l$ be the corresponding good big rectangle with base α.*

Let $\mathcal{C}$ be a good path disjoint from $\mathcal{C}^1$ and $\mathcal{C}^2$ which begins on α. Then either the route $\mathcal{R}(\mathcal{C})$ extends $\mathcal{R}(\mathcal{C}^1) = \mathcal{R}(\mathcal{C}^2)$, or the other way around.

Proof. Assume $\mathcal{C}$ is not contained in the rectangle P. Then it must exit P through the roof ω. Let e be the first point of intersection of $\mathcal{C}$ with ω. Then the initial segment $\mathcal{C}^*$ of $\mathcal{C}$ ending at e is a vertical path in P. By Lemma 2.9, $\mathcal{R}(\mathcal{C}^*) = \mathcal{R}(\mathcal{C}^1)$, so that $\mathcal{R}(\mathcal{C})$ extends $\mathcal{R}(\mathcal{C}^1)$.

Assume now that $\mathcal{C} \subset P$. Let us consider the biggest $j \leq l$ such that $\mathcal{C}$ intersects the roof ω_j of the good big rectangle P_j, and let $e_j \in \mathcal{C} \cap \omega_j$ be the first intersection point. Then the initial segment $\mathcal{C}_j$ of $\mathcal{C}$ with endpoint e_j is a vertical path in P_j. By Lemma 2.9, it has the same route as $\mathcal{C}_j^1$. In particular, it crosses all the archipelagos A_i, $i = 1, \ldots, j$.

But in fact, $\mathcal{C} = \mathcal{C}_j$, for otherwise $\mathcal{C}$ (being good) would end at some archipelago A_i with $i > j$. For $i > l$ this is impossible since those archipelagos are disjoint from P. For $i \in [j+1, l]$ this is impossible for otherwise $\mathcal{C}$ would exit the rectangle $\operatorname{int} P_i$ and hence would cross the roof ω_i.

We conclude that $\mathcal{R}(\mathcal{C}^1)$ is an extension of $\mathcal{R}(\mathcal{C}_j) = \mathcal{R}(\mathcal{C})$. □

Let us now consider two disjoint vertical curves Γ^1 and Γ^2 in a good rectangle P. Together with appropriate base and roof arcs, they bound a truncated good rectangle $\tilde{P} \subset P$.

Lemma 2.11. *For the associated sequence of little rectangles, $\tilde{Q}_j \subset Q_j$.*

Proof. By Lemma 2.9, Γ^1 and Γ^2 have the same route as P. We consider the associated sequences of good curves Γ_j^1 and Γ_j^2, $j = 1, \ldots, l$ and let $\tilde{\delta}_j^1$ and $\tilde{\delta}_j^2$ be the terminal paths in $S \smallsetminus \bigcup A_j$ of these curves. By definition, $\tilde{Q}_j$ is the rectangle bounded by these two paths, together with two appropriate horizontal arcs. By Lemma 2.2, the $\tilde{\delta}_j^i$ are vertical paths in the little rectangle Q_j. Hence $\tilde{Q}_j \subset Q_j$. □

Finally, we have the following important disjointness property:

Proposition 2.12. *Let P and P' be two good rectangles with disjoint vertical boundaries. Assume that some associated little rectangles, Q_j and Q'_k, have a non-trivial overlap. Then they represent the same proper homotopy class in $S \smallsetminus \Lambda$ (up to orientation). If their orientations match, then one of the routes, $\mathcal{R}(P)$ or $\mathcal{R}(P')$, is an extension of the other, and $j = k$.*

Proof. Since the overlapping little rectangles Q_j and Q'_k have disjoint vertical boundaries, one of the vertical boundary components, say $\delta'_k \subset \partial Q'_k$, must be a vertical path in the other rectangle, Q_j, which implies the first assertion.

Assume the vertical orientation of Q_j and Q'_k match. Let $\mathcal{C}'$ be the vertical boundary component of P' containing the path δ'_k, and let $\mathcal{C}'_k$ be the associated good curve ending with the path δ'_k.

Let us consider the (associated with P) good big rectangle P_j (with the little rectangle Q_j just under its roof ω_j). Since the path δ'_k is positively oriented in Q_j, it ends on the roof ω_j. Thus, the whole curve $\mathcal{C}'_k$ also ends on ω_j. But since $\mathcal{C}'_k$ is good, its interior does not cross ω_j. Neither can it cross the vertical boundary of P_j (by the assumption). Hence $\mathcal{C}'_k$ is trapped in P_j, and must begin on the base α_j of P_j.

Thus, $\mathcal{C}'_k$ is a vertical curve in P_j. By Lemma 2.9, $\mathcal{C}'_k$ and P_j have the same height, so that $k = j$. By Lemma 2.10, the route $\mathcal{R}(\mathcal{C}') = \mathcal{R}(P')$ is either an extension of $\mathcal{R}(P)$, or the other way around. □

2.4. *Harmonic foliations.* Let now $\mathbf{S}$ be a compact Riemann surface with boundary, and let S be obtained from $\mathbf{S}$ by making finitely many punctures $p_k \in \operatorname{int} \mathbf{S}$. We let $\partial S = \partial \mathbf{S}$.

By making a few artificial punctures (depending only on the topological complexity of the family of archipelagos), we can ensure that *no component of $S \smallsetminus A_j$ is an annulus* (see our convention after Lemma 2.1 and Figure 2.4). Note that making extra punctures does not change extremal lengths of the path families in question.

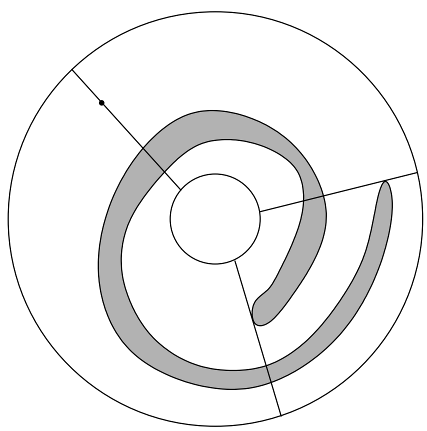

Figure 2.4: Long Island. On this picture, S is an annulus with one island on it. Without an artificial puncture, all the leaves of the harmonic foliation would be in the same parallel class. With the puncture, the leaves are decomposed into three parallel classes that form three rectangles.

Let us consider the harmonic measure $\omega_j(z) = \omega_{S \smallsetminus A_j}(\partial A_j, z)$ of ∂A_j in the Riemann surface $\mathbf{S} \smallsetminus A_j$ (see [A]). It is the unique harmonic function on $\operatorname{int}(\mathbf{S} \smallsetminus A_j)$ equal to 1 on ∂A_j and vanishing on $\partial \mathbf{S}$. For instance, if $\mathbf{S}$ and A_j

are disks, then ω_j is the height function on the annulus $\mathbf{S} \setminus A_j$ uniformized by the flat cylinder C_j with height 1 in such a way that $\partial\mathbf{S}$ is the base of it.

