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Curvature of vector bundles associated to holomorphic fibrations

By Bo Berndtsson

Abstract

Let L be a (semi)-positive line bundle over a Kähler manifold, X, fibered over a complex manifold Y. Assuming the fibers are compact and nonsingular we prove that the hermitian vector bundle E over Y whose fibers over points y are the spaces of global sections over X_y to $L \otimes K_{X/Y}$, endowed with the L^2-metric, is (semi)-positive in the sense of Nakano. We also discuss various applications, among them a partial result on a conjecture of Griffiths on the positivity of ample bundles.

1. Introduction

Let us first consider a domain $D = U \times \Omega$ in $\mathbb{C}^m \times \mathbb{C}^n$ and a function ϕ, plurisubharmonic in D. We also assume for simplicity that ϕ is smooth up to the boundary . Then, for each t in U, $\phi^t(\cdot) := \phi(t, \cdot)$ is plurisubharmonic in Ω and we denote by A^2_t the Bergman spaces of holomorphic functions in Ω with norm

$$\|h\|^2 = \|h\|^2_t = \int_\Omega |h|^2 e^{-\phi^t}.$$

The spaces A^2_t are then all equal as vector spaces but have norms that vary with t. The - infinite rank - vector bundle E over U with fiber $E_t = A^2_t$ is therefore trivial as a bundle but is equipped with a nontrivial metric. The first result of this paper is the following theorem.

Theorem 1.1. *If ϕ is (strictly) plurisubharmonic, then the hermitian bundle $(E, \|\cdot\|_t)$ is (strictly) positive in the sense of Nakano.*

Of the two main differential geometric notions of positivity (see §2, where these matters are reviewed in the slightly nonstandard setting of bundles of infinite rank), positivity in the sense of Nakano is the stronger one and implies the weaker property of positivity in the sense of Griffiths. On the other hand the Griffiths notion of positivity has nicer functorial properties and implies in particular that the dual bundle is negative (in the sense of Griffiths). This latter property is in turn equivalent to the condition that if ξ is any nontrivial local holomorphic section of the dual bundle, then the function

$$\log \|\xi\|^2$$

is plurisubharmonic. We can obtain such holomorphic sections to the dual bundle from point evaluations. More precisely, let f be a holomorphic map from U to Ω and define ξ_t by its action on a local section of E

$$\langle \xi_t, h_t \rangle = h_t(f(t)).$$

Since the right-hand side here is a holomorphic function of t, ξ is indeed a holomorphic section of E^*. The norm of ξ at a point is given by

$$\|\xi_t\|^2 = \sup_{\|h_t\| \leq 1} |h_t(f(t))|^2 = K_t(f(t), f(t)),$$

where $K_t(z, z)$ is the Bergman kernel function for A_t^2. It therefore follows from Theorem 1.1 that $\log K_t(z, z)$ is plurisubharmonic (or identically equal to $-\infty$) in D, which is the starting point of the results in [1]. The point here is of course that $\log K_t(z, z)$ is plurisubharmonic with respect to the parameter t, and even with respect to all the variables (t, z) jointly.

In [1] it is proved that this subharmonicity property of the Bergman kernel persists if D is a general pseudoconvex domain in $\mathbb{C}^m \times \mathbb{C}^n$, for general plurisubharmonic weight functions. In this case the spaces A_t^2 are the Bergman spaces for the slices of D, $D_t = \{z; (t, z) \quad \text{lies in} \quad D\}$. This more general case should also lead to a positively curved vector bundle. The main problem in proving such an extension of Theorem 1.1 is not to prove the inequalities involved, but rather to define the right notion of vector bundle in this case. In general, the spaces A_t^2 will not be identical as vector spaces, so the bundle in question is not locally trivial.

There is however a natural analog of Theorem 1.1 for holomorphic fibrations with compact fibers. Consider a complex manifold X of dimension $n+m$ which is smoothly fibered over another connected complex m-dimensional manifold Y. There is then a holomorphic map, p, from X to Y with surjective differential, and all the fibers $X_t = p^{-1}(t)$ are assumed compact. This implies, see [32], that the fibers are all diffeomorphic, but they are in general not biholomorphic to each other. We next need a substitute for the assumption on pseudoconvexity in Theorem 1.1. What first comes to mind is that X is a projective fibration, i.e. that there is a strictly positive line bundle on X. This would mean that X contains a divisor A such that $X \setminus A$ is Stein, so that we would be almost in the Stein case, which is quite similar to the case in Theorem 1.1. It turns out however that all we need to assume is that X is Kähler.

Let L be a holomorphic, hermitian line bundle over the total space X. Our substitute for the Bergman spaces A_t^2 is now the space of global sections over each fiber of $L \otimes K_{X_t}$,

$$E_t = \Gamma(X_t, L|X_t \otimes K_{X_t}),$$

where K_{X_t} is the canonical bundle of, i.e. the bundle of forms of bidegree $(n, 0)$ on, each fiber. We assume that L is semipositive so that the hermitian metric on L has nonnegative curvature form. Fix a point y in Y and choose local coordinates $t = (t_1, \dots t_m)$ near y with $t(y) = 0$. We consider the coordinates as functions on X by identifying t with $t \circ p$, and let $dt = dt_1 \wedge \dots dt_m$. The canonical bundle of a fiber X_t can then be identified with the restriction of K_X, the canonical bundle of the total space, to X_t by mapping a local section u to K_{X_t} to $u \wedge dt$. This map is clearly injective, and it is also surjective since any section w to K_X can locally be written

$$w = u \wedge dt,$$

and the restriction of u to X_t is independent of the choice of u. With this identification, any global holomorphic section of $L \otimes K_{X_t}$ over a fiber can be holomorphically extended to a holomorphic section of $L \otimes K_{X_s}$, for s near t. When L is trivial this follows from the Kähler assumption, by invariance of Hodge numbers; see [32]. When L is semipositive it follows from a variant of the Ohsawa-Takegoshi extension theorem, discussed in an appendix . Starting from a basis for $\Gamma(X_t, L|X_t \otimes K_{X_t})$ we therefore get a local holomorphic frame for E, so that E has a natural structure as a holomorphic vector bundle. Moreover, elements of E_t can be naturally integrated over the fiber and we obtain in this way a metric, $\|\cdot\|$ on E in complete analogy with the plane case. We then get the same conclusion as before:

THEOREM 1.2. *If the total space X is Kähler and L is (semi)positive over X, then $(E, \|\cdot\|)$ is (semi)positive in the sense of Nakano.*

This can be compared to results of Fujita, [13], Kawamata, [18], and Kollár, [20], who proved positivity properties for E when L is trivial. Kawamata also extended these results to multiples of the canonical bundle, and, in [19], to nontrivial L equipped with a special, singular, semipositive metric. Related work is also due to Tsuji, see [29] and the references therein. The proofs in these papers are based on results of Griffiths on variations of Hodge structures, whereas our proofs use techniques related to Hörmander-like L^2-estimates for $\bar\partial$. This seems to be the main novelty in our approach. Among the advantages are that this permits us to work directly in the twisted context (i.e. with nontrivial L), it gives Nakano positivity and not just Griffiths positivity and it also works in the noncompact case (like Theorem 1.1). Moreover it gives an explicit interesting lower bound for the curvature operator; see Section 6.

After a first version of this paper was posted on the ArXiv, Tsuji also announced a version of Theorem 1.2 in [30]. In this paper he indicates a proof that E is Griffiths positive, assuming the fibrations is projective, by a reduction to the case of a locally trivial fibration.

Theorems 1.1 and 1.2 have applications, or at least illustrations, in a number of different contexts. One concerns subharmonicity properties of Bergman kernels depending on a parameter, which follow from Theorem 1.1 as explained above; see [1]. This can be seen as a complex version of the Brunn-Minkowski inequality. The link to Brunn-Minkowski theory lies in the fact that the inverse of the volume of a domain in $\mathbb{R}^n$ is the Bergman kernel for the space of constant functions on the domain. The classical Brunn-Minkowski inequality is therefore a convexity property, with respect to parameters, of the Bergman kernel for the nullspace of the d-operator, whereas here we have plurisubharmonicity of the Bergman kernel for the nullspace of the $\bar\partial$-operator.

Just as in the case of Theorem 1.1, Theorem 1.2 also has as a consequence a result on plurisubharmonicity of a Bergman kernel. In this case however, the Bergman kernel is not a function, but transforms as a metric on the *relative canonical bundle* of the fibration

$$K_{X/Y} := K_X \otimes p^*(K_Y)^{-1},$$

twisted with L. In forthcoming work with Mihai Paun, [4] and [5], we show how Theorem 1.2 implies that this Bergman kernel has semipositive curvature or is identically equal to 0. In particular $L + K_{X/Y}$ is pseudoeffective if it has a nontrivial square integrable holomorphic section over at least one fiber. In [4], [5] we also extend this result to the case when L has a singular metric and the map p is not necessarily a smooth fibration. Moreover, these methods can be extended to twisted multiples of the canonical bundle, generalising to the twisted case the work of Kawamata quoted above.

