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The geometry of fronts

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Abstract

We shall introduce the *singular curvature function* on cuspidal edges of surfaces, which is related to the Gauss-Bonnet formula and which characterizes the shape of cuspidal edges. Moreover, it is closely related to the behavior of the Gaussian curvature of a surface near cuspidal edges and swallowtails.

Introduction

Let M^2 be an oriented 2-manifold and $f\colon M^2 \to \boldsymbol{R}^3$ a C^∞-map. A point $p \in M^2$ is called a *singular point* if f is not an immersion at p. A singular point is called a *cuspidal edge* or *swallowtail* if it is locally diffeomorphic to

$$(1) \qquad f_C(u,v) := (u^2, u^3, v) \quad \text{or} \quad f_S(u,v) := (3u^4 + u^2v, 4u^3 + 2uv, v)$$

at $(u,v) = (0,0)$, respectively. These two types of singular points characterize the generic singularities of wave fronts (cf. [AGV]; for example, parallel surfaces of immersed surfaces in $\boldsymbol{R}^3$ are fronts), and we have a useful criterion (Fact 1.5; cf. [KRSUY]) for determining them. It is of interest to investigate these singularities from the viewpoint of differential geometry. In this paper, we shall distinguish two types of cuspidal edges as in Figure 1. More precisely, we shall define the singular curvature function κ_s along cuspidal edges. The left-hand figure in Figure 1 is positively curved and the right-hand figure is negatively curved (see Corollary 1.18).

The definition of the singular curvature function does not depend on the orientation nor on the co-orientation of the front. It is closely related to the following two Gauss-Bonnet formulas given by Langevin-Levitt-Rosenberg and

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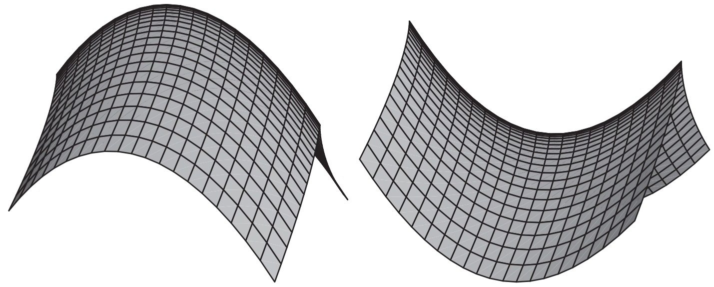

Figure 1: Positively and negatively curved cuspidal edges (Example 1.9).

Kossowski when M^2 is compact:

$$2\deg(\nu) = \chi(M_+) - \chi(M_-) + \#S_+ - \#S_- \qquad \text{([LLR], [Kos1])} \tag{2}$$

$$2\pi\chi(M^2) = \int_{M^2} K\,dA + 2\int_{\text{Singular set}} \kappa_s\,ds \qquad \text{([Kos1])}, \tag{3}$$

where $\deg(\nu)$ is the degree of the Gauss map ν, $\#S_+$ and $\#S_-$ are the numbers of positive and negative swallowtails respectively (see §2), and M_+ (resp. M_-) is the open submanifold of M^2 to which the co-orientation is compatible (resp. not compatible) with respect to the orientation. In the proofs of these formulas in [LLR] and [Kos1], the singular curvature implicitly appeared as a form $\kappa_s\,ds$. (Formula (2) is stated in [LLR]. Proofs for both (2) and (3) are in [Kos1].)

Recently, global properties of fronts were investigated via flat surfaces in hyperbolic 3-space H^3 ([KUY], [KRSUY], [KRUY]), via maximal surfaces in Minkowski 3-space ([UY]), and via constant mean curvature one surfaces in de Sitter space ([F]; see also [FRUYY] and [LY]). Such surfaces satisfy certain Osserman type inequalities for which equality characterizes the proper embeddedness of their ends. We note that Martínez [Mar] and Ishikawa and Machida [IM] investigated properties of improper affine spheres with singularities which are related to flat fronts in H^3. Special linear Weingarten surfaces having singularities are also investigated in [GMM], [Kok], and [IST].

The purpose of this paper is to give geometric meaning to the singular curvature function and investigate its properties. For example, it diverges to $-\infty$ at swallowtails (Corollary 1.14). Moreover, we shall investigate behavior of the Gaussian curvature K near singular points. For example, the Gaussian curvature K is generically unbounded near cuspidal edges and swallowtails, and will take different signs from the left-hand side to the right-hand side of a singular curve. However, on the special occasions that K is bounded, the shape of these singularities is very restricted. For example, *singular curvature is nonpositive if the Gaussian curvature is nonnegative* (Theorem 3.1). A similar phenomena holds for the case of hypersurfaces (§5).

The paper is organized as follows: In Section 1, we define the singular curvature, and give its fundamental properties. In Section 2, we generalize the two Gauss-Bonnet formulas (2) and (3) to fronts which admit finitely many corank one “peak” singularities. In Section 3, we investigate behavior of Gaussian curvature. Section 4 is devoted to formulating a topological invariant of closed fronts called the “zigzag number” (introduced in [LLR]) from the viewpoint of differential geometry. In Section 5, we shall generalize the results of Section 3 to hypersurfaces. Finally, in Section 6, we introduce an intrinsic formulation of the geometry of fronts.

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1. Singular curvature

Let M^2 be an oriented 2-manifold and (N^3, g) an oriented Riemannian 3-manifold. The unit cotangent bundle $T_1^*N^3$ has the canonical contact structure and can be identified with the unit tangent bundle T_1N^3. A smooth map $f\colon M^2 \to N^3$ is called a *front* if there exists a unit vector field ν of N^3 along f such that $L := (f, \nu)\colon M^2 \to T_1N^3$ is a Legendrian immersion (which is also called an isotropic immersion); that is, the pull-back of the canonical contact form of T_1N^3 vanishes on M^2. This condition is equivalent to the following orthogonality condition:

$$g(f_*X, \nu) = 0 \qquad (X \in TM^2), \tag{1.1}$$

where f_* is the differential map of f. The vector field ν is called the *unit normal vector* of the front f. The *first fundamental form* ds^2 and the *second fundamental form* h of the front are defined in the same way as for surfaces:

$$ds^2(X,Y) := g(f_*X, f_*Y),\ h(X,Y) := -g(f_*X, D_Y\nu) \qquad \bigl(X, Y \in TM^2\bigr), \tag{1.2}$$

where D is the Levi-Civita connection of (N^3, g).

We denote the Riemannian volume element of (N^3, g) by μ_g. Let $f\colon M^2 \to N^3$ be a front and let ν be the unit normal vector of f. Set

$$d\hat{A} := f^*(\iota_\nu \mu_g) = \mu_g(f_u, f_v, \nu)\, du \wedge dv \quad \left(f_u = f_*\left(\frac{\partial}{\partial u}\right), f_v = f_*\left(\frac{\partial}{\partial v}\right)\right), \tag{1.3}$$

called the *signed area form*, where (u, v) is a local coordinate system of M^2 and ι_ν is the interior product with respect to $\nu \in TN^3$. Suppose now that (u, v) is compatible to the orientation of M^2. Then the function

$$\lambda(u, v) := \mu_g(f_u, f_v, \nu) \tag{1.4}$$

is called the (*local*) *signed area density function*. We also set

$$dA := |\mu_g(f_u, f_v, \nu)|\, du \wedge dv = \sqrt{EG - F^2}\, du \wedge dv = |\lambda|\, du \wedge dv \qquad (1.5)$$
$$\big(E := g(f_u, f_u), F := g(f_u, f_v), G := g(f_v, f_v)\big),$$

which is independent of the choice of orientation-compatible coordinate system (u, v) and is called the (*absolute*) *area form* of f. Let M_+ (resp. M_-) be the open submanifolds where the ratio $(d\hat{A})/(dA)$ is positive (resp. negative). If (u, v) is a coordinate system compatible to the orientation of M^2, the point (u, v) belongs to M_+ (resp. M_-) if and only if $\lambda(u, v) > 0$ ($\lambda(u, v) < 0$), where λ is the signed area density function.

Definition 1.1. Let $f\colon M^2 \to N^3$ be a front. A point $p \in M^2$ is called a *singular point* if f is not an immersion at p. We call the set of singular points of f the *singular set* and denote it by $\Sigma_f := \{p \in M^2 \,|\, p \text{ is a singular point of } f\}$. A singular point $p \in \Sigma_f$ is called *nondegenerate* if the derivative $d\lambda$ of the signed area density function does not vanish at p. This condition does not depend on choice of coordinate systems.

It is well-known that a front can be considered locally as a projection of a Legendrian immersion $L\colon U^2 \to P(T^*N^3)$, where U^2 is a domain in $\boldsymbol{R}^2$ and $P(T^*N^3)$ is the projective cotangent bundle. The canonical contact structure of the unit cotangent bundle $T_1^*N^3$ is the pull-back of that of $P(T^*N^3)$. Since the contact structure on $P(T^*N^3)$ does not depend on the Riemannian metric, the definition of a front does not depend on the choice of the Riemannian metric g and is invariant under diffeomorphisms of N^3.

Definition 1.2. Let $f\colon M^2 \to N^3$ be a front and $TN^3|_{M^2}$ the restriction of the tangent bundle of N^3 to $f(M^2)$. The subbundle $\mathcal{E}$ of rank 2 on M^2 that is perpendicular to the unit normal vector field ν of f is called the *limiting tangent bundle* with respect to f.

There exists a canonical vector bundle homomorphism

$$\psi\colon TM^2 \ni X \longmapsto f_*X \in \mathcal{E}.$$

The nondegenerateness in Definition 1.1 is also independent of the choice of g and can be described in terms of the limiting tangent bundle:

PROPOSITION 1.3. *Let $f\colon U^2 \to N^3$ be a front defined on a domain U^2 in $\boldsymbol{R}^2$ and $\mathcal{E}$ the limiting tangent bundle. Let $\mu\colon (U^2; u, v) \to \mathcal{E}^* \wedge \mathcal{E}^*$ be an arbitrary fixed nowhere vanishing section, where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$. Then a singular point $p \in M^2$ is nondegenerate if and only if the derivative $d\tilde{\lambda}$ of the function $\tilde{\lambda} := \mu\big(\psi(\partial/\partial u), \psi(\partial/\partial v)\big)$ does not vanish at p.*

Proof. Let μ_0 be the 2-form that is the restriction of the 2-form $\iota_\nu \mu_g$ to M^2, where ι_ν denotes the interior product and μ_g is the volume element of g. Then μ_0 is a nowhere vanishing section on $\mathcal{E}^* \wedge \mathcal{E}^*$, and the local signed area density function λ is given by $\lambda = \mu_0(\psi(\partial/\partial u), \psi(\partial/\partial v))$.

On the other hand, let $\mu\colon (U^2; u, v) \to \mathcal{E}^* \wedge \mathcal{E}^*$ be an arbitrary fixed nowhere vanishing section. Then there exists a smooth function $\tau\colon U^2 \to \boldsymbol{R} \setminus \{0\}$ such that $\mu = \tau\mu_0$ (namely $\tilde{\lambda} = \tau\lambda$) and

$$d\tilde{\lambda}(p) = d\tau(p) \cdot \lambda(p) + \tau(p) \cdot d\lambda(p) = \tau(p) \cdot d\lambda(p),$$

since $\lambda(p) = 0$ for each singular point p. Then $d\tilde{\lambda}$ vanishes if and only if $d\lambda$ does as well. □

Remark 1.4. A C^∞-map $f : U^2 \to N^3$ is called a *frontal* if it is a projection of isotropic map $L : U^2 \to T_1^*N^3$; i.e., the pull-back of the canonical contact form of $T_1^*N^3$ by L vanishes on U^2. The definition of nondegenerate singular points and the above lemma do not use the properties that L is an immersion. Thus they hold for any frontals. (See §1 of [FSUY].)

Let $p \in M^2$ be a nondegenerate singular point. Then by the implicit function theorem, the singular set near p consists of a regular curve in the domain of M^2. This curve is called the *singular curve* at p. We denote the singular curve by

$$\gamma\colon (-\varepsilon, \varepsilon) \ni t \longmapsto \gamma(t) \in M^2 \qquad (\gamma(0) = p).$$

For each $t \in (-\varepsilon, \varepsilon)$, there exists a 1-dimensional linear subspace of $T_{\gamma(t)}M^2$, called the *null direction*, which is the kernel of the differential map f_*. A nonzero vector belonging to the null direction is called a *null vector*. One can choose a smooth vector field $\eta(t)$ along $\gamma(t)$ such that $\eta(t) \in T_{\gamma(t)}M^2$ is a null vector for each t, which is called a *null vector field*. The tangential 1-dimensional vector space of the singular curve $\gamma(t)$ is called the *singular direction*.

FACT 1.5 (Criteria for cuspidal edges and swallowtails [KRSUY]). *Let p be a nondegenerate singular point of a front f, γ the singular curve passing through p, and η a null vector field along γ. Then*

(a) *$p = \gamma(t_0)$ is a cuspidal edge (that is, f is locally diffeomorphic to f_C of* (1) *in the introduction) if and only if the null direction and the singular direction are transversal; i.e.,* $\det\bigl(\gamma'(t), \eta(t)\bigr)$ *does not vanish at $t = t_0$, where* det *denotes the determinant of* 2×2 *matrices and where we identify the tangent space in $T_{\gamma(t_0)}M^2$ with $\boldsymbol{R}^2$.*

(b) $p = \gamma(t_0)$ *is a swallowtail* (*that is*, f *is locally diffeomorphic to* f_S *of* (1) *in the introduction*) *if and only if*

$$\det\bigl(\gamma'(t_0), \eta(t_0)\bigr) = 0 \qquad \textit{and} \qquad \frac{d}{dt}\bigg|_{t=t_0} \det\bigl(\gamma'(t), \eta(t)\bigr) \neq 0$$

hold.

For later computation, it is convenient to take a local coordinate system (u, v) centered at a given nondegenerate singular point $p \in M^2$ as follows:

- the coordinate system (u, v) is compatible with the orientation of M^2,
- the u-axis is the singular curve, and
- there are no singular points other than the u-axis.

We call such a coordinate system (u, v) an *adapted coordinate system* with respect to p. In these coordinates, the signed area density function $\lambda(u, v)$ vanishes on the u-axis. Since $d\lambda \neq 0$, λ_v never vanishes on the u-axis. This implies that

(1.6) the signed area density function λ changes sign on singular curves;

that is, the singular curve belongs to the boundary of M_+ and M_-.

Now we suppose that a singular curve $\gamma(t)$ on M^2 consists of cuspidal edges. Then we can choose the null vector fields $\eta(t)$ such that $\bigl(\gamma'(t), \eta(t)\bigr)$ is a positively oriented frame field along γ. We then define the *singular curvature function* along $\gamma(t)$ as follows:

$$\kappa_s(t) := \operatorname{sgn}\bigl(d\lambda(\eta)\bigr) \frac{\mu_g\bigl(\hat\gamma'(t), \hat\gamma''(t), \nu\bigr)}{|\hat\gamma'(t)|^3}. \tag{1.7}$$

Here, we denote $|\hat\gamma'(t)| = g\bigl(\hat\gamma'(t), \hat\gamma'(t)\bigr)^{1/2}$,

$$\hat\gamma(t) = f(\gamma(t)), \qquad \hat\gamma'(t) = \frac{d\hat\gamma(t)}{dt}, \quad \text{and} \quad \hat\gamma''(t) = D_t\hat\gamma'(t), \tag{1.8}$$

where D is the Levi-Civita connection and μ_g is the volume element of (N^3, g).

