Annals of Mathematics, **169** (2009), 139–227

Stable manifolds for an orbitally unstable nonlinear Schrödinger equation

By W. Schlag*

1. Introduction

We consider the cubic nonlinear Schrödinger equation in $\mathbb{R}^3$

$$i\partial_t\psi + \triangle\psi = -|\psi|^2\psi. \tag{1}$$

This equation is locally well-posed in $H^1(\mathbb{R}^3) = W^{1,2}(\mathbb{R}^3)$. Let $\phi = \phi(\cdot, \alpha)$ be the ground state of

$$-\triangle\phi + \alpha^2\phi = \phi^3. \tag{2}$$

By this we mean that $\phi > 0$ and that $\phi \in C^2(\mathbb{R}^3)$. It is a classical fact (see Coffman [10]) that such solutions exist and are unique for the cubic nonlinearity. Moreover, they are radial and smooth. Similar facts are known for more general nonlinearities; see e.g., Strauss [45] and Berestycki and Lions [5] for existence and Kwon [30] for uniqueness in greater generality.

Clearly, $\psi = e^{it\alpha^2}\phi$ solves (1). We seek an H^1-solution ψ of the form $\psi = W + R$ where

$$W(t,x) = e^{i\theta(t,x)}\phi(x - y(t), \alpha(t)), \tag{3}$$

$$\theta(t,x) = v(t)\cdot x - \int_0^t (|v(s)|^2 - \alpha^2(s))\,ds + \gamma(t), \tag{4}$$

$$y(t) = 2\int_0^t v(s)\,ds + D(t), \tag{5}$$

is the usual soliton with a moving set of parameters $\pi(t) := (\gamma(t), v(t), D(t), \alpha(t))$, and R is a small perturbation. Performing a Galilei transform, we may assume that $W(0,x) = \phi(x,\alpha) = \alpha\phi(\alpha x, 1)$ for some $\alpha > 0$. The final equality holds because of the cubic nonlinearity.

*The author was partially supported by the NSF grant DMS-0300081 and a Sloan Fellowship. He wishes to thank Marius Beceanu, Joachim Krieger, Francois Ledrappier, Dimitry Pelinovsky, Igor Rodnianski, Michael Sigal, Barry Simon, and Avy Soffer for helpful comments on a preliminary version of this paper.

With each $\alpha > 0$ we associate the matrix operator

$$\mathcal{H} = \mathcal{H}(\alpha) = \begin{pmatrix} -\triangle + \alpha^2 - 2\phi^2(\cdot,\alpha) & -\phi^2(\cdot,\alpha) \\ \phi^2(\cdot,\alpha) & \triangle - \alpha^2 + 2\phi^2(\cdot,\alpha) \end{pmatrix}. \tag{6}$$

This operator is closed on the domain $W^{2,2}(\mathbb{R}^3) \times W^{2,2}(\mathbb{R}^3)$ and its spectrum is known to be located on $\mathbb{R} \cup i\mathbb{R}$ with essential spectrum equal to $(-\infty, -\alpha^2] \cup [\alpha^2, \infty)$. As proved by Weinstein [49] and [50], $\mathcal{H}(\alpha)$ has a rootspace of dimension eight at zero, and $\ker(\mathcal{H}^3) = \ker(\mathcal{H}^2) \neq \ker(\mathcal{H})$. In fact, $\dim\ker(\mathcal{H}) = 4$. On the other hand, any discrete spectrum different from zero is known to consist entirely of eigenvalues whose geometric and algebraic multiplicities coincide. Due to the L^2 supercritical nature of the problem, $\mathcal{H}(\alpha)$ *does* have purely imaginary eigenvalues; see Grillakis, Shatah, Strauss [24] as well as Section 4 below. Moreover, due to the standard symmetries of the spectrum (which follow from the commutation properties of $\mathcal{H}$ with the Pauli matrices) we know that these purely imaginary eigenvalues are symmetric with respect to the real axis, together with their multiplicities.

The orbital stability question (for Klein-Gordon, NLS, as well as many other classes of PDE) was addressed by Shatah [40], Shatah, Strauss [41], Weinstein [49], [50], Grillakis, Shatah, Strauss [24], [25] (who developed an "abstract" theory of orbital stability), Grillakis [22], [23], Comech, Pelinovsky [11]. As for the question of *asymptotic stability* (which is much closer to the present paper), see Soffer, Weinstein [43], [44], Buslaev, Perelman [6], [7], Cuccagna [13], Rodnianski, Schlag, Soffer [38], [39], Perelman [32], [33],[34], Fröhlich, Jonsson, Gustafson, Sigal [18], Fröhlich, Tsai, Yau [19]. For surveys of some of this material, see Strauss [46], and C. Sulem and P.-L. Sulem [47].

It is well-known that the *supercritical equation* (1) is *orbitally unstable*; see Berestycki and Cazenave [4]. This is in contrast to the *orbital stability* of the *subcritical equations* that was proved by Cazenave and Lions [8] and Weinstein [49], [50].

For our main theorem we need to impose the following spectral condition:

$\mathcal{H}(\alpha)$ *does not have any embedded eigenvalues in the essential spectrum.*

Then we prove the following:

THEOREM 1. *Impose*[1] *the spectral conditions for all* $\alpha > 0$ *and fix any* $\alpha_0 > 0$. *Then there exist a real-linear subspace* $\mathcal{S} \subset W^{1,2}(\mathbb{R}^3) \cap W^{1,1}(\mathbb{R}^3)$ *of co-dimension nine and a small* $\delta > 0$ *with the following properties*: *Let*

$$\mathcal{B} := \Big\{ R_0 \in L^2(\mathbb{R}^3) \mid \|\!|R_0|\!\| := \|R_0\|_{W^{1,2}\cap W^{1,1}} < \delta \Big\} \tag{7}$$

[1]By scaling invariance, if they hold for one $\alpha > 0$, then they hold for all $\alpha > 0$. This is due to the monomial nonlinearity.

and let $\Sigma := \{f \in L^2(\mathbb{R}^3) \mid \|\!|f|\!\|< \infty\}$. *Then there exists a map* $\Phi : \mathcal{B} \cap \mathcal{S} \to \Sigma$ *with the properties*

$$\|\!|\Phi(R_0)|\!\| \lesssim \|\!|R_0|\!\|^2 \qquad \forall R_0 \in \mathcal{B} \cap \mathcal{S}, \tag{8}$$

$$\|\!|\Phi(R_0) - \Phi(\tilde{R}_0)|\!\| \lesssim \delta \|\!|R_0 - \tilde{R}_0|\!\| \qquad \forall R_0, \tilde{R}_0 \in \mathcal{B} \cap \mathcal{S}, \tag{9}$$

and so for any $R_0 \in \mathcal{B} \cap \mathcal{S}$ *the nonlinear Schrödinger equation* (1) *has a global* H^1 *solution* $\psi(t)$ *for* $t \geq 0$ *with initial condition* $\psi(0) = \phi(\cdot, \alpha_0) + R_0 + \Phi(R_0)$. *Moreover,*

$$\psi(t) = W(t, \cdot) + R(t)$$

where W *as in* (3) *is governed by a path* $\pi(t)$, *of parameters, which converges to some terminal vector* $\pi(\infty)$ *such that* $\sup_{t\geq 0} |\pi(t) - \pi(\infty)| \lesssim \delta^2$ *and so that*

$$\|R(t)\|_{W^{1,2}} \lesssim \delta, \qquad \|R(t)\|_\infty \lesssim \delta t^{-\frac{3}{2}}$$

for all $t > 0$. *Finally, there is scattering*:

$$R(t) = e^{it\triangle} f_0 + o_{L^2}(1) \quad \text{as } t \to \infty$$

for some $f_0 \in L^2(\mathbb{R}^3)$.

Concerning the spectral condition, it is well-known that imbedded eigenvalues are unstable under perturbations. See Grillakis [23], as well as the recent work by Cuccagna, Pelinovsky, and Vougalter [15], [14] for precise statements to this effect for matrix Schrödinger operators, as well as Costin, Soffer [12] for the scalar case. The proof of Theorem 1 does not rely too heavily on the specific structure of the cubic nonlinearity and applies to other supercritical nonlinearities as well.[2] Hence it is possible to formulate Theorem 1 without any spectral conditions for a "generic perturbation" (in a suitable sense) of the cubic nonlinearity. However, we have chosen to present Theorem 1 as stated in order not to obscure the main ideas. We plan to return to the issue of generic perturbations elsewhere.

On the other hand, it is to be expected that there are no imbedded eigenvalues in the essential spectrum, at least for the linearization around a ground state soliton. Indeed, in [29] Krieger and the author prove the analogue of Theorem 1 in one dimension without any spectral condition for supercritical monomial nonlinearities. Rather, in that case the absence of imbedded eigenvalues can be proved by adapting some arguments of Perelman [33]. It is to be expected that the same property also holds in $\mathbb{R}^3$, although this yet needs to be proved.

[2] In contrast to the one-dimensional case [29], however, we cannot cover the entire L^2-supercritical range here. This has to do with the numerical work [16] where it was found that one needs to be close to the cubic NLS to guarantee some crucial spectral property.

The method of proof of Theorem 1 also extends to the case of more derivatives, i.e., $R_0 \in W^{k,1} \cap W^{k,2}$, for $k \geq 2$, but we do not elaborate on this here. We will refer to the fact that R_0 needs to satisfy

$$\|R_0\|_{W^{1,2}} + \|R_0\|_{W^{1,1}} < \delta \tag{10}$$

as the *smallness condition.* It is not hard to see from a close inspection of our proof that $W^{1,1}$ can be improved to some $W^{1,p}$, with p close to 1, but we choose $p = 1$ for simplicity.

To understand the origin of $\mathcal{S}$, we need to introduce the Riesz projections $P_{\mathrm{s}}(\alpha)$ and $\mathrm{Id} - P_{\mathrm{s}}(\alpha)$ (the index s here stands for *stable*). They are invariant under $\mathcal{H}(\alpha)$ and

$$\begin{aligned} \operatorname{spec}(\mathcal{H}(\alpha)P_{\mathrm{s}}(\alpha)) &= (-\infty, -\alpha^2] \cup [\alpha^2, \infty), \\ \operatorname{spec}(\mathcal{H}(\alpha)(\mathrm{Id} - P_{\mathrm{s}}(\alpha)) &= \{\pm i\sigma\} \cup \{0\}. \end{aligned}$$

Here $\pm i\sigma$ are precisely the unique pair of simple, purely imaginary eigenvalues of $\mathcal{H}(\alpha)$, $\sigma > 0$. Finally, let $P_u^+(\alpha)$ be the Riesz projection such that

$$\operatorname{spec}(\mathcal{H}(\alpha)P_u^+(\alpha)) = \{0\} \cup \{i\sigma\}.$$

The notation P_u^+ is meant to indicate the unstable modes as $t \to +\infty$. The real-linear, finite-codimensional subspace $\mathcal{S}$ above is precisely the set of R_0 so that

$$P_u^+(\alpha_0)\begin{pmatrix} R_0 \\ \bar{R}_0 \end{pmatrix} = 0. \tag{11}$$

The codimension of $\mathcal{S}$ is simply the number of unstable (or non-decaying) modes of the linearization: eight in the root space and one exponentially unstable mode. The stable manifold $\mathcal{M}$ is the surface described by the parametrization $R_0 \mapsto R_0 + \Phi(R_0)$ where R_0 belongs to a small ball $\mathcal{B} \cap \mathcal{S}$ inside of $\mathcal{S}$. The inequality (8) means that $\mathcal{S}$ is the tangent space to $\mathcal{M}$ at zero, whereas (9) expresses that $\mathcal{M}$ is given in terms of a Lipschitz parametrization. It is easy to see that it is also the graph of a Lipschitz map $\tilde{\Phi} : \mathcal{S} \cap \mathcal{B} \to \Sigma$. Indeed, define $\tilde{\Phi}$ as

$$R_0 + P_{\mathcal{S}}\Phi(R_0) \mapsto R_0 + \Phi(R_0),$$

where $P_{\mathcal{S}}$ is the projection onto $\mathcal{S}$ which is induced by the Riesz-projection $I - P_u^+(\alpha_0)$ (the latter operates on $L^2 \times L^2$, whereas we need only the first coordinate of this projection; see Remark 14 below for the details of this). The left-hand side is clearly in $\mathcal{S}$. Moreover, to see that this map is well-defined as well as Lipschitz, note that (9) implies that

$$\begin{aligned} (1 - C\delta)\|R_1 - R_0\| &\leq \|R_1 - R_0 + \Phi(R_1) - \Phi(R_0)\| \leq (1 + C\delta)\|R_1 - R_0\|, \\ (1 - C\delta)\|R_1 - R_0\| &\leq \|R_1 - R_0 + P_{\mathcal{S}}\Phi(R_1) - P_{\mathcal{S}}\Phi(R_0)\| \leq (1 + C\delta)\|R_1 - R_0\|. \end{aligned}$$

Theorem 1 should be understood as follows: The instability result of Berestycki and Cazenave [4] shows that one can have finite time blow-up for initial data $\psi_0 = \phi(\cdot, \alpha) + R_0$ where R_0 can be made arbitrarily small in any reasonable norm. On the other hand, one may ask what the obstruction for global existence and even stronger, for asymptotic stability, is in the orbitally unstable case. Naturally, the first guess is the unstable subspace of the linearized evolution $e^{-it\mathcal{H}}$ with $\mathcal{H}$ as in (6). This refers to the finite-dimensional subspaces of those $f \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for which $e^{-it\mathcal{H}} f$ does not decay locally as $t \to \infty$. Clearly, this subspace contains all the (generalized) eigenspaces of all eigenvalues of $\mathcal{H}(\alpha)$ that lie on $i\mathbb{R}^+ \cup \{0\}$. Conversely, Erdogan and the author [17] proved that (for much more general systems than (6))

$$\sup_{t \geq 0} \left\| e^{-it\mathcal{H}(\alpha)} (I - P_u^+(\alpha)) \right\|_{2 \to 2} < \infty. \tag{12}$$

While this bound was proved by Weinstein [49] and [50] in the *subcritical case*, in which $I - P_u^+ = P_{\mathrm{s}}$ only projects out the rootspace at zero, we are not aware of a reference for (12). Moreover, adapting the method of proof ([21]) from the scalar case considered there to the matrix case needed here, we show that

$$\left\| e^{-it\mathcal{H}(\alpha)} (I - P_u^+(\alpha)) \right\|_{1 \to \infty} \lesssim t^{-\frac{3}{2}} \tag{13}$$

for all $t > 0$. In view of (12) and (13), it is conceivable that at least to first approximation, the condition (11) should ensure stability. On the other hand, since it is based on linearization one expects quadratic corrections. This is precisely the content of (8). So the statement of the theorem is that after quadratic corrections, (11) gives the desired asymptotic stability.

In the subcritical (monomial, say) case the linearized operator (6) has a root space of dimension eight and no imaginary eigenvalues. Since there is asymptotic stability in this case, one would naturally expect that the root space should not contribute to the "bad directions" in the supercritical case, i.e., that the codimension of the true stable manifold should really be one for (1). This can indeed be achieved by letting all symmetries of the NLS equation act on the manifold $\mathcal{M}$ from Theorem 1. In this way we regain eight dimensions (six from the Galilei transforms, one from modulation, and one from scaling) provided we show that these symmetries act transversally to $\mathcal{M}$. This is done in the following theorem.

THEOREM 2. *Impose the spectral conditions for all $\alpha > 0$ and fix any $\alpha_0 > 0$. Then there exist a small $\delta > 0$ and a Lipschitz manifold $\mathcal{N}$ of size*[3] *δ inside of Σ and codimension one so that $\phi(\cdot, \alpha_0) \in \mathcal{N}$ with the following property*: *for any choice of initial data $\psi(0) \in \mathcal{N}$ the nonlinear Schrödinger*

[3]This means that $\mathcal{N}$ is the graph of a Lipschitz map Ψ with domain $\mathcal{B} \cap \tilde{\mathcal{S}}$ where $\tilde{\mathcal{S}}$ is a subspace of codimension one, with $\mathcal{B}$ as in (7).

equation (1) *has a global* H^1 *solution* $\psi(t)$ *for* $t \geq 0$. *Moreover,*

$$\psi(t) = W(t, \cdot) + R(t)$$

where W *as in* (3) *is governed by a path* $\pi(t)$ *of parameters so that*

$$|\pi(0) - (0, 0, 0, \alpha_0)| \lesssim \delta.$$

Also, $\pi(t)$ *converges to some terminal vector* $\pi(\infty)$ *such that*

$$\sup_{t \geq 0} |\pi(t) - \pi(\infty)| \lesssim \delta.$$

Finally,

$$\|R(t)\|_{W^{1,2}} \lesssim \delta, \qquad \|R(t)\|_\infty \lesssim \delta t^{-\frac{3}{2}}$$

for all $t > 0$ *and there is scattering*:

$$R(t) = e^{it\triangle} f_0 + o_{L^2}(1) \quad \textit{as } t \to \infty$$

for some $f_0 \in L^2(\mathbb{R}^3)$.

This result raises the question of deciding the behavior of solutions with initial data $\phi(0) \in \mathcal{B} \setminus \mathcal{N}$. One reasonable possibility would be that data from one half of $\mathcal{B} \setminus \mathcal{N}$ yield scattering solutions, whereas data from the other half lead to blow-up in finite time. Such a dichotomy was obtained by Kenig and Merle [28] for the H^1 critical focusing nonlinear Schrödinger equation.

The motivation for studying these questions was two-fold. First, it is natural to seek stable manifolds for unstable problems. There is a substantial ODE literature in this context, but the PDE case is much less studied although partial results exist. Bates and Jones [3] studied the questions of stable manifolds for evolution equations in great generality by means of the method of invariant cones. They proved the existence of stable, unstable, and center stable manifolds (which are also Lipschitz) for abstract evolution equations under certain conditions on the linear part of the evolution. Later Gesztesy, Jones, Latushkin, and Stanislavova [20] verified these conditions for a class of nonlinear Schrödinger equations which are obtained by linearizing around standing waves. Thus, they obtain the existence of such a dynamical splitting of the space of initial data close to a standing wave. However, in contrast to our theorems, no asymptotic convergence or scattering statements are made. In fact, the method of proof of [3] does not seem to yield global solutions. On the other hand, their manifolds in [3] are locally invariant in time. This property cannot be addressed in the context of our methods since the underlying topology is not locally invariant in time (due to the L^1 component).

For yet more related results, see the paper by Pillet and Wayne, as well as the book by Li and Wiggins [31]. Tsai and Yau [48] investigated the question of stable manifolds for NLS equations, with a potential, which admit excited states. These are standing waves which are generated by bifurcations off bound states of the linear Schrödinger operator. Such excited states are unstable as they will tend to collapse to the ground state. Tsai and Yau obtain conditional stability of these excited states.

Second, there is a large literature concerning *asymptotic stability* questions for *subcritical equations*; see the papers by Soffer, Weinstein [43], [44], Pillet, Wayne [35], and Buslaev, Perelman [6], [7], as well as Cuccagna [13]. Recently, there has also been some work on the multi-soliton case; see Rodnianski, Soffer, and the author [38], [39], as well as Perelman [34]. Most of these papers are based on a Lyapunov-Schmidt reduction, i.e., on splitting the evolution into a finite dimensional part and a complementary part on which the linearized evolution needs to be dispersive. In the subcritical case the finite dimensional part is exactly the root space, assuming as one usually does, that there are no other eigenvalues than zero. The dimension of this finite dimensional part then coincides with the number of parameters in our soliton (namely eight). This is natural, since both are intimately related to the family of symmetries of the nonlinear Schrödinger equation (1); see [49]. This fact allows one to write down a system of ODEs for the parameter paths called the modulation equations which ensure that the finite dimensional part is not present at all. In the context of asymptotic stability of solitons this method was first implemented by Soffer and Weinstein [43], [44]. Our second motivation for Theorem 1 was the question to what extent these asymptotic stability methods also apply to the L^2 supercritical case, which is orbitally unstable. As explained before, for the case of supercritical monomial nonlinearities the time evolution of the linearized problem has exponentially growing solutions. Needless to say, these modes cannot be controlled by the modulation equations. Rather, they need to be eliminated by a different mechanism. To first order, the unstable modes of the linearization need to be removed from the initial conditions. This is the origin of (11). However, this is only an approximation and quadratic corrections need to be made.

2. The linearization, Galilei transforms, and $\mathcal{J}$-invariance

As in [39] we require the soliton paths in (3) to be *admissible*. The constant δ which appears in the following definition is the same small constant as in Theorem 1. It will be specified later.

Definition 3. We say that a path $\pi : [0,\infty) \to \mathbb{R}^8$ with $\pi(t) := (\gamma(t), v(t), D(t), \alpha(t))$ is *admissible* provided it belongs to $\mathrm{Lip}([0,\infty),\mathbb{R}^8)$, the limit

$$\lim_{t\to\infty} (\gamma(t), v(t), D(t), \alpha(t)) =: (\gamma(\infty), v(\infty), D(\infty), \alpha(\infty))$$

exists, and is such that the entire path lies within a δ-neighborhood of those limiting values for all times $t \geq 0$. Moreover, we assume that

$$|v(t) - v(\infty)| = o(t^{-1}) \quad \text{as} \quad t \to \infty,$$
$$\int_0^\infty \int_s^\infty (|\dot v(\sigma)| + |\dot\alpha(\sigma)|)\, d\sigma\, ds < \infty.$$

Under these conditions, define a constant parameter vector $\pi_\infty = (\gamma_\infty, v_\infty, D_\infty, \alpha_\infty)$ as

$$\begin{aligned} \gamma_\infty &:= \gamma(\infty) + 2\int_0^\infty \int_s^\infty (v(\sigma)\cdot \dot v(\sigma) - \alpha(\sigma)\dot\alpha(\sigma))\, d\sigma\, ds, \\ v_\infty &:= v(\infty), \\ D_\infty &:= D(\infty) - 2\int_0^\infty \int_s^\infty \dot v(\sigma)\, d\sigma\, ds, \\ \alpha_\infty &:= \alpha(\infty). \end{aligned}$$

With these parameters, define the usual *Galilei transform* to be

$$\mathfrak{g}_\infty(t) = e^{i(\gamma_\infty + v_\infty\cdot x - t|v_\infty|^2)}\, e^{-i(2tv_\infty + D_\infty)\cdot\vec p}, \tag{14}$$

where $\vec p := -i\nabla$.

The action of $\mathfrak{g}_\infty(t)$ on functions is

$$(\,\mathfrak{g}_\infty(t) f)(x) = e^{i(\gamma_\infty + v_\infty\cdot x - t|v_\infty|^2)} f(x - 2tv_\infty - D_\infty),$$

they are unitary on L^2, isometries on all L^p, and the commutation property $e^{it\triangle}\,\mathfrak{g}_\infty(0) = \mathfrak{g}_\infty(t) e^{it\triangle}$ holds. The inverse of $\mathfrak{g}_\infty(t)$ is

$$\begin{aligned} \mathfrak{g}_\infty(t)^{-1} &= e^{i(2tv_\infty + D_\infty)\cdot\vec p}\, e^{-i(\gamma_\infty + v_\infty\cdot x - t|v_\infty|^2)} \\ &= e^{-i(\gamma_\infty + v_\infty\cdot D_\infty + v_\infty\cdot x + t|v_\infty|^2)}\, e^{i(2tv_\infty + D_\infty)\cdot\vec p}. \end{aligned}$$

Moreover, the Galilei transform (14) generates an eight-parameter family of solitons: Let $\phi(\cdot, \alpha_\infty)$ be the ground state of (2) with $\alpha = \alpha_\infty$. Then

$$W_\infty(t,\cdot) = \mathfrak{g}_\infty(t)[e^{it\alpha_\infty^2}\phi(\cdot, \alpha_\infty)] \tag{15}$$

solves (1), where W_∞ is a soliton as introduced in (3) but with the constant parameter path π_∞. For future reference, let

$$y_\infty(t) := 2tv_\infty + D_\infty,\ \theta_\infty(t,x) := v_\infty\cdot x - t(|v_\infty|^2 - \alpha_\infty^2) + \gamma_\infty. \tag{16}$$

We will use repeatedly that

$$\theta_\infty(t, x + y_\infty) = t(|v_\infty|^2 + \alpha_\infty^2) + v_\infty\cdot(x + D_\infty) + \gamma_\infty. \tag{17}$$

With this notation, W_∞ in (15) takes the form

$$W_\infty(t,x) = e^{i\theta_\infty(t,x)}\phi(x - y_\infty(t), \alpha_\infty).$$

Lemma 4. *Suppose π is an admissible path and let θ, y and θ_∞, y_∞ be as in* (4), (5), *and* (16), *respectively. Define*

$$\rho_\infty(t,x) := \theta(t, x+y_\infty) - \theta_\infty(t, x+y_\infty). \tag{18}$$

Then

$$\rho_\infty(t,x) = (1+|x|)o(1), \quad y(t) - y_\infty(t) = o(1)$$

as $t \to \infty$.

Proof. In view of the definition of π_∞,

$$\begin{aligned} &\theta(t, x+y_\infty) - \theta_\infty(t, x+y_\infty) \\ &\quad = v(t)\cdot(x+2tv_\infty + D_\infty) - \int_0^t (|v(s)|^2 - \alpha^2(s))\,ds + \gamma(t) \\ &\qquad - v_\infty\cdot(x+2tv_\infty+D_\infty) + t(|v_\infty|^2 - \alpha_\infty^2) - \gamma_\infty \\ &\quad = (v(t)-v_\infty)\cdot(x+2tv_\infty+D_\infty) + 2\int_0^\infty\int_s^\infty (v\cdot\dot v - \alpha\dot\alpha)(\sigma)\,d\sigma ds \\ &\qquad - \gamma_\infty + \gamma(t) - 2\int_t^\infty\int_s^\infty (v\cdot\dot v - \alpha\dot\alpha)(\sigma)\,d\sigma ds \\ &\quad = (v(t)-v_\infty)\cdot(x+2tv_\infty+D_\infty) - 2\int_t^\infty\int_s^\infty (v\cdot\dot v - \alpha\dot\alpha)(\sigma)\,d\sigma ds \\ &\qquad - \gamma(\infty) + \gamma(t). \end{aligned} \tag{19}$$

Definition 3 implies the desired bound on ρ_∞. As for $y(t)-y_\infty(t)$, the definition of D_∞ implies that

$$\begin{aligned} y_\infty(t) - y(t) &= 2tv_\infty + D_\infty - 2\int_0^t v(s)\,ds - D(t) \\ &= D(\infty) - D(t) - 2\int_t^\infty\int_s^\infty \dot v(\sigma)\,d\sigma\,ds, \end{aligned} \tag{20}$$

which goes to zero as $t\to\infty$. □

Recall from Section 1 that we seek an H^1 solution $\psi(t)$ of the cubic nonlinear Schrödinger equation (1) of the form $\psi = W + R$. The following standard lemma derives the equation for R, or rather for the vector $\binom{R}{\bar R}$.

