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Holomorphic curves into algebraic varieties

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Abstract

This paper establishes a defect relation for algebraically nondegenerate holomorphic mappings into an arbitrary nonsingular complex projective variety V (rather than just the projective space) intersecting possible nonlinear hypersurfaces, extending the result of H. Cartan.

1. Introduction and statements

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic map, and $H_j, 1 \le j \le q$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. In 1933, H. Cartan [Ca] proved the defect relation(or a Second Main Theorem) $\sum_{j=1}^{q} \delta_f(H_j) \le n+1$. Since then, researches of higher dimensional Nevanlinna theory have been carried out along these two directions: (i) study the algebraically nondegenerate holomorphic mappings into an arbitrary nonsingular complex projective variety V; (ii) replace targets of the hyperplanes appearing in Cartan's result by curvilinear divisors. Recently, the author (see [Ru3]) established a defect relation for algebraically nondegenerate holomorphic curves $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ intersecting curvilinear hypersurfaces, which settled a long-standing conjecture of B. Shiffman (see [Shi]). This paper further extends the above mentioned result to holomorphic curves $f : \mathbb{C} \to V$ intersecting hypersurfaces, where V is an arbitrary nonsingular complex projective variety.

To get a more precise statement, we first introduce some standard notation in Nevanlinna theory: Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic map. Let $\tilde{f} = (f_0, \ldots, f_N)$ be a reduced representative of f, where $f_0, \ldots, f_N$ are entire functions on $\mathbb{C}$ and have no common zeros. The Nevanlinna-Cartan character-

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istic function $T_f(r)$ is defined by

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log ||\tilde{f}(re^{i\theta})|| d\theta$$

where

$$\|\tilde{f}(z)\| = \max\{|f_0(z)|, \ldots, |f_N(z)|\}.$$

The above definition is independent, up to an additive constant, of the choice of the reduced representation of f. Let D be a hypersurface in $\mathbb{P}^N(\mathbb{C})$ of degree d. Let Q be the homogeneous polynomial (form) of degree d defining D. The proximity function $m_f(r, D)$ is defined as

$$m_f(r, D) = \int_0^{2\pi} \log \frac{\|\tilde{f}(re^{i\theta})\|^d \|Q\|}{|Q(\tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi},$$

where $\|Q\|$ is the maximum of the absolute values of the coefficients of Q. To define the counting function, let $n_f(r, D)$ be the number of zeros of $Q \circ \tilde{f}$ in the disk $|z| < r$, counting multiplicity. The counting function is then defined by

$$N_f(r, D) = \int_0^r \frac{n_f(t, D) - n_f(0, D)}{t} dt + n_f(0, D) \log r.$$

The Poisson-Jensen formula implies:

THE FIRST MAIN THEOREM. *Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic map, and let D be a hypersurface in $\mathbb{P}^N(\mathbb{C})$ of degree d. If $f(\mathbb{C}) \not\subset D$, then for every real number r with $0 < r < \infty$,*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$, where $q > n$. Also, $D_1, \ldots, D_q$ are said to be *in general position in* V if for every subset $\{i_0, \ldots, i_n\} \subset \{1, \ldots, q\}$,

$$V \cap \operatorname{supp} D_{i_0} \cap \cdots \cap \operatorname{supp} D_{i_n} = \emptyset, \tag{1.1}$$

where $\operatorname{supp} D$ means the support of the divisor D. A map $f : \mathbb{C} \to V$ is said to be *algebraically nondegenerate* if the image of f is not contained in any proper subvarieties of V. The main result of this paper is as follows.

THEOREM (The main result). *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of degree d_j, located in general position in V. Let $f : \mathbb{C} \to V$ be an algebraically nondegenerate holomorphic map. Then, for every $\varepsilon > 0$,*

$$\sum_{j=1}^q d_j^{-1} m_f(r, D_j) \leq (n + 1 + \varepsilon) T_f(r),$$

where the inequality holds for all $r \in (0, +\infty)$ except for a possible set E with finite Lebesgue measure.

Define the defect of f, with respect to a hypersurface D of degree d,

$$\delta_f(D) = \liminf_{r \to +\infty} \frac{m_f(r, D)}{dT_f(r)}.$$

Then we have the following defect relation.

Corollary (Defect relation). *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety. Let $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$, located in general position in V. Let $f : \mathbb{C} \to V$ be an algebraically nondegenerate holomorphic map. Then*

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \dim V + 1.$$

Note that our result could be easily extended to meromorphic maps $f : \mathbb{C}^m \to V$.