Not surprisingly, the curvature of the bundle E in Theorem 1.2 can be zero at some point and in some direction only if the curvature of the line bundle L also degenerates. In Section 5 we prove a result that indicates that conditions for degeneracy of the curvature of E are much more restrictive than that: When X is a product, null vectors for the curvature can only come from infinitesimal automorphisms of the fiber.

In Section 6 we discuss some, largely philosophical, relations between Theorem 1.2 and recent work on the variation of Kähler metrics. This corresponds to the case when X is a product $U \times Z$ with one-dimensional base U, and when L is the pull-back of a bundle on Z under the second projection map. The variation of the metric on L that we get from the fibration then gives a path in the space of Kähler metrics on Z and the lower bound that we get for the curvature operator in this case is precisely the Toepliz operator defined by the geodesic curvature of this path. This theme is further developed in [3].

Another example of the situation in Theorem 1.2 arises naturally if we start with a (finite rank) holomorphic vector bundle V over Y and let $\mathbb{P}(V)$ be the associated bundle of projective spaces of the dual bundle V^*. This is then clearly an - even locally trivial - holomorphic fibration and there is a naturally defined line bundle L over the total space

$$L = O_{\mathbb{P}(V)}(1)$$

that restricts to the hyperplane section bundle over each fiber. The global holomorphic sections of this bundle over each fiber are now the linear forms on V^*, i.e. the elements of V. To obtain sections to our bundle E defined above, we take tensor products with the canonical bundle. We therefore replace L by

$$L^{r+1} = O_{\mathbb{P}(V)}(r+1)$$

(with r being the rank of V). Since the canonical bundle of a fiber is $O(-r)$ we see that on each fiber $L \otimes K_{X_t} = O(1)$ so that its space of global sections is again equal to V_t. Define as before

$$E_t = \Gamma(X_t, L^{r+1}|X_t \otimes K_{X_t}).$$

One can then verify that, globally, E is isomorphic to $V \otimes \det V$. The condition that L is positive is now equivalent to $O_{\mathbb{P}(V)}(1)$ being positive which is the same as saying that V is *ample* in the sense of Hartshorne, [16]. We therefore obtain (in §7) the following result as a corollary of Theorem 1.2.

THEOREM 1.3. *Let V be a (finite rank) holomorphic vector bundle over a complex manifold which is ample in the sense of Hartshorne. Then $V \otimes \det V$ has a smooth hermitian metric which is strictly positive in the sense of Nakano.*

Replacing $O_{\mathbb{P}(V)}(r+1)$ by $O_{\mathbb{P}(V)}(r+m)$, we also get that $S^m(V) \otimes \det V$ is Nakano-positive for any nonnegative m, where $S^m(V)$ is the m:th symmetric power of V.

It is a well known conjecture of Griffiths, [14], that an ample vector bundle is positive in the sense of Griffiths. Theorem 1.3 can perhaps be seen as indirect evidence for this conjecture, since by a a theorem of Demailly, [10], $V \otimes \det V$ is Nakano positive if V itself is Griffiths positive. It seems that not so much is known about Griffiths' conjecture in general, except that it does hold when Y is a compact curve (see [31], [7]).

After the first version of this mansucript was completed I received a preprint by C. Mourougane and S. Takayama, [25]. There they prove that $V \otimes \det V$ is positive in the sense of Griffiths, assuming the base manifold is projective. The method of proof is quite different from this paper, as is the metric they find.

We end this introduction with a brief discussion of the proofs. The proof of Theorem 1.1 is based on regarding the bundle E as a holomorphic subbundle of the hermitian bundle F with fibers

$$F_t = L^2(\Omega, e^{-\phi^t}) =: L^2_t.$$

By definition, the curvature of F is a $(1,1)$-form

$$\sum \Theta^F_{jk} dt_j \wedge d\bar{t}_k$$

whose coefficients are operators on F_t. By direct and simple computation,

$$\Theta^F$$

is the operator of multiplication with $\partial_t \bar{\partial}_t \phi$, so it is positive as soon as ϕ is plurisubharmonic of t for z fixed. By a formula of Griffiths, the curvature of the holomorphic subbundle E is obtained from the curvature of F by subtracting the *second fundamental form* of E, and the crux of the proof is to control this term by the curvature of F. For this we note that the second fundamental form is given by the square of the norm of an element in the orthogonal complement of A^2_t in L^2_t. This element is therefore the minimal solution of a certain $\bar{\partial}$-equation, and the needed inequality follows from an application of Hörmander's L^2-estimate.

We have not been able to generalize this proof to the situation of Theorem 1.2. The proof does generalize to the case of a holomorphically trivial fibration, but in the general case we have not been able to find a natural complex structure on the space of all (not necessarily holomorphic) $(n,0)$ forms, extending the complex structure on E. We therefore compute directly the Chern connection of the bundle E itself, and compute the curvature from there, much as one proves Griffiths' formula. In these computations appears also the Kodaira-Spencer class of the fibration, [32]. This class plays somewhat the role of another second fundamental form, but this time of a quotient bundle, arising when we restrict $(n,0)$-forms to the fiber. The Kodaira-Spencer class therefore turns out to give a positive contribution to the curvature. This proof could also be adapted to give Theorem 1.1 by using fiberwise complete Kähler metrics, but we have chosen not to do so since the first proof seems conceptually clearer.

Finally, I would like to thank Sebastien Boucksom for pointing out the relation between Theorem 1.1 and the Griffiths conjecture, Jean-Pierre Demailly for encouraging me to treat also the case of a general nontrivial fibration and Yum-Tong Siu and Mihai Paun for helpful discussions. Last but not least, thanks are due to Hiroshi Yamaguchi, whose work on plurisubharmonicity of the Robin function [33] and Bergman kernels, [24] was an important source of inspiration for this work.

2. Curvature of finite and infinite rank bundles

Let E be a holomorphic vector bundle with a hermitian metric over a complex manifold Y. By definition this means that there is a holomorphic projection map p from E to Y and that every point in Y has a neighbourhood U such that $p^{-1}(U)$ is isomorphic to $U \times W$, where W is a vector space equipped with a smoothly varying hermitian metric. In our applications it is important to be able to allow this vector space to have infinite dimension, in which case we assume that the metrics are also complete, so that the fibers are Hilbert spaces.

Let $t = (t_1, \dots, t_m)$ be a system of local coordinates on Y. The Chern connection, D_{t_j} is now given by a collection of differential operators acting on smooth sections to $U \times W$ and satisfying

$$\partial_{t_j}(u, v) = (D_{t_j} u, v) + (u, \bar\partial_{t_j} v),$$

with $\partial_{t_j} = \partial/\partial_{t_j}$ and $\bar\partial_{t_j} = \partial/\partial\bar t_j$. The curvature of the Chern connection is a $(1,1)$-form of operators

$$\Theta^E = \sum \Theta^E_{jk} dt_j \wedge d\bar t_k,$$

where the coefficients Θ^E_{jk} are densily defined operators on W. By definition these coefficients are the commutators

$$\Theta^E_{jk} = [D_{t_j}, \bar\partial_{t_k}].$$

The vector bundle is said to be positive in the sense of Griffiths if for any section u of E and any vector v in $\mathbb{C}^m$

$$\sum (\Theta^E_{jk} u, u) v_j \bar v_k \geq \delta \|u\|^2 |v|^2$$

for some positive δ. E is said to be positive in the sense of Nakano if for any m-tuple $(u_1, \dots, u_m)$ of sections of E

$$\sum (\Theta^E_{jk} u_j, u_k) \geq \delta \sum \|u_j\|^2.$$

Taking $u_j = u v_j$ we see that Nakano positivity implies positivity in the sense of Griffiths.

The dual bundle of E is the vector bundle E^* whose fiber at a point t in Y is the Hilbert space dual of E_t. There is therefore a natural antilinear isometry between E^* and E, which we will denote by J. If u is a local section of E, ξ is a local section of E^*, and $\langle \cdot, \cdot \rangle$ denotes the pairing between E^* and E, we have

$$\langle \xi, u \rangle = (u, J\xi).$$

Under the natural holomorphic structure on E^*,

$$\bar\partial_{t_j} \xi = J^{-1} D_{t_j} J \xi,$$

and the Chern connection on E^* is given by

$$D^*_{t_j}\xi = J^{-1}\bar\partial_{t_j} J\xi.$$

It follows that

$$\bar\partial_{t_j}\langle\xi, u\rangle = \langle\bar\partial_{t_j}\xi, u\rangle + \langle\xi, \bar\partial_{t_j}u\rangle,$$

and

$$\partial_{t_j}\langle\xi, u\rangle = \langle D^*_{t_j}\xi, u\rangle + \langle\xi, D_{t_j}u\rangle,$$

and hence

$$0 = \left[\partial_{t_j}, \bar\partial_{t_k}\right]\langle\xi, u\rangle = \langle\Theta^{E^*}_{jk}\xi, u\rangle + \langle\xi, \Theta^E_{jk}u\rangle,$$

if we let Θ^{E^*} be the curvature of E^*. If ξ_j is an r-tuple of sections to E^*, and $u_j = J\xi_j$, we thus see that

$$\sum(\Theta^{E^*}_{jk}\xi_j, \xi_k) = -\sum(\Theta^E_{jk}u_k, u_j).$$

Notice that the order between u_k and u_j on the right-hand side is opposite to the order between the ξ's on the left-hand side. Therefore E^* is negative in the sense of Griffiths if and only if E is positive in the sense of Griffiths, but we cannot draw the same conclusion in the the case of Nakano positivity.