Lemma 5. *Assume that $\pi(t) = (\gamma(t), v(t), D(t), \alpha(t))$ is admissible*; *see Definition* 3, *and let $W = W(t,x)$ be as in* (3). *Let $0 < T \le \infty$. Then $\psi \in C([0,T), H^1(\mathbb{R}^3)) \cap C^1([0,T), H^{-1}(\mathbb{R}^3))$ solves* (1) *with $\psi = W+R$ if and only if $Z = \binom{R}{\bar R}$ solves the equation*

$$
\begin{aligned}
(21)\qquad i\partial_t Z + \begin{pmatrix} \triangle + 2|W|^2 & W^2 \\ -\bar{W}^2 & -\triangle - 2|W|^2 \end{pmatrix} Z
&= \dot{v}\begin{pmatrix} -xe^{i\theta}\phi(\cdot - y, \alpha) \\ xe^{-i\theta}\phi(\cdot - y, \alpha) \end{pmatrix} + \dot{\gamma}\begin{pmatrix} -e^{i\theta}\phi(\cdot - y, \alpha) \\ e^{-i\theta}\phi(\cdot - y, \alpha) \end{pmatrix} \\
&\quad + i\dot{\alpha}\begin{pmatrix} e^{i\theta}\partial_\alpha\phi(\cdot - y, \alpha) \\ e^{-i\theta}\partial_\alpha\phi(\cdot - y, \alpha) \end{pmatrix} + i\dot{D}\begin{pmatrix} -e^{i\theta}\nabla\phi(\cdot - y, \alpha) \\ -e^{-i\theta}\nabla\phi(\cdot - y, \alpha) \end{pmatrix} \\
&\quad + \begin{pmatrix} -2|R|^2 W - \bar{W}R^2 - |R|^2 R \\ 2|R|^2\bar{W} + W\bar{R}^2 + |R|^2\bar{R} \end{pmatrix}
\end{aligned}
$$

in the sense of $C([0,T), H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)) \cap C^1([0,T), H^{-1}(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3))$. *Here* y *and* θ *are the functions from* (5) *and* (4), *and* $\alpha = \alpha(t)$. *For future reference, we denote the matrix operator on the left-hand side of* (21) *by* $-\mathcal{H}(\pi(t))$, *i.e.*,

$$
(22)\qquad \mathcal{H}(\pi(t)) := \begin{pmatrix} -\triangle - 2|W|^2 & -W^2 \\ \bar{W}^2 & \triangle + 2|W|^2 \end{pmatrix}.
$$

Proof. Let $\phi = \phi(\cdot, \alpha(t))$ for ease of notation. Direct differentiation shows that $W(t,x)$ satisfies

$$
i\partial_t W + \triangle W = -|W|^2 W - W(\dot{v}x + \dot{\gamma}) - ie^{i\theta}\nabla\phi \cdot \dot{D} + ie^{i\theta}\dot{\alpha}\partial_\alpha\phi.
$$

Hence $W + R$ is a solution of (1) if and only if

$$
\begin{aligned}
i\partial_t R + \triangle R = &-2|W|^2 R - 2|R|^2 W - |R|^2 R - W^2\bar{R} - \bar{W}R^2 \\
&- e^{i\theta}\phi(\dot{v}x + \dot{\gamma}) - ie^{i\theta}\nabla\phi \cdot \dot{D} + ie^{i\theta}\dot{\alpha}\partial_\alpha\phi.
\end{aligned}
$$

Joining this equation with its conjugate leads to the system (21). Conversely, if $Z(0)$ is of the form

$$
Z(0) = \begin{pmatrix} Z_1(0) \\ \overline{Z_1(0)} \end{pmatrix},
$$

and $Z(t)$ solves (21), then $Z(t)$ remains of this form for all times. This is simply the statement that the system (21) is invariant under the transformation

$$
(23)\qquad \mathcal{J} : f \mapsto \overline{Jf}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ \overline{f_1} \end{pmatrix},
$$

which can be checked by direct verification. This fact allows us to go back from the system to the scalar equation. □

The issue of $\mathcal{J}$-invariance is of great importance. The $\mathcal{J}$-invariant vectors in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ form a real-linear subspace, namely

$$
\left\{ \begin{pmatrix} f \\ \bar{f} \end{pmatrix} \;\middle|\; f \in L^2(\mathbb{R}^3) \right\}.
$$

Writing $f = f_1 + if_2$ it can be seen to be isomorphic to the subspace

$$\left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \;\Big|\; f_1, f_2 \in L^2(\mathbb{R}^3),\ f_1, f_2 \text{ are real-valued} \right\},$$

which is clearly linear, but only over $\mathbb{R}$. We need to insure that all vectorial solutions we construct belong to this subspace. Only then is it possible to revert to the scalar nonlinear Schrödinger equation (1).

As usual, it will be convenient to transform (21) to a stationary frame. In addition, a modulation will be performed. The details are as follows.

LEMMA 6. *Let $\pi(t)$ be an admissible path and let π_∞ be the constant vector associated with it as in Definition 3. Given a vector $Z = \binom{Z_1}{Z_2}$, introduce U, as well as $M(t), \mathcal{G}_\infty(t)$ by means of*

$$U(t) = \begin{pmatrix} e^{i\omega(t)} & 0 \\ 0 & e^{-i\omega(t)} \end{pmatrix} \begin{pmatrix} \mathfrak{g}_\infty(t)^{-1} Z_1(t) \\ \overline{\mathfrak{g}_\infty(t)^{-1} \overline{Z_2(t)}} \end{pmatrix} = M(t)\mathcal{G}_\infty(t)Z(t), \tag{24}$$

where $\omega(t) = -t\alpha_\infty^2$. Then $Z(t)$ solves (21) in the H^1 sense if and only if $U = \binom{U_1}{U_2}$ as in (24) satisfies the following PDE *in the H^1 sense (with $\phi_\infty = \phi(\cdot, \alpha_\infty)$):*

$$i\dot{U}(t) + \begin{pmatrix} \triangle + 2\phi_\infty^2 - \alpha_\infty^2 & \phi_\infty^2 \\ -\phi_\infty^2 & -\triangle - 2\phi_\infty^2 + \alpha_\infty^2 \end{pmatrix} U = -i\dot{\pi}\partial_\pi \widetilde{W}(\pi) + N(U,\pi) + VU \tag{25}$$

where we use the formal notation

(26)

$$V = V(t) := \begin{pmatrix} 2(\phi_\infty^2(x) - \phi^2(x + y_\infty - y)) & \phi_\infty^2(x) - e^{2i\rho_\infty}\phi^2(x + y_\infty - y) \\ -\phi_\infty^2(x) + e^{-2i\rho_\infty}\phi^2(x + y_\infty - y) & -2(\phi_\infty^2(x) - \phi^2(x + y_\infty - y)) \end{pmatrix},$$

(27)

$$-\dot{\pi}\partial_\pi \widetilde{W}(\pi) := \dot{v}\begin{pmatrix} -(x + y_\infty)e^{i\rho_\infty}\phi(x + y_\infty - y) \\ (x + y_\infty)e^{-i\rho_\infty}\phi(x + y_\infty - y) \end{pmatrix} + \dot{\gamma}\begin{pmatrix} -e^{i\rho_\infty}\phi(x + y_\infty - y) \\ e^{-i\rho_\infty}\phi(x + y_\infty - y) \end{pmatrix} + i\dot{\alpha}\begin{pmatrix} e^{i\rho_\infty}\partial_\alpha\phi(x + y_\infty - y) \\ e^{-i\rho_\infty}\partial_\alpha\phi(x + y_\infty - y) \end{pmatrix} + i\dot{D}\begin{pmatrix} -e^{i\rho_\infty}\nabla\phi(x + y_\infty - y) \\ -e^{-i\rho_\infty}\nabla\phi(x + y_\infty - y) \end{pmatrix},$$

(28)

$$N(U,\pi) := \begin{pmatrix} -2|U_1|^2 e^{i\rho_\infty}\phi(x + y_\infty - y) - e^{-i\rho_\infty}\phi(x + y_\infty - y)U_1^2 - |U_1|^2 U_1 \\ 2|U_2|^2 e^{-i\rho_\infty}\phi(x + y_\infty - y) + e^{i\rho_\infty}\phi(x + y_\infty - y)U_2^2 + |U_2|^2 U_2 \end{pmatrix}.$$

Here $\rho_\infty = \rho_\infty(t,x)$ is as in Lemma 4, $\phi(x + y_\infty - y) = \phi(x + y_\infty(t) - y(t), \alpha(t))$, and $\omega = \omega(t)$ is as in (24). Finally, Z is $\mathcal{J}$-invariant if and only if U is $\mathcal{J}$-invariant, and U is $\mathcal{J}$-invariant if and only if $U(0)$ is $\mathcal{J}$-invariant.

Proof. Throughout this proof we will adhere to the convention that $\phi = \phi(\cdot, \alpha(t))$ whereas $\phi_\infty = \phi(\cdot, \alpha_\infty))$. Write the equation (21) for Z in the form

$$i\partial_t Z - \mathcal{H}_\infty Z = F + (\mathcal{H}(\pi(t)) - \mathcal{H}_\infty)Z \tag{29}$$

where

$$\mathcal{H}_\infty = \begin{pmatrix} -\triangle - 2|W_\infty|^2 & -W_\infty^2 \\ \bar{W}_\infty^2 & \triangle + 2|W_\infty|^2 \end{pmatrix}; \tag{30}$$

see (15) and (16). With $\mathcal{G}_\infty(t)$ defined as in (24), and with $p = -i\nabla$,

$$\begin{aligned} i\frac{d}{dt}\mathcal{G}_\infty(t)f &= \begin{pmatrix} i\,\dot{\mathfrak{g}}_\infty(t)^{-1}f_1 \\ i\,\dot{\mathfrak{g}}_\infty(t)^{-1}\overline{f_2} \end{pmatrix} = \begin{pmatrix} -(2v_\infty\cdot p + |v_\infty|^2)\,\mathfrak{g}_\infty(t)^{-1}f_1 \\ -(2v_\infty\cdot p - |v_\infty|^2)\,\overline{\mathfrak{g}_\infty(t)^{-1}\overline{f_2(t)}} \end{pmatrix} \\ &= \begin{pmatrix} -(2v_\infty\cdot p + |v_\infty|^2) & 0 \\ 0 & -(2v_\infty\cdot p - |v_\infty|^2) \end{pmatrix}\mathcal{G}_\infty(t)f \end{aligned} \tag{31}$$

for any $f = \binom{f_1}{f_2}$. Furthermore,

(32)

$$\begin{aligned} &M(t)\mathcal{G}_\infty(t)\mathcal{H}_\infty\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= \begin{pmatrix} e^{i\omega(t)} & 0 \\ 0 & e^{-i\omega(t)} \end{pmatrix}\begin{pmatrix} -(\triangle + 2\phi_\infty^2)\,\mathfrak{g}_\infty(t)^{-1}f_1 - \phi_\infty^2 e^{2i\theta_\infty(t,x+y_\infty)}\,\overline{\mathfrak{g}_\infty(t)^{-1}f_2} \\ \phi_\infty^2 e^{-2i\theta_\infty(t,x+y_\infty)}\,\overline{\mathfrak{g}_\infty(t)^{-1}\overline{f_1}} + (\triangle + 2\phi_\infty^2)\,\overline{\mathfrak{g}_\infty(t)^{-1}\overline{f_2}} \end{pmatrix} \\ &\quad - \begin{pmatrix} e^{i\omega(t)} & 0 \\ 0 & e^{-i\omega(t)} \end{pmatrix}\begin{pmatrix} -|v_\infty|^2 + 2iv_\infty\cdot\nabla & 0 \\ 0 & |v_\infty|^2 + 2iv_\infty\cdot\nabla \end{pmatrix}\mathcal{G}_\infty(t)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= \begin{pmatrix} e^{i\omega(t)} & 0 \\ 0 & e^{-i\omega(t)} \end{pmatrix} \\ &\times\begin{pmatrix} -(\triangle+2\phi_\infty^2)\,\mathfrak{g}_\infty(t)^{-1}f_1 - \phi_\infty^2 e^{2i[\theta_\infty(t,x+y_\infty)-(t|v_\infty|^2+v_\infty\cdot(x+D_\infty)+\gamma_\infty)]}\,\overline{\mathfrak{g}_\infty(t)^{-1}\overline{f_2}} \\ \phi_\infty^2 e^{-2i[\theta_\infty(t,x+y_\infty)-(t|v_\infty|^2+v_\infty\cdot(x+D_\infty)+\gamma_\infty)]}\,\mathfrak{g}_\infty(t)^{-1}f_1 + (\triangle+2\phi_\infty^2)\,\overline{\mathfrak{g}_\infty(t)^{-1}\overline{f_2}} \end{pmatrix} \\ &\quad - \begin{pmatrix} e^{i\omega(t)} & 0 \\ 0 & e^{-i\omega(t)} \end{pmatrix}\begin{pmatrix} -|v_\infty|^2 + 2iv_\infty\cdot\nabla & 0 \\ 0 & |v_\infty|^2 + 2iv_\infty\cdot\nabla \end{pmatrix}\mathcal{G}_\infty(t)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \end{aligned}$$

Now

$$\theta_\infty(t, x + y_\infty) - (t|v_\infty|^2 + v_\infty\cdot(x + D_\infty) + \gamma_\infty) = t\alpha_\infty^2;$$

see (17). Hence, by the definition of $\omega(t)$ (and dropping the argument t from M and $\mathcal{G}_\infty$ for simplicity),

(33)

$$\begin{aligned} (32) &= \begin{pmatrix} e^{i\omega(t)} & 0 \\ 0 & e^{-i\omega(t)} \end{pmatrix}\begin{pmatrix} -(\triangle + 2\phi_\infty^2) & -\phi_\infty^2 e^{2it\alpha_\infty^2} \\ \phi_\infty^2 e^{-2it\alpha_\infty^2} & \triangle + 2\phi_\infty^2 \end{pmatrix}\mathcal{G}_\infty f \\ &\quad - \begin{pmatrix} e^{i\omega(t)} & 0 \\ 0 & e^{-i\omega(t)} \end{pmatrix}\begin{pmatrix} -|v_\infty|^2 - 2v_\infty\cdot p & 0 \\ 0 & |v_\infty|^2 - 2v_\infty\cdot p \end{pmatrix}\mathcal{G}_\infty f \end{aligned}$$

$$= \begin{pmatrix} -\triangle - 2\phi_\infty^2 & -\phi_\infty^2 \\ \phi_\infty^2 & \triangle + 2\phi_\infty^2 \end{pmatrix} M\mathcal{G}_\infty f$$
$$- \begin{pmatrix} -|v_\infty|^2 - 2v_\infty \cdot p & 0 \\ 0 & |v_\infty|^2 - 2v_\infty \cdot p \end{pmatrix} \mathcal{G}_\infty f.$$

Denote the first matrix operator in (33) by $\mathcal{H}_\phi$. Hence, in combination with (31) one concludes from (29) that

$$\begin{aligned} i\dot{U} &= i\dot{M}\mathcal{G}_\infty Z + iM\dot{\mathcal{G}}_\infty Z + M\mathcal{G}_\infty \mathcal{H}_\infty + M\mathcal{G}_\infty(F + (\mathcal{H}(\pi(t)) - \mathcal{H}_\infty)Z) \\ &= \begin{pmatrix} -\dot{\omega} & 0 \\ 0 & \dot{\omega} \end{pmatrix} M\mathcal{G}_\infty Z \\ &\quad + \begin{pmatrix} -(2v_\infty \cdot p + |v_\infty|^2) & 0 \\ 0 & -(2v_\infty \cdot p - |v_\infty|^2) \end{pmatrix} M\mathcal{G}_\infty Z + \mathcal{H}_\phi M\mathcal{G}_\infty Z \\ &\quad + \begin{pmatrix} |v_\infty|^2 + 2v_\infty \cdot p & 0 \\ 0 & -|v_\infty|^2 + 2v_\infty \cdot p \end{pmatrix} M\mathcal{G}_\infty Z \\ &\quad + M\mathcal{G}_\infty(F + (\mathcal{H}(\pi(t)) - \mathcal{H}_\infty)Z) \\ &= \begin{pmatrix} -\triangle + \alpha_\infty^2 - 2\phi_\infty^2 & -\phi_\infty^2 \\ \phi_\infty^2 & \triangle - \alpha_\infty^2 + 2\phi_\infty^2 \end{pmatrix} U(t) \\ &\quad + M\mathcal{G}_\infty(F + (\mathcal{H}(\pi(t)) - \mathcal{H}_\infty)\mathcal{G}_\infty^{-1} M^{-1} U). \end{aligned}$$

It remains to compute the terms

$$\dot{\pi}\partial_\pi \widetilde{W}(\pi) + N(U, \pi) = M(t)\mathcal{G}_\infty(t)F(t), \tag{34}$$
$$V = M(t)\mathcal{G}_\infty(t)(\mathcal{H}(\pi(t)) - \mathcal{H}_\infty)\mathcal{G}_\infty(t)^{-1}M(t)^{-1} \tag{35}$$

In view of (21), one has

$$\begin{aligned} F = \dot{v}\begin{pmatrix} -xe^{i\theta}\phi(x-y) \\ xe^{-i\theta}\phi(x-y) \end{pmatrix} + \dot{\gamma}\begin{pmatrix} -e^{i\theta}\phi(x-y) \\ e^{-i\theta}\phi(x-y) \end{pmatrix} \\ + i\dot{\alpha}\begin{pmatrix} e^{i\theta}\partial_\alpha\phi(x-y) \\ e^{-i\theta}\partial_\alpha\phi(x-y) \end{pmatrix} + i\dot{D}\begin{pmatrix} -e^{i\theta}\nabla\phi(x-y) \\ -e^{-i\theta}\nabla\phi(x-y) \end{pmatrix} \\ + \begin{pmatrix} -2|R|^2W - \bar{W}R^2 - |R|^2R \\ 2|R|^2\bar{W} + W\bar{R}^2 + |R|^2\bar{R} \end{pmatrix}. \end{aligned}$$

Now

$$\begin{aligned} \theta(t, x + y_\infty) - (\alpha_\infty^2 t + v_\infty \cdot (x + D_\infty) + t|v_\infty|^2 + \gamma_\infty) \\ = \theta(t, x + y_\infty) - \theta_\infty(t, x + y_\infty) = \rho_\infty(t, x); \end{aligned}$$

see (17) and Lemma 4. Thus, the first term of $M\mathcal{G}F$ is

$$\begin{aligned} \dot{v}\begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix}\begin{pmatrix} -(x + y_\infty)e^{i\theta(t,x+y_\infty)}\, e^{-i(t|v_\infty|^2 + v_\infty\cdot(x+D_\infty)+\gamma_\infty)}\phi(x + y_\infty - y) \\ (x + y_\infty)e^{-i\theta(t,x+y_\infty)}\, e^{i(t|v_\infty|^2 + v_\infty\cdot(x+D_\infty)+\gamma_\infty)}\phi(x + y_\infty - y) \end{pmatrix} \\ = \dot{v}\begin{pmatrix} -(x + y_\infty)e^{i\rho_\infty(t,x)}\phi(x + y_\infty - y) \\ (x + y_\infty)e^{-i\rho_\infty(t,x)}\phi(x + y_\infty - y) \end{pmatrix}. \end{aligned}$$

This gives the $\dot{v}$ term in (27). The other terms involving $\dot{\alpha}, \dot{\gamma}$, and $\dot{D}$ are treated similarly, and we skip the details. The cubic term in (21) is also easily transformed, and it leads to the nonlinear term $N(U, \pi)$ in (28). We skip that calculation as well. Finally, it remains to transform $\mathcal{H}(\pi(t)) - \mathcal{H}_\infty$:

$$\begin{aligned}&\mathcal{H}(\pi(t)) - \mathcal{H}_\infty\\ &= \begin{pmatrix} 2(\phi_\infty^2(\cdot - y_\infty) - \phi^2(\cdot - y)) & e^{2i\theta_\infty}\phi_\infty^2(\cdot - y_\infty) - e^{2i\theta}\phi^2(\cdot - y) \\ e^{-2i\theta}\phi^2(\cdot - y) - e^{-2i\theta_\infty}\phi_\infty^2(\cdot - y_\infty) & -2(\phi_\infty^2(\cdot - y_\infty) - \phi^2(\cdot - y)) \end{pmatrix}\end{aligned}$$

where $\phi_\infty = \phi(\cdot - y_\infty(t), \alpha_\infty)$, $\phi = \phi(\cdot - y(t), \alpha(t))$ for simplicity. It is easy to check that

$$\begin{aligned}&\mathcal{G}_\infty(t)(\mathcal{H}(\pi(t)) - \mathcal{H}_\infty)\\ &\qquad = \begin{pmatrix} 2(\phi_\infty^2(x) - \phi^2(x + y_\infty - y)) & * \\ -e^{-2it\alpha_\infty^2}(\phi_\infty^2(x) - e^{-2i\rho_\infty}\phi^2(\cdot + y_\infty - y)) & * \end{pmatrix}\mathcal{G}_\infty(t).\end{aligned}$$

After conjugation by the matrix $M(t)$ this takes the desired form (26) and we are done. For the final statements concerning $\mathcal{J}$-invariance, observe first that the transformation (24) from Z to U preserves $\mathcal{J}$-invariance. Second, the equation (25) is $\mathcal{J}$-invariant, which shows that it suffices to assume the $\mathcal{J}$-invariance of $U(0)$ to guarantee it for all $t \geq 0$. To check the $\mathcal{J}$-invariance of (25), note that the right-hand side of (25) transforms like

$$\mathcal{J}[-i\dot{\pi}\partial_\pi\widetilde{W}(\pi) + N(U, \pi) + VU] = -[-i\dot{\pi}\partial_\pi\widetilde{W}(\pi) + N(\mathcal{J}U, \pi) + V\mathcal{J}U],$$

while the left-hand side transforms as follows:

$$\begin{aligned}&\mathcal{J}[i\dot{U}(t) + \begin{pmatrix} \triangle + 2\phi_\infty^2 - \alpha_\infty^2 & \phi_\infty^2 \\ -\phi_\infty^2 & -\triangle - 2\phi_\infty^2 + \alpha_\infty^2 \end{pmatrix} U]\\ &\qquad = -i\dot{\mathcal{J}}U(t) - \begin{pmatrix} \triangle + 2\phi_\infty^2 - \alpha_\infty^2 & \phi_\infty^2 \\ -\phi_\infty^2 & -\triangle - 2\phi_\infty^2 + \alpha_\infty^2 \end{pmatrix}\mathcal{J}U.\end{aligned}$$

Combining these statements yields the desired $\mathcal{J}$-invariance of (25). □

In what follows, we will need to bound the nonlinear term $N(U, \pi)$ in various norms. For future reference we therefore include the following lemma.

LEMMA 7. *Let π be an admissible path and let $N(U, \pi)$ be as in* (28). *Then*

$$\|N(U, \pi)\|_1 \lesssim \min(\|U\|_\infty^2, \|U\|_2^2) + \|U\|_3^3, \tag{36}$$

$$\|N(U, \pi)\|_2 \lesssim \min(\|U\|_\infty^2, \|U\|_4^2) + \|U\|_6^3, \tag{37}$$

$$\|\nabla N(U, \pi)\|_2 \lesssim \min(\|U\|_\infty^2, \|U\|_4^2) + \|U\nabla U\|_2 + \||U|^2\nabla U\|_2, \tag{38}$$

$$\|\nabla N(U, \pi)\|_1 \lesssim \min(\|U\|_\infty^2, \|U\|_2^2) + \|U\nabla U\|_1 + \||U|^2\nabla U\|_1. \tag{39}$$

Proof. Direct estimation of the terms of the matrix on the right-hand side of (28). □

3. The linearized problem and the root spaces at zero

Recall that $\phi = \phi(\cdot, \alpha)$ is the ground state of $-\triangle\phi + \alpha^2\phi = \phi^3$. Define

$$\mathcal{H}(\alpha) := \begin{pmatrix} -\triangle - 2\phi^2 + \alpha^2 & -\phi^2 \\ \phi^2 & \triangle + 2\phi^2 - \alpha^2 \end{pmatrix}. \tag{40}$$

Hence the matrix operator on the left-hand side of (25) is equal to $-\mathcal{H}(\alpha_\infty)$; i.e., (25) can be rewritten as

$$i\partial_t U - \mathcal{H}(\alpha_\infty)U = -i\dot{\pi}\partial_\pi \widetilde{W}(\pi) + N(U,\pi) + VU$$

or

$$i\partial_t U - \mathcal{H}(t)U = -i\dot{\pi}\partial_\pi \widetilde{W}(\pi) + N(U,\pi),$$

where $\mathcal{H}(t) := \mathcal{H}(\alpha_\infty) + V(t)$. The main goal of this section and the following one is to characterize the entire discrete spectrum of $\mathcal{H}(\alpha)$. Fix some $\alpha > 0$. One has the representation (see for example [6], [38], [17])

$$L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) = \mathcal{N} + \mathcal{L} + (\mathcal{N}^* + \mathcal{L}^*)^\perp$$

where $\mathcal{L}, \mathcal{L}^*$ are the sum of the eigenspaces corresponding to the purely imaginary eigenvalues of $\mathcal{H}(\alpha)$ and $\mathcal{H}(\alpha)^*$, respectively, and $\mathcal{N}, \mathcal{N}^*$ are the root spaces of $\mathcal{H}(\alpha)$ and $\mathcal{H}(\alpha)^*$, respectively, i.e.,

$$\mathcal{N} = \bigcup_{n=1}^\infty \ker(\mathcal{H}(\alpha)^n), \quad \mathcal{N}^* = \bigcup_{n=1}^\infty \ker((\mathcal{H}(\alpha)^*)^n).$$

The sum here is direct but not orthogonal. In particular, this representation shows that

$$\operatorname{Ran}(P_{\rm s}(\alpha)) = (\mathcal{N}^* + \mathcal{L}^*)^\perp \tag{41}$$

where $I - P_{\rm s}(\alpha)$ is the Riesz projection corresponding to the discrete spectrum of $\mathcal{H}(\alpha)$. In [49], Weinstein showed that the root spaces $\mathcal{N}(\alpha)$ and $\mathcal{N}^*(\alpha)$ of $\mathcal{H}(\alpha)$ and $\mathcal{H}^*(\alpha)$, respectively, are (with $\phi = \phi(\cdot, \alpha)$)

$$\mathcal{N} = \mathcal{N}(\alpha) = \operatorname{span}\Big\{ \begin{pmatrix} i\phi \\ -i\phi \end{pmatrix}, \begin{pmatrix} \partial_\alpha\phi \\ \partial_\alpha\phi \end{pmatrix}, \begin{pmatrix} \partial_j\phi \\ \partial_j\phi \end{pmatrix}, \begin{pmatrix} ix_j\phi \\ -ix_j\phi \end{pmatrix} \Big| \; 1 \le j \le 3 \Big\}, \tag{42}$$

$$\mathcal{N}^* = \mathcal{N}(\alpha)^* = \operatorname{span}\Big\{ \begin{pmatrix} \phi \\ \phi \end{pmatrix}, \begin{pmatrix} i\partial_\alpha\phi \\ -i\partial_\alpha\phi \end{pmatrix}, \begin{pmatrix} i\partial_j\phi \\ -i\partial_j\phi \end{pmatrix}, \begin{pmatrix} x_j\phi \\ x_j\phi \end{pmatrix} \Big| \; 1 \le j \le 3 \Big\}. \tag{43}$$

Showing that the root spaces contain the sets on the right-hand side is just a matter of direct computation. The difficulty lies with showing the equality. Moreover, Weinstein showed that $\ker(\mathcal{H}^2(\alpha)) = \ker(\mathcal{H}^3(\alpha))$ (his argument only applies to certain nonlinearities, which include the cubic nonlinear Schrödinger equation in $\mathbb{R}^3$).