The *harmonic foliation* $\mathcal{F}_j$ on $\mathbf{S}$ is the phase portrait of the gradient flow γ_j^t of ω_j. It has finitely many saddle type singularities (with finitely many incoming and outgoing separatricies), where the punctures are considered to be singularities as well. It is oriented according to the direction of the gradient flow. Each non-singular leaf of $\mathcal{F}_j$ begins on $\partial\mathbf{S}$ and ends on ∂A_j. In the case when $\mathbf{S}$ is a topological annulus, $\mathcal{F}_j$ is the genuinely vertical foliation on the uniformizing cylinder C_j.

Let us remove from $S \setminus A_j$ all separatricies O^k of the foliation $\mathcal{F}_j$ and take the components of $S \setminus (A_j \cup \bigcup O^k)$. We obtain finitely many rectangles $\Pi = \Pi_j^m$ foliated by the harmonic leaves. Indeed, take some component λ of $\partial S \setminus \bigcup O^k$. The gradient flow brings every point $z \in \lambda$ in time 1 to some archipelago A_j, and these trajectories fill in some component Π of $S \setminus A_j \setminus \bigcup O^k$. The map

$$(z,t) \to (z, \gamma_j^t(z)), \quad z \in \lambda,\ t \in [0,1],$$

provides us with the rectangular structure on Π. (Since every annular component of $\mathbf{S} \setminus A_j$ contains a puncture, there are no annuli among the Π_i's.)

The conjugate harmonic function ω_j^* induces the natural transverse measure on the Π_j^m. In fact, the map $\omega_j + i\omega_j^*$ provides us with the uniformization of Π_j^m by a standard rectangle of height 1.

Every rectangle Π_j^m represents some non-trivial proper homotopy class of paths in $S \setminus A_j$. Moreover, different rectangles represent different classes. Indeed, if two leaves, γ and γ', of $\mathcal{F}_j$ are properly homotopic in $S \setminus A_j$, then by Lemma 2.1 they bound a rectangle Q in $S \setminus A_j$. The conjugate harmonic functions ω_j and ω_j^* are well defined on Q, and ω_j is constant on its horizontal sides, while ω_j^* is constant on the vertical sides. Hence $\omega_j + i\omega_j^*$ is a conformal map of Q onto a standard rectangle, so that neither ω_j nor ω_j^* has critical points in Q. It follows that Q is contained in one of the rectangles Π_j^m.

A *harmonic rectangle* in S is a subrectangle of some Π_j^m saturated by the leaves of $\mathcal{F}_j$.

Any non-singular leaf $\mathcal{C}$ of a harmonic foliation $\mathcal{F}_j$ represents a good path in S. Notice that the route $\mathcal{R}(\mathcal{C})$ determines the proper homotopy class of $\mathcal{C}$ in $S \setminus A_j$, and hence determines the foliation $\mathcal{F}_j$ and the rectangle Π_j^m containing $\mathcal{C}$. These remarks, together with Lemma 2.9 imply that the leaves with the same route, $\mathcal{R}(\mathcal{C}) = \boldsymbol{\alpha}$, form a (non-closed) harmonic rectangle $P(\boldsymbol{\alpha})$ in S. By Lemma 2.8, there are at most $s(\mathrm{Top}, N)$ such routes $\boldsymbol{\alpha}$. Therefore there are at most Ns routes for the harmonic foliations to all of the N archipelagoes.

Associated big and little rectangles, $P_j(\boldsymbol{\alpha})$ and $Q_j(\boldsymbol{\alpha})$, $j = 1, \dots l$, come together with any harmonic rectangle $P(\boldsymbol{\alpha})$.

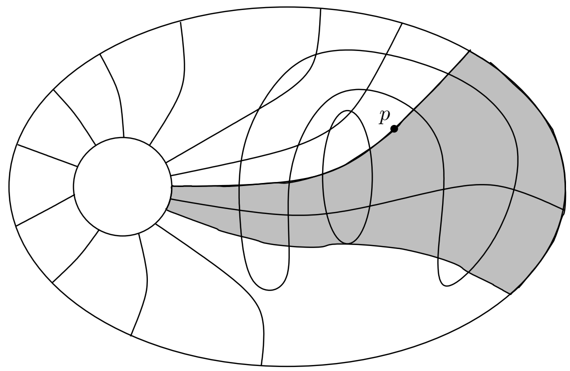

Figure 2.5: Harmonic foliation $\mathcal{F}_i$. Here $\mathbf{S}$ and all A_j are disks. The artificial puncture p is made in $\mathbf{S}$ to ensure that $S \smallsetminus A_i$ is not an annulus. One harmonic rectangle is shaded.

2.5. *Buffers and the small-overlapping principle.* We are going to make use of an important principle saying that *two wide path families have a relatively small overlap.*

A path family Λ on a rectangle P is called a *genuinely vertical lamination* if the paths of Λ are genuinely vertical in R, and the union of these paths, $\operatorname{supp} \Lambda$, is measurable. The projection to the horizontal side of P (after uniformization by a standard rectangle) induces a transverse measure ν on Λ (defined up to scaling). If P is embedded into a Riemann surface S and γ is a path on S, we say that γ *intersects less than an ε-portion of the total width of* Λ if

$$\nu\{\lambda \in \Lambda : \ \lambda \cap \gamma \neq \emptyset\} < \varepsilon \nu(\Lambda)$$

(note that this condition does not depend on the normalization of ν). The same discussion applies to the case of an annulus.

LEMMA 2.13. *Let $\kappa \geq 1$. Let us consider a genuinely vertical lamination Λ on some annulus or rectangle $R \subset S$, and let $\mathcal{G}$ be another path family on S. If $\mathcal{W}(\Lambda) > \kappa$ and $\mathcal{W}(\mathcal{G}) \geq \kappa$, then there exists a path $\gamma \in \mathcal{G}$ that intersects less than $1/\kappa$-portion of the total width of Λ. In particular, if $\kappa = 1$ then there is a path $\gamma \in \mathcal{G}$ that does not cross some leaf of Λ.*

Proof. Assume, to be definite, that R is a rectangle. Let $\phi : E \to R$ be the uniformization of R by a standard rectangle $E = [0, a] \times [0, h]$ normalized so that the projection of $\phi^*\Lambda$ (which is a genuinely vertical lamination in E) onto $[0, a]$ has length κ. Let us use the Euclidean metric μ on E to bound $\mathcal{W}(\Lambda)$:

$$\mathcal{W}(\Lambda) \leq \frac{\operatorname{area}(\phi^*\Lambda)}{\mu(\phi^*\Lambda)^2} = \frac{\kappa}{h}$$

(where area$(\phi^*\Lambda)$ stands for the area of supp $\phi^*\Lambda$). Since $\mathcal{W}(\Lambda) > \kappa$, we conclude that $h < 1$, and thus area$(\phi^*\Lambda) < \kappa$.