If u is a holomorphic section of E we also find that

$$\frac{\partial^2}{\partial t_j\partial\bar t_k}(u, u) = (D_{t_j}u, D_{t_k}u) - (\Theta^E_{jk}u, u)$$

and it follows after a short computation that E is (strictly) negative in the sense of Griffiths if and only if $\log\|u\|^2$ is (strictly) plurisubharmonic for any nonvanishing holomorphic section u.

We next briefly recapitulate the Griffiths formula for the curvature of a subbundle. Assume E is a holomorphic subbundle of the bundle F, and let π be the fiberwise orthogonal projection from F to E. We also let $\pi_\perp$ be the orthogonal projection on the orthogonal complement of E. By the definition of Chern connection we have

$$D^E = \pi D^F.$$

Let $\bar\partial_{t_j}\pi$ be defined by

$$\bar\partial_{t_j}(\pi u) = (\bar\partial_{t_j}\pi)u + \pi(\bar\partial_{t_j}u). \tag{2.1}$$

Computing the commutators occurring in the definition of curvature we see that

$$\Theta^E_{jk}u = -(\bar\partial_{t_k}\pi)D^F_{t_j}u + \pi\Theta^F_{jk}u, \tag{2.2}$$

if u is a section of E. By (2.1) $(\bar\partial\pi)v = 0$ if v is a section of E, so that

$$(\bar\partial\pi)D^F u = (\bar\partial\pi)\pi_\perp D^F u. \tag{2.3}$$

Since $\pi\pi_\perp = 0$ it also follows that

$$(\bar\partial\pi)\pi_\perp D^F u = -\pi\bar\partial(\pi_\perp D^F u).$$

Hence, if v is also a section of E,

$$\begin{aligned}((\bar\partial_{t_k}\pi)D^F_{t_j}u, v) &= -(\bar\partial_{t_k}(\pi_\perp D^F_{t_j}u), v)\\ &= ((\pi_\perp D^F_{t_j}u), D^F_{t_k}v) = (\pi_\perp(D^F_{t_j}u), \pi_\perp(D^F_{t_k}v)).\end{aligned}$$

Combining with (2.2) we finally get that if u and v are both sections to E then

$$(\Theta^F_{jk}u, v) = (\pi_\perp(D^F_{t_j}u), \pi_\perp(D^F_{t_k}v)) + (\Theta^E_{jk}u, v), \tag{2.4}$$

and thus

$$\sum(\Theta^F_{jk}u_j, u_k) = \|\pi_\perp \sum D^F_{t_j}u_j\|^2 + \sum(\Theta^E_{jk}u_j, u_k).$$

which is the starting point for the proof of Theorem 1.1.

For the proof of Theorem 1.2 we finally describe another way of computing the curvature form of a vector bundle. Fix a point y in Y and choose local coordinates t centered at y. Any point u_0 in the fiber E_0 over y can be extended to a holomorphic section u of E near 0. Modifying u by a linear combination $\sum t_j v_j$ for suitably chosen local holomorphic sections v_j we can also arrange things so that $Du = 0$ at $t = 0$. Let u and v be two local sections with this property and compute

$$\partial_{\bar t_k}\partial_{t_j}(u, v) = \partial_{\bar t_k}(D_{t_j}u, v) = (\partial_{\bar t_k}D_{t_j}u, v) = -(\Theta^E_{jk}u, v).$$

Let u_j be an m-tuple of holomorphic sections to E, satisfying $Du_j = 0$ at 0. Put

$$T_u = \sum(u_j, u_k)\widehat{dt_j \wedge d\bar t_k}.$$

Here $\widehat{dt_j \wedge d\bar t_k}$ denotes the wedge product of all dt_i and $d\bar t_i$ except dt_j and $d\bar t_k$, multiplied by a constant of absolute value 1, chosen so that T_u is a positive form. Then

$$i\partial\bar\partial T_u = -\sum(\Theta^E_{jk}u_j, u_k)dV_t, \tag{2.5}$$

so that E is Nakano-positive at a given point if and only if this expression is negative for any choice of holomorphic sections u_j satisfying $Du_j = 0$ at the point.

3. The proof of Theorem 1.1

We consider the setup described before the statement of Theorem 1.1 in the introduction. Thus E is the vector bundle over U whose fibers are the Bergman spaces A^2_t equipped with the weighted L^2 metrics induced by $L^2(\Omega, e^{-\phi^t})$. We also let F be the vector bundle with fiber $L^2(\Omega, e^{-\phi^t})$, so that

E is a trivial subbundle of the trivial bundle F with a metric induced from a nontrivial metric on F. From the definition of the Chern connection we see that

$$D^F_{t_j} = \partial_{t_j} - \phi_j,$$

where the last term on the right-hand side should be interpreted as the operator of multiplication by the (smooth) function $-\phi_j = -\partial_{t_j}\phi^t$. (In the sequel we use the letters j, k for indices of the t-variables, and the letters λ, μ for indices of the z-variables.) For the curvature of F we therefore get

$$\Theta^F_{jk} = \phi_{jk},$$

the operator of multiplication with the complex Hessian of ϕ with respect to the t-variables. We shall now apply formula (2.4), and so we let u_j be smooth sections of E. This means that u_j are functions that depend smoothly on t and holomorphically on z. To verify the positivity of E in the sense of Nakano we need to estimate from below the curvature of E acting on the k-tuple u,

$$\sum (\Theta^E_{jk} u_j, u_k).$$

By (2.4) this means that we need to estimate from above

$$\sum (\pi_\perp(\phi_j u_j), \pi_\perp(\phi_k u_k)) = \|\pi_\perp(\sum \phi_j u_j)\|^2.$$

Put $w = \pi_\perp(\sum \phi_j u_j)$. For fixed t, w solves the $\bar\partial_z$-equation

$$\bar\partial w = \sum u_j \phi_{j\lambda} d\bar z_\lambda,$$

since the u_j's are holomorphic in z. Moreover, since w lies in the orthogonal complement of A^2, w is the minimal solution to this equation.

We shall next apply Hörmander's weighted L^2-estimates for the $\bar\partial$-equation. The precise form of these estimates that we need says that if f is a $\bar\partial$-closed form in a psedudoconvex domain Ω, and if ψ is a smooth strictly plurisubharmonic weight function, then the minimal solution w to the equation $\bar\partial v = f$ satisfies

$$\int_\Omega |w|^2 e^{-\psi} \leq \int_\Omega \sum \psi^{\lambda\mu} f_\lambda \bar f_\mu e^{-\psi},$$

where $(\psi^{\lambda\mu})$ is the inverse of the complex Hessian of ψ (see [9]).

In our case this means that

$$\int_\Omega |w|^2 e^{-\phi^t} \leq \int_\Omega \sum \phi^{\lambda\mu} \phi_{j\lambda} u_j \overline{\phi_{k\mu} u_k} e^{-\phi^t}.$$

Inserting this estimate in formula (2.4) together with the formula for the curvature of F we find

$$\sum (\Theta^E_{jk} u_j, u_k) \geq \int_\Omega \sum_{jk} \left(\phi_{jk} - \sum_{\lambda\mu} \phi^{\lambda\mu} \phi_{j\lambda} \bar\phi_{k\mu} \right) u_j \bar u_k e^{-\phi^t}. \tag{3.1}$$

We claim that the expression

$$D_{jk} =: \left(\phi_{jk} - \sum_{\lambda\mu} \phi^{\lambda\mu} \phi_{j\lambda} \bar{\phi}_{k\mu} \right),$$

in the integrand is a positive definite matrix at any fixed point . In the proof of this, we may by a linear change of variables in t, of course, assume that the vector u that D acts on equals $(1, 0 \ldots 0)$. The positivity of D then follows from a computation in [27], but we will give the short argument here too. Let $\Phi = i\partial\bar{\partial}\phi$ where the $\partial\bar{\partial}$-operator acts on t_1 and the z-variables, the remaining t-variables being fixed. Then

$$\Phi = \Phi_{11} + i\alpha \wedge d\bar{t}_1 + i dt_1 \wedge \bar{\alpha} + \Phi',$$

where Φ_{11} is of bidegree $(1,1)$ in t_1, α is of bidegree $(1,0)$ in z, and Φ' is of bidegree $(1,1)$ in z. Then

$$\Phi_{n+1} = \Phi^{n+1}/(n+1)! = \Phi_{1,1} \wedge \Phi'_n - i\alpha \wedge \bar{\alpha} \wedge \Phi'_{n-1} \wedge i dt_1 \wedge d\bar{t}_1.$$

Both sides of this equation are forms of maximal degree that can be written as certain coefficients multiplied by the Euclidean volume form of $\mathbb{C}^{n+1}$. The coefficient of the left-hand side is the determinant of the complex Hessian of ϕ with respect to t_1 and z together. Similarily, the coefficient of the first term on the right-hand side is ϕ_{11} times the Hessian of ϕ with respect to the z-variables only. Finally, the coefficient of the last term on the right-hand side is the norm of the $(0,1)$ form in z,

$$\bar{\partial}_z \partial_{t_1} \phi,$$

measured in the metric defined by Φ', multiplied by the volume form of the same metric. Dividing by the coefficient of Φ'_n we thus see that the matrix D acting on a vector u as above equals the Hessian of ϕ with respect to t_1 and z divided by the Hessian of ϕ with respect to the z-variables only. This expression is therefore positive so that the proof of Theorem 1.1 is complete.

