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Nontangential limits in $\mathcal{P}^t(\mu)$-spaces and the index of invariant subspaces

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Abstract

Let μ be a finite positive measure on the closed disk $\overline{\mathbb{D}}$ in the complex plane, let $1 \leq t < \infty$, and let $P^t(\mu)$ denote the closure of the analytic polynomials in $L^t(\mu)$. We suppose that $\mathbb{D}$ is the set of analytic bounded point evaluations for $P^t(\mu)$, and that $P^t(\mu)$ contains no nontrivial characteristic functions. It is then known that the restriction of μ to $\partial\mathbb{D}$ must be of the form $h|dz|$. We prove that every function $f \in P^t(\mu)$ has nontangential limits at $h|dz|$-almost every point of $\partial\mathbb{D}$, and the resulting boundary function agrees with f as an element of $L^t(h|dz|)$.

Our proof combines methods from James E. Thomson's proof of the existence of bounded point evaluations for $P^t(\mu)$ whenever $P^t(\mu) \neq L^t(\mu)$ with Xavier Tolsa's remarkable recent results on analytic capacity. These methods allow us to refine Thomson's results somewhat. In fact, for a general compactly supported measure ν in the complex plane we are able to describe locations of bounded point evaluations for $P^t(\nu)$ in terms of the Cauchy transform of an annihilating measure.

As a consequence of our result we answer in the affirmative a conjecture of Conway and Yang. We show that for $1 < t < \infty$ dim $\mathcal{M}/z\mathcal{M} = 1$ for every nonzero invariant subspace $\mathcal{M}$ of $P^t(\mu)$ if and only if $h \not\equiv 0$.

We also investigate the boundary behaviour of the functions in $P^t(\mu)$ near the points $z \in \partial\mathbb{D}$ where $h(z) = 0$. In particular, for $1 < t < \infty$ we show that there are interpolating sequences for $P^t(\mu)$ that accumulate nontangentially almost everywhere on $\{z : h(z) = 0\}$.

1. Introduction, statement of main results and preliminaries

Let μ be a compactly supported finite positive measure on the complex plane $\mathbb{C}$, and $t \in [1, \infty)$ with conjugate exponent $t' = \frac{t}{t-1}$. We denote by $\mathcal{P}$

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the set of polynomials in the complex variable z, and by $\mathcal{P}^t(\mu)$ the closure of $\mathcal{P}$ in $L^t(\mu)$. Multiplication by z defines a bounded linear operator on $\mathcal{P}^t(\mu)$ which we will denote by S. An *invariant subspace* of $\mathcal{P}^t(\mu)$ is a closed linear subspace $M \subseteq \mathcal{P}^t(\mu)$ such that $SM \subseteq M$. For $\lambda \in \mathbb{C}$ we denote evaluation on $\mathcal{P}$ at λ by e_λ, i.e. $e_\lambda(p) = p(\lambda)$ for $p \in \mathcal{P}$. If e_λ is bounded in the $L^t(\mu)$ norm then it extends to a bounded linear functional on $\mathcal{P}^t(\mu)$, which we will also denote by e_λ. With a slight abuse of English we refer to such a λ as a *bounded point evaluation* (bpe) for $\mathcal{P}^t(\mu)$, and we set $M_\lambda = \|e_\lambda\|_{\mathcal{P}^t(\mu)^*}$.

If $\lambda_0 \in \mathbb{C}$ is such that there is a neighborhood U of λ_0 consisting entirely of bpe's for $\mathcal{P}^t(\mu)$ with $\lambda \mapsto e_\lambda(f)$ analytic in U for all $f \in \mathcal{P}^t(\mu)$, then we say that λ_0 is an *analytic bounded point evaluation* (abpe) for $\mathcal{P}^t(\mu)$. We denote the set of abpe's for $\mathcal{P}^t(\mu)$ by $\operatorname{abpe}(\mathcal{P}^t(\mu))$. A simple argument using the Uniform Boundedness Principle and the functionals $\frac{1}{\lambda-\zeta}(e_\lambda - e_\zeta)$ shows that the map $\lambda \mapsto M_\lambda$ is continuous, in fact locally Lipschitz, on $\operatorname{abpe}(\mathcal{P}^t(\mu))$.

We will be chiefly concerned in this paper with the case when $\mathcal{P}^t(\mu)$ can be identified with a space of analytic functions in the open unit disk $\mathbb{D} = B(0,1)$ in $\mathbb{C}$. In fact, we will assume that $\operatorname{spt}\mu \subseteq \operatorname{clos}\mathbb{D}$, $\operatorname{abpe}(\mathcal{P}^t(\mu)) = \mathbb{D}$, and $\mathcal{P}^t(\mu)$ is *irreducible*, by which we mean that $\mathcal{P}^t(\mu)$ contains no nontrivial characteristic functions. If $f \in \mathcal{P}^t(\mu)$ then clearly $f(\lambda) = e_\lambda(f)$ for μ-almost all $\lambda \in \mathbb{D}$, so that no confusion will result if we write $f(\lambda) = e_\lambda(f)$ for *all* $\lambda \in \mathbb{D}$, thus thinking of f as being defined in all of $\mathbb{D}$ and μ-almost everywhere on $\partial\mathbb{D}$.

It is well known that irreducibility implies that $\mu|\partial\mathbb{D}$ is absolutely continuous with respect to Lebesgue measure m on $\partial\mathbb{D}$ — this follows easily from a theorem of F. and M. Riesz [RR16] (see also [Gar81, p. 125]) that says that if $E \subseteq \partial\mathbb{D}$ is a compact set with $m(E) = 0$, then there is a function f, continuous on $\operatorname{clos}\mathbb{D}$ and analytic in $\mathbb{D}$, such that $f \equiv 1$ on E and $|f(z)| < 1$ for $z \in (\operatorname{clos}\mathbb{D}) \setminus E$. We will write $\mu|\partial\mathbb{D} = \frac{1}{2\pi} h \cdot m$.

It is also well known that if $K \subseteq \mathbb{D}$ is compact then $\mathcal{P}^t(\mu|\operatorname{clos}\mathbb{D} \setminus K) = \mathcal{P}^t(\mu)$ with equivalence of norms (see also part of the proof of our Lemma 4.5). This easily implies that $\mathcal{P}^t(\mu)$ has the *division property* at all $\lambda \in \mathbb{D}$, i.e., if $f \in \mathcal{P}^t(\mu)$ and $\lambda \in \mathbb{D}$ with $f(\lambda) = 0$, then $\frac{f(z)}{z-\lambda} \in \mathcal{P}^t(\mu)$. It also implies that S is bounded below, so that if M is an invariant subspace of $\mathcal{P}^t(\mu)$ then SM is closed. We define the *index* of an invariant subspace M to be the dimension of M/SM.

In this paper we will be studying the boundary behavior of functions in $\mathcal{P}^t(\mu)$, the index of invariant subspaces, and related questions. In order to discuss boundary behavior we must first review the notions of nontangential limits and nontangential cluster points. For $0 < \sigma < 1$ and $z \in \partial\mathbb{D}$ we define the nontangential approach region $\Gamma_\sigma(z)$ to be the interior of the convex hull of $\{z\} \cup B(0,\sigma)$. We say a function f in $\mathbb{D}$ has a *nontangential limit* $f^*(z)$ at z if $\lim_{\substack{\lambda \to z \\ \lambda \in \Gamma_\sigma(z)}} f(\lambda) = f^*(z)$ for some $\sigma \in (0,1)$. We say a point $z \in \partial\mathbb{D}$ is a *nontangential cluster point* of a set $\Lambda \subseteq \mathbb{D}$ if for some $\sigma \in (0,1)$, $\Gamma_\sigma(z) \cap (\Lambda \setminus B(0,r)) \neq \varnothing$ for all $r \in (0,1)$. It is well known that the existence of nontangential limits and nontangential cluster points on a set $E \subseteq \partial\mathbb{D}$ is independent of σ up to sets of m-measure zero, and so most of the time we will standardize to $\sigma = \frac{1}{2}$ and write $\Gamma(z)$ for $\Gamma_{\frac{1}{2}}(z)$.

The two most familiar examples of the type of $\mathcal{P}^t(\mu)$-spaces we are considering are when $\mu = \frac{1}{2\pi}m$ is normalized Lebesgue measure on $\partial\mathbb{D}$ and when $\mu = \frac{1}{\pi}A$ is normalized Lebesgue measure on $\mathbb{D}$. These two cases satisfy respectively $\mu(\partial\mathbb{D}) > 0$ and $\mu(\partial\mathbb{D}) = 0$, and they illustrate well the sort of phenomena we are interested in.

$\mu = \frac{1}{2\pi}m$: The spaces $\mathcal{P}^t(\mu)$ are the *Hardy spaces* H^t. Every $f \in H^t$ has a nontangential limit $f^*(z)$ at m-almost all $z \in \partial\mathbb{D}$, and $f^* = f$ as elements of $L^t\left(\frac{1}{2\pi}m\right)$. It follows that $f^* = 0$ implies $f = 0$, but in fact a stronger result is true: if $f^* = 0$ on a set of positive m-measure then $f = 0$.

A celebrated theorem of Arne Beurling [Beu48] describes the invariant subspaces of H^2, and shows in particular that every nonzero invariant subspace of H^2 has index 1. It is well-known that this fact extends to all values of t.

For these and other basic facts about H^t, see e.g., [Dur70], [Gar81], [Koo98].

$\mu = \frac{1}{\pi}A$: The spaces $\mathcal{P}^t(\mu)$ are the *Bergman spaces* L^t_a. There are functions in L^t_a that have nontangential limits at no point of $\partial\mathbb{D}$. For any integer n, or even for $n = \infty$, there are invariant subspaces of L^t_a having index n.

The first of these facts has been known for a long time. The second was discovered somewhat later and in view of the H^t case came as a bit of a surprise. The case $t = 2$ was done by Constantin Apostol, Hari Bercovici, Ciprian Foias, and Carl Pearcy [ABFP85]. This was extended to the case $1 < t < \infty$ by Jörg Eschmeier [Esc94], and to all t, $1 \leq t < \infty$ by Haakan Hedenmalm in [Hed93] and by Hedenmalm, the second-named author of the present paper, and Kristian Seip in [HRS96].

In our context it is important to note that the proofs of Apostol, Bercovici, Foias, and Pearcy and of Eschmeier just depend on the fact that $\|S^n f\| \to 0$ for all f, and hence their results hold for $\mathcal{P}^t(\mu)$ whenever $\mu(\partial\mathbb{D}) = 0$ and $1 < t < \infty$.

We are led to study the case when $\mu(\partial\mathbb{D}) > 0$. In this case we can talk about a boundary function $f|\partial\mathbb{D}$ for $f \in \mathcal{P}^t(\mu)$, simply as an element of $L^t(\mu|\partial\mathbb{D})$. The relation between $f|\partial\mathbb{D}$ and possible boundary values of the analytic function f in $\mathbb{D}$ is however not immediately clear. Work by a number of researchers has suggested a strong connection between this type of question and questions about the index of invariant subspaces for $\mathcal{P}^t(\mu)$. The above examples suggest the following questions in the case $\mu(\partial\mathbb{D}) > 0$:

(1) Is $f|\partial\mathbb{D}$ the boundary value function of f in some suitable sense?
(2) Is $f|\partial\mathbb{D} = f^*$ $\mu|\partial\mathbb{D}$-almost everywhere?
(3) Is f determined by $f|\partial\mathbb{D}$, i.e., does $f|\partial\mathbb{D} = 0$ imply that $f = 0$?
(4) Is the index of every nonzero invariant subspace equal to 1?

An answer to Question (1) was given by the two first-named authors of the present paper together with William T. Ross in [ARR98]: if $f \in \mathcal{P}^t(\mu)$ then $f_r \to f|\partial\mathbb{D}$ in measure, where $f_r(z) = f(rz)$. This type of convergence is however not strong enough to deal with Question (3). On the other hand it is not hard to see that if $f \equiv 0$ on $\mathbb{D}$ then $f = 0$; so by Privalov's Theorem ([Gar81, p. 94]) an affirmative answer to Question (2) implies an affirmative answer to Question (3).

The first author whose work suggested a connection between Question (4) and the existence of boundary values of functions was Liming Yang in [Yan95b]. Later, in [CY98], John B. Conway and Liming Yang conjectured that the answer to Question (4) is affirmative, specifically in the case $t = 2$. This conjecture has been supported by numerous partial results by various authors, both prior to and subsequent to Conway and Yang's paper, and these results also give more evidence of a connection between Questions (2) and (4). We mention here the work of J. Akeroyd [Ake01], [Ake02], [Ake03], A. Aleman and S. Richter [AR97], T.L. Miller and R.C. Smith [MS90], T.L. Miller, W. Smith, and L. Yang [MSY99], R. F. Olin and J. E. Thomson [OT80], J.E Thomson and L. Yang [TY95], T. Trent [Tre79b], [Tre79a], Z. Wu and L. Yang [WY98], L. Yang [Yan95a].

Our first result answers all of the above questions in the affirmative.

Theorem A. *Suppose that μ is supported in* $\operatorname{clos}\mathbb{D}$ *and is such that* $\operatorname{abpe}\big(\mathcal{P}^t(\mu)\big) = \mathbb{D}$ *and $\mathcal{P}^t(\mu)$ is irreducible, and that $\mu(\partial\mathbb{D}) > 0$. Then*:

a) *If $f \in \mathcal{P}^t(\mu)$ then the nontangential limit $f^*(z)$ of f exists for $\mu|\partial\mathbb{D}$-almost all z, and $f^* = f|\partial\mathbb{D}$ as elements of $L^t(\mu|\partial\mathbb{D})$.*

b) *Every nonzero invariant subspace of $\mathcal{P}^t(\mu)$ has index* 1.

Away from the part of $\partial\mathbb{D}$ where μ has mass we might expect that the boundary behavior of $\mathcal{P}^t(\mu)$ functions could be wild. To see just how wild, we first recall the notion of interpolating sequences. Let $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ be a sequence of distinct points in $\mathbb{D}$. We consider the weighted sequence space $l^t_\Lambda(\mu)$, defined as the set of complex sequences $\{a_n\}$ for which

$$\|\{a_n\}\|_{l^t_\Lambda(\mu)} = \left[\sum_n \left(\frac{|a_n|}{M_{\lambda_n}}\right)^t\right]^{1/t} < \infty,$$

and we consider also the map T_Λ taking a function f on $\mathbb{D}$ to the sequence $T_\Lambda f = \{f(\lambda_n)\}$. We say that Λ is an *interpolating sequence* for $\mathcal{P}^t(\mu)$ if $T_\Lambda(\mathcal{P}^t(\mu)) = l^t_\Lambda(\mu)$. We can now state our second main result.

THEOREM B. *Suppose that μ is supported in* $\operatorname{clos}\mathbb{D}$ *and is such that* $\operatorname{abpe}\left(\mathcal{P}^t(\mu)\right)=\mathbb{D}$ *and $\mathcal{P}^t(\mu)$ is irreducible. Let $E\subseteq\partial\mathbb{D}$ with $\mu(E)=0$. Then*

If $t\in(1,\infty)$: there is an interpolating sequence for $\mathcal{P}^t(\mu)$ that clusters nontangentially at m-almost every point of E.

If $t=1$: there is a sequence Λ clustering nontangentially at m-almost every point of E such that $T_\Lambda\left(\mathcal{P}^1(\mu)\right)\supseteq l^1_\Lambda(\mu)$.

It is clear that in either case one obtains functions in $\mathcal{P}^t(\mu)$ that have nontangential limits at m-almost no points of E, since using the test function 1 we see that $M_\lambda\geq\|\mu\|^{-1/t}$ for all $\lambda\in\mathbb{D}$. In the case where $\mu(\partial\mathbb{D})=0$ and $t\in(1,\infty)$ we see that Theorem B gives us the existence of an interpolating sequence for $\mathcal{P}^t(\mu)$ that nontangentially clusters at m-almost every point of $\partial\mathbb{D}$. The argument used to prove Proposition 7.3 in [ARS02] then shows the existence of invariant subspaces of $\mathcal{P}^t(\mu)$ of index greater than 1, thus giving another proof of the result of Apostol, Bercovici, Foias, and Pearcy, and that of Eschmeier mentioned above.