This paper is motivated by the analogy between Nevanlinna theory and Diophantine approximation, discovered by C. Osgood, P. Vojta and S. Lang. etc.. It is now known that Cartan's Second Main Theorem corresponds to Schmidt's subspace theorem. In 1994, G. Faltings and G. Wüstholz [FW] extended Schmidt's result to the systems of Diophantine inequalities to be solved in algebraic points of an arbitrary projective variety. Whereas Schmidt's proof of his subspace theorem is based on techniques from Diophantine approximation and the geometry of numbers, Faltings and Wüstholz developed a totally different method, based on Faltings' Product Theorem (cf. [FW, Ths. 3.1, 3.3]). Moreover they introduced a probability measure on $\mathbb{R}$ whose expected value is the crucial tool in the proof of their main result. R. G. Ferretti (see [F1], [F2]) later observed that their expected value can be reformulated in terms of the Chow weight of X (or Mumford's degree of contact). In fact, for every N-tuple $\mathbf{c} = (c_0, \dots, c_N)$ where $c_0, \dots, c_N$ are integers with $c_0 \geq \cdots \geq c_N$, R.G. Ferretti observed that $E_{\mathbf{c},\infty} = \dfrac{e_{\mathbf{c}}(X)}{(\dim(X) + 1)\deg(X)}$, where $E_{\mathbf{c},\infty}$ is the Faltings-Wüstholz expected value with respect to $\mathbf{c}$ and $e_{\mathbf{c}}(X)$ is the Chow weight of X with respect to $\mathbf{c}$. Ferretti's observation brought the geometric invariant theory (Mumford's degree of contact is a birational invariant often considered in Geometric Invariant Theory (see [Mu], [Mo])) into the study of Diophantine approximation. Later, J.H. Evertse and R. Ferretti (cf. [EF1], [EF2]) further developed this technique and derived a quantitative version of Faltings and Wüstholz's result directly from Schmidt's (quantitative) subspace theorem. They also extended Schmidt's subspace theorem with polynomials of arbitrary degree (see also [CZ]). The Main Theorem in this paper can be viewed as the counterpart of Corollary 1.2 of [EF2] in Nevanlinna theory.

2. Chow weights and Hilbert weights

Let $X \subset \mathbb{P}^N$ be a projective variety (i.e., a geometrically irreducible Zariski-closed subset) of dimension n and degree $\triangle$. In this section, we shall give the definition of Chow weight of X and the definition of the m-th Hilbert weight of X. We then recall a theorem due to Evertse and Ferretti (see Theorem 4.1 in [EF1]) which gives an explicit lower bound of the m-th normalized Hilbert weight of X in terms of the normalized Chow weight of X (see Theorem 2.1 below). This lower bound is sufficient for our purpose in proving the Main Theorem.

2.1. To X we can associate, up to a constant scalar, a unique polynomial

$$F_X(\mathbf{u}_0, \ldots, \mathbf{u}_n) = F_X(u_{00}, \ldots, u_{0N}; \ldots; u_{n0}, \ldots, u_{nN})$$

in $n+1$ blocks of variables $\mathbf{u}_i = (u_{i0}, \ldots, u_{iN}), i = 0, \ldots, n$, which is called the (*Cayley-Bertini-van der Waerden-*)*Chow form* of X, with the following properties: F_X is irreducible in $\mathbb{C}[u_{00}, \ldots, u_{nN}]$; F_X is homogeneous of degree $\triangle$ in each block $\mathbf{u}_i, i = 0, \ldots, n,$; and $F_X(\mathbf{u}_0, \ldots, \mathbf{u}_n) = 0$ if and only if $X \cap H_{\mathbf{u}_0} \cap \cdots \cap H_{\mathbf{u}_n} \neq \emptyset$, where $H_{\mathbf{u}_i}, i = 0, \ldots, n$, are the hyperplanes given by $\mathbf{u}_i \cdot \mathbf{x} = u_{i0}x_0 + \cdots + u_{iN}x_N = 0$.

2.2. Let F_X be the Chow form associated to X. Let $\mathbf{c} = (c_0, \ldots, c_N)$ be a tuple of reals. Let t be an auxiliary variable. We consider the decomposition

$$\begin{aligned} F_X(t^{c_0}u_{00}, \ldots, t^{c_N}u_{0N}; \ldots; t^{c_0}u_{n0}, \ldots, t^{c_N}u_{nN}) \\ = t^{e_0}G_0(\mathbf{u}_0, \ldots, \mathbf{u}_n) + \cdots + t^{e_r}G_r(\mathbf{u}_0, \ldots, \mathbf{u}_n), \end{aligned} \tag{2.1}$$

with $G_0, \ldots, G_r \in \mathbb{C}[u_{00}, \ldots, u_{0N}; \ldots; u_{n0}, \ldots, u_{nN}]$ and $e_0 > e_1 > \cdots > e_r$. We define the *Chow weight of X with respect to* $\mathbf{c}$ by