4. Kähler fibrations with compact fibers

Let X be a Kähler manifold of dimension $m + n$, fibered over a complex m-dimensional manifold Y. This means that we have a holomorphic map p from X to Y with surjective differential at all points. All our computations will be local, and so we may as well assume that $Y = U$ is a ball or polydisk in $\mathbb{C}^m$. For each t in U we let

$$X_t = p^{-1}(t)$$

be the fiber of X over t. We shall assume that all fibers are compact.

Next, we let L be a holomorphic hermitian line bundle over X. Our standing assumption on L is that it is semipositive, i.e. that it is equipped with a smooth hermitian metric of nonnegative curvature. For each fiber X_t we are interested in the space of holomorphic L-valued $(n,0)$-forms on X_t,

$$\Gamma(X_t, L|_{X_t} \otimes K_{X_t}) =: E_t.$$

For each t, E_t is a finite-dimensional vector space and we claim that

$$E := \bigcup \{t\} \times E_t$$

has a natural structure as a holomorphic vector bundle.

To see this we need to study how E_t varies with t. First note that K_{X_t} is isomorphic to $K_X|_{X_t}$, the restriction of the canonical bundle of the total space to X_t, via the map that sends a section u to K_{X_t} to

$$\tilde{u} := u \wedge dt,$$

where $dt = dt_1 \wedge \dots dt_m$. It is clear that this map is injective. Conversely, any local section $\tilde{u}$ to K_X can be locally represented as $\tilde{u} := u \wedge dt$, and even though u is not uniquely determined, the restriction of u to each fiber is uniquely determined. We thus have two ways of thinking of an element u in E_t: as a holomorphic L-valued $(n,0)$-form on X_t or as a section $\tilde{u} = u \wedge dt$ of K_X over X_t. It turns out to be convenient for the computations later on to have yet another interpretation: as the restriction of and $(n,0)$-form, u', on X to X_t (here we understand by restriction the pullback to X_t under the inclusion map from X_t to X). Clearly, u' is not uniquely determined by u. Indeed u' restricts to 0 on X_{t_0} precisely when $u' \wedge dt = 0$ vanishes for $t = t_0$, which in turn is equivalent to saying that

$$u' = \sum \gamma_j \wedge dt_j.$$

We will refer to a choice of u' as a representative of u. When t varies, a smooth section of E is then represented by a smooth $(n,0)$-form on X. To avoid cumbersome notation we will in the sequel use the same letter to denote an element in E_t and any representative of it.

The semipositivity of L, and the assumption that X is Kähler, implies that any holomorphic section u to K_{X_t} for one fixed t can be locally extended in the sense that there is a holomorphic section $\tilde{u}$ to K_X over $p^{-1}(W)$ for some neighbourhood W of t whose restriction to X_t maps to u under the isomorphism above. In case L is trivial this follows from the fact that Hodge numbers are locally constant; see [32]. For general semipositive bundles L it follows from a result of Ohsawa-Takegoshi type, to be discussed in an appendix.

Taking a basis for E_t for one fixed t and extending as above we therefore get a local frame for the bundle E. We define a complex structure on E by saying that an $(n,0)$-form over $p^{-1}(W)$, u, whose restriction to each fiber is holomorphic, defines a holomorphic section of E if $u \wedge dt$ is a holomorphic section of K_X. The frame we have constructed is therefore holomorphic.

Note that this means that u is holomorphic if and only if $\bar\partial u \wedge dt = 0$, which means that $\bar\partial u$ can be written

$$\bar\partial u = \sum \eta^j \wedge dt_j, \tag{4.1}$$

with η^j smooth forms of bidegree $(n-1,1)$. Again, the η^j are not uniquely determined, but their restrictions to fibers are.

Remark. Even though we will not use it, it is worth mentioning the connection between the forms η_j and the Kodaira-Spencer map of the fibration; see [32], [21].

The Kodaira-Spencer map at a point t in the base, is a map from the holomorphic tangent space of U to the first Dolbeault cohomology group,

$$H^{0,1}(X_t, T^{1,0}(X_t)),$$

of X_t with values in the holomorphic tangent space of X_t, i.e., as t varies it is a $(1,0)$-form, $\sum \theta_j dt_j$ on U with values in $H^{0,1}(X_t, T^{1,0}(X_t))$.The classes θ_j can be represented by $\bar\partial$-closed $(0,1)$-forms, ϑ_j, on X_t whose coefficients are vector fields of type $(1,0)$ tangent to the fiber. Such representatives can be found as follows. Let V_j be some choice of smooth $(1,0)$ vector fields on X, such that $dp(V_j) = \partial/\partial t_j$. Then $dp(\bar\partial V_j) = 0$ so that the $\bar\partial V_j$ are $\bar\partial$-closed forms with values in the bundle of vectors tangent to fibers. It is not hard to check that they represent the classes θ_j, by use of the definition in [21].

Letting the vectorfield in the coefficients of ϑ_j act on forms by contraction, we obtain maps

$$v \mapsto \vartheta_j \rfloor v$$

from (p,q)-forms on X_t to $(p-1,q+1)$-forms. We claim that the forms

$$\eta^j,$$

restricted to fibers X_t, is what we obtain when we let these map operate on u. Different representatives of u correspond to different representatives of the same cohomology class.

To prove the claim, we need to verify that $\eta_j = \vartheta_j \rfloor u$ on each fiber, where ϑ_j is some representative on X_t of the class θ_j. Let $\widehat{dt_j}$ be the wedge product of all differentials dt_k, except dt_j, with the right ordering, and let $\tilde u = u \wedge dt$. Then

$$V_j \rfloor \tilde u = (V_j \rfloor u) \wedge dt + u \wedge \widehat{dt_j}.$$

Hence

$$\bar\partial V_j \rfloor \tilde u = (\bar\partial(V_j \rfloor u)) \wedge dt + \eta_j \wedge dt.$$

Since $\bar\partial V_j \rfloor dt = \bar\partial(V_j \rfloor dt) = 0$, it follows that

$$(\bar\partial V_j \rfloor u) \wedge dt = (\bar\partial(V_j \rfloor u)) \wedge dt + \eta_j \wedge dt.$$

Therefore

$$\bar\partial V_j \rfloor u = \bar\partial(V_j \rfloor u) + \eta_j$$

on fibers, which proves our claim. □

Let u be a smooth local section of E. This means that u can be represented by a smooth L-valued form of bidegree $(n, 0)$ over $p^{-1}(W)$ for some W open in U, such that the restriction of u to each fiber is holomorphic. Then $\bar\partial u \wedge dt \wedge d\bar t = 0$, so that

$$\bar\partial u = \sum d\bar t_j \wedge \nu^j + \sum \eta_j \wedge dt_j,$$

where the ν^j define sections to E. We define the $(0, 1)$-part of the connection D on E by letting

$$D^{0,1} u = \sum \nu^j d\bar t_j.$$

Sometimes we write

$$\nu^j = \bar\partial_{t_j} u$$

with the understanding that this refers to the $\bar\partial$ operator on E. Note that $D^{0,1} u = 0$ for $t = t_0$ if and only if each ν^j vanishes when restricted to X_{t_0}, i.e. if $\bar\partial u \wedge dt = 0$, which is consistent with the definition of holomorphicity given earlier. Note also that if we choose another $(n, 0)$-form u' to represent the same section of E, then $u - u'$ vanishes when restricted to each fiber. Hence $u - u' = \sum a_j \wedge dt_j$ and it follows that $D^{0,1}$ is well defined.

The bundle E has a naturally defined hermitian metric, induced by the metric on L. To define the metric, let u_t be an element of E_t. Locally, with respect to a local trivialization of L, u_t is given by a scalar valued $(n, 0)$-form, u', and the metric on L is given by a smooth weight function ϕ'. Put

$$[u_t, u_t] = c_n u' \wedge \overline{u'} e^{-\phi'},$$

with $c_n = i^{n^2}$ chosen to make this (n, n)-form positive. Clearly this definition is independent of the trivialization, so that $[u_t, u_t]$ is globally defined. The metric on E_t is now defined as

$$\|u_t\|^2 = \int_{X_t} [u_t, u_t],$$

and the associated scalar product is

$$(u_t, v_t)_t = \int_{X_t} [u_t, v_t]. \tag{4.2}$$

In the sequel we will, abusively, write $[u, v] = c_n u \wedge \bar v e^{-\phi}$. When t varies we suppress the dependence on t and get a smooth hermitian metric on E. For local sections u and v to E the scalar product is then a function of t and it will be convenient to write this function as

$$(u, v) = p_*([u, v]) = p_*(c_n u \wedge \bar v e^{-\phi}),$$

where u and v are forms on X that represent the sections. Here p_* denotes the direct image, or push-forward, of a form, defined by

$$\int_U p_*(\alpha) \wedge \beta = \int_X \alpha \wedge p^*(\beta),$$

if α is a form on X and β is a form on U.