An important part of the proof of Theorem A consists of getting estimates on M_λ for $\lambda\in\mathbb{D}$. If we use the test function

$$f_\lambda(w)=\frac{1}{(1-\bar\lambda w)^{2/t}}$$

we see that

$$M_\lambda\geqslant\left[\left(1-|\lambda|^2\right)^{1/t}\left(\int\frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}\,d\mu(w)\right)^{1/t}\right]^{-1}. \tag{1.1}$$

In their study of the boundary behavior of M_λ [KT76], Thomas Kriete and Tavan Trent prove the following, which we will list as Lemma 1.1 for reference purposes:

LEMMA 1.1 (Kriete and Trent). *Let ω be a measure on $\mathbb{D}$. Then*

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}}\int\frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}\,d\omega(w)=0$$

for m-almost all $z\in\partial\mathbb{D}$. □

Now suppose μ satisfies the hypotheses of Theorem A and recall that we are writing $\mu|\partial\mathbb{D}=\frac{1}{2\pi}h\cdot m$. On $\partial\mathbb{D}$, $\frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}$ is just the Poisson kernel, so that combining Lemma 1.1 with a standard property of Poisson integrals, we see

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}}\int\frac{1-|\lambda|^2}{\left|1-\bar\lambda w\right|^2}\,d\mu(w)=h(z) \tag{1.2}$$

for m-almost all $z \in \partial\mathbb{D}$. Kriete and Trent observe that (1.1) and (1.2) combine to show that

$$\varliminf_{\substack{\lambda \to z \\ \lambda \in \Gamma(z)}} \left(1-|\lambda|^2\right)^{1/t} M_\lambda \geqslant \frac{1}{h(z)^{1/t}} \tag{1.3}$$

for m-almost all $z \in \partial\mathbb{D}$.

Our proof of Theorem A will depend on proving the converse estimate

$$\varlimsup_{\substack{\lambda \to z \\ \lambda \in \Gamma(z)}} \left(1-|\lambda|^2\right)^{1/t} M_\lambda \leqslant \frac{C}{h(z)^{1/t}} \tag{1.4}$$

for m-almost all $z \in \partial\mathbb{D}$, where C is some constant. We will prove this estimate in Section 3. These two inequalities are indicative of H^t behavior near points z for which $h(z) \neq 0$, since as is well known $M_\lambda = \frac{1}{(1-|\lambda|^2)^{1/t}}$ for H^t. It is thus of interest to note that at least for $t > 1$ we can refine the estimates (1.3) and (1.4) to an asymptotic equality:

Theorem C. *Under the hypotheses of Theorem* A, *if* $t > 1$,

$$\lim_{\substack{\lambda \to z \\ \lambda \in \Gamma(z)}} \left(1-|\lambda|^2\right)^{1/t} M_\lambda = \frac{1}{h(z)^{1/t}}$$

for m-almost all $z \in \partial\mathbb{D}$.

Our results have applications to general $\mathcal{P}^t(\mu)$-spaces. In his remarkable paper [Tho91] James E. Thomson describes the structure of $\mathcal{P}^t(\mu)$ where μ is any compactly supported positive measure on $\mathbb{C}$.

Thomson's Theorem. *There is a Borel partition $\{\Delta_i\}_{i=0}^\infty$ of* $\operatorname{spt}\mu$ *such that for $i \geqslant 1$ the space $\mathcal{P}^t(\mu|\Delta_i)$ contains no nontrivial characteristic functions and*

$$\mathcal{P}^t(\mu) = L^t(\mu|\Delta_0) \oplus \left\{ \bigoplus_{i=1}^{\infty} \mathcal{P}^t(\mu|\Delta_i) \right\}.$$

Furthermore, *if U_i is the open set of analytic bounded point evaluations for $\mathcal{P}^t(\mu|\Delta_i)$ for $i \geqslant 1$, then U_i is a simply connected region and the closure of U_i contains Δ_i.* □

In particular if $\mathcal{P}^t(\mu) \neq L^t(\mu)$ then there exist bpe's for $\mathcal{P}^t(\mu)$. Indeed the proof of this fact is the most difficult part of, and the chief motivation for, Thomson's work. We also note that Thomson's Theorem together with Theorem 4.11 of the same paper shows that the bpe's for $\mathcal{P}^t(\mu)$ are either abpe's or point masses for μ.

Because of Thomson's decomposition, the study of general $\mathcal{P}^t(\mu)$-spaces can be reduced to the case where $\mathcal{P}^t(\mu)$ contains no nontrivial characteristic functions and $\operatorname{abpe}\big(\mathcal{P}^t(\mu)\big)$ is a nonempty simply connected open set whose closure contains $\operatorname{spt}\mu$. One might expect to be able to use the Riemann Mapping Theorem to further reduce to the case where $\operatorname{abpe}\big(\mathcal{P}^t(\mu)\big) = \mathbb{D}$. This can in fact be done, but it is far from trivial; it was accomplished by Robert F. Olin and Liming Yang in [OY95]. As a consequence, there are versions of our results valid in the case when $\operatorname{abpe}\big(\mathcal{P}^t(\mu)\big)$ is an arbitrary bounded simply connected region.

An important motivation in the study of $\mathcal{P}^t(\mu)$-spaces comes from the theory of subnormal operators — see e.g. [Con81]. By the Spectral Theorem any cyclic subnormal operator is unitarily equivalent to the operator S on some $\mathcal{P}^2(\mu)$-space — see J. Bram [Bra55]. Thus our results have implications in the study of subnormal operators — indeed it was in this context that Conway and Yang posed the conjecture that we discussed above.

We now outline the remainder of the paper. The main difficulty in the proof of Theorem A is the proof of the inequality (1.4). In order to prove this we must develop a method for estimating norms of point evaluations. We do this in Section 2, where the main result, Theorem 2.1, provides us with the needed tool. The proof of Theorem 2.1 will be accomplished by combining part of Thomson's proof of the existence of bpe's in [Tho91] with Xavier Tolsa's remarkable recent results on analytic capacity ([Tol03]). Our methods here are quite similar to those developed independently by James E. Brennan in his proof of Thomson's Theorem using Tolsa's work ([Bre06]). Besides its importance in the proof of the inequality (1.4), Theorem 2.1 also allows us to prove a refinement of Thomson's result (Corollary 2.2).

In Section 3 we show how Theorem 2.1 implies the inequality (1.4) and using this inequality we prove Theorems A and C. Section 4 is devoted to the proof of Theorem B, and in Section 5 we show how the work of Olin and Yang allows us to apply our results in general $\mathcal{P}^t(\mu)$-spaces.

We have had very stimulating and useful conversations regarding the material in this paper with John Akeroyd, Daniel Luecking, Jim Thomson, and Alexander Volberg, and we would like to express our gratitude to them.

2. Bounded point evaluations

The main difficulty in the proof of Theorem A comes in the proof of the estimate (1.4), which depends in turn on developing a method for estimating the norms of point evaluations. We begin our efforts in this direction by outlining Thomson's proof of the existence of bpe's in [Tho91], but we must first deal with some preliminaries.

Let ω be a compactly supported measure on $\mathbb{C}$. The *Cauchy transform* of ω is defined by

$$\mathrm{C}\,\omega(z) = \int \frac{d\omega(w)}{w-z}$$

for all $z \in \mathbb{C}$ for which $\int \frac{d|\omega|(w)}{|w-z|} < \infty$. A standard application of Fubini's Theorem shows that $\mathrm{C}\,\omega \in L^s_{\mathrm{loc}}(\mathbb{C})$ for $s \in (0,2)$, in particular that it is defined for A-almost all z, and clearly $\mathrm{C}\,\omega$ is analytic in $\mathbb{C}_\infty \setminus \operatorname{spt}\omega$, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. It is also standard that if ω is a compactly supported bounded function times area measure, then $\mathrm{C}\,\omega$ is continuous.

Now suppose that ν is a compactly supported measure on $\mathbb{C}$ that annihilates the polynomials $\mathcal{P}$. If $p \in \mathcal{P}$ and $w \in \mathbb{C}$ then the function q defined by

$$q(z) = \frac{p(z) - p(w)}{z - w}$$

is again in $\mathcal{P}$; hence

$$0 = \int q\, d\nu = \int \frac{p(z) - p(w)}{z - w}\, d\nu(z).$$

Rearranging, we see that we have shown that

$$\mathrm{C}\,\nu(w)p(w) = \int \frac{p(z)\, d\nu(z)}{z - w} \tag{2.1}$$

for A-almost all w whenever $p \in \mathcal{P}$ and ν is a compactly supported measure on $\mathbb{C}$ that annihilates $\mathcal{P}$.

We turn to Thomson's proof. Suppose that μ is a compactly supported positive measure on $\mathbb{C}$ for which $\mathcal{P}^t(\mu) \neq L^t(\mu)$. There is then a non-zero function $G \in \mathcal{P}^t(\mu)^\perp \subseteq L^{t'}(\mu)$. We set $\nu = G\mu$ so that ν annihilates $\mathcal{P}$ and (2.1) applies. If $\alpha \in \mathbb{C}$ and U is an annulus centered at α and such that $|\mathrm{C}\,\nu|$ is bounded below on U, then (2.1) can easily be used to show that α is a bpe for $\mathcal{P}^1(|\nu|)$ and hence, by Hölder's inequality, for $\mathcal{P}^t(\mu)$. This observation was used by Brennan in [Bre79] to prove the existence of bpe's in some special cases, and it is the starting point for Thomson's proof. Thomson devised a coloring scheme on dyadic squares which, given a point $a \in \mathbb{C}$ and a positive integer m, starts with a dyadic square of side length 2^{-m} containing a and either terminates at some finite stage or produces an infinite sequence of annuli surrounding a. These annuli are made up of dyadic squares colored red, on which the mean value of $|\mathrm{C}\,\nu|$ is controlled from below. He then used an approximation technique due to Scott Brown [Bro78] to show that this sequence of annuli could replace the single annulus in the above argument to show that a is a bpe for $\mathcal{P}^1(|\nu|)$, with a bound that just depends on the integer m. Thomson finishes his argument by showing that if his coloring scheme were to terminate for all $a \in \mathbb{C}$ and all positive integers m, then a contradiction would result. His ingenious proof, utilizing the Vitushkin localization operators, is somewhat notorious for giving no direct information on the location of the bpe's or bounds on the norms of the point evaluations.

Later in this section (in the proof of Lemma B below) we will be following the first part of Thomson's proof outlined above almost verbatim, but since it is exactly the estimates on the norms of point evaluations that we are concerned with, we must replace the second part of the argument by one that will allow us to make such estimates.

Our argument at this point will make essential use of Xavier Tolsa's recent work on analytic capacity, which we now discuss. For a compact $K \subseteq \mathbb{C}$ we define the *analytic capacity* of K by

$$\gamma(K) = \sup |f'(\infty)|$$

where the sup is taken over those functions f analytic in $\mathbb{C}_\infty \setminus K$ for which $|f(z)| \leqslant 1$ for all $z \in \mathbb{C}_\infty \setminus K$, and

$$f'(\infty) = \lim_{z\to\infty} z[f(z) - f(\infty)].$$

The analytic capacity of a general $E \subseteq \mathbb{C}$ is defined to be $\gamma(E) = \sup\{\gamma(K) : K \subseteq E,\ K \text{ compact}\}$. A good source for basic information about analytic capacity is [Gar72].

A related capacity, γ_+, is defined for $E \subset \mathbb{C}$ by

$$\gamma_+(E) = \sup \|\mu\|$$

where now the sup is taken over positive measures μ with compact support contained in E for which $\| \mathrm{C}\,\mu\|_{L^\infty(\mathbb{C})} \leqslant 1$. Since $\mathrm{C}\,\mu$ is analytic in $\mathbb{C}_\infty \setminus \operatorname{spt} \mu$ and $(\mathrm{C}\,\mu)'(\infty) = -\|\mu\|$ we have

$$\gamma_+(E) \leqslant \gamma(E)$$

for all $E \subseteq \mathbb{C}$. In 2001, Tolsa proved the astounding result that γ_+ and γ are actually equivalent [Tol03]:

Tolsa's Theorem. *There is an absolute constant A_T such that*

$$\gamma(E) \leqslant A_T\gamma_+(E)$$

for all $E \subseteq \mathbb{C}$. □

Tolsa's result answered several important basic questions about analytic capacity that had been open for at least fifty years.

We are going to obtain the estimates we need by using Thomson's coloring scheme together with Tolsa's Theorem to prove the following, the main result of this section:

Theorem 2.1. *There are constants $\epsilon_0 > 0$ and $C_0 < \infty$ such that the following is true. If ν is a compactly supported measure in $\mathbb{C}$ that annihilates* *the polynomials $\mathcal{P}$, and $\nu = \nu_1 + \nu_2$ where ν_1 and ν_2 are compactly supported measures in $\mathbb{C}$ with*

$$\operatorname{Re} \mathrm{C}\,\nu_1 \geqslant 1 \qquad a.e.\ [A]\ in\ \operatorname{clos}\mathbb{D}$$

and

$$\|\nu_2\| < \epsilon_0,$$

then

$$|p(0)| \leqslant C_0 \int |p(w)| d|\nu|(w) \qquad \text{for all } p \in \mathcal{P}.$$

Theorem 2.1 will follow in turn from the next two lemmas.

Lemma A. *Suppose ω is a compactly supported bounded function times area measure. We then have the following weak-type inequality for analytic capacity,*

$$\gamma([\operatorname{Re} \mathrm{C}\,\omega \geqslant a]) \leqslant \frac{A_T}{a}\|\omega\| \qquad \text{for all } a > 0,$$

where A_T is Tolsa's constant.

Remark. For a general compactly supported measure ω, $\mathrm{C}\,\omega$ is only defined A-almost everywhere, so that $\gamma([\operatorname{Re} \mathrm{C}\,\omega \geqslant a])$ might not even make sense. The restriction we have put on ω avoids this problem since it implies that $\mathrm{C}\,\omega$ is continuous.

Lemma B. *There are absolute constants $\epsilon_1 > 0$ and $C_1 < \infty$ with the following property. Let $E \subset \operatorname{clos}\mathbb{D}$ with $\gamma(E) < \epsilon_1$. Then*

$$|p(0)| \leqslant C_1 \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p| \frac{dA}{\pi} \qquad \text{for all } p \in \mathcal{P}.$$

As we will see, Lemma A is an immediate consequence of Tolsa's Theorem via a standard capacity-type argument. The proof of Lemma B will be the main difficulty. Before proving these lemmas we will show how Theorem 2.1 follows from them, and then show how the theorem implies a refinement of Thomson's result on the existence of bpe's (Corollary 2.2).

Let ν, ν_1, ν_2 satisfy the hypotheses of Theorem 2.1 with $\epsilon_0 = \epsilon_1/2A_T$. By convolving with $\frac{n^2}{\pi}\chi_{B(0,\frac{1}{n})}$ and taking limits as $n \to \infty$, we see that we may assume that the measures ν, ν_1, ν_2 are all compactly supported bounded functions times area measures, so that $\mathrm{C}\,\nu$, $\mathrm{C}\,\nu_1$, $\mathrm{C}\,\nu_2$ are continuous. Set $E = [-\operatorname{Re} \mathrm{C}\,\nu_2 \geqslant \frac{1}{2}]$. We apply Lemma A with $a = \frac{1}{2}$ to $-\nu_2$ to get

$$\gamma(E) \leqslant 2A_T\|\nu_2\| < \epsilon_1. \tag{2.2}$$

For $w \in (\operatorname{clos}\mathbb{D}) \setminus E$ we have

$$|\,\mathrm{C}\,\nu(w)| \geqslant \operatorname{Re} \mathrm{C}\,\nu(w) > 1 - \frac{1}{2} = \frac{1}{2}. \tag{2.3}$$

By (2.2) E satisfies the hypotheses of Lemma B; hence for $p \in \mathcal{P}$ we can apply that lemma together with (2.1) and (2.3) to obtain

$$\begin{aligned} |p(0)| &\leqslant C_1 \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p| \frac{dA}{\pi} \\ &= C_1 \int_{w\in(\operatorname{clos}\mathbb{D})\setminus E} \left| \frac{1}{\mathrm{C}\,\nu(w)} \int \frac{p(z)d\nu(z)}{z-w} \right| \frac{dA(w)}{\pi} \\ &\leqslant 2C_1 \int |p(z)| \int_{w\in\operatorname{clos}\mathbb{D}} \frac{1}{|z-w|} \frac{dA(w)}{\pi} d|\nu|(z) \\ &\leqslant 4C_1 \int |p(z)| d|\nu|(z), \end{aligned}$$

where we have used the inequality

$$\int_{w\in\Delta} \frac{1}{|z-w|} \frac{dA(w)}{\pi} \leqslant 2\sqrt{\frac{A(\Delta)}{\pi}} \tag{2.4}$$

for $z \in \mathbb{C}$, $\Delta \subseteq \mathbb{C}$ (see [Gar72, pp. 2, 3]). This proves Theorem 2.1 with $C_0 = 2C_1$. □

As discussed above, Thomson's proof of the existence of bpe's does not result in a criterion for a point to be a bpe. Using Theorem 2.1 we can give such a criterion.