In order to apply the linear dispersive $L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ estimates from Section 7 to (25), one needs to project U onto $\operatorname{Ran}(P_{\rm s})$. Following common practice, see Soffer, Weinstein [43], [44], and Buslaev, Perelman [6], we will make an appropriate choice of the path $\pi(t)$ in order to insure that $U(t)$ is perpendicular to $\mathcal{N}^*$. However, for technical reasons it is advantageous to impose an orthogonality condition onto a *time-dependent* family of functions rather than $\mathcal{N}^*$ itself. We introduce this family in the following definition. In view of Lemma 4, it approaches $\mathcal{N}^*$ in the limit $t \to \infty$.

Definition 8. Assume that π is an admissible path and let y, θ be as in (5), (4), y_∞, θ_∞ as in (16), and ρ_∞ as in (18). With these functions, define

$$\begin{aligned}
\xi_1(t) &:= \begin{pmatrix} e^{i\rho_\infty}\phi(\cdot + y_\infty - y, \alpha(t)) \\ e^{-i\rho_\infty}\phi(\cdot + y_\infty - y, \alpha(t)) \end{pmatrix}, \\
\xi_2(t) &:= \begin{pmatrix} ie^{i\rho_\infty}\partial_\alpha\phi(\cdot + y_\infty - y, \alpha(t)) \\ -ie^{-i\rho_\infty}\partial_\alpha\phi(\cdot + y_\infty - y, \alpha(t)) \end{pmatrix}, \\
\xi_{2+\ell}(t) &:= \begin{pmatrix} e^{i\rho_\infty}(x_\ell + y_\infty - y)\phi(\cdot + y_\infty - y, \alpha(t)) \\ e^{-i\rho_\infty}(x_\ell + y_\infty - y)\phi(\cdot + y_\infty - y, \alpha(t)) \end{pmatrix}, \\
\xi_{5+\ell}(t) &:= \begin{pmatrix} ie^{i\rho_\infty}\partial_\ell\phi(\cdot + y_\infty - y, \alpha(t)) \\ -ie^{-i\rho_\infty}\partial_\ell\phi(\cdot + y_\infty - y, \alpha(t)) \end{pmatrix}
\end{aligned}$$

for $\ell = 1, 2, 3$. We also introduce another family $\{\eta_j\}_{j=1}^8$ by

$$\eta_j = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \xi_j \quad \text{for any} \quad 1 \le j \le 8. \tag{44}$$

By inspection, $\mathcal{J}\xi_j = \xi_j$ for $1 \le j \le 8$ and we chose η_j in such a way that $\mathcal{J}\eta_j = \eta_j$ for each j. Clearly, while the ξ_j correspond to $\mathcal{H}^*$, the η_j correspond to $\mathcal{H}$, cf. (42) and (43). Let U be as in Lemma 6. We refer to the condition that

$$\langle U(t), \xi_j(t) \rangle = 0 \tag{45}$$

for all $t \ge 0$, $1 \le j \le 8$ as the *orthogonality condition*. As usual, the orthogonality condition (45) leads to an ODE for the path $\pi(t)$. Following [13], we first modify the γ parameter.

Lemma 9. *Let $\pi(t)$ be an admissible path as in Definition 3. Set*

$$\dot{\tilde{\gamma}}(t) := \dot{\gamma}(t) + \dot{v}(t) \cdot y(t) \tag{46}$$

and $\tilde{\gamma}(\infty) := 0$; i.e.,

$$\tilde{\gamma}(t) := -\int_t^\infty \Big[\dot{\gamma}(s) + \dot{v}(s) \cdot y(s)\Big]\, ds.$$

Then the function $\dot{\pi}\partial_\pi \widetilde{W}(\pi)$ on the right-hand side of (25) satisfies

$$\dot{\pi}\partial_\pi \widetilde{W}(\pi) = \sum_{\ell=1}^{3} (\dot{D}_\ell \eta_{5+\ell} + \dot{v}_\ell \eta_{2+\ell}) - \dot{\alpha}\eta_2 + \dot{\tilde{\gamma}}\eta_1$$

where the functions $\{\eta_j\}_{j=1}^8$ are as in (44).

Proof. By inspection. □

The following lemma records some useful facts about the two families in Definition 8.

LEMMA 10. *Let $\phi = \phi(\cdot, \alpha(t))$ be the ground state of* (2) *and let $\{\xi_j\}_{j=1}^8$ and $\{\eta_j\}_{j=1}^8$ be as in Definition* 8. *Then*

$$\begin{aligned}
\langle \xi_1, \eta_j\rangle &= 2\langle \partial_\alpha \phi, \phi\rangle && \text{if } j=2 \text{ and } =0 \text{ else},\\
\langle \xi_2, \eta_j\rangle &= -2\langle \partial_\alpha \phi, \phi\rangle && \text{if } j=1 \text{ and } =0 \text{ else},\\
\langle \xi_{2+\ell}, \eta_j\rangle &= -2\langle \phi, \phi\rangle && \text{if } j=5+\ell \text{ and } =0 \text{ else},\\
\langle \xi_{5+\ell}, \eta_j\rangle &= 2\langle \phi, \phi\rangle && \text{if } j=2+\ell \text{ and } =0 \text{ else}.
\end{aligned}$$

Here $\partial_\alpha\langle\phi,\phi\rangle = 2\langle\partial_\alpha\phi,\phi\rangle = -\alpha^{-1}\|\phi\|_2^2$. Moreover, let $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and

$$E := [-|v(t) - v_\infty|^2 - 2i(v(t) - v_\infty)\cdot\nabla + \alpha_\infty^2 - \alpha(t)^2]\sigma_3. \tag{47}$$

Also, $\mathcal{H}(t) = \mathcal{H}(\alpha_\infty) + V(t)$, see (40) *and* (26). *Then*

$$\begin{aligned}
\mathcal{H}(t)^*\xi_1 &= E\xi_1,\\
\mathcal{H}(t)^*\xi_2 &= -2i\alpha(t)\xi_1 + E\xi_2,\\
\mathcal{H}(t)^*\xi_{2+\ell} &= -2i\xi_{5+\ell} + E\xi_{2+\ell},\\
\mathcal{H}(t)^*\xi_{5+\ell} &= E\xi_{5+\ell}
\end{aligned}$$

for any $\ell = 1, 2, 3$.

Proof. The statements about the scalar products are checked by direct verification. That $\partial_\alpha\langle\phi,\phi\rangle = -\alpha^{-1}\|\phi\|_2^2$ follows from the fact that the ground states $\phi(\cdot,\alpha)$ of (2) satisfy $\phi(\cdot,\alpha) = \alpha\phi(\alpha x, 1)$. The matrix $\mathcal{H}(t)^*$ is explicitly given by

$$\mathcal{H}(t)^* = \begin{pmatrix} -\triangle + \alpha_\infty^2 - 2\phi^2(\cdot + y_\infty(t) - y(t), \alpha(t)) & e^{2i\rho_\infty}\phi^2(\cdot + y_\infty(t) - y(t), \alpha(t)) \\ -e^{-2i\rho_\infty}\phi^2(\cdot + y_\infty(t) - y(t), \alpha(t)) & \triangle - \alpha_\infty^2 + 2\phi^2(x + y_\infty(t) - y(t), \alpha(t)) \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
&\mathcal{H}(t)^*\xi_1(t)\\
&= \begin{pmatrix} -\triangle + \alpha(t)^2 - 2\phi^2(\cdot + y_\infty(t) - y(t), \alpha(t)) & e^{2i\rho_\infty}\phi^2(\cdot + y_\infty(t) - y(t), \alpha(t)) \\ -e^{-2i\rho_\infty}\phi^2(\cdot + y_\infty(t) - y(t), \alpha(t)) & \triangle - \alpha(t)^2 + 2\phi^2(x + y_\infty(t) - y(t), \alpha(t)) \end{pmatrix}\xi_1\\
&\quad + \sigma_3(\alpha_\infty^2 - \alpha(t)^2)\xi_1(t)\\
&= \sigma_3(\alpha_\infty^2 - \alpha(t)^2)\xi_1(t)\\
&\quad + \begin{pmatrix} -\triangle\Big(e^{i\rho_\infty}\phi(\cdot + y_\infty(t) - y(t), \alpha(t))\Big) + e^{i\rho_\infty}\triangle\phi(\cdot + y_\infty(t) - y(t), \alpha(t)) \\ \triangle\Big(e^{-i\rho_\infty}\phi(\cdot + y_\infty(t) - y(t), \alpha(t))\Big) - e^{-i\rho_\infty}\triangle\phi(\cdot + y_\infty(t) - y(t), \alpha(t)) \end{pmatrix}.
\end{aligned}$$

Here we used that

$$(-\triangle + \alpha(t)^2 - \phi^2(\cdot + y_\infty(t) - y(t), \alpha(t)))\phi(\cdot + y_\infty(t) - y(t), \alpha(t)) = 0,$$

which is the ground state equation. An explicit calculation shows that

$$\begin{aligned}&\begin{pmatrix} -\triangle\Big(e^{i\rho_\infty}\phi(\cdot + y_\infty(t) - y(t), \alpha(t))\Big) + e^{i\rho_\infty}\triangle\phi(\cdot + y_\infty(t) - y(t), \alpha(t)) \\ \triangle\Big(e^{-i\rho_\infty}\phi(\cdot + y_\infty(t) - y(t), \alpha(t))\Big) - e^{-i\rho_\infty}\triangle\phi(\cdot + y_\infty(t) - y(t), \alpha(t)) \end{pmatrix} \\ &= \begin{pmatrix} -i\triangle\rho_\infty - |\nabla\rho_\infty|^2 - 2i\nabla\rho_\infty\cdot\nabla & 0 \\ 0 & -i\triangle\rho_\infty + |\nabla\rho_\infty|^2 - 2i\nabla\rho_\infty\cdot\nabla \end{pmatrix}\xi_1.\end{aligned}$$

The conclusion is that (see (19))

$$\begin{aligned}\mathcal{H}(t)^*\xi_1 &= \sigma_3(\alpha_\infty^2 - \alpha(t)^2)\xi_1 \\ &\quad + \begin{pmatrix} -i\triangle\rho_\infty - |\nabla\rho_\infty|^2 - 2i\nabla\rho_\infty\cdot\nabla & 0 \\ 0 & -i\triangle\rho_\infty + |\nabla\rho_\infty|^2 - 2i\nabla\rho_\infty\cdot\nabla \end{pmatrix}\xi_1 \\ &= E\xi_1,\end{aligned}$$

as claimed. The calculation for ξ_ℓ, $2 \le \ell \le 8$ is analogous, and we skip it. □

We can now derive the usual modulation equations for the admissible path π under the orthogonality assumption (45).

LEMMA 11. *Assume that π is an admissible path and that U is an H^1 solution of* (25) *with an initial condition $U(0)$ which satisfies the orthogonality assumptions* (45) *at time $t = 0$. Then U satisfies the orthogonality assumptions* (45) *for all times if and only if π satisfies the modulation equations* (*with E as in* (47) *and with $\phi = \phi(\cdot, \alpha(t))$*)

$$\begin{aligned}\dot\alpha\alpha^{-1}\|\phi\|_2^2 &= \langle U, \dot\xi_1\rangle - i\langle U, E\xi_1\rangle - i\langle N(U,\pi), \xi_1\rangle, \\ \dot{\tilde\gamma}\alpha^{-1}\|\phi\|_2^2 &= \langle U, \dot\xi_2\rangle - i\langle U, E\xi_2\rangle - i\langle N(U,\pi), \xi_2\rangle, \\ 2\dot D_\ell\|\phi\|_2^2 &= \langle U, \dot\xi_{2+\ell}\rangle - i\langle U, E\xi_{2+\ell}\rangle - i\langle N(U,\pi), \xi_{2+\ell}\rangle, \\ 2\dot v_\ell\|\phi\|_2^2 &= \langle U, \dot\xi_{5+\ell}\rangle - i\langle U, E\xi_{5+\ell}\rangle - i\langle N(U,\pi), \xi_{5+\ell}\rangle\end{aligned}$$

for all $1 \le \ell \le 3$.

Proof. Clearly, for any $1 \le j \le 8$,

$$\langle U(t), \xi_j(t)\rangle = 0 \quad \text{for all} \quad t \ge 0$$

is equivalent to

$$\langle U(0), \xi_j(0)\rangle = 0 \quad \text{and} \quad \langle \partial_t U, \xi_j\rangle = -\langle U, \dot\xi_j\rangle \quad \text{for all} \quad t \ge 0.$$

Starting from

$$i\partial_t U - \mathcal{H}(t)U = -i\dot\pi\partial_\pi\widetilde{W}(\pi) + N(U,\pi),$$

the modulation equations now follow from the previous two lemmas. □

Later it will be important to have a family of functions that plays the same role for $Z(t)$ as $\{\xi_j\}_{j=1}^8$ does for $U(t)$. The following lemma introduces this family and establishes some statements for it analogous to the ones we just obtained for $\{\xi_j\}_{j=1}^8$.

Lemma 12. *Fix an admissible path π and let θ, y be as in (4) and (5), respectively. Define*

$$\begin{aligned}
\tilde{\xi}_1(t,x) &:= \begin{pmatrix} e^{i\theta(t,x)}\phi(x-y(t),\alpha(t)) \\ e^{-i\theta(t,x)}\phi(x-y(t),\alpha(t)) \end{pmatrix}, \\
\tilde{\xi}_2(t,x) &:= \begin{pmatrix} ie^{i\theta(t,x)}\partial_\alpha\phi(x-y(t),\alpha(t)) \\ -ie^{-i\theta(t,x)}\partial_\alpha\phi(x-y(t),\alpha(t)) \end{pmatrix}, \\
\tilde{\xi}_{2+\ell}(t) &:= \begin{pmatrix} e^{i\theta(t,x)}(x_\ell - y_\ell(t))\phi(x-y(t),\alpha(t)) \\ e^{-i\theta(t,x)}(x_\ell - y_\ell(t))\phi(x-y(t),\alpha(t)) \end{pmatrix}, \\
\tilde{\xi}_{5+\ell}(t) &:= \begin{pmatrix} ie^{i\theta(t,x)}\partial_\ell\phi(x-y(t),\alpha(t)) \\ -ie^{-i\theta(t,x)}\partial_\ell\phi(x-y(t),\alpha(t)) \end{pmatrix}
\end{aligned}$$

for $\ell = 1,2,3$. Then

$$\xi_j(t) = M(t)\mathcal{G}_\infty(t)\tilde{\xi}_j(t) \quad \textit{for all} \quad 1 \le j \le 8.$$

Also, let U and Z be related by (24). Then U satisfies the orthogonality condition (45) if and only if $Z(t)$ satisfies

$$\langle Z(t), \tilde{\xi}_j(t)\rangle = 0 \textit{ for all } \ 1 \le j \le 8,\ t \ge 0.$$

Finally, introduce $\{\tilde{\eta}_j(t)\}_{j=1}^8$ as in (44), i.e.,

$$\tilde{\eta}_j = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\tilde{\xi}_j \quad \textit{for any} \quad 1 \le j \le 8.$$

Then the same scalar product relations hold as in Lemma 10. Indeed,

$$\begin{aligned}
\langle\tilde{\xi}_1, \tilde{\eta}_j\rangle &= 2\langle\partial_\alpha\phi, \phi\rangle && \textit{if } j = 2 \textit{ and } = 0 \textit{ else}, \\
\langle\tilde{\xi}_2, \tilde{\eta}_j\rangle &= -2\langle\partial_\alpha\phi, \phi\rangle && \textit{if } j = 1 \textit{ and } = 0 \textit{ else}, \\
\langle\tilde{\xi}_{2+\ell}, \tilde{\eta}_j\rangle &= -2\langle\phi, \phi\rangle && \textit{if } j = 5+\ell \textit{ and } = 0 \textit{ else}, \\
\langle\tilde{\xi}_{5+\ell}, \tilde{\eta}_j\rangle &= 2\langle\phi, \phi\rangle && \textit{if } j = 2+\ell \textit{ and } = 0 \textit{ else}.
\end{aligned}$$

Finally, with $\mathcal{H}(\pi(t))$ as in (22) we have the relations (with ∂_α acting on ϕ but not on θ):

(48)

$$i\partial_t\tilde{\xi}_1(t) - \mathcal{H}^*(\pi(t))\tilde{\xi}_1(t) = -i\left(\dot{\tilde{\gamma}}\tilde{\eta}_1 + \sum_{\ell=1}^3(\dot{D}_\ell\tilde{\eta}_{5+\ell} + \dot{v}_\ell\tilde{\eta}_{2+\ell}\right) - \dot{\alpha}\tilde{\eta}_2) =: i\dot{\tilde{\pi}}\mathcal{S}_1(t),$$

$$i\partial_t\tilde{\xi}_2(t) - \mathcal{H}^*(\pi(t))\tilde{\xi}_2(t) = 2i\alpha\tilde{\xi}_1 + \partial_\alpha\left(\dot{\tilde{\gamma}}\tilde{\eta}_1 + \sum_{\ell=1}^{3}(\dot{D}_\ell\tilde{\eta}_{5+\ell} + \dot{v}_\ell\tilde{\eta}_{2+\ell}) - \dot{\alpha}\tilde{\eta}_2\right)$$
$$=: 2i\alpha\tilde{\xi}_1 + i\dot{\tilde{\pi}}\mathcal{S}_2(t),$$

as well as for all $1 \le \ell \le 3$,

$$\begin{aligned}
&i\partial_t\tilde{\xi}_{2+\ell}(t) - \mathcal{H}^*(\pi(t))\tilde{\xi}_{2+\ell}(t)\\
&\quad = 2\tilde{\xi}_{5+\ell} + (x_\ell - y_\ell(t))\left(\dot{\tilde{\gamma}}\tilde{\eta}_1 + \sum_{k=1}^{3}(\dot{D}_k\tilde{\eta}_{5+k} + \dot{v}_k\tilde{\eta}_{2+k}) - \dot{\alpha}\tilde{\eta}_2\right)\\
&\quad =: 2\tilde{\xi}_{5+\ell} + i\dot{\tilde{\pi}}\mathcal{S}_{2+\ell}(t),\\
&i\partial_t\tilde{\xi}_{5+\ell}(t) - \mathcal{H}^*(\pi(t))\tilde{\xi}_{5+\ell}(t)\\
&\quad = \partial_x\left(\dot{\tilde{\gamma}}\tilde{\eta}_1 + \sum_{k=1}^{3}(\dot{D}_k\tilde{\eta}_{5+k} + \dot{v}_k\tilde{\eta}_{2+k}) - \dot{\alpha}\tilde{\eta}_2\right) - v_\ell\dot{\tilde{\gamma}}\tilde{\xi}_1\\
&\qquad - v_\ell\sum_{k=1}^{3}(\dot{D}_k\tilde{\xi}_{5+k} + \dot{v}_k\tilde{\xi}_{2+k}) + \dot{\alpha}v_\ell\tilde{\xi}_2 + i\dot{v}_\ell\tilde{\xi}_1 =: i\dot{\tilde{\pi}}\mathcal{S}_{5+\ell}(t),
\end{aligned}$$

where the matrices $\mathcal{S}_j$ *are defined via these relations.*

Proof. This can be read off from the definitions of $M(t)$, $\mathcal{G}_\infty(t)$. □

We now make two remarks. The first one concerns how to insure the orthogonality condition for the transformed solution $U(t)$ (which depends on some path) at time $t = 0$ by a condition which is path independent. The second one concerns the $\mathcal{J}$-invariance of eigenfunctions.

Remark 13. Let $R_0 \in L^2(\mathbb{R}^3)$ be such that $Z_0 = \binom{R_0}{\bar{R}_0}$ satisfies $Z_0 \in \mathcal{N}^{*\perp}$. According to Lemma (25) the transformed initial condition is

$$U(0) = M(0)\mathcal{G}_\infty(0)Z_0.$$

We claim that then $\langle U(0), \xi_j(0)\rangle = 0$ for all $1 \le j \le 8$, which is precisely the condition of the previous lemma. Here $\{\xi_j\}_{j=1}^8$ are the functions from Definition 8 defined relative to *any admissible path* π as long as it starts at $\pi(0) = (\alpha_0, 0, 0, 0)$ as required by Theorem 1 (this is of course no restriction, since the initial soliton in the theorem is as good as any other modulo a Galilei transform). We verify this claim for $\xi_1(0)$, the other cases being similar. First, since $\pi(0) = (\alpha_0, 0, 0, 0)$, one checks directly from the definitions that

$$\begin{aligned}
\xi_1(0) &= \begin{pmatrix} e^{-i(v_\infty\cdot(x+D_\infty)+\gamma_\infty)}\phi(\cdot + D_\infty, \alpha_0) \\ e^{i(v_\infty\cdot(x+D_\infty)+\gamma_\infty)}\phi(\cdot + D_\infty, \alpha_0)\end{pmatrix}\\
&= \mathcal{G}_\infty(0)\begin{pmatrix}\phi(\cdot, \alpha_0)\\ \phi(\cdot, \alpha_0)\end{pmatrix}.
\end{aligned}$$

Also since $M(0) = \mathrm{Id}$, it therefore follows that

$$\langle U(0), \xi_1(0)\rangle = \Big\langle \mathcal{G}_\infty(0)M(0)Z_0, \mathcal{G}_\infty(0)\begin{pmatrix}\phi(\cdot,\alpha_0)\\ \phi(\cdot,\alpha_0)\end{pmatrix}\Big\rangle = \Big\langle Z_0, \begin{pmatrix}\phi(\cdot,\alpha_0)\\ \phi(\cdot,\alpha_0)\end{pmatrix}\Big\rangle = 0,$$

by unitarity of $\mathcal{G}_\infty(0)$ and the assumption on Z_0.

Remark 14. By inspection, all root spaces in this section are $\mathcal{J}$-invariant. This is a general fact. Indeed, one checks easily that $J\mathcal{H}(\alpha)J = -\mathcal{H}(\alpha)$. Therefore, if $\mathcal{H}(\alpha)f = i\sigma f$ with $\sigma \in \mathbb{R}$, it follows that $\mathcal{H}(\alpha)Jf = -i\sigma Jf$ where as usual $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence

$$\mathcal{J}\ker(\mathcal{H} - i\sigma I) = \overline{J\ker(\mathcal{H} - i\sigma I)} = \ker(\mathcal{H} - i\sigma I)$$

for any $\sigma \in \mathbb{R}$. A similar argument shows that the root spaces at zero are also $\mathcal{J}$-invariant. In particular, one concludes from this that the Riesz projections $P_{\mathrm{s}}, P_{\mathrm{root}}, P_{\mathrm{im}}$ preserve the space of $\mathcal{J}$-invariant functions in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. This can also easily be seen directly: Let P be any Riesz projection corresponding to an eigenvalue of $\mathcal{H}(\alpha)$ on $i\mathbb{R}$, i.e.,

$$P = \frac{1}{2\pi i}\oint_\gamma (zI - \mathcal{H}(\alpha))^{-1}\,dz$$

where γ is a small positively oriented circle centered at that eigenvalue. Since $J\mathcal{H}(\alpha)J = -\mathcal{H}(\alpha)$, one concludes that

$$JPJ = \frac{1}{2\pi i}\oint_\gamma J(zI - \mathcal{H}(\alpha))^{-1}J\,dz = \frac{1}{2\pi i}\oint_\gamma (\mathcal{H}(\alpha) + zI)^{-1}\,dz.$$

Thus, if $F = \binom{F_1}{F_2}$, then $-\bar{\gamma} = \gamma$ (in the sense of oriented curves) implies that

$$\overline{JPF} = -\frac{1}{2\pi i}\oint_\gamma (\mathcal{H}(\alpha) + \bar{z}I)^{-1}\,d\bar{z}\,\overline{JF} = \frac{1}{2\pi i}\oint_\gamma (zI - \mathcal{H}(\alpha))^{-1}\,dz\,\overline{JF} = P\mathcal{J}F,$$

so that $\mathcal{J} \circ P = P \circ \mathcal{J}$, as claimed.

Another important issue related to $\mathcal{J}$-invariance is whether or not a solution $\pi(t)$ of the system of modulation equations is real-valued. This is of course crucial, and is indeed the case if $U(t)$ is $\mathcal{J}$-invariant for all $t \geq 0$ and if $\pi(0) \in \mathbb{R}^8$. However, we will not take up this issue here, but rather when we start solving the modulation equations by means of a contraction scheme; see Lemma 22 below.

4. The linearized problem and the discrete spectrum

In this section we describe the entire discrete spectrum of the linearized Hamiltonian obtained from the cubic nonlinear Schrödinger equation (1). Recall that the nonlinearity $|\psi|^{2\beta}\psi$ has two scalar elliptic operators associated with it, namely,

$$L_- := -\triangle + \alpha^2 - \phi(\cdot,\alpha)^{2\beta}, \quad L_+ := -\triangle + \alpha^2 - (2\beta+1)\phi(\cdot,\alpha)^{2\beta}$$

where $\phi(\cdot,\alpha)$ is a ground state of the equation

$$-\triangle\phi + \alpha^2\phi = \phi^{2\beta}.$$

The meaning of L_-, L_+ is that the linearized operator of the nonlinear Schrödinger equation

$$i\partial_t\psi + \triangle\psi + |\psi|^{2\beta}\psi = 0$$

takes the form

$$\begin{pmatrix} 0 & iL_- \\ -iL_+ & 0 \end{pmatrix}$$

provided the perturbation is written as $R = u + iv$ and this matrix acts on $\binom{u}{v}$ (in contrast, (40) acts on $\binom{R}{\bar{R}}$). We are interested in the range $\frac{2}{3} < \beta \leq 1$, which is supercritical. The restriction $\beta \leq 1$ has to do with Weinstein's work [49], where it is imposed. We recall that it is known that L_- has zero as lowest eigenvalue (with ϕ as ground state), whereas L_+ has a unique negative eigenvalue E_0, and a kernel spanned by $\partial_j\phi$, $1 \leq j \leq 3$. In one dimension, it is known that L_- and L_+ do not have eigenvalues in $(0,\alpha^2]$ and no resonance at the edge α^2; see [29]. These properties hold in the entire supercritical range. In dimension three it can be checked numerically that L_+ and L_- also do not have eigenvalues in $(0,\alpha^2]$ and no resonance at the edge α^2. This is accomplished by showing that the associated Birman-Schwinger kernels

$$K_-(x,y) := \frac{\phi(x)^\beta\phi(y)^\beta}{4\pi|x-y|} \quad \text{for the case of} \quad L_-,$$

$$K_+(x,y) := \frac{(2\beta+1)\phi(x)^\beta\phi(y)^\beta}{4\pi|x-y|} \quad \text{for the case of} \quad L_+,$$

have the corresponding number of eigenvalues in $(1-\varepsilon,\infty)$: One for K_- and four for K_+. The details of this work can be found in the paper of Demanet and the author [16]. In contrast to the one-dimensional case it is shown there that a restriction $\beta_* < \beta$ is needed where $\frac{2}{3} < \beta_* < 1$. Hence, the method of proof of this section does *not* apply to the entire super-critical range $\beta > 2/3$.

We start by showing that zero is the only point of the discrete spectrum of the matrix operator on the real axis. Since any such point would have to be an eigenvalue, we just need to show that zero is the only eigenvalue in the interval $(-\alpha^2,\alpha^2)$. The following lemma is somewhat stronger, since it proves this for the closed interval $[-\alpha^2,\alpha^2]$. The argument is an adaptation of Proposition 2.1.2 in Perelman [33]. It is based on the fact that L_+ does not have any eigenvalues in $(0,\alpha^2)$.