With the metric and the $\bar\partial$ operator defined on E we can now proceed to find the $(1,0)$-part of the Chern connection. Let u be a form on X with values in L. Locally, with respect to a trivialization of L, u is given by a scalar-valued form u' and the metric on L is given by a function ϕ'. Let

$$\partial^{\phi'} u' = e^{\phi'} \partial(e^{-\phi'} u').$$

One easily verifies that this expression is invariantly defined, and we will, somewhat abusively, write $\partial^{\phi'} u' = \partial^{\phi} u$, using ϕ to indicate the metric on L. In particular, let u be of bidegree $(n,0)$ such that the restrictions of u to fibers are holomorphic. As $\partial^\phi u$ is of bidegree $(n+1,0)$ we can write

$$\partial^\phi u = \sum dt_j \wedge \mu^j,$$

where μ^j are smooth $(n,0)$-forms whose restrictions to fibers are uniquely defined. These restrictions are in general not holomorphic and so we let

$$P(\mu^j)$$

be the orthogonal projection of μ^j on the space of holomorphic forms on each fiber.

LEMMA 4.1. *The* $(1,0)$*-part of the Chern connection on* E *is given by*

$$D^{1,0} u = \sum P(\mu^j) dt_j.$$

Proof. Even though it will follow implicitly from the proof below, we will first prove that $D^{1,0}u$ is well defined, i.e. independent of the choice of a representative of u. Let therefore u be a form that restricts to 0 on X_t for t in some open set. Then we can write $u = \sum \gamma_j \wedge dt_j$ there. Hence

$$\partial^\phi u = \sum \partial^\phi \gamma_j \wedge dt_j,$$

so that $\mu_j = \partial^\phi \gamma_j$ on any fiber. It then follows from the definition of the scalar product on X_t, and Stokes theorem, that μ_j is orthogonal to all holomorphic forms on the fiber. In other words, $P(\mu_j) = 0$, so that $D^{1,0}u = 0$, which is what we wanted to prove.

To prove the lemma it suffices, by the definition of Chern connection, to verify that

$$\partial_{t_j}(u,v) = (P(\mu^j), v) + (u, \bar\partial_{t_j} v) = (\mu^j, v) + (u, \bar\partial_{t_j} v) \tag{4.3}$$

if u and v are smooth sections to E. But

$$\begin{aligned}\partial(u,v) = \partial p_*([u,v]) &= c_n(p_*(\partial^\phi u\wedge\bar v\, e^{-\phi}) + (-1)^n p_*(u\wedge\overline{\bar\partial v}\, e^{-\phi}))\\ &= c_n(p_*(\sum \mu_j\wedge\bar v\wedge dt_j\, e^{-\phi}) + p_*(u\wedge\bar\nu^j\wedge dt_j\, e^{-\phi})).\end{aligned}$$

This equals

$$\sum((\mu^j,v)+(u,\nu^j))dt_j,$$

and so we have proved 4.2. □

We will write $P(\mu^j) = D_{t_j}u$. We are now ready to verify the Nakano positivity of the bundle E. For this we will use the recipe given at the end of Section 2. Let u_j be an m-tuple of holomorphic sections to E that satisfy $D^{1,0}u_j = 0$ at a given point that we take to be equal to 0. Let

$$T_u = \sum(u_j,u_k)\widehat{dt_j\wedge d\bar t_k}.$$

Here $\widehat{dt_j\wedge d\bar t_k}$ denotes the product of all differentials dt_i and $d\bar t_i$, except dt_j and $d\bar t_k$ multiplied by a number of modulus 1, so that T_u is nonnegative. We need to verify that

$$i\partial\bar\partial T_u$$

is negative. Represent the u_j's by smooth forms on X, and put

$$\hat u = \sum u_j\wedge\widehat{dt_j}.$$

Then, with $N = n+m-1$,

$$T_u = c_N p_*(\hat u\wedge\bar{\hat u}\, e^{-\phi}).$$

Thus

$$\bar\partial T_u = c_N(p_*(\bar\partial\hat u\wedge\bar{\hat u}e^{-\phi}) + (-1)^N p_*(\hat u\wedge\overline{\partial^\phi\hat u}e^{-\phi})).$$

Since each u_j is holomorphic we have seen that

$$\bar\partial u_j = \sum\eta_j^l\wedge dt_l.$$

Therefore each term in the form

$$p_*(\bar\partial\hat u\wedge\bar{\hat u}e^{-\phi})$$

contains a factor dt. On the other hand, the push forward of an $(n+m-1, n+m)$-form is of bidegree $(m-1,m)$, and so we conclude that

$$p_*(\bar\partial\hat u\wedge\bar{\hat u}e^{-\phi}) = 0.$$

Thus

$$\partial\bar\partial T_u = c_N((-1)^N p_*(\partial^\phi\hat u\wedge\overline{\partial^\phi\hat u}e^{-\phi}) + p_*(\hat u\wedge\overline{\bar\partial\partial^\phi\hat u})).$$

We rewrite the last term, using

$$\bar\partial\partial^\phi + \partial^\phi\bar\partial = \partial\bar\partial\phi.$$

Since

$$p_*(\hat{u} \wedge \overline{\bar\partial \hat{u}} e^{-\phi})$$

vanishes identically we find that

$$(-1)^N p_*(\hat{u} \wedge \overline{\partial^\phi \bar\partial \hat{u}} e^{-\phi}) + p_*(\bar\partial \hat{u} \wedge \overline{\bar\partial \hat{u}} e^{-\phi}) = 0,$$

so all in all

$$\partial\bar\partial T_u = c_N \Big((-1)^N p_*(\partial^\phi \hat{u} \wedge \overline{\partial^\phi \hat{u}} e^{-\phi}) - p_*(\hat{u} \wedge \overline{\hat{u}} \wedge \partial\bar\partial\phi e^{-\phi}) + (-1)^N p_*(\bar\partial \hat{u} \wedge \overline{\bar\partial \hat{u}} e^{-\phi}) \Big). \tag{4.4}$$

So far, the computations hold for any choice of representative of our sections u_j. We shall next choose our representatives in a careful way.

PROPOSITION 4.2. *Let u be a section of E over an open set U containing the origin, such that*

$$D^{0,1} u = 0,$$

in U and

$$D^{1,0} u = 0$$

at $t = 0$. Then u can be represented by a smooth $(n, 0)$-form, still denoted u, such that

$$\bar\partial u = \sum \eta^k \wedge dt_k, \tag{4.5}$$

where η^k is **primitive** *on X_0, i.e. satisfies $\eta^k \wedge \omega = 0$ on X_0, and furthermore*

$$\partial^\phi u \wedge \widehat{dt_j} = 0, \tag{4.6}$$

at $t = 0$ for all j.

To prove the proposition we need two lemmas.

LEMMA 4.3. *Let u be an $(n, 0)$-form on X, representing a holomorphic section of E, and write*

$$\bar\partial u = \sum \eta^k \wedge dt_k.$$

Then $\eta^k \wedge \omega$ are $\bar\partial$-exact on any fiber.

Proof. Since $u \wedge \omega$ is of bidegree $(n + 1, 1)$ we can write locally

$$u \wedge \omega = \sum u^k \wedge dt_k.$$

The coefficients u^k here are not unique, but their restrictions to fibers are unique. This follows since $\sum u^k \wedge dt_k = 0$ implies

$$\sum u^k \wedge dt_k \wedge \widehat{dt_k} = u^k \wedge dt = 0,$$

which implies that u^k vanishes when restricted to any fiber.

Hence in particular the u^k are well defined global forms on any fiber. Moreover

$$\sum \eta^k \wedge \omega \wedge dt_k = \bar\partial u \wedge \omega = \sum \bar\partial u^k \wedge dt_k,$$

so that

$$\sum (\eta^k \wedge \omega - \bar\partial u^k) \wedge dt_k = 0.$$

Again, wedging with $\widehat{dt}_k$, we see that

$$\eta^k \wedge \omega = \bar\partial u^k$$

on fibers, so that $\eta^k \wedge \omega$ is exact on fibers. □

LEMMA 4.4. *Let μ be an $(n, 0)$-form on a compact n-dimensional Kähler manifold Z, with values in a hermitian holomorphic line bundle L. Assume μ is orthogonal to the space of holomorphic L-valued forms under the scalar product* (4.2). *Let ξ be a $\bar\partial$-exact $(n, 2)$-form on Z, with values in L. Then there is an L-valued form γ of bidegree $(n-1, 0)$ such that*

$$\partial^\phi \gamma = \mu,$$

and

$$\bar\partial \gamma \wedge \omega = \xi.$$

Proof. Since ξ is exact we can solve $\bar\partial \chi = \xi$. Then $\mu - \bar\partial^* \chi$ is orthogonal to holomorphic forms, and so we can solve

$$\bar\partial^* \alpha = \mu - \bar\partial^* \chi,$$

with $\bar\partial \alpha = 0$. This follows since the range of $\bar\partial^*$ is closed on a compact manifold. When $\alpha' = \alpha + \chi$,

$$\mu = \bar\partial^* \alpha'$$

and $\bar\partial \alpha' = \xi$. Write $\alpha' = \gamma \wedge \omega$, where ω is the Kahler form. Then γ satisfies the conditions in the lemma (possibly up to a sign). □

We are now ready to prove Proposition 4.2.