To bound $\mathcal{W}(\mathcal{G})$, let us use the push-forward metric $\rho = \phi_*(\mu|\Lambda)$ on S. If a curve $\gamma \in \mathcal{G}$ intersects at least $1/\kappa$-portion of the total width of Λ, then the projection of $\phi^{-1}(\gamma) \subset E$ to $[0, a]$ has length at least 1, and hence

$$\rho(\gamma) = \mu(\phi^{-1}(\gamma)) \geq 1.$$

If this happened for every $\gamma \in \mathcal{G}$ then we would have

$$\mathcal{W}(\mathcal{G}) \leq \text{area}_\rho(\Lambda) = \text{area}(\phi^*\Lambda) < \kappa,$$

contradicting the assumption. □

Take some number $M > 8$. Given a harmonic rectangle $P(\boldsymbol{\alpha})$ of width greater than M, let us define two *buffers*, $B^l(\boldsymbol{\alpha}) \subset P(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha}) \subset P(\boldsymbol{\alpha})$, as harmonic rectangles of width $M/2$ attached to the vertical sides of $P(\boldsymbol{\alpha})$.

LEMMA 2.14. *Consider two harmonic rectangles $P(\boldsymbol{\alpha})$ and $P(\boldsymbol{\beta})$ of width greater than M. Then there are four disjoint vertical leaves, one from each of the corresponding four buffers.*

Proof. Let Λ be the vertical foliation in $B^l(\boldsymbol{\alpha}) \cup B^r(\boldsymbol{\alpha})$, and let $\mathcal{S}$ be the vertical foliation of $B_l(\boldsymbol{\beta})$. Applying the previous lemma to these data, we conclude that there is a vertical leaf $\Gamma^l(\boldsymbol{\beta})$ in $\mathcal{S}$ that crosses less than $1/4$ of the total width of Λ. Hence it crosses less than $1/2$ of the total width of each $B^l(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha})$.

Similarly, there is a vertical leaf $\Gamma^r(\boldsymbol{\beta})$ that crosses less than $1/2$ of the total width of each $B^l(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha})$. Together, $\Gamma^l(\boldsymbol{\beta})$ and $\Gamma^r(\boldsymbol{\beta})$ cross less than the full width of each $B^l(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha})$. Hence each $B^l(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha})$ contains a vertical leaf, $\Gamma^l(\boldsymbol{\alpha})$ and $\Gamma^r(\boldsymbol{\alpha})$ respectively, disjoint from both $\Gamma^l(\boldsymbol{\beta})$ and $\Gamma^r(\boldsymbol{\beta})$. □

2.6. *Truncated rectangles and the disjointness property.* Let us remove the buffers from our harmonic rectangles:

$$\tilde{P}(\boldsymbol{\alpha}) = \text{cl}(P(\boldsymbol{\alpha}) \smallsetminus (B^l(\boldsymbol{\alpha}) \cup B^r(\boldsymbol{\alpha}))).$$

The associated truncated big and little rectangles will be naturally marked with tildes: $\tilde{P}_j(\boldsymbol{\alpha})$ and $\tilde{Q}_j(\boldsymbol{\alpha})$.

We can now formulate the key disjointness property for the truncated rectangles:

LEMMA 2.15. *If two associated truncated little rectangles $\tilde{Q}_j(\boldsymbol{\alpha})$ and $\tilde{Q}_k(\boldsymbol{\beta})$ overlap then they represent the same proper homotopy class in $S \smallsetminus \Lambda$ (up to orientation). If their orientations match, then one of the routes, $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, is an extension of the other, and $j = k$.*

Proof. Let us select in the buffers of $P_j(\boldsymbol{\alpha})$ and $P_k(\boldsymbol{\beta})$ two disjoint pairs of leaves (by Lemma 2.14) and consider the rectangles $\mathbf{P}_j(\boldsymbol{\alpha}) \subset P_j(\boldsymbol{\alpha})$ and $\mathbf{P}_k(\boldsymbol{\beta}) \subset P_k(\boldsymbol{\beta})$ bounded by the corresponding pairs. By Lemma 2.11, their associated little rectangles, $\mathbf{Q}_j(\boldsymbol{\alpha})$ and $\mathbf{Q}_k(\boldsymbol{\beta})$, contain the respective little rectangles $\tilde{Q}_j(\boldsymbol{\alpha})$ and $\tilde{Q}_k(\boldsymbol{\beta})$. Hence $\mathbf{Q}_j(\boldsymbol{\alpha})$ and $\mathbf{Q}_k(\boldsymbol{\beta})$ overlap as well. Since the big rectangles $\mathbf{P}_j(\boldsymbol{\alpha})$ and $\mathbf{P}_k(\boldsymbol{\beta})$ have disjoint vertical boundaries, we can apply Lemma 2.12 and complete the proof. □

COROLLARY 2.16. *For any route $\boldsymbol{\alpha}$, the little rectangles $\tilde{Q}_i(\boldsymbol{\alpha})$ are pairwise disjoint.*

Proof. Assume $Q_i(\boldsymbol{\alpha}) \cap Q_j(\boldsymbol{\alpha}) \neq \emptyset$ for some $i < j$. Then by the first assertion of the previous lemma, one component of $\partial Q_i(\boldsymbol{\alpha})$ would lie on ∂A_j, which is impossible. □

COROLLARY 2.17. *Suppose that $\tilde{Q}_j(\boldsymbol{\alpha})$ and $\tilde{Q}_k(\boldsymbol{\beta})$ overlap with matched vertical orientation. Then $j = k$; moreover, if $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, then $\boldsymbol{\alpha} = \boldsymbol{\beta}$.*

Fix your favorite $\delta \in (0, 1)$, e.g., $\delta = 1 - \sqrt{2/3}$. The total width of the rectangles $P(\boldsymbol{\alpha})$ is equal to the modulus Y (by definition (1.1), Example 4.1 and the Parallel Law). For every route $\boldsymbol{\alpha}$, we find that $\mathcal{W}(\tilde{P}(\boldsymbol{\alpha})) \geq \mathcal{W}(P(\boldsymbol{\alpha})) - M$. The number of routes $\boldsymbol{\alpha}$ is bounded by $Ns = Ns(\text{Top}, N)$. Therefore, if $Y > MNs/\delta$ then

$$\sum_{\boldsymbol{\alpha}} \mathcal{W}(\tilde{P}(\boldsymbol{\alpha})) > (1 - \delta)Y. \tag{2.1}$$

2.7. *a- and b-moduli.* We let

$$a_k = \sum_{|\boldsymbol{\alpha}|=k} \mathcal{W}(\tilde{P}(\boldsymbol{\alpha})), \qquad b_i^k = \sum_{|\boldsymbol{\alpha}|=k} \mathcal{W}(\tilde{Q}_i(\boldsymbol{\alpha})),$$

and $b_i = \max_{k \geq i} b_i^k$, $a = \sum_k a_k$, and $b = \sum_i b_i$.

As introduced in the Appendix (§4.2), $x \oplus y$ stands for the harmonic sum of x and y.