$$e_X(\mathbf{c}) := e_0. \tag{2.2}$$

For each subset $J = \{j_0 \ldots, j_n\}$ of $\{0, \ldots, N\}$ with $j_0 < j_1 < \cdots < j_n$ we define the *bracket*

$$[J] = [J](\mathbf{u}_0, \ldots, \mathbf{u}_n) := \det(u_{ij_k})_{i,k=0,\ldots,n}, \tag{2.3}$$

where again $\mathbf{u}_i = (u_{i0}, \ldots, u_{iN})$ denotes the blocks of $N+1$ variables. Let $J_1, \ldots, J_\beta$ with $\beta = \binom{N+1}{n+1}$ be all subsets of $\{0, \ldots, N\}$ of cardinality $n+1$. Then, from [HP, p. 41, Th. IV], the Chow form F_X of X can be written as a homogeneous polynomial of degree $\triangle$ in $[J_1], \dots, [J_\beta]$. It is easy to show that, for $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$ and for any J among $J_1, \dots, J_\beta$,

$$(2.4)\quad [J](t^{c_0}u_{00}, \dots, t^{c_N}u_{0N}; \dots; t^{c_0}u_{n0}, \dots, t^{c_N}u_{nN}) \\ = t^{\sum_{j \in J} c_j}[J](u_{00}, \dots, u_{0N}; \dots; u_{n0}, \dots, u_{nN}).$$

2.3. Denote by $\mathbb{Z}_{\geq 0}^{N+1}$, $\mathbb{R}_{\geq 0}^{N+1}$ the set of $(N+1)$-tuples consisting of nonnegative integers, nonnegative reals, respectively. For $\mathbf{a} = (a_0, \dots, a_N) \in \mathbb{Z}_{\geq 0}^{N+1}$ we write $\mathbf{x}^{\mathbf{a}}$ for the monomial $x_0^{a_0} \cdots x_N^{a_N}$. Let $I = I_X$ be the prime ideal in $\mathbb{C}[x_0, \dots, x_N]$ defining X. Let $\mathbb{C}[x_0, \dots, x_N]_m$ denote the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_N]$ of degree m (including 0). Put $I_m := \mathbb{C}[x_0, \dots, x_N]_m \cap I$ and define the *Hilbert function* H_X of X by, for $m = 1, 2, \dots,$

$$(2.5)\qquad H_X(m) := \dim\left(\mathbb{C}[x_0, \dots, x_N]_m / I_m\right).$$

By the usual theory of Hilbert polynomials,

$$(2.6)\qquad H_X(m) = \triangle \cdot \frac{m^n}{n!} + O(m^{n-1}).$$

2.4. We define the m-th *Hilbert weight* $S_X(m, \mathbf{c})$ *of* X *with respect to a tuple* $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$ by

$$(2.7)\qquad S_X(m, \mathbf{c}) := \max\left(\sum_{i=1}^{H_X(m)} \mathbf{a}_i \cdot \mathbf{c}\right),$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_{H_X(m)}}$ whose residue classes modulo I form a basis of $\mathbb{C}[x_0, \dots, x_N]_m / I_m$.

2.5. According to Mumford [Mu, Prop. 2.11],

$$S_X(m, \mathbf{c}) = e_X(\mathbf{c}) \cdot \frac{m^{n+1}}{(n+1)!} + O(m^n).$$

Together with (2.6), this implies that

$$(2.8)\qquad \lim_{m \to \infty} \frac{1}{mH_X(m)} \cdot S_X(m, \mathbf{c}) = \frac{1}{(n+1)\triangle} \cdot e_X(\mathbf{c}).$$

We call $\frac{1}{mH_X(m)} S_X(m, \mathbf{c})$ the m-th *normalized Hilbert weight* and $\frac{1}{(n+1)\triangle} e_X(\mathbf{c})$ the *normalized Chow weight of* X *with respect to* $\mathbf{c}$.

2.6. Mumford's identity above is not sufficient for our purpose. To prove our main result, we need to compute the explicit constants appearing in Mumford's identity. However, a lower of $S_X(m, \mathbf{c})$ with explicit constants should be sufficient for our purpose.

Theorem 2.1. *Let $X \subset \mathbb{P}^N(\mathbb{C})$ be an algebraic variety of dimension n and degree $\triangle$. Let $m > \triangle$ be an integer and let $\mathbf{c} = (c_0, \ldots, c_N) \in \mathbb{R}^{N+1}_{\geq 0}$. Then*

$$\frac{1}{mH_X(m)} S_X(m, \mathbf{c}) \geq \frac{1}{(n+1)\triangle} e_X(\mathbf{c}) - \frac{(2n+1)\triangle}{m} \cdot \left(\max_{i=0,\ldots,N} c_i \right).$$

Theorem 2.1 here is the special case when $K = \mathbb{C}$ of Theorem 4.1 of [EF1]. Readers may consult with [EF1] for the proof.