Recall that u can in any case be represented by a form satisfying

$$\bar\partial u = \sum \eta^k \wedge dt_k,$$

over U and

$$\partial^\phi u = \sum \mu^k \wedge dt_k,$$

where the restriction of μ^k is orthogonal to holomorphic forms on X_0. By Lemmas 4.3 and 4.4 there are forms γ^k on X_0 such that

$$\mu^k = \partial^\phi \gamma^k$$

and

$$\bar\partial \gamma^k \wedge \omega = \eta^k \wedge \omega$$

on X_0. Extend γ^k smoothly to a neighbourhood of X_0 (i.e. find a form that restricts to γ^k), and put

$$u' = u - \sum \gamma^k \wedge dt_k.$$

Then u' is a form of bidegree $(n, 0)$ that represents the same section of E as u. Then

$$\partial^\phi u' \wedge \widehat{dt}_j = \partial^\phi u \wedge \widehat{dt}_j - \partial^\phi \gamma^j \wedge dt = (\mu_j - \partial^\phi \gamma^j) \wedge dt = 0$$

at $t = 0$, since

$$\mu_j = \partial^\phi \gamma^j$$

on X_0. Moreover

$$\bar\partial u' = \sum (\eta^k - \bar\partial \gamma^k) \wedge dt_k,$$

and

$$((\eta^k - \bar\partial \gamma^k) \wedge \omega = 0$$

on X_0. Hence u' satisfies all the requirements and the propositions is proved.

We now return to the proof of Theorem 1.2. Note that with the choice of representatives of our sections u_j furnished by Proposition 4.2, formula 4.4 simplifies at $t = 0$ to

$$\partial\bar\partial T_u = c_N \left(-p_*(\hat u \wedge \overline{\hat u} \wedge \partial\bar\partial \phi e^{-\phi}) + (-1)^N p_*(\bar\partial \hat u \wedge \overline{\bar\partial \hat u} e^{-\phi}) \right). \tag{4.7}$$

The first term on the right-hand side obviously gives a (semi)negative contribution to $i\partial\bar\partial T$. To analyse the last term write

$$\bar\partial u_j = \sum \eta_j^k \wedge dt_k,$$

and

$$\bar\partial \hat u = \sum \eta_j^j \wedge dt =: \eta \wedge dt.$$

Then the last term equals

$$c_n \int_{X_0} \eta \wedge \bar\eta \; e^{-\phi} \, dV_t.$$

In general the quadratic form in η appearing here is indefinite. In our case however, all the η_j^k are primitive on X_0, and it is well known that

$$c_n \eta \wedge \bar\eta = -|\eta|^2$$

if η is primitive. (This is easily checked by hand at a point by choosing coordinates that are orthogonal at the point.) Hence

$$c_n \int_{X_0} \eta \wedge \bar\eta e^{-\phi} \, dV_t = -\int_{X_0} |\eta|^2 \, dV_t.$$

and we get

$$i\partial\bar{\partial}T = c_N\left(-p_*(\hat{u}\wedge\bar{\hat{u}}\wedge i\partial\bar{\partial}\phi e^{-\phi})\right) - \int_{X_0}|\eta|^2\,dV_t. \tag{4.8}$$

By 4.4, this means that $i\partial\bar{\partial}T_u \leq 0$, and so E is at least seminegative in the sense of Nakano. If $i\partial\bar{\partial}\phi$ is strictly positive, it is clear that the curvature term alone gives a strictly negative contribution to $i\partial\bar{\partial}T$. Therefore E is strictly positive if L is strictly positive and so we have proved Theorem 1.2. In the next section we shall see that even when L is only semipositive, equality can hold in our estimates only in very special cases.

We want to add one remark on the relation between the proof of Theorem 1.2 in this section and the proof of Theorem 1.1 in Section 3. The proof in Section 3 is easily adapted to the case of a trivial fibration (so that X is a global product). It may then seem that the proof here is quite different since it does not use the Hörmander-Kodaira L^2-estimates at all. The two proofs are however really quite similar, the difference being that in this section we basically reprove the special case of the L^2-estimates that we need as we go along.

5. Semipositive vector bundles

In this section we will discuss when equality holds in the inequalities of Theorem 1.2, i.e. when the bundle E is not strictly positive. As we have already seen in the last section, this can only happen if the line bundle L is not strictly positive. More precisely, provided the components u_j of $\hat{u}$ are chosen to satisfy the conditions in Proposition 4.2, equality holds if and only if

$$\eta = 0$$

and

$$\hat{u}\wedge\bar{\hat{u}}\wedge i\partial\bar{\partial}\phi = 0.$$

Since $i\partial\bar{\partial}\phi \geq 0$, the last condition is equivalent to

$$\hat{u}\wedge i\partial\bar{\partial}\phi = 0. \tag{5.1}$$

For simplicity we assume from now on that the base domain U is one-dimensional, so that we do not need to discuss degeneracy in different directions, and that the fibration we consider is locally holomorphically trivial, i.e. that $X = U \times Z$, where Z is a compact n-dimensional complex manifold. Moreover, we assume that the curvature of our metric ϕ on L, Θ^L, is strictly positive along each fiber $X_t \simeq Z$. Then we can also assume that we have chosen our Kähler metric ω on X so that $\omega_t := \omega|X_t = i\Theta^L|X_t$ on each fiber.

Since X now is a global product we can decompose Θ^L according to its degree in t and z, where z is any local coordinate on Z. In particular, there is a well defined $(0,1)$-form θ^L on X such that $dt \wedge \theta^L$ is the component of Θ^L of degree 1 in dt. Expressed in invariant language,

$$\theta^L = \delta_{\partial/\partial t}\Theta^L,$$

where $\delta.$ means contraction with a vector field. By the formulas

$$\partial\delta_V + \delta_V\partial = \mathcal{L}_V, \tag{5.2}$$

where $\mathcal{L}$ is the holomorphic Lie derivative, and

$$\bar\partial\delta_V + \delta_V\bar\partial = 0, \tag{5.3}$$

if V is a holomorhic vector field we see that θ^L is $\bar\partial$-closed on X, and that

$$\partial\theta^L = \mathcal{L}_{\partial/\partial t}\Theta^L$$

on X. On the other hand, on each fiber X_t there is a unique $(1,0)$ vector field V_t, defined by

$$\delta_{V_t}\omega_t = \theta^L.$$

Our key obeservation is contained in the next lemma.

LEMMA 5.1. *Assume that for some $u \neq 0$ in E_0, $(\Theta^E u, u) = 0$. Then V_0 is a holomorphic vector field on X_0.*

Proof. Since the base U is one dimensional, $\hat{u} = u$ and since $X = Z \times U$ is a global product we can decompose

$$u = u_0 + dt \wedge v,$$

where u_0 does not contain dt, i.e. $\delta_{\partial/\partial t}u_0 = 0$. Here u is chosen to satisfy the conditions of Proposition 4.2,

$$\bar\partial u = \eta \wedge dt,$$

where $\eta \wedge \omega = 0$ on X_0, and

$$\partial^\phi u = 0$$

for $t = 0$. Since u is holomorphic on X_0, $\bar\partial u_0 = 0$, so that $\eta = \bar\partial v$ on X_0. But we have seen above that if u is a null vector for the curvature of E at $t = 0$, then $\eta = 0$ so the restriction of v to X_0 is a holomorphic $(n-1,0)$-form.

We also know from (5.1) that

$$u \wedge \Theta^L = 0$$

for $t = 0$. Applying $\delta_{\partial/\partial t}$, we get

$$v \wedge \Theta^L + (-1)^n u \wedge \theta^L = 0$$

for $t = 0$. Restriction to X_0 yields

$$v \wedge \Theta^L = (-1)^{n+1} u \wedge \delta_{V_0}\Theta^L$$

on X_0. But, $u \wedge \Theta^L = 0$ on X_0 (for reasons of bidegree) so that

$$\delta_{V_0} u \wedge \Theta^L + (-1)^n u \wedge \delta_{V_0} \Theta^L = 0.$$

Hence v equals $\delta_{V_0} u$ on X_0, so $\delta_{V_0} u$ is a holomorphic form on X_0. Since u is also holomorphic it follows that V_0 must be holomorphic too, except possibly where u vanishes. But since V is smooth, V must actually be holomorphic everywhere by Riemann's theorem on removable singularities. □

This lemma could also have been proved using the approach via L^2-estimates. It is then strongly related to the following proposition that we state explicitly since we feel it has independent interest.