Corollary 2.2. *Let μ be a compactly supported positive measure on $\mathbb{C}$, let $1 \le t < \infty$, let $G \in \mathcal{P}^t(\mu)^\perp$, and set $\nu = G\mu$. Suppose z_0 is a Lebesgue point for $\mathrm{C}\,\nu$ at which $\mathrm{C}\,\nu \neq 0$. Suppose also that z_0 is a point of lower linear density zero for ν, i.e. $\underline{\lim}_{r\to 0^+} \frac{1}{r}|\nu|(B(z_0,r)) = 0$. Then z_0 is a* bpe *for $\mathcal{P}^t(\mu)$.*

In particular, A-almost every point at which $\mathrm{C}\,\nu \neq 0$ is a bpe *for $\mathcal{P}^t(\mu)$.*

Proof. We may clearly assume $z_0 = 0$. Set $\mathrm{C}\,\nu(0) = a \neq 0$. For $r > 0$ to be determined set $\nu_1 = \nu|\mathbb{C} \setminus B(0,r)$, $\nu_2 = \nu|B(0,r)$. Now,

$$\begin{aligned} \int_{B(0,r)} |\mathrm{C}\,\nu_2| \frac{dA}{\pi} &= \int_{|z|<r} \left| \int_{|w|<r} \frac{d\nu(w)}{w-z} \right| \frac{dA(z)}{\pi} \\ &\leqslant \int_{|w|<r} \int_{|z|<r} \frac{dA(z)}{\pi|w-z|} d|\nu|(w) \\ &\leqslant 2r|\nu|(B(0,r)), \end{aligned}$$

where we have used (2.4). Hence by hypothesis

$$\begin{aligned} \int_{B(0,r)} |\mathrm{C}\,\nu_1 - a| \frac{dA}{\pi} &\leqslant \int_{B(0,r)} |\mathrm{C}\,\nu - a| \frac{dA}{\pi} + \int_{B(0,r)} |\mathrm{C}\,\nu_2| \frac{dA}{\pi} \\ &\leqslant o\left(r^2\right) + 2r|\nu|(B(0,r)). \end{aligned} \tag{2.5}$$

The Bergman space estimate ([Vuk93])

$$|f(z)| \leqslant \frac{1}{(1-|z|^2)^2} \int_{\mathbb{D}} |f| \frac{dA}{\pi},$$

valid for f analytic in $\mathbb{D}$ and $z \in \mathbb{D}$, rescales to

$$|f(z)| \leqslant \frac{r^2}{(r^2 - |z|^2)^2} \int_{B(0,r)} |f| \frac{dA}{\pi}$$

for f analytic in $B(0,r)$ and $z \in B(0,r)$. We apply this with $f = \mathrm{C}\,\nu_1 - a$ to get

$$|\,\mathrm{C}\,\nu_1(z) - a| \leqslant \frac{16}{9r^2} \int_{B(0,r)} |\,\mathrm{C}\,\nu_1 - a| \frac{dA}{\pi}$$

for $|z| \leqslant \frac{1}{2}r$, and combining this with (2.5) we obtain that

$$|\,\mathrm{C}\,\nu_1(z) - a| \leqslant o(1) + \frac{32}{9} \frac{|\nu|(B(0,r))}{r}$$

uniformly in $|z| \leqslant \frac{1}{2}r$. From this and our hypotheses we see that we can pick an $r > 0$ such that

$$\frac{1}{r}\|\nu_2\| < \frac{1}{4}|a|\epsilon_0 \tag{2.6}$$

and

$$|\,\mathrm{C}\,\nu_1(z) - a| < \frac{1}{2}|a| \qquad \text{for } |z| \leqslant \frac{1}{2}r. \tag{2.7}$$

We now define measures $\hat{\nu}$, $\hat{\nu}_1$, $\hat{\nu}_2$ by the formulas

$$\begin{aligned} \hat{\nu}(E) &= \frac{4}{ar}\nu\left(\frac{r}{2}E\right) \\ \hat{\nu}_1(E) &= \frac{4}{ar}\nu_1\left(\frac{r}{2}E\right) \\ \hat{\nu}_2(E) &= \frac{4}{ar}\nu_2\left(\frac{r}{2}E\right). \end{aligned}$$

A calculation shows that

$$\mathrm{C}\,\hat{\nu}_1(z) = \frac{2}{a}\,\mathrm{C}\,\nu_1\left(\frac{r}{2}z\right).$$

From (2.6) and (2.7) we now see that

$$\|\hat{\nu}_2\| < \epsilon_0 \tag{2.8}$$

and

$$|\,\mathrm{C}\,\hat{\nu}_1(z) - 2| < 1 \qquad \text{for } |z| \leqslant 1. \tag{2.9}$$

Clearly $\hat{\nu}$ annihilates the polynomials and $\hat{\nu} = \hat{\nu}_1 + \hat{\nu}_2$. By (2.8) and (2.9) we thus see that $\hat{\nu}$, $\hat{\nu}_1$, $\hat{\nu}_2$ satisfy the hypotheses of Theorem 2.1, so that

$$|p(0)| \leqslant C_0 \int |p(w)| d|\hat{\nu}|(w) \qquad \text{for } p \in \mathcal{P}.$$

It follows that

$$|p(0)| \leqslant \frac{4C_0}{|a|r} \int |p(w)| d|\nu|(w) \qquad \text{for } p \in \mathcal{P},$$

and the corollary now follows from Hölder's inequality. □

The bpe's produced by Thomson's coloring scheme are actually abpe's for $\mathcal{P}^t(\mu)$, and so the same is true of the bpe's we produce in Corollary 2.2. It follows that this corollary implies Theorem 4.10 in Thomson's paper: if $\mathcal{P}^t(\mu)$ is pure (i.e. contains no nontrivial L^t-summands) then the closure of the set of abpe's for $\mathcal{P}^t(\mu)$ contains the support of μ.

It remains to prove Lemmas A and B.

Proof of Lemma A. With a constant C different from A_T we in fact have from Proposition 2.1 of [Tol02] and Tolsa's Theorem that

$$\gamma([|\operatorname{C}\nu| \geqslant a]) \leqslant \frac{C}{a}\|\nu\|.$$

However, we can also give an easy short proof that just uses Tolsa's Theorem.

Let $a > 0$ and $E = [\operatorname{Re}\operatorname{C}\nu \geqslant a]$. Since $\operatorname{C}\nu$ is a continuous function vanishing at ∞, E is compact. Let μ be a positive measure supported in E for which $\|\operatorname{C}\mu\|_{L^\infty(\mathbb{C})} \leqslant 1$. The continuity of the function $z \mapsto \int \frac{d|\nu|(w)}{|w-z|}$ can be used to justify the use of Fubini's Theorem needed to show that

$$\int \operatorname{C}\mu \cdot d\nu = -\int \operatorname{C}\nu \cdot d\mu. \tag{2.10}$$

Thus

$$\left|\int \operatorname{C}\nu \cdot d\mu\right| = \left|\int_E \operatorname{C}\nu \cdot d\mu\right| \geqslant \operatorname{Re}\int_E \operatorname{C}\nu \cdot d\mu \geqslant a\|\mu\|,$$

and clearly our hypothesis on ν implies that

$$\left|\int \operatorname{C}\mu \cdot d\nu\right| \leqslant \|\operatorname{C}\mu\|_{L^\infty(\mathbb{C})}\|\nu\| \leqslant \|\nu\|.$$

Hence by (2.10) we see that

$$\|\mu\| \leqslant \frac{1}{a}\|\nu\|.$$

This shows that $\gamma_+([\operatorname{Re}\operatorname{C}\nu \geqslant a]) \leqslant \frac{1}{a}\|\nu\|$, so by Tolsa's Theorem we are done. $\square$

Proof of Lemma B. We may clearly assume that E is closed. Our proof will consist of another application of Tolsa's Theorem together with Thomson's coloring scheme, with some minor modifications, and his application of the Scott Brown-type approximation scheme. We will sketch our version of Thomson's ideas and refer the reader to his paper for details.

Let E be a closed subset of $\operatorname{clos}\mathbb{D}$. For each integer $k \geqslant 3$ let $\{S_{kj}\}$ be an enumeration of the closed squares contained in $\mathbb{C}$ with edges of length 2^{-k} parallel to the coordinate axes, and corners at the points whose coordinates are both integral multiples of 2^{-k}. Denote by $S_0(k)$ one of the S_{kj}'s that contains 0. We say S_{kj} is *light* if

$$A((S_{kj} \setminus E) \cap \mathbb{D}) \leq \frac{1}{10}A(S_{kj}),$$

and *heavy* if it is not light. We consider Thomson's coloring scheme described in Section 2 of his paper, but with our slightly different definitions of light and heavy squares — the role of Thomson's Φ is played here by $\chi_{(\operatorname{clos}\mathbb{D})\setminus E}$ — and we always have $a = 0$. The scheme depends on an integer $m \geqslant 3$ and starts by coloring $S_0(m)$ yellow.

Two things can happen; which of these occurs may depend on m:

Case I. The scheme terminates — in our setup this means Thomson's scheme reaches a square that is not contained in $\operatorname{clos}\mathbb{D}$.

Case II. The scheme does not terminate.

We consider Case II first. With our Φ, the Scott Brown-type approximation argument in Section 4 of Thomson's paper ([Tho91, Lemmas 4.2–4.6]) shows that in this case there is a function H that is zero off $(\operatorname{clos}\mathbb{D}) \setminus E$ with a bound on $\|H\|_{L^\infty(\mathbb{C})}$ that depends only on m, for which

$$p(0) = \int pH\, dA \qquad \text{for } p \in \mathcal{P}.$$

In fact with our definition of heavy squares the estimate on c that Thomson uses on the bottom of page 493 can be replaced by $c \leqslant 10 \cdot 2^{2n}$, and it then follows that we can get the estimate $\|H\|_{L^\infty(\mathbb{C})} \leqslant 10 \cdot 2^{2(m+1)}$. Of course this implies that

$$|p(0)| \leqslant 10 \cdot 2^{2(m+1)} \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p|\, dA \qquad \text{for } p \in \mathcal{P},$$

so that Lemma B will be proved if we can show that if $\gamma(E)$ is small enough then Case I cannot occur for sufficiently high m.

This brings us to pages 482–484 of Thomson's paper. The argument there shows us that if the scheme terminates for some $m \geqslant m_0$, where $m_0 \geqslant 3$ is some absolute constant, then there exists a finite collection of light squares $S_j \subseteq \operatorname{clos}\mathbb{D}$ such that $\gamma\left(\bigcup S_j\right) > \delta_0$, where δ_0 is an absolute constant. Since each S_j is light we have that $A(E \cap S_j) > \frac{9}{10} A(S_j)$. We are going to complete the proof of Lemma B by using Tolsa's Theorem together with a Vitushkin-type construction to show that the lower bound $\gamma\left(\bigcup S_j\right) > \delta_0$ implies a lower bound on $\gamma(E)$.

The basic building block for our construction is given in the next lemma. We use the following notation: if f is analytic in a neighborhood of ∞ in $\mathbb{C}_\infty$ and $z_0 \in \mathbb{C}$ then, writing the Laurent expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{-n}$, we define $\beta(f, z_0) = a_2$ (of course $a_0 = f(\infty)$ and $a_1 = f'(\infty)$).

Lemma 2.3. *Suppose $S, Q, R \subseteq \mathbb{C}$ are closed squares with sides parallel to the coordinate axes, all with center z_0 and with edge lengths l, $\frac{7}{6}l$, $\frac{2}{3}l$ respectively*;

$\varphi \in C^\infty(\mathbb{C})$ with $0 \leqslant \varphi \leqslant 1$, $\operatorname{spt}\varphi \subseteq \operatorname{int} Q$, $\varphi \equiv 1$ on R, and $\|\nabla\varphi\|_\infty \leqslant 50/l$;

E is a closed subset of S with $A(E) > \frac{9}{10}A(S)$; and

μ is a compactly supported positive measure on $\mathbb{C}$ with $\|\mathrm{C}\,\mu\|_{L^\infty(\mathbb{C})} \leqslant 1$.

Then

(i) $\|\mathrm{C}(\varphi\mu)\|_{L^\infty(\mathbb{C})} \leqslant 100$

and there is a measure η such that

(ii) $\operatorname{spt}\eta \subseteq E \cap R$,

(iii) $\|\eta\| \leqslant 30\|\varphi\mu\|$,

(iv) $\|\mathrm{C}\,\eta\|_{L^\infty(\mathbb{C})} \leqslant 15,000$,

(v) $(\mathrm{C}\,\eta)'(\infty) = (\mathrm{C}(\varphi\mu))'(\infty)$, *and*

(vi) $\beta(\mathrm{C}\,\eta, z_0) = \beta(\mathrm{C}(\varphi\mu), z_0)$.

Proof. We record here two well known and easily proven facts:

(2.11) If ν is a compactly supported measure on $\mathbb{C}$ and ψ is a compactly supported C^∞ function, then

$$\mathrm{C}(\psi\nu)(z) = \psi(z)\,\mathrm{C}\,\nu(z) + \frac{1}{\pi}\int \frac{\mathrm{C}\,\nu(w)\bar{\partial}\psi(w)}{w-z}\,dA(w)$$

for all $z \in \mathbb{C}$ for which $\int \frac{d|\nu|(w)}{|w-z|} < \infty$.

and

(2.12) If ν is a compactly supported measure on $\mathbb{C}$ for which $\int \frac{d|\nu|(w)}{|w-z|}$ is uniformly bounded in z, and $\rho = \mathrm{C}\,\nu \cdot \nu$, then $\mathrm{C}\,\rho = \frac{1}{2}(\mathrm{C}\,\nu)^2$.

We see that (i) is an immediate consequence of our hypotheses, (2.11), and (2.4).

Now set $\nu = A|E \cap R$, so that

$$\|\nu\| = A(E \cap R) > \frac{1}{3}l^2 \tag{2.13}$$

and, again by (2.4),

$$\|\mathrm{C}\,\nu\|_{L^\infty(\mathbb{C})} < 3l, \tag{2.14}$$

and set $\rho = C\nu \cdot \nu$. Clearly if $\eta = a\nu + b\rho$, with $a, b \in \mathbb{C}$, then (ii) is satisfied. We pick a and b so that (v) and (vi) are also satisfied. From (2.12) it is easy to see that $\beta(\mathrm{C}\,\rho, z_0) = \frac{1}{2}\left((\mathrm{C}\,\nu)'(\infty)\right)^2$ and we then see that

$$a = \frac{\|\varphi\mu\|}{A(E \cap R)}$$

and

$$b = \frac{2}{A(E \cap R)^2} \left[\frac{\|\varphi\mu\|}{A(E \cap R)} \int (z - z_0)\, d\nu(z) - \int (z - z_0)\varphi(z)\, d\mu(z) \right].$$

It follows easily that $|b| \le 3l \frac{\|\varphi\mu\|}{A(E \cap R)^2}$, and (iii) now follows from (2.13) and (2.14).

We now estimate $\| \mathrm{C}\,\eta\|_{L^\infty(\mathbb{C})}$. Since by (2.12) $\mathrm{C}\,\eta = a\,\mathrm{C}\,\nu + \frac{1}{2} b(\mathrm{C}\,\nu)^2$, we see by (2.13), (2.14), our expression for a, and our estimate on $|b|$ that

$$\| \mathrm{C}\,\eta\|_{L^\infty(\mathbb{C})} < 150 \frac{\|\varphi\mu\|}{l}. \tag{2.15}$$

It is a well known consequence of Schwarz's Lemma that the analytic capacity of a disk of radius r is r, hence $\gamma(Q) \leqslant \frac{7}{12}\sqrt{2}l$. From this and (i) we see that $\|\varphi\mu\| \leqslant \| \mathrm{C}(\varphi\mu)\|_{L^\infty(\mathbb{C})}\gamma(Q) \leqslant 100l$. Now (iv) follows from (2.15). □

We return to the proof of Lemma B. Let $\{S_j\}$ be the finite collection of light squares discussed above, so that $\gamma(\bigcup S_j) > \delta_0$. For each j let

z_j be the center of S_j,

l_j be the edge length of S_j,

Q_j, R_j be the closed squares with center z_j and sides parallel to the coordinate axes of lengths $\frac{7}{6}l_j = \delta_j$, $\frac{2}{3}l_j$ respectively.