We take an adapted coordinate system (u, v) and write the null vector field $\eta(t)$ as

$$\eta(t) = a(t)\frac{\partial}{\partial u} + e(t)\frac{\partial}{\partial v}, \tag{1.9}$$

where $a(t)$ and $e(t)$ are C^∞-functions. Since (γ', η) is a positive frame, we have $e(t) > 0$. Here,

$$\lambda_u = 0 \qquad \text{and} \qquad \lambda_v \neq 0 \qquad \text{(on the u-axis)} \tag{1.10}$$

hold, and then $d\lambda\bigl(\eta(t)\bigr) = e(t)\lambda_v$. In particular, we have

$$\operatorname{sgn}\bigl(d\lambda(\eta)\bigr) = \operatorname{sgn}(\lambda_v) = \begin{cases} +1 & \text{if the left-hand side of γ is M_+,} \\ -1 & \text{if the left-hand side of γ is M_-.} \end{cases} \tag{1.11}$$

Thus, we have the following expression: in an adapted coordinate system (u, v),

$$\kappa_s(u) := \operatorname{sgn}(\lambda_v)\frac{\mu_g(f_u, f_{uu}, \nu)}{|f_u|^3}, \tag{1.12}$$

where $f_{uu} = D_u f_u$ and $|f_u| = g(f_u, f_u)^{1/2}$.

THEOREM 1.6 (Invariance of the singular curvature). *The definition* (1.7) *of the singular curvature does not depend on the parameter t, the orientation of M^2, the choice of ν, nor the orientation of the singular curve.*

Proof. If the orientation of M^2 reverses, then λ and η both change sign. If ν is changed to $-\nu$, so does λ. If γ changes orientation, both γ' and η change sign. In all cases, the sign of κ_s is unchanged. □

Remark 1.7. We have the following expression:

$$\kappa_s = \operatorname{sgn}\bigl(d\lambda(\eta)\bigr)\,\frac{\mu_0(\hat{\gamma}'', \nu, \hat{\gamma}'/|\hat{\gamma}'|)}{|\hat{\gamma}'|^2} = \operatorname{sgn}\bigl(d\lambda(\eta)\bigr)\,\frac{g(\hat{\gamma}'', n)}{|\hat{\gamma}'|^2} \qquad \left(n := \nu \times_g \frac{\hat{\gamma}'}{|\hat{\gamma}'|}\right).$$

Here, the vector product operation $\times_g$ in T_xN^3 is defined by $a \times_g b := *(a \wedge b)$, under the identification $TN^3 \ni X \leftrightarrow g(X, \ ) \in T^*N^3$, where $*$ is the Hodge $*$-operator. If $\gamma(t)$ is not a singular curve, $n(t)$ is just the conormal vector of γ. We call $n(t)$ the *limiting conormal vector*, and $\kappa_s(t)$ can be considered as the limiting geodesic curvature of (regular) curves with the singular curve on their right-hand sides.

PROPOSITION 1.8 (Intrinsic formula for the singular curvature). *Let p be a point of a cuspidal edge of a front f. Let (u, v) be an adapted coordinate system at p such that $\partial/\partial v$ gives the null direction. Then the singular curvature is given by*

$$\kappa_s(u) = \frac{-F_vE_u + 2EF_{uv} - EE_{vv}}{E^{3/2}\lambda_v},$$

where $E = g(f_u, f_u), F = g(f_u, f_v), G = g(f_v, f_v)$, and where λ is the signed area density function with respect to (u, v).

Proof. Fix $v > 0$ and denote the u-curve by $\gamma(u) = (u, v)$. Then the unit vector

$$n(u) = \frac{1}{\sqrt{E}\sqrt{EG - F^2}}\left(-F\frac{\partial}{\partial u} + E\frac{\partial}{\partial v}\right)$$

gives the conormal vector such that $\bigl(\gamma'(u), n(u)\bigr)$ is a positive frame. Let ∇ be the Levi-Civita connection on $\{v > 0\}$ with respect to the induced metric $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, and $s = s(u)$ the arclength parameter of $\gamma(u)$. Then

$$\nabla_{\gamma'(s)}\gamma'(s) = \frac{1}{\sqrt{E}}\nabla_{\partial/\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\right) \equiv \frac{\Gamma_{11}^2}{E}\frac{\partial}{\partial v} \mod \frac{\partial}{\partial u},$$

where Γ_{11}^2 is the Christoffel symbol given by

$$\Gamma_{11}^2 = \frac{-FE_u + 2EF_u - EE_v}{2(EG - F^2)}.$$

Since $\lambda^2 = EG - F^2$ and $g(f_u, n) = 0$, the geodesic curvature of γ is given by

$$\kappa_g = g\bigl(\nabla_{\gamma'(s)}\gamma'(s), n(s)\bigr) = \frac{\sqrt{EG - F^2}\,\Gamma_{11}^2}{E^{3/2}} = \frac{-FE_u + 2EF_u - EE_v}{|\lambda| E^{3/2}}.$$

Hence, by Remark 1.7, the singular curve of the u-axis is

$$\kappa_s = \operatorname{sgn}(\lambda_v) \lim_{v\to 0} \kappa_g = \operatorname{sgn}(\lambda_v) \lim_{v\to 0} \frac{-FE_u + 2EF_u - EE_v}{|\lambda| E^{3/2}}.$$

It is clear that all of λ, F and F_u tend to zero as $v \to 0$. Moreover,

$$E_v = 2g(D_v f_u, f_u) = 2g(D_u f_v, f_u) = 2\frac{\partial}{\partial v} g(f_v, f_u) - 2g(f_v, D_u f_u) \to 0$$

as $v \to 0$, and the right differential $|\lambda|_v$ is equal to $|\lambda_v|$ since $\lambda(u, 0) = 0$. By L'Hospital's rule,

$$\kappa_s = \operatorname{sgn}(\lambda_v) \frac{-F_v E_u + 2EF_{uv} - EE_v}{|\lambda|_v E^{3/2}} = \frac{-F_v E_u + 2EF_{uv} - EE_v}{\lambda_v E^{3/2}},$$

which is the desired conclusion. □

Example 1.9 (Cuspidal parabolas). Define a map f from $\boldsymbol{R}^2$ to the Euclidean 3-space $(\boldsymbol{R}^3, g_0)$ as

$$f(u, v) = (au^2 + v^2, bv^2 + v^3, u) \qquad (a, b \in \boldsymbol{R}). \tag{1.13}$$

Then we have $f_u = (2au, 0, 1)$, $f_v = (2v, 2bv + 3v^2, 0)$. This implies that the u-axis is the singular curve, and the v-direction is the null direction. The unit normal vector and the signed area density $\lambda = \mu_{g_0}(f_u, f_v, \nu)$ are given by

$$\nu = \frac{1}{\delta}\bigl(-3v - 2b, 2, 2au(3v + 2b)\bigr), \qquad \lambda = v\delta, \tag{1.14}$$
$$\text{where} \qquad \delta = \sqrt{4 + (1 + 4a^2u^2)(4b^2 + 12bv + 9v^2)}.$$

In particular, since $d\nu(\partial/\partial v) = \nu_v \neq 0$ on the u-axis, $(f, \nu)\colon \boldsymbol{R}^2 \to \boldsymbol{R}^3 \times S^2 = T_1\boldsymbol{R}^3$ is an immersion; i.e. f is a front, and each point of the u-axis is a cuspidal edge. The singular curvature is given by

$$\kappa_s(u) = \frac{2a}{(1 + 4a^2u^2)^{3/2}\sqrt{1 + b^2(1 + 4a^2u^2)}}. \tag{1.15}$$

When $a > 0$ (resp. $a < 0$), that is when the singular curvature is positive (resp. negative), we shall call f a *cuspidal elliptic* (resp. *hyperbolic*) *parabola* since the figure looks like an elliptic (resp. hyperbolic) parabola, as seen in Figure 1 in the introduction.

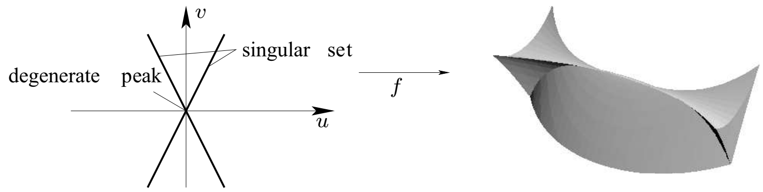


Figure 2: A double swallowtail (Example 1.11).

Definition 1.10 (Peaks). A singular point $p \in M^2$ (which is not a cuspidal edge) is called a *peak* if there exists a coordinate neighborhood $(U^2; u, v)$ of p such that

(1) There are no singular points other than cuspidal edges on $U^2 \setminus \{p\}$;
(2) The rank of the derivative $f_*\colon T_pM^2 \to T_{f(p)}N^3$ at p is equal to 1; and
(3) The singular set of U^2 consists of finitely many (possibly empty) C^1-regular curves starting at p. The number $2m(p)$ of these curves is called the *number of cuspidal edges* starting at p.

If a peak is a nondegenerate singular point, it is called a *nondegenerate peak*.

Swallowtails are examples of nondegenerate peaks. A front which admits cuspidal edges and peaks is called *a front which admits at most peaks*. There are degenerate singular points which are not peaks. Typical examples are cone-like singularities which appear in rotationally symmetric surfaces in $\boldsymbol{R}^3$ of positive constant Gaussian curvature. However, since generic fronts (in the local sense) have only cuspidal edges and swallowtails, the set of fronts which admits at most peaks covers a sufficiently wide class of fronts.

Example 1.11 (A double swallowtail). Define a map $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}^3$ as

$$f(u,v) := (2u^3 - uv^2, 3u^4 - u^2v^2, v).$$

Then

$$\nu = \frac{1}{\sqrt{1 + 4u^2(1 + u^2v^2)}}(-2u, 1, -2u^2v)$$

is the unit normal vector to f. The pull-back of the canonical metric of $T_1\boldsymbol{R}^3 = \boldsymbol{R}^3 \times S^2$ by $(f,\nu)\colon \boldsymbol{R}^2 \to \boldsymbol{R}^3 \times S^2$ is positive definite. Hence f is a front. The signed area density function is $\lambda = (v^2 - 6u^2)\sqrt{1 + 4u^2(1 + u^2v^2)}$, and then the singular set is $\Sigma_f = \{v = \sqrt{6}u\} \cup \{v = -\sqrt{6}u\}$. In particular, $d\lambda = 0$ at $(0,0)$. The first fundamental form of f is expressed as $ds^2 = dv^2$ at the origin, which is of rank one. Hence the origin is a degenerate peak (see Figure 2).

To analyze the behavior of the singular curvature near a peak, we prepare the following proposition.

Proposition 1.12 (Boundedness of the singular curvature measure). *Let $f\colon M^2 \to (N^3, g)$ be a front with a peak p. Take $\gamma\colon [0, \varepsilon) \to M^2$ as a singular curve of f starting from the singular point p. Then $\gamma(t)$ is a cuspidal edge for $t > 0$, and the singular curvature measure $\kappa_s\, ds$ is continuous on $[0, \varepsilon)$, where ds is the arclength-measure. In particular, the limiting tangent vector $\lim_{t\to 0} \hat\gamma'(t)/|\hat\gamma'(t)|$ exists, where $\hat\gamma = f \circ \gamma$.*

Proof. Let ds^2 be the first fundamental form of f. Since p is a peak, the rank ds^2 is 1 at p. Thus one of the eigenvalues is 0, and the other is not. Hence the eigenvalues of ds^2 are of multiplicity one on a neighborhood of p. Therefore one can choose a local coordinate system (u, v) around p such that each coordinate curve is tangent to an eigendirection of ds^2. In particular, we can choose (u, v) such that $\partial/\partial v$ is the null vector field on γ. In such a coordinate system, $f_v = 0$ and $D_t f_v = 0$ hold on γ. Then the derivatives of $\hat\gamma = f \circ \gamma$ are

$$\hat\gamma' = u' f_u, \qquad D_t\hat\gamma' = u'' f_u + u' D_t f_u \qquad \left({}' = \frac{d}{dt}\right),$$

where $\gamma(t) = \bigl(u(t), v(t)\bigr)$. Hence

$$\kappa_s = \pm\frac{\mu_g(\hat\gamma', D_t\hat\gamma', \nu)}{|\hat\gamma'|^3} = \pm\frac{\mu_g(f_u, D_t f_u, \nu)}{|u'|\,|f_u|^3}, \tag{1.16}$$

where $|X|^2 = g(X, X)$ for $X \in TN^3$. Since $ds = |\hat\gamma'|\, dt = |u'|\,|f_u|\, dt$ and $f_u \neq 0$,

$$\kappa_s\, ds = \pm\frac{\mu_g(f_u, D_t f_u, \nu)}{|f_u|^2}\, dt$$

is continuous along $\gamma(t)$ at $t = 0$. □

To analyze the behavior of the singular curvature near a nondegenerate peak, we give another expression of the singular curvature measure:

Proposition 1.13. *Let (u, v) be an adapted coordinate system of M^2. Suppose that $(u, v) = (0, 0)$ is a nondegenerate peak. Then the singular curvature measure has the expression*

$$\kappa_s(u) ds = \operatorname{sgn}(\lambda_v)\frac{\mu_g(f_v, f_{uv}, \nu)}{|f_v|^2} du, \tag{1.17}$$

where ds is the arclength-measure and $f_{uv} := D_u f_v = D_v f_u$. In particular, the singular curvature measure is smooth along the singular curve.

Proof. We can take the null direction $\eta(u) = a(u)(\partial/\partial u) + e(u)(\partial/\partial v)$ as in (1.9). Since the peak is not a cuspidal edge, $\eta(0)$ must be proportional to $\partial/\partial u$. In particular, we can multiply $\eta(u)$ by a nonvanishing function and may assume that $a(u) = 1$. Then $f_u + e(u)f_v = 0$, and by differentiation, $f_{uu} + e_u f_v + e f_{uv} = 0$; that is,

$$f_u = -ef_v, \quad f_{uu} = -e_u f_v - e f_{uv}.$$

Substituting them into (1.12), we have (1.17) using the relation $ds = |\hat{\gamma}'|dt = |f_u|dt$. □

COROLLARY 1.14 (Behavior of the singular curvature near a nondegenerate peak). *At a nondegenerate peak, the singular curvature diverges to* $-\infty$.

Proof. We take an adapted coordinate (u, v) centered at the peak. Then

$$\kappa_s(u) = \operatorname{sgn}(\lambda_v)\frac{\mu_g(f_v, f_{uv}, \nu)}{|e(u)|\,|f_v|^3}.$$

On the other hand,

$$\mu_g(f_v, f_{uv}, \nu) = \mu_g(f_v, f_u, \nu)_v - \mu_g(f_{vv}, f_u, \nu) = (-\lambda)_v - \mu_g(f_{vv}, f_u, \nu).$$

Since $f_u(0, 0) = 0$,

$$\operatorname{sgn}(\lambda_v)\frac{\mu_g(f_v, f_{uv}, \nu)}{|f_v|^3}\bigg|_{(u,v)=(0,0)} = -\frac{|\lambda_v(0,0)|}{|f_v(0,0)|^3} < 0.$$

Since $e(u) \to 0$ as $u \to 0$, we have the assertion. □

Example 1.15. The typical example of peaks is a swallowtail. We shall compute the singular curvature of the swallowtail $f(u, v) = (3u^4 + u^2 v, 4u^3 + 2uv, v)$ at $(u, v) = (0, 0)$ given in the introduction, which is the discriminant set $\{(x, y, z) \in \boldsymbol{R}^3\,;\, F(x, y, z, s) = F_s(x, y, z, s) = 0 \text{ for } s \in \boldsymbol{R}\}$ of the polynomial $F := s^4 + zs^2 + ys + x$ in s. Since $f_u \times f_v = 2(6u^2 + v)(1, -u, u^2)$, the singular curve is $\gamma(t) = (t, -6t^2)$ and the unit normal vector is given by $\nu = (1, -u, u^2)/\sqrt{1 + u^2 + u^4}$. We have

$$\kappa_s(t) = \frac{\det(\hat{\gamma}', \hat{\gamma}'', \nu)}{|\hat{\gamma}'|^3} = -\frac{\sqrt{1 + t^2 + t^4}}{6|t|(1 + 4t^2 + t^4)^{3/2}},$$

which shows that the singular curvature tends to $-\infty$ when $t \to 0$.