3. Proof of the main result

To prove our main result, we need the following general form of the Second Main Theorem for holomorphic curves intersecting hyperplanes, due to P. Vojta (see [V2]). The theorem is also stated and proved in [Ru1, Th. 2.1].

Theorem 3.1. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map whose image is not contained in any proper linear subspace of $\mathbb{P}^n(\mathbb{C})$. Let $H_1, \ldots, H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Let $L_j, 1 \leq j \leq q$, be the linear forms defining $H_1, \ldots, H_q$. Then, for every $\varepsilon > 0$,*

$$\int_0^{2\pi} \max_K \log \prod_{j \in K} \frac{\|\tilde{f}(re^{i\theta})\| \|L_j\|}{|L_j(\tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi} \leq (n+1+\varepsilon) T_f(r), \tag{3.1}$$

where the inequality holds for all r outside of a set E with finite Lebesgue measure, $\tilde{f} = (f_0, \ldots, f_n)$ is a reduced representation of f, the maximum is taken over all subsets K of $\{1, \ldots, q\}$ such that $\#K = n+1$ and the linear forms $L_j, j \in K$, are linearly independent, and $\|L_j\|$ is the maximum of the absolute values of the coefficients in L_j.

We now prove our main result.

Proof. Let $D_1, \ldots, D_q$ be the hypersurfaces in $\mathbb{P}^N(\mathbb{C})$, located in general position on V. Let $Q_j, 1 \leq j \leq q$, be the homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_N]$ of degree d_j defining D_j. Replacing Q_j by Q_j^{d/d_j} if necessary, where d is the l.c.m of $d_j's$, we can assume that $Q_1, \ldots, Q_q$ have the same degree of d. For every $\mathbf{b} = [b_0 : \cdots : b_N] \in \mathbb{P}^N(\mathbb{C})$, consider the function

$$\|\mathbf{b}, D_j\| = \frac{|Q_j(\mathbf{b})|}{\|\mathbf{b}\|^d \|Q_j\|},$$

where $\|\mathbf{b}\| = \max_{0 \leq j \leq N} |b_j|$ and $\|Q_j\|$ is the maximum of the absolute values of the coefficients of Q_j. At each point $\mathbf{b} \in V$, by the "in general position" condition, $\|\mathbf{b}, D_j\|$ can be zero for no more than n indicies $j \in \{1, \ldots, q\}$. For the remaining indices j, we have $\|\mathbf{b}, D_j\| > 0$ and by the continuity of these functions and the compactness of V, there exists $C > 0$ such that $\|\mathbf{b}, D_j\| > C$

for all $\mathbf{b} \in V$ and all D_j, except for at most n of them. Hence, for any holomorphic map $f : \mathbb{C} \to V$,

$$\begin{aligned} (3.2) \qquad \sum_{j=1}^{q} m_f(r, D_j) &= \int_0^{2\pi} \sum_{j=1}^{q} \log \frac{1}{\|f(re^{i\theta}), D_j\|} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \log \prod_{j=1}^{q} \frac{\|\tilde{f}(re^{i\theta})\|^d \|Q_j\|}{|Q_j(\tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} \max_{\{i_0,\ldots,i_n\}} \left\{ \log \prod_{k=0}^{n} \frac{\|\tilde{f}(re^{i\theta})\|^d \|Q_{i_k}\|}{|Q_{i_k}(\tilde{f})(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} + O(1), \end{aligned}$$

where $\tilde{f} = (f_0, \ldots, f_N)$ is a reduced representation of f. Define a map $\phi : \mathbf{x} \in V \mapsto [Q_1(\mathbf{x}) : \cdots : Q_q(\mathbf{x})] \in \mathbb{P}^{q-1}(\mathbb{C})$ and let $Y = \phi(V)$. By the "in general position" assumption, ϕ is a finite morphism on V and Y is a complex projective subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$. We also have $\dim Y = n$ and $\deg Y =: \triangle \leq d^n \deg(V)$. For every $\mathbf{a} = (a_1, \ldots, a_q) \in \mathbb{Z}^q_{\geq 0}$, denote $\mathbf{y}^{\mathbf{a}} = y_1^{a_1} \cdots y_q^{a_q}$. Let m be a positive integer. Put