The collection $\{S_j\}$ has the following properties:

(2.16) No point lies in more than four $Q_j's$.

(2.17) There are C^∞ functions φ_j with $0 \leqslant \varphi_j \leqslant 1$, $\operatorname{spt}\varphi_j \subseteq \operatorname{int} Q_j$, $\|\nabla\varphi_j\|_\infty \leqslant 50/l_j$, $\varphi_j \equiv 1$ on R_j, and $\sum \varphi_j \equiv 1$ on $\bigcup S_j$.

(2.18) For each $z \in \mathbb{C}$

$$\sum \min\left\{1, \frac{\delta_j^3}{|z - z_j|^3}\right\} \leqslant 10{,}000.$$

These statements correspond to part of Proposition 2.6 of Thomson's paper and we refer the reader there for the proofs of (2.16) and (2.18). We need a bit more than is given in the usual partition of unity argument, so we give the proof of (2.17). Let $\{\psi_j\}$ be C^∞ functions with $0 \leqslant \psi_j \leqslant 1$, $\operatorname{spt}\psi_j \subseteq \operatorname{int} Q_j$, $\psi_j \equiv 1$ on S_j, and $\|\nabla\psi_j\|_\infty \leqslant 25/2l_j$. We use these to construct a partition of unity in the standard way (see e.g. [Rud91, p. 148]):

$$\varphi_1 = \psi_1$$

and for $j > 1$

$$\varphi_j = \psi_j(1 - \psi_1) \cdots (1 - \psi_{j-1}).$$

Then for all n, $\sum_{j=1}^n \varphi_j = 1 - \prod_{j=1}^n (1-\psi_j)$, which shows that $\sum \varphi_j \equiv 1$ on $\bigcup S_j$, and (2.16) implies that $\|\nabla \varphi_j\|_\infty \leqslant 50/l_j$. Because of the yellow "buffer" squares in Thomson's scheme, if Q_k intersects S_j then $l_k = l_j$, and it follows that $\operatorname{int} Q_k \cap R_j = \varnothing$ if $j \neq k$. Since $\sum \varphi_j \equiv 1$ on $\bigcup S_j$, this implies that $\varphi_j \equiv 1$ on R_j, and the proof of (2.17) is complete.

Now suppose that μ is a positive measure supported in $\bigcup S_j$ with $\|\operatorname{C}\mu\|_{L^\infty(\mathbb{C})} \leqslant 1$. For each j we let z_j, S_j, Q_j, R_j, l_j, $E \cap S_j$, φ_j play the respective roles of z_0, S, Q, R, l, E, φ in Lemma 2.3, and construct the corresponding measure η_j as in that lemma. Then

$$\|\operatorname{C}(\varphi_j\mu)\|_{L^\infty(\mathbb{C})} \leqslant 100, \tag{2.19}$$

$$\operatorname{spt}\eta_j \subseteq E \cap R_j, \tag{2.20}$$

$$\|\eta_j\| \leqslant 30\|\varphi_j\mu\|, \tag{2.21}$$

$$\|\operatorname{C}\eta_j\|_{L^\infty(\mathbb{C})} \leqslant 15,000, \tag{2.22}$$

$$(\operatorname{C}\eta_j)'(\infty) = (\operatorname{C}(\varphi_j\mu))'(\infty), \tag{2.23}$$

and

$$\beta(\operatorname{C}\eta_j, z_j) = \beta(\operatorname{C}(\varphi_j\mu), z_j). \tag{2.24}$$

From (2.23) and (2.24) we see that the function

$$[\operatorname{C}\eta_j(z) - \operatorname{C}(\varphi_j\mu)(z)](z-z_j)^3$$

is analytic in $\mathbb{C}_\infty \setminus Q_j$, and so using (2.19), (2.22), and the Maximum Modulus Theorem, if $z \in \mathbb{C}\setminus Q_j$, then

$$\begin{aligned} &|\operatorname{C}\eta_j(z) - \operatorname{C}(\varphi_j\mu)(z)| \cdot |z-z_j|^3 \\ &\qquad \leqslant \sup_{w\in\partial Q_j} |\operatorname{C}\eta_j(w) - \operatorname{C}(\varphi_j\mu)(w)| \cdot |w-z_j|^3 \leqslant 10,000\delta_j^3. \end{aligned} \tag{2.25}$$

Set $\eta = \sum \eta_j$. By (2.17), $\mu = \sum \varphi_j\mu$. Together with (2.18), (2.19), (2.22), and (2.25) this tells us that for A-almost all $z \in \mathbb{C}$

$$\begin{aligned} |\operatorname{C}\eta(z)| &\leqslant \sum_{|z-z_j|\leqslant\delta_j} |\operatorname{C}\eta_j(z)| \\ &\quad + \sum_{|z-z_j|>\delta_j} |\operatorname{C}\eta_j(z) - \operatorname{C}(\varphi_j\mu)(z)| \\ &\quad + \sum_{|z-z_j|\leqslant\delta_j} |\operatorname{C}(\varphi_j\mu)(z)| + |\operatorname{C}\mu(z)| \\ &\leqslant 15,000\cdot 10,000 + 10,000\cdot 10,000 + 100\cdot 10,000 + 1 \\ &< 10^9. \end{aligned}$$

We can now finish the proof of Lemma B. By Tolsa's Theorem and the fact that $\gamma(\bigcup S_j) > \delta_0$ we can find a positive measure supported in $\bigcup S_j$ for which $\|\operatorname{C}\mu\|_{L^\infty(\mathbb{C})} \leqslant 1$ and $\|\mu\| \geqslant \frac{1}{A_T}\delta_0$. The measure η produced by the above construction is supported in E, and we have just shown that $\|\mathrm{C}\,\eta\|_{L^\infty(\mathbb{C})} \leqslant 10^9$. We also have by (2.23) that

$$\begin{aligned}(\mathrm{C}\,\eta)'(\infty) &= \sum (\mathrm{C}\,\eta_j)'(\infty)\\ &= \sum (\mathrm{C}(\varphi_j \mu))'(\infty)\\ &= (\mathrm{C}\,\mu)'(\infty) = -\|\mu\|.\end{aligned}$$

Thus $\gamma(E) \geqslant \frac{1}{10^9 A_T}\delta_0$. This gives the desired lower bound on $\gamma(E)$, and completes the proof of Lemma B. □

3. Asymptotic estimates on point evaluations, nontangential limits, and the index of invariant subspaces

In this section we will prove Theorems A and C. Throughout, μ will be a fixed measure satisfying the hypotheses of Theorem A, and we write $\mu|\partial\mathbb{D} = \frac{1}{2\pi} hm$.

Our first step will be to show how Theorem A follows from the inequality (1.4):

$$\varlimsup_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \left(1-|\lambda|^2\right)^{1/t} M_\lambda \leqslant \frac{C}{h(z)^{1/t}}$$

for m-almost all $z \in \partial\mathbb{D}$. To do this we will need some basic properties of the Hardy spaces on general regions in $\mathbb{C}$.

Let $\Omega \subseteq \mathbb{C}$ be a nonempty connected open set. The Hardy space $H^t(\Omega)$ is then defined to be the space of functions f analytic in Ω for which $|f|^t$ has a harmonic majorant in Ω. We norm $H^t(\Omega)$ by

$$\|f\|_{H^t(\Omega)} = u_f(a)^{1/t}$$

where u_f is the least harmonic majorant of $|f|^t$ and a is some fixed base point. As is well known, this definition is conformally invariant and for the case $\Omega = \mathbb{D}$ and $a = 0$ agrees with the classical Hardy spaces H^t we discussed in Section 1 – see e.g. [Dur70, Ch. 10] or [Gar81, Ch. 2].

We now specialize to the case

$$\Omega = \Gamma(E) \underset{\text{def}}{=} \bigcup_{z\in E} \Gamma(z) \text{ for some compact } E \subseteq \partial\mathbb{D} \text{ with } m(E) > 0,$$

and $a = 0$. Clearly, $\Gamma(E)$ is simply connected. Let φ be the Riemann map taking $\mathbb{D}$ onto $\Gamma(E)$ with $\varphi(0) = 0$, $\varphi'(0) > 0$. If $f \in H^t(\Gamma(E))$ then $f \circ \varphi \in H^t$ and we thus have the estimate

$$|f\circ\varphi(w)| \leqslant \frac{1}{(1-|w|^2)^{1/t}}\|f\circ\varphi\|_{H^t}. \tag{3.1}$$

The boundary of $\Gamma(E)$ is clearly a rectifiable Jordan curve, and it follows from well-known results of Caratheodory and F. and M. Riesz that φ extends to a homeomorphism of $\operatorname{clos}\mathbb{D}$ onto $\operatorname{clos}\Gamma(E)$, that $\varphi' \in H^1$ and if $F \subset \partial\Gamma(E)$ then

$$\operatorname{length}(F) = \int_{\varphi^{-1}(F)} |\varphi'(z)|\, dm(z).$$

Also, φ is conformal at m-almost every point of $\partial\mathbb{D}$ (see the discussion on page 93 of [Gar81]).

It follows from these facts and basic properties of H^t that if $f \in H^t(\Gamma(E))$ then the nontangential limit $f^*(z)$ exists for m-almost every $z \in E$, and that if $f_n \to f$ in $H^t(\Gamma(E))$ then there is a subsequence f_{n_j} such that $f^*_{n_j}(z) \to f^*(z)$ for m-almost all $z \in E$.

It also follows from these facts together with (3.1) that for a.e. $z \in E$,

$$\varlimsup_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \left(1-|\lambda|^2\right)^{1/t} \sup\left\{|f(\lambda)| :\ f \in H^t(\Gamma(E)),\ \|f\|_{H^t(\Gamma(E))} \leqslant 1\right\} \leqslant 1.$$

We can reverse this argument in $\mathcal{P}^t(\mu)$.

THEOREM 3.1. *Let $1 \le t < \infty$. Suppose $E \subseteq \partial\mathbb{D}$ is compact with $m(E) > 0$ and there is a constant $C < \infty$ such that*

$$\left(1-|\lambda|^2\right)^{1/t} M_\lambda \leqslant C \qquad \text{for } \lambda \in \Gamma(E).$$

Then $f \in \mathcal{P}^t(\mu)$ implies that $f|\Gamma(E) \in H^t(\Gamma(E))$ and $\|f|\Gamma(E)\|_{H^t(\Gamma(E))} \leqslant C\|f\|_{\mathcal{P}^t(\mu)}$.

It follows that if $f \in \mathcal{P}^t(\mu)$ then the nontangential limit $f^*(z)$ exists for m-almost all $z \in E$, and that $f^*(z) = f(z)$ for μ-almost all $z \in E$.

Proof. For $f \in \mathcal{P}^t(\mu)$ and $\lambda \in \Gamma(E)$ set

$$g(w) = \frac{f(w)}{\left(1-\bar\lambda w\right)^{2/t}}.$$

Then $g \in \mathcal{P}^t(\mu)$, so

$$\begin{aligned}
|f(\lambda)|^t &= \left(1-|\lambda|^2\right)^2 |g(\lambda)|^t \\
&\leqslant \left(1-|\lambda|^2\right)^2 M_\lambda^t \|g\|^t_{\mathcal{P}^t(\mu)} \\
&= \left(1-|\lambda|^2\right)^2 M_\lambda^t \int \frac{|f(w)|^t}{\left|1-\bar\lambda w\right|^2}\, d\mu(w) \\
&\leqslant C^t \int \frac{1-|\lambda|^2}{\left|1-\bar\lambda w\right|^2} |f(w)|^t\, d\mu(w) \\
&\leqslant C^t \int \frac{1-\left|\bar\lambda w\right|^2}{\left|1-\bar\lambda w\right|^2} |f(w)|^t\, d\mu(w).
\end{aligned}$$

Since $\frac{1-|\bar{\lambda}w|^2}{|1-\bar{\lambda}w|^2} = \operatorname{Re}\frac{1+\bar{\lambda}w}{1-\bar{\lambda}w}$ we see that this last integral is a harmonic function of λ that equals $\|f\|^t_{\mathcal{P}^t(\mu)}$ when $\lambda = 0$, and the first conclusion of the theorem follows.

To prove the second conclusion let $f \in \mathcal{P}^t(\mu)$ and let $p_n \in \mathcal{P}$ with $p_n \to f$ in $L^t(\mu)$. Then $p_n|\Gamma(E) \to f|\Gamma(E)$ in $H^t(E)$; so from the discussion preceding the statement of the theorem we may assume, passing to a subsequence if necessary, that $p_n \to f^*$ m-almost everywhere on E. We can also assume that $p_n \to f$ a.e. $[\mu]$. By (1.3) we have $h(z) \neq 0$ for m-almost all $z \in E$, and the conclusion follows. □

Our next result establishes a connection between nontangential limits and indices of invariant subspaces.

THEOREM 3.2. *Let $1 \le t < \infty$. Suppose there is an $E \subseteq \partial\mathbb{D}$ with $m(E) > 0$ and $h > 0$ on E, such that every $f \in \mathcal{P}^t(\mu)$ has a nontangential limit $f^*(z)$ which equals $f(z)$ for μ-almost all $z \in E$. Then every nonzero invariant subspace of $\mathcal{P}^t(\mu)$ has index 1.*

Proof. Let M be a nonzero invariant subspace of $\mathcal{P}^t(\mu)$. We must show that $\dim M/SM = 1$. If n is the greatest integer such that every $f \in M$ has a zero of order at least n at 0, we consider the invariant subspace

$$\tilde{M} = \left\{ f \in \mathcal{P}^t(\mu) : \ S^n f \in M \right\}.$$

Using the division property of $\mathcal{P}^t(\mu)$ it is not hard to show that $S^n|\tilde{M}$ induces an isomorphism of $\tilde{M}/S\tilde{M}$ onto M/SM, and the division property implies that $\tilde{M}$ contains a function that does not vanish at 0. The upshot of this is that we may assume that there is a $g \in M$ such that $g(0) \neq 0$.

We will prove the theorem by showing that if $f \in M$ and $f(0) = 0$ then $\frac{f(z)}{z} \in M$ — in other words by showing that M has the division property at 0. Let ϕ be an arbitrary function in $M^\perp \subset L^{t'}(\mu)$ and set

$$\Phi(\lambda) = \int \frac{f(w) - \frac{f(\lambda)}{g(\lambda)} g(w)}{w - \lambda} \phi(w)\, d\mu(w),$$

so that Φ is a meromorphic function in $\mathbb{D}$. We will be done if we show that

$$\Phi(0) = \int \frac{f(w)}{w} \phi(w)\, d\mu(w) = 0,$$

and this we will do by showing that $\Phi \equiv 0$.