LEMMA 2.18. *The a- and b-moduli are related by the Series Inequality*:

$$a_k \leq \bigoplus_{i=1}^{k} b_i.$$

Proof . By Lemma 2.9, for each $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = k$, every vertical path of $\tilde{P}(\boldsymbol{\alpha})$ overflows each of the little rectangles $\tilde{Q}_i(\boldsymbol{\alpha})$, with $1 \leq i \leq k$. Moreover, by Corollary 2.16, the $\tilde{Q}_i(\boldsymbol{\alpha})$ are disjoint. Therefore, by Proposition 4.2,

$$\sum_{|\boldsymbol{\alpha}|=k} \mathcal{W}(\tilde{P}(\boldsymbol{\alpha})) \leq \bigoplus_{i=1}^{k} \sum_{|\boldsymbol{\alpha}|=k} \mathcal{W}(\tilde{Q}_i(\boldsymbol{\alpha})),$$

and the lemma follows. □

Let us now relate the a- and b-moduli to the geometric moduli X,Y and Z in the Quasi-Additivity Law (see the Introduction). By (2.1),

$$a \geq (1-\delta)Y, \tag{2.2}$$

provided $Y > MNs/\delta$. Furthermore,

LEMMA 2.19. *$b_1 \leq X$.*

Proof. We need to show that $b_1^k \leq X$ for every k. Let us therefore fix k. By Corollary 2.17, the $\tilde{Q}_1(\boldsymbol{\alpha})$ for $|\boldsymbol{\alpha}| = k$ are all disjoint, so that the union of the associated vertical path families has width equal to

$$\sum_{|\boldsymbol{\alpha}|=k} \mathcal{W}(\tilde{Q}_1(\boldsymbol{\alpha})) = b_1^k.$$

On the other hand, this union is a subfamily of the family of paths connecting ∂S and $\partial\Lambda$ in $S \smallsetminus \Lambda$ (recall that $\Lambda = \bigcup A_j$). Therefore

$$\sum_{|\boldsymbol{\alpha}|=k} \mathcal{W}(\tilde{Q}_1(\boldsymbol{\alpha})) \leq \mathcal{W}(S,\Lambda) = X. \qquad \square$$

Finally,

LEMMA 2.20. *$b \leq Z$.*

Proof. We arbitrarily label the archipelagoes $\{A_1, \dots, A_n\}$ and let $\boldsymbol{\alpha}[i]$ denote the label of the i^{th} archipelago visited on the route $\boldsymbol{\alpha}$. Now, let

$$b_i^k(l) = \sum_{|\boldsymbol{\alpha}|=k;\, \boldsymbol{\alpha}[i]=l} \mathcal{W}(\tilde{Q}_i(\boldsymbol{\alpha})),$$

so that $b_i^k = \sum_l b_i^k(l)$. Let $k : \mathbb{N} \to \mathbb{N}$ be such that $b_i = b_i^{k(i)}$.

We claim that

$$\sum_i b_i^{k(i)}(l) \leq \mathcal{W}(S \smallsetminus \bigcup_{k \neq l} A_k, A_l);$$

this would imply (by summing over l) that $b \leq Z$. To show the claim, first note that the $\tilde{Q}_i(\boldsymbol{\alpha})$ for $|\boldsymbol{\alpha}| = k(i)$ and $\boldsymbol{\alpha}[i] = l$ are disjoint (where l is fixed and i is arbitrary). Indeed, any two such rectangles have the same roof, and so they have the same vertical orientation if they overlap; then by Corollary 2.17, they have the same height i and therefore the same route $\boldsymbol{\alpha}$. Moreover the vertical paths of these $\tilde{Q}_i(\boldsymbol{\alpha})$ all connect $\partial(S \smallsetminus \bigcup_{k\neq l} A_k)$ to ∂A_l in $S \smallsetminus \Lambda$; the claim follows. $\square$

2.8. *An arithmetic inequality.*

LEMMA 2.21. *Consider two sequences of positive numbers,* $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^n$*, such that* $a_1 = b_1$, $a_i \leq \bigoplus_{k=1}^{i} b_k$*. Then*

$$\left(\sum_{i=1}^n a_i\right)^2 \leq \frac{4}{3}\, b_1 \sum_{i=1}^n b_i. \tag{2.3}$$

Proof. Without loss of generality, we can assume

$$a_i = \bigoplus_{k=1}^{i} b_k = a_{i-1} \oplus b_i.$$

Let

$$a = \sum_{i=1}^n a_i, \quad b = \sum_{i=1}^n b_i.$$

We have, for $i > 1$,

$$b_i = \frac{a_i a_{i-1}}{a_{i-1} - a_i} = a_i + \frac{a_i^2}{a_{i-1} - a_i}.$$

and therefore

$$\begin{aligned}
b - b_1 &= \sum_{i=2}^n \left(a_i + \frac{a_i^2}{a_{i-1} - a_i}\right) \\
&= a - a_1 + \sum_{i=2}^n \frac{a_i^2}{a_{i-1} - a_i} \\
&\geq a - a_1 + \frac{\left(\sum_{i=2}^n a_i\right)^2}{\sum_{i=2}^n (a_{i-1} - a_i)} \qquad (2.4) \\
&= a - a_1 + \frac{(a - a_1)^2}{a_1 - a_n} \\
&\geq a - a_1 + \frac{(a - a_1)^2}{a_1},
\end{aligned}$$

where inequality (2.4) follows from the Cauchy-Schwarz inequality written as follows:

$$\left(\sum x_j\right)^2 \leq \sum \frac{x_j^2}{y_j} \sum y_j.$$

Therefore, because $a_1 = b_1$,

$$\frac{b_1 b}{a^2} \geq 1 - \frac{a_1}{a} + \left(\frac{a_1}{a}\right)^2 \geq \frac{3}{4}. \qquad \square$$

2.9. *Completion of the proof of the* QA *Law.* Let us consider the a- and b-moduli from §2.7. Lemma 2.18 puts us into a position to apply estimate (2.3) to these moduli. Incorporating (2.2) and Lemmas 2.19 and 2.20 into (2.3), we obtain:

$$(1-\delta)^2 Y^2 \leq \frac{4}{3} XZ,$$

provided $Y > MNs/\delta$, and we are done. □

2.10. QA *Law*: *variations.* We will now formulate several variations and special cases of the QA Law suitable for the dynamical applications.

2.10.1. *Fractal archipelagos.* A compact set $A \subset \operatorname{int} S$ is called a set of *finite type*, or a (*closed*) *archipelago*, if $A = \cap U_i$ where U_i is a nested sequence of open archipelagos of bounded topological complexity (equivalently, $S \smallsetminus A$ is a Riemann surface of finite type). In this case, we let

$$\operatorname{Top}_S(A) = \liminf \operatorname{Top}_S(U_i).$$

(If there is a finite family of disjoint closed archipelagos A_j, we let $\operatorname{Top}_S\{A_j\} = \operatorname{Top}_S(\bigcup A_j)$.)