Definition 1.16 (Null curves). Let $f\colon M^2 \to N^3$ be a front. A regular curve $\sigma(t)$ in M^2 is called a *null curve* of f if $\sigma'(t)$ is a null vector at each singular point. In fact, $\hat{\sigma}(t) = f\big(\sigma(t)\big)$ looks like the curve (virtually) transversal to the cuspidal edge, in spite of $\hat{\sigma}' = 0$, and $D_t\hat{\sigma}'$ gives the "tangential" direction of the surface at the singular point.

THEOREM 1.17 (A geometric meaning for the singular curvature). *Let p be a cuspidal edge, $\gamma(t)$ a singular curve parametrized by the arclength t with $\gamma(0) = p$, and $\sigma(s)$ a null curve passing through $p = \sigma(0)$. Then the sign of*

$$g\big(\ddot{\hat{\sigma}}(0), \hat{\gamma}''(0)\big)$$

coincides with that of the singular curvature at p*, where* $\hat\sigma = f\circ\sigma$*,* $\hat\gamma = f\circ\gamma$*,*

$$\dot{\hat\sigma} = \frac{d\hat\sigma}{ds}, \qquad \hat\gamma' = \frac{d\hat\gamma}{dt}, \qquad \ddot{\hat\sigma} = D_s\left(\frac{d\hat\sigma}{ds}\right), \quad \textit{and} \quad \hat\gamma'' = D_t\left(\frac{d\hat\gamma}{dt}\right).$$

Proof. We can take an adapted coordinate system (u, v) around p such that $\eta := \partial/\partial v$ is a null vector field on the u-axis. Then $f_v = f_*\eta$ vanishes on the u-axis, and it holds that $f_{uv} := D_v f_u = D_u f_v = 0$ on the u-axis. Since the u-axis is parametrized by the arclength, we have

$$g(f_{uu}, f_u) = 0 \qquad \text{on the } u\text{-axis} \qquad (f_{uu} := D_u f_u)\,. \tag{1.18}$$

Now let $\sigma(s) = \bigl(u(s), v(s)\bigr)$ be a null curve such that $\sigma(0) = (0, 0)$. Since $\dot\sigma(0)$ is a null vector, $\dot u(0) = 0$, where $\dot{} = d/ds$. Moreover, since $f_v(0, 0) = 0$ and $f_{uv}(0, 0) = 0$,

$$\begin{aligned}\ddot{\hat\sigma}(0) &= D_s(\dot u f_u + \dot v f_v) = \ddot u f_u + \ddot v f_v + \dot u^2 D_u f_u + 2\dot u\dot v D_u f_v + \dot v^2 D_v f_v\\ &= \ddot u f_u + \dot v^2 D_v f_v = \ddot u f_u(0, 0) + \dot v^2 f_{vv}(0, 0),\end{aligned}$$

and by (1.18),

$$g\bigl(\ddot{\hat\sigma}(0)), \hat\gamma''(0)\bigr) = g\bigl(f_{uu}(0, 0), \ddot u f_u + \dot v^2 f_{vv}(0, 0)\bigr) = \dot v^2 g\bigl(f_{uu}(0, 0), f_{vv}(0, 0)\bigr).$$

Now we can write $f_{vv} = a f_u + b(f_u \times_g \nu) + c\nu$, where $a, b, c \in \boldsymbol{R}$. Then

$$\begin{aligned}c &= g(f_{vv}, \nu) = g(f_v, \nu)_v - g(f_v, \nu_v) = 0,\\ b &= g(f_{vv}, f_u \times_g \nu) = g(f_v, f_u \times_g \nu)_v = -\lambda_v,\end{aligned}$$

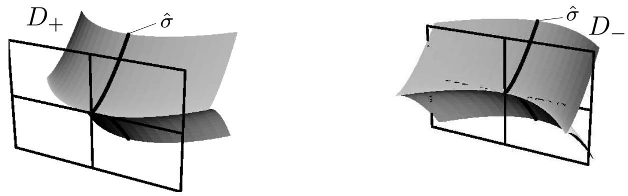
where we apply the scalar triple product formula $g(X, Y \times_g Z) = \mu_g(X, Y, Z)$ for $X, Y, Z \in T_{f(0,0)}N^3$. Thus

$$\begin{aligned}g\bigl(\ddot{\hat\sigma}(0), \hat\gamma''(0)\bigr) &= \dot v^2 g\bigl(f_{uu}, a f_u - \lambda_v(f_u \times_g \nu)\bigr) = -\dot v^2\lambda_v\, g(f_{uu}, f_u \times_g \nu)\\ &= \dot v^2\lambda_v\,\mu_g(\hat\gamma', \hat\gamma'', \nu) = \dot v^2|\lambda_v|\,\kappa_s(0).\end{aligned}$$

This proves the assertion. □

In the case of fronts in the Euclidean 3-space $\boldsymbol{R}^3 = (\boldsymbol{R}^3, g_0)$, positively curved cuspidal edges and negatively curved cuspidal edges look like cuspidal elliptic parabola or hyperbolic parabola (see Example 1.9 and Figure 1), respectively. More precisely, we have the following:

COROLLARY 1.18. *Let* $f\colon M^2 \to (\boldsymbol{R}^3, g_0)$ *be a front,* $p \in M^2$ *a cuspidal edge point, and* γ *a singular curve with* $\gamma(0) = p$*. Let* T *be the rectifying plane of the singular curve* $\hat\gamma = f\circ\gamma$ *at* p*, i.e., the plane perpendicular to the principal normal vector of* $\hat\gamma$*. When the singular curvature at* p *is positive* (*resp. negative*)*, every null curve* $\sigma(s)$ *passing through* $\sigma(0) = p$ *lies on the same region* D_+ (*resp. the opposite region* D_-) *bounded by* T *as the principal normal vector of* $\hat\gamma$ *at* p *for sufficiently small* s*. Moreover, if the singular curvature is positive, the image of the neighborhood of* p *itself lies in* D_+ (*see Figures* 1 *and* 3).



positively curved negatively curved
The principal half-spaces are behind the rectifying plane.

Figure 3: The principal half-spaces of cuspidal edges.

Definition 1.19. The half-space in Corollary 1.18, bounded by the rectifying plane of the singular curve and in which the null curves lie, is called the *principal half-space* at the cuspidal edge. The surface lies mostly in this half-space. When the singular curvature is positive, the surface is locally contained in the closure of the principal half-space.

Proof of Corollary 1.18. Let (u, v) be the same coordinate system at p as in the proof of Proposition 1.17 and assume that $f(0,0) = 0$. Since $N^3 = \boldsymbol{R}^3$, with $f_{uu} = \partial^2 f/\partial u^2$ etc., we have the following Taylor expansion:

$$f(u,v) = f_u(0,0) + \frac{1}{2}\bigl(f_{uu}(0,0)u^2 + f_{vv}(0,0)v^2\bigr) + o(u^2+v^2). \tag{1.19}$$

Here, u is the arclength parameter of $\hat{\gamma}(u) = f(u,0)$. Then $g_0(f_u, f_{uu}) = 0$ holds on the u-axis. Thus

$$g_0\bigl(f(u,v), f_{uu}(0,0)\bigr) = \frac{1}{2}u^2|f_{uu}(0,0)|^2 + \frac{1}{2}v^2 g_0\bigl(f_{vv}(0,0), f_{uu}(0,0)\bigr) + o(u^2+v^2).$$

If the singular curvature is positive, Theorem 1.17 implies $g_0\bigl(f(u,v), f_{uu}(0,0)\bigr) > 0$ on a neighborhood of p. Since $f_{uu}(0,0)$ is the principal curvature vector of $\hat{\gamma}$ at p, $f(u,v)$ lies in the same side of T as the principal normal.

Next we suppose that the singular curvature is negative at p. We can choose a coordinate system in which the null curve is written as $\sigma(v) = (0, v)$. Then by (1.19) and Theorem 1.17,

$$g_0\bigl(f(0,v), f_{uu}(0,0)\bigr) = v^2 g_0\bigl(f_{vv}(0,0), f_{uu}(0,0)\bigr) + o(v^2) < 0$$

for sufficiently small v. Hence we have the conclusion. □

Example 1.20 (Fronts with Chebyshev net). A front $f\colon M^2 \to \boldsymbol{R}^3$ is said to be of *constant Gaussian curvature* -1 if the set $W = M^2 \setminus \Sigma_f$ of regular points are dense in M^2 and f has constant Gaussian curvature -1 on W. Then f is a projection of the Legendrian immersion $L_f\colon M^2 \to T_1\boldsymbol{R}^3$, and the pull-back $d\sigma^2 = |df|^2 + |d\nu|^2$ of the canonical metric on $T_1\boldsymbol{R}^3$ by L_f is flat. Thus for each $p \in M^2$, there exists a coordinate neighborhood $(U^2; u, v)$ such that $d\sigma^2 = 2(du^2 + dv^2)$. The two different families of asymptotic curves on W are all geodesics of $d\sigma^2$, giving two foliations of W. Moreover, they are mutually orthogonal with respect to $d\sigma^2$. Then one can choose the u-curves and v-curves all to be asymptotic curves on $W \cap U^2$. For such a coordinate system (u, v), the first and second fundamental forms are

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad h = 2\sin\theta\, du\, dv, \tag{1.20}$$

where $\theta = \theta(u, v)$ is the angle between the two asymptotic curves. The coordinate system (u, v) as in (1.20) is called the *asymptotic Chebyshev net* around p. The sine-Gordon equation $\theta_{uv} = \sin\theta$ is the integrability condition of (1.20); that is, if θ satisfies the sine-Gordon equation, then there exists a corresponding front $f = f(u, v)$.

For such a front, we can choose the unit normal vector ν such that $f_u \times f_v = \sin\theta\, \nu$ holds, i.e. $\lambda = \sin\theta$. The singular sets are characterized by $\theta \in \pi\boldsymbol{Z}$. We write $\varepsilon = e^{\pi i\theta} = \pm 1$ at a singular point. A given singular point is nondegenerate if and only if $d\theta \neq 0$. Moreover, the cuspidal edges are characterized by $\theta_u - \varepsilon\theta_v \neq 0$, and the swallowtails are characterized by $\theta_u + \varepsilon\theta_v \neq 0$, $\theta_u - \varepsilon\theta_v = 0$, and $\theta_{uu} + \theta_{vv} \neq 0$. By a straightforward calculation applying Proposition 1.8, we have

$$\kappa_s = -\varepsilon \frac{\theta_u \theta_v}{|\theta_u - \varepsilon\theta_v|} \qquad (\varepsilon = e^{\pi i\theta}).$$

Recently Ishikawa-Machida [IM] showed that the generic singularities of such fronts are cuspidal edges or swallowtails, as an application of Fact 1.5.

2. The Gauss-Bonnet theorem

In this section, we shall generalize the two types of Gauss-Bonnet formulas mentioned in the introduction to compact fronts which admit at most peaks.

PROPOSITION 2.1. *Let $f\colon M^2 \to (N^3, g)$ be a front, and K the Gaussian curvature of f which is defined on the set of regular points of f. Then $K\, d\hat{A}$ can be continuously extended as a globally defined* 2*-form on M^2, where $d\hat{A}$ is the signed area form as in* (1.3).

Proof. Let (u, v) be a local coordinate system compatible to the orientation of M^2, and $S = (S^i_j)$ the (matrix representation of) the shape operator of f which is defined on the set of regular points $M^2 \setminus \Sigma_f$. That is, the Weingarten equation holds:

$$\nu_u = -S^1_1 f_u - S^2_1 f_v, \quad \nu_v = -S^1_2 f_u - S^2_2 f_v, \quad \text{where } \nu_u = D_u\nu,\ \nu_v = D_v\nu.$$

Since the extrinsic curvature is defined as $K_{\text{ext}} = \det S$, we have

$$\mu_g(\nu_u, \nu_v, \nu) = (\det S)\, \mu_g(f_u, f_v, \nu) = K_{\text{ext}}\, \lambda,$$

where λ is the signed area density. Thus,

$$K_{\text{ext}}\, d\hat{A} = K_{\text{ext}}\, \lambda\, du \wedge dv = \mu_g(\nu_u, \nu_v, \nu)\, du \wedge dv$$

is a well-defined smooth 2-form on M^2.

By the Gauss equation, the Gaussian curvature K satisfies

$$K = c_{N^3} + K_{\text{ext}}, \tag{2.1}$$

where c_{N^3} is the sectional curvature of (N^3, g) with respect to the tangent plane. Since $f_* T_p M^2 \subset T_{f(p)} N^3$ is the orthogonal complement of the normal vector $\nu(p)^\perp$, the tangent plane is well-defined on all of M^2. Thus c_{N^3} is a smooth function, and

$$K\, d\hat{A} = c_{N^3}\, d\hat{A} + K_{\text{ext}}\, d\hat{A}$$

is a smooth 2-form defined on M^2. □

Remark 2.2. On the other hand,

$$K\, dA = \begin{cases} K\, d\hat{A} & \text{(on } M_+\text{)}, \\ -K\, d\hat{A} & \text{(on } M_-\text{)} \end{cases}$$

is bounded, and extends continuously to the closure of M_+ and also to the closure of M_-. (However, $K\, dA$ cannot be extended continuously to all of M^2.)

Now we suppose that M^2 is compact and $f\colon M^2 \to \boldsymbol{R}^3$ is a front which admits at most peak singularities. Then the singular set coincides with $\partial M_+ = \partial M_-$, and ∂M_+ and ∂M_- are piecewise C^1-differentiable because all singularities are at most peaks, and the limiting tangent vector of each singular curve starting at a peak exists by Proposition 1.12.

For a given peak p, let $\alpha_+(p)$ (resp. $\alpha_-(p)$) be the sum of all the interior angles of $f(M_+)$ (resp. $f(M_-)$) at p. Then by definition,

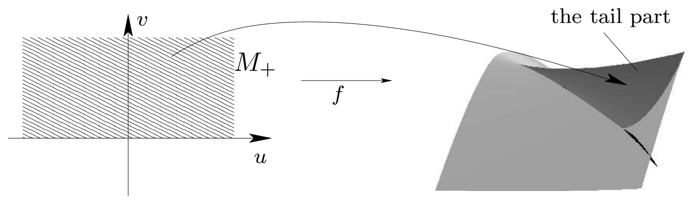
$$\alpha_+(p) + \alpha_-(p) = 2\pi. \tag{2.2}$$

Moreover, since the rank of f_* is one at p (see [SUY, Th. A]),

$$\alpha_+(p),\ \alpha_-(p) \in \{0, \pi, 2\pi\}. \tag{2.3}$$

For example, $\alpha_+(p) = \alpha_-(p) = \pi$ when p is a cuspidal edge. If p is a swallowtail, $\alpha_+(p) = 2\pi$ or $\alpha_-(p) = 2\pi$. If $\alpha_+(p) = 2\pi$, p is called a *positive swallowtail*, and if $\alpha_-(p) = 2\pi$, p is called a *negative swallowtail* (see Figure 4). Since $K\, dA$, $K\, d\hat{A}$, and $\kappa_s\, ds$ all are bounded, we get two Gauss-Bonnet formulas as follows:

THEOREM 2.3 (Gauss-Bonnet formulas for compact fronts). *Let M^2 be a compact oriented 2-manifold and let $f\colon M^2 \to (N^3, g)$ be a front which admits*



If the image of M_+ is the tail part, $\alpha_+(p) = 0$.