From our hypothesis it is easy to see that there is a compact $F \subset E$ with $m(F) > 0$ and constants $C < \infty$, $\kappa \in (0,1)$ such that if $\lambda \in \Gamma(F)$ and $|\lambda| > \kappa$ then $\left|\frac{f(\lambda)}{g(\lambda)}\right| \leqslant C$ and $|f(w)|$, $|g(w)|$ are bounded by C for $\rho(w,\lambda) \leqslant \frac{1}{2}$ — here $\rho(w,\lambda) = \left|\frac{w-\lambda}{1-\bar{\lambda}w}\right|$ is the familiar pseudohyperbolic metric. Now let $\lambda \in \mathbb{D}$ be such that $g(\lambda) \neq 0$. Then the function of w given by

$$\frac{f(w) - \frac{f(\lambda)}{g(\lambda)} g(w)}{w - 1/\bar{\lambda}}$$

is in M, so that

$$\begin{aligned}\Phi(\lambda) &= \int \left(\frac{1}{w-\lambda} - \frac{1}{w - 1/\bar{\lambda}} \right) \left(f(w) - \frac{f(\lambda)}{g(\lambda)} g(w) \right) \phi(w)\, d\mu(w) \\ &= \int \frac{1 - |\lambda|^2}{(w-\lambda)(1-\bar{\lambda} w)} \left(f(w) - \frac{f(\lambda)}{g(\lambda)} g(w) \right) \phi(w)\, d\mu(w).\end{aligned}$$

Hence

$$|\Phi(\lambda)| \leqslant \left[\int_{\rho(w,\lambda) \leqslant \frac{1}{2}} + \int_{\frac{1}{2} < \rho(w,\lambda) < 1} \right] \frac{1 - |\lambda|^2}{|1 - \bar{\lambda} w|} \left| \frac{f(w) - \frac{f(\lambda)}{g(\lambda)} g(w)}{w - \lambda} \right| |\phi(w)|\, d\mu(w)$$

$$+ \left| \int_{w \in \partial\mathbb{D}} \frac{1 - |\lambda|^2}{\left|1 - \bar{\lambda} w\right|^2} \bar{w} \left[f(w) - \frac{f(\lambda)}{g(\lambda)} g(w) \right] \phi(w)\, d\mu(w) \right| = I_1(\lambda) + I_2(\lambda) + I_3(\lambda).$$

Suppose $\lambda \in \Gamma(F)$ with $|\lambda| > \kappa$. If $\rho(w,\lambda) \leqslant \frac{1}{2}$ then

$$\left| \frac{f(w) - \frac{f(\lambda)}{g(\lambda)} g(w)}{w - \lambda} \right| \leqslant \max_{\rho(\zeta,\lambda) = \frac{1}{2}} \left| \frac{f(\zeta) - \frac{f(\lambda)}{g(\lambda)} g(\zeta)}{\zeta - \lambda} \right| \leqslant \frac{3\left(C + C^2\right)}{1 - |\lambda|^2}$$

and $\frac{1-|\lambda|^2}{\left|1 - \bar{\lambda} w\right|} \geqslant \frac{1}{2}$. It follows that

$$\begin{aligned}I_1(\lambda) &\leqslant \int_{\rho(w,\lambda) \leqslant \frac{1}{2}} \frac{2\left(1 - |\lambda|^2\right)}{\left|1 - \bar{\lambda} w\right|} \frac{1 - |\lambda|^2}{\left|1 - \bar{\lambda} w\right|} \frac{3\left(C + C^2\right)}{1 - |\lambda|^2} |\phi(w)|\, d\mu(w) \\ &\leqslant 6\left(C + C^2\right) \int_{w \in \mathbb{D}} \frac{1 - |\lambda|^2}{\left|1 - \bar{\lambda} w\right|^2} |\phi(w)|\, d\mu(w).\end{aligned}$$

We also see that

$$\begin{aligned}I_2(\lambda) &\leqslant \int_{\frac{1}{2} < \rho(w,\lambda) < 1} 2 \left| \frac{w - \lambda}{1 - \bar{\lambda} w} \right| \frac{1 - |\lambda|^2}{\left|1 - \bar{\lambda} w\right|} \left| \frac{f(w) - \frac{f(\lambda)}{g(\lambda)} g(w)}{w - \lambda} \right| |\phi(w)|\, d\mu(w) \\ &\leqslant 2 \int_{w \in \mathbb{D}} \frac{1 - |\lambda|^2}{\left|1 - \bar{\lambda} w\right|^2} [|f(w)| + C|g(w)|] |\phi(w)|\, d\mu(w).\end{aligned}$$

Then, by Lemma 1.1, for m-almost all $z \in F$,

$$\lim_{\substack{\lambda \to z \\ \lambda \in \Gamma(z)}} I_i(\lambda) = 0, \qquad i = 1, 2.$$

To deal with $I_3(\lambda)$ we observe that by a standard property of Poisson integrals

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \int_{w\in\partial\mathbb{D}} \frac{1-|\lambda|^2}{\left|1-\bar{\lambda}w\right|^2}\bar{w}f(w)\phi(w)\,d\mu(w) = \bar{z}f(z)\phi(z)h(z)$$

and

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \int_{w\in\partial\mathbb{D}} \frac{1-|\lambda|^2}{\left|1-\bar{\lambda}w\right|^2}\bar{w}g(w)\phi(w)\,d\mu(w) = \bar{z}g(z)\phi(z)h(z)$$

for m-almost all $z \in \partial\mathbb{D}$. Our hypotheses and Privalov's theorem also imply that

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \frac{f(\lambda)}{g(\lambda)} = \frac{f(z)}{g(z)}$$

for m-almost all $z \in E$. It now follows that $\lim\limits_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} I_3(\lambda) = 0$ for m-almost all $z \in F$.

We have shown that for m-almost all $z \in F$, $\lim\limits_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \Phi(\lambda) = 0$, so we are done by Privalov's Theorem. $\square$

We note that the above proof shows more; if μ is any measure satisfying the hypothesis of Theorem 3.2, if $1 \le t < \infty$, and if $f, g \in \mathcal{P}^t(\mu)$ are nonzero analytic functions such that the meromorphic function f/g has nontangential limits on a set $E \subseteq \partial\mathbb{D}$ of positive Lebesgue measure, then the index of the invariant subspace generated by f and g is one.

From Theorems 3.1 and 3.2, the continuity of $\lambda \mapsto M_\lambda$, and an elementary measure-theoretic argument, we see that Theorem A will follow from the inequality (1.4). We will prove this inequality by reducing it to Theorem 2.1 by use of a conformal translation.

For $\lambda \in \mathbb{D}$ we set $\varphi_\lambda(z) = \frac{z+\lambda}{1+\bar{\lambda}z}$. To a measure η on $\operatorname{clos}\mathbb{D}$ and $\lambda \in \mathbb{D}$ we associate the measure η_λ given by

$$d\eta_\lambda(z) = \frac{\left|1+\bar{\lambda}z\right|^2}{1-|\lambda|^2}\, d(\eta\circ\varphi_\lambda)(z).$$

It is easy to check that

$$\int F(z)\,d\eta_\lambda(z) = \int F\left(\frac{z-\lambda}{1-\bar{\lambda}z}\right)\frac{1-|\lambda|^2}{\left|1-\bar{\lambda}z\right|^2}\,d\eta(z) \tag{3.2}$$

for all F for which either side makes sense. In particular if η is a measure supported in $\partial\mathbb{D}$ of the form

$$\eta = \frac{1}{2\pi}k\cdot m,$$

then

$$\eta_\lambda = \frac{1}{2\pi} k \circ \varphi_\lambda \cdot m. \tag{3.3}$$

We apply this transformation to our measure μ and use (3.2) to see that if

$$|p(\lambda)|^t \leqslant \frac{C^t}{1-|\lambda|^2} \int |p|^t \, d\mu \qquad \text{for all } p \in \mathcal{P},$$

then for all $q \in \mathcal{P}$,

$$\begin{aligned} \int |q|^t \, d\mu_\lambda &= \int \left| q\left(\frac{z-\lambda}{1-\bar{\lambda}z}\right) \left(\frac{1-|\lambda|^2}{\left(1-\bar{\lambda}z\right)^2}\right)^{1/t} \right|^t d\mu(z) \\ &\geqslant \frac{1-|\lambda|^2}{C^t} \left| q(0) \left(\frac{1}{1-|\lambda|^2}\right)^{1/t} \right|^t = \frac{|q(0)|^t}{C^t}. \end{aligned}$$

This argument is clearly reversible, so we have shown

$$|p(\lambda)|^t \leqslant \frac{C^t}{1-|\lambda|^2} \int |p|^t \, d\mu \qquad \forall p \in \mathcal{P} \tag{3.4}$$

if and only if

$$|q(0)|^t \leqslant C^t \int |q|^t \, d\mu_\lambda \qquad \forall q \in \mathcal{P}.$$

It is an immediate consequence of the proof of Lemma VII.1.7 in [Con81] that there is a $G \in \mathcal{P}^t(\mu)^\perp \subset L^{t'}(\mu)$ such that $G(z) \neq 0$ for μ-almost every $z \in \partial\mathbb{D}$. We let $P[\cdot]$ denote the Poisson integral and we set $\omega = \mu|\mathbb{D}$. By Lemma 1.1 and standard properties of Poisson integrals, for μ-almost all $z \in \partial\mathbb{D}$

$$G(z)h(z) \neq 0, \tag{3.5}$$

$$\lim_{\substack{\lambda \to z \\ \lambda \in \Gamma(z)}} \int \frac{1-|\lambda|^2}{\left|1-\bar{\lambda}w\right|^2} |G(w)| \, d\omega(w) = 0, \tag{3.6}$$

$$\lim_{\substack{\lambda \to z \\ \lambda \in \Gamma(z)}} P[|Gh - G(z)h(z)|](\lambda) = 0, \tag{3.7}$$

and if $t > 1$

$$\lim_{\substack{\lambda \to z \\ \lambda \in \Gamma(z)}} \int \frac{1-|\lambda|^2}{\left|1-\bar{\lambda}w\right|^2} |G(w)|^{t'} \, d\mu(w) = |G(z)|^{t'} h(z). \tag{3.8}$$

Fix such a z and set $\alpha = G(z)$, $\beta = h(z)$, so that $\alpha \neq 0$ and $\beta > 0$, and let $\epsilon > 0$ be a small number to be chosen presently. By (3.6), (3.7), and (3.8) if $\lambda \in \Gamma(z)$ is sufficiently close to z we have

$$\int \frac{1-|\lambda|^2}{\left|1-\bar{\lambda}w\right|^2}|G(w)|\,d\omega(w) < \epsilon|\alpha|\beta, \tag{3.9}$$

$$P[|Gh - \alpha\beta|](\lambda) < \frac{1}{16}|\alpha|\beta \tag{3.10}$$

and

$$\text{if } t>1, \quad \int \frac{1-|\lambda|^2}{\left|1-\bar{\lambda}w\right|^2}|G(w)|^{t'}\,d\mu(w) < 2^{t'}|\alpha|^{t'}\beta, \tag{3.11a}$$

$$\text{if } t=1, \quad \|G\|_{L^\infty(\mu)} < 2|\alpha|. \tag{3.11b}$$

We can get (3.11b) by multiplying G by a suitable outer factor.

Using (3.2) it is easy to check that if

$$G_\lambda(w) = G\left(\frac{w+\lambda}{1+\bar{\lambda}w}\right)\frac{1+\bar{\lambda}w}{1+\lambda\bar{w}} = G\circ\varphi_\lambda(w)\chi_\lambda(w)$$

then $G_\lambda \in \mathcal{P}^t(\mu_\lambda)^\perp$, and from (3.3) we see that

$$G_\lambda\mu_\lambda|\partial\mathbb{D} = \frac{1}{2\pi}G\circ\varphi_\lambda\cdot\chi_\lambda\cdot h\circ\varphi_\lambda\cdot m. \tag{3.12}$$

The inequalities (3.9), (3.10), (3.11a), (3.11b) translate to

$$\int |G_\lambda|\,d\omega_\lambda < \epsilon|\alpha|\beta, \tag{3.13}$$

$$\int |G_\lambda\cdot h\circ\varphi_\lambda - \alpha\beta\chi_\lambda|\frac{dm}{2\pi} < \frac{1}{16}|\alpha|\beta \tag{3.14}$$

and

$$\text{if } t>1, \quad \|G_\lambda\|_{L^{t'}(\mu_\lambda)} < 2|\alpha|\beta^{1/t'} \tag{3.15a}$$

$$\text{if } t=1, \quad \|G_\lambda\|_{L^\infty(\mu_\lambda)} < 2|\alpha|. \tag{3.15b}$$

From (3.14) we see that

$$\left|\mathrm{C}\left(\frac{1}{2\pi}G_\lambda\cdot h\circ\varphi_\lambda\cdot m\right)(w) - \alpha\beta\,\mathrm{C}\left(\frac{1}{2\pi}\chi_\lambda m\right)(w)\right| < \frac{1}{8}|\alpha|\beta$$

if $|w| \leqslant \frac{1}{2}$. A calculation shows that $\mathrm{C}\left(\frac{1}{2\pi}\chi_\lambda m\right) \equiv \bar{\lambda}$ in $\mathbb{D}$, so if we restrict our attention to $|\lambda| \geqslant \frac{1}{2}$ we see that

$$\left|\frac{1}{\alpha\beta\bar{\lambda}}\,\mathrm{C}\left(\frac{1}{2\pi}G_\lambda\cdot h\circ\varphi_\lambda\cdot m\right) - 1\right| < \frac{1}{4} \tag{3.16}$$

for $|w| \leqslant \frac{1}{2}$. Set $\nu = \frac{1}{\alpha\beta\bar{\lambda}}G_\lambda\mu_\lambda = \nu_1+\nu_2$ where $\nu_1 = \nu|\partial\mathbb{D} = \frac{1}{2\pi}\frac{1}{\alpha\beta\bar{\lambda}}G_\lambda\cdot h\circ\varphi_\lambda\cdot m$ and $\nu_2 = \nu|\mathbb{D} = \frac{1}{\alpha\beta\bar{\lambda}}G_\lambda\omega_\lambda$. By (3.13) and (3.16), as long as $|\lambda| \geqslant \frac{1}{2}$ we have that $\|\nu_2\| < 2\epsilon$ and $\mathrm{Re}\,\mathrm{C}\,\nu_1(w) > \frac{3}{4}$ for $|w| \leqslant \frac{1}{2}$.

We now choose $\epsilon = \frac{3}{16}\epsilon_0$, where ϵ_0 is as in the statement of Theorem 2.1. By that theorem and a rescaling argument as at the end of the proof of Corollary 2.2 we see that

$$|p(0)| \leqslant \frac{8}{3} C_0 \int |p| \frac{1}{|\alpha|\beta|\lambda|} |G_\lambda| \, d\mu_\lambda \qquad \text{for } p \in \mathcal{P}. \tag{3.17}$$

Using (3.15a) or (3.15b) as appropriate together with Hölder's inequality we see that (3.17) implies

$$|p(0)| \leqslant \frac{16}{3|\lambda|} C_0 \frac{1}{h(z)^{1/t}} \|p\|_{L^t(\mu_\lambda)} \qquad \text{for } p \in \mathcal{P}. \tag{3.18}$$

Now (3.4) and (3.18) imply

$$|p(\lambda)| \leqslant \frac{16}{3|\lambda|} C_0 \frac{1}{h(z)^{1/t}} \frac{1}{\left(1 - |\lambda|^2\right)^{1/t}} \|p\|_{L^t(\mu)} \qquad \text{for } p \in \mathcal{P}.$$

This completes the proof of the inequality (1.4) and with it the proof of Theorem A. □

We turn now to the proof of Theorem C. As discussed in Section 1, this amounts to showing that the C in the inequality (1.4) can be replaced by 1 if $t > 1$. To this end, let $p \in \mathcal{P}$ and $G \in \mathcal{P}^t(\mu)^\perp \subseteq L^{t'}(\mu)$. We then have for $\lambda \in \mathbb{D}$

$$\int \frac{1 - \bar{\lambda} w}{w - \lambda} \left[p(w) - p(\lambda) \left(1 - |\lambda|^2\right)^{1/t} \left(\frac{1 - |\lambda|^2}{\left(1 - \bar{\lambda} w\right)^2} \right)^{1/t} \right] \times \left(\frac{1 - |\lambda|^2}{\left(1 - \bar{\lambda} w\right)^2} \right)^{1/t'} G(w) \, d\mu(w) = 0,$$

which we rewrite as

$$\begin{aligned} \left(1 - |\lambda|^2\right)^{1/t} p(\lambda) \int_{w \in \partial\mathbb{D}} & \frac{1 - \bar{\lambda} w}{w - \lambda} \frac{1 - |\lambda|^2}{\left(1 - \bar{\lambda} w\right)^2} G(w) h(w) \frac{dm(w)}{2\pi} \\ &= \int_{w \in \partial\mathbb{D}} \frac{1 - \bar{\lambda} w}{w - \lambda} p(w) \left(\frac{1 - |\lambda|^2}{\left(1 - \bar{\lambda} w\right)^2} \right)^{1/t'} G(w) h(w) \frac{dm(w)}{2\pi} \\ &\quad + \int_{w \in \mathbb{D}} \frac{1 - \bar{\lambda} w}{w - \lambda} \left[p(w) - p(\lambda) \left(1 - |\lambda|^2\right)^{1/t} \left(\frac{1 - |\lambda|^2}{\left(1 - \bar{\lambda} w\right)^2} \right)^{1/t} \right] \\ &\quad \times \left(\frac{1 - |\lambda|^2}{\left(1 - \bar{\lambda} w\right)^2} \right)^{1/t'} G(w) d\mu(w). \end{aligned}$$

For $w \in \partial\mathbb{D}$ we have that $\frac{1-\bar\lambda w}{w-\lambda}\frac{1-|\lambda|^2}{(1-\bar\lambda w)^2} = \frac{(1-|\lambda|^2)\bar w}{|1-\bar\lambda w|^2}$ and $\left|\frac{1-\bar\lambda w}{w-\lambda}\right| = 1$. Using Hölder's inequality we then see that

(3.19)
$$\begin{aligned}
&\left(1-|\lambda|^2\right)^{1/t}|p(\lambda)|\left|\int_{w\in\partial\mathbb{D}}\frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}\bar w G(w)h(w)\frac{dm(w)}{2\pi}\right|\\
&\leqslant \|p\|_{L^t(\mu)}\left[\int_{w\in\partial\mathbb{D}}\frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}|G(w)|^{t'}h(w)\frac{dm(w)}{2\pi}\right]^{1/t'}\\
&\quad+\left[\int_{w\in\mathbb{D}}\left|\frac{1-\bar\lambda w}{w-\lambda}\right|^t\left|p(w)-p(\lambda)\left(1-|\lambda|^2\right)^{1/t}\left(\frac{1-|\lambda|^2}{(1-\bar\lambda w)^2}\right)^{1/t}\right|^t d\mu(w)\right]^{1/t}\\
&\qquad\times\left[\int_{w\in\mathbb{D}}\frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}|G(w)|^{t'}\,d\mu(w)\right]^{1/t'}.
\end{aligned}$$

We will now estimate the first factor of the second summand of the right-hand side of this inequality. We will denote this factor by $I(\lambda)^{1/t}$. Our methods here will be very close to methods we used in the proof of Theorem 3.2.