By an approximation argument, the QA *Law is valid for these more general archipelagos as well.*

2.10.2. *Collars.* Let A'_j be a topological disk such that

$$A_j \subset A'_j \subset S \smallsetminus \bigcup_{k \neq j} A_k.$$

If $\operatorname{mod}(A'_j, A_j) \geq \eta \operatorname{mod}(S, A_j) > 0$, then we call A'_j an η-*collar* around A_j. If all the archipelagos A_j have η-collars, we say that the archipelagos satisfy the η-*Collar Assumption*. Under this assumption, they are η^{-1}-separated (since $Z \leq \sum \mathcal{W}(A'_j, A_j)$). Thus, we obtain:

QA Law with collars. *Under the η-Collar Assumption, there exists K depending only on η and $\operatorname{Top}_S\{A_j\}$ such that*:

$$Y \geq K \Rightarrow Y \leq 2\eta^{-1} X.$$

One can also allow general *holomorphic collars* instead of embedded ones. Precisely speaking, assume A_j is embedded into an abstract conformal disk A'_j which in turn is mapped into $S \smallsetminus \bigcup_{k\neq j} A_k$ holomorphically by some map i such that $i|A_j = \mathrm{id}$ and $i^{-1}(A_j) = A_j$. If $\operatorname{mod}(A'_j, A_j) \geq \eta \operatorname{mod}(S, A_j) > 0$, then we call A'_j a *holomorphic* η-collar around A_j. Since every path connecting A_j to the rest of the boundary of $S \smallsetminus \bigcup A_k$ can be lifted to a vertical path in $A'_j \smallsetminus A_j$, Corollary 4.4 yields: $Z \leq \sum \mathcal{W}(A'_j, A_j)$. Thus, the *$\eta$-Collar Assumption for holomorphic collars implies η^{-1}-separation of the archipelagos as well.*

2.10.3. *Comparable terms.* In further applications in holomorphic dynamics, we will often encounter the situation when the individual terms that appear in the moduli Y and Z are all comparable. Here is the user-friendly version of the Quasi-Additivity Law in this situation:

QA LAW WITH COMPARABLE TERMS. *Fix some $\eta \in (0,1)$. Let $W \Subset \operatorname{int} U$ and $D_i' \Subset \operatorname{int} W$, $i = 1, \dots, N$, be topological disks such that the closures of D_i' are pairwise disjoint, and let $D_i \Subset D_i'$ be smaller disks. Then there exists a $\delta_0 > 0$ (depending on η and N) such that: If for some $\delta \in (0, \delta_0)$ and for all i,*

$$\eta\delta < \operatorname{mod}(D_i' \smallsetminus D_i) \le \operatorname{mod}(U \smallsetminus D_i) < \delta,$$

then

$$\operatorname{mod}(U \smallsetminus W) < \frac{2\eta^{-1}\delta}{N}.$$

Of course, this version is a particular case of the QA Law with collars.

3. Quasi-invariance law

In this section, we will prove a general transformation law for conformal moduli under covering maps. To this end, we will make use of the following well-known result:

PROPOSITION 3.1. *Let $f : U \to V$ be a branched cover of Riemann surfaces of degree N. Then there is a Galois branched cover $g : S \to V$ of degree at most $N!$ that factors as $g = f \circ h$ for some $h : S \to U$. Moreover, g is ramified only over critical values of f.*

The proof uses a lemma that is a simple exercise in group theory:

LEMMA 3.2. *Suppose that H is a subgroup of a group G, and $[G : H] = N$. Then there is a normal subgroup L of G such that $L < H$, and $[G : L] \le N!$.*

Proof. The coset action of G on G/H provides a homomorphism from G to the group of permutations of G/H, which has order at most $N!$. We let L be the kernel of this homomorphism; it has the desired properties. □

Proof of Proposition 3.1. Let O be the set of critical values of f, and let $E = f^{-1}(O)$. Then $f : U \smallsetminus E \to V \smallsetminus O$ is an unbranched cover of degree N. Hence $f_*\pi_1(U \smallsetminus E)$ has index N in $\pi_1(V \smallsetminus O)$, so that by Lemma 3.2 we can find a subgroup of $f_*\pi_1(U \smallsetminus E)$ that is a normal subgroup of $\pi_1(V \smallsetminus O)$ of degree at most $N!$. There is then the corresponding cover $g : S' \to V \smallsetminus O$ which we can complete to a branched cover $g : S \to V$ with the desired properties. □

We say that a closed set $K \subset S$ is a *hull* if it is a full connected non-degenerate continuum.

Given a holomorphic map $f : S \to S'$, and two closed subsets $K \subset S$, $K' \subset S'$ such that $f(K) \subset K'$, we say that the restriction $f : K \to K'$ is a *branched covering of degree* d if:

- For any $x \in K$, there exists a neighborhood $U \ni x$ such that $K \cap U = f^{-1}(K') \cap U$;
- For any regular value $x' \in K'$ of f, $\#(f|K)^{-1}(x) = d$.

Let us consider a Riemann surface S with several archipelagos B_j contained in hulls B'_j, and several marked points v_i (some of them may belong to the archipelagos or the hulls). Let $B = \bigcup B_j$. For each k, let us consider two families $\mathcal{G}'_k$ and $\mathcal{G}''_k$ of proper curves $\gamma \subset S \smallsetminus B$ that begin on B_k and satisfy one of the following conditions:

- $\gamma \in \mathcal{G}'_k$ ends on another archipelago B_j, $j \neq k$, or on ∂S;
- $\gamma \in \mathcal{G}''_k$ ends on the same B_k, does not pass through the marked points v_i, and is *non-trivial* in the sense that it cannot be homotopic in $S \smallsetminus (B \cup \{v_i\})$ to an arbitrary small neighborhood of the hull B'_k.[2]

Under these circumstances, we let

$$Z_S\{B_j, v_i\} \equiv Z_S\{B_j, B'_j, v_i\} = \sum_k (\mathcal{W}(\mathcal{G}'_k) + 2\mathcal{W}(\mathcal{G}''_k)).$$

Remark. In the case when $CV \subset B \subset B'$ and the B_j are connected, the family $\mathcal{G}'_k \cup \mathcal{G}''_k$ is the family of all non-trivial proper curves $\gamma \subset S \smallsetminus B$ that begin on B_k.

General quasi-invariance law. *Consider the following data:*

- *Two Riemann surfaces of finite type, U and V;*
- *Two closed sets $\Lambda' = \bigcup_{j=1}^{p} \Lambda'_j \subset U$ and $B' = \bigcup_{j=1}^{p} B'_j \subset V$ whose connected components, Λ'_j and B'_j respectively, are hulls;*
- *Two families of compact archipelagos, $\Lambda_j \subset \Lambda'_j$ and $B_j \subset B'_j$;*
- *A branched covering $f : U \to V$ of degree D that restricts to branched coverings $f : \Lambda'_j \to B'_j$ of degree $d_j \le d$. Suppose Λ_j is the union of some components of $f^{-1}(B_j)$, and let* CV *stand for the set of critical values of f.*

Then there exists K depending on $\mathrm{Top}_V\{B_j\}$ and D such that

$$Y_U\{\Lambda_j\} > K \Rightarrow Y_U\{\Lambda_j\}^2 \le 2d^2\, X_V\{B_j\}\, Z_V\{B_j, B'_j, \mathrm{CV}\}.$$

[2]Notice that a trivial γ is allowed to have arbitrary complexity in $B'_k \smallsetminus B_k$.

Proof. If we replace the archipelagos Λ_j with $\Lambda_j = (f|\Lambda'_j)^{-1}(B_j)$ we make the left-hand side bigger without changing the right-hand side. So, we can assume without loss of generality that $\Lambda_j = (f|\Lambda'_j)^{-1}(B_j)$.