LEMMA 15. *The only eigenvalue of $\mathcal{H}(\alpha)$ in the interval $[-\alpha^2,\alpha^2]$ is zero.*

Proof. Suppose not. Then $\mathcal{H}(\alpha)^2$ has an eigenvalue $E \in (0, \alpha^4]$. For simplicity and without loss of generality, let us choose $\alpha = 1$. Then there is $\psi \in L^2(\mathbb{R}^3)$, $\psi \neq 0$, such that

$$L_- L_+ \psi = E\psi$$

with $0 < E \leq 1$. Clearly, $\psi \perp \phi$ and $\psi \in H^4_{\mathrm{loc}}(\mathbb{R}^3)$ by elliptic regularity. Define $A := PL_+P$ where P is the projection orthogonal to ϕ. Since

$$\ker(L_+) = \mathrm{span}\{\partial_j \phi \mid 1 \leq j \leq 3\},$$

and $\langle \phi, \partial_\alpha \phi \rangle \neq 0$, we conclude that

$$\ker(A) = \mathrm{span}\{\partial_j \phi,\ \phi \mid 1 \leq j \leq 3\}.$$

Moreover, let $E_0 < 0$ be the unique negative eigenvalue of L_+. Then consider (as before) the function

$$g(\lambda) := \langle (L_+ - \lambda)^{-1} \phi, \phi \rangle$$

which is differentiable on the interval $(E_0, 1)$ due to the orthogonality of ϕ to the kernel of L_+. Moreover,

$$g'(\lambda) = \langle (L_+ - \lambda)^{-2} \phi, \phi \rangle > 0,\ g(0) = -\frac{1}{2} \langle \phi, \partial_\alpha \phi \rangle > 0.$$

The final inequality here is due to the supercritical nature of our problem. Since also $g(\lambda) \to -\infty$ as $\lambda \to E_0$, it follows that $g(\lambda_1) = 0$ for some $E_0 < \lambda_1 < 0$. Moreover, this is the only zero of $g(\lambda)$ with $E_0 < \lambda < 1$. If we set

$$\eta := (L_+ - \lambda_1)^{-1} \phi,$$

then

$$A\eta = \lambda_1 \eta, \qquad \langle \eta, \phi \rangle = 0.$$

Conversely, if

$$Af = \lambda f$$

for some $-\infty < \lambda < 1$, $\lambda \neq 0$, and $f \in L^2(\mathbb{R}^3)$, then $f \perp \phi$ and

$$(PL_+P - \lambda) f = (A - \lambda) f = 0.$$

Since also

$$E_0 \langle f, f \rangle \leq \langle L_+ f, f \rangle = \lambda \langle f, f \rangle$$

it follows that $\lambda \geq E_0$. If $\lambda = E_0$, then f would necessarily have to be the ground state of L_+ and thus of definite sign. But then $\langle f, \phi \rangle \neq 0$, which is impossible. Hence $E_0 < \lambda < 1$. But then $g(\lambda) = 0$ implies that $\lambda = \lambda_1$ is unique. In summary, A has eigenvalues λ_1 and 0 in $(-\infty, 1)$, with λ_1 being a simple eigenvalue and 0 being an eigenvalue of multiplicity four. Now define

$$\mathcal{F} := \mathrm{span}\{\psi,\ \eta,\ \partial_j \phi,\ \phi \mid 1 \leq j \leq 3\}.$$

We claim that

$$\dim(\mathcal{F}) = 6. \tag{49}$$

Since ϕ is perpendicular to the other functions, it suffices to show that

$$c_1\psi + c_2\eta + \sum_{j=3}^{5} c_j\partial_j\phi = 0$$

can only be the trivial linear combination. Apply L_+. Then

$$c_1 L_+\psi + c_2 L_+\eta = 0$$

and therefore

$$\begin{aligned} c_1\langle L_+\psi, \psi\rangle + c_2\langle L_+\eta, \psi\rangle &= 0, \\ c_1\langle L_+\psi, \eta\rangle + c_2\langle L_+\eta, \eta\rangle &= 0. \end{aligned}$$

This is the same as

$$\begin{aligned} c_1 E\langle L_-^{-1}\psi, \psi\rangle + c_2\lambda_1\langle \eta, \psi\rangle &= 0, \\ c_1\lambda_1\langle \psi, \eta\rangle + c_2\lambda_1\langle \eta, \eta\rangle &= 0. \end{aligned}$$

The determinant of this system is

$$E\lambda_1\langle L_-^{-1}\psi, \psi\rangle\langle \eta, \eta\rangle - \lambda_1^2|\langle \eta, \psi\rangle|^2 < 0.$$

Hence $c_1 = c_2 = 0$ and therefore also $c_3 = c_4 = c_5 = 0$, as desired. Thus, (49) holds. Finally, we claim that

$$\sup_{\|f\|_2=1,\, f\in\mathcal{F}} \langle Af, f\rangle < 1. \tag{50}$$

If this is true, then by the min-max principle and (49) we would obtain that the number of eigenvalues of A in the interval $(-\infty, 1)$ (counted with multiplicity) would have to be at least six. On the other hand, we showed before that this number is exactly five, leading to a contradiction. Hence, the lemma will follow once we verify (50). Since $\langle PL_-^{-1}Pf, f\rangle < \langle f, f\rangle$ for all $f \neq 0$, and since $E \leq 1$ by assumption, this in turn follows from the stronger claim that

$$\langle Af, f\rangle \leq E\langle PL_-^{-1}Pf, f\rangle \tag{51}$$

for all $f = a\psi + b\phi + \vec{c}\cdot\nabla\phi + d\eta$. Clearly, we can take $b = 0$. Then the left-hand side of (51) is equal to

(52)
$$\begin{aligned} &\langle L_+(a\psi), a\psi + \vec{c}\cdot\nabla\phi + d\eta\rangle + \langle L_+(\vec{c}\cdot\nabla\phi + d\eta), a\psi + \vec{c}\cdot\nabla\phi + d\eta\rangle \\ &= E\langle L_-^{-1}(a\psi), a\psi + \vec{c}\cdot\nabla\phi + d\eta\rangle + E\langle \vec{c}\cdot\nabla\phi + d\eta, L_-^{-1}(a\psi)\rangle + \langle L_+(d\eta), d\eta\rangle \\ &= E\langle L_-^{-1}(a\psi), a\psi + \vec{c}\cdot\nabla\phi + d\eta\rangle + E\langle \vec{c}\cdot\nabla\phi + d\eta, L_-^{-1}(a\psi)\rangle + \lambda_1\|d\eta\|_2^2, \end{aligned}$$

whereas the right-hand side of (51) is

$$(53)\qquad = E\langle L_-^{-1}(a\psi), a\psi + \vec{c}\cdot\nabla\phi + d\eta\rangle \\ + E\langle \vec{c}\cdot\nabla\phi + d\eta, L_-^{-1}(a\psi)\rangle + E\langle L_-^{-1}(\vec{c}\cdot\nabla\phi + d\eta), \vec{c}\cdot\nabla\phi + d\eta\rangle.$$

Since

$$\lambda_1\|d\eta\|_2^2 \le 0, \quad E\langle L_-^{-1}(\vec{c}\cdot\nabla\phi + d\eta), \vec{c}\cdot\nabla\phi + d\eta\rangle \ge 0,$$

we see that (53) does indeed dominate (52), and (51) follows. □

Next, we turn to the issue of resonances of $\mathcal{H}(\alpha)$ at the edges of the essential spectrum. A "resonance" at α^2 (or $-\alpha^2$) here refers to the existence of a solution f to $\mathcal{H}(\alpha)f = \alpha^2 f$ (or $= -\alpha^2 f$) so that $f \notin L^2(\mathbb{R}^3)$, but such that

$$(54)\qquad \int_{\mathbb{R}^3} |f(x)|^2(1+|x|)^{-2\gamma}\,dx < \infty$$

for all $\gamma > \frac{1}{2}$. If $\pm\alpha^2$ are neither resonances nor eigenvalues (we have already excluded the latter), then $(\mathcal{H}(\alpha) \mp \alpha^2)^{-1}$ is bounded on suitable weighted L^2 spaces; see [17]. This will be important in order to establish the dispersive estimates for $e^{it\mathcal{H}(\alpha)}$. The proof of the following lemma is similar to that of Lemma 15, and is an adaptation of the argument in Appendix 1 of Perelman's paper [33]. It shows that if the *scalar* operator L_- does not have a resonance at α^2 (the edge of its continuous spectrum), then the *matrix* operator $\mathcal{H}(\alpha)$ does not have a resonance at $\pm\alpha^2$. As already mentioned, one can verify numerically that L_- has this property; see [16].

LEMMA 16. *Suppose that L_- has neither an eigenvalue nor a resonance at α^2. Then the edges $\pm\alpha^2$ are not resonances of $\mathcal{H}(\alpha)$, i.e., there do not exist solutions f of $\mathcal{H}(\alpha)f = \pm\alpha^2 f$ which satisfy* (54) *but are not in L^2.*

Proof. We again set $\alpha = 1$. By symmetry, it suffices to consider the right edge α^2. Suppose then there is such a solution f with $\mathcal{H}(\alpha)f = f$. Write $f = \binom{\psi}{\tilde\psi}$. Then $iL_-\tilde\psi = \psi$ and $-iL_+\psi = \tilde\psi$. In particular, $L_-L_+\psi = \psi$ and

$$\int_{\mathbb{R}^3} |\psi(x)|^2\,dx = \infty, \quad \int_{\mathbb{R}^3} |\psi(x)|^2(1+|x|)^{-2\gamma}\,dx < \infty$$

for all $\gamma > \frac{1}{2}$. Clearly, $\langle\psi,\phi\rangle = 0$, the latter inner product being well-defined because of the rapid decay of ϕ and (54). Furthermore, $\psi \in H^4_{\rm loc}(\mathbb{R}^3)$ by elliptic regularity. Pick a smooth cut-off $\chi \ge 0$ which is a constant $= 1$ around zero, and compactly supported. Define for any $0 < \varepsilon < 1$

$$\psi^\varepsilon := \psi\chi(\varepsilon\cdot) + \mu(\varepsilon)\phi, \quad \mu(\varepsilon) := -\frac{\langle\psi\chi(\varepsilon\cdot),\phi\rangle}{\langle\phi,\phi\rangle}.$$

Clearly, $\langle \psi^\varepsilon, \phi \rangle = 0$ and $|\mu(\varepsilon)| = o(1)$ as $\varepsilon \to 0$ (in fact, like $e^{-C/\varepsilon}$). It follows that

$$\|\psi^\varepsilon\|_2^2 = M_0(\varepsilon) + o(1), \quad M_0(\varepsilon) := \int_{\mathbb{R}^3} |\psi(x)|^2 \chi(\varepsilon x)^2 \, dx$$

with $M_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$. We now claim that

$$\langle L_+ \psi^\varepsilon, \psi^\varepsilon \rangle = \|\psi^\varepsilon\|_2^2 + \langle (L_+ - 1)\psi, \psi \rangle + o(1) \tag{55}$$

as $\varepsilon \to 0$. We will need to justify that

$$M := \langle (L_+ - 1)\psi, \psi \rangle$$

is a finite expression. We first show that this justification, as well as (55) can be reduced to showing that $\nabla\psi \in L^2(\mathbb{R}^3)$. Write $L_- = -\triangle + 1 + V_1$ and $L_+ = -\triangle + 1 + V_2$, with Schwartz functions V_1, V_2 (they are of course explicitly given in terms of ϕ, but we are not going to use that now). We start from the evident expression

$$\langle L_+ \psi^\varepsilon, \psi^\varepsilon \rangle = \|\psi^\varepsilon\|_2^2 + \langle (L_+ - 1)\psi^\varepsilon, \psi^\varepsilon \rangle = \|\psi^\varepsilon\|_2^2 + \langle (-\triangle + V_2)\psi^\varepsilon, \psi^\varepsilon \rangle.$$

By the rapid decay of V_2 and (54),

$$\langle (-\triangle + V_2)\psi^\varepsilon, \psi^\varepsilon \rangle = \int_{\mathbb{R}^3} |\nabla \psi^\varepsilon(x)|^2 \, dx + \int_{\mathbb{R}^3} V_2(x) |\psi(x)|^2 \, dx + o(1).$$

Assuming $\nabla\psi \in L^2$, we calculate further that

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla \psi^\varepsilon(x)|^2 \, dx &= \int_{\mathbb{R}^3} \Big| \nabla\psi(x)\chi(\varepsilon x) + \varepsilon\psi(x)\nabla\chi(\varepsilon x) \Big|^2 \, dx \\
&= \int_{\mathbb{R}^3} |\nabla\psi(x)|^2 \, dx + \int_{\mathbb{R}^3} |\nabla\psi(x)|^2 (\chi(\varepsilon x)^2 - 1) \, dx \\
&\quad + 2\varepsilon \int_{\mathbb{R}^3} \psi(x)\chi(\varepsilon x)\nabla\psi(x) \cdot \nabla\chi(\varepsilon x) \, dx \\
&\quad + \varepsilon^2 \int_{\mathbb{R}^3} \psi(x)^2 |\nabla\chi(\varepsilon x)|^2 \, dx \\
&= \int_{\mathbb{R}^3} |\nabla\psi(x)|^2 \, dx + o(1).
\end{aligned}$$

To pass to the last line, estimate the error terms using $\nabla\psi \in L^2$ and (54) (any $\gamma < 1$ works here). This proves (55) provided we interpret $\langle (L_+ - 1)\psi, \psi \rangle$ as

$$\int_{\mathbb{R}^3} [|\nabla\psi(x)|^2 + V_2(x)|\psi(x)|^2] \, dx.$$

To prove $\nabla\psi \in L^2$, we start from the definition, i.e.,

$$(-\triangle + 1 + V_1)(-\triangle + 1 + V_2)\psi = \psi$$

which can be written as

$$(\triangle^2 - 2\triangle)\psi + (-\triangle + 1)V_2\psi + V_1(-\triangle + 1)\psi + V_1 V_2 \psi = 0. \tag{56}$$

At least formally, integrating by parts against ψ yields that

$$\|\triangle\psi\|_2^2+2\|\nabla\psi\|_2^2\le\int(|V_1|+|V_2|+|V_1V_2|)|\psi(x)|^2\,dx+\|V_2\psi\|_2^2+\|V_1\psi\|_2^2+\frac{1}{2}\|\triangle\psi\|_2^2,$$

and thus, in particular, $\nabla\psi\in L^2$. To make this precise, we write the defining equations as $-iL_+\psi=\tilde{\psi}$ and $iL_-\tilde{\psi}=\psi$. Then

$$-\triangle\psi=i\tilde{\psi}-\psi-V_2\psi$$

which implies that (via (54))

$$\int_{\mathbb{R}^3}|\triangle\psi(x)|^2\langle x\rangle^{-2\gamma}\,dx<\infty\qquad\forall\,\gamma>\frac{1}{2}. \tag{57}$$

It is now a simple matter to deduce from this and (54) that

$$\int_{\mathbb{R}^3}|\nabla\psi(x)|^2\langle x\rangle^{-2\gamma}\,dx<\infty\qquad\forall\,\gamma>\frac{1}{2}. \tag{58}$$

This can be done in various ways. For example, it follow from Gauss' integral theorem applied to $\mathrm{div}(u\nabla u\langle x\rangle^{-2\gamma})$ that for all $u\in H^2$ with compact support

$$\int_{\mathbb{R}^3}|\nabla u(x)|^2\langle x\rangle^{-2\gamma}\,dx\le C\int_{\mathbb{R}^3}(|u(x)|^2+|\triangle u(x)|^2)\langle x\rangle^{-2\gamma}\,dx.$$

Setting $u=\psi^\varepsilon$, and letting $\varepsilon\to0$ yields the desired inequality (58) for ∇u. To make our heuristic argument leading to $\nabla\psi\in L^2$ precise, we pair (56) with $\chi(\varepsilon x)^4\psi$ and integrate by parts. This yields

$$\begin{aligned}&\langle\triangle\psi,\triangle(\chi(\varepsilon\cdot)^4\psi)\rangle+2\langle\nabla\psi,\nabla(\chi(\varepsilon\cdot)^4\psi)\rangle\\&=-\langle V_2\psi,(-\triangle+1)\chi(\varepsilon\cdot)^4\psi\rangle-\langle(-\triangle+1)\psi,V_1\chi(\varepsilon\cdot)^4\psi\rangle-\langle V_1V_2\psi,\chi(\varepsilon\cdot)^4\psi\rangle.\end{aligned}$$

The terms on the right-hand side are uniformly bounded as $\varepsilon\to0$ due to (54), (57) and (58) and the rapid decay of V_1,V_2 and their derivatives. The second term on the left-hand side satisfies

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla\psi(x)(4\varepsilon\nabla\chi(\varepsilon x)\chi(\varepsilon x)^3\psi(x)+\chi(\varepsilon x)^4\nabla\psi(x))\,dx\\&\quad\ge\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\psi(x)|^2\chi(\varepsilon x)^4\,dx-C\varepsilon^2\int_{\mathbb{R}^3}|\psi(x)|^2|\nabla\chi(\varepsilon x)|^2\chi(\varepsilon x)^2\,dx\\&\quad\ge\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\psi(x)|^2\chi(\varepsilon x)^4\,dx-O(1)\end{aligned}$$

by (54). Similarly,

$$\begin{aligned}&\langle\triangle\psi,\triangle(\chi(\varepsilon\cdot)^4\psi)\rangle\\&\quad=\int_{\mathbb{R}^3}\triangle\psi(x)\Big((4\varepsilon^2\triangle\chi(\varepsilon x)\chi(\varepsilon x)^3+12\varepsilon^2|\nabla\chi(\varepsilon x)|^2\chi(\varepsilon x)^2)\psi(x)\\&\qquad+8\varepsilon\nabla\chi(\varepsilon x)\cdot\nabla\psi(x)\chi(\varepsilon x)^3+\chi(\varepsilon x)^4\triangle\psi(x)\Big)\,dx\end{aligned}$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^3} |\triangle\psi(x)|^2\chi(\varepsilon x)^4\,dx - C\varepsilon^2\int_{\mathbb{R}^3} |\nabla\psi(x)|^2\chi(\varepsilon x)^2\,dx$$
$$- C\varepsilon^4\int_{\mathbb{R}^3} |\psi(x)|^2[|\nabla\chi(\varepsilon x)|^4 + \chi(\varepsilon x)^2]\,dx$$
$$\geq \frac{1}{2}\int_{\mathbb{R}^3} |\triangle\psi(x)|^2\chi(\varepsilon x)^4\,dx - O(1)$$

by (54) and (58). Combining these estimates and invoking the monotone convergence theorem yields $\triangle\psi \in L^2$ and $\nabla\psi \in L^2$. It is easy to see that the previous argument allows a better conclusion than L^2, namely that $\langle x\rangle^b\triangle\psi \in L^2$ for any $b < \frac{1}{2}$ and similarly for $\nabla\psi$. In fact, a much stronger conclusion is possible for $\triangle\psi$: recall that $g_1 := \psi + i\tilde{\psi}$ and $g_2 := \psi - i\tilde{\psi}$ satisfy

$$\begin{pmatrix} -\triangle + 1 + W_1 & W_2 \\ -W_2 & \triangle - 1 - W_1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

where W_1, W_2 are again exponentially decaying potentials. This implies that

$$\triangle g_1 = W_1 g_1 + W_2 g_2 \quad \text{and} \quad \triangle g_2 = 2g_2 + W_1 g_2 + W_2 g_1.$$

Hence $\langle x\rangle^b\triangle g_1 \in L^2$ for all $b > 0$. Similarly,

$$g_2 = (\triangle - 2)^{-1}[W_1 g_2 + W_2 g_1]$$

is exponentially decaying, which implies that $\langle x\rangle^b\triangle g_2 \in L^2$ for all $b > 0$. Consequently, the same estimate holds for $\triangle\psi$ as well as for $f := (L_+ - 1)\psi$. Hence $\langle (L_+ - 1)\psi, \psi\rangle = \langle f, \psi\rangle$ is well-defined as a usual scalar product. Moreover, one has

$$L_- f = -(L_- - 1)\psi \;\text{ or }\; \psi = -(L_- - 1)^{-1}L_- f = -f - (L_- - 1)^{-1}f.$$

We conclude that

$$\langle f + \psi, f\rangle = -\langle (L_- - 1)^{-1}f, f\rangle < 0, \tag{59}$$

where the final inequality follows from $L_- \geq 1$ on $\{\phi\}^\perp$, as well as our assumption that L_- has neither an eigenvalue nor a resonance at $\alpha^2 = 1$. Recall that this insures that for any $\tau > 0$

$$\|\langle x\rangle^{-1-\tau}(L_- - 1)^{-1}h\|_2 \leq C\|\langle x\rangle^{1+\tau}h\|_2$$

for all h for which the right-hand side is finite, in particular for $h = f$. The inequality (59) will play a crucial role in estimating a quadratic form as in Lemma 15. To see this, let

$$\mathcal{F}_\varepsilon := \operatorname{span}\{\psi^\varepsilon, \partial_1\phi, \partial_2\phi, \partial_3\phi, \eta, \phi\}.$$

As in the proof of Lemma 15 one shows that $\dim \mathcal{F}_\varepsilon = 6$, as least if $\varepsilon > 0$ is sufficiently small (use that $\langle L_+\psi^\varepsilon, \psi^\varepsilon\rangle \to \infty$ as $\varepsilon \to 0$). It remains to show that for small $\varepsilon > 0$

$$\max_{f\in\mathcal{F}_\varepsilon} \frac{\langle PL_+Pf, f\rangle}{\langle f, f\rangle} < 1 \tag{60}$$

where P is the projection orthogonal to ϕ. If so, then this would imply that $A = PL_+P$ has at least six eigenvalues (with multiplicity) in $(-\infty, 1)$. However, we have shown in the proof of Lemma 15 that there are exactly five such eigenvalues. To prove (60), it suffices to consider the case $f \perp \phi$. Compute

$$\begin{aligned}
&\frac{\langle L_+(a\psi^\varepsilon + \underline{v}\cdot\nabla\phi + d\eta), a\psi^\varepsilon + \underline{v}\cdot\nabla\phi + d\eta\rangle}{\|a\psi^\varepsilon + \underline{v}\cdot\nabla\phi + d\eta\|^2} \\
&= \frac{|a|^2(\|\psi^\varepsilon\|_2^2 + M + o(1)) + 2\Re\lambda_1\langle a\psi^\varepsilon, d\eta\rangle + \lambda_1\|d\eta\|_2^2}{|a|^2\|\psi^\varepsilon\|_2^2 + 2\Re\langle a\psi^\varepsilon, \underline{v}\cdot\nabla\phi + d\eta\rangle + \|\underline{v}\cdot\nabla\phi + d\eta\|_2^2} \\
&\le \max_{x\in\mathbb{C}^5} \frac{|x_1|^2(1 + \delta^2 M + o(\delta^2)) + 2\delta\lambda_1\Re\langle x_1\psi^\varepsilon, x_5 e_4\rangle + \lambda_1|x_5|^2}{|x_1|^2 + 2\delta\Re\langle x_1\psi^\varepsilon, x_2e_1 + x_3e_2 + x_4e_3 + x_5e_4\rangle + \|x_2e_1 + x_3e_2 + x_4e_3 + x_5e_4\|_2^2}
\end{aligned}$$

where we have set $\delta^2 := \|\psi^\varepsilon\|_2^2$ and

$$e_1 = \frac{\partial_1\phi}{\|\partial_1\phi\|_2}, \quad e_2 = \frac{\partial_2\phi}{\|\partial_2\phi\|_2}, \quad e_3 = \frac{\partial_3\phi}{\|\partial_3\phi\|_2}, \quad e_4 = \frac{\eta}{\|\eta\|_2}.$$

Note that η is a radial function, since it is given by $(L_+ - \lambda_1)^{-1}\phi$ and both ϕ and the kernel of $(L_+ - 1)^{-1}$ are radial. Hence $e_j \perp e_4$ for $1 \le j \le 3$. Set

$$b_j^\varepsilon := \langle \psi^\varepsilon, e_j\rangle \quad \text{for} \quad 1 \le j \le 4.$$

Then $b_j^\varepsilon \to b_j^0 := \langle \psi, e_j\rangle$ as $\varepsilon \to 0$ by the exponential decay of the e_j. Let $B^\varepsilon, C^\varepsilon$ (which depend on ε) be 5×5 Hermitian matrices so that

$$C_{11}^\varepsilon := 1 + \delta^2 M + o(\delta^2),\ C_{15}^\varepsilon = C_{51}^\varepsilon := \lambda_1\delta b_4^\varepsilon,\ C_{55}^\varepsilon := \lambda_1$$

and $C_{ij}^\varepsilon = 0$ else,

$$B_{1j}^\varepsilon = B_{j1}^\varepsilon := \delta b_{j-1}^\varepsilon \quad \text{for} \quad 2 \le j \le 5$$

and $B_{ij}^\varepsilon = 0$ else. In view of the preceding,

$$\max_{f\in\mathcal{F}_\varepsilon} \frac{\langle PL_+Pf, f\rangle}{\langle f, f\rangle} \le \max_{x\in\mathbb{C}^5} \frac{\langle Cx, x\rangle}{\langle (I+B)x, x\rangle}.$$

Clearly, the right-hand side equals the largest eigenvalue of the Hermitian matrix

$$(I+B^\varepsilon)^{-\frac12}C^\varepsilon(I+B^\varepsilon)^{-\frac12} = C - \frac12(BC+CB) + \frac38(B^2C+CB^2) + \frac14 BCB + O(\delta^3),$$

where we have dropped the ε in the notation on the right-hand side. With some patience one can check that the right-hand side equals the matrix D which is given by (dropping ε from the notation)

$$\begin{bmatrix}
1+\delta^2 M_1 & -\frac{\delta}{2}b_1 & -\frac{\delta}{2}b_2 & -\frac{\delta}{2}b_3 & \frac{\delta}{2}(\lambda_1-1)b_4 \\
-\frac{\delta}{2}b_1 & \frac{\delta^2}{4}b_1^2 & \frac{\delta^2}{4}b_1b_2 & \frac{\delta^2}{4}b_1b_3 & \frac{\delta^2}{4}(1-\frac12\lambda_1)b_1b_4 \\
-\frac{\delta}{2}b_2 & \frac{\delta^2}{4}b_1b_2 & \frac{\delta^2}{4}b_2^2 & \frac{\delta^2}{4}b_2b_3 & \frac{\delta^2}{4}(1-\frac12\lambda_1)b_2b_4 \\
-\frac{\delta}{2}b_3 & \frac{\delta^2}{4}b_1b_3 & \frac{\delta^2}{4}b_2b_3 & \frac{\delta^2}{4}b_3^2 & \frac{\delta^2}{4}(1-\frac12\lambda_1)b_3b_4 \\
\frac{\delta}{2}(\lambda_1-1)b_4 & \frac{\delta^2}{4}(1-\frac12\lambda_1)b_1b_4 & \frac{\delta^2}{4}(1-\frac12\lambda_1)b_2b_4 & \frac{\delta^2}{4}(1-\frac12\lambda_1)b_3b_4 & \lambda_1+\frac{\delta^2}{4}(1-\lambda_1)b_4^2
\end{bmatrix} + o(\delta^2)$$

where $M_1 := M - \frac{3}{4}\lambda_1 b_4^2 + \frac{3}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2)$. When $\delta = 0$, this matrix has eigenvalues 1, 0, $\lambda_1 < 0$, and 0 has multiplicity three. When $\delta \neq 0$ but very small, the largest eigenvalue will be close to one, of the form $1+x$ with x small. We need to see that $x < 0$. Collecting powers[4] of x in $\det(D - (1+x)I)$ we arrive at the condition

$$\begin{aligned}(1-\lambda_1)x &= \delta^2[M(1-\lambda_1) + (b_1^2 + b_2^2 + b_3^2)(1-\lambda_1) + b_4^2(1-\lambda_1)^2] + o(\delta^2)\\ &= \delta^2[M(1-\lambda_1) + ((b_1^0)^2 + (b_2^0)^2 + (b_3^0)^2)(1-\lambda_1) + (b_4^0)^2(1-\lambda_1)^2] + o(\delta^2).\end{aligned}$$

Now,

$$b_j^0 = \langle \psi, e_j \rangle = -\langle (L_+ - 1)\psi, e_j \rangle = -\langle f, e_j \rangle \quad \text{for} \quad 1 \le j \le 3.$$

On the other hand,

$$b_4^0 = \langle \psi, e_4 \rangle = -\langle f, e_4 \rangle + \langle \psi, L_+ e_4 \rangle = -\langle f, e_4 \rangle + \lambda_1 b_4^0$$

and thus,

$$b_4^0 = -(1-\lambda_1)^{-1}\langle f, e_4 \rangle.$$

Since $\lambda_1 < 0$ in the supercritical case, we obtain that

$$\begin{aligned}(1-\lambda_1)x &\le (1-\lambda_1)\delta^2[M + \sum_{j=1}^4 \langle f, e_j \rangle^2] + o(\delta^2) \le (1-\lambda_1)\delta^2[M + \langle f, f \rangle] + o(\delta^2)\\ &= (1-\lambda_1)\delta^2\langle f + \psi, f \rangle + o(\delta^2) = -(1-\lambda_1)\delta^2\langle (L_- - 1)^{-1} f, f \rangle + o(\delta^2)\end{aligned}$$

which yields that $x < 0$ for δ small. But $\varepsilon > 0$ small implies that δ is small and we are done. □

In the subcritical case the proof of Lemma 15 breaks down. In fact, the statement is false: there has to be at least one pair $\pm\lambda$ of real eigenvalues with $0 < \lambda < \alpha^2$ in the subcritical case. It is reasonable to expect that there should be exactly one such pair, but we do not address that here.