Write

$$\begin{aligned}
I(\lambda) = &\left[\int_{\rho(w,\lambda)\leqslant\frac12}+\int_{\frac12<\rho(w,\lambda)<1}\right]\left|\frac{1-\bar\lambda w}{w-\lambda}\right|^t\times\\
&\times\left|p(w)-p(\lambda)\left(1-|\lambda|^2\right)^{1/t}\left(\frac{1-|\lambda|^2}{(1-\bar\lambda w)^2}\right)^{1/t}\right|^t d\mu(w) = I_1(\lambda)+I_2(\lambda).
\end{aligned}$$

It is easy to see that for any $z \in \partial\mathbb{D}$, $\lambda \in \Gamma(z)$ and $\rho(w,\lambda) \leqslant \frac12$ together imply that $w \in \Gamma_{\sigma_0}(z)$ for some fixed $\sigma_0 \in \left(\frac12, 1\right)$. The nontangential approach region in the inequality (1.4) was fixed at $\Gamma(z) = \Gamma_{\frac12}(z)$ for convenience, but any $\Gamma_\sigma(z)$ would have worked as well. Using the continuity of $\lambda \mapsto M_\lambda$ it is then easy to see that there is a closed subset $E \subseteq [h>0] \subseteq \partial\mathbb{D}$ such that $m([h>0]\setminus E)$ is arbitrarily small and $\left(1-|w|^2\right)^{1/t}M_w$ is bounded on $\Gamma_{\sigma_0}(E) \underset{\text{def}}{=} \bigcup_{z\in E}\Gamma_{\sigma_0}(z)$. Say

$$\left(1-|w|^2\right)^{1/t}M_w \leqslant \tilde C \qquad \text{for } w \in \Gamma_{\sigma_0}(E). \tag{3.20}$$

By Lemma 1.1 we may also assume that

$$\int_{w\in\mathbb{D}}\frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}\,d\mu(w) \leqslant \tilde C \qquad \text{for } \lambda \in \Gamma(E). \tag{3.21}$$

Now suppose $\lambda \in \Gamma(E)$. It is easy to show that if $\rho(\zeta,\lambda) \leqslant \frac12$ then

$$|\zeta| \leqslant \frac{|\lambda|+\frac12}{1+\frac12|\lambda|} \qquad \text{and} \qquad \frac{1-|\lambda|^2}{|1-\bar\lambda\zeta|} \leqslant \frac32.$$

Hence using (3.20) we see that if $\rho(w,\lambda) \leqslant \frac{1}{2}$, then

$$\begin{aligned}
&\left| \frac{1-\bar\lambda w}{w-\lambda} \left(p(w) - p(\lambda)\left(1-|\lambda|^2\right)^{1/t} \left(\frac{1-|\lambda|^2}{(1-\bar\lambda w)^2}\right)^{1/t}\right)\right| \\
&\qquad \leqslant \max_{\rho(\zeta,\lambda)=\frac{1}{2}} \left| \frac{1-\bar\lambda \zeta}{\zeta-\lambda} \left(p(\zeta) - p(\lambda)\left(1-|\lambda|^2\right)^{1/t} \left(\frac{1-|\lambda|^2}{(1-\bar\lambda \zeta)^2}\right)^{1/t}\right)\right| \\
&\qquad \leqslant 2\left[\frac{3^{1/t}\tilde C}{(1-|\lambda|^2)^{1/t}} + \frac{\left(\frac{3}{2}\right)^{1/t}\tilde C}{(1-|\lambda|^2)^{1/t}}\right]\|p\|_{L^t(\mu)}.
\end{aligned}$$

We also have, using (3.21) and the fact that $\frac{1-|\lambda|^2}{|1-\bar\lambda w|} \geqslant \frac{1}{2}$ if $\rho(w,\lambda) \leqslant \frac{1}{2}$, that

$$\int_{\rho(w,\lambda)\leqslant\frac{1}{2}} d\mu(w) \leqslant 4\int_{w\in\mathbb{D}} \frac{\left(1-|\lambda|^2\right)^2}{|1-\bar\lambda w|^2}\, d\mu(w) \leqslant 4\left(1-|\lambda|^2\right)\tilde C.$$

These last two inequalities imply that

$$I_1(\lambda) \leqslant 2^t\left(3^{1/t} + \left(\frac{3}{2}\right)^{1/t}\right)^t \tilde C^t \cdot 4\tilde C\|p\|^t_{L^t(\mu)}.$$

To estimate $I_2(\lambda)$ we use Minkowski's inequality together with (3.21):

$$\begin{aligned}
I_2(\lambda) \leqslant 2^t &\left[\left(\int |p(w)|^t\, d\mu(w)\right)^{1/t}\right. \\
&\qquad \left. + \left(1-|\lambda|^2\right)^{1/t}|p(\lambda)|\left(\int_{w\in\mathbb{D}} \frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}\, d\mu(w)\right)^{1/t}\right]^t \\
\leqslant 2^t &\left(1+\tilde C^{1+\frac{1}{t}}\right)^t \|p\|^t_{L^t(\mu)}.
\end{aligned}$$

Our estimates on $I_1(\lambda)$ and $I_2(\lambda)$ combine to give us

$$I(\lambda) \leqslant \tilde{\tilde{C}}^t\|p\|^t_{L^t(\mu)}.$$

We put this estimate into (3.19), set $\|p\|_{\mathcal{P}^t(\mu)} = 1$ on the right-hand side, and sup over the corresponding $|p(\lambda)|$ on the left-hand side to get

$$\begin{aligned}
\text{(3.22)}\qquad &\left(1-|\lambda|^2\right)^{1/t} M_\lambda \left|\int_{w\in\partial\mathbb{D}} \frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}\bar w G(w)h(w)\frac{dm(w)}{2\pi}\right| \\
&\qquad \leqslant \left[\int_{w\in\partial\mathbb{D}} \frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}|G(w)|^{t'}h(w)\frac{dm(w)}{2\pi}\right]^{1/t'} \\
&\qquad\quad + \tilde{\tilde{C}}\left[\int_{w\in\mathbb{D}} \frac{1-|\lambda|^2}{|1-\bar\lambda w|^2}|G(w)|^{t'}\, d\mu(w)\right]^{1/t'}
\end{aligned}$$

for $\lambda \in \Gamma(E)$.

We have for m-almost all $z \in \partial\mathbb{D}$

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \int_{w\in\partial\mathbb{D}} \frac{1-|\lambda|^2}{|1-\bar{\lambda}w|^2}\bar{w}G(w)h(w)\frac{dm(w)}{2\pi} = \bar{z}G(z)h(z)$$

and

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \int_{w\in\partial\mathbb{D}} \frac{1-|\lambda|^2}{|1-\bar{\lambda}w|^2}|G(w)|^{t'}h(w)\frac{dm(w)}{2\pi} = |G(z)|^{t'}h(z).$$

By Lemma 1.1 we also have for m-almost all $z \in \partial\mathbb{D}$ that

$$\lim_{\substack{\lambda\to z\\ \lambda\in\Gamma(z)}} \int_{w\in\mathbb{D}} \frac{1-|\lambda|^2}{|1-\bar{\lambda}w|^2}|G(w)|^{t'}\,d\mu(w) = 0.$$

Theorem C now follows from (3.22), (1.3), and the fact that G may be chosen so that $G(z) \neq 0$ a.e. on $\{z : h(z) > 0\}$. $\square$

4. Interpolating sequences

In this section we will prove Theorem B. Throughout, μ will be a fixed measure satisfying the hypotheses of the theorem. We will also use the notation

$$M(r) = \sup_{|\lambda|=r} M_\lambda, \qquad \text{for } r \in (0,1).$$

By the continuity of $\lambda \mapsto M_\lambda$ in $\mathbb{D}$, $M(r) < \infty$.

We will need some basic results about interpolating sequences for H^∞, the usual Hardy space of bounded analytic functions in $\mathbb{D}$. Let $\Lambda = \{\lambda_n\}$ be a sequence of distinct points in $\mathbb{D}$. We say Λ is an *interpolating sequence for* H^∞ if T_Λ maps H^∞ onto l^∞. The interpolating sequences for H^∞ were characterized by Lennart Carleson in [Car58], and it was subsequently shown by Per Beurling [Car63] that for such a sequence there is a bounded linear operator $l^\infty \to H^\infty$ that acts as a right inverse for T_Λ. We summarize these results in the form that will be most useful to us. We use the

Notation. If $\lambda \in \mathbb{D}$ then

$$I_\lambda = \{z \in \partial\mathbb{D} : \ \lambda \in \Gamma(z)\}$$
$$S_\lambda = \{rz : \ z \in I_\lambda, \ |\lambda| < r < 1\}.$$

THEOREM 4.1 (Carleson and P. Beurling). *The following are equivalent for a sequence $\Lambda = \{\lambda_n\}$ of distinct points in $\mathbb{D}$:*

a) Λ *is an interpolating sequence for H^∞.*

b) *There are numbers* $\delta_b > 0$, $C_b < \infty$ *such that*

$$\left| \frac{\lambda_m - \lambda_n}{1 - \bar{\lambda}_m \lambda_n} \right| \geqslant \delta_b \qquad \textit{for } m \neq n$$

and

$$\sum_{\lambda_n \in S_\lambda} \left(1 - |\lambda_n|^2\right) \leqslant C_b \left(1 - |\lambda|^2\right) \qquad \textit{for all } \lambda \in \mathbb{D}.$$

c) *There is a number* $C_c < \infty$ *and functions* φ_n *analytic in* $\mathbb{D}$ *such that* $\varphi_n(\lambda_m) = \delta_{nm}$ *and*

$$\sum |\varphi_n(z)| \leqslant C_c \qquad \textit{for all } z \in \mathbb{D}.$$

Furthermore, C_c *can be bounded above by a constant depending only on* δ_b *and* C_b*, and* δ_b *can be bounded below and* C_b *can be bounded above by constants depending only on* C_c*.* □

For a proof see e.g. [Gar81, Ch. 7]. Of course if c) is satisfied, the function $\sum a_n \varphi_n$ is an H^∞ function interpolating the values $\{a_n\}$ at $\{\lambda_n\}$. The second inequality in b) says that $\sum \left(1 - |\lambda_n|^2\right) \delta_{\lambda_n}$ is a *Carleson measure.*

Subsequent to Carleson's work it was shown by Harold S. Shapiro and Allen Shields [SS61] that the interpolating sequences for H^∞ were the same as the interpolating sequences for H^t. It is of interest in its own right, and is closely related to ideas that will be important in the proof of Theorem B, that at least half of this equivalence holds for our $\mathcal{P}^t(\mu)$. This will be an easy consequence of Theorem 4.1 and the next lemma, a special case of the classical Schur's Lemma.

Lemma 4.2. *Let* κ *be a positive measure on a measure space* X*, let* $s \in [1, \infty]$*, and suppose that* $\{\varphi_n\}$ *is a sequence of measurable functions on* X *such that* $\sum |\varphi_n(x)| \leqslant C$ *for* κ*-almost all* x*.*

Then if $\{f_n\}$ *is a sequence of measurable functions on* X *such that*

$$\left\{ \|f_n\|_{L^s(\kappa)} \right\} \in l^s,$$

we have

$$\sum \varphi_n f_n \in L^s(\kappa)$$

and

$$\left\| \sum \varphi_n f_n \right\|_{L^s(\kappa)} \leqslant C \left\| \left\{ \|f_n\|_{L^s(\kappa)} \right\} \right\|_{l^s}.$$

Proof. For $s = 1, \infty$ the assertions are obvious. If $s \in (1, \infty)$ let $s' = \frac{s}{s-1}$ be the conjugate exponent to s. Then

$$\begin{aligned}\int \left|\sum \varphi_n f_n\right|^s d\kappa &\leqslant \int \left(\sum |f_n|^s |\varphi_n|\right) \cdot \left(\sum |\varphi_n|\right)^{s/s'} d\kappa \\ &\leqslant C^{1+\frac{s}{s'}} \int \sum |f_n|^s \, d\kappa \\ &= C^s \left\|\left\{\|f_n\|_{L^s(\kappa)}\right\}\right\|_{l^s}^s . \end{aligned}$$

□

COROLLARY 4.3. *Let* $1 \leqslant t < \infty$. *Suppose* $\{\lambda_n\}$ *is an interpolating sequence for* H^∞. *Then* $\{\lambda_n\}$ *is an interpolating sequence for* $\mathcal{P}^t(\mu)$.

Proof. Let $\{\varphi_n\}$ be the Per Beurling functions given by c) of Theorem 4.1. We can find a sequence $\{g_n\}$ of polynomials such that $g_n(\lambda_n) = 1$ and $\|g_n\|_{L^t(\mu)} \leqslant \frac{2}{M_{\lambda_n}}$ for all n. Now suppose that $\{a_n\}$ is a sequence of complex numbers such that $\sum \left(\frac{|a_n|}{M_{\lambda_n}}\right)^t < \infty$. By Lemma 4.2 with $s = t$, $g = \sum a_n \varphi_n g_n \in \mathcal{P}^t(\mu)$ with

$$\|g\|_{\mathcal{P}^t(\mu)} \leqslant 2C_c \left[\sum \left(\frac{|a_n|}{M_{\lambda_n}}\right)^t\right]^{1/t},$$

and of course $g(\lambda_n) = a_n$ for all n. This shows that T_λ maps $\mathcal{P}^t(\mu)$ onto $l^t_\Lambda(\mu)$.

For the other half, find a sequence $\{H_n\}$ of functions in $L^{t'}(\mu)$ such that

$$\int f H_n \, d\mu = f(\lambda_n) \qquad \text{for } f \in \mathcal{P}^t(\mu)$$

and $\|H_n\|_{L^{t'}(\mu)} = M_{\lambda_n}$. Let $\{\gamma_n\}$ be a sequence of complex numbers such that $\{\gamma_n M_{\lambda_n}\} \in l^{t'}$. By Lemma 4.2 with $s = t'$ we see that

$$H = \sum \gamma_n \varphi_n H_n \in L^{t'}(\mu)$$

and

$$\|H\|_{L^{t'}(\mu)} \leqslant C_c \|\{\gamma_n M_{\lambda_n}\}\|_{l^{t'}}.$$

Since $\int fH \, d\mu = \sum f(\lambda_n)\gamma_n$ for $f \in \mathcal{P}^t(\mu)$, a duality argument now shows that T_Λ maps $\mathcal{P}^t(\mu)$ into $l^t_\Lambda(\mu)$. □

We need a few more preliminaries. We have already defined the arc I_λ for $\lambda \in \mathbb{D}$. We now define $\tilde{I}_\lambda$ to be the open arc with the same center as I_λ and one-third its length, implicitly assuming that $I_\lambda \neq \partial\mathbb{D}$, i.e. that $|\lambda| > \frac{1}{2}$.