Let $E = f^{-1}(\mathrm{CV}) \subset U$. By Proposition 3.1, there exists a branched covering $h : S \to U$ of degree at most $(D-1)!$ with critical values in E such that $g = f \circ h : S \to U$ is a Galois branched covering. Let Γ be the Galois group of the covering g acting on S.

Let $A'_j(i) \subset S$ be the connected components of $g^{-1}(B'_j)$ labeled in such a way that $h(A'_j(1)) = \Lambda'_j$, and let $A'_j = A'_j(1)$. For any given j, these components are transitively permuted by Γ. We let L_j be the number of these components.

Also, consider the corresponding archipelagos

$$A_j(i) = (g|\, A'_j(i))^{-1}(B_j), \quad A_j \equiv A_j(1),$$
$$A = \bigcup A_j(i) = g^{-1}(B).$$

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ stand respectively for the X-, Y- and Z-moduli for this family of archipelagos. By Lemma 4.7 from the appendix, we have:

$$\mathcal{X} = |\Gamma|\, X_V\{B_j\}. \tag{3.1}$$

Let $m_j = \deg(h : A'_j \to \Lambda'_j) \equiv \deg(h : A_j \to \Lambda_j)$. Then the stabilizer of A'_j in Γ consists of $d_j m_j$ elements, and hence the Γ-orbit of A_j consists of $L_j = |\Gamma|/d_j m_j$ archipelagos $A_j(i)$. Since for each j, these archipelagos are symmetric in S, we have:

$$\mathcal{Y} = \sum_j \frac{|\Gamma|}{d_j m_j} \mathcal{W}(S, A_j) \geq \frac{|\Gamma|}{d} \sum_j \mathcal{W}(U, \Lambda_j) = \frac{|\Gamma|}{d} Y_U\{\Lambda_j\} \tag{3.2}$$

where the middle inequality follows from Lemma 4.6.

We will now show that

$$\mathcal{Z} \leq |\Gamma|\, Z_V\{B_j, \mathrm{CV}\} + C, \tag{3.3}$$

where C depends only on $\mathrm{Top}_U\{\Lambda_j\}$ (which in turn depends only on $\mathrm{Top}_V\{B_j\}$ and D).

For any $k \in [1, p]$, we consider the harmonic foliation $\mathcal{F}_k$ that measures the extremal width between A_k and the rest of the boundary of $S \smallsetminus A$ (see §2.4). Then $S \smallsetminus A$ is tiled by the harmonic rectangles Π_k^n, $n = 1, \ldots, s_k$. Their total number $\sum s_k$ depends only on $\mathrm{Top}_S\{A_j(i)\}$. Applying the group Γ, we obtain a family of harmonic rectangles Π_{ji}^n (connecting the $A_j(i)$ to the rest of the boundary of $S \smallsetminus A$) that are permuted by the Γ-action.

Let $\tilde{\Pi}_{ji}^n$ be the truncated rectangle obtained by removing two buffers of width four each from Π_{ji}^n (as in §2.6). They are also permuted by Γ. Since these rectangles represent different homotopy classes in $S \smallsetminus A$, Lemma 2.14 implies that the truncated rectangles are pairwise disjoint.

Since the fibers of g coincide with the orbits of Γ, each $\tilde{\Pi}_k^n$ projects injectively onto some proper rectangle $\tilde{Q}_k^l$ in $V \setminus B$, and these rectangles are either pairwise disjoint or coincide. Moreover, there are $d_k m_k$ rectangles $\tilde{\Pi}_k^n$ that project onto $\tilde{Q}_k^l$ representing a curve $\gamma \in \mathcal{G}'_k$, and twice as many rectangles that project onto $\tilde{Q}_k^l$ representing $\gamma \in \mathcal{G}''_k$ (corresponding to two possible orientations of such a γ). Let $\mathcal{Q}'_k$ and $\mathcal{Q}''_k$ denote these two families of rectangles.

The foliation $\mathcal{F}_k$ on $\bigcup_n \tilde{\Pi}_k^n$ descends to a foliation $\mathcal{H}_k$ supported on $\bigcup_l \tilde{Q}_k^l$. The leaves of this foliation belong to the family of curves defining the modulus $Z_V\{B_j, \mathrm{CV}\}$. (Indeed, if some leaf γ connecting B_k to itself were trivial then it would lift to a path connecting A'_k to itself.) Hence

$$\sum_{\mathcal{Q}'_k} \mathcal{W}(\tilde{Q}_k^l) \le \mathcal{W}(\mathcal{G}'_k), \qquad \sum_{\mathcal{Q}''_k} \mathcal{W}(\tilde{Q}_k^l) \le \mathcal{W}(\mathcal{G}''_k),$$

and we obtain:

$$\begin{aligned}\mathcal{W}(S \setminus \bigcup_{j\neq k} A_j,\, A_k) &= \sum_n \mathcal{W}(\tilde{\Pi}_k^n) + 8s_k \\ &= d_k m_k \Big(\sum_{\mathcal{Q}'_k} \mathcal{W}(\tilde{Q}_k^l) + 2 \sum_{\mathcal{Q}''_k} \mathcal{W}(\tilde{Q}_k^l) \Big) + 8s_k \\ &\le d_k m_k \left(\mathcal{W}(\mathcal{G}'_k) + 2\mathcal{W}(\mathcal{G}''_k) \right) + 8s_k.\end{aligned}$$

(Here $8s_k$ appears as the total width of the buffers removed.) Multiplying the last estimate by L_k and summing up over k (making use of the symmetry and of $|\Gamma| = L_k d_k m_k$), we obtain (3.3).

By the Quasi-Additivity Law, $\mathcal{Y}^2 \le 1.5\mathcal{X}\mathcal{Z}$. Together with (3.1) and (3.2) and (3.3) it implies the desired estimate, provided $\mathcal{Z}$ is sufficiently big (which is certainly the case when $Y_U\{\Lambda_j\}$ is sufficiently big). □

3.1. QI *Law*: *Variations.* We now list several variations and special cases of the General QI Law. In what follows, the setting of the General QI Law is assumed, and we let $Y_U = Y_U\{\Lambda_j\}$, $X_V = X_V\{B_j\}$, $Z_V = Z_V\{B_j, B'_j, \mathrm{CV}\}$.

3.1.1. *Separation.* In the context of the QI Law, the ξ-Separation Assumption should be formulated as follows:

$$Z_V \le \xi Y_U.$$

QI LAW WITH SEPARATION. *If the archipelagos B_j are ξ-separated, then there exists K depending only on ξ, $\mathrm{Top}_V\{B_j\}$ and D such that*:

$$Y_U \ge K \Rightarrow Y_U \le 2\xi d^2 X_V.$$

3.1.2. *Collars.* The definition of η-collars should also be adjusted in this more general context. Namely, a disk $\mathcal{B}_j \supset B'_j$ is called an *η-collar* of B_j if

$\mathcal{B}_j \smallsetminus B'_j \subset V \smallsetminus (\bigcup_{k\neq j} B_k \cup \mathrm{CV})$ and

$$\mathrm{mod}(\mathcal{B}_j, B_j) \geq \eta \,\mathrm{mod}(U, \Lambda_j). \tag{3.4}$$

More generally, one can define a *holomorphic η-collar* $\mathcal{B}_j$ as an abstract conformal disk $\mathcal{B}_j$ such that B'_j is embedded into $\mathcal{B}_j$ and there is a holomorphic map $i : \mathcal{B}_j \to V$ such that $i|B'_j = \mathrm{id}$,

$$i(\mathcal{B}_j \smallsetminus B'_j) \subset V \smallsetminus (B \cup \mathrm{CV}),$$

and (3.4) is satisfied.