Figure 4: A negative swallowtail.

at most peak singularities, and Σ_f *the singular set of* f. *Then*

$$\int_{M^2} K\,dA + 2\int_{\Sigma_f} \kappa_s\,ds = 2\pi\chi(M^2), \tag{2.4}$$

$$\int_{M^2} K\,d\hat{A} - \sum_{p:peak} \bigl(\alpha_+(p) - \alpha_-(p)\bigr) = 2\pi\bigl(\chi(M_+) - \chi(M_-)\bigr) \tag{2.5}$$

hold, where ds is the arclength measure on the singular set.

Remark 2.4. The integral $\int_{M^2} K\,d\hat{A}$ is 2π times the Euler number $\chi_{\mathcal{E}}$ of the limiting tangent bundle $\mathcal{E}$ (see (6.3)). When $N^3 = \boldsymbol{R}^3$, $\chi_{\mathcal{E}}/2$ is equal to the degree of the Gauss map.

Remark 2.5. These formulas are generalizations of the two Gauss-Bonnet formulas in the introduction. If the surface is regular, the limiting tangent bundle $\mathcal{E}$ coincides with the tangent bundle, and the two Gauss-Bonnet formulas are the same.

Proof of Theorem 2.3. Although ∂M_+ and ∂M_- are the same set, their orientations are opposite. The singular curvature κ_s does not depend on the orientation of the singular curve and coincides with the limit of the geodesic curvature if we take the conormal vector in the positive direction with respect to the velocity vector of the singular curve. Thus we have

$$\int_{\partial M_+} \kappa_s\,ds + \int_{\partial M_-} \kappa_s\,ds = 2\int_{\Sigma_f} \kappa_s\,ds. \tag{2.6}$$

Then, by the classical Gauss-Bonnet theorem (see also [SUY]),

$$2\pi\chi(M_+) = \int_{M_+} K\,dA + \int_{\partial M_+} \kappa_s ds + \sum_{p:\text{peak}} \bigl(\pi m(p) - \alpha_+(p)\bigr),$$

$$2\pi\chi(M_-) = \int_{M_-} K\,dA + \int_{\partial M_-} \kappa_s\,ds + \sum_{p:\text{peak}} \bigl(\pi m(p) - \alpha_-(p)\bigr),$$

where $2m(p)$ is the number of cuspidal edges starting at p (see Definition 1.10). Hence by (2.6),

$$2\pi\chi(M^2) = \int_{M^2} K\,dA + 2\int_{\Sigma_f} \kappa_s\,ds,$$
$$2\pi\bigl(\chi(M_+) - \chi(M_-)\bigr) = \int_{M^2} K\,d\hat{A} - \sum_{p:\mathrm{peak}} \bigl(\alpha_+(p) - \alpha_-(p)\bigr),$$

where we used (2.2), and $\chi(M^2) = \chi(M_+) + \chi(M_-) - \sum_{p:\mathrm{peak}}\bigl(m(p)-1\bigr)$. □

We shall now define the completeness of fronts and give Gauss-Bonnet formulas for noncompact fronts. As defined in [KUY], a front $f\colon M^2 \to N^3$ is called *complete* if the singular set is compact and there exists a symmetric tensor T with compact support such that $ds^2 + T$ gives a complete Riemannian metric on M^2, where ds^2 is the first fundamental form of f. On the other hand, as defined in [KRSUY], [KRUY], a front $f\colon M^2 \to N^3$ is called *weakly complete* if the pull-back of the canonical metric of T_1N^3 (that is, the sum of the first and the third fundamental forms) by the Legendrian lift $L_f\colon M^2 \to T_1N^3$ is a complete Riemannian metric. Completeness implies weak completeness.

Let $f\colon M^2 \to N^3$ be a complete front with finite absolute total curvature. Then there exists a compact 2-manifold $\overline{M}^2$ without boundary, and finitely many points $p_1,\dots,p_k$, such that M^2 is diffeomorphic to $\overline{M}^2 \setminus \{p_1,\dots,p_k\}$. We call the p_i's the *ends* of the front f. According to Theorem A of Shiohama [S], we define the limiting area growth ratio of disks of radius r

$$a(p_i) = \lim_{r\to\infty} \frac{\mathrm{Area}\bigl(B_o(r)\cap E_i\bigr)}{\mathrm{Area}\bigl(B_{\boldsymbol{R}^2}(r)\cap E_i\bigr)}, \tag{2.7}$$

where E_i is the punctured neighborhood of p_i in $\overline{M}^2$, and $o \in M^2$ is a base point.

Theorem 2.6 (Gauss-Bonnet formulas for complete fronts). *Let $f\colon M^2 \to (N^3, g)$ be a complete front with finite absolute total curvature, which has at most peak singularities, and write $M^2 = \overline{M}^2 \setminus \{p_1,\dots,p_k\}$. Then*

$$\int_{M^2} K\,dA + 2\int_{\Sigma_f} \kappa_s\,ds + 2\pi\sum_{i=1}^{k} a(p_i) = 2\pi\chi(M^2), \tag{2.8}$$

(2.9)
$$\int_{M^2} K\,d\hat{A} - \sum_{p:peak} \bigl(\alpha_+(p) - \alpha_-(p)\bigr) + 2\pi\sum_{i=1}^{k} \varepsilon(p_i)a(p_i) = 2\pi\bigl(\chi(M_+) - \chi(M_-)\bigr)$$

hold, where $\varepsilon(p_i) = 1$ (resp. $\varepsilon(p_i) = -1$) if the neighborhood E_i of p_i is contained in M_+ (resp. M_-).

Example 2.7 (Pseudosphere). Define $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}^3$ as

$$f(x,y) := (\operatorname{sech} x\, \cos y, \operatorname{sech} x\, \sin y, x - \tanh x)\,.$$

If we set $\nu := (\tanh x\, \cos y, \tanh x\, \sin y, \operatorname{sech} x)$, then ν is the unit normal vector and f is a front whose singular set $\{x = 0\}$ consists of cuspidal edges. The Gaussian curvature of f is -1, and the coordinate system (u, v) defined as $x = u - v$, $y = u + v$ is the asymptotic Chebyshev net (see Example 1.20) with $\theta = 4 \arctan \exp(u - v)$.

Since $f(x, y+2\pi) = f(x, y)$, f induces a smooth map f_1 from the cylinder $M^2 = \boldsymbol{R}^2/\{(0, 2\pi m)\,;\, m \in \boldsymbol{Z}\}$ into $\boldsymbol{R}^3$. The front $f_1\colon M^2 \to \boldsymbol{R}^3$ has two ends p_1, p_2 with growth order $a(p_j) = 0$. Hence by Theorem 2.6, we have

$$2 \int_{\Sigma_{f_1}} \kappa_s\, ds = \operatorname{Area}(M^2) = 8\pi.$$

In fact, the singular curvature is positive.

Example 2.8 (Kuen's surface). The smooth map $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}^3$ defined as

$$f(x,y) = \left(\frac{2(\cos x + x \sin x)\cosh y}{x^2 + \cosh^2 y}, \frac{2(\sin x - x\cos x)\cosh y}{x^2 + \cosh^2 y}, y - \frac{2\cosh y \sinh y}{x^2 + \cosh^2 y}\right)$$

is called *Kuen's surface*, which is considered as a weakly complete front with the unit normal vector

$$\nu(x,y) = \left(\frac{2x \cos x + \sin x(x^2 - \cosh^2 y)}{x^2 + \cosh^2 y}, \frac{2x \sin x - \cos x(x^2 - \cosh^2 y)}{x^2 + \cosh^2 y}, \frac{-2x \sinh y}{x^2 + \cosh^2 y}\right)$$

and has Gaussian curvature -1. The coordinate system (u, v) given by $x = u - v$ and $y = u + v$ is the asymptotic Chebyshev net with $\theta = -4 \arctan(x \operatorname{sech} y)$. Since the singular set $\Sigma_f = \{x = 0\} \cap \{x = \pm \cosh y\}$ is noncompact, f is not complete.

Example 2.9 (Cones). Define $f\colon \boldsymbol{R}^2 \setminus \{(0,0)\} \to \boldsymbol{R}^3$ as

$$f(x,y) = (\log r \cos\theta, \log r \sin\theta, a \log r) \qquad (x,y) = (r\cos\theta, r\sin\theta),$$

where $a \neq 0$ is a constant. Then f is a front with

$$\nu = (a\cos\theta, a\sin\theta, -1)/\sqrt{1+a^2}.$$

The singular set is $\Sigma_f = \{r = 1\}$, which corresponds to the single point $(0,0,0) \in \boldsymbol{R}^3$. All points in Σ_f are nondegenerate singular points. The image of the singular points is a cone of angle $\mu = 2\pi/\sqrt{1+a^2}$ and the area growth order of the two ends are $1/\sqrt{1+a^2}$. Theorem 2.6 cannot be applied to this example because the singularities are not peaks. However, this example suggests that it might be natural to define the "singular curvature measure" at a cone-like singularity as the *cone angle*.

3. Behavior of the Gaussian curvature

First, we shall prove the following assertion, which says that the shape of singular points is very restricted when the Gaussian curvature is bounded.

THEOREM 3.1. *Let $f\colon M^2 \to (N^3, g)$ be a front, $p \in M^2$ a singular point, and $\gamma(t)$ a singular curve consisting of nondegenerate singular points with $\gamma(0) = p$ defined on an open interval $I \subset \boldsymbol{R}$. Then the Gaussian curvature K is bounded on a sufficiently small neighborhood of $\gamma(I)$ if and only if the second fundamental form vanishes on $\gamma(I)$.*

Moreover, if the extrinsic curvature K_{ext} (i.e. the product of the principal curvatures) is nonnegative on $U^2 \setminus \gamma(I)$ for a neighborhood of U^2 of p, then the singular curvature is nonpositive. Furthermore, if K_{ext} is bounded below by a positive constant on $U^2 \setminus \gamma(I)$ then the singular curvature at p takes a strictly negative value.

In particular, when $(N^3, g) = (\boldsymbol{R}^3, g_0)$, the singular curvature is nonpositive if the Gaussian curvature K is nonnegative near the singular set.

Proof of the first part of Theorem 3.1. We shall now prove the first part of the theorem. Take an adapted coordinate system (u, v) (see §1) such that the singular point p corresponds to $(0, 0)$, and write the second fundamental form of f as

(3.1)
$$h = L\,du^2 + 2\,M\,du\,dv + N\,dv^2 \quad \begin{pmatrix} L= -g(f_u, \nu_u),\ N = -g(f_v, \nu_v), \\ M= -g(f_v, \nu_u) = -g(f_u, \nu_v) \end{pmatrix}.$$

Since f_u and f_v are linearly dependent on the u-axis, $LN - (M)^2$ vanishes on the u-axis as well as the area density function $\lambda(u, v)$. Then by the Malgrange preparation theorem (see [GG, p. 91]), there exist smooth functions $\varphi(u, v)$, $\psi(u, v)$ such that

$$\lambda(u, v) = v\varphi(u, v) \qquad \text{and} \qquad LN - (M)^2 = v\psi(u, v). \tag{3.2}$$

Since (1.10), $\lambda_v \neq 0$ holds. Hence $\varphi(u, v) \neq 0$ on a neighborhood of the origin.

First we consider the case where p is a cuspidal edge point. Then we can choose (u, v) so that $\partial/\partial v$ gives the null direction. Since $f_v = 0$ holds on the u-axis, $M = N = 0$. By (2.1) and (3.2), we have that $K = c_{N^3} + \psi(u, v)/(v\varphi(u, v)^2)$. Thus the Gaussian curvature is bounded if and only if

$$L(u, 0)N_v(u, 0) = \bigl(LN - (M)^2\bigr)_v\big|_{v=0} = \psi(u, 0) = 0$$

holds on the u-axis. To prove the assertion, it is sufficient to show that $N_v(0,0) \neq 0$. Since $\lambda_v = \mu_g(f_u, f_{vv}, \nu) \neq 0$, $\{f_u, f_{vv}, \nu\}$ is linearly independent. Here,

$$2g(\nu_v, \nu) = g(\nu, \nu)_v = 0 \qquad \text{and} \qquad g(\nu_v, f_u)|_{v=0} = -M = 0.$$

Thus $\nu_v = 0$ if and only if $g(\nu_v, f_{vv}) = 0$. On the other hand, $\nu_v(0,0) \neq 0$ holds, since f is a front and $f_v = 0$. Thus,

$$N_v(0,0) = g(f_v, \nu_v)_v = g(\nu_v, f_{vv}) \neq 0. \tag{3.3}$$

Hence the first part of Theorem 3.1 is proved for cuspidal edges.

Next we consider the case that p is not a cuspidal edge point. Under the same notation as in the previous case, $f_u(0,0) = 0$ holds because p is not a cuspidal edge. Then, $M(0,0) = L(0,0) = 0$; thus the Gaussian curvature is bounded if and only if

$$L_v(u,0)N(u,0) = \left(LN - (M)^2\right)_v\big|_{v=0} = \psi(u,0) = 0$$

holds on the u-axis. Thus, to prove the assertion, it is sufficient to show that $L_v(0,0) \neq 0$. Since $\lambda_v = \mu_g(f_{uv}, f_v, \nu)$ does not vanish, $\{f_{uv}, f_v, \nu\}$ is linearly independent. On the other hand, $\nu_u(0,0) \neq 0$, because f is a front and $f_u(0,0) = 0$. Since $g(\nu, \nu)_v = 0$ and $g(\nu_u, f_v) = -M = 0$,

$$L_v(0,0) = g(\nu_u, f_u)_v = g(\nu_u, f_{uv}) \neq 0. \tag{3.4}$$

Hence the first part of the theorem is proved. □

Before proving the second part of Theorem 3.1, we prepare the following lemma:

Lemma 3.2 (Existence of special adapted coordinates along cuspidal edges). *Let $p \in M^2$ be a cuspidal edge of a front $f\colon M^2 \to (N^3, g)$. Then there exists an adapted coordinate system (u,v) around p satisfying the following properties*:

(1) $g(f_u, f_u) = 1$ *on the u-axis*,
(2) f_v *vanishes on the u-axis*,
(3) $\lambda_v = 1$ *holds on the u-axis*,
(4) $g(f_{vv}, f_u)$ *vanishes on the u-axis, and*
(5) $\{f_u, f_{vv}, \nu\}$ *is a positively oriented orthonormal basis along the u-axis.*

We shall call such a coordinate system (u,v) a *special adapted coordinate system.*

Proof of Lemma 3.2. One can easily take an adapted coordinate system (u,v) at p satisfying (1) and (2). Since $\lambda_v \neq 0$ on the u-axis, we can choose

(u,v) as $\lambda_v > 0$ on the u-axis. In this case, $r := \sqrt{\lambda_v}$ is a smooth function on a neighborhood of p. Now we set

$$u_1 = u, \qquad v_1 = \sqrt{\lambda_v(u,0)}\, v.$$

Then the Jacobian matrix is given by

$$\frac{\partial(u_1,v_1)}{\partial(u,v)} = \begin{pmatrix} 1 & 0 \\ r'(u) & r(u) \end{pmatrix}, \quad \text{where } r(u) := \sqrt{\lambda_v(u,0)}.$$

Thus,

$$(f_{u_1}, f_{v_1})|_{v=0} = (f_u, f_v) \begin{pmatrix} 1 & 0 \\ \frac{r'(u)}{r(u)} v & \frac{1}{r(u)} \end{pmatrix}\Bigg|_{v=0} = (f_u, f_v) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r(u)} \end{pmatrix}.$$

This implies that $f_{u_1} = f_u$ and $f_{v_1} = 0$ on the u-axis. Thus the new coordinates (u_1, v_1) satisfy (1) and (2). The signed area density function with respect to (u_1, v_1) is given by $\lambda_1 := \mu_g(f_{u_1}, f_{v_1}, \nu)$. Since $f_{v_1} = 0$ on the u-axis, we have

$$(\lambda_1)_{v_1} := \mu_g(f_{u_1}, D_{v_1} f_{v_1}, \nu). \tag{3.5}$$

On the other hand,

$$f_{v_1} = \frac{f_v}{r(u)} \quad \text{and} \quad D_{v_1} f_{v_1} = \frac{D_v f_{v_1}}{r(u)} = \frac{f_{vv}}{r^2} = \frac{f_{vv}}{\lambda_v} \tag{3.6}$$

on the u_1-axis. By (3.5) and (3.6), $(\lambda_1)_{v_1} = \lambda_v/\lambda_v = 1$ and we have shown that (u_1, v_1) satisfies (1), (2) and (3).