It is geometrically obvious that if $\{\lambda_n\} \subseteq \mathbb{D}$ is such that the intervals $\tilde{I}_{\lambda_n}$ are pairwise disjoint, then the conditions b) of Theorem 4.1 are met, with bounds. We thus get from that theorem:

LEMMA 4.4. *There is a constant C_I with the following property. If $\lambda_n \in \mathbb{D}$ are such that $\tilde{I}_{\lambda_n} \cap \tilde{I}_{\lambda_m} = \varnothing$ for $m \neq n$, then there are functions φ_n analytic in $\mathbb{D}$ such that $\varphi_n(\lambda_m) = \delta_{nm}$ and $\sum |\varphi_n(z)| \leqslant C_I$ for all $z \in \mathbb{D}$.* $\square$

We have already mentioned in Section 1 the well-known fact that if $K \subseteq \operatorname{abpe}\big(\mathcal{P}^t(\mu)\big)$ is compact, then $\mathcal{P}^t(\mu|\mathbb{C}\setminus K) = \mathcal{P}^t(\mu)$ with equivalence of norms. Our next result shows that this removal of K makes no difference in the asymptotics of M_λ.

LEMMA 4.5. *Let $r \in (0,1)$ and $\epsilon > 0$. Then there exists a $\rho \in (r,1)$ such that if $\lambda \in \mathbb{D}$ with $|\lambda| \geqslant \rho$ then*

$$|f(\lambda)| \leqslant (1+\epsilon) M_\lambda \left[\int_{r<|w|\leqslant 1} |f(w)|^t \, d\mu(w)\right]^{1/t} \qquad \text{for } f \in \mathcal{P}^t(\mu).$$

Proof. Let $N \in \mathbb{N}$, then for $\lambda \in \mathbb{D}$ and $f \in \mathcal{P}^t(\mu)$

$$\begin{aligned} |\lambda^N f(\lambda)|^t &\leqslant M_\lambda^t \int |z^N f(z)|^t d\mu(z) \\ &\leqslant M_\lambda^t \left(r^{Nt} \int_{|z|\leqslant r} |f(z)|^t d\mu(z) + \int_{r<|z|\leqslant 1} |f(z)|^t d\mu(z) \right). \end{aligned}$$

Hence, if $0 < \rho < 1$ and $|\lambda| \geqslant \rho$, then

$$|f(\lambda)|^t \leqslant \left(\frac{M_\lambda}{\rho^N}\right)^t \left(r^{Nt} \int_{|z|\leqslant r} |f(z)|^t d\mu(z) + \int_{r<|z|\leqslant 1} |f(z)|^t d\mu(z) \right) \tag{4.1}$$

and for $|\lambda| = \rho$

$$|f(\lambda)|^t \leqslant \left(\frac{M(\rho)}{\rho^N}\right)^t \left(r^{Nt} \int_{|z|\leqslant r} |f(z)|^t d\mu(z) + \int_{r<|z|\leqslant 1} |f(z)|^t d\mu(z) \right). \tag{4.2}$$

We will now show that there must be a constant $C_r > 0$ such that

$$\int_{|z|\leqslant r} |f(z)|^t d\mu(z) \leqslant C_r \int_{r<|z|\leqslant 1} |f(z)|^t d\mu(z) \qquad \forall f \in \mathcal{P}^t(\mu). \tag{4.3}$$

Indeed, suppose there is a sequence $\{f_n\}$ of functions in $\mathcal{P}^t(\mu)$ such that $\int_{|z|\leqslant r} |f_n(z)|^t d\mu(z) = 1$ for each n and $\int_{r<|z|\leqslant 1} |f_n(z)|^t d\mu(z) \to 0$ as $n \to \infty$. We fix ρ, $r < \rho < 1$. Then by (4.2) we have for $|\lambda| = \rho$ and all n and N

$$|f_n(\lambda)|^t \leqslant M(\rho)^t (\frac{r}{\rho})^{Nt} + (\frac{M(\rho)}{\rho^N})^t \int_{r<|z|\leqslant 1} |f_n(z)|^t d\mu(z). \tag{4.4}$$

Given $\delta > 0$ choose N so large that $M(\rho)^t(\frac{r}{\rho})^{Nt} < \delta$. Using this N in (4.4) we see that $\limsup_{n\to\infty} |f_n(\lambda)|^t < \delta$ uniformly in $|\lambda| = \rho$. Hence it follows that $|f_n(\lambda)| \to 0$ uniformly on $|\lambda| = \rho$ and the maximum principle shows that the convergence is uniform on $|\lambda| \leqslant \rho$. This implies that $\int_{|z|\leqslant r} |f_n(z)|^t d\mu(z) \to 0$, and it thus contradicts the choice of $\{f_n\}$, proving (4.3).

Next we substitute (4.3) into (4.1), take t-th roots, and obtain for all $N \in \mathbb{N}$, $0 < \rho < 1$, $|\lambda| \geqslant \rho$, and $f \in \mathcal{P}^t(\mu)$ that

$$|f(\lambda)| \leqslant M_\lambda \frac{(1 + C_r r^{Nt})^{1/t}}{\rho^N} \left(\int_{r<|z|\leqslant 1} |f(z)|^t d\mu(z) \right)^{1/t}.$$

Now we choose N and then ρ such that $\frac{(1+C_r r^{Nt})^{1/t}}{\rho^N} < 1 + \varepsilon$. This completes the proof of the lemma. $\square$

Before we undertake the proof of Theorem B we make one more observation. If $K \subset \partial\mathbb{D}$ is compact, then for any $0 < s < 1$ we have

$$\bigcup_{\lambda \in \Gamma(K), |\lambda| = s} \tilde{I}_\lambda \supset K.$$

Hence by Lemma 7.3 in [Rud87] (the familiar Vitali-type covering lemma due to Wiener) there are

(4.5) $\lambda_1, \ldots, \lambda_n \in \Gamma(K)$ such that $|\lambda_k| = s$ for all $k = 1, \ldots, n$, $\tilde{I}_k \cap \tilde{I}_l = \varnothing$ if $k \neq l$ and $\bigcup_{k=1}^n I_k \supset K$, where $I_k = I_{\lambda_k}$.

We turn to the proof of Theorem B. Let $E \subset \partial\mathbb{D}$ with $\mu(E) = 0$. We will construct a sequence Λ that clusters nontangentially at m-almost every point of E and show that T_Λ maps $\mathcal{P}^t(\mu)$ onto $l^t_\Lambda(\mu)$ for all $t \in [1, \infty)$ and into $l^t_\Lambda(\mu)$ if $t \in (1, \infty)$. We will actually just deal with the case $t > 1$, leaving the easy alterations for $t = 1$ to the reader.

Let Φ be the outer function with

$$|\Phi| = \begin{cases} 1 & \text{on } E \\ e^{-1} & \text{on } \partial\mathbb{D} \setminus E. \end{cases}$$

By (1.2) and Egoroff's theorem we can find compact sets $E_0 \subseteq E_1 \subseteq \cdots \subseteq E$ such that $m(E_j) \nearrow m(E)$ and for each $j = 1, 2, \ldots$ $\int \frac{1-|\lambda|^2}{|1-\bar{\lambda} w|^2}\, d\mu(w) \to 0$ and $|\Phi(\lambda)| \to 1$ uniformly as $|\lambda| \to 1$, $\lambda \in \Gamma(E_j)$. Set $N_0 = 0$, $B_0 \equiv 1$, $r_0 = 0$. Using Lemmas 4.4 and 4.5, duality, and the observation (4.5) we inductively find:

- numbers r_j for $j = 0, 1, \ldots$ such that $0 = r_0 < r_1 < r_2 < \cdots < 1$ and $r_j \to 1$;
- integers N_j for $j = 0, 1, \ldots$ such that $0 = N_0 < N_1 < \cdots$;
- points $\lambda_{jk} \in \mathbb{D}$ for $j \geqslant 0$, $1 \leqslant k \leqslant n_j$, such that

$$r_j < |\lambda_{jk}| < r_{j+1} \tag{4.6}$$

$$\tilde{I}_{jk} \cap \tilde{I}_{jl} = \varnothing \qquad \text{if } k \neq l \tag{4.7}$$

and

$$\bigcup_{k=1}^{n_j} I_{jk} \supset E_j; \tag{4.8}$$

- functions $\varphi_{j1}, \dots, \varphi_{jn_j}$ corresponding to the points $\lambda_{j1}, \dots, \lambda_{jn_j}$, as in Lemma 4.4, which we may assume to be polynomials;
- polynomials g_{jk} such that

$$g_{jk}(\lambda_{jk}) = 1 \tag{4.9}$$

and

$$\|g_{jk}\|_{L^t(\mu)} \leqslant \frac{2}{M_{\lambda_{jk}}}; \tag{4.10}$$

- functions $H_{jk} \in L^{t'}(\mu)$ such that (here we use Lemma 4.5 with $1+\epsilon = 2$)

$$H_{jk}(w) = 0 \qquad \text{if } |w| < r_j, \tag{4.11}$$

$$\int f H_{jk}\, d\mu = f(\lambda_{jk}) \qquad \text{for } f \in \mathcal{P}^t(\mu), \tag{4.12}$$

and

$$\|H_{jk}\|_{L^{t'}(\mu)} \leqslant 2M_{\lambda_{jk}} \tag{4.13}$$

with the following properties: setting for $j \geqslant 1$,

$$B_j = \text{the Blaschke product with zeros } \{\lambda_{j'k}\}_{\substack{j'<j \\ 1\leqslant k\leqslant n_{j'}}},$$

and for $j \geqslant 0$,

$$M_j = \sup\left\{|g(w)| : \ g \in \operatorname{span}\{\varphi_{jk} g_{jk}\}_{k=1}^{n_j},\ \|g\|_{L^t(\mu)\leqslant 1},\ w \in \mathbb{D}\right\},$$

we have

$$\left(\sum_{j<j'} M_j^{t'}\right)^{t/t'} \mu([r_{j'} \leqslant |w| < 1]) < \frac{1}{2^{j'}} \quad \forall j' > 0, \tag{4.14}$$

(4.15) if $H \in \operatorname{span}\{\varphi_{jk} H_{jk} : \ 1 \leqslant k \leqslant n_j\}$ with $\|H\|_{L^{t'}(\mu)} \leqslant 1$, and $j' > j$, then $\int_{r_{j'} \leqslant |w| < 1} |H(w)|^{t'}\, d\mu(w) < \frac{1}{2^{j'}}$ (this is the point with no analogue for $t = 1$),

$$e^{-N_j} \leqslant \min\left\{\frac{1}{2^{j/t'}}, \frac{1}{2^{j/t}}\right\}, \tag{4.16}$$

$$r_j^{N_j} M(r_j) < \frac{1}{2^{j/t'}} \quad \forall j > 0, \tag{4.17}$$

$$4C_I \left(\sum_{j'<j} M_{j'}^{t'}\right)^{\frac{t}{t'}} \int \frac{1 - |\lambda_{jk}|^2}{|1 - \bar{\lambda}_{jk} w|^2}\, d\mu(w) < \frac{1}{100} \frac{1}{2^j} \ \forall j > 0, 1 \leq k \leq n_j, \tag{4.18}$$

and

$$\left|\lambda_{jk}^{N_j} B_j(\lambda_{jk})\Phi(\lambda_{jk})^{N_j}\right| \geqslant \frac{1}{2} \quad \forall j,k. \tag{4.19}$$

The sequence $\Lambda = \{\lambda_{jk}\}_{j,k}$ is our desired sequence. From (4.8) it is clear that Λ clusters nontangentially at m-almost every point of E.

To see that T_Λ maps $\mathcal{P}^t(\mu)$ onto $l^t_\Lambda(\mu)$, suppose that $\{a_{jk}\}$ are complex numbers such that $\sum_{j,k}\left(\frac{|a_{jk}|}{M_{\lambda_{jk}}}\right)^t \leqslant 1$. For $j \geqslant 0$ define $A_j = \left[\sum_{k=1}^{n_j}\left(\frac{|a_{jk}|}{M_{\lambda_{jk}}}\right)^t\right]^{1/t}$. Now set

$$\begin{aligned} F(w) &= \sum_{j,k} \frac{a_{jk}}{\lambda_{jk}^{N_j} B_j(\lambda_{jk})\Phi(\lambda_{jk})^{N_j}} w^{N_j} B_j(w)\Phi(w)^{N_j}\varphi_{jk}(w) g_{jk}(w) \\ &= \sum_j F_j(w) \end{aligned}$$

so that $F_j(w) = w^{N_j} B_j(w)\Phi(w)^{N_j} G_j(w)$ where

$$G_j = \sum_{k=1}^{n_j} \frac{a_{jk}}{\lambda_{jk}^{N_j} B_j(\lambda_{jk})\Phi(\lambda_{jk})^{N_j}} \varphi_{jk} g_{jk}.$$

We will show that the series defining F converges in $\mathcal{P}^t(\mu)$ and that F almost does the desired interpolation.

Using Lemma 4.2 as in the proof of Corollary 4.3, (4.10), and (4.19), we see that

$$\|G_j\|_{\mathcal{P}^t(\mu)} \leqslant 4C_I A_j. \tag{4.20}$$

It follows that

$$|F_j(w)| \leqslant 4C_I A_j M_j \qquad \text{for all } w \in \mathbb{D}, \tag{4.21}$$

$$\|F_j\|_{\mathcal{P}^t(\mu)} \leqslant 4C_I A_j, \tag{4.22}$$

and

$$|F_j(w)| \leqslant 4C_I A_j r_j^{N_j} M(r_j) \qquad \text{for } |w| \leqslant r_j. \tag{4.23}$$

Now

$$\begin{aligned} &\int \left(\sum_j |F_j(w)|\right)^t d\mu(w) \\ &\qquad = \left(\int_{|w|=1} + \sum_{j'=0}^{\infty} \int_{r_{j'} \leqslant |w| < r_{j'+1}}\right) \left(\sum_j |F_j(w)|\right)^t d\mu(w). \end{aligned}$$

We estimate the two types of terms separately. First:

$$\begin{aligned}
\int_{|w|=1}\left(\sum_j |F_j(w)|\right)^t d\mu(w) &= \int_{(\partial\mathbb{D})\setminus E}\left(\sum_j |F_j(w)|\right)^t d\mu(w)\\
&= \int_{(\partial\mathbb{D})\setminus E}\left(\sum_j e^{-N_j}|G_j(w)|\right)^t d\mu(w)\\
&\leqslant \int_{(\partial\mathbb{D})\setminus E}\left(\sum_j e^{-N_j t'}\right)^{t/t'} \sum_j |G_j(w)|^t\, d\mu(w)\\
&\leqslant 2^{t/t'}\cdot 4^t C_I^t \sum_j A_j^t \qquad \text{(by (4.16) and (4.20))}\\
&\leqslant 2^{t/t'}\cdot 4^t C_I^t.
\end{aligned}$$

Second:

$$\begin{aligned}
&\int_{r_{j'}\leqslant|w|<r_{j'+1}}\left(\sum_j |F_j(w)|\right)^t d\mu(w)\\
&\quad\leqslant 3^{t-1}\int_{r_{j'}\leqslant|w|<r_{j'+1}}\left[\sum_{j<j'}|F_j(w)|^t + |F_{j'}(w)|^t + \left(\sum_{j>j'}|F_j(w)|\right)^t\right] d\mu(w)\\
&\quad\leqslant 3^{t-1}\left[\left(\sum_{j<j'} 4C_I A_j M_j\right)^t \mu([r_{j'}\leqslant|w|<1]) + 4^t C_I^t A_{j'}^t\right.\\
&\qquad\qquad\left. + \left(\sum_{j>j'} 4C_I A_j r_j^{N_j} M(r_j)\right)^t \|\mu\|\right] \text{(by (4.21), (4.22), and (4.23))}\\
&\quad\leqslant 3^{t-1}4^t C_I^t\left[\left(\sum_{j<j'}A_j^t\right)\left(\sum_{j<j'}M_j^{t'}\right)^{t/t'} \mu([r_{j'}\leqslant|w|<1])\right.\\
&\qquad\qquad\left. + A_{j'}^t + \left(\sum_{j>j'}A_j^t\right)\left(\sum_{j>j'} r_j^{N_j t'} M(r_j)^{t'}\right)^{t/t'} \|\mu\|\right].
\end{aligned}$$