QI LAW WITH COLLARS. *If all the archipelagos B_j have holomorphic η-collars then there exists K depending only on η, $\mathrm{Top}_V\{B_j\}$ and D such that*:

$$Y_U \geq K \Rightarrow Y_U \leq 2\eta^{-1} d^2 X_V.$$

3.1.3. *Covering lemma.* The Basic Covering Lemma stated in the Introduction is a special case of the General QI Law with embedded collars when both Riemann surfaces, U and V, are conformal disks, and the archipelagos Λ and B consist of a single Jordan island each. In the following variation the collars are allowed to be holomorphic:

COVERING LEMMA WITH HOLOMORPHIC COLLARS. *Fix some $\eta \in (0, 1)$. Consider two topological disks U and V, two hulls $\Lambda' \subset U$ and $B' \subset V$, and two compact hulls $\Lambda \subset \Lambda'$ and $B \subset B'$.*

Let $f : U \to V$ be a branched covering of degree D such that Λ' is a component of $f^{-1}(B')$, and Λ is a component of $f^{-1}(B)$. Let $d = \deg(f : \Lambda' \to B')$.

Assume B' is also embedded into a holomorphic η-collar $\mathcal{B}'$; i.e., there is a holomorphic map $i : \mathcal{B} \to V$ such that $i|\, B' = \mathrm{id}$, $i^{-1}(B') = B'$, $i(\mathcal{B}) \smallsetminus B'$ does not contain the critical values of f, and

$$\mathrm{mod}(\mathcal{B}, B) > \eta \,\mathrm{mod}(U, \Lambda).$$

Then

$$\mathrm{mod}(U, \Lambda) < \varepsilon(\eta, D) \Rightarrow \mathrm{mod}(V, B) < 2\eta^{-1} d^2 \,\mathrm{mod}(U, \Lambda).$$

The Basic Covering Lemma stated in the Introduction is used in [KL1], the Covering Lemma with holomorphic collars is used in [KL2], [KL3], while the QI Law with all critical values in B is used in [K].

4. Appendix: Extremal length and width

There is a wealth of sources containing background material on extremal length; see, e.g., the book of Ahlfors [A]. We will briefly summarize the necessary minimum.

4.1. *Definitions.* Let $\mathcal{G}$ be a family of curves on a Riemann surface U. Given a (measurable) conformal metric $\mu = \mu(z)|dz|$ on U, let

$$\mu(\mathcal{G}) = \inf_{\gamma \in \mathcal{G}} \mu(\gamma),$$

where $\mu(\gamma)$ stands for the μ-length of γ. The length of $\mathcal{G}$ with respect to μ is defined as

$$\mathcal{L}_\mu(\mathcal{G}) = \frac{\mu(\mathcal{G})^2}{\mathrm{area}_\mu(U)},$$

where area_μ is the area corresponding to the form $\mu^2 = \mu(z)^2 dx \wedge dy$. Taking the supremum over all conformal metrics μ, we obtain the *extremal length* $\mathcal{L}(\mathcal{G})$ of the family $\mathcal{G}$.

The *extremal width* is the inverse of the extremal length:

$$\mathcal{W}(\mathcal{G}) = \mathcal{L}^{-1}(\mathcal{G}).$$

It can be also defined as follows. Consider all conformal metrics μ such that $\mu(\gamma) \geq 1$ for any $\gamma \in \mathcal{G}$. Then $\mathcal{W}(\mathcal{G})$ is the infimum of the areas $\mu^2(U)$ of all such metrics.

Example 4.1. For a standard rectangle $P = I \times [0, h]$, let $\mathcal{G}$ be the family of vertical curves, and let Λ be the genuinely vertical foliation. Then

$$\mathcal{L}(\mathcal{G}) = \mathcal{L}(\Lambda) = \frac{h}{|I|} \equiv \operatorname{mod} P.$$

Similar formulas hold for the standard cylinder $C = \mathbb{T} \times [0, h]$.

4.2. *Electric circuits laws.* We say that a family $\mathcal{G}$ of curves *overflows* a family $\mathcal{H}$ if any curve of $\mathcal{G}$ contains some curve of $\mathcal{H}$. Also, two families, $\mathcal{G}_1$ and $\mathcal{G}_2$, are *disjoint* if any two curves, $\gamma_1 \in \mathcal{G}_1$ and $\gamma_2 \in \mathcal{G}_2$, are disjoint.

We let $x \oplus y = (x^{-1} + y^{-1})^{-1}$ be the *harmonic sum* of x and y (it is conjugate to the usual sum by the inversion map $x \mapsto x^{-1}$).

The following crucial properties of the extremal length and width show that the former behaves like the resistance in electric circuits, while the latter behaves like conductance.

Series law/Grötzsch inequality. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two disjoint families of curves, and let $\mathcal{G}$ be a third family that overflows both $\mathcal{G}_1$ and $\mathcal{G}_2$. Then*

$$\mathcal{L}(\mathcal{G}) \geq \mathcal{L}(\mathcal{G}_1) + \mathcal{L}(\mathcal{G}_2),$$

or equivalently,

$$\mathcal{W}(\mathcal{G}) \leq \mathcal{W}(\mathcal{G}_1) \oplus \mathcal{W}(\mathcal{G}_2).$$

Parallel law. *For any two families* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *of curves we have*:

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) \le \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2).$$

If $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are disjoint then*

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2)$$

Note that the Parallel Law inequality implies the estimate $X \le Y$ between the moduli from the Introduction.

From the Series and Parallel Laws we can derive the following more general result:

Proposition 4.2. *Suppose that* Δ^i_λ, Γ_λ *for* $i = 1 \dots k$, $\lambda \in \Lambda$ (*where* Λ *is finite*) *are path families supported on a Riemann surface* S. *Assume for each* $\lambda \in \Lambda$, *the* Δ^i_λ *have disjoint support*, *and* Γ_λ *overflows each of the* Δ^i_λ. *Then*

$$\sum_\lambda \mathcal{W}(\Gamma_\lambda) \le \bigoplus_{i=1}^{k} \sum_\lambda \mathcal{W}(\Delta^i_\lambda).$$

Proof. We form path families $\hat\Delta^i_\lambda$ and $\hat\Gamma_\lambda$ on the Riemann surface $S \times \Lambda$ by putting Δ^i_λ and Γ_λ on the copy of S labeled by λ. Let $\hat\Gamma = \bigcup_\lambda \hat\Gamma_\lambda$ and $\hat\Delta^i = \bigcup_\lambda \hat\Delta^i_\lambda$. By the Parallel Law,

$$\mathcal{W}(\hat\Gamma) = \sum_\lambda \mathcal{W}(\hat\Gamma_\lambda), \quad \mathcal{W}(\hat\Delta^i) = \sum_\lambda \mathcal{W}(\hat\Delta^i_\lambda).$$