Next, we set

$$u_2 := u_1 + v_1^2\, s(u_1), \qquad v_2 := v_1,$$

where $s(u_1)$ is given smooth function in u_1. Then,

$$\frac{\partial(u_2,v_2)}{\partial(u_1,v_1)} = \begin{pmatrix} 1 + v_1^2 s' & 2v_1 s(u_1) \\ 0 & 1 \end{pmatrix},$$

and

$$\frac{\partial(u_1,v_1)}{\partial(u_2,v_2)}\Bigg|_{v_1=0} = \frac{1}{1+v_1^2 s'} \begin{pmatrix} 1 & -2v_1 s(u_2) \\ 0 & 1 + v_1^2 s' \end{pmatrix}\Bigg|_{v_1=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the new coordinates (u_2, v_2) satisfy (1) and (2). On the other hand, the area density function $\lambda_2 := \mu_g(f_{u_2}, f_{v_2}, \nu)$ satisfies

$$(\lambda_2)_{v_2} = \mu_g(f_{u_2}, f_{v_2}, \nu)_{v_2} = \mu_g(f_{u_2}, D_{v_2} f_{v_2}, \nu).$$

On the u_2-axis, $f_{u_2} = f_{u_1}$ and

$$f_{v_2} = \frac{-2v_1 s}{1 + v_1^2 s'} f_{u_1} + f_{v_1}, \tag{3.7}$$

$$g(D_{v_2} f_{v_2}) = D_{v_1} f_{v_2} = \frac{-2s}{1 + v_1^2 s'} f_{u_1} + D_{v_1} f_{v_1}. \tag{3.8}$$

Thus, it is easy to check that $(\lambda_2)_{v_2} = 1$ on the u-axis. By (3.8), $g(f_{u_2}, D_{v_2} f_{v_2}) = -2s + g(f_{u_1}, D_{v_1} f_{v_1})$. Hence, if we set

$$s(u_1) := \frac{1}{2} g\bigl(f_{u_1}(u_1, 0), (D_{v_1} f_{v_1})(u_1, 0)\bigr),$$

then the coordinate (u_2, v_2) satisfies (1), (2), (3) and (4). Since $g\bigl((f_{v_2})_{v_2}, \nu\bigr) = -g(f_{v_2}, \nu_{v_2}) = 0$, $f_{v_2 v_2}(u_2, 0)$ is perpendicular to both ν and f_{u_2}. Moreover, on the u_2-axis,

$$1 = (\lambda_2)_{v_2} = \mu_g(f_{u_2}, f_{v_2}, \nu)_{v_2} = \mu_g(f_{u_2}, D_{v_2} f_{v_2}, \nu) = g(D_{v_2} f_{v_2}, f_{u_2} \times_g \nu),$$

and we can conclude that $D_{v_2} f_{v_2}$ is a unit vector. Thus (u_2, v_2) satisfies (5). $\square$

Using the existence of a special adapted coordinate system, we shall show the second part of the theorem.

Proof of the second part of Theorem 3.1. We suppose $K \geq c_{N^3}$, where c_{N^3} is the sectional curvature of (N^3, g) with respect to the tangent plane. Then by (2.1), $K_{\text{ext}} \geq 0$ holds.

If a given nondegenerate singular point p is not a cuspidal edge, the singular curvature is negative by Corollary 1.14. Hence it is sufficient to consider the case that p is a cuspidal edge. Thus we may take a special adapted coordinate system as in Lemma 3.2. We take smooth functions φ and ψ as in (3.2).

Since K is bounded, $\psi(u, 0) = 0$ holds, as seen in the proof of the first part. Again, by the Malgrange preparation theorem, $LN - M^2 = v^2 \psi_1(u, v)$, which gives the expression $K_{\text{ext}} = \psi_1/\varphi^2$. Since $K_{\text{ext}} \geq 0$, $\psi_1(u, 0) \geq 0$. Moreover, if $K_{\text{ext}} \geq \delta > 0$ on a neighborhood of p, then $\psi_1(u, 0) > 0$. Since $L = M = N = 0$ on the u-axis,

$$0 \leq 2\psi_1(u, 0) = \bigl(LN - (M)^2\bigr)_{vv} = L_v N_v - (M_v)^2 \leq L_v N_v. \tag{3.9}$$

Here, $\{f_u, f_{vv}, \nu\}$ is an orthonormal basis, and $g(f_{uu}, f_u) = 0$ and $L = g(f_{vv}, \nu) = 0$ on the u-axis. Hence

$$f_{uu} = g(f_{uu}, f_{vv}) f_{vv} + g(f_{uu}, \nu)\nu = g(f_{uu}, f_{vv}) f_{vv}.$$

Similarly, since $2g(\nu_v, \nu) = g(\nu, \nu)_v = 0$ and $g(\nu_v, f_u) = -M = 0$,

$$\nu_v = g(\nu_v, f_{vv}) f_{vv}.$$

Since $\lambda_v = 1 > 0$ and $|f_u| = 1$, the singular curvature is given by

$$\kappa_s = \mu_g(f_u, f_{uu}, \nu) = g(f_{uu}, f_{vv}) \mu_g(f_u, f_{vv}, \nu) = g(f_{uu}, f_{vv}) = \frac{g(f_{uu}, \nu_v)}{g(f_{vv}, \nu_v)}. \tag{3.10}$$

On the other hand, on the u-axis

$$-L_v = g(f_u, \nu_u)_v = g(f_{uv}, \nu_u) + g(f_u, \nu_{uv}) = g(f_u, \nu_{uv}),$$

because $g(f_{uv}, \nu_u)|_{v=0} = -M_u(u, 0) = 0$. Moreover,

$$\nu_{uv} = D_v D_u \nu = D_u D_v \nu + R(f_v, f_u)\nu = D_u D_v \nu = \nu_{vu}$$

since $f_v = 0$, where R is the Riemannian curvature tensor of (N^3, g). Thus,

$$L_v = -g(f_u, \nu_{uv}) = -g(f_u, \nu_v)_u + g(f_{uu}, \nu_v) = M_u + g(f_{uu}, \nu_v) = g(f_{uu}, \nu_v)$$

holds. Since

$$-N_v = g(f_v, \nu_v)_v = g(f_{vv}, \nu_v) + g(f_v, \nu_{vv}) = g(f_{vv}, \nu_v)$$

on the u-axis, (3.10) and (3.9) imply that

$$\kappa_s = -\frac{L_v}{N_v} = -\frac{L_v N_v}{N_v^2} \leq 0.$$

If $K_{\text{ext}} \geq \delta > 0$, (3.9) becomes $0 < L_v N_v$, and we have $\kappa_s < 0$. □

Remark 3.3. Let $f\colon M^2 \to \boldsymbol{R}^3$ be a compact front with positive Gaussian curvature. For example, parallel surfaces of compact immersed constant mean curvature surfaces (e.g. Wente tori) give such examples. In this case, we have the following opposite of the Cohn-Vossen inequality by Theorem 2.3:

$$\int_{M^2} K dA > 2\pi\chi(M^2).$$

On the other hand, the total curvature of a compact 2-dimensional Alexandrov space is bounded from above by $2\pi\chi(M^2)$ (see Machigashira [Mac]). This implies that a front with positive curvature cannot be a limit of Riemannian 2-manifolds with Gaussian curvature bounded below by a constant.

Example 3.4 (Fronts of constant positive Gaussian curvature). Let $f_0\colon M^2 \to \boldsymbol{R}^3$ be an immersion of constant mean curvature $1/2$ and ν the unit normal vector of f_0. Then the parallel surface $f := f_0 + \nu$ gives a front of constant Gaussian curvature 1. If we take isothermal principal curvature coordinates (u, v) on M^2 with respect to f_0, the first and second fundamental forms of f are given by

$$ds^2 = dz^2 + 2\cosh\theta\, dz d\bar{z} + d\bar{z}^2, \qquad h = 2\sinh\theta\, dz d\bar{z},$$

where $z = u + iv$ and θ is a real-valued function in (u, v). This is called the *complex Chebyshev net.* The sinh-Gordon equation $\theta_{uu} + \theta_{vv} + 4\sinh\theta = 0$ is the integrability condition. In this case, the singular curve is characterized by $\theta = 0$, and the condition for nondegenerate singular points is given by $d\theta \neq 0$. Moreover, the cuspidal edges are characterized by $\theta_v \neq 0$, and the swallowtails are characterized by $\theta_u \neq 0, \theta_v = 0$ and $\theta_{vv} \neq 0$. The singular curvature on cuspidal edges is given by

$$\kappa_s = -\frac{(\theta_u)^2 + (\theta_v)^2}{4|\theta_v|} < 0.$$

The negativity of κ_s has been shown in Theorem 3.1. Like the case of fronts of constant negative curvature, Ishikawa-Machida [IM] also showed that the generic singularities of fronts of constant positive Gaussian curvature are cuspidal edges or swallowtails.

Here we remark on the behavior of mean curvature function near the nondegenerate singular points.

COROLLARY 3.5. *Let $f : M^2 \to (N^3, g)$ be a front and $p \in M^2$ a nondegenerate singular point. Then the mean curvature function of f is unbounded near p.*

Proof. The mean curvature function H is given by

$$2H := \frac{EN - 2FM + GL}{EG - F^2} = \frac{EN - 2FM + GL}{2\lambda^2}.$$

We assume that u-axis is a singular curve. By applying L'Hospital's rule, we have

$$\lim_{v\to 0} H = \lim_{v\to 0} \frac{E_vN + EN_v - 2F_vM - 2FM_v + G_vL - GL_v}{2\lambda\lambda_v}.$$

First, consider the case that $(0,0)$ is a cuspidal edge. Then by the proof of the first part of Theorem 3.1,

$$F(0,0) = G(0,0) = M(0,0) = N(0,0) = G_v(0,0) = 0.$$

Thus

$$\lim_{v\to 0} H = \lim_{v\to 0} \frac{EN_v}{2\lambda\lambda_v}.$$

Since $\lambda(0,0) = 0$ and $N_v(0,0) \neq 0$ as shown in the proof of Theorem 3.1, H diverges.

Next, consider the case that $(0,0)$ is not a cuspidal edge. When p is not a cuspidal edge, by the proof of the first part of Theorem 3.1,

$$E(0,0) = F(0,0) = L(0,0) = M(0,0) = E_v(0,0) = 0, \qquad L_v(0,0) \neq 0.$$

Thus

$$\lim_{v\to 0} H = -\lim_{v\to 0} GL_v/(2\lambda\lambda_v)$$

diverges, since $\lambda(0,0) = 0$ and $L_v(0,0) \neq 0$. □

Generic behavior of the curvature near cuspidal edges. As an application of Theorem 3.1, we shall investigate the generic behavior of the Gaussian curvature near cuspidal edges and swallowtails in $(\boldsymbol{R}^3, g_0)$.

We call a given cuspidal edge $p \in M^2$ *generic* if the second fundamental form does not vanish at p. We remark that the genericity of cuspidal edges and the sign of the Guassian curvature are both invariant under projective transformations. Theorem 3.1 implies that fronts with bounded Gaussian curvature have only nongeneric cuspidal edges. In the proof of the theorem for cuspidal edges, $L = 0$ if and only if f_{uu} is perpendicular to both ν and f_u, which implies that the osculating plane of the image of the singular curve coincides with the limiting tangent plane, and we get the following:

COROLLARY 3.6. *Let $f\colon M^2 \to \boldsymbol{R}^3$ be a front. Then a cuspidal edge $p \in M^2$ is generic if and only if the osculating plane of the singular curve does not coincide with the limiting tangent plane at p. Moreover, the Gaussian curvature is unbounded and changes sign between the two sides of a generic cuspidal edge.*

Proof. By (2.1) and (3.2), $K = \psi/\bigl(v\varphi^2\bigr)$, where $\psi(0,0) \neq 0$ if $(0,0)$ is generic. Hence K is unbounded and changes sign between the two sides along the generic cuspidal edge. □

We shall now determine which side has positive Gaussian curvature: Let γ be a singular curve of f consisting of cuspidal edge points, and let $\hat{\gamma} = f \circ \gamma$. Define

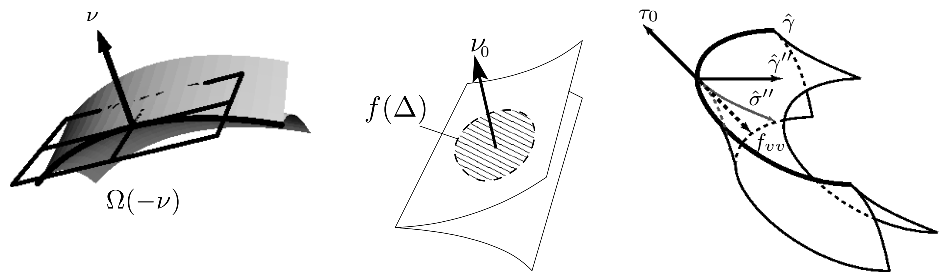
$$\kappa_\nu := \frac{g_0(\hat{\gamma}'', \nu)}{|\hat{\gamma}'|^2} \tag{3.11}$$

on the singular curve, which is independent of the choice of parameter t. We call it the *limiting normal curvature* along $\gamma(t)$. Then one can easily check that *p is a generic cuspidal edge if and only if $\kappa_\nu(p)$ does not vanish.* Let $\Omega(\nu)$ (resp. $\Omega(-\nu)$) be the closed half-space bounded by the limiting tangent plane such that ν (resp. $-\nu$) points into $\Omega(\nu)$ (resp. $\Omega(-\nu)$). Then the singular curve lies in $\Omega(\nu)$ if $\kappa_\nu(p) > 0$ and lies in $\Omega(-\nu)$ if $\kappa_\nu(p) < 0$. Namely, the closed *half-space H containing the singular curve* at $f(p)$ coincides with $\Omega(\nu)$ if $\kappa_\nu(p) > 0$, and with $\Omega(-\nu)$ if $\kappa_\nu(p) < 0$. We call $\Omega(\nu)$ (resp. $\Omega(-\nu)$) the *half-space containing the singular curve* at the cuspidal edge point p. This half-space H is different from the principal half-space in general (see Definition 1.19 and Figure 5).

We set

$$\begin{aligned}\operatorname{sgn}_0(\nu) &:= \operatorname{sgn}(\kappa_\nu)\\ &= \begin{cases} 1 & (\text{if } \Omega(\nu) \text{ is the half-space containing the singular curve}),\\ -1 & (\text{if } \Omega(-\nu) \text{ is the half-space containing the singular curve}).\end{cases}\end{aligned}$$

On the other hand, one can choose the *outward normal vector* ν_0 near a given cuspidal edge p, as in the middle figure of Figure 5. Let $\Delta \subset M^2$ be a sufficiently small domain, consisting of regular points sufficiently close to p, that lies only to one side of the singular curve. For a given unit normal vector ν of the front, we define its sign $\operatorname{sgn}_\Delta(\nu)$ by $\operatorname{sgn}_\Delta(\nu) = 1$ (resp. $\operatorname{sgn}_\Delta(\nu) = -1$) if ν



the half-space containing the singular curve is $\Omega(-\nu)$ | The outward normal | the vector τ_0

Figure 5: The half-space containing the singular curve, Theorem 3.7.