$$(3.3) \qquad n_m := H_Y(m) - 1, \quad q_m := \binom{q+m-1}{m} - 1.$$

Consider the Veronese embedding

$$(3.4) \qquad \phi_m : \mathbb{P}^{q-1}(\mathbb{C}) \hookrightarrow \mathbb{P}^{q_m}(\mathbb{C}) : \mathbf{y} \mapsto [\mathbf{y}^{\mathbf{a}_0} : \cdots : \mathbf{y}^{\mathbf{a}_{q_m}}],$$

where $\mathbf{y}^{\mathbf{a}_0}, \ldots, \mathbf{y}^{\mathbf{a}_{q_m}}$ are the monomials of degree m in $y_1, \ldots, y_q$, in some order. Denote by Y_m the smallest linear subvariety of $\mathbb{P}^{q_m}(\mathbb{C})$ containing $\phi_m(Y)$. Then, clearly, a linear form $\sum_{i=0}^{q_m} \gamma_i z_i$ vanishes identically on Y_m if and only if $\sum_{i=0}^{q_m} \gamma_i \mathbf{y}^{\mathbf{a}_i}$, as a polynomial of degree m, vanishes identically on Y. In other words, there is an isomorphism

$$\mathbb{C}[y_1, \ldots, y_q]_m / (I_Y)_m \simeq Y_m^{\vee} : \mathbf{y}^{\mathbf{a}_i} \mapsto z_i, \quad i = 0, \ldots, q_m,$$

where I_Y is the prime ideal in $\mathbb{C}[y_1, \ldots, y_q]$ defining Y, $(I_Y)_m$ is the vector space of homogeneous polynomials of degree m in I_Y, and $Y_m^{\vee}$ is the vector space of linear forms in $\mathbb{C}[z_0, \ldots, z_{q_m}]$ modulo the linear forms vanishing identically on Y_m. Hence Y_m is an n_m-dimensional linear subspace of $\mathbb{P}^{q_m}(\mathbb{C})$ where $n_m = H_Y(m) - 1$. Since Y_m is an n_m-dimensional linear subspace of $\mathbb{P}^{q_m}(\mathbb{C})$, there are linear forms $L_0, \ldots, L_{q_m} \in \mathbb{C}[w_0, \ldots, w_{n_m}]$ such that the map

$$(3.5) \qquad \psi_m : \mathbf{w} \in \mathbb{P}^{n_m}(\mathbb{C}) \mapsto [L_0(\mathbf{w}) : \cdots : L_{q_m}(\mathbf{w})] \in Y_m$$

is a linear isomorphism from $\mathbb{P}^{n_m}(\mathbb{C})$ to Y_m. Thus $\psi_m^{-1} \circ \phi_m$ is an injective map from Y into $\mathbb{P}^{n_m}(\mathbb{C})$. Let $f : \mathbb{C} \to V$ be the given holomorphic map and let $F = \psi_m^{-1} \circ \phi_m \circ \phi \circ f : \mathbb{C} \to \mathbb{P}^{n_m}(\mathbb{C})$. Then F is a holomorphic map. Furthermore, since f is algebraically nondegenerate, F is linearly nondegenerate. For every $z \in \mathbb{C}$, let $\mathbf{c}_z = (c_{1,z}, \ldots, c_{q,z})$ where

$$c_{j,z} := \log \frac{\|\tilde{f}(z)\|^d \|Q_j\|}{|Q_j(\tilde{f})(z)|}, \qquad j = 1, \ldots, q. \tag{3.6}$$

Obviously, $c_{j,z} \geq 0$ for $j = 1, \ldots, q$. By the definition of the Hilbert weight, for every $z \in \mathbb{C}$, there is a subset I_z of $\{0, \ldots, q_m\}$ with $\# I_z = n_m + 1 = H_Y(m)$ such that $\{\mathbf{y}^{\mathbf{a}_i} : i \in I_z\}$ is a basis of $\mathbb{C}[y_1, \ldots, y_q]_m/(I_Y)_m$ and

$$S_Y(m, \mathbf{c}_z) = \sum_{i \in I_z} \mathbf{a}_i \cdot \mathbf{c}_z. \tag{3.7}$$

Note that, for every $\mathbf{w} \in \mathbb{P}^{n_m}(\mathbb{C})$, we have $L_j(\mathbf{w}) = \mathbf{y}^{\mathbf{a}_j}$, $j = 0, \ldots, q_m$, where $L_j, 0 \leq j \leq q_m$, are the linear forms obtained in (3.5). Hence, by the definition of F and (3.7), we have