LEMMA 17. *The discrete spectrum of the linearized operator* $\mathcal{H}(\alpha)$ *consists of zero and a single pair of imaginary eigenvalues* $\pm i\sigma$, $\sigma > 0$, *each of which is simple. Moreover,*

$$\mathcal{H}(\alpha) f^{\pm}(\alpha) = \pm i\sigma(\alpha) f^{\pm}(\alpha)$$

where $f^{\pm}(\alpha)$ *are exponentially decreasing,* C^∞ *functions with* $\|f^{\pm}(\alpha)\|_2 = 1$.

Proof. It was shown above that there is no other real discrete spectrum than zero. As for the existence of an imaginary discrete spectrum, which then necessarily has to consist of eigenvalues, we refer the reader to the works of Grillakis, Shatah, Strauss [24], and Grillakis [22], [23]. The general results

[4] this was done by means of Maple

of these papers imply that there is exactly one such pair $\pm i\sigma$ counted with multiplicity for the cubic NLS. In fact, it is relatively easy to see by means of variational arguments that there needs to be at least one such pair – such arguments hinge on the orbital instability condition $\partial_\alpha \|\phi(\cdot,\alpha)\|_2 < 0$ which is of course satisfied for all focusing nonlinearities $|\psi|^\beta \psi$ with $\beta > \frac{2}{3}$. Then one observes that the imaginary spectrum needs to move continuously into the root-space as $\beta \to \frac{2}{3}+$. Since the latter was shown by M. Weinstein to have dimension exactly eight for the range $\frac{2}{3} < \beta \le 1$, and exactly ten for the critical case $\beta = \frac{2}{3}$ it follows that there has to be exactly one such pair $\pm i\sigma(\beta)$ for each $1 \ge \beta > \frac{2}{3}$ as claimed. The symmetry $\pm i\sigma$ is a well-known consequence of the commutation properties of $\mathcal{H}(\alpha)$ with the Pauli matrices. That the associated eigenfunctions decay exponentially was proved by Hundertmark and Lee [27] by a variation of Agmon's technique [2], whereas the smoothness follows by elliptic regularity since the potential is smooth. □

We now present a simple continuity statement which will be important in the following two sections.

Corollary 18. *We can choose the $f^\pm(\alpha)$ in the previous lemma to be $\mathcal{J}$-invariant, i.e., $\mathcal{J} f^\pm(\alpha) = f^\pm(\alpha)$. Since $\|f^\pm(\alpha)\|_2 = 1$, they are therefore unique up to a sign. Choose this sign consistently, i.e., so that $f^\pm(\alpha)$ varies continuously with α. In that case there is the bound*

$$|\sigma(\alpha_1) - \sigma(\alpha_2)| + \|f^\pm(\alpha_1) - f^\pm(\alpha_2)\|_2 \le C(\alpha_1)|\alpha_1 - \alpha_2| \tag{61}$$

for all $\alpha_1, \alpha_2 > 0$ which are sufficiently close. Let $P^\pm_{\mathrm{im}}(\alpha)$ denote the Riesz projection onto $f^\pm(\alpha)$, respectively. Then one has, relative to the operator norm on $L^2 \times L^2$,

$$\|P^\pm_{\mathrm{im}}(\alpha_1) - P^\pm_{\mathrm{im}}(\alpha_2)\| \le C(\alpha_1)|\alpha_1 - \alpha_2| \tag{62}$$

for all α_1, α_2 as above. Moreover, the Riesz projections admit the explicit representation

$$P^\pm_{\mathrm{im}}(\alpha) = f^\pm(\alpha)\langle \cdot, \tilde{f}^\pm(\alpha)\rangle, \tag{63}$$

where $\mathcal{H}(\alpha)^* \tilde{f}^\pm(\alpha) = \mp i\sigma \tilde{f}^\pm(\alpha)$, and $\|\tilde{f}^\pm(\alpha)\|_2 = 1$.

Proof. By Remark 14, $\ker(\mathcal{H}(\alpha) \mp i\sigma)$ is $\mathcal{J}$-invariant. Thus, $\mathcal{J} f^\pm(\alpha) = \lambda f^\pm(\alpha)$ for some $\lambda \in \mathbb{C}$. It is easy to see that this requires that $|\lambda|^2 = 1$. Let $e^{2i\beta} = \lambda$. It follows that $\mathcal{J}(e^{i\beta} f^\pm(\alpha)) = e^{i\beta} f^\pm(\alpha)$, leading to our choice of the $\mathcal{J}$-invariant eigenfunction. Using the well-known fact that

$$\ker[\mathcal{H}(\alpha) \mp i\sigma(\alpha)] = \ker[(\mathcal{H}(\alpha) \mp i\sigma(\alpha))^2],$$

see [6] and [17], one easily obtains (by means of the Riesz projections) that

$$\|(\mathcal{H}(\alpha) - z)^{-1}\| \lesssim |z \mp i\sigma(\alpha)|^{-1} \quad \text{provided} \quad |z \mp i\sigma(\alpha)| < r_0(\alpha).$$

In conjunction with the resolvent identity, this yields

$$|\sigma(\alpha_1) - \sigma(\alpha_2)| \le C(\alpha_1)|\alpha_1 - \alpha_2|,$$

as well as (62). However, the latter clearly implies the remaining bound in (61). Finally, by the Riesz representation theorem, we necessarily have that (63) holds with some choice of $\tilde f^{\pm}(\alpha) \in L^2 \times L^2$. Since $P^{\pm}_{\mathrm{im}}(\alpha)^2 = P^{\pm}_{\mathrm{im}}(\alpha)$, one checks that

$$P^{\pm}_{\mathrm{im}}(\alpha)^* \tilde f^{\pm}(\alpha) = \tilde f^{\pm}(\alpha).$$

However, writing down $P^{\pm}_{\mathrm{im}}(\alpha)$ explicitly shows that

$$\begin{aligned} P^{+}_{\mathrm{im}}(\alpha)^* &= \Big(\frac{1}{2\pi i}\oint_\gamma (-\mathcal{H}(\alpha) + zI)^{-1}\,dz\Big)^* = -\frac{1}{2\pi i}\oint_\gamma (-\mathcal{H}(\alpha)^* + \bar z I)^{-1}\,d\bar z \\ &= \frac{1}{2\pi i}\oint_{-\bar\gamma} (-\mathcal{H}(\alpha)^* + zI)^{-1}\,dz \end{aligned}$$

which is equal to the Riesz projection corresponding to the eigenvalue $-i\sigma$ of $\mathcal{H}(\alpha)^*$. Here γ is a small, positively oriented, circle around $i\sigma$. A similar calculation applies to $P^{-}_{\mathrm{im}}(\alpha)$. Hence $\mathcal{H}(\alpha)^* \tilde f^{\pm}(\alpha) = \mp i\sigma(\alpha)\tilde f^{\pm}(\alpha)$, as claimed. In view of (63),

$$\|\tilde f^{\pm}(\alpha)\|_2^2 = \|P^{+}_{\mathrm{im}}(\alpha)\tilde f^{\pm}(\alpha)\|_2 \le \|\tilde f^{\pm}(\alpha)\|_2.$$

which implies that $\|\tilde f^{\pm}(\alpha)\|_2 \le 1$. On the other hand,

$$1 = \|f^{\pm}(\alpha)\|_2 = \|P^{+}_{\mathrm{im}}(\alpha) f^{\pm}(\alpha)\|_2 \le \|f^{\pm}(\alpha)\|_2 \|\tilde f^{\pm}(\alpha)\|_2 = \|\tilde f^{\pm}(\alpha)\|_2,$$

and we are done. □

5. The contraction scheme: Part I

We now set up the contraction map that will lead to a proof of Theorem 1. According to Lemmas 5 and 6, in order to solve the cubic nonlinear Schrödinger equation (1) with $\psi(t) = W(t) + R(t)$, we need to find an admissible path $\pi(t)$ as well as a function

$$Z \in C([0,\infty), H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)) \cap C^1([0,\infty), H^{-1}(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3))$$

so that $Z(t)$ is $\mathcal{J}$-invariant and such that $(\pi(t), Z(t))$ together satisfy (21). As initial conditions we will choose, with R_0 satisfying (10) and (11) as well as with some $\alpha = \alpha(R_0)$,

$$\pi(0) := (\alpha_0, 0, 0, 0), \qquad Z(0) := \begin{pmatrix} R_0 \\ \bar R_0 \end{pmatrix} + h f^+(\alpha) + \sum_{j=1}^{8} a_j \eta_j(\alpha) \tag{64}$$

where $h \in \mathbb{R}$, $f^+(\alpha)$ is an eigenvector of $\mathcal{H}(\alpha)$ with eigenvalue $i\sigma$, $a_j \in \mathbb{R}$, and $\mathcal{N}(\alpha) = \{\eta_j(\alpha)\}_{j=1}^{8}$ is the rootspace of $\mathcal{H}(\alpha)^*$. The contraction argument will be set in the following space. The parameter $\alpha_0 > 0$ is fixed.

Definition 19. Let $q > 2$ be large and fixed. For any sufficiently small $\delta > 0$ define

$$X_\delta := \Big\{(\pi, Z) \in \mathrm{Lip}([0,\infty), \mathbb{R}^8) \\ \times \big[L^\infty((0,\infty),(H^1(\mathbb{R}^3))^2) \cap L^\infty_{\mathrm{loc}}((0,\infty),(Y_q(\mathbb{R}^3))^2)\big] \;\big|\; \\ \text{conditions (65)–(68) are valid}\Big\}$$

where $Y_q = \{f \in H^1(\mathbb{R}^3) \mid \nabla f \in L^4 + L^q\}$ and for a.e. $t \geq 0$,

$$|\dot\alpha(t)| + |\dot v(t)| + |\dot{\tilde\gamma}(t)| + |\dot D(t)| \leq \delta^2 \langle t\rangle^{-3} \tag{65}$$

$$\|Z(t)\|_2 + \|\nabla Z(t)\|_2 \leq c_0\,\delta \tag{66}$$

$$t^{\frac{3}{2}}\|Z(t)\|_\infty \leq c_0\,\delta \tag{67}$$

$$t^{\frac{3}{4}}\|\nabla Z(t)\|_{L^4+L^q} \leq \delta. \tag{68}$$

Here $\langle t\rangle = (1+t^2)^{\frac{1}{2}}$. We also require that $\pi(0) = (\alpha_0, 0, 0, 0)$. Here c_0 is a sufficiently small constant and $\tilde\gamma$ is defined as in Lemma 9. Finally, we require that for a.e. $t \geq 0$

$$Z(t) = \mathcal{J} Z(t) = \overline{J Z(t)} \tag{69}$$

where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Note that any path in X_δ is admissible for small δ. In (68) one would like to take $q = \infty$, but for technical reasons it is better to take finite but very large q. We assume that some such large q was chosen and it will be kept fixed. Note that $Y_q \hookrightarrow L^\infty(\mathbb{R}^3)$, so that (67) is meaningful.

In what follows, we will need to deal with several paths simultaneously. Therefore, our notation will need to indicate which paths Galilei transforms, root spaces, etc. are defined. For example, $\mathcal{G}_\infty(\pi)(t)$ will mean the (vector) Galilei transform from (24) defined in terms of π, and $\{\xi_j(\pi)(t)\}_{j=1}^8$ will be the set of functions from Definition 8 which are obtained from π.

The contraction scheme is based on the linearized equation (21). Indeed, given $(\pi^{(0)}, Z^{(0)}) \in X_\delta$ with $Z^{(0)} = \binom{R^{(0)}}{\bar R^{(0)}}$, we solve for

$$i\partial_t Z(t) + \begin{pmatrix} \triangle + 2|W(\pi^{(0)})|^2 & W^2(\pi^{(0)}) \\ -\bar W^2(\pi^{(0)}) & -\triangle - 2|W(\pi^{(0)})|^2 \end{pmatrix} Z(t) \\ = \dot v \begin{pmatrix} -x e^{i\theta(\pi^{(0)})(t)}\phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \\ x e^{-i\theta(\pi^{(0)})(t)}\phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \end{pmatrix} \\ + \dot\gamma \begin{pmatrix} -e^{i\theta(\pi^{(0)})(t)}\phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \\ e^{-i\theta(\pi^{(0)})(t)}\phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \end{pmatrix} \tag{70}$$

$$+ i\dot{\alpha}\begin{pmatrix} e^{i\theta(\pi^{(0)})(t)}\partial_\alpha\phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \\ e^{-i\theta(\pi^{(0)})(t)}\partial_\alpha\phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \end{pmatrix}$$
$$+ i\dot{D}\begin{pmatrix} -e^{i\theta(\pi^{(0)})(t)}\nabla\phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \\ -e^{-i\theta(\pi^{(0)})(t)}\nabla\phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \end{pmatrix}$$
$$+ \begin{pmatrix} -2|R^{(0)}|^2 W(\pi^{(0)})(t) - \bar{W}(\pi^{(0)})(t)(R^{(0)})^2 - |R^{(0)}|^2 R^{(0)} \\ 2|R^{(0)}|^2 \bar{W}(\pi^{(0)})(t) + W(\pi^{(0)})(t)(\bar{R}^{(0)})^2 + |R^{(0)}|^2 \bar{R}^{(0)} \end{pmatrix}$$

with initial condition (64). The vector $\dot{\pi}$ here will be determined by means of the orthogonality condition $\langle Z(t), \dot{\tilde{\xi}}_j\rangle = 0$ for all $1 \le j \le 8$, $t \ge 0$; cf. Lemma 12. In this section it will be convenient to work on the level of the transformed solutions $U^{(0)}, U$ and the following definition makes this precise. The reader should note that (71)–(72) are the same as (70), whereas (73) is related to the aforementioned orthogonality condition on $Z(t)$.

Definition 20. Suppose $(\pi^{(0)}, Z^{(0)}) \in X_\delta$ and set

$$U^{(0)}(t) := M(\pi^{(0)})(t)\mathcal{G}_\infty(\pi^{(0)})(t)Z^{(0)}(t);$$

cf. (24). Let $\pi_\infty^{(0)}$ be the constant vector associated with

$$\pi^{(0)}(t) = (\alpha^{(0)}(t), v^{(0)}(t), D^{(0)}(t), \gamma^{(0)}(t))$$

as in Definition 3. Let (π, Z) be defined as the solutions of the linear problems

$$Z(t) := \mathcal{G}_\infty(\pi^{(0)})(t)^{-1}M(\pi^{(0)})(t)^{-1}U(t) \tag{71}$$

$$i\partial_t U - \mathcal{H}(\alpha_\infty^{(0)})U = -i\dot{\pi}\partial_\pi\widetilde{W}(\pi^{(0)}) + N(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})U \tag{72}$$

$$\begin{aligned} i\langle\dot{\pi}\partial_\pi\widetilde{W}(\pi^{(0)}), \xi_j(\pi^{(0)})\rangle &= i\langle U, \dot{\xi}_j(\pi^{(0)})\rangle + \langle U, E(\pi^{(0)})\xi_j(\pi^{(0)})\rangle \\ &\quad + \langle N(U^{(0)}, \pi^{(0)}), \xi_j(\pi^{(0)})\rangle \end{aligned} \tag{73}$$

for $1 \le j \le 8$. The notation on the right-hand side of (72) is analogous to that in (26), (27), (28), and the matrix operators $E(\pi^{(0)})$ are those from (47). The initial conditions are, with R_0 satisfying the smallness condition (10),

$$U(0) = \mathcal{G}_\infty(\pi^{(0)})(0)Z(0) = \mathcal{G}_\infty(\pi^{(0)})(0)\left[\begin{pmatrix} R_0 \\ \bar{R}_0 \end{pmatrix} + hf^+(\alpha_\infty^{(0)}) + \sum_{j=1}^8 a_j\eta_j(\alpha_\infty^{(0)})\right] \tag{74}$$

$$\pi(0) = (\alpha_0, 0, 0, 0) \tag{75}$$

where $h, \{a_j\}_{j=1}^8 \in \mathbb{C}$ are constants (later we will make a unique choice of these constants in terms of the data $(\pi^{(0)}, U^{(0)})$, and in fact they will be chosen real-valued). Here, for any $\alpha > 0$ we set $\mathcal{N}(\alpha) = \{\eta_j(\alpha)\}_{j=1}^8$, and we define $f^\pm(\alpha)$ via

$$\mathcal{H}(\alpha)f^\pm(\alpha) = \pm i\sigma f^\pm(\alpha), \qquad \sigma > 0.$$

We are assuming for simplicity that there is a unique pair $\{f^{\pm}(\alpha)\}$ of simple eigenvectors of $\mathcal{H}(\alpha)$ with imaginary eigenvalues.

The main point of this section as well as the next is to show that the map

$$\Psi : (\pi^{(0)}, Z^{(0)}) \mapsto (\pi, Z), \tag{76}$$

as given by (71)–(73), defines a contraction on X_δ relative to a suitable norm provided the parameters $h, \{a_j\}_{j=1}^{8}$ are chosen correctly. As a first step, we show in this section that $\Psi : X_\delta \to X_\delta$ for $\delta > 0$ small provided h is chosen properly. Before doing so, we add some clarifying remarks on Definition 20. In particular, we need to prove the existence of solutions to (72), (73).

We start with a simple technical statement that improves on Lemma 4 by means of the stronger assumptions (65).

LEMMA 21. *Let θ, y and θ_∞, y_∞ be as in* (4), (5), *and* (16), *respectively. Let ρ_∞ be as in Lemma* 4. *Under the conditions of Definition* 19 *the bounds*

$$|\rho_\infty(t,x)| \lesssim \delta^2(1+|x|)\langle t\rangle^{-1}, \quad |y(t) - y_\infty(t)| \lesssim \delta^2\langle t\rangle^{-1}$$

hold for all $t \ge 0$. Moreover,

$$|\dot\rho_\infty(t,x)| \lesssim \delta^2(1+|x|)\langle t\rangle^{-2}, \quad |\dot y(t) - \dot y_\infty(t)| \lesssim \delta^2\langle t\rangle^{-2} \tag{77}$$

for all $t \ge 0$. In particular, one has the bounds

$$\|V(t)\|_{L^1\cap L^\infty} \lesssim \delta^2\langle t\rangle^{-1}, \quad \|\dot\xi_j(t)\|_{L^1\cap L^\infty} \lesssim \delta^2\langle t\rangle^{-2},$$
$$\|E(t)\xi_j(t)\|_{L^1\cap L^\infty} \lesssim \delta^2\langle t\rangle^{-2},$$

where $V(t)$ is the matrix from (26), *and $E(t)$ is the matrix operator from* (47).

Proof. In view of the definitions,

$$\theta(t, x+y_\infty) - \theta_\infty(t, x+y_\infty) = (v(t) - v_\infty)\cdot(x + 2tv_\infty + D_\infty) + \gamma(t) - \gamma(\infty)$$
$$- 2\int_t^\infty \int_s^\infty (v\cdot\dot v - \alpha\dot\alpha)(\sigma)\, d\sigma ds.$$

Now $|\dot\gamma(t)| \le \delta^2\langle t\rangle^{-2}$ because of (46). Definition 19 therefore implies the desired bound on ρ_∞. As for $y(t) - y_\infty(t)$, the definition of D_∞ implies that

$$y_\infty(t) - y(t) = 2tv_\infty + D_\infty - 2\int_0^t v(s)\, ds - D(t)$$
$$= D(\infty) - D(t) - 2\int_t^\infty \int_s^\infty \dot v(\sigma)\, d\sigma\, ds,$$

which is no larger than $C\delta^2\langle t\rangle^{-1}$, as claimed. □

We will make frequent use of the following simple observation: If $U(t)$ and $Z(t)$ are related by (71), then $U(t)$ satisfies (66)–(68) if and only if $Z(t)$ does (possibly at the loss of a small multiplicative constant).

LEMMA 22. *Let* $R_0 \in H^1(\mathbb{R}^3)$ *satisfy* $\binom{R_0}{\bar{R}_0} \perp \mathcal{N}(\alpha_0)^*$. *Given* $(\pi^{(0)}, Z^{(0)}) \in X_\delta$ *and any* $h, \{a_j\}_{j=1}^8 \in \mathbb{C}$, *there exist unique solutions*

$$(\pi, Z) \in \mathrm{Lip}([0,\infty), \mathbb{C}^8) \\ \times \big[C([0,\infty), H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)) \cap C^1([0,\infty), H^{-1}(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3))\big]$$

of (71)–(73) *with initial conditions* (74), (75). *Moreover, if* $\delta > 0$ *is sufficiently small, then for any value of* $h \in \mathbb{C}$, *there is a unique choice of* $\{a_j\}_{j=1}^8 = \{a_j(h)\}_{j=1}^8 \in \mathbb{C}^8$ *so that* $U(t)$ *satisfies the orthogonality conditions*

$$\langle U(t), \xi_j(\pi^{(0)})(t)\rangle = 0 \quad \textit{for all} \quad t \geq 0,\ 1 \leq j \leq 8; \tag{78}$$

cf. (45). *Moreover, if* $h \in \mathbb{R}$ *then also* $\{a_j(h)\}_{j=1}^8 \in \mathbb{R}^8$ *and in that case* $U(t)$, *and therefore also* $Z(t)$, *is* $\mathcal{J}$*-invariant and* $\pi(t) \in \mathbb{R}^8$ *for all* $t \geq 0$.

Proof. For the existence statement, solve (73) for $\dot{\pi}$, which can be done as in Lemma 11. Plugging the result into (72) leads to a linear equation for U, which takes the following form:

$$i\partial_t U - \mathcal{H}(\alpha_\infty^{(0)})U = \mathcal{L}(U, \pi^{(0)}) + \tilde{N}(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})U. \tag{79}$$

Here $\mathcal{L}(U, \pi^{(0)})$ is the linear term which is obtained by replacing $\dot{\pi}\partial_\pi \widetilde{W}(\pi^{(0)})$ on the right-hand side of (72) with the expressions that result by solving (73) for $\dot{\pi}$. See Lemmas 9 and 11 for the details of this process. Moreover, in this way one picks up the final term in (73) which leads to the modified nonlinear term $\tilde{N}(U^{(0)}, \pi^{(0)})$ in (79). We will need to bound this nonlinear term. For this purpose, we record the estimate

$$\|\tilde{N}(U^{(0)}, \pi^{(0)})\|_{W^{k,p}} \lesssim \|N(U^{(0)}, \pi^{(0)})\|_{W^{k,p}} \\ + \min(\|N(U^{(0)}, \pi^{(0)})\|_1, \|N(U^{(0)}, \pi^{(0)})\|_2). \tag{80}$$

Viewed as a linear operator in U, $\mathcal{L}(\cdot, \pi^{(0)})$ has finite rank and co-rank. In fact, both its range and co-kernel are spanned by eight exponentially decreasing, smooth functions (which depend on time). Moreover, by Lemma 21 it satisfies the bound

$$\|\mathcal{L}(U, \pi^{(0)})\|_{W^{k,p}} \leq C_{k,p}\, \delta^2 \langle t\rangle^{-2} \|U\|_2 \tag{81}$$

for any integer $k \geq 0$, and $1 \leq p \leq \infty$. The equations (73) are chosen precisely in order to ensure that

$$\frac{d}{dt}\langle U(t), \xi_j(\pi^{(0)})(t)\rangle = 0 \quad \text{for all} \quad t \geq 0,\ 1 \leq j \leq 8.$$

On the other hand, in Remark 13 we showed that

$$\langle U(0), \xi_1(\pi^{(0)})(0)\rangle = 0$$

is the same as (with $\mathcal{N}(\alpha_0)^* = \{\xi_j(\alpha_0)\}_{j=1}^8$, see (43)),

$$\begin{aligned}0 &= \Big\langle \mathcal{G}_\infty(\pi^{(0)})(0)\Big[\binom{R_0}{\bar{R}_0} + hf^+(\alpha_\infty^{(0)}) + \sum_{j=1}^8 a_j\eta_j(\alpha_\infty^{(0)})\Big], \mathcal{G}_\infty(\pi^{(0)})(0)\binom{\phi(\cdot,\alpha_0)}{\phi(\cdot,\alpha_0)}\Big\rangle \\ &= \Big\langle \binom{R_0}{\bar{R}_0}, \binom{\phi(\cdot,\alpha_0)}{\phi(\cdot,\alpha_0)}\Big\rangle + \Big\langle hf^+(\alpha_\infty^{(0)}) + \sum_{j=1}^8 a_j\eta_j(\alpha_\infty^{(0)}), \binom{\phi(\cdot,\alpha_0)}{\phi(\cdot,\alpha_0)}\Big\rangle \\ &= h\langle f^+(\alpha_\infty^{(0)}), \xi_1(\alpha_0)\rangle + \sum_{j=1}^8 a_j\langle \eta_j(\alpha_\infty^{(0)}), \xi_1(\alpha_0)\rangle\end{aligned}$$

and similarly for $\{\xi_k(\pi^{(0)})(0)\}_{k=2}^8$. Here we used that $\binom{R_0}{\bar{R}_0} \in \mathcal{N}^*(\alpha_0)^\perp$ by assumption, as well as that $\mathcal{G}_\infty(\pi^{(0)})(0)$ is unitary. Hence (78) holds for all times if and only if for any $h \in \mathbb{C}$ we can find $\{a_j\}_{j=1}^8 \in \mathbb{C}^8$ such that

$$0 = h\langle f^+(\alpha_\infty^{(0)}), \xi_\ell(\alpha_0)\rangle + \sum_{j=1}^8 a_j\langle \eta_j(\alpha_\infty^{(0)}), \xi_\ell(\alpha_0)\rangle \quad \text{for all } \ 1 \le \ell \le 8. \tag{82}$$

However, $\|\eta_j(\alpha_\infty^{(0)}) - \eta_j(\alpha_0)\| \lesssim \delta^2$ because $|\alpha_\infty^{(0)} - \alpha_0| \lesssim \delta^2$. Together with Lemma 10 this implies that the matrix

$$\mathcal{M} := \{\langle \eta_j(\alpha_\infty^{(0)}), \xi_\ell(\alpha_0)\rangle\}_{j,\ell=1}^8$$

is invertible with norm $\lesssim 1$. Hence $\{a_j\}_{j=1}^8 = \{a_j(h)\}_{j=1}^8 \in \mathbb{C}^8$ is uniquely determined for any $h \in \mathbb{C}$. For future reference, we note the estimate

$$\sum_{j=1}^8 |a_j(h)| \lesssim \delta^2\,|h|, \tag{83}$$

which follows from the fact that $\langle f^+(\alpha_\infty^{(0)}), \xi_\ell(\alpha_\infty^{(0)})\rangle = 0$ and thus

$$|\langle f^+(\alpha_\infty^{(0)}), \xi_\ell(\alpha_0)\rangle| = |\langle f^+(\alpha_\infty^{(0)}), \xi_\ell(\alpha_0) - \xi_\ell(\alpha_\infty^{(0)})\rangle| \lesssim \delta^2.$$

It is important to realize that the assumption $\binom{R_0}{\bar{R}_0} \perp \mathcal{N}(\alpha_0)^*$ is precisely used in (83); if we drop this assumption, then $a_j(h) \not\to 0$ as $h \to 0$. Finally, we note that for any $\mathcal{J}$-invariant functions f, g one has $\langle f, g\rangle \in \mathbb{R}$. Hence for $h \in \mathbb{R}$ both the matrix $\mathcal{M}$ as well as the vector

$$\{h\langle f^+(\alpha_\infty^{(0)}), \xi_\ell(\alpha_0)\rangle\}_{\ell=1}^8$$

are real-valued so that in fact $\{a_j(h)\}_{j=1}^8 \in \mathbb{R}^8$ (recall that $\mathcal{G}_\infty(\pi^{(0)})(0)$ preserves $\mathcal{J}$-invariance). In view of the preceding, any solution of (79) with initial condition (74) and this choice of $a_j(h)$ will satisfy the orthogonality condition (78) on its interval of existence.