PROPOSITION 5.2. *Let L be a positive line bundle over a compact complex manifold Z. Give Z the Kähler metric defined by the curvature form of L. Let μ be the L^2-minimal solution to $\bar\partial\mu = f$, where f is an L-valued $(n,1)$-form on Z. Then equality holds in Hörmander's estimate*; *i.e.*,

$$\int_Z |\mu|^2 = \int_Z |f|^2$$

if and only if $\gamma = *f$ is a holomorphic form.

Proof. Let ϕ be the metric on L. By Lemma 4.4

$$\mu = \partial^\phi \gamma,$$

for some $\bar\partial$-closed $(n-1,0)$-form γ. Thus

$$\int_Z |\mu|^2 = \int_Z f \wedge \bar\gamma e^{-\phi} \le \|f\| \|\gamma\|,$$

with equality only if $*f$ is proportional to γ. By the Hörmander-Kodaira-Nakano identity,

$$\int_Z \gamma \wedge \bar\gamma \wedge i\partial\bar\partial\phi \, e^{-\phi} + \int_Z |\bar\partial\gamma|^2 = \int_Z |\mu|^2.$$

The first term on the left-hand side here is the norm squared of γ and so it follows that

$$\|\gamma\|^2 \le \|\mu\|^2,$$

with equality only if $\bar\partial\gamma = 0$, and combined with our previous estimate

$$\|\mu\|^2 \le \|f\|^2$$

with equality only if $\bar\partial\gamma = 0$ and $*f$ proportional to γ. Hence $*f$ must be holomorphic. The argument is easily seen to be reversible. □

We are now ready to state the main theorem of this section.

THEOREM 5.3. *Assume that Z has no nonzero global holomorphic vector field. Suppose that*

(i) *X is locally a product $U \times Z$ where U is an open set in $\mathbb{C}$,*

(ii) *L is semipositive on X, and that*

(iii) *L restricted to each fiber is strictly positive.*

Let ω_t, the Kähler metric, be the fiber X_t induced by the curvature of L. Then, if for each t in U there is some element u_t in E_t such that

$$(\Theta^E u_t, u_t) = 0,$$

it follows that

$$\omega_t = \omega_0$$

for t in U.

Proof. By Lemma 5.1 the restriction of θ^L to each fiber X_t is zero. Hence

$$\partial\theta^L = \mathcal{L}_{\partial/\partial t}\Theta^L$$

also vanishes on fibers, which means that

$$\tfrac{d}{dt}\omega_t = 0. \qquad \square$$

6. The space of Kähler metrics

In this section we will specify the situation even more, and assume that $X = U \times Z$ is a product, and that moreover the line bundle L is the pullback of a bundle on Z under the projection on the second factor. Intuitively this means that not only are all fibers the same, but also the line bundle on them, so it is only the metric that varies. Fix one metric ϕ_0 on L, that we can take to be the pullback of a metric on the bundle on Z, i.e. independent of the t-variable. Then any other metric on L can be written

$$\phi = \phi_0 + \psi,$$

where ψ is a function on X. We also continue to assume that U is a domain in $\mathbb{C}$. Let u be an element in E_t.

In this situation we have an explicit lower bound for the curvature form operating on u, generalizing 3.1:

$$(\Theta^E u, u) \geq \int_{X_t} \left(\psi_{t\bar{t}} - |\bar{\partial}_z \psi_t|^2_\phi\right) [u, u]. \tag{6.1}$$

Here the expression $|f|_\phi$ means the norm of the form f with respect to the metric $\omega := i\partial\bar{\partial}_z\phi$ on X_t. This can be proved, either by adapting the method of

Section 3 (note that we may replace any t-derivative of ϕ by the corresponding derivative of ψ since ϕ_0 is independent of t) or from the more complicated proof in Section 4. To see how it follows from the formulas in Section 4 we again decompose

$$u = u_0 + dt \wedge v$$

as in the previous section, and also write

$$i\partial\bar\partial\phi = \omega - 2\mathrm{Re}\, idt \wedge \bar\partial\psi_t + \psi_{t\bar t} idt \wedge d\bar t.$$

Then

$$c_N u \wedge \bar u \wedge i\partial\bar\partial\phi = \left(\psi_{t\bar t} c_n u_0 \wedge \bar u_0 + c_{n-1} v \wedge \bar v \wedge \omega - 2\mathrm{Re}\, \bar\partial\psi_t \wedge u_0 \wedge \bar v\right) \wedge idt \wedge d\bar t.$$

By Cauchy's inequality

$$2\mathrm{Re}\, \bar\partial\psi_t \wedge u_0 \wedge \bar v \leq c_{n-1} v \wedge \bar v \wedge \omega + |\bar\partial\psi_t|^2_\phi c_n u_0 \wedge \bar u_0$$

so that

$$c_N u \wedge \bar u \wedge i\partial\bar\partial\phi \geq \left(\psi_{t\bar t} - |\bar\partial\psi_t|^2_\phi\right) c_n u_0 \wedge \bar u_0,$$

and (6.1) follows from (4.8) and (2.5). Notice that we have used nowhere that Θ is positive on the total space X, just that the restrictions to fibers are positive. Therefore (6.1) holds for any metric on L which is strictly positive along the fibers, even though, of course, this does not imply that E is positive in general.

The expression occurring in the integrand in 6.1,

$$C(\psi) = \left(\psi_{t\bar t} - |\bar\partial_z\psi_t|^2_\phi\right)$$

plays a crucial role in the recent work on variations of Kähler metrics on compact manifolds; see [27], [23], [11], [12], [26] and [8], to cite just a few. Fixing a line bundle L on Z, these papers consider the space $\mathcal{K}(L)$ of all Kähler metrics whose Kähler form is cohomologous to the Chern class of L. This means precisely that the Kähler form can be written

$$i\partial\bar\partial\phi = i\partial\bar\partial\phi_0 + i\partial\bar\partial\psi,$$

for some function ψ, and so the set up we described above, where ψ depends on t, corresponds to a path in $\mathcal{K}(L)$.

The tangent space of $\mathcal{K}(L)$ at a point ϕ is a space of functions $\dot\psi$ and a Riemannian metric on the tangent space is given by the L^2-norm

$$|\dot\psi|^2 = \int_Z |\dot\psi|^2 (i\partial\bar\partial\phi)^n/n!.$$

In this way, $\mathcal{K}(L)$ becomes an infinite-dimensional Riemannian manifold.

Now consider our space X above and let $U = \{|\mathrm{Re}\, t| < 1\}$ be a strip, and consider functions ψ that depend only on $\mathrm{Re}\, t$. Then

$$4C(\psi) = \ddot\psi - |\bar\partial_z\dot\psi|^2_\phi$$

if we use dots to denote derivatives with respect to $\mathrm{Re}\, t$. The link between Theorem 1.2 and the papers cited above lies in the fact that, by the results in [27], the right-hand side here is the geodesic curvature of the path in $\mathcal{K}(L)$ determined by ψ.

A basic idea in the papers cited above is to consider the spaces

$$E_t = \Gamma(X_t, K_{X_t} \otimes L)$$

with the induced L^2-metric as a finite-dimensional approximation or "quantization" of the manifold Z with metric $\phi_t = \phi_0 + \psi(t, \cdot)$. (Actually this is not quite true. In the papers cited above one does not take the tensor product with the canonical bundle, but instead integrates with respect to the volume element $(i\partial\bar{\partial}\phi_t)^n/n!$.) Here one also replaces L by L^k - with k^{-1} playing the role of Planck's constant - and studies the asymptotic behaviour as k goes to infinity.

Under this "quantization" map, functions, χ, on Z correspond to the induced Toepliz operator, T_χ, on E_t. This Toepliz operator is defined by

$$(T_\chi u, u)_{E_t} = \int_{\{t\}\times Z} \chi[u, u],$$

if u is any element in E_t. Note that the right-hand side in our estimate for the curvature (6.1) equals (4 times) $(T_\chi u, u)$, with χ equal to the geodesic curvature of the path in $\mathcal{K}$. Thus the inequality 6.1 can be formulated as saying that "the curvature of the quantization is greater than the quantization of the curvature", i.e., that the curvature operator of the vector bundle corresponding to a path in $\mathcal{K}(L)$ is greater than the Toepliz operator defined by the geodesic curvature of the path. Moreover, Theorem 5.3 implies that if Z has no nonzero global holomorphic vector fields, then equality holds only for a constant path.