(3.8)
$$\begin{aligned}
\log \prod_{i \in I_z} \frac{\|L_i\|}{|L_i(\tilde{F})(z)|} &= \log \prod_{i \in I_z} \frac{1}{|Q_1(\tilde{f})(z)|^{a_{i,1}} \cdots |Q_q(\tilde{f})(z)|^{a_{i,q}}} + O(H_Y(m)) \\
&= \log \prod_{i \in I_z} \left[\left(\frac{\|\tilde{f}(z)\|^d \|Q_1\|}{|Q_1(\tilde{f})(z)|} \right)^{a_{i,1}} \cdots \left(\frac{\|\tilde{f}(z)\|^d \|Q_q\|}{|Q_q(\tilde{f})(z)|} \right)^{a_{i,q}} \right] \\
&\quad - dH_Y(m) m \log \|\tilde{f}(z)\| + O(H_Y(m)) \\
&= \sum_{i \in I_z} \mathbf{a}_i \cdot \mathbf{c}_z - dmH_Y(m) \log \|\tilde{f}(z)\| + O(H_Y(m)) \\
&= S_Y(m, \mathbf{c}_z) - dmH_Y(m) \log \|\tilde{f}(z)\| + O(H_Y(m)),
\end{aligned}$$

where $\tilde{F}$ is a reduced representation of F, and the term $O(H_Y(m))$ does not depend on z. Hence

(3.9)
$$\begin{aligned}
S_Y(m, \mathbf{c}_z) &\leq \max_J \log \prod_{j \in J} \frac{\|L_j\|}{|L_j(\tilde{F})(z)|} + dmH_Y(m) \log \|\tilde{f}(z)\| + O(H_Y(m)) \\
&= \max_J \log \prod_{j \in J} \frac{\|\tilde{F}(z)\| \|L_j\|}{|L_j(\tilde{F})(z)|} + dmH_Y(m) \log \|\tilde{f}(z)\| \\
&\quad - (n_m + 1) \log \|\tilde{F}(z)\| + O(H_Y(m)),
\end{aligned}$$

where the maximum is taken over all $J \subset \{0, \ldots, q_m\}$ such that $\# J = n_m + 1$ and $L_j, j \in J$, are linearly independent. By Theorem 2.1,

$$\frac{1}{mH_Y(m)} S_Y(m, \mathbf{c}_z) \geq \frac{1}{(n+1)\triangle} e_Y(\mathbf{c}_z) - \frac{(2n+1)\triangle}{m} \left(\max_{1 \leq i \leq q} c_{i,z} \right). \tag{3.10}$$

Hence

(3.11)

$$\frac{1}{(n+1)\triangle}e_Y(\mathbf{c}_z) \le \frac{1}{mH_Y(m)}\left(\max_J \log \prod_{j\in J}\frac{\|\tilde{F}(z)\|\|L_j\|}{|L_j(\tilde{F})(z)|} - (n_m+1)\log\|\tilde{F}(z)\|\right)$$
$$+d\log\|\tilde{f}(z)\| + \frac{(2n+1)\triangle}{m}\left(\max_{1\le i\le q} c_{i,z}\right) + O(1/m).$$

Our next step is to derive a lower bound for the Chow weight $e_Y(\mathbf{c})$. To do so, we need the following lemma.

LEMMA 3.2. *Let Y be a subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$ of dimension n and degree $\triangle$. Let $\mathbf{c} = (c_1, \dots, c_q)$ be a tuple of positive reals. Let $\{i_0, \dots, i_n\}$ be a subset of $\{1, \dots, q\}$ such that*

$$Y \cap \{y_{i_0} = 0, \dots, y_{i_n} = 0\} = \emptyset. \tag{3.12}$$

Then

$$e_Y(\mathbf{c}) \ge (c_{i_0} + \cdots + c_{i_n}) \cdot \triangle. \tag{3.13}$$

Lemma 3.2 is nearly identical to Lemma 5.1 of [EF2], except that the base field here is $\mathbb{C}$ instead of $\mathbb{Q}^a$. Since the proof is short, we enclose it here for the sake of completeness.

Proof. For a subset $J = \{j_0 \dots, j_n\}$ of $\{1, \dots, q\}$ with $j_0 < j_1 < \cdots < j_n$ recall that we defined (see §2.2) the *bracket*

$$[J] = [J](\mathbf{u}_0, \dots, \mathbf{u}_n) := \det(u_{i,j_k})_{i,k=0,\dots,n},$$

where again $\mathbf{u}_i = (u_{i1}, \dots, u_{iq})$ denotes the blocks of q variables. Let $J_1, \dots, J_\beta$ with $\beta = \binom{q}{n+1}$ be all subsets of $\{1, \dots, q\}$ of cardinality $n+1$. Then the Chow form F_Y of Y can be written as a homogeneous polynomial of degree $\triangle$ in $[J_1], \dots, [J_\beta]$:

$$F_Y = \sum_{\mathbf{a}\in A} C(\mathbf{a})[J_1]^{a_1}\cdots[J_\beta]^{a_\beta}, \tag{3.14}$$

where A is the set of tuples of nonnegative integers $\mathbf{a} = (a_1, \dots, a_\beta)$ with $a_1 + \cdots + a_\beta = \triangle$. For each bracket $[J]$,

$$[J](t^{c_1}u_{01}, \dots, t^{c_q}u_{0q}; \dots; t^{c_1}u_{n1}, \dots, t^{c_q}u_{nq})]$$
$$= t^{\sum_{j\in J} c_j}[J](u_{01}, \dots, u_{0q}; \dots; u_{n1}, \dots, u_{nq}).$$