To prove global existence of solutions to the linear problem (79), we perform a contraction argument in $C([0,T], L^2)$ on some finite time interval $[0,T]$

(one can take $T = 1$). Given any initial condition $U(0) \in H^1 \times H^1$, and any $\tilde U \in C([0,T], H^1(\mathbb{R}^3))$, we solve

$$i\partial_t U - \mathcal{H}(\alpha_\infty^{(0)})U = \mathcal{L}(\tilde U, \pi^{(0)}) + \tilde N(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})\tilde U$$

via the evolution $e^{-it\mathcal{H}(\alpha_\infty^{(0)})}$; i.e., write the solution as

$$\begin{aligned} U(t) &= e^{-it\mathcal{H}(\alpha_\infty^{(0)})}U(0) \\ &\quad -i\int_0^t e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}[\mathcal{L}(\tilde U, \pi^{(0)}) + \tilde N(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})\tilde U](s)\, ds \end{aligned} \tag{84}$$

for all times $t \ge 0$. In addition to the bounds in (81), we note the following two bounds: First,

$$\sup_{s\ge 0} \|\tilde N(U^{(0)}, \pi^{(0)})(s)\|_{L^2} \lesssim \delta^2, \tag{85}$$

which follows from (80), (37) of Lemma 7, and (66) applied to $U^{(0)}$, and second

$$\sup_{s\ge 0} \|V(\pi^{(0)})(t)\tilde U(s)\|_{H^1} \lesssim \delta^3, \tag{86}$$

which follows from Lemma 21 and again (66). Apply the linear estimate on the evolution $e^{-it\mathcal{H}(\alpha_\infty^{(0)})}$ given by Theorem 34. Note that in contrast to these estimates, here we are including the entire discrete spectrum, which possibly leads to exponential growth. However, on a time interval of length $T = 1$, say, we can always see that the map $\tilde U \mapsto U$ is a contraction in the norm of $C([0,T], L^2 \times L^2)$ for small δ. Since the size of this δ does not depend on the size of the initial condition, we can restart this procedure and thus obtain a global solution of (84) that belongs to $C([0,\infty), L^2 \times L^2)$. Typically, this solution will grow exponentially.

Next, we wish to estimate the first derivative of (84) by means of the L^2 bound in Theorems 34 which will lead to the improved statement that

$$U \in C([0,\infty), H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)) \cap C^1([0,\infty), H^{-1}(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3))$$

solves (79) in the strong sense. Inserting this solution U into equation (73) then yields the path π. Indeed, simply integrate in time using the initial condition (75). It remains to show that for $T = 1$,

$$\sup_{0\le t\le T} \Big\| \nabla \int_0^t e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}\, \tilde N(U^{(0)}, \pi^{(0)})\, ds \Big\|_2 \lesssim \delta^2. \tag{87}$$

Here we omitted the other terms on the right-hand side of (79), i.e., $\mathcal{L}(U, \pi^{(0)})$ and $V(\pi^{(0)})\tilde U$, since they satisfy the bounds (81) and (86), respectively, and thus yield the desired L^2 estimate on the derivative (for the issue of interchanging the evolution with a gradient, see Corollary 39 below). In view of (80) and (38) one has

(88)

$$\begin{aligned}
&\|\nabla\tilde{N}(U^{(0)},\pi^{(0)})(t)\|_2\\
&\quad\lesssim \min(\|U^{(0)}(t)\|_\infty^2,\|U^{(0)}(t)\|_4^2)+\|U^{(0)}\nabla U^{(0)}(t)\|_2+\||U^{(0)}|^2\nabla U^{(0)}(t)\|_2\\
&\quad\lesssim \delta^2+\|U^{(0)}(t)\|_{L^4\cap L^{\frac{2q}{q-2}}}\|\nabla U^{(0)}(t)\|_{4+q}+\|\,|U^{(0)}|^2\nabla U^{(0)}(t)\|_2\\
&\quad\lesssim \delta^2+\delta^2 t^{-\frac34}+\|\,|U^{(0)}|^2\nabla U^{(0)}(t)\|_2.
\end{aligned}$$

The first two terms in (88) contribute a finite amount to (87), as desired. The final term in (88), however, is too singular at $t=0$ and we therefore need to invoke the Strichartz estimates from Corollary 39 to control it. More precisely, we split $|U|^2U(s)$ into $P_{\rm s}(\alpha_\infty^{(0)})|U|^2U(s)$ and $(I-P_{\rm s}(\alpha_\infty^{(0)}))|U|^2U(s)$. The latter does not present a problem, since the range of $I-P_{\rm s}(\alpha_\infty^{(0)})$ is spanned by finitely many Schwartz functions. Thus,

$$\begin{aligned}
&\sup_{0\le t\le T}\Big\|\nabla\int_0^t e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}(I-P_{\rm s}(\alpha_\infty^{(0)}))|U^{(0)}|^2(s)U^{(0)}(s)\,ds\Big\|_2\\
&\lesssim e^{CT}\sup_{s\ge 0}\|\nabla(I-P_{\rm s}(\alpha_\infty^{(0)}))|U^{(0)}|^2(s)U^{(0)}(s)\|_2\lesssim e^{CT}\sup_{s\ge0}\|U^{(0)}(s)\|_3^3\lesssim e^{CT}\delta^3,
\end{aligned}$$

as desired. For $P_{\rm s}(\alpha_\infty^{(0)})|U|^2U(s)$ we use the following Strichartz estimate:

$$\begin{aligned}
&\sup_{0\le t\le T}\Big\|\nabla\int_0^t e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}\,P_{\rm s}(\alpha_\infty^{(0)})|U^{(0)}|^2U^{(0)}(s)\,ds\Big\|_2\\
&\lesssim\Big(\int_0^T\|\,|U^{(0)}|^2U^{(0)}(s)\|_{L^{\frac43}(\mathbb{R}^3)}^{\frac85}\,ds\Big)^{\frac58}+\Big(\int_0^T\|\,|U^{(0)}|^2\nabla U^{(0)}(s)\|_{L^{\frac43}(\mathbb{R}^3)}^{\frac85}\,ds\Big)^{\frac58}.
\end{aligned}$$

It will suffice to deal with the term on the right-hand side containing $\nabla U^{(0)}$, since the one without any derivatives is easier. The corresponding integrand is estimated in terms of (66) and (68) as follows: For all $0<s\le T$,

(89)

$$\begin{aligned}
\||U^{(0)}|^2\nabla U^{(0)}(s)\|_{L^{\frac43}(\mathbb{R}^3)}&\lesssim\|\nabla U^{(0)}(s)\|_2\|U^{(0)}(s)\|_8^2\lesssim\delta\|U^{(0)}(s)\|_6^{\frac32}\|U^{(0)}\|_\infty^{\frac12}\\
&\lesssim\delta\,\delta^{\frac32}(\delta s^{-\frac34}+\delta)^{\frac12}\le C(T)\delta^3 s^{-\frac38},
\end{aligned}$$

where we used the Sobolev embedding bound

$$\|U^{(0)}(s)\|_\infty\lesssim\|\nabla U^{(0)}(s)\|_{4+q}+\|U^{(0)}(s)\|_2\lesssim\delta s^{-\frac34}+\delta.$$

Since $s^{-\frac38}\in L^{\frac85}(0,T)$ we are done. The conclusion is that $U\in L^\infty([0,T],H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3))$. The continuity in t relative to the H^1 norm is implicit in the above argument, and we skip it. Finally, time-stepping extends the H^1-statement to all times.

Finally, if $h\in\mathbb{R}$ and $a_j(h)\in\mathbb{R}$ are as above, then the initial condition (74) is $\mathcal{J}$-invariant by Remark 14. Also, we assume that $\pi(0)\in\mathbb{R}^8$. It remains to derive the system of equations which $(\bar\pi,\mathcal{J}U)(t)$ obeys. By the assumption that $\pi^{(0)}(t) \in \mathbb{R}^8$ and $\mathcal{J}U^{(0)}(t) = U^{(0)}(t)$ for all $t \geq 0$, one checks that (72) implies that

$$i\partial_t \mathcal{J}U - \mathcal{H}(\alpha_\infty^{(0)})\mathcal{J}U = -i\dot{\bar{\pi}}\partial_\pi \widetilde{W}(\pi^{(0)}) + N(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})U^{(0)}; \tag{90}$$

see the proof of Lemma 6 for more details here. On the other hand, as in Lemma 11, one obtains the following system which is equivalent to (73), with $E(t)$ as in (47) and with $\phi = \phi(\cdot, \alpha^{(0)}(t))$:

(91)
$$\begin{aligned}
\dot{\alpha}\alpha^{-1}\|\phi\|_2^2 &= \langle U, \dot{\xi}_1(\pi^{(0)})\rangle - i\langle U, E(\pi^{(0)})\xi_1(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_1(\pi^{(0)})\rangle \\
\dot{\tilde{\gamma}}\alpha^{-1}\|\phi\|_2^2 &= \langle U, \dot{\xi}_2(\pi^{(0)})\rangle - i\langle U, E(\pi^{(0)})\xi_2(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_2(\pi^{(0)})\rangle \\
2\dot{D}_\ell\|\phi\|_2^2 &= \langle U, \dot{\xi}_{2+\ell}(\pi^{(0)})\rangle - i\langle U, E(\pi^{(0)})\xi_{2+\ell}(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_{2+\ell}(\pi^{(0)})\rangle \\
2\dot{v}_\ell\|\phi\|_2^2 &= \langle U, \dot{\xi}_{5+\ell}(\pi^{(0)})\rangle - i\langle U, E(\pi^{(0)})\xi_{5+\ell}(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_{5+\ell}(\pi^{(0)})\rangle
\end{aligned}$$

for all $1 \leq \ell \leq 3$. This is based on the observation of Lemma 9, namely that

$$\dot{\pi}\partial_\pi \widetilde{W}(\pi^{(0)}) = \sum_{\ell=1}^{3} (\dot{D}_\ell\, \eta_{5+\ell}(\pi^{(0)}) + \dot{v}_\ell\, \eta_{2+\ell}(\pi^{(0)})) - \dot{\alpha}\, \eta_2(\pi^{(0)}) + \dot{\tilde{\gamma}}\, \eta_1(\pi^{(0)}).$$

Note that $\overline{JE(\pi^{(0)})J} = -E(\pi^{(0)})$; see (47). Taking complex conjugates of the $\dot{\alpha}$ equation (91) yields

$$\begin{aligned}
\dot{\bar{\alpha}}\bar{\alpha}^{-1}\|\phi\|_2^2 &= \overline{\langle JU, J\dot{\xi}_1(\pi^{(0)})\rangle} \\
&\quad + i\, \overline{\langle JU, JE(\pi^{(0)})JJ\xi_1(\pi^{(0)})\rangle} + i\, \overline{\langle JN(U^{(0)}, \pi^{(0)}), J\xi_1(\pi^{(0)})\rangle} \\
&= \langle \overline{JU}, \overline{J\dot{\xi}_1(\pi^{(0)})}\rangle \\
&\quad + i\langle \overline{JU}, \overline{JE(\pi^{(0)})JJ\xi_1(\pi^{(0)})}\rangle + i\langle \overline{JN(U^{(0)}, \pi^{(0)})}, \overline{J\xi_1(\pi^{(0)})}\rangle \\
&= \langle \mathcal{J}U, \dot{\xi}_1(\pi^{(0)})\rangle - i\langle \mathcal{J}U, E(\pi^{(0)})\xi_1(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_1(\pi^{(0)})\rangle.
\end{aligned}$$

Taking complex conjugates one therefore derives the following system from the preceding one; see (91),

$$\begin{aligned}
\dot{\bar{\alpha}}\bar{\alpha}^{-1}\|\phi\|_2^2 &= \langle \mathcal{J}U, \dot{\xi}_1(\pi^{(0)})\rangle - i\langle \mathcal{J}U, E(\pi^{(0)})\xi_1(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_1(\pi^{(0)})\rangle, \\
\dot{\bar{\tilde{\gamma}}}\bar{\alpha}^{-1}\|\phi\|_2^2 &= \langle \mathcal{J}U, \dot{\xi}_2(\pi^{(0)})\rangle - i\langle \mathcal{J}U, E(\pi^{(0)})\xi_2(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_2(\pi^{(0)})\rangle, \\
2\dot{\bar{D}}_\ell\|\phi\|_2^2 &= \langle \mathcal{J}U, \dot{\xi}_{2+\ell}(\pi^{(0)})\rangle - i\langle \mathcal{J}U, E(\pi^{(0)})\xi_{2+\ell}(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_{2+\ell}(\pi^{(0)})\rangle, \\
2\dot{\bar{v}}_\ell\|\phi\|_2^2 &= \langle \mathcal{J}U, \dot{\xi}_{5+\ell}(\pi^{(0)})\rangle - i\langle \mathcal{J}U, E(\pi^{(0)})\xi_{5+\ell}(\pi^{(0)})\rangle - i\langle N(U^{(0)}, \pi^{(0)}), \xi_{5+\ell}(\pi^{(0)})\rangle
\end{aligned}$$

for all $1 \leq \ell \leq 3$. Combining this system with (90) shows that $(\bar{\pi}, \mathcal{J}U)$ solves the same equations as (π, U), namely (72), (73). Since their initial conditions agree if $h \in \mathbb{R}$, we conclude that they agree for all times. □

Next we present a rather simple lemma about bounded solutions to hyperbolic ODE. This will be the mechanism to determine the unique value of h in (74) so that the solution $U(t)$ constructed in Lemma 22 remains bounded in L^2 for all times.

LEMMA 23. *Consider the two-dimensional* ODE

$$\dot{x}(t) - A_0 x(t) = f(t), \qquad x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

where $f = \binom{f_1}{f_2} \in L^\infty([0,\infty), \mathbb{C}^2)$ *and* $A_0 = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$ *where* $\sigma > 0$. *Then* $x(t) = \binom{x_1(t)}{x_2(t)}$ *remains bounded for all times if and only if*

$$0 = x_1(0) + \int_0^\infty e^{-\sigma t} f_1(t)\, dt. \tag{92}$$

Moreover, in that case

$$x_1(t) = -\int_t^\infty e^{-(s-t)\sigma} f_1(s)\, ds, \quad x_2(t) = e^{-t\sigma} x_2(0) + \int_0^t e^{-(t-s)\sigma} f_2(s)\, ds \tag{93}$$

for all $t \geq 0$.

Proof. Clearly, $x_1(t) = e^{t\sigma} x_1(0) + \int_0^t e^{(t-s)\sigma} f_1(s)\, ds$ and $x_2(t) = e^{-t\sigma} x_2(0) + \int_0^t e^{-(t-s)\sigma} f_2(s)\, ds$. If $\lim_{t\to\infty} e^{-t\sigma} x_1(t) = 0$, then $0 = x_1(0) + \int_0^\infty e^{-s\sigma} f_1(s)\, ds$, which is (92). Conversely, if this holds, then $x_1(t) = -e^{t\sigma} \int_t^\infty e^{-s\sigma} f_1(s)\, ds$, and the lemma is proved. □

LEMMA 24. *There exists a small constant* c_1 (*depending on the constant* c_0 *in* (66), (67)) *so that the following holds*: *With* $\delta > 0$ *small, let* $R_0 \in W^{1,1}(\mathbb{R}^3) \cap W^{1,2}(\mathbb{R}^3)$ *satisfy* $P_u^+(\alpha_0)\binom{R_0}{\bar{R}_0} = 0$ *and*

$$\|R_0\|_{W^{1,1}\cap W^{1,2}} \leq c_1 \delta. \tag{94}$$

Furthermore, let (π, Z) *be the solution from Lemma 22 for a given* $(\pi^{(0)}, Z^{(0)}) \in X_\delta$ *and* $h \in \mathbb{C}$, $a_j = a_j(h)$. *Then there exists a unique value of* $h \in \mathbb{R}$ *so that* $(\pi, Z) \in X_\delta$. *In other words, under the assumption* (94), *there exists the map* $\Psi : X_\delta \to X_\delta$; *see* (76). *Moreover, as a function of* $(R_0, \pi^{(0)}, Z^{(0)})$, $h = h(R_0, \pi^{(0)}, Z^{(0)})$ *satisfies*

$$|h(R_0, \pi^{(0)}, Z^{(0)})| \leq C_0 \, \|R_0\|^2_{W^{1,1}\cap W^{1,2}} \tag{95}$$

with a universal constant C_0 *as well as*

$$|h(R_0, \pi^{(0)}, Z^{(0)}) - h(R_1, \pi^{(0)}, Z^{(0)})| \leq \|R_0 - R_1\|_2 \tag{96}$$

for any R_0, R_1 *as above.*

Proof. Let $(\pi^{(0)}, U^{(0)}) \in X_\delta$ be fixed and let (π, U) be the solutions constructed in Lemma 22, with $h \in \mathbb{R}$ arbitrary and $\{a_j(h)\}_{j=1}^8 \in \mathbb{R}^8$ the unique choice that guarantees the orthogonality condition (78). Moreover, π is real-valued, and $\mathcal{J}U = U$. We start by decomposing the function $U(t)$ into three pieces $U(t) = U_{\mathrm{dis}}(t) + U_{\mathrm{root}}(t) + U_{\mathrm{hyp}}(t)$ where

$$U_{\mathrm{dis}}(t) = P_{\mathrm{s}}(\alpha_\infty^{(0)})U(t), \quad U_{\mathrm{root}}(t) = P_{\mathrm{root}}(\alpha_\infty^{(0)})U(t), \quad U_{\mathrm{hyp}}(t) = P_{\mathrm{im}}(\alpha_\infty^{(0)})U(t).$$

Here $P_{\mathrm{root}}(\alpha)$ and $P_{\mathrm{im}}(\alpha)$ are the Riesz projections corresponding to the spectrum at $\{0\}$, and $\{\pm i\sigma\}$ of $\mathcal{H}(\alpha)$, respectively. For ease of notation, let the elements of the rootspaces of $\mathcal{H}(\alpha_\infty^{(0)})$ and $\mathcal{H}(\alpha_\infty^{(0)})^*$ be

$$\mathcal{N}(\alpha_\infty^{(0)}) = \{\eta_j^{(0)}\}_{j=1}^8, \qquad \mathcal{N}(\alpha_\infty^{(0)})^* = \{\xi_j^{(0)}\}_{j=1}^8,$$

respectively, and write accordingly

$$U_{\mathrm{root}}(t) = \sum_{j=1}^8 \tilde{a}_j(t)\eta_j^{(0)}, \qquad U_{\mathrm{hyp}}(t) = b^+(t)f^+(\alpha_\infty^{(0)}) + b^-(t)f^-(\alpha_\infty^{(0)}). \tag{97}$$

Since U_{root} and U_{hyp} are $\mathcal{J}$-invariant, see Remark 14, it follows that $\{\tilde{a}_j\}_{j=1}^8$ and b^+, b^- are real. Moreover, because of the orthogonality condition (78), for all $1 \le k \le 8$,

$$\begin{aligned} 0 = \sum_{j=1}^8 \tilde{a}_j(t)\langle \eta_j^{(0)}, \xi_k(\pi^{(0)})(t)\rangle + b^+(t)\langle f^+(\alpha_\infty^{(0)}), \xi_k(\pi^{(0)})(t)\rangle \\ + b^-(t)\langle f^-(\alpha_\infty^{(0)}), \xi_k(\pi^{(0)})(t)\rangle + \langle U_{\mathrm{dis}}(t), \xi_k(\pi^{(0)})(t)\rangle \end{aligned} \tag{98}$$

for all times $t \ge 0$. For small δ this allows one to solve for $\tilde{a}_j(t)$. Indeed, by Definition 8 and Lemma 21,

$$\sup_{t\ge 0} \max_{1\le k\le 8} \|\xi_k^{(0)} - \xi_k(\pi^{(0)})(t)\|_2 \lesssim \delta^2 \langle t\rangle^{-1}.$$

Also, by Lemma 10, for each j there is $k(j)$ so that $|\langle \eta_j(\pi^{(0)})(t), \xi_k(\pi^{(0)})(t)\rangle| \asymp 1$ if $k = k(j)$ and $= 0$ otherwise. Hence, $|\langle \eta_j^{(0)}, \xi_k(\pi^{(0)})(t)\rangle| \asymp 1$ if $k = k(j)$, but $|\langle \eta_j^{(0)}, \xi_k(\pi^{(0)})(t)\rangle| \lesssim \delta^2$ if $k \ne k(j)$. Since $j \to k(j)$ is a permutation, it follows that the matrix $\{\langle \eta_j^{(0)}, \xi_k(\pi^{(0)})(t)\rangle\}_{j,k=1}^8$ is invertible with norm of the inverse $\lesssim 1$. Consequently, there exist uniformly bounded functions $c^\pm(t)$, $c_{jk}(t)$ and $d_{jk}(t)$ so that for all $t \ge 0$,

$$\begin{aligned} \tilde{a}_j(t) &= b^+(t)c_j^+(t) + b^-(t)c_j^-(t) + \sum_{k=1}^8 d_{jk}(t)\langle U_{\mathrm{dis}}(t), \xi_k(\pi^{(0)})(t)\rangle \\ &= \sum_{k=1}^8 c_{jk}(t)\langle U_{\mathrm{hyp}}(t), \xi_k(\pi^{(0)})(t)\rangle + \sum_{k=1}^8 d_{jk}(t)\langle U_{\mathrm{dis}}(t), \xi_k(\pi^{(0)})(t)\rangle \end{aligned} \tag{99}$$

and therefore, in particular,

$$\|U_{\mathrm{root}}(t)\|_{1\cap\infty} \le C(\|U_{\mathrm{dis}}(t)\|_{1+\infty} + \|U_{\mathrm{hyp}}(t)\|_{1+\infty}), \tag{100}$$

for all $t \ge 0$ with a constant C that does not depend on time t. Hence, the solution $U(t)$ is completely determined by $U_{\mathrm{dis}}(t)$ and $U_{\mathrm{hyp}}(t)$, and in fact, for all $t \ge 0$,

$$\|U\|_2 \le C(\|U_{\mathrm{dis}}(t)\|_2 + \|U_{\mathrm{hyp}}(t)\|_2), \tag{101}$$

with a constant C that does not depend on time t. Clearly, (100) remains correct with derivatives on the left-hand side (without derivatives on the right-hand side), and (101) therefore remains true with derivatives and/or other L^p-norms. For example, it follows that

$$\|U\|_{2+\infty} \leq C(\|U_{\mathrm{dis}}(t)\|_{2+\infty} + \|U_{\mathrm{hyp}}(t)\|_{2+\infty}). \tag{102}$$