Using (4.14), (4.17), and the fact that $\sum A_j^t \leqslant 1$, we see that this is bounded by

$$3^{t-1}4^t C_I^t\left[\frac{1}{2^{j'}} + A_{j'}^t + \left(\frac{1}{2^{j'}}\right)^{t/t'}\|\mu\|\right]$$

(where the first term in the brackets is absent if $j'=0$), and it follows that

$$\sum_{j'=0}^{\infty}\int_{r'_j\leqslant|w|<r_{j'+1}}\left(\sum_j|F_j(w)|\right)^t d\mu(w)\leqslant 3^{t-1}4^tC_I^t\left[1+1+\frac{2^{t/t'}}{2^{t/t'}-1}\|\mu\|\right].$$

Hence $F=\sum F_j$ converges in $\mathcal{P}^t(\mu)$ and

$$\|F\|^t_{\mathcal{P}^t(\mu)}\leqslant 2^{t/t'}\cdot 4^tC_I^t\|\mu\|+3^{t-1}4^tC_I^t\left[2+\frac{2^{t/t'}}{2^{t/t'}-1}\|\mu\|\right]=C_1^t.$$

We now show that F almost does the desired interpolation. Clearly $F_j(\lambda_{jk}) = a_{jk}$. Since $F_{j'}(\lambda_{jk})=0$ for $j'>j$, we see that $F(\lambda_{0k})=a_{0k}$, and for $j>0$,

$$\begin{aligned}|F(\lambda_{jk})-a_{jk}|&\leqslant\sum_{j'<j}|F_{j'}(\lambda_{jk})|\\&\leqslant\sum_{j'<j}4C_IA_{j'}M_{j'}\qquad\text{(by (4.21))}\\&\leqslant 4C_I\left(\sum_{j'<j}A_{j'}^t\right)^{1/t}\left(\sum_{j'<j}M_{j'}^{t'}\right)^{1/t'}\\&\leqslant\frac{1}{100^{1/t}}\frac{1}{2^{j/t}}\frac{1}{\left(\int\frac{1-|\lambda_{jk}|^2}{|1-\bar{\lambda}_{jk}w|^2}\,d\mu(w)\right)^{1/t}}\qquad\text{(by (4.18)).}\end{aligned}$$

Hence, using the inequality (1.1), we get

$$\sum_{j,k}\left(\frac{|F(\lambda_{jk})-a_{jk}|}{M_{\lambda_{jk}}}\right)^t\leqslant\frac{1}{100}\sum_{j=1}^{\infty}\sum_{k=1}^{n_j}\frac{1}{2^j}\left(1-|\lambda_{jk}|^2\right).$$

Our choice of the λ_{jk}'s implies that

$$\sum_{k=1}^{n_j}\left(1-|\lambda_{jk}|^2\right)<50\qquad\text{for all } j,$$

and then

$$\sum_{j,k}\left(\frac{|F(\lambda_{jk})-a_{jk}|}{M_{\lambda_{jk}}}\right)^t\leqslant\frac{1}{2}.$$

Summing up and rescaling, we have shown that if $a_{jk}\in\mathbb{C}$ are such that $\sum_{j,k}\left(\frac{|a_{jk}|}{M_{\lambda_{jk}}}\right)^t<\infty$, then there is an $F\in\mathcal{P}^t(\mu)$ such that

$$\|F\|_{\mathcal{P}^t(\mu)}\leqslant C_1\left[\sum_{j,k}\left(\frac{|a_{jk}|}{M_{\lambda_{jk}}}\right)^t\right]^{1/t}$$

and

$$\left[\sum_{j,k}\left(\frac{|F(\lambda_{jk})-a_{jk}|}{M_{\lambda_{jk}}}\right)^t\right]^{1/t} \leqslant \frac{1}{2}\left[\sum_{j,k}\left(\frac{|a_{jk}|}{M_{\lambda_{jk}}}\right)^t\right]^{1/t}.$$

A standard exercise in Functional Analysis now shows that T_Λ maps $\mathcal{P}^t(\mu)$ onto $l^t_\Lambda(\mu)$.

We turn to the into-ness of T_Λ when $t > 1$. Suppose $\gamma_{jk} \in \mathbb{C}$ with $\sum_{j,k} |\gamma_{jk}|^{t'} M^{t'}_{\lambda_{jk}} \leqslant 1$. Set

$$\Gamma_j = \left(\sum_{k=1}^{n_j} |\gamma_{jk}|^{t'} M^{t'}_{\lambda_{jk}}\right)^{1/t'}$$

and define

$$H = \sum_{j=0}^{\infty} \Phi^{N_j} \sum_{k=1}^{n_j} \frac{\gamma_{jk}\varphi_{jk}}{\Phi(\lambda_{jk})^{N_j}} H_{jk} = \sum_{j=0}^{\infty} \Phi^{N_j} H_j.$$

Using Lemma 4.2 as in the proof of Corollary 4.3 together with (4.13) and (4.19), we see that

$$\|H_j\|_{L^{t'}(\mu)} \leqslant 4C_I\Gamma_j. \tag{4.24}$$

As above, we write

$$\begin{aligned}\int \left(\sum_j \left|\Phi^{N_j} H_j\right|\right)^{t'} d\mu \\ = \left[\int_{|w|=1} + \sum_{j'=0}^{\infty} \int_{r_{j'} \leqslant |w| < r_{j'+1}}\right] \left(\sum_j \left|\Phi(w)^{N_j} H_j(w)\right|\right)^{t'} d\mu(w)\end{aligned}$$

and estimate the two types of terms separately. First:

$$\begin{aligned}\int_{|w|=1} \left(\sum_j \left|\Phi(w)^{N_j} H_j(w)\right|\right)^{t'} d\mu(w) &= \int_{(\operatorname{clos}\mathbb{D})\setminus E} \left(\sum_j e^{-N_j}|H_j|\right)^{t'} d\mu \\ &\leqslant \left(\sum_j e^{-N_j t}\right)^{t'/t} \int \sum_j |H_j|^{t'}\, d\mu \\ &\leqslant \left(\sum_j \frac{1}{2^j}\right)^{t'/t} \sum_j 4^{t'} C_I^{t'} \Gamma_j^{t'} \\ &\qquad \text{(by (4.16) and (4.24))} \\ &\leqslant 2^{t'/t} 4^{t'} C_I^{t'}.\end{aligned}$$

Second:

$$\begin{aligned}
&\sum_{j'=0}^{\infty}\int_{r_{j'}\leqslant|w|<r_{j'+1}}\left(\sum_j\left|\Phi(w)^{N_j}H_j(w)\right|\right)^{t'}d\mu(w)\\
&\leqslant\sum_{j'=0}^{\infty}\int_{r_{j'}\leqslant|w|<r_{j'+1}}\left(|H_{j'}(w)|+\sum_{j<j'}|H_j(w)|\right)^{t'}d\mu(w)\\
&\leqslant\sum_{j'=0}^{\infty}2^{t'-1}\left[\|H_{j'}\|_{L^{t'}(\mu)}^{t'}+(j')^{t'-1}\int_{r_{j'}\leqslant|w|<1}\sum_{j<j'}|H_j(w)|^{t'}\,d\mu(w)\right]\\
&\leqslant\sum_{j'=0}^{\infty}2^{t'-1}\left[4^{t'}C_I^{t'}\Gamma_j^{t'}+(j')^{t'-1}\sum_{j<j'}\frac{4^{t'}C_I^{t'}\Gamma_j^{t'}}{2^{j'}}\right]\text{(by (4.15) and (4.24))}\\
&\leqslant 2^{t'-1}4^{t'}C_I^{t'}\left[1+\sum_{j'=1}^{\infty}\frac{(j')^{t'-1}}{2^{j'}}\right]=C_2.
\end{aligned}$$

This shows that

$$H=\sum_j\Phi^{N_j}H_j\quad\text{converges in } L^{t'}(\mu)$$

with $\|H\|_{L^{t'}(\mu)}\leqslant\left(2^{t'/t}4^{t'}C_I^{t'}+C_2\right)^{1/t'}=C_3$, and it is then clear from (4.12) that

$$\int fH\,d\mu=\sum_{j,k}\gamma_{jk}f(\lambda_{jk})\qquad\text{for } f\in\mathcal{P}^t(\mu).$$

By a duality argument it follows that

$$\left[\sum_{j,k}\left(\frac{|f(\lambda_{jk})|}{M_{\lambda_{jk}}}\right)^t\right]^{1/t}\leqslant C_3\|f\|_{\mathcal{P}^t(\mu)}\qquad\text{for all } f\in\mathcal{P}^t(\mu).$$

This completes the proof of Theorem B. □

5. General simply connected regions

As Thomson shows in Example 5.10 of his paper, given any bounded simply connected region $\Omega\subseteq\mathbb{C}$ there is a positive measure μ with $\operatorname{spt}\mu\subseteq\operatorname{clos}\Omega$, $\mathcal{P}^t(\mu)$ irreducible, and $\operatorname{abpe}(\mathcal{P}^t(\mu))=\Omega$. We wish to extend our results to this general case. This extension will be an easy consequence of results of Olin and Yang ([OY95]), which we will now outline.

In this section we use the "hat" notation of Thomson and of Olin and Yang, writing $\hat{f}(\lambda)$ for $e_\lambda(f)$ when $\lambda\in\Omega$. It will be convenient to normalize to $\|\mu\| = 1$. Thomson shows in Theorem 5.8 of [Tho 91] that the map $f \mapsto \hat{f}$ induces an isometric algebra isomorphism of $\mathcal{P}^t(\mu) \cap L^\infty(\mu)$ onto $H^\infty(\Omega)$, the space of bounded analytic functions on Ω. We denote its inverse by "$\tilde{}$". Let ω denote harmonic measure on $\partial\Omega$ relative to Ω, at some fixed base point $\lambda_0 \in \Omega$, let φ be a Riemann map of Ω onto $\mathbb{D}$ taking λ_0 to 0, and let $\psi = \varphi^{-1}$. Then $\tilde{\varphi} \in \mathcal{P}^t(\mu) \cap L^\infty(\mu)$ and so we can define the measure ν supported in $\operatorname{clos} \mathbb{D}$ as the pullback of μ by $\tilde{\varphi}$, i.e. $\nu = \mu \circ \tilde{\varphi}^{-1}$. Olin and Yang show that $\mathcal{P}^t(\nu)$ is irreducible with $\operatorname{abpe}\big(\mathcal{P}^t(\nu)\big) = \mathbb{D}$. We can therefore use the corresponding maps "$\hat{}$" and "$\tilde{}$" between $\mathcal{P}^t(\nu) \cap L^\infty(\nu)$ and $H^\infty = H^\infty(\mathbb{D})$.

It is well known that ω is the pullback of normalized Lebesgue measure on $\partial\mathbb{D}$ by the nontangential limit function ψ^* of ψ. Olin and Yang show that

$$\mu = \nu \circ \tilde{\psi}^{-1} \ll \omega \quad \text{on } \partial\Omega$$

and

> there is a set $E \subseteq \partial\mathbb{D}$ with $\nu(\partial\mathbb{D} \setminus E) = 0$ such that ψ^* exists and is $1-1$ on E and $\mu\,(\partial\Omega \setminus \psi^*(E)) = 0$.

They further show that the map $U : p \mapsto p \circ \tilde{\psi}$ for $p \in \mathcal{P}$ extends to an isometric isomorphism of $\mathcal{P}^t(\mu)$ onto $\mathcal{P}^t(\nu)$ that intertwines S with the operator of multiplication by $\hat{\psi}$ on $\mathcal{P}^t(\nu)$. Since $\mu = \nu \circ \hat{\psi}^{-1}$, it is not hard to show that $Uf = f \circ \tilde{\psi}$ ν-almost everywhere.

Following Olin and Yang, we now define the nontangential approach region $\Gamma^\Omega(z)$ for $z \in F = \psi^*(E)$ by $\Gamma^\Omega(z) = \psi(\Gamma(w))$, where w is the unique element of E such that $\psi^*(w) = z$. We use these approach regions to define the notions of nontangential limits and nontangential cluster points.

Using these ideas and results we can extend our Theorems A and B to this general case. A *hyperinvariant subspace* of $\mathcal{P}^t(\mu)$ is a closed linear subspace $M \subseteq \mathcal{P}^t(\mu)$ such that $TM \subseteq M$ for any operator T that commutes with S, i.e. any operator of multiplication by an $H^\infty(\Omega)$ function. We use the isomorphism of $\mathcal{P}^t(\mu) \cap L^\infty(\mu)$ with $H^\infty(\Omega)$ to make sense of such operators.

THEOREM 5.1. *With notation and conventions as above*:

(i) *If $f \in \mathcal{P}^t(\mu)$ then the nontangential limit of $\hat{f}$ exists ω-almost everywhere on F and equals $f|\partial\Omega$ $\mu|\partial\Omega$-almost everywhere.*

(ii)
$$\varlimsup_{\substack{\lambda \to z \\ \lambda \in \Gamma^\Omega(z)}} \left(1 - |\varphi(\lambda)|^2\right)^{1/t} M_\lambda < \infty$$

or equivalently

$$\varlimsup_{\substack{\lambda \to z \\ \lambda \in \Gamma^\Omega(z)}} \big(\operatorname{dist}(\lambda, \partial\Omega) \cdot |\varphi'(\lambda)|\big)^{1/t} M_\lambda < \infty$$

for ω-almost all $z \in F$.

(iii) *If* $\mu(\partial\Omega) > 0$ *then* $\dim M \Big/ (\tilde{\psi} - \lambda_0)\, M = 1$ *for any nonzero hyperinvariant subspace* M *of* $\mathcal{P}^t(\mu)$.

(iv) *If* $G \subset F$ *with* $\mu(G) = 0$ *then there exists a sequence* $\Lambda \subset \operatorname{abpe}(\mathcal{P}^t(m))$ *clustering nontangentially at* ω*-almost every point of* G *for which* T_Λ *maps* $\mathcal{P}^t(\mu)$ *onto* $l^t_\Lambda(\mu)$. *If* $t > 1$ *we can also get* T_Λ *to map into* $l^t_\Lambda(\mu)$, *so that* Λ *is an interpolating sequence.*

(v) *If* $\mu(\partial\Omega) = 0$ *and* $t > 1$, *there exist hyperinvariant subspaces* M *of* $\mathcal{P}^t(\mu)$ *for which* $\dim M/(\tilde{\psi} - \lambda_0)M > 1$.

Proof. The proof of (i) follows from the results of Olin and Yang discussed above and a) of our Theorem A, given that μ and ω are mutually absolutely continuous on F.

As Olin and Yang show, $\|e_\lambda\|_{\mathcal{P}^t(\mu)^*} = \|e_{\varphi(\lambda)}\|_{\mathcal{P}^t(\nu)^*}$ for $\lambda \in \Omega$. Now (ii) follows from the inequality (1.4). We have also used the inequalities

$$\operatorname{dist}(\lambda, \partial\Omega)|\varphi'(\lambda)| \leqslant 1 - |\varphi(\lambda)|^2 \leqslant 8 \operatorname{dist}(\lambda, \partial\Omega)|\varphi'(\lambda)|,$$

which are well-known consequences of Schwarz's Lemma and the Koebe $\frac{1}{4}$-Theorem, respectively.

By the properties of the map U, (iii), (iv), and (v) follow from b) of our Theorem A and our Theorem B. □

An example. Let $\Omega = \mathbb{D} \setminus (-1, 0]$, pick $\lambda_0 = \frac{1}{2}$, and pick the Riemann map $\varphi : \Omega \to \mathbb{D}$ so that $\varphi'\left(\frac{1}{2}\right) > 0$. Then $\psi = \varphi^{-1}$ maps two arcs J_+ and J_- onto $(-1, 0]$, where J_+ lies in the upper unit semicircle and J_- is the complex conjugate of J_+. Olin and Yang construct a positive measure μ that is essentially Lebesgue measure on $(-1, 0]$ plus a weighted area measure on Ω, such that $\mathcal{P}^t(\mu)$ is irreducible and $\operatorname{abpe}\big(\mathcal{P}^t(\mu)\big) = \Omega$. In this example $E = J_+$ and $F = (-1, 0]$ and one sees that if $f \in \mathcal{P}^t(\mu)$ then nontangential limits of $\hat{f}$ on $(-1, 0]$ exist almost everywhere from above, but not from below. This example explains the necessity of the care taken in the definition of nontangential limits.

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