This, together with (3.14), implies that

$$F_Y(t^{c_1}u_{01},\dots,t^{c_q}u_{0q};\dots;t^{c_1}u_{n1},\dots,t^{c_q}u_{nq}) = \sum_{\mathbf{a}\in A} C(\mathbf{a})t^{\sum_{j=1}^{\beta} a_j(\sum_{i\in J_j} c_i)}[J_1]^{a_1}\cdots[J_\beta]^{a_\beta}. \tag{3.15}$$

Put $\mathbf{e}_1 := (1,0,\dots,0)$, $\mathbf{e}_2 := (0,1,\dots,0),\dots,$ $\mathbf{e}_q := (0,0,\dots,1)$. Write $\{i_0,\dots,i_n\} =: J_1$. By (3.12) we have $F_Y(\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_n}) \neq 0$. Further,

$$[J_1](\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_n}) = 1, \qquad [J](\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_n}) = 0, \quad \text{for} \quad J \neq J_1.$$

Hence in expression (3.14) there is a term $C\cdot[J_1]^{\triangle}$ with $C\neq 0$, and if we substitute $\mathbf{u}_j = \mathbf{e}_{i_j}, j=0,\dots,n$, in (3.15) we obtain $C\cdot t^{(c_{i_0}+\cdots+c_{i_n})\cdot\triangle}$. That is, one of the numbers e_i in (2.1) is equal to $(c_{i_0}+\cdots+c_{i_n})\cdot\triangle$. Hence $e_Y(\mathbf{c}) \geq (c_{i_0}+\cdots+c_{i_n})\cdot\triangle$. This proves Lemma 3.2

Now, we continue the proof of the Main Theorem. By Lemma 3.2, for any $\{i_0,\dots,i_n\}\subset\{1,\dots,q\}$, since $D_1,\dots,D_q$ are in general position in V (so that (3.12) is satisfied),

$$e_Y(\mathbf{c}_z) \geq (c_{i_0,z}+\cdots+c_{i_n,z})\cdot\triangle. \tag{3.16}$$

By the definition of $\mathbf{c}_z$, we get

$$c_{i_0,z}+\cdots+c_{i_n,z} = \log\left(\frac{\|\tilde f(z)\|^d\|Q_{i_0}\|}{|Q_{i_0}(\tilde f)(z)|}\cdots\frac{\|\tilde f(z)\|^d\|Q_{i_n}\|}{|Q_{i_n}(\tilde f)(z)|}\right). \tag{3.17}$$

Combining (3.11), (3.16) and (3.17) gives

(3.18)

$$\begin{aligned}
&\log\left(\frac{\|\tilde f(z)\|^d\|Q_{i_0}\|}{|Q_{i_0}(\tilde f)(z)|}\cdots\frac{\|\tilde f(z)\|^d\|Q_{i_n}\|}{|Q_{i_n}(\tilde f)(z)|}\right)\\
&\quad\leq \frac{(n+1)}{mH_Y(m)}\left(\max_J \log\prod_{j\in J}\frac{\|\tilde F(z)\|\|L_j\|}{|L_j(\tilde F)(z)|} - (n_m+1)\log\|\tilde F(z)\|\right)\\
&\qquad + d(n+1)\log\|\tilde f(z)\| + \frac{(2n+1)(n+1)\triangle}{m}\max_{1\leq i\leq q} c_{i,z} + O(1/m)\\
&\quad= \frac{(n+1)}{mH_Y(m)}\left(\max_J \log\prod_{j\in J}\frac{\|\tilde F(z)\|\|L_j\|}{|L_j(\tilde F)(z)|} - (n_m+1)\log\|\tilde F(z)\|\right) + O\left(\frac{1}{m}\right)\\
&\qquad + d(n+1)\log\|\tilde f(z)\| + \frac{(2n+1)(n+1)\triangle}{m}\left(\max_{1\leq j\leq q}\log\frac{\|\tilde f(z)\|\|Q_j\|}{|Q_j(\tilde f)(z)|}\right).
\end{aligned}$$