In Lemma 22 we showed that the system (72), (73) is equivalent to the single equation

$$i\partial_t U - \mathcal{H}(\alpha_\infty^{(0)})U = \mathcal{L}(U, \pi^{(0)}) + \tilde{N}(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})U;$$

see (79). This equation is $\mathcal{J}$-invariant in the sense that $\mathcal{J}U$ satisfies the identical equation. We now split this equation into two equations, one for U_{dis} and the other for U_{hyp}. This yields (with $P_{\mathrm{s}} = P_{\mathrm{s}}(\alpha_\infty^{(0)})$ and $P_{\mathrm{im}} = P_{\mathrm{im}}(\alpha_\infty^{(0)})$),

$$\begin{aligned} i\partial_t U_{\mathrm{dis}} - \mathcal{H}(\alpha_\infty^{(0)})U_{\mathrm{dis}} &= P_{\mathrm{s}}\Big[\mathcal{L}_1(U_{\mathrm{dis}}, \pi^{(0)}) + \mathcal{L}_2(U_{\mathrm{hyp}}, \pi^{(0)}) \\ &\qquad + \tilde{N}(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})U_{\mathrm{dis}} + V(\pi^{(0)})U_{\mathrm{hyp}}\Big] \\ i\partial_t U_{\mathrm{hyp}} - \mathcal{H}(\alpha_\infty^{(0)})U_{\mathrm{hyp}} &= P_{\mathrm{im}}\Big[\mathcal{L}_1(U_{\mathrm{dis}}, \pi^{(0)}) + \mathcal{L}_2(U_{\mathrm{hyp}}, \pi^{(0)}) \\ &\qquad + \tilde{N}(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})U_{\mathrm{dis}} + V(\pi^{(0)})U_{\mathrm{dis}}\Big], \end{aligned} \tag{103}$$

with initial conditions $U_{\mathrm{dis}}(0) = P_{\mathrm{s}}(\alpha_\infty^{(0)})U(0)$ and $U_{\mathrm{hyp}}(0) = P_{\mathrm{im}}(\alpha_\infty^{(0)})U(0)$; see (74). Here the linear terms $\mathcal{L}_1$ and $\mathcal{L}_2$ are derived from $\mathcal{L}$ by expressing U_{root} as a linear combination of (projections of) U_{dis} and U_{hyp}; see (98). More precisely, write

$$\begin{aligned} &\mathcal{L}(U, \pi^{(0)}) + V(\pi^{(0)})U \\ &\quad = \mathcal{L}(U_{\mathrm{dis}}, \pi^{(0)}) + \mathcal{L}(U_{\mathrm{hyp}}, \pi^{(0)}) \\ &\qquad + \sum_{j=1}^{8} \tilde{a}_j(t)[\mathcal{L}(\eta_j^{(0)}, \pi^{(0)}) + V(\pi^{(0)})\eta_j^{(0)}] + V(\pi^{(0)})U_{\mathrm{dis}} + V(\pi^{(0)})U_{\mathrm{hyp}} \\ &\quad =: \mathcal{L}_1(U_{\mathrm{dis}}, \pi^{(0)}) + \mathcal{L}_2(U_{\mathrm{hyp}}, \pi^{(0)}) + V(\pi^{(0)})U_{\mathrm{dis}} + V(\pi^{(0)})U_{\mathrm{hyp}}, \end{aligned}$$

where the second line follows from the first by means of (99). Since the functions $\tilde{a}_j(t)$ have the explicit expression in (99), $\mathcal{L}_1, \mathcal{L}_2$ satisfy the following estimates as linear operators in the variable U,

$$\|\mathcal{L}_1(U, \pi^{(0)})\|_2 + \|\mathcal{L}_2(U, \pi^{(0)})\|_2 \lesssim \delta^2 \langle t\rangle^{-1}\|U\|_2; \tag{104}$$

see (81). Moreover, they are small as well as of finite rank and co-rank with ranges spanned by smooth, exponentially decreasing functions. Hence, the estimate (104) holds with any number of derivatives. In particular, we record the estimate

$$\|\nabla\mathcal{L}_1(U, \pi^{(0)})\|_{1\cap\infty} + \|\nabla\mathcal{L}_2(U, \pi^{(0)})\|_{1\cap\infty} \lesssim \delta^2 \langle t\rangle^{-1}\|U\|_{1+\infty} \tag{105}$$

for future use. Because of the small parameter δ^2 in (104), we shall solve for $U_{\rm dis}, U_{\rm hyp}$ by means of a contraction. However, recall that we yet need to determine the value of h. Thus, fix $\tilde U_{\rm dis}, \tilde U_{\rm hyp} \in C([0,\infty), L^2(\mathbb{R}^3)+L^\infty(\mathbb{R}^3))$, with

$$\sup_{t\ge 0} \langle t\rangle^{\frac32}\big[\|\tilde U_{\rm dis}(t)\|_{2+\infty} + \|\tilde U_{\rm hyp}(t)\|_{2+\infty}\big] \le \delta \tag{106}$$

and so that $\mathcal{J}\tilde U_{\rm dis} = \tilde U_{\rm dis}$ and $\mathcal{J}\tilde U_{\rm hyp} = \tilde U_{\rm hyp}$, and set

$$\begin{aligned} F_1(\tilde U_{\rm dis}, \tilde U_{\rm hyp}) := P_{\rm s}\big[&\mathcal{L}_1(\tilde U_{\rm dis}, \pi^{(0)}) + \mathcal{L}_2(\tilde U_{\rm hyp}, \pi^{(0)}) + \tilde N(U^{(0)}, \pi^{(0)}) \\ &+ V(\pi^{(0)})\tilde U_{\rm dis} + V(\pi^{(0)})\tilde U_{\rm hyp}\big] \end{aligned} \tag{107}$$

$$\begin{aligned} F_2(\tilde U_{\rm dis}, \tilde U_{\rm hyp}) := P_{\rm im}\big[&\mathcal{L}_1(\tilde U_{\rm dis}, \pi^{(0)}) + \mathcal{L}_2(\tilde U_{\rm hyp}, \pi^{(0)}) + \tilde N(U^{(0)}, \pi^{(0)}) \\ &+ V(\pi^{(0)})\tilde U_{\rm dis} + V(\pi^{(0)})\tilde U_{\rm hyp}\big]. \end{aligned} \tag{108}$$

In view of the definition (28), (80) and Lemma 7, the assumptions on $(U^{(0)}, \pi^{(0)})$ in Definition 19, as well as Lemma 21, the following bounds hold: If $t > 1$, then

$$\begin{aligned} \|\tilde N(U^{(0)}, \pi^{(0)})(t)\|_{1\cap 2} &\lesssim \|U^{(0)}(t)\|_\infty^2 + \|\,|U^{(0)}|^2 U^{(0)}(t)\,\|_{1\cap 2} \\ &\lesssim \delta^2\langle t\rangle^{-3} + \delta^2\langle t\rangle^{-\frac32}\|U^{(0)}(t)\|_2 \lesssim \delta^2\langle t\rangle^{-\frac32}. \end{aligned} \tag{109}$$

On the other hand, if $0 < t < 1$, then by Sobolev embedding,

$$\begin{aligned} \|\tilde N(U^{(0)}, \pi^{(0)})(t)\|_{1\cap 2} &\lesssim \|U^{(0)}(t)\|_4^2 + \|\,|U^{(0)}|^2 U^{(0)}(t)\,\|_2 \\ &\lesssim \|U^{(0)}(t)\|_{H^1}^2 + \|U^{(0)}(t)\|_{H^1}^3 \lesssim \delta^2, \end{aligned}$$

so that the bound in (109) holds for all $t \ge 0$. We therefore conclude from (106), (104) that for all $t \ge 0$

(110)
$$\max_{j=1,2}\|F_j(\tilde U_{\rm dis}, \tilde U_{\rm hyp})(t)\|_{1\cap 2} \lesssim \delta^2\langle t\rangle^{-\frac32} + \delta^2\langle t\rangle^{-1}\big[\|\tilde U_{\rm dis}(t)\|_{2+\infty} + \|\tilde U_{\rm hyp}(t)\|_{2+\infty}\big],$$

(111)
$$\begin{aligned} &\max_{j=1,2}\|F_j(\tilde U^{(1)}_{\rm dis}, \tilde U^{(1)}_{\rm hyp})(t) - F_j(\tilde U^{(2)}_{\rm dis}, \tilde U^{(2)}_{\rm hyp})(t)\|_{1\cap 2} \\ &\qquad \lesssim \delta^2\langle t\rangle^{-1}(\|(\tilde U^{(1)}_{\rm dis} - \tilde U^{(2)}_{\rm dis})(t)\|_{2+\infty} + \|(\tilde U^{(1)}_{\rm hyp} - \tilde U^{(2)}_{\rm hyp})(t)\|_{2+\infty}). \end{aligned}$$

Since the system (103) is $\mathcal{J}$-invariant in the usual sense, it follows that

$$\mathcal{J}F_1(\tilde U_{\rm dis}, \tilde U_{\rm hyp}) = -F_1(\tilde U_{\rm dis}, \tilde U_{\rm hyp}),\ \mathcal{J}F_2(\tilde U_{\rm dis}, \tilde U_{\rm hyp}) = -F_2(\tilde U_{\rm dis}, \tilde U_{\rm hyp}).$$

We now solve

$$i\partial_t U_{\rm dis} - \mathcal{H}(\alpha_\infty^{(0)})U_{\rm dis} = F_1(\tilde U_{\rm dis}, \tilde U_{\rm hyp}), \quad U_{\rm dis}(0) = P_{\rm s}(\alpha_\infty^{(0)})U(0), \tag{112}$$

$$i\partial_t U_{\rm hyp} - \mathcal{H}(\alpha_\infty^{(0)})U_{\rm hyp} = F_2(\tilde U_{\rm dis}, \tilde U_{\rm hyp}), \quad U_{\rm hyp}(0) = P_{\rm im}(\alpha_\infty^{(0)})U(0). \tag{113}$$

We can rewrite (113) in the equivalent form (using the basis $f^{\pm}(\alpha_\infty^{(0)})$)

$$\frac{d}{dt}\binom{b^+}{b^-} + \begin{pmatrix} -\sigma(\alpha_\infty^{(0)}) & 0 \\ 0 & \sigma(\alpha_\infty^{(0)}) \end{pmatrix}\binom{b^+}{b^-} = \binom{g^+}{g^-}$$

where $g^{\pm} \in \mathbb{R}$ satisfy

$$\sup_{t\geq 0} \langle t\rangle^{\frac{3}{2}} |g^{\pm}(t)| \lesssim \delta^2;$$

see (110) and (106). We impose the *stability condition* from Lemma 23; i.e.,

$$0 = b^+(0) + \int_0^\infty e^{-\sigma(\alpha_\infty^{(0)})s} g^+(s)\, ds. \tag{114}$$

We conclude from the bound on g^+ and (114) that

$$|b^+(0)| \lesssim \delta^2. \tag{115}$$

Recall that $b^+(0)$ is the coefficient of $f^{\pm}(\alpha_\infty^{(0)})$ in (97). Hence, in view of (74), we need to choose $h = h(\tilde{U}_{\rm dis}, \tilde{U}_{\rm hyp})$ so that

(116)

$$b^+(0) f^{\pm}(\alpha_\infty^{(0)}) = P^+_{\rm im}(\alpha_\infty^{(0)}) U(0)$$

$$= P^+_{\rm im}(\alpha_\infty^{(0)}) \mathcal{G}_\infty(\pi^{(0)})(0)\Big[\binom{R_0}{\bar{R}_0} + h f^+(\alpha_\infty^{(0)}) + \sum_{j=1}^{8} a_j \eta_j(\alpha_\infty^{(0)})\Big]. \tag{117}$$

We claim that (115) implies that $|h| \lesssim \delta^2$. To see this, we of course need to use the assumption that $P^+_{\rm im}(\alpha_0)\binom{R_0}{\bar{R}_0} = 0$. Thus, using the notation and estimates of Corollary 18 we conclude that

$$\begin{aligned}
&\Big\| P^+_{\rm im}(\alpha_\infty^{(0)}) \mathcal{G}_\infty(\pi^{(0)})(0)\binom{R_0}{\bar{R}_0}\Big\|_{1\cap 2} \\
&\quad = \Big\| P^+_{\rm im}(\alpha_0)\binom{R_0}{\bar{R}_0} - P^+_{\rm im}(\alpha_\infty^{(0)}) \mathcal{G}_\infty(\pi^{(0)})(0)\binom{R_0}{\bar{R}_0}\Big\|_{1\cap 2} \\
&\quad = \Big\| f^+(\alpha_0)\Big\langle \binom{R_0}{\bar{R}_0}, \tilde{f}^+(\alpha_0)\Big\rangle - f^+(\alpha_\infty^{(0)})\Big\langle \binom{R_0}{\bar{R}_0}, \mathcal{G}_\infty(\pi^{(0)})(0)^* \tilde{f}^+(\alpha_\infty^{(0)})\Big\rangle \Big\|_{1\cap 2} \\
&\quad \lesssim \|f^+(\alpha_0) - f^+(\alpha_\infty^{(0)})\|_2 \|R_0\|_2 \\
&\qquad + \|R_0\|_2 \big\|\mathcal{G}_\infty(\pi^{(0)})(0)^* \tilde{f}^+(\alpha_0) - \tilde{f}^+(\alpha_0)\big\|_2 \lesssim \delta^3.
\end{aligned}$$

To pass to the final inequality, we invoke the bound

$$\|\mathcal{G}_\infty(\pi^{(0)})(0)^* f - f\|_2 \lesssim \delta^2(\|f\|_{H^1} + \|\langle x\rangle f\|_2).$$

The appearance of the weight here is the reason we did not estimate the difference between $\mathcal{G}_\infty(\pi^{(0)})(0)\binom{R_0}{\bar{R}_0}$ and $\binom{R_0}{\bar{R}_0}$. We conclude from (116), (115), and (83) that

$$|h| \lesssim \delta^2, \quad \sum_{j=1}^{8} |a_j(h)| \lesssim \delta^4. \tag{118}$$

Note that this estimate requires the full strength of the assumption $P_u^+(\alpha_0)\binom{R_0}{\bar R_0} = 0$. In particular, (95) holds. It is now easy to prove the Lipschitz bound (96). Indeed, if $h^{(1)}$, $h^{(0)}$ are associated with R_0, R_1, respectively, then

$$\begin{aligned}(h^{(1)} - h^{(0)}) & P_{\rm im}^+(\alpha_\infty^{(0)})\mathcal{G}_\infty(\pi^{(0)})(0)f^+(\alpha_\infty^{(0)}) \\ &= P_{\rm im}^+(\alpha_\infty^{(0)})\mathcal{G}_\infty(\pi^{(0)})(0)\Big[\begin{pmatrix} R_0 \\ \bar R_0\end{pmatrix} - \begin{pmatrix} R_1 \\ \bar R_1\end{pmatrix}\Big] \\ &\quad + P_{\rm im}^+(\alpha_\infty^{(0)})\mathcal{G}_\infty(\pi^{(0)})(0)\sum_{j=1}^8 [a_j(h^{(1)}) - a_j(h^{(0)})]\tilde\eta_j(\alpha_\infty^{(0)}),\end{aligned}$$

where we used (82) to write $a_j = a_j(h)$. Moreover,

$$|a_j(h^{(0)}) - a_j(h^{(1)})| \lesssim \delta^2 |h^{(1)} - h^{(0)}|,$$

and (96) follows by taking L^2-norms. For simplicity, let $a_j(\tilde U_{\rm dis}, \tilde U_{\rm hyp}) := a_j(h(\tilde U_{\rm dis}, \tilde U_{\rm hyp}))$. Define the map $\Psi_0 : (\tilde U_{\rm dis}, \tilde U_{\rm hyp}) \mapsto (U_{\rm dis}, U_{\rm hyp})$ by means of

(119)
$$\begin{aligned} U_{\rm dis}(t) &= e^{-it\mathcal{H}(\alpha_\infty^{(0)})}U_{\rm dis}(0) - i\int_0^t e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}F_1(\tilde U_{\rm dis}, \tilde U_{\rm hyp})(s)\,ds, \\ U_{\rm dis}(0) &= P_{\rm s}(\alpha_\infty^{(0)})\mathcal{G}_\infty(\pi^{(0)})(0) \\ &\quad \times\Big[\begin{pmatrix} R_0 \\ \bar R_0\end{pmatrix} + h(\tilde U_{\rm dis}, \tilde U_{\rm hyp})f^+(\alpha_\infty^{(0)}) + \sum_{j=1}^8 a_j(\tilde U_{\rm dis}, \tilde U_{\rm hyp})\eta_j(\alpha_\infty^{(0)})\Big],\end{aligned}$$

(120)
$$\begin{aligned} U_{\rm hyp}(t) &= e^{-it\mathcal{H}(\alpha_\infty^{(0)})}U_{\rm hyp}(0) - i\int_0^t e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}F_2(\tilde U_{\rm dis}, \tilde U_{\rm hyp})(s)\,ds, \\ U_{\rm hyp}(0) &= P_{\rm im}(\alpha_\infty^{(0)})\mathcal{G}_\infty(\pi^{(0)})(0) \\ &\quad \times\Big[\begin{pmatrix} R_0 \\ \bar R_0\end{pmatrix} + h(\tilde U_{\rm dis}, \tilde U_{\rm hyp})f^+(\alpha_\infty^{(0)}) + \sum_{j=1}^8 a_j(\tilde U_{\rm dis}, \tilde U_{\rm hyp})\eta_j(\alpha_\infty^{(0)})\Big].\end{aligned}$$

By (118) and (94),

$$\|U_{\rm dis}(0)\|_{1\cap 2} + \|U_{\rm hyp}(0)\|_{1\cap 2} \lesssim \delta_0 + \delta^2,$$

where $\delta_0 := c_1\delta$. We claim that, with c_0 being the small constant from (66), (67),

$$\sup_{t\geq 0}\langle t\rangle^{\frac32}\big[\|U_{\rm dis}(t)\|_{2+\infty} + \|U_{\rm hyp}(t)\|_{2+\infty}\big] \leq c_0\delta. \tag{121}$$

To verify this claim, we use the linear bound of Theorem 34 and 35 on $U_{\rm dis}$. Because of (110) this leads to

$$\begin{aligned}\|U_{\rm dis}(t)\|_{2+\infty} &\lesssim \langle t\rangle^{-\frac32}(\delta_0 + \delta^2) + \int_0^t \delta^2\langle t-s\rangle^{-\frac32}\langle s\rangle^{-\frac32}\,ds \\ &\lesssim \langle t\rangle^{-\frac32}(c_1\delta + \delta^2) \leq c_0\frac{\delta}{2}\langle t\rangle^{-\frac32}\end{aligned}$$

for all $t \geq 0$ provided c_1 was chosen small enough. Similarly, because of our choice of h, see (93), we obtain that for all $t \geq 0$

$$\|U_{\rm hyp}(t)\|_2 \lesssim \int_t^\infty e^{-\sigma(\alpha_\infty^{(0)})(s-t)}\delta^2\langle s\rangle^{-\frac{3}{2}}\,ds + e^{-\sigma(\alpha_\infty^{(0)})t}(\delta_0+\delta^2) + \int_0^t e^{-\sigma(\alpha_\infty^{(0)})(t-s)}\delta^2\langle s\rangle^{-\frac{3}{2}}\,ds \leq c_0\frac{\delta}{2},$$

and (121) follows. Next, we claim that the map Ψ_0 is a contraction in the space of $\mathcal{J}$-invariant functions satisfying (106). To see this, we first remark that there is the Lipschitz bound

$$|h(\tilde{U}_{\rm dis}^{(1)},\tilde{U}_{\rm hyp}^{(1)}) - h(\tilde{U}_{\rm dis}^{(2)},\tilde{U}_{\rm hyp}^{(2)})| \lesssim \delta^2 \sup_{t\geq 0}\langle t\rangle^{\frac{3}{2}}(\|(\tilde{U}_{\rm dis}^{(1)}-\tilde{U}_{\rm dis}^{(2)})(t)\|_{2+\infty} + \|(\tilde{U}_{\rm hyp}^{(1)}-\tilde{U}_{\rm hyp}^{(2)})(t)\|_{2+\infty}). \tag{122}$$

This is a consequence of (111) and the explicit expressions for $b^+(0)$ and h in (114) and (116). Since the coefficients a_j are linear in h, they satisfy the exact same bounds. Let $(U_{\rm dis}^{(j)}, U_{\rm hyp}^{(j)}) = \Psi_0(\tilde{U}_{\rm dis}^{(j)},\tilde{U}_{\rm hyp}^{(j)})$ for $j=1,2$. Subtracting the two equations in (119) for $j=1,2$ with the corresponding difference of initial conditions, and applying Theorems 34, 35 lead to

$$\sup_{t\geq 0}\langle t\rangle^{\frac{3}{2}}\|U_{\rm dis}^{(1)}(t)-U_{\rm dis}^{(2)}(t)\|_{2+\infty} \lesssim \delta^2\sup_{s\geq 0}\langle s\rangle^{\frac{3}{2}}(\|(\tilde{U}_{\rm dis}^{(1)}-\tilde{U}_{\rm dis}^{(2)})(s)\|_{2+\infty} + \|(\tilde{U}_{\rm hyp}^{(1)}-\tilde{U}_{\rm hyp}^{(2)})(s)\|_{2+\infty}).$$

Note that the difference $(U_{\rm hyp}^{(1)}-U_{\rm hyp}^{(2)})(t)$ is potentially dangerous, since we cannot adjust the initial condition to make sure that the stability condition holds. The point is, however, that this condition holds automatically since

$$\sup_{t\geq 0}\|(U_{\rm hyp}^{(1)}-U_{\rm hyp}^{(2)})(t)\|_2 < \infty.$$

Lemma 23 therefore guarantees that both (92) and (93) hold for $U_{\rm hyp}^{(1)}-U_{\rm hyp}^{(2)}$. In particular, one concludes that in this case as well

$$\sup_{t\geq 0}\langle t\rangle^{\frac{3}{2}}\|(U_{\rm hyp}^{(1)}-U_{\rm hyp}^{(2)})(t)\|_2 \lesssim \delta^2\sup_{s\geq 0}\langle s\rangle^{\frac{3}{2}}(\|(\tilde{U}_{\rm dis}^{(1)}-\tilde{U}_{\rm dis}^{(2)})(s)\|_{2+\infty} + \|(\tilde{U}_{\rm hyp}^{(1)}-\tilde{U}_{\rm hyp}^{(2)})(s)\|_{2+\infty}),$$

and we have shown that Ψ_0 is indeed a contraction. The conclusion is that there exist $\mathcal{J}$-invariant functions $(U_{\rm dis}, U_{\rm hyp})$ satisfying (121) as well the system (103). In addition, there exist $h, a_j(h) \in \mathbb{R}$ as in (118) determining the initial conditions (74).

Next, observe that the solution $(U_{\rm dis}, U_{\rm hyp})$ which we just constructed also satisfies the bound

$$\sup_{t\ge 0}(\|U_{\rm dis}(t)\|_2 + \|U_{\rm hyp}(t)\|_2) \le c_0\,\delta. \tag{123}$$

To see this, it suffices to deal with $U_{\rm dis}(t)$. Applying Theorem 34 to (119) yields

$$\begin{aligned}\sup_{t\ge 0}\|U_{\rm dis}(t)\|_2 &\lesssim \|U_{\rm dis}(0)\|_2\\ &\quad + \int_0^\infty \Big[\delta^2\langle s\rangle^{-1}(\|U_{\rm dis}(s)\|_{2+\infty} + \|U_{\rm hyp}(s)\|_{2+\infty}) + \delta^2\langle s\rangle^{-\frac32}\\ &\quad + \|V(\pi^{(0)})(s)\|_{2\cap\infty}\|(U_{\rm dis}+U_{\rm hyp})(s)\|_{2+\infty}\Big]\,ds\\ &\lesssim (\delta_0+\delta^2) + \int_0^\infty \Big[\delta^3\langle s\rangle^{-\frac52} + \delta^2\langle s\rangle^{-\frac32} + \delta^2\langle s\rangle^{-\frac52}\Big]\,ds\\ &\ll c_0\delta,\end{aligned}$$

as desired. Retracing our steps we now reintroduce $U_{\rm root}$ via (98) which leads to a (weak, i.e., Duhamel) solution $(\pi(t), U(t))$ of the system (72), (73) with initial conditions (74), (75). Moreover, $U(t)$ is $\mathcal{J}$-invariant, and $\pi(t)\in\mathbb{R}^8$ for all $t\ge 0$, and the orthogonality condition (78) holds.

Note that (121) insures that

$$\sup_{t\ge 0}\langle t\rangle^{-\frac32}\|U(t)\|_{2+\infty} \le c_0\delta,\ \sup_{t\ge 0}\|U(t)\| \le c_0\delta.$$

Estimating the two terms involving U on the right-hand side of (73) by means of this bound and the bounds from Lemma 21 leads to the estimate

$$|\dot\alpha(t)| + |\dot v(t)| + |\dot{\tilde\gamma}(t)| + |\dot D(t)| \le \delta^2\langle t\rangle^{-3}$$

for all $t\ge 0$ (this is where we need to use the small c_0 in (66), (67)). This is precisely (65). Strictly speaking, (65) can be improved by a small factor of $\asymp c_0^2$ on the right-hand side. However, here and in what follows we ignore this improvement.

It remains to show that our solution $U(t)$ satisfies the other bounds in (66)–(68). Moreover, we have only shown that $(U_{\rm dis}, U_{\rm hyp})$ satisfies the system (103) in the weak (i.e., Duhamel) sense. However, once we prove

$$\sup_{t\ge 0}\Big[\|\nabla U_{\rm dis}(t)\|_2 + \|\nabla U_{\rm hyp}(t)\|_2\Big] \le c_0\delta \tag{124}$$

it will follow that (66) holds and that $(U_{\rm dis}, U_{\rm hyp})$ solves (103) in the strong sense, i.e., in

$$C([0,\infty), H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)) \cap C^1([0,\infty), H^{-1}(\mathbb{R}^3)\times H^{-1}(\mathbb{R}^3)). \tag{125}$$

The details of (124) are as follows: Clearly, we need to show that the conditions (66)–(68) are consistent with our contraction scheme. Thus, in addition to (106) we assume that $\tilde U_{\rm dis}, \tilde U_{\rm hyp}$ satisfy these assumptions and then check that $U_{\rm dis}, U_{\rm hyp}$ satisfy them as well First, by the nature of $\mathrm{Ran}(P_{\rm im})$, see Lemma 17,

$$\|U_{\rm hyp}(t)\|_{H^1} \lesssim \|U_{\rm hyp}(t)\|_{L^2} \quad \text{for all} \quad t \geq 0,$$

so that it suffices to deal with $U_{\rm dis}$. Second, by the Strichartz estimates of Corollary 39, as well as (80) and Lemma 7 we obtain

(126)
$$\sup_{0\le t}\Big\|\nabla \int_0^t e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}\, F_1(\tilde U_{\rm dis}, \tilde U_{\rm hyp})(s)\, ds\Big\|_2 \lesssim \int_0^\infty \Big(\|\mathcal{L}_1(\tilde U_{\rm dis}, \pi^{(0)})(s)\|_{H^1} + \|\mathcal{L}_2(\tilde U_{\rm hyp}, \pi^{(0)})(s)\|_{H^1} + \|V(\pi^{(0)})\tilde U_{\rm hyp}(s)\|_{H^1}\Big)\, ds$$

(127)
$$+ \int_0^\infty \Big(\|N(U^{(0)}, \pi^{(0)})(s)\|_1 + \|U^{(0)}(s)\|_4^2 + \|U^{(0)}\nabla U^{(0)}(s)\|_2 + \|V(\pi^{(0)})\tilde U_{\rm dis}(s)\|_{H^1}\Big)\, ds$$

(128)
$$+ \Big(\int_0^\infty \|\,|U^{(0)}|^2 U^{(0)}(s)\|_{L^{\frac43}(\mathbb{R}^3)}^{\frac85}\, ds\Big)^{\frac58} + \Big(\int_0^\infty \|\,|U^{(0)}|^2 \nabla U^{(0)}(s)\|_{L^{\frac43}(\mathbb{R}^3)}^{\frac85}\, ds\Big)^{\frac58}.$$

In view of (105) and (106), the contribution of (126) is

$$\lesssim \int_0^\infty \delta^2 \langle s\rangle^{-1} (\|\tilde U_{\rm dis}(s)\|_{2+\infty} + \|\tilde U_{\rm hyp}(s)\|_{2+\infty})\, ds \lesssim \delta^3.$$

By (36), (66) and (67),

$$\begin{aligned}\|N(U^{(0)}, \pi^{(0)})(s)\|_1 &\lesssim \min(\|U^{(0)}(s)\|_\infty^2, \|U^{(0)}(s)\|_2^2) + \|U^{(0)}(s)\|_3^3 \\ &\lesssim \delta^2\langle s\rangle^{-3} + \delta^2 \langle s\rangle^{-\frac32} \lesssim \delta^2\langle s\rangle^{-\frac32}.\end{aligned}$$

Furthermore, if $0 < s < 1$, then we estimate

$$\begin{aligned}\|U^{(0)}(s)\|_4^2 + \|U^{(0)}\nabla U^{(0)}(s)\|_2 &\lesssim \|U^{(0)}(s)\|_{H^1}^2 + \|U^{(0)}(s)\|_{L^4\cap L^{\frac{2q}{q-2}}} \|\nabla U^{(0)}(s)\|_{4+q} \\ &\lesssim \delta^2 + \delta^2 s^{-\frac34} \lesssim \delta^2 s^{-\frac34},\end{aligned}$$

whereas for $s > 1$ we have

$$\begin{aligned}\|U^{(0)}(s)\|_4^2 + \|U^{(0)}\nabla U^{(0)}(s)\|_2 &\lesssim \|U^{(0)}(s)\|_2 \|U^{(0)}(s)\|_\infty + \|U^{(0)}(s)\|_\infty \|\nabla U^{(0)}(s)\|_2 \\ &\lesssim \delta^2 s^{-\frac32} + \delta^2 s^{-\frac32} \lesssim \delta^2 s^{-\frac32}.\end{aligned}$$

Hence the first three terms in (127) are integrable and their contribution is $\lesssim \delta^2$. As far as the final term in (127) is concerned, note that

$$\begin{aligned}\|V(\pi^{(0)})\tilde U_{\rm dis}(s)\|_{H^1} &\lesssim \|V(\pi^{(0)})\tilde U_{\rm dis}(s)\|_2 \\ &\quad + \|(\nabla V)(\pi^{(0)})\tilde U_{\rm dis}(s)\|_2 + \|V(\pi^{(0)})\nabla \tilde U_{\rm dis}(s)\|_2\end{aligned}$$

$$\lesssim \delta^3 \langle s\rangle^{-\frac{5}{2}} + \|V(\pi^{(0)})\|_{L^4\cap L^{\frac{2q}{q-2}}} \|\nabla \tilde{U}_{\rm dis}(s)\|_{4+q}$$
$$\lesssim \delta^3 \langle s\rangle^{-\frac{5}{2}} + \delta^2 \langle s\rangle^{-1} \delta s^{-\frac{3}{4}},$$

which contributes $\lesssim \delta^2$ to (127). Here we used (66)-(68), as well as the bound from Lemma 21. Previously, we derived the bound

$$\| \, |U^{(0)}|^2 \nabla U^{(0)}(s)\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \lesssim \delta^3 s^{-\frac{3}{8}},$$

for $0 < s < 1$; see (89). On the other hand, if $s > 1$, then

$$\begin{aligned}\| \, |U^{(0)}|^2 \nabla U^{(0)}(s)\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} &\lesssim \|\nabla U^{(0)}(s)\|_2 \|U^{(0)}(s)\|_8^2 \\ &\lesssim \delta \|U^{(0)}(s)\|_\infty^{\frac{3}{2}} \|U^{(0)}(s)\|_2^{\frac{1}{2}} \lesssim \delta^3 s^{-\frac{9}{4}}.\end{aligned}$$

Hence,

$$\Big(\int_0^\infty \| \, |U^{(0)}|^2 \nabla U^{(0)}(s)\|^{\frac{8}{5}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} \, ds\Big)^{\frac{5}{8}} \lesssim \delta^3,$$

and similarly for the term without a gradient in (128). We have proved (124) and therefore also the gradient part of (66).