(resp. $-\nu$) coincides with the outward normal ν_0 on Δ. The following assertion holds:

THEOREM 3.7. *Let $f\colon M^2 \to (\boldsymbol{R}^3, g_0)$ be a front with unit normal vector field ν, p a cuspidal edge, and $\Delta(\subset M^2)$ a sufficiently small domain consisting of regular points sufficiently close to p that lies only to one side of the singular curve. Then $\operatorname{sgn}_\Delta(\nu)$ coincides with the sign of the function $g_0(\hat{\sigma}'', \hat{\nu}')$ at p, namely*

$$\operatorname{sgn}_\Delta(\nu) = \operatorname{sgn} g_0(\hat{\sigma}'', \hat{\nu}') \qquad \left({}' = \frac{d}{ds},\ {}'' = D_s\frac{d}{ds}\right), \tag{3.12}$$

where $\sigma(s)$ is an arbitrarily fixed null curve starting at p and moving into Δ, $\hat{\sigma}(s) = f(\sigma(s))$, and $\hat{\nu}(s) = \nu(\sigma(s))$. Moreover, if p is a generic cuspidal edge, then

$$\operatorname{sgn}_0(\nu) \cdot \operatorname{sgn}_\Delta(\nu)$$

coincides with the sign of the Gaussian curvature on Δ.

Proof. We take a special adapted coordinate system (u, v) as in Lemma 3.2 at the cuspidal edge. The vector $\tau_0 := -f_{vv} = f_u \times \nu$ lies in the limiting tangent plane and points in the opposite direction of the image of the null curve (see Figure 5, right side).

Without loss of generality, we may assume that $\Delta = \{v > 0\}$. The unit normal ν is the outward normal on $f(\Delta)$ if and only if $g_0(\nu_v, \tau_0) > 0$, namely $N_v = -g_0(f_{vv}, \nu_v) > 0$. Thus we have $\operatorname{sgn}_{\{v>0\}}(\nu) = \operatorname{sgn}(N_v)$, which proves (3.12). Since p is generic, we have $\kappa_\nu(p) \neq 0$ and $\kappa_\nu(p) = L$ holds. On the other hand, the sign of K on $v > 0$ is equal to the sign of (cf. (3.2))

$$\left(LN - (M)^2\right)_v\Big|_{v=0} = L(u,0)N_v(u,0),$$

which proves the assertion. $\square$

Example 3.8. Consider again the cuspidal parabola $f(u,v)$ as in Example 1.9. Then (u,v) gives an adapted coordinate system so that $\partial/\partial v$ gives a null direction, and we have

$$L = g_0(f_{uu},\nu) = \frac{-2ab}{\sqrt{1+b^2(1+4a^2u^2)}}, \qquad N_v = \frac{6}{\sqrt{1+b^2(1+4a^2u^2)}} > 0.$$

The cuspidal edges are generic if and only if $ab \neq 0$. In this case, let Δ be a domain in the upper half-plane $\{(u,v)\,;\, v>0\}$. Then the unit normal vector (1.14) is the outward normal on $f(\Delta)$, that is, $\operatorname{sgn}_\Delta(\nu) = +1$. The limiting normal curvature as in (3.11) is computed as

$$\kappa_\nu = -ab/\bigl(2|a|^2\sqrt{1+b^2(1+4a^2u^2)}\bigr),$$

and hence $\operatorname{sgn}_0(\nu) = -\operatorname{sgn}(ab)$. Then $\operatorname{sgn}(K) = -\operatorname{sgn}(ab)$ holds on the upper half-plane. In fact, the Gaussian curvature is computed as

$$K = \frac{-12(ab+3av)}{v\bigl(4+\bigl(1+4a^2u^2\bigr)(4b^2+12bv+9v^2)\bigr)}.$$

On the other hand, the Gaussian curvature is bounded if $b=0$. In this case, the Gaussian curvature is positive if $a<0$. In this case the singular curvature is negative when $a<0$, as stated in Theorem 3.1.

Generic behavior of the curvature near swallowtails. We call a given swallowtail $p \in M^2$ of a front $f\colon M^2 \to (\boldsymbol{R}^3, g_0)$ *generic* if the second fundamental form does not vanish at p. As in the case of the cuspidal edge, the notion of genericity belongs to the projective geometry.

PROPOSITION 3.9. *Let $f\colon M^2 \to (\boldsymbol{R}^3, g_0)$ be a front and p a generic swallowtail. Then we can take a closed half-space $H \subset \boldsymbol{R}^3$ bounded by the limiting tangent plane such that any null curve at p lies in H near p (see Figure 6).*

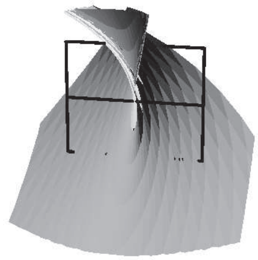
We shall call H the *half-space containing the singular curve* at the generic swallowtail. At the end of this section, we shall see that the singular curve is in fact contained in this half-space for a neighborhood of the swallowtail (see Figure 6 and Corollary 3.13). For a given unit normal vector ν of the front, we set $\operatorname{sgn}_0(\nu)=1$ (resp. $\operatorname{sgn}_0(\nu)=-1$) if ν points (resp. does not point) into the half-space containing the singular curve.

Proof of Proposition 3.9. Take an adapted coordinate system (u,v) and assume $f(0,0)=0$ by translating in $\boldsymbol{R}^3$ if necessary. Write the second fundamental form as in (3.1). Since $f_u(0,0)=0$, we have $L(0,0)=M(0,0)=0$, and we have the following Taylor expansion:

$$g_0\bigl(f(u,v),\nu\bigr) = \frac{v^2}{2}g_0\bigl(f_{vv}(0,0),\nu(0,0)\bigr)+o(u^2+v^2) = \frac{1}{2}N(0,0)v^2+o(u^2+v^2).$$

Thus the assertion holds. Moreover, □

$$\operatorname{sgn}(N) = \operatorname{sgn}_0(\nu). \tag{3.13}$$



the swallowtail f_+

the swallowtail f_-

The half-space containing the singular curve is the closer side of the limiting tangent plane for the left-hand figure, and the farther side for the right-hand figure.

Figure 6: The half-space containing the singular curve for generic swallowtails (Example 3.12).

COROLLARY 3.10. *Let $\sigma(s)$ be an arbitrary curve starting at the swallowtail such that $\sigma'(0)$ is transversal to the singular direction. Then*

$$\operatorname{sgn}_0(\nu) = \operatorname{sgn} g_0(\hat{\sigma}''(0), \nu(0,0))$$

holds, where $\hat{\sigma} = f \circ \sigma$.

We let $\Delta(\subset M^2)$ be a sufficiently small domain consisting of regular points sufficiently close to a swallowtail p. The domain Δ is called the *tail part* if $f(\Delta)$ lies in the opposite side of the self-intersection of the swallowtail (see Figure 4). We define $\operatorname{sgn}_\Delta(\nu)$ by $\operatorname{sgn}_\Delta(\nu) = 1$ (resp. $\operatorname{sgn}_\Delta(\nu) = -1$) if ν is (resp. is not) the outward normal of Δ. Now we have the following assertion:

THEOREM 3.11. *Let $f\colon M^2 \to (\boldsymbol{R}^3, g_0)$ be a front with unit normal vector field ν, p a generic swallowtail and $\Delta(\subset M^2)$ a sufficiently small domain consisting of regular points sufficiently close to p. Then the Gaussian curvature is unbounded and changes sign between the two sides along the singular curve. Moreover, $\operatorname{sgn}_0(\nu)\operatorname{sgn}_\Delta(\nu)$ coincides with the sign of the Gaussian curvature on Δ.*

Proof. If we change Δ to the opposite side, $\operatorname{sgn}_\Delta(\nu)\operatorname{sgn}_\Delta(K)$ does not change sign. So we may assume that Δ is the tail part. We take an adapted coordinate system (u,v) with respect to p and write the null vector field as $\eta(u) = (\partial/\partial u) + e(u)(\partial/\partial v)$, where $e(u)$ is a smooth function. Then

$$f_u(u,0) + e(u) f_v(u,0) = 0 \quad \text{and} \quad f_{uu}(u,0) + e_u(u) f_v(u,0) + e(u) f_{uv}(u,0) = 0$$

hold. Since $(0,0)$ is a swallowtail, $e(0) = 0$ and $e'(0) \neq 0$ hold, where $' = d/du$.

It is easy to check that the vector $f_{uu}(0,0)$ points toward the tail part $f(\Delta)$. Thus $f_v(0,0)$ points toward $f(\Delta)$ if and only if $g_0(f_v, f_{uu})$ is positive. Since $f_u = -e(u)f_v$ and $e(0) = 0$, we have $f_{uu}(0,0) = e'(0)f_v(0,0)$ and

$$g_0\big(f_{uu}(0,0), f_v(0,0)\big) = -e'(0)\, g_0\big(f_v(0,0), f_v(0,0)\big).$$

Thus $g_0\big(f_{uu}(0,0), f_v(0,0)\big)$ is positive (that is, the tail part is $v > 0$) if and only if $e'(0) < 0$.

Changing v to $-v$ if necessary, we assume $e'(0) > 0$, that is, the tail part lies in $v > 0$. For each fixed value of $u \neq 0$, we take a curve

$$\sigma(s) = \big(u + \varepsilon s, s|e(u)|\big) = \big(u + \varepsilon s, \varepsilon e(u)\, s\big) \qquad \varepsilon = \operatorname{sgn} e(u)$$

and let $\hat\sigma = f \circ \sigma$. Then the null curve σ passes through $(u,0)$ and is traveling into the upper half-plane $\{v > 0\}$; that is, $\hat\sigma$ is traveling into $f(\Delta)$. Here,

$$\hat\sigma'(0) = \varepsilon\big(f_u(u,0) + e(u)f_v(u,0)\big) = 0,$$

and

$$\begin{aligned}\hat\sigma''(0) &= \varepsilon\big(\varepsilon(f_u + ef_v)_u + \varepsilon e(f_u + ef_v)_v\big)\big|_{v=0}\\ &= e(u)\big(f_{uv}(u,0) + e(u)f_{vv}(u,0)),\end{aligned}$$

where $' = d/ds$. Then by Theorem 3.7,

$$\operatorname{sgn}_\Delta(\nu) = \lim_{u\to 0} \operatorname{sgn} g_0(\hat\sigma''_u(s), \hat\nu'(s)).$$

Here, the derivative of $\hat\nu(s) = \nu(\sigma(s))$ is computed as $\hat\nu' = \varepsilon\{\nu_u(u,0) + e(u)\nu_v(u,0)\}$. Since $e(0) = 0$,

$$g\big(\hat\sigma''(s), \hat\nu'(s)\big)\big|_{s=0} = |e(u)|g_0\big(f_{uv}(u,0), \nu_u(u,0)\big) + \{e(u)\}^2\varphi(u),$$

where $\varphi(u)$ is a smooth function in u. Then

$$\operatorname{sgn}_\Delta(\nu) = \lim_{u\to 0} \operatorname{sgn} g_0(\hat\sigma''(s), \hat\nu'(s)) = \operatorname{sgn} g_0(f_{uv}(0,0), \nu_u(0,0)).$$

Here, $L_v(0,0) = -g_0(f_u, \nu_u)_v = -g_0(f_{uv}, \nu_u)$ because $f_u(0,0) = 0$, which implies that

$$\operatorname{sgn}_\Delta(\nu) = \operatorname{sgn}(L_v(0,0)).$$

On the other hand, the sign of K on $v > 0$ is equal to the sign of

$$\big(LN - (M)^2\big)_v\big|_{v=0} = N(0,0)L_v(0,0).$$

Then (3.13) implies the assertion. □

Example 3.12. Let

$$f_\pm(u,v) = \frac{1}{12}(3u^4 - 12u^2v \pm (6u^2 - 12v)^2, 8u^3 - 24uv, 6u^2 - 12v).$$

Then one can see that $f_\pm$ is a front and $(0,0)$ is a swallowtail with the unit normal vector

$$\nu_\pm = \frac{1}{\delta}\bigl(1, u, u^2 \pm 12(2v-u^2)\bigr) \qquad \left(\delta = \sqrt{1+u^2+145u^4+576v(v-u^2) \pm 24u^2(2v-u^2)}\right).$$

In particular, (u,v) is an adapted coordinate system. Since the second fundamental form is $\pm 24dv^2$ at the origin, the swallowtail is generic and $\operatorname{sgn}_0(\nu_\pm) = \pm 1$ because of (3.13). The images of $f_\pm$ are shown in Figure 6. Moreover, since $L_v = \pm 2$ at the origin, by Theorem 3.11, the Gaussian curvature of the tail part of f_+ (resp. f_-) is positive (resp. negative).

Summing up the previous two theorems, we get the following:

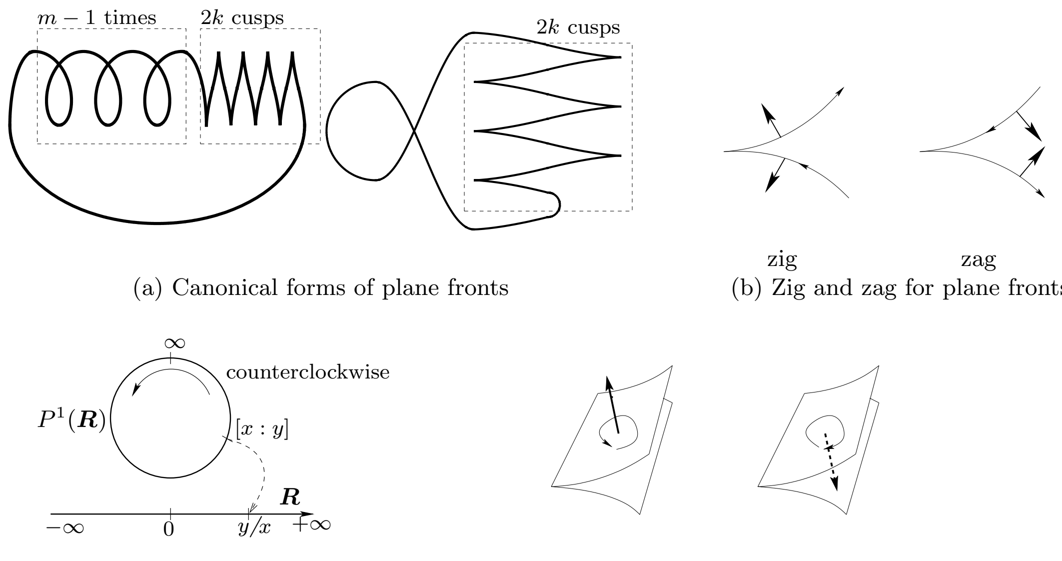
Corollary 3.13. *Let $\gamma(t)$ be a singular curve such that $\gamma(0)$ is a generic swallowtail. Then the half-space containing the singular curve at $\hat\gamma(t)$ converges to the half-space at the swallowtail $\hat\gamma(0)$ as $t \to 0$.*

4. Zigzag numbers

In this section, we introduce a geometric formula for a topological invariant called the *zigzag number*. We remark that Langevin, Levitt, and Rosenberg [LLR] gave topological upper bounds of zigzag numbers for generic compact fronts in $\boldsymbol{R}^3$. Several topological properties of zigzag numbers are given in [SUY2].