Therefore

(3.19)
$$\max_{i_0,\dots,i_n} \log\left(\frac{\|\tilde f(z)\|^d\|Q_{i_0}\|}{|Q_{i_0}(\tilde f)(z)|}\cdots\frac{\|\tilde f(z)\|^d\|Q_{i_n}\|}{|Q_{i_n}(\tilde f)(z)|}\right)$$
$$\le \frac{(n+1)}{mH_Y(m)}\left(\max_J \log\prod_{j\in J}\frac{\|\tilde F(z)\|\|L_j\|}{|L_j(\tilde F)(z)|}-(n_m+1)\log\|\tilde F(z)\|\right)+O\left(\frac{1}{m}\right)$$
$$+d(n+1)\log\|\tilde f(z)\|+\frac{(2n+1)(n+1)\triangle}{m}\left(\max_{1\le j\le q}\log\frac{\|\tilde f(z)\|\|Q_j\|}{|Q_j(\tilde f)(z)|}\right).$$

Applying integration on the both sides of (3.19) and using the First Main theorem yield

(3.20)
$$\int_0^{2\pi}\max_{i_0,\dots,i_n} \log\left(\frac{\|\tilde f(re^{i\theta})\|^d\|Q_{i_0}\|}{|Q_{i_0}(\tilde f)(re^{i\theta})|}\cdots\frac{\|\tilde f(re^{i\theta})\|^d\|Q_{i_n}\|}{|Q_{i_n}(\tilde f)(re^{i\theta})|}\right)\frac{d\theta}{2\pi}$$
$$\le \frac{(n+1)}{mH_Y(m)}\left(\int_0^{2\pi}\max_J \log\prod_{j\in J}\frac{\|\tilde F(re^{i\theta})\|\|L_j\|}{|L_j(\tilde F)(re^{i\theta})|}\frac{d\theta}{2\pi}-(n_m+1)T_F(r)\right)$$
$$+d(n+1)T_f(r)+\frac{(2n+1)(n+1)\triangle}{m}\sum_{1\le j\le q}m_f(r,D_j)+O\left(\frac{1}{m}\right)$$
$$\le \frac{(n+1)}{mH_Y(m)}\left(\int_0^{2\pi}\max_J \log\prod_{j\in J}\frac{\|\tilde F(re^{i\theta})\|\|L_j\|}{|L_j(\tilde F)(re^{i\theta})|}\frac{d\theta}{2\pi}-(n_m+1)T_F(r)\right)$$
$$+d(n+1)T_f(r)+\frac{(2n+1)(n+1)q\triangle}{m}T_f(r)+O\left(\frac{1}{m}\right),$$

here we note that various constants in the " $O\left(\frac{1}{m}\right)$" term above depend only on $Q_1,\dots,Q_q$, not on f and z. For the $\varepsilon>0$ given in the Main Theorem, take m large enough so that

$$\frac{(n+1)}{H_Y(m)}<\frac{\varepsilon}{3d},\quad \text{and}\quad \frac{(2n+1)(n+1)q\triangle}{m}<\varepsilon/3. \tag{3.21}$$

Fix such an m. Applying Theorem 3.1 with $\varepsilon=1$ to holomorphic map F and linear forms $L_0,\dots,L_{q_m}$, we obtain that

$$\int_0^{2\pi}\max_J \log\prod_{j\in J}\frac{\|\tilde F(re^{i\theta})\|\|L_j\|}{|L_j(\tilde F)(re^{i\theta})|}\frac{d\theta}{2\pi}\le(n_m+2)T_F(r) \tag{3.22}$$

holds for all r outside of a set E with finite Lebesgue measure. By combining (3.2), (3.20), (3.21) and (3.22), we get

$$\sum_{j=1}^{q} m_f(r, D_j) \leq \frac{\varepsilon}{3dm} T_F(r) + d(n+1)T_f(r) + (\varepsilon/3)T_f(r) + O(1) \tag{3.23}$$

where the inequality holds for all r outside of a set E with finite Lebesgue measure. By the definition of the characteristic function, we have $T_F(r) \leq dmT_f(r)$. Hence (3.23) becomes

$$\sum_{j=1}^{q} m_f(r, D_j) \leq (d(n+1) + 2\varepsilon/3)T_f(r) + C \tag{3.24}$$

where the inequality holds for for all r outside of a set E with finite Lebesgue measure, and where C is a constant, independent of r. Take r big enough so that we can make $C \leq (\varepsilon/3)T_f(r)$. Thus we have

$$\sum_{j=1}^{q} m_f(r, D_j) \leq (d(n+1) + \varepsilon)T_f(r)$$

where the inequality holds for all r outside of a set E with finite Lebesgue measure. This proves our main result. $\square$

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