7. Bundles of projective spaces

Let V be a holomorphic vector bundle of finite rank r over a complex manifold Y, and let V^* be its dual bundle. We let $\mathbb{P}(V)$ be the fiber bundle over Y whose fiber at each point t of the base is the projective space of lines in V_t^*, $\mathbb{P}(V_t^*)$. Then $\mathbb{P}(V)$ is a holomorphically locally trivial fibration. There is a naturally defined line bundle $O_{\mathbb{P}(V)}(1)$ over $\mathbb{P}(V)$ whose restriction to any fiber $\mathbb{P}(V_t^*)$ is the hyperplane section bundle (see [22]). One way to define this bundle is to first consider the tautological line bundle $O_{\mathbb{P}(V)}(-1)$. The total space of this line bundle, with the zero section removed, is just the total space of V^* with the zero section removed, and the projection to $\mathbb{P}(V)$ is the map that sends a nonzero point in V_t^* to its image in $\mathbb{P}(V_t^*)$. The bundle $O_{\mathbb{P}(V)}(1)$ is then defined as the dual of $O_{\mathbb{P}(V)}(-1)$. The global holomorphic sections of this bundle over any fiber are in one-to-one correspondence with the linear forms on V_t^*, i.e. the elements of V. More generally, $O_{\mathbb{P}(V)}(1)^l = O_{\mathbb{P}(V)}(l)$ has as global holomorphic sections over each fiber the homogeneous polynomials on V_t^* of degree l, i.e. the elemets of the l:th symmetric power of V. We shall apply Theorem 1.2 to the line bundles

$$L(l) =: O_{\mathbb{P}(V)}(l).$$

Let $E(l)$ be the vector bundle whose fiber over a point t in Y is the space of global holomorphic sections of $L(l) \otimes K_{\mathbb{P}(V_t^*)}$. If $l < r$ there is only the zero section, so that we assume from now on that l is greater than or equal to r.

We claim that

$$E(r) = \det V,$$

the determinant bundle of V. To see this, note that $L(r) \otimes K_{\mathbb{P}(V_t^*)}$ is trivial on each fiber, since the canonical bundle of $(r-1)$-dimensional projective space is $O(-r)$. The space of global sections is therefore one dimensional. A convenient basis element is

$$\sum_1^r z_j \widehat{dz_j},$$

if z_j are coordinates on V_t^*. Here $\widehat{dz_j}$ is the wedge product of all differentials dz_k except dz_j with a sign chosen so that $dz_j \wedge \widehat{dz_j} = dz_1 \wedge \ldots dz_r$. If we make a linear change of coordinates on V_t^*, this basis element gets multiplied with the determinant of the matrix giving the change of coordinates, so the bundle of sections must transform as the determinant of V. Since

$$L(r+1) \otimes K_{\mathbb{P}(V_t^*)} = O_{\mathbb{P}(V)}(1) \otimes L(r) \otimes K_{\mathbb{P}(V_t^*)},$$

it also follows that

$$E(r+1) = V \otimes \det V.$$

In the same way

$$E(r+m) = S^m(V) \otimes \det V,$$

where $S^m(V)$ is the mth symmetric power of V.

Let us now assume that V is ample in the sense of Hartshorne, see [16]. By a theorem of Hartshorne, [16], V is ample if and only if $L(1)$ is ample, i.e. has a metric with strictly positive curvature. Theorem 1.2 then implies that the L^2-metric on each of the bundles $E(r+m)$ for $m \geq 0$ has curvature which is strictly positive in the sense of Nakano; thus, we obtain:

THEOREM 7.1. *Let V be a vector bundle (of finite rank) over a complex manifold. Assume V is ample in the sense of Hartshorne. Then for any $m \geq 0$ the bundle*

$$S^m(V) \otimes \det V$$

has an hermitian metric with curvature which is (strictly) positive in the sense of Nakano.

8. Appendix

In this section we will state and prove an extension result of Ohsawa-Takegoshi type which in particular implies that the bundles E discussed in this paper really are vector bundles. The proof follows the method of [2]. See also [28] for a closely related result.

THEOREM 8.1. *Let X be a Kähler manifold fibered over the unit ball U in $\mathbb{C}^m$, with compact fibers X_t. Let L be a holomorphic line bundle on X with a smooth hermitian metric with semipositive curvature. Let u be a holomorphic section of $K_{X_0}\otimes L$ over X_0 such that*

$$\int_{X_0}[u,u]\leq 1.$$

Then there is a holomorphic section, $\tilde u$ to K_X over X such that $\tilde u = u\wedge dt$ for $t=0$ and

$$\int_X[\tilde u,\tilde u]\leq C$$

where C is an absolute constant.

Proof. We assume $m=1$. We see that the general case follows in the same way, extending with respect to one variable at a time. At first we also assume that the metric on L is smooth. The proof follows closely the method in [2] so that we may be somewhat sketchy.

Let $f = u\wedge[X_0]/(2\pi i)$, where $[X_0]$ is the current of integration on X_0. Then $\bar\partial f = 0$ and if v is any solution to $\bar\partial v = f$ then $\tilde u = tv$ is a section of $K_X\otimes L$ that extends u in the sense described. To find a v with L^2-estimates we need to estimate

$$\int_X (f,\alpha)$$

for any compactly supported test form α of bidegree $(n+1,1)$ on X. For α given, decompose $\alpha = \alpha^1+\alpha^2$, where α^1 is $\bar\partial$-closed, and α^2 is orthogonal to the kernel of $\bar\partial$. This means that α^2 can be written

$$\alpha^2 = \bar\partial^*\beta$$

for some β. By the regularity of the $\bar\partial$-Neumann problem α^i and β are all smooth up to the boundary. We first claim that

$$\int (f,\alpha^2) = 0. \tag{8.1}$$

This is not surprising since f is $\bar\partial$-closed, but it is not quite evident since f is not in L^2. To prove it, extend u smoothly to X. Then $\bar\partial u\wedge d\bar t = 0$ for $t=0$. Let χ be a smooth cut-off function equal to one near the origin in $\mathbb{R}$, and put

$$\chi_\epsilon(t) = \chi(|t|^2/\epsilon).$$

Then

$$f=\chi_\epsilon f=\bar\partial(u\wedge\frac{dt}{t})\chi_\epsilon-\bar\partial u\wedge\frac{dt}{t}\chi_\epsilon$$
$$=\bar\partial(u\wedge\frac{dt}{t}\chi_\epsilon)-u\wedge\frac{dt}{t}\wedge\bar\partial\chi_\epsilon-\bar\partial u\wedge\frac{dt}{t}\chi_\epsilon=:I+II+III.$$

Clearly the scalar product between I and α^2 vanishes. It is also clear that the scalar product between III and α^2 goes to zero as ϵ goes to zero. The scalar product between II and α^2 equals, up to signs

$$\int\chi'(\bar\partial u\wedge dt\wedge d\bar t,\beta)/\epsilon,$$

which is easily seen to tend to zero as well since $\bar\partial u\wedge d\bar t$ vanishes for $t=0$. Hence 8.1 follows. Therefore

$$\Big|\int_X(f,\alpha)\Big|^2=\Big|\int_X(f,\alpha^1)\Big|^2\leq\int_{X_0}\gamma\wedge\bar\gamma e^{-\phi},$$

where γ is the Hodge-* of α^1. The form γ satifies $\omega\wedge\gamma=\alpha^1$ and $\bar\partial^*\alpha=\bar\partial^*\alpha^1=\partial^\phi\gamma$. To estimate this we apply the Siu $\partial\bar\partial$-Bochner formula (see [2]): If w is any nonnegative function smooth up to the boundary of X, then

$$-\int i\partial\bar\partial w\wedge c_n\gamma\wedge\bar\gamma e^{-\phi}+\int i\partial\bar\partial\phi\wedge c_n\gamma\wedge\bar\gamma e^{-\phi}w$$
$$\leq 2c_n\mathrm{Re}\int\bar\partial\partial^\phi\gamma\wedge\bar\gamma e^{-\phi}w=2\int|\bar\partial^*\alpha|^2w+2\mathrm{Re}\int(\bar\partial^*\alpha,\partial w\wedge\gamma).\tag{8.2}$$

Now choose $w=(1/2\pi)\log(1/|t|^2)$. (Although w is not smooth it can be approximated by the smooth functions $(1/2\pi)\log(1/(|t|^2+\epsilon))$, so formula (8.2) still holds.) If $i\partial\bar\partial\phi$ is nonnegative we then find that

$$\int_{X_0}\gamma\wedge\bar\gamma e^{-\phi}\leq C\int|\bar\partial^*\alpha|^2(\log(1/|t|^2+1/|t|)+C\int_X idt\wedge d\bar t\wedge\gamma\wedge\bar\gamma e^{-\phi}(1/|t|).$$

To take care of the last term we repeat the last argument once more, this time choosing $w=(1-|t|)$ and finally obtain an estimate

$$\Big|\int_X(f,\alpha)\Big|^2\leq C\int|\bar\partial^*\alpha|^2(1/|t|).$$

This implies that there is some function v on X such that

$$\int_X(f,\alpha)=\int(v,\bar\partial^*\alpha),$$

for all test forms α, satisfying

$$\int|v|^2|t|\leq C.$$

Then $\tilde u:=tv$ satisfies the conclusion of the theorem. □

Chalmers University of Technology and the University of Göteborg, Göteborg, Sweden
E-mail address: bob@math.chalmers.se

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