Zigzag number for fronts in the plane. First, we mention the Maslov index (see [A]; which is also called the zigzag number) for fronts in the Euclidean plane $(\boldsymbol{R}^2, g_0)$. Let $\gamma\colon S^1 \to \boldsymbol{R}^2$ be a generic front, that is, all self-intersections and singularities are double points and 3/2-cusps. Let ν be the unit normal vector field of γ. Then γ is Legendrian isotropic (isotropic as the Legendrian lift $(\gamma,\nu)\colon S^1 \to T_1\boldsymbol{R}^2 \simeq \boldsymbol{R}^2 \times S^1$) to one of the fronts in Figure 7 (a). The nonnegative integer m is called the *rotation number*, which is the rotational index of the unit normal vector field $\nu\colon S^1 \to S^1$. The number k is called the *Maslov index* or *zigzag number*. We shall give a precise definition and a formula to calculate the number: a 3/2-cusp $\gamma(t_0)$ of γ is called *zig* (resp. *zag*) if the leftward normal vector of γ points to the outside (resp. inside) of the cusp (see Figure 7 (b)). We define a C^∞-function λ on S^1 as $\lambda := \det(\gamma',\nu)$, where $' = d/dt$. Then the leftward normal vector is given by $(\operatorname{sgn}\lambda)\nu_0$. Since $\gamma''(t_0)$ points to the inside of the cusp, t_0 is zig (resp. zag) if and only if

$$\operatorname{sgn}\bigl(\lambda' g_0(\gamma'',\nu')\bigr) < 0 \qquad (\text{resp. } > 0). \tag{4.1}$$



(a) Canonical forms of plane fronts

(b) Zig and zag for plane fronts

(c) Counterclockwise in $P^1(\boldsymbol{R})$

(d) Zig and zag for cuspidal edges

Figure 7: Zigzag.

Let $\{t_0, t_1, \dots, t_l\}$ be the set of singular points of γ ordered by their appearance, and define $\zeta_j = a$ (resp. $= b$) if $\gamma(t_j)$ is zig (resp. zag). Set $\zeta_\gamma := \zeta_0\zeta_1 \dots \zeta_l$, which is a word consisting of the letters a and b. The projection of ζ_γ to the free product $\boldsymbol{Z}_2 * \boldsymbol{Z}_2$ (reduction with the relation $a^2 = b^2 = 1$) is of the form $(ab)^k$ or $(ba)^k$. The nonnegative integer $k_\gamma := k$ is called the *zigzag number* of γ. We shall give a geometric formula for the zigzag number via the curvature map given in [U]:

Definition 4.1. Let $\gamma\colon S^1 \to \boldsymbol{R}^2$ be a front with unit normal vector ν. The *curvature map* of γ is the map

$$\kappa_\gamma\colon S^1 \setminus \Sigma_\gamma \ni t \longmapsto \bigl[g_0(\gamma', \gamma') : g_0(\gamma', \nu')\bigr] \in P^1(\boldsymbol{R}),$$

where $' = d/dt$, $\Sigma_\gamma \subset S^1$ is the set of singular points of γ, and $[\ :\ ]$ denotes the homogeneous coordinates of $P^1(\boldsymbol{R})$.

Proposition 4.2. *Let γ be a generic front with unit normal vector ν. Then the curvature map κ_γ can be extended to a smooth map on S^1. Moreover, the rotation number of κ_γ is the zigzag number of γ.*

Proof. Let t_0 be a singular point of γ. Since γ is a front, $\nu'(t) \neq 0$ holds on a neighborhood of t_0. As ν' is perpendicular to ν, we have $\det(\nu, \nu') \neq 0$.

Here, using $\lambda = \det(\gamma', \nu)$, we have $\gamma' = -(\lambda/\det(\nu, \nu'))\nu'$. Hence

$$\kappa_\gamma = [g_0(\gamma', \gamma') : g_0(\gamma', \nu')] = \left[\lambda : -\frac{g_0(\nu', \nu')}{\det(\nu, \nu')}\right] = [\gamma' : \nu']$$

is well-defined on a neighborhood of t_0. Moreover, $\kappa_\gamma(t) = [0 : 1](= \infty)$ if and only if t is a singular point. Here, we choose an inhomogeneous coordinate of $[x : y]$ as y/x.

Since $g_0(\gamma', \nu')' = g_0(\gamma'', \nu')$ holds at a singular point t_0, κ_γ passes through $[0 : 1]$ with counterclockwise (resp. clockwise) direction if $g_0(\gamma'', \nu') > 0$ (resp. < 0); see Figure 7 (c).

Let t_0 and t_1 be two adjacent zigs, and suppose $\lambda'(t_0) > 0$. Since λ changes sign on each cusp, $\lambda'(t_1) < 0$. Then by (4.3), $g_0(\gamma'', \nu')(t_0) > 0$ and $g_0(\gamma'', \nu')(t_1) < 0$. Hence κ_γ passes through $[0 : 1]$ in the counterclockwise direction at t_0, and the clockwise direction at t_1. Thus, this interval does not contribute to the rotation number of κ_γ. On the other hand, if t_0 and t_1 are zig and zag respectively, κ_γ passes through $[0 : 1]$ counterclockwise at both t_0 and t_1. Then the rotation number of κ_γ is 1 on the interval $[t_0, t_1]$. In summary, the proposition holds. □

Zigzag number for fronts in Riemannian 3*-manifolds.* Let M^2 be an oriented manifold and $f\colon M^2 \to N^3$ a front with unit normal vector ν into an oriented Riemannian 3-manifold (N^3, g). Let $\Sigma_f \subset M^2$ be the singular set, and ν_0 the unit normal vector field of f defined on $M^2 \setminus \Sigma_f$ which is compatible with the orientations of M^2 and N^3; that is, $\nu_0 = (f_u \times_g f_v)/|f_u \times_g f_v|$, where (u, v) is a local coordinate system on M^2 compatible to the orientation. Then $\nu_0(p)$ is $\nu(p)$ if $p \in M_+$ and $-\nu(p)$ if $p \in M_-$.

We assume all singular points of f are nondegenerate. Then each connected component $C \subset \Sigma_f$ must be a regular curve on M^2. Let $p \in C$ be a cuspidal edge. According to [LLR], p is called *zig* (resp. *zag*) if ν_0 points towards the outward (resp. inward) side of the cuspidal edge (see Figure 7 (d)). As this definition does not depend on $p \in C$, we call C *zig* (resp. *zag*) if $p \in C$ is zig (resp. zag).

Now, we define the zigzag number for loops (i.e., regular closed curves) on M^2. Take a *null loop* $\sigma\colon S^1 \to M^2$, that is, the intersection of $\sigma(S^1)$ and Σ_f consists of cuspidal edges and σ' points in the null direction at each singular point. Note that there exists a null loop in each homotopy class. Later, we will use the nullity of loops to define the normal curvature map. Let $Z_\sigma = \{t_0, \dots, t_l\} \subset S^1$ be the set of singular points of $\hat{\sigma} := f \circ \sigma$ ordered by their appearance along the loop. Define $\zeta_j = a$ (resp. b) if $\sigma(t_j)$ is zig (resp. zag), and set $\zeta_\sigma := \zeta_0\zeta_1 \dots \zeta_l$, which is a term consisting of the letters a and b. The projection of ζ_σ to the free product $\boldsymbol{Z}_2 * \boldsymbol{Z}_2$ (reduction with the relation $a^2 = b^2 = 1$) is of the form $(ab)^k$ or $(ba)^k$. The nonnegative integer $k_\sigma := k$ is called the *zigzag number* of σ.

It is known that the zigzag number is a homotopy invariant, and the greatest common divisor k_f of $\{k_\sigma \,|\sigma$ is a null loop on $M^2\}$ is the *zigzag number of* f (see [LLR]).

In this section, we shall give a geometric formula for zigzag numbers of loops. First, we define the normal curvature map, similar to the curvature map for fronts in $\boldsymbol{R}^2$:

Definition 4.3 (Normal curvature map). Let $f\colon M^2 \to (N^3, g)$ be a front with unit normal vector ν and let $\sigma\colon S^1 \to M^2$ be a null loop. The *normal curvature map* of σ is the map

$$\kappa_\sigma\colon S^1 \setminus Z_\gamma \ni t \longmapsto \bigl[g(\hat\sigma', \hat\sigma') : g(\hat\sigma', \hat\nu')\bigr] \in P^1(\boldsymbol{R}),$$

where $\hat\sigma = f\circ\sigma$, $\hat\nu = \nu\circ\sigma$, $' = d/dt$, $Z_\sigma \subset S^1$ is the set of singular points of σ, and $[\ :\ ]$ denotes the homogeneous coordinates of $P^1(\boldsymbol{R})$.

Then we have the following:

THEOREM 4.4 (Geometric formula for zigzag numbers). *Let $f\colon M^2 \to (N^3, g)$ be a front with unit normal vector ν, whose singular points are all nondegenerate, and let $\sigma\colon S^1 \to M^2$ be a null loop. Then the normal curvature map κ_σ can be extended to S^1, and the rotation number of κ_σ is equal to the zigzag number of σ.*

Proof. Let $\sigma(t_0)$ be a singular point of f, and take a special adapted coordinate system (u, v) of M^2 on a neighborhood U^2 of the cuspidal edge point $\sigma(t_0)$ as in Lemma 3.2. Then $f_v = 0$ and $f_{vv} \neq 0$ holds on the u-axis, and by the Malgrange preparation theorem, there exists a smooth function α such that $g(f_v, f_v) = v^2\alpha(u, v)$ and $\alpha(u, 0) \neq 0$. On the other hand, $g(f_v, \nu_v) = -N$ vanishes and $N_v \neq 0$ on the u-axis. Hence there exists a function β such that $g(f_v, \nu_v) = v\beta(u, v)$ and $\beta(u, 0) \neq 0$. Thus

$$\kappa_\sigma = [g(f_v, f_v) : g(f_v, \nu_v))] = [v^2\alpha(u, v) : v\beta(u, v)] = [v\alpha(u, v) : \beta(u, v)] \tag{4.2}$$

can be extended to the singular point $v = 0$. Namely, $\kappa_\sigma(t_0) = [0 : 1](= \infty)$, where we choose an inhomogeneous coordinate y/x for $[x : y]$. Moreover, $g(\hat\sigma', \hat\sigma') \neq 0$ on regular points, and $\kappa_\sigma(t) = [0 : 1]$ if and only if t is a singular point.

Since $\nu = (\operatorname{sgn}\lambda)\nu_0$, a singular point t_0 is zig (resp. zag) if and only if

$$\operatorname{sgn}(\lambda)\operatorname{sgn}_\Delta(\nu) > 0 \qquad (\text{resp. } < 0),$$

where ε is a sufficiently small number and Δ is a domain containing $\sigma(t_0 + \varepsilon)$ which lies only to one side of the singular curve. By Theorem 3.7, $\operatorname{sgn}_\Delta(\nu) = \operatorname{sgn} g(\hat\sigma'', \hat\nu')$, and t_0 is zig (resp. zag) if and only if

$$\operatorname{sgn}\bigl(\hat\lambda' g(\hat\sigma'', \hat\nu')\bigr) > 0 \qquad (\text{resp. } < 0), \tag{4.3}$$

where $\hat\lambda = \lambda\circ\sigma$. Since $g(\hat\sigma', \hat\nu') = g(\hat\sigma'', \hat\nu')$ holds at singular points, we have

- If t_0 is zig and $\hat{\lambda}'(t_0) > 0$ (resp. < 0), then κ_σ passes through $[0:1]$ counterclockwisely (resp. clockwisely).
- If t_0 is zag and $\hat{\lambda}'(t_0) > 0$ (resp. < 0), then κ_σ passes through $[0:1]$ clockwisely (resp. counterclockwisely).

Let $Z_\sigma = \{t_0, \dots, t_l\}$ be the set of singular points. Since the function λ has alternative sign on the adjacent domains, $\hat{\lambda}'(t_j)$ and $\hat{\lambda}'(t_{j+1})$ have opposite signs. Thus, if both t_j and t_{j+1} are zigs and $\hat{\lambda}(t_j) > 0$, κ_σ passes through $[0:1]$ counterclockwisely (resp. clockwisely) at $t = t_j$ (resp. t_{j+1}). Hence the interval $[t_j, t_{j+1}]$ does not contribute to the rotation number of κ_σ. Similarly, two consecutive zags do not affect the rotation number. On the other hand, if t_j is zig and t_{j+1} is zag and $\hat{\lambda}(t_j) > 0$, κ_σ passes through $[0:1]$ counterclockwisely at both t_j and t_{j+1}. Hence the rotation number of κ_σ on the interval $[t_j, t_{j+1}]$ is 1. Similarly, two consecutive zags increases the rotation number by 1. Hence we have the conclusion. □

5. Singularities of hypersurfaces

In this section, we shall investigate the behavior of sectional curvature on fronts that are hypersurfaces. Let $U^n\,(n \geq 3)$ be a domain in $(\boldsymbol{R}^n; u_1, u_2, \dots, u_n)$ and let

$$f\colon U^n \longrightarrow (\boldsymbol{R}^{n+1}, g_0)$$

be a front; i.e., there exists a unit vector field ν (called the *unit normal vector*) such that $g_0(f_*X, \nu) = 0$ for all $X \in TU^n$ and $(f, \nu)\colon U^n \to \boldsymbol{R}^{n+1} \times S^n$ is an immersion. We set

$$\lambda := \det(f_{u_1}, \dots, f_{u_n}, \nu),$$

and call it the *signed volume density function*. A point $p \in U^n$ is called a *singular point* if f is not an immersion at p. Moreover, if $d\lambda \neq 0$ at p, we call p a *nondegenerate singular point*. On a sufficiently small neighborhood of a nondegenerate singular point p, the singular set is an $(n-1)$-dimensional submanifold called the *singular submanifold*. The 1-dimensional vector space at the nondegenerate singular point p which is the kernel of the differential map $(f_*)_p\colon T_pU^n \to \boldsymbol{R}^{n+1}$ is called the *null direction*. We call $p \in U^n$ a *cuspidal edge* if the null direction is transversal to the singular submanifold. Then, by a similar argument to the proof of Fact 1.5 in [KRSUY], one can prove that a cuspidal edge is an A_2-singularity, that is, locally diffeomorphic at the origin to the front $f_C(u_1, \dots, u_n) = (u_1^2, u_1^3, u_2, \dots, u_n)$. In fact, criteria for A_k-singularites for fronts are given in [SUY2].

Theorem 5.1. *Let $f\colon U^n \to (\boldsymbol{R}^{n+1}, g_0)$ $(n \geq 3)$ be a front whose singular points are all cuspidal edges. If the sectional curvature K at the regular points is bounded, then the second fundamental form h of f vanishes on the singular* *submanifold. Moreover, if K is positive everywhere on the regular set, the sectional curvature of the singular submanifold is nonnegative. Furthermore, if $K \geq \delta(> 0)$, then the sectional curvature of the singular submanifold is positive.*

Remark 5.2. Theorem 3.1 is deeper than Theorem 5.1. When $n \geq 3$, we can consider sectional curvature on the singular set, but when $n = 2$, the singular set is 1-dimensional. Thus we cannot define the sectional curvature. Instead, one defines the singular curvature. We do not define here singular curvature for fronts when $n \geq 3$.