Moreover, $\hat\Gamma$ overflows each of the $\hat\Delta^i$, and the $\hat\Delta^i$ are disjoint. Therefore, by the Series Law,

$$\mathcal{W}(\hat\Gamma) \le \bigoplus_{i=1}^{k} \mathcal{W}(\hat\Delta^i),$$

and the result follows. □

4.3. *Transformation rules.* Both extremal length and extremal width are conformal invariants. More generally, we have:

Lemma 4.3. *Let* $f : U \to V$ *be a holomorphic map between two Riemann surfaces*, *and let* $\mathcal{G}$ *be a family of curves on* U. *Then*

$$\mathcal{L}(f(\mathcal{G})) \ge \mathcal{L}(\mathcal{G}).$$

Moreover, *if* f *is at most* $d - to - 1$, *then*

$$\mathcal{L}(f(\mathcal{G})) \le d \cdot \mathcal{L}(\mathcal{G}).$$

Proof. Let μ be a conformal metric on U. Let us push-forward the area form μ^2 by f. We obtain the area form $\nu^2 = f_*(\mu^2)$ of some conformal metric ν on V. Then $\text{area}_\nu(V) = \text{area}_\mu(U)$ and $f^*(\nu) \geq \mu$. It follows that

$$\mathcal{L}_\mu(\mathcal{G}) \leq \mathcal{L}_\nu(f(\mathcal{G})) \leq \mathcal{L}(f(\mathcal{G})).$$

Taking the supremum over μ completes the proof of the first assertion.

For the second assertion, let us consider a conformal metric ν on V and pull it back to U, $\mu = f^*\nu$. Then $\mu(\gamma) = \nu(f(\gamma))$ for any $\gamma \in \mathcal{G}$, while $\text{area}_\mu(U) \leq d \cdot \text{area}_\nu(V)$. Hence

$$\mathcal{L}(\mathcal{G}) \geq \mathcal{L}_\mu(\mathcal{G}) \geq \frac{1}{d}\mathcal{L}_\nu(f(\mathcal{G})),$$

and taking the supremum over ν completes the proof. □

COROLLARY 4.4. *Under the circumstances of the previous lemma, let $\mathcal{H}$ be a family of curves in V satisfying the following lifting property: any curve $\gamma \in \mathcal{H}$ contains an arc that lifts to some curve in $\mathcal{G}$. Then $\mathcal{L}(\mathcal{H}) \geq \mathcal{L}(\mathcal{G})$.*

Proof. The lifting property means that the family $\mathcal{H}$ overflows the family $f(\mathcal{G})$. Hence $\mathcal{L}(\mathcal{H}) \geq \mathcal{L}(f(\mathcal{G}))$, and the conclusion follows. □

4.4. *Extremal distance and the Dirichlet integral.* Given a compact subset $K \subset \text{int}\, U$, the *extremal distance*

$$\mathcal{L}(U, K) \equiv \text{mod}(U, K)$$

(between ∂U and K) is defined as $\mathcal{L}(\mathcal{G})$, where $\mathcal{G}$ is the family of curves connecting ∂U and K. In the case when U is a topological disk and K is connected, we obtain the usual modulus $\text{mod}(U \smallsetminus K)$ of the annulus $U \smallsetminus K$.

Remark. $\mathcal{L}(U, K)$ can also be defined as $\mathcal{L}(\mathcal{G}')$ where $\mathcal{G}'$ is the family of curves *in* $U \smallsetminus K$ connecting ∂U to K. Indeed, since $\mathcal{G} \supset \mathcal{G}'$, $\mathcal{L}(\mathcal{G}) \leq \mathcal{L}(\mathcal{G}')$. Since each curve of $\mathcal{G}$ overflows some curve of $\mathcal{G}'$, $\mathcal{L}(\mathcal{G}) \geq \mathcal{L}(\mathcal{G}')$. One can also compromise and use the intermediate family of curves in U connecting ∂U to K.

We let $\mathcal{W}(U, K) = \mathcal{L}^{-1}(U, K)$.

LEMMA 4.5. *Let $f : U \to V$ be a branched covering of degree N between two compact Riemann surfaces with boundary. Let A be a compact subset of $\text{int}\, U$ and let $B = f(A)$. Then*

$$\text{mod}(U, A) \leq \text{mod}(V, B) \leq N\ \text{mod}(U, A).$$

Proof. Let $\mathcal{G}$ be the family of curves in U connecting ∂U to A, and let $\mathcal{H}$ be the similar family in V. Notice that every curve $\gamma \in \mathcal{H}$ lifts to a curve in $\mathcal{G}$: begin the lifting on A; it must end on ∂U since $f : U \to V$ is proper. Thus, $\mathcal{H} = f(\mathcal{G})$, and Lemma 4.3 completes the proof. □

Extremal width $\mathcal{W}(U, A)$ can be explicitly expressed as the *Dirichlet integral of the harmonic measure* (see [A, §4–9]):

$$\mathcal{W}(U, A) = 4 \int_{U \smallsetminus A} |\partial h|^2$$

where $h : U \smallsetminus A \to \mathbb{R}$ is the harmonic function equal to 1 on ∂A and vanishing on ∂U, and $|\partial h|^2$ is the area form associated with the holomorphic differential $\partial h = (\partial h/\partial z)dz$.

4.5. *More transformation rules.* The Dirichlet integral formulation allows us to sharpen the lower bound in Lemma 4.5:

LEMMA 4.6. *Let $f : U \to V$ be a branched covering between two compact Riemann surfaces with boundary. Let A be an archipelago in U, $B = f(A)$, and assume that $f : A \to B$ is a branched covering of degree d. Then*

$$\operatorname{mod}(V, B) \geq d \operatorname{mod}(U, A).$$

Proof. The Riemann surface $V \smallsetminus B$ is decomposed into finitely many rectangles saturated by the leaves of the harmonic flow (see §2.4). Slit these rectangles by the leaves containing the critical values of f. We obtain finitely many foliated rectangles Π_i such that

$$\sum \mathcal{W}(\Pi_i) = \mathcal{W}(V, B).$$

Each of these rectangles lifts to d properly embedded rectangles P_i^j in $U \smallsetminus A$ (with the horizontal sides on ∂U and ∂A). Moreover, $\mathcal{W}(P_i^j) = \mathcal{W}(\Pi_i)$. Hence

$$\mathcal{W}(U, A) \geq \sum \mathcal{W}(P_i^j) = d\, \mathcal{W}(V, B). \qquad \square$$

Remark. A similar estimate is still valid for an arbitrary compact set A, and can be proved by approximating A by archipelagos.

Putting the above two lemmas together (or using directly that the Dirichlet integral is transformed as the area under branched coverings) we obtain:

LEMMA 4.7. *Let (U, A) and (V, B) be as above, and let $f : U \smallsetminus A \to V \smallsetminus B$ be a branched covering of degree N. Then*

$$\operatorname{mod}(V, B) = N \operatorname{mod}(U, A).$$

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