Proof of Theorem 5.1. Without loss of generality, we may assume that the singular submanifold of f is the $(u_1, \dots, u_{n-1})$-plane, and $\partial_n := \partial/\partial u_n$ is the null direction. To prove the first assertion, it is sufficient to show that $h(X, X) = 0$ for an arbitrary fixed tangent vector of the singular submanifold. By changing coordinates if necessary, we may assume that $X = \partial_1 = \partial/\partial u_1$. The sectional curvature $K(\partial_1 \wedge \partial_n)$, with respect to the 2-plane spanned by $\{\partial_1, \partial_n\}$, is given by

$$K(\partial_1 \wedge \partial_n) = \frac{h_{11}h_{nn} - (h_{1n})^2}{g_{11}g_{nn} - (g_{1n})^2} \qquad \bigl(g_{ij} = g_0(\partial_i, \partial_j),\ h_{ij} = h(\partial_i, \partial_j)\bigr).$$

By the same reasoning as in the proof of Theorem 3.1, the boundedness of $K(\partial_1 \wedge \partial_n)$ implies

$$0 = \bigl(h_{11}h_{nn} - (h_{1n})^2\bigr)_{u_n}\Big|_{u_n=0} = h_{11}\,\frac{\partial h_{nn}}{\partial u_n}\bigg|_{u_n=0} = h_{11}\ g_0(D_{u_n}f_{u_n}, \nu_{u_n})|_{u_n=0}\,.$$

To show $h_{11} = h(X, X) = 0$, it is sufficient to show that $g_0(D_{u_n}f_{u_n}, \nu_{u_n})$ does not vanish when $u_n = 0$. Since f is a front with nondegenerate singularities, we have

$$0 \neq \lambda_{u_n} = \det(f_{u_1}, \dots, f_{u_{n-1}}, D_{u_n}f_{u_n}, \nu),$$

which implies that $\{f_{u_1}, \dots, f_{u_{n-1}}, D_{u_n}f_{u_n}, \nu\}$ is a basis when $u_n = 0$. Then ν_{u_n} can be written as a linear combination of the basis. Since f is a front, $\nu_{u_n} \neq 0$ holds when $u_n = 0$. On the other hand, we have $2g_0(\nu_{u_n}, \nu) = g_0(\nu, \nu)_{u_n} = 0$, and

$$g_0(\nu_{u_n}, f_{u_j}) = -g_0(\nu, D_{u_n}f_{u_j}) = g_0(\nu_{u_j}, f_{u_n}) = 0 \qquad (j = 1, \dots, n-1).$$

Thus we have the fact that $g_0(D_{u_n}f_{u_n}, \nu_{u_n})$ never vanishes at $u_n = 0$.

Next we show the nonnegativity of the sectional curvature K_S of the singular manifold S. It is sufficient to show $K_S(\partial_1 \wedge \partial_2) \geq 0$ at $u_n = 0$. Since the sectional curvature K is nonnegative, by the same argument as in the proof of Theorem 3.1,

$$\frac{\partial^2}{(\partial u_n)^2}(h_{11}h_{22} - (h_{12})^2)\bigg|_{u_n=0} \geq 0. \tag{5.1}$$

Since the restriction of f to the singular manifold is an immersion, the Gauss equation yields that

$$K_S(\partial_1 \wedge \partial_2) = \frac{g_0(\alpha_{11}, \alpha_{22}) - g_0(\alpha_{12}, \alpha_{12})}{g_{11}g_{22} - (g_{12})^2},$$

where α is the second fundamental form of the singular submanifold in $\boldsymbol{R}^{n+1}$ and $\alpha_{ij} = \alpha(f_{u_i}, f_{u_j})$.

On the other hand, since the second fundamental form h of f vanishes, $g_0(\nu_{u_n}, f_{u_j}) = 0$ holds for $j = 1, \dots, n$, that is, ν and ν_{u_n} are linearly independent vectors. Moreover,

$$\begin{aligned}\alpha_{ij} &= g_0(\alpha_{ij}, \nu)\nu + \frac{1}{|\nu_{u_n}|^2} g_0(\alpha_{ij}, \nu_{u_n})\nu_{u_n} \\ &= h_{ij}\nu + \frac{1}{|\nu_{u_n}|^2} g_0(\alpha_{ij}, \nu_{u_n})\nu_{u_n} = \frac{1}{|\nu_{u_n}|^2}(h_{ij})_{u_n}\nu_{u_n},\end{aligned}$$

since the second fundamental form h of f vanishes and

$$g_0(\alpha_{ij}, \alpha_{kl}) = \frac{1}{|\nu_{u_n}|^2}(h_{ij})_{u_n}(h_{kl})_{u_n} = \frac{1}{|\nu_{u_n}|^2}\frac{\partial^2}{(\partial u_n)^2}(h_{ij}h_{kl}) \qquad (\text{on } u_n = 0)$$

for $i, j, k, l = 1, \dots, n-1$. By (5.1),

$$K_S(\partial_1 \wedge \partial_2) = \frac{1}{g_{11}g_{22} - (g_{12})^2} \frac{\partial^2}{(\partial u_n)^2}(h_{11}h_{22} - (h_{12})^2)\bigg|_{u_n=0} \geq 0. \qquad \square$$

Example 5.3. We set

$$f(u, v, w) := (v, w, u^2 + av^2 + bw^2, u^3 + cu^2) : \boldsymbol{R}^3 \to \boldsymbol{R}^4,$$

which gives a front with the unit normal vector

$$\nu = \frac{1}{\delta}\bigl(2av(2c + 3u), 2bw(2c + 3u), -2c - 3u, 2\bigr),$$
$$\text{where} \quad \delta = \sqrt{4 + (3u + 2c)^2(1 + 4a^2v^2 + 4b^2w^2)}.$$

The singular set is the vw-plane and the u-direction is the null direction. Then all singular points are cuspidal edges. The second fundamental form is given by $h = \delta^{-1}\{6u\,du^2 - 2(3u + 2c)(a\,dv^2 + b\,dw^2)\}$, which vanishes on the singular set if $ac = bc = 0$.

On the other hand, the sectional curvatures are computed as

$$K(\partial_u \wedge \partial_v) = \frac{12a(3u + 2c)}{u\delta^2\bigl(4 + (3u + 2c)^2(1 + 4a^2v^2)\bigr)},$$
$$K(\partial_u \wedge \partial_w) = \frac{12b(3u + 2c)}{u\delta^2\bigl(4 + (3u + 2c)^2(1 + 4b^2w^2)\bigr)},$$

which are bounded in a neighborhood of the singular set if and only if $ac = bc = 0$. If $ac = bc = 0$, $K \geq 0$ if and only if $a \geq 0$ and $b \geq 0$, which implies $K_S = 4ab(3u + 2c)^2/(\delta^2|\partial_v \wedge \partial_w|^2) \geq 0$.

6. Intrinsic formulation

The Gauss-Bonnet theorem is intrinsic in nature, and it it quite natural to formulate the singularities of wave fronts intrinsically. We can characterize the limiting tangent bundles of the fronts and can give the following abstract definition:

Definition 6.1. Let M^2 be a 2-manifold. An orientable vector bundle $\mathcal{E}$ of rank 2 with a metric $\langle\ ,\ \rangle$ and a metric connection D is called *an abstract limiting tangent bundle* or *a coherent tangent bundle* if there is a bundle homomorphism

$$\psi\colon TM^2 \longrightarrow \mathcal{E}$$

such that

$$D_X\psi(Y) - D_Y\psi(X) = \psi([X,Y]), \tag{6.1}$$

where X, Y are vector fiels on M^2.

In this setting, the pull-back of the metric $ds^2 := \psi^* \langle\ ,\ \rangle$ is called *the first fundamental form* of $\mathcal{E}$. A point $p \in M^2$ is called a *singular point* if the first fundamental form is not positive definite. Since $\mathcal{E}$ is orientable, there exists a skew-symmetric bilinear form $\mu_p\colon \mathcal{E}_p \times \mathcal{E}_p \to \boldsymbol{R}$ for each $p \in M^2$, where $\mathcal{E}_p$ is the fiber of $\mathcal{E}$ at p, such that $\mu(e_1, e_2) = \pm 1$ for any orthonormal frame $\{e_1, e_2\}$ on $\mathcal{E}$.

A frame $\{e_1, e_2\}$ is called positive if $\mu(e_1, e_2) = 1$. A singular point p is called *nondegenerate* if the derivative $d\lambda$ of the function

$$\lambda := \mu\left(\psi\left(\frac{\partial}{\partial u}\right), \psi\left(\frac{\partial}{\partial v}\right)\right) \tag{6.2}$$

does not vanish at p, where $(U^2; u, v)$ is a local coordinate system of M^2 at p. On a neighborhood of a nondegenerate singular point, the singular set consists of a regular curve, called the *singular curve*. The tangential direction of the singular curve is called the *singular direction*, and the direction of the kernel of ψ is called the *null direction*. Then we can define *intrinsic cuspidal edges* and *intrinsic swallowtails* according to Fact 1.5. For a given singular curve $\gamma(t)$ consisting of intrinsic cuspidal edge points, the singular curvature function is defined by

$$\kappa_s(t) := \operatorname{sgn}\bigl(d\lambda(\eta)\bigr)\hat{\kappa}_g(t),$$

where $\hat{\kappa}_g(t) := \langle D_t\psi(\gamma'(t)), n(t)\rangle$ is the *limiting geodesic curvature*, $n(t) \in \mathcal{E}_{\gamma(t)}$ is a unit vector such that $\mu\bigl(\psi(\gamma'(t)), n(t)\bigr) = 1$, and $\eta(t)$ is the null direction such that $\bigl(\gamma'(t), \eta(t)\bigr)$ is a positive frame on M^2. Then Theorem 1.6 and Proposition 1.8 hold. Let $(U^2; e_1, e_2)$ be an orthonormal frame field of $\mathcal{E}$ such that $\mu(e_1, e_2) = 1$. Then there exists a unique 1-form α on U such that

$$D_X e_1 = -\alpha(X)e_2, \qquad D_X e_2 = \alpha(X)e_1 \qquad (X \in TM^2),$$

which is called the *connection form*. Moreover, the exterior derivative $d\alpha$ does not depend on the choice of a positive frame $(U^2; e_1, e_2)$ and gives a (globally defined) 2-form on M^2. When M^2 is compact, the integration

$$\chi_{\mathcal{E}} := \frac{1}{2\pi}\int_{M^2} d\alpha \tag{6.3}$$

is an integer called the *Euler number* of $\mathcal{E}$. Modifying the classical proof of the Gauss-Bonnet Theorem, intrinsically we get the formulas (2) and (3) in the introduction. (See [SUY] for details.) This intrinsic formulation is meaningful if we consider the following examples:

Example 6.2 (Cuspidal cross caps). A map $f\colon M^2 \to \boldsymbol{R}^3$ is called a *frontal* if there exists a unit normal vector field ν such that f_*X is perpendicular to ν for all $X \in TM^2$. A frontal is a front if $(f,\nu)\colon M^2 \to \boldsymbol{R}^3 \times S^2$ is an immersion. A cuspidal cross cap is a singular point locally diffeomorphic to the map $(u,v) \mapsto (u, v^2, uv^3)$ and is a frontal but not a front. In [FSUY], a useful criterion for cuspidal cross caps is given. Though a cuspidal cross cap is not a cuspidal edge, the limiting tangent bundle is well-defined and the singular point is an intrinsic cuspidal edge. In particular, our Gauss-Bonnet formulas hold for a frontal that admits only cuspidal edges, swallowtails and cuspidal cross caps.

Example 6.3 (Singularities with higher codimensions). A smooth map $f\colon M^2 \to \boldsymbol{R}^n$ defined on a 2-manifold M^2 into $\boldsymbol{R}^n$ $(n \geq 3)$ is called *an admissible map* if there exists a map $\nu\colon M^2 \to G_2(\boldsymbol{R}^n)$ into the oriented 2-plane Grassman manifold $G_2(\boldsymbol{R}^n)$, such that it coincides with the Gauss map of f on regular points of f. For an admissible map, the limiting tangent bundle is canonically defined and we can apply our intrinsic formulation to it.

The realization of first fundamental forms with singularities has been treated in [Kos2].

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References

[A] V. I. Arnol'd, *Topological Invariants of Plane Curves and Caustics*, *University Lecture Series* **5**, Amer. Math. Soc., Providence, RI (1994).

[AGV] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of Differentiable Maps*, Vol. 1, *Monographs in Math.* **82**, Birkhäuser, Boston (1985).

[BG] J. W. Bruce and P. J. Giblin, *Curves and Singularities*, Cambridge University Press, Cambridge, 1984.

[F] S. Fujimori, Spacelike CMC 1 surfaces with elliptic ends in de Sitter 3-space, *Hokkaido Math. J.* **35** (2006), 289–320.

[FRUYY] S. Fujimori, W. Rossman, M. Umehara, K. Yamada, and S.-D. Yang, Spacelike mean curvature one surfaces in de Sitter 3-space, preprint, arXiv:0706.0973.

[FSUY] S. Fujimori, K. Saji, M. Umehara, and K. Yamada, Singularities of maximal surfaces, *Math. Z.* **259** (2008), 827–848.

[GG] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, *Graduate Texts in Math.* **14**, Springer-Verlag, New York, 1973.

[GMM] J. A. Gálvez, A. Martínez, and F. Milán, Complete linear Weingarten surfaces of Bryant type. A Plateau problem at infinity, *Trans. Amer. Math. Soc.* **356** (2004), 3405–3428.

[IM] G. Ishikawa and Y. Machida, Singularities of improper affine spheres and surfaces of constant Gaussian curvature, *Intern. J. Math.* **17** (2006), 269–293.

[IST] S. Izumiya, K. Saji, and M. Takahashi, Horospherical flat surfaces in hyperbolic 3-space, preprint.

[Kok] M. Kokubu, Surfaces and fronts with harmonic-mean curvature one in hyperbolic three-space, preprint, arXiv:math/0504124.

[KRSUY] M. Kokubu, W. Rossman, K. Saji, M. Umehara, and K. Yamada, Singularities flat fronts in hyperbolic 3-space, *Pacific J. of Math.* **221** (2005), 303–351.

[KUY] M. Kokubu, M. Umehara, and K. Yamada, Flat fronts in hyperbolic 3-space, *Pacific J. Math.* **216** (2004), 149–175.

[KRUY] M. Kokubu, W. Rossman, M. Umehara, and K. Yamada, Flat fronts in hyperbolic 3-space and their caustics, *J. Math. Soc. Japan* **59** (2007), 265–299.

[Kos1] M. Kossowski, The Boy-Gauss-Bonnet theorems for C^∞-singular surfaces with limiting tangent bundle, *Ann. Global Anal. Geom.* **21** (2002), 19–29.

[Kos2] ———, Realizing a singular first fundamental form as a nonimmersed surface in Euclidean 3-space, *J. Geom.* **81** (2004), 101–113.

[LLR] R. Langevin, G. Levitt, and H. Rosenberg, Classes d'homotopie de surfaces avec rebroussements et queues d'aronde dans $\mathbb{R}^3$, *Canad. J. Math.* **47** (1995), 544–572.

[LY] S. Lee and S.-D. Yang, A spinor representation for spacelike surfaces of constant mean curvature -1 in de Sitter three-space, *Osaka J. Math.* **43** (2006), 641–663.

[Mac] Y. Machigashira, The Gaussian curvature of Alexandrov spaces, *J. Math. Soc. Japan* **50** (1998) 859–878.

[Mar] A. Martínez, Improper affine maps, *Math. Z.* **249** (2005), 755–766.

[S] K. Shiohama, Total curvatures and minimal areas of complete open surfaces, *Proc. Amer. Math. Soc.* **94** (1985), 310–316.

[SUY] K. Saji, M. Umehara, and K. Yamada, Behavior of corank one singular points on wave fronts, *Kyushu J. of Math.* **62** (2008), 259–280.

[SUY2] ———, A_k singularities of wave fronts, preprint, *Math. Proc. Camb. Phil. Soc.*, Nov. 2008, arXiv:0804.0701.

[U] M. Umehara, Geometry of curves and surfaces with singularities, in *Mathematics in the 21st Century—Unscaled Peaks of Geometry* (R. Miyaoka and M. Kotani, eds.), Nihon-Hyoronsha, 2004 (in Japanese).

[UY] M. Umehara and K. Yamada, Maximal surfaces with singularities in Minkowski space, *Hokkaido Math. J.* **35** (2006), 13–40.

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