For the remainder of the proof we will subscribe to the somewhat imprecise practice of replacing the term $V(\pi^{(0)})U(t)$ with $V(\pi^{(0)})U^{(0)}(t)$ in (72). This will allow us to avoid working with $\tilde{U}_{\rm dis}, \tilde{U}_{\rm hyp}$ and instead will make it possible to estimate $U(t)$ directly. The logic here is that we will only use the bounds (66)–(68) to estimate $U^{(0)}$, just as we would in order to show that the contraction scheme is consistent with the remaining conditions (67), (68).

Thus we turn to proving $\|P_{\rm s} U(t)\|_\infty \le \delta t^{-\frac{3}{2}}$ for $t > 0$. It will be necessary to bound various terms in $L^1(\mathbb{R}^3)$. One such term is, see (36) and (65),

$$\begin{aligned}&\| -i\dot{\pi}\partial_\pi \widetilde{W}(\pi^{(0)}) + \tilde{N}(U^{(0)}, \pi^{(0)})\|_1 \\ &\quad \lesssim |\dot{\alpha}(s)| + |\dot{v}(s)| + |\dot{\tilde{\gamma}}(s)| + |\dot{D}(s)| + \|U^{(0)}(s)\|_\infty^2 + \|U^{(0)}(s)\|_3^3 \\ &\quad \lesssim \delta^2 s^{-3} + \|U^{(0)}(s)\|_\infty^2 + \|U^{(0)}(s)\|_2^2 \|U^{(0)}(s)\|_\infty \lesssim \delta^2 s^{-\frac{3}{2}},\end{aligned}$$

provided $s \ge 1$. If $0 \le s \le 1$, then one argues similarly. More precisely, using (66) and Sobolev embedding instead of (67), we obtain

$$\| -i\dot{\pi}\partial_\pi \widetilde{W}(\pi^{(0)})(s) + \tilde{N}(U^{(0)}, \pi^{(0)})(s)\|_1 \lesssim \delta^2.$$

Another term is

$$\|V(\pi^{(0)})U^{(0)}(s)\|_1 \lesssim \delta^2 \langle s\rangle^{-\frac{5}{2}},$$

valid for all $s \ge 0$. This follows from Lemma 21, and (66), (67). We now rewrite (72) via the Duhamel formula. Let us first consider the case $t \ge 1$. Then, by the embedding $W^{1,4} \hookrightarrow L^\infty$,

$$\begin{aligned}\|P_{\rm s} U(t)\|_\infty \le &\|e^{-it\mathcal{H}(\alpha_\infty^{(0)})} P_{\rm s} U(0)\|_\infty \\ &+ \int_0^t \|e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})} P_{\rm s}(-i\dot{\pi}(s)\partial_\pi \widetilde{W}(\pi^{(0)})(s) + \tilde{N}(U^{(0)}, \pi^{(0)})(s)\end{aligned}$$

$$
\begin{aligned}
&+V(\pi^{(0)})U^{(0)}(s))\|_\infty\,ds\\
&\lesssim t^{-\frac32}\|U(0)\|_1+\int_0^{t-\frac12}(t-s)^{-\frac32}(\|-i\dot\pi(s)\partial_\pi\widetilde{W}(\pi^{(0)})(s)\\
&+\tilde N(U^{(0)},\pi^{(0)})(s)\|_1+\|V(\pi^{(0)})U^{(0)}(s)\|_1)\,ds\\
&+\int_{t-\frac12}^t\|e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}P_{\rm s}(-i\dot\pi(s)\partial_\pi\widetilde{W}(\pi^{(0)})(s)\\
&+\tilde N(U^{(0)},\pi^{(0)})(s)+V(\pi^{(0)})U^{(0)})(s))\|_{W^{1,4}}\,ds.
\end{aligned}
$$

Invoking the L^1 bounds which we just derived on the right-hand side yields

(129)

$$
\begin{aligned}
\|P_{\rm s}U(t)\|_\infty &\lesssim t^{-\frac32}\|U(0)\|_1+\int_0^{t-\frac12}(t-s)^{-\frac32}\delta^2\langle s\rangle^{-\frac32}\,ds\\
&+\int_{t-\frac12}^t(t-s)^{-\frac34}\|-i\dot\pi(s)\partial_\pi\widetilde{W}(\pi^{(0)})(s)\\
&+\tilde N(U^{(0)},\pi^{(0)})(s)+V(\pi^{(0)})U^{(0)})(s)\|_{W^{1,\frac43}}\,ds\\
&\lesssim t^{-\frac32}\|U(0)\|_1+\delta^2t^{-\frac32}\\
&+\int_{t-\frac12}^t(t-s)^{-\frac34}\Big[\|\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)})\|_{W^{1,\frac43}}+\||U^{(0)}|^2\phi\|_{\frac43}+\|U^{(0)}\nabla U^{(0)}\phi\|_{\frac43}\\
&+\||U^{(0)}|^2\nabla\phi\|_{\frac43}+\|\,|U^{(0)}|^2U^{(0)}\|_{\frac43}+\|\,|U^{(0)}|^2\nabla U^{(0)}\|_{\frac43}\\
&+\|V(\pi^{(0)})U^{(0)}(s)\|_{\frac43}+\|U^{(0)}\nabla V(\pi^{(0)})\|_{\frac43}+\|V(\pi^{(0)})\nabla U^{(0)}\|_{\frac43}\Big](s)\,ds.
\end{aligned}
$$

Here we have used the slightly formal notation $|U^{(0)}|^2\phi$ for the quadratic part of the nonlinearity $N(U^{(0)},\pi^{(0)})$. In view of our assumptions (65)-(67) on $U^{(0)}$,

$$
\begin{aligned}
&\|\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)})(s)\|_{W^{1,\frac43}}+\||U^{(0)}|^2(s)\phi\|_{\frac43}+\||U^{(0)}|^2(s)\nabla\phi\|_{\frac43}\lesssim\delta^2\langle s\rangle^{-3},\\
&\|U^{(0)}\nabla U^{(0)}(s)\phi\|_{\frac43}\lesssim\|U^{(0)}(s)\|_\infty\|\nabla U^{(0)}(s)\|_2\lesssim\delta^2s^{-\frac32}\quad\text{provided}\quad s\ge1,\\
&\|U^{(0)}\nabla U^{(0)}(s)\phi\|_{\frac43}\lesssim\|U^{(0)}(s)\|_4\|\nabla U^{(0)}(s)\|_2\lesssim\delta^2\quad\text{provided}\quad 0<s\le1,\\
&\|\,|U^{(0)}|^2U^{(0)}(s)\|_{\frac43}\lesssim\|U^{(0)}(s)\|_2^{\frac32}\|U^{(0)}(s)\|_\infty^{\frac32}\lesssim\delta^3s^{-\frac94}\quad\text{if}\quad s\ge1,\\
&\|\,|U^{(0)}|^2U^{(0)}(s)\|_{\frac43}=\|U^{(0)}(s)\|_4^3\lesssim\delta^3\quad\text{if}\quad 0<s<1.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\|\,|U^{(0)}|^2\nabla U^{(0)}(s)\|_{\frac43}\le\|U^{(0)}(s)\|_8^2\|\nabla U^{(0)}(s)\|_2\lesssim\delta^3s^{-\frac94}\quad\text{if}\quad s\ge\frac12,\\
&\|\,|U^{(0)}|^2\nabla U^{(0)}(s)\|_{\frac43}\le\|U^{(0)}(s)\|_6^{\frac32}\|U^{(0)}(s)\|_\infty^{\frac12}\|\nabla U^{(0)}(s)\|_2\\
&\qquad\qquad\lesssim\delta^3s^{-\frac34}\quad\text{if}\quad 0<s<\frac12,\\
&\|V(\pi^{(0)})U^{(0)}(s)\|_{\frac43}+\|U^{(0)}(s)\nabla V(\pi^{(0)})(s)\|_{\frac43}\lesssim\delta^3\langle s\rangle^{-\frac52}.
\end{aligned}
$$

It remains to consider the bounds on $\|V(\pi^{(0)})\nabla U^{(0)}(s)\|_{\frac43}$ for all $s>0$. Note that the latter is always $\lesssim \delta^3\langle s\rangle^{-1}$, by (65) and (66), but this is insufficient. At this point we need to use (68) to generate more decay. Indeed,

$$\|V(\pi^{(0)})\nabla U^{(0)}(s)\|_{\frac43} \lesssim \|V(\pi^{(0)})\|_{L^2\cap L^{\frac{4q}{3q-4}}}\|\nabla U^{(0)}(s)\|_{L^4+L^q} \lesssim \delta^3\langle s\rangle^{-1}s^{-\frac34}.$$

Inserting these bounds into (129) leads to (recall $t\ge 1$),

$$\|P_{\mathrm{s}}U(t)\|_\infty \lesssim (\delta_0+\delta^2)t^{-\frac32}+\delta^2 t^{-\frac32}+\int_{t-\frac12}^t (t-s)^{-\frac34}\delta^2 s^{-\frac32}\,ds \le c_0\frac{\delta}{2}t^{-\frac32},$$

provided $c_1 \ll c_0$ and δ are sufficiently small. To deal with the range $0<t<1$, we perform a similar estimate, using now the small time cases of the previous bounds:

$$\begin{aligned}
&(130)\\
&\|P_{\mathrm{s}}U(t)\|_\infty \lesssim \|e^{-it\mathcal{H}(\alpha_\infty^{(0)})}P_{\mathrm{s}}U(0)\|_\infty\\
&+\int_0^t \|\nabla e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}P_{\mathrm{s}}(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)})+\tilde N(U^{(0)},\pi^{(0)})+V(\pi^{(0)})U^{(0)})(s)\|_4\,ds\\
&+\int_0^t \|e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}P_{\mathrm{s}}(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)})+\tilde N(U^{(0)},\pi^{(0)})+V(\pi^{(0)})U^{(0)})(s)\|_2\,ds\\
&(131)\\
&\quad\lesssim t^{-\frac32}\|U(0)\|_1+\delta^2+\int_0^t (t-s)^{-\frac34}\Big[\|\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)})\|_{W^{1,\frac43}}+\||U^{(0)}|^2\phi\|_{\frac43}\\
&\qquad+\|U^{(0)}\nabla U^{(0)}\phi\|_{\frac43}+\|\,|U^{(0)}|^2U^{(0)}\|_{\frac43}+\|\,|U^{(0)}|^2\nabla U^{(0)}\|_{\frac43}+\|V(\pi^{(0)})U^{(0)}(s)\|_{\frac43}\\
&\qquad+\|U^{(0)}\nabla V(\pi^{(0)})\|_{\frac43}+\|V(\pi^{(0)})\nabla U^{(0)}\|_{\frac43}\Big](s)\,ds\\
&\quad\lesssim t^{-\frac32}\|U(0)\|_1+\delta^2\int_0^t (t-s)^{-\frac34}s^{-\frac34}\,ds \lesssim t^{-\frac32}(\delta_0+\delta^2)+\delta^2 t^{-\frac12} \le c_0\frac{\delta}{2}t^{-\frac32},
\end{aligned}$$

provided c_1,δ are small. Here (130) comes about because of the Sobolev embedding bound

$$\|f\|_{L^\infty} \lesssim \|\nabla f\|_4+\|f\|_2.$$

Since $t<1$ it makes the harmless contribution δ^2 to the following line.

The only remaining bound on the infinite dimensional evolution $P_{\mathrm{s}}U(t)$ is (68). Here q is chosen very large so that the dispersive $L^{q'}(\mathbb{R}^3)\to L^q(\mathbb{R}^3)$ decay is $t^{-\frac32+}$. The reason we do not take $q=\infty$ can be found in Corollary 38 below. Thus, with $(t-\frac12)_+=\max(t-\frac12,0)$,

$$\begin{aligned}\|\nabla P_{\rm s}U(t)\|_{L^4+L^q} \le \|\nabla e^{-it\mathcal{H}(\alpha_\infty^{(0)})}P_{\rm s}U(0)\|_{L^4} \\ + \int_0^{(t-\frac12)_+} \big\|\nabla e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}P_{\rm s}(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)}) \\ +\tilde N(U^{(0)},\pi^{(0)}) + V(\pi^{(0)})U^{(0)})(s)\big\|_q\, ds \\ + \int_{t-\frac12}^t \big\|\nabla e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}P_{\rm s}(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)}) \\ +\tilde N(U^{(0)},\pi^{(0)}) + V(\pi^{(0)})U^{(0)})(s)\big\|_4\, ds\ \chi_{[t\ge1]} \\ + \int_0^t \big\|\nabla e^{-i(t-s)\mathcal{H}(\alpha_\infty^{(0)})}P_{\rm s}(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)}) \\ +\tilde N(U^{(0)},\pi^{(0)}) + V(\pi^{(0)})U^{(0)})(s)\big\|_4\, ds\ \chi_{[0<t<1]}.\end{aligned}$$

Interchanging ∇ with the evolution as before, and invoking the dispersive estimate yield

(132)
$$\begin{aligned}&\|\nabla P_{\rm s}U(t)\|_{L^4+L^q} \lesssim t^{-\frac34}\|U(0)\|_{W^{1,\frac43}} \\ &+ \int_0^{(t-\frac12)_+} (t-s)^{-\frac32+}\|(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)}) + \tilde N(U^{(0)},\pi^{(0)}) + V(\pi^{(0)})U^{(0)})(s)\|_{W^{1,q'}}\, ds \\ &+ \int_{t-\frac12}^t (t-s)^{-\frac34}\|(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)}) + \tilde N(U^{(0)},\pi^{(0)}) + V(\pi^{(0)})U^{(0)})(s)\|_{W^{1,\frac43}}\, ds\ \chi_{[t\ge1]} \\ &+ \int_0^t (t-s)^{-\frac34}\|(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)}) + \tilde N(U^{(0)},\pi^{(0)}) + V(\pi^{(0)})U^{(0)})(s)\|_{W^{1,\frac43}}\, ds\ \chi_{[0<t<1]}.\end{aligned}$$

The terms involving the $W^{1,\frac43}$ arose already in (129) and (131) above. Invoking these terms shows that the two final integrals in (132) contribute

$$\begin{aligned}(133)\qquad &\lesssim \int_{t-\frac12}^t (t-s)^{-\frac34}\delta^2\langle s\rangle^{-\frac32}\, ds\ \chi_{[t\ge1]} + \int_0^t (t-s)^{-\frac34}\delta^2 s^{-\frac34}\, ds\ \chi_{[0<t<1]} \\ &\lesssim \delta^2 t^{-\frac32}\chi_{[t\ge1]} + \delta^2 t^{-\frac12}\chi_{[0<t<1]} \lesssim \delta^2 t^{-\frac34},\end{aligned}$$

as desired. It remains to bound the integral involving the $W^{1,q'}$ norm in (132). First, we have

$$\begin{aligned}\|(-i\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)}) + \tilde N(U^{(0)},\pi^{(0)}) + V(\pi^{(0)})U^{(0)})(s)\|_{L^{q'}} \\ \lesssim \delta^2\langle s\rangle^{-3} + \|U^{(0)}\|_{3q'}^3 \lesssim \delta^2\langle s\rangle^{-\frac32}.\end{aligned}$$

This bound is a small variation of previous ones, and we skip the details. Second, we derive the following variant of the $L^{\frac43}$ bounds obtained above: In view of our assumptions (65)-(68) on $U^{(0)}$,

$$\begin{aligned}&\|\dot\pi\partial_\pi\widetilde{W}(\pi^{(0)})(s)\|_{W^{1,1}} + \||U^{(0)}|^2(s)\phi\|_1 + \||U^{(0)}|^2(s)\nabla\phi\|_1 \lesssim \delta^2\langle s\rangle^{-3}, \\ &\|U^{(0)}\nabla U^{(0)}(s)\phi\|_1 \lesssim \|U^{(0)}(s)\|_\infty\|\nabla U^{(0)}(s)\|_2 \lesssim \delta^2 s^{-\frac32} \quad \text{provided} \quad s\ge 1, \\ &\|U^{(0)}\nabla U^{(0)}(s)\phi\|_1 \lesssim \|U^{(0)}(s)\|_2\|\nabla U^{(0)}(s)\|_2 \lesssim \delta^2 \quad \text{provided} \quad 0<s\le 1,\end{aligned}$$

$$\| |U^{(0)}|^2 U^{(0)}(s)\|_1 \lesssim \|U^{(0)}(s)\|_2^2 \|U^{(0)}(s)\|_\infty \lesssim \delta^3 s^{-\frac{3}{2}} \quad \text{if} \quad s \geq 1,$$
$$\| |U^{(0)}|^2 U^{(0)}(s)\|_1 = \|U^{(0)}(s)\|_3^3 \lesssim \delta^3 \quad \text{if} \quad 0 < s < 1.$$

Furthermore,

$$\| |U^{(0)}|^2 \nabla U^{(0)}(s)\|_1 \leq \|U^{(0)}(s)\|_4^2 \|\nabla U^{(0)}(s)\|_2 \lesssim \delta^3 s^{-\frac{3}{2}} \quad \text{if} \quad s \geq \frac{1}{2},$$
$$\| |U^{(0)}|^2 \nabla U^{(0)}(s)\|_1 \leq \|U^{(0)}(s)\|_4^2 \|\nabla U^{(0)}(s)\|_2 \lesssim \delta^3 \quad \text{if} \quad 0 < s < \frac{1}{2},$$
$$\|V(\pi^{(0)}) U^{(0)}(s)\|_1 + \|U^{(0)}(s) \nabla V(\pi^{(0)})(s)\|_1 \lesssim \delta^3 \langle s\rangle^{-\frac{5}{2}},$$
$$\|V(\pi^{(0)}) \nabla U^{(0)}(s)\|_1 \lesssim \|V(\pi^{(0)})\|_{L^{\frac{4}{3}} \cap L^{q'}} \|\nabla U^{(0)}(s)\|_{L^4 + L^q} \lesssim \delta^3 \langle s\rangle^{-1} s^{-\frac{3}{4}}.$$

We performed these estimates on L^1 rather than $L^{q'}$ for simplicity. However, the $L^{q'}$ case is an interpolation of the $L^{\frac{4}{3}}$ bounds above and the L^1 bounds which we just derived. Thus, the first integral in (132) which involves the $W^{1,q'}$ norm is no larger than

$$\lesssim \int_0^{(t-\frac{1}{2})_+} (t-s)^{-\frac{3}{2}+} \delta^2 \, \langle s\rangle^{-\frac{3}{2}} \, ds.$$

In conjunction with (133) we finally arrive at

$$\begin{aligned}
&\|\nabla P_{\mathrm{s}} U(t)\|_{L^4+L^q} \\
&\lesssim t^{-\frac{3}{4}} \|U(0)\|_{W^{1,\frac{4}{3}}} + \int_0^{(t-\frac{1}{2})_+} (t-s)^{-\frac{3}{2}+} \delta^2 \langle s\rangle^{-\frac{3}{2}+} \, ds \\
&\quad + \int_{t-\frac{1}{2}}^t (t-s)^{-\frac{3}{4}} \delta^2 \langle s\rangle^{-\frac{3}{2}} \, ds \; \chi_{[t \geq 1]} + \int_0^t (t-s)^{-\frac{3}{4}} \delta^2 s^{-\frac{3}{4}} \, ds \; \chi_{[0<t<1]} \\
&\lesssim t^{-\frac{3}{4}} (\delta_0 + \delta^2) \leq \frac{\delta}{2} t^{-\frac{3}{4}},
\end{aligned}$$

provided c_1, δ are small.

The conclusion is that $P_{\mathrm{s}} U(t)$ satisfies (67) and (68). As far as $P_{\mathrm{im}} U(t)$ is concerned, we claim that it satisfies the stronger estimate

$$\|P_{\mathrm{im}} U(t)\|_\infty + \|\nabla P_{\mathrm{im}} U(t)\|_\infty \ll \delta \langle t\rangle^{-\frac{3}{2}} \tag{134}$$

for all $t \geq 0$. To see this, return to the equation

$$i\partial_t U_{\mathrm{hyp}} - \mathcal{H}(\alpha_\infty^{(0)}) U_{\mathrm{hyp}} = F_2(U_{\mathrm{dis}}, U_{\mathrm{hyp}}),$$

see (103), which governs the evolution of $P_{\mathrm{im}} U(t)$. Here F_2 satisfies (110), i.e.,

$$\begin{aligned}
&\|F_2(U_{\mathrm{dis}}, U_{\mathrm{hyp}})(t)\|_2 \\
&\qquad \lesssim \delta^2 \langle t\rangle^{-\frac{3}{2}} + \delta^2 \langle t\rangle^{-1} (\|U_{\mathrm{dis}}(t)\|_{2+\infty} + \|U_{\mathrm{hyp}}(t)\|_{2+\infty}) \lesssim \delta^2 \langle t\rangle^{-\frac{3}{2}}.
\end{aligned}$$

Writing $U_{\mathrm{hyp}}(t) = b^+(t) f^+(\alpha_\infty^{(0)}) + b^-(t) f^-(\alpha_\infty^{(0)})$, see (97), we conclude from Lemma 23 that in fact

$$|b^+(t)| \lesssim \delta^3 \langle t\rangle^{-\frac{3}{2}}, \quad |b^-(t)| \lesssim e^{-\sigma(\alpha_\infty^{(0)}) t} \delta_0 + \delta^3 \langle t\rangle^{-\frac{3}{2}},$$

which implies that

$$\|U_{\rm hyp}(t)\|_{W^{k,p}} \ll \delta \langle t\rangle^{-\frac{3}{2}}$$

since the functions $f^{\pm}(\alpha_\infty^{(0)})$ are smooth and decay at infinity. In particular, (134) holds. Finally, inserting the bounds on $U_{\rm dis}$ and $U_{\rm hyp}$ into (99) yields

$$\|P_{\rm root}U(t)\|_\infty + \|\nabla P_{\rm root}U(t)\|_\infty \ll \delta\langle t\rangle^{-\frac{3}{2}},$$

and the lemma is proved. □

6. The contraction scheme: Part II

It remains to check that Ψ is a contraction. One guess would be to prove this property in the norm

$$\|(\pi, U)\| = \|\pi\| + \|U\|_{L^\infty([0,\infty),L^2)},$$

where

$$\|\pi\| := \sup_{t\ge 0}\langle t\rangle^3 (|\dot\alpha(t)| + |\dot v(t)| + |\dot{\tilde\gamma}(t)| + |\dot D(t)|).$$

Since the paths are all required to start at the same point $(\alpha_0, 0, 0, 0)$, it suffices to control their derivatives, which is what this norm does. Moreover, it is easy to check that the set X_δ is a complete metric space in this norm. Unfortunately, Ψ does not contract here. To see this, suppose we are given two different data $(\pi^{(0)}, Z^{(0)}) \in X_\delta$ and $(\pi^{(1)}, Z^{(1)}) \in X_\delta$. Set $(\pi^{(2)}, Z^{(2)}) := \Psi(\pi^{(0)}, Z^{(0)})$, $(\pi^{(3)}, Z^{(3)}) := \Psi(\pi^{(1)}, Z^{(1)})$. Then the evolutions of $Z^{(2)}$ and $Z^{(3)}$ are governed by the reference Hamiltonians $\mathcal{H}(\alpha_\infty^{(0)})$ and $\mathcal{H}(\alpha_\infty^{(1)})$, respectively. These Hamiltonians have different spectra, namely their thresholds are $\pm(\alpha_\infty^{(0)})^2$ and $\pm(\alpha_\infty^{(1)})^2$, respectively. For this reason one cannot hope to obtain a favorable estimate for $\|Z^{(2)}(t) - Z^{(3)}(t)\|_2$, at least for long times. As a model problem, consider the ODEs

$$i\dot u - \alpha_1^2 u = 0, \quad i\dot v - \alpha_2^2 v = 0, \quad u(0) = v(0) \neq 0$$

with $\alpha_1 \neq \alpha_2$. Evidently, $|u(t) - v(t)|$ will be as large as $|u(0)|$ infinitely often for large t. In contrast to this example, our solutions do disperse at the rate $t^{-\frac{3}{2}}$. Hence, we need to contract in a dispersive norm and the best decay we can hope for is $t^{-\frac{1}{2}}$, as can be seen from

$$|e^{it\alpha_1^2} - e^{it\alpha_2^2}|t^{-\frac{3}{2}} \lesssim t^{-\frac{1}{2}}|\alpha_1 - \alpha_2|.$$

Since we incur this loss of t in the Z-norm, we also end up losing t over the decay of $\dot\pi$. The actual norm is a bit technical, and we introduce it now.

Definition 25. For any $(\pi, Z) \in X_\delta$ set

$$\|(\pi, Z)\| := \sup_{0<t\le 1} t^{\frac{3}{4}}|\dot{\tilde\pi}(t)| + \sup_{t\ge 1} t^2|\dot{\tilde\pi}(t)| + \sup_{0<t\le 1} t^{\frac{3}{4}}\|Z(t)\|_{L^4(\mathbb{R}^3)} + \sup_{t\ge 1} t^{\frac{1}{2}}\|Z(t)\|_{L^4+L^\infty}. \tag{135}$$

The suprema here are *essential suprema*.

The appearance of L^4 rather than the perhaps more natural L^2 has to do with the cubic nonlinearity. We first make a routine check that X_δ is indeed complete in this metric.

LEMMA 26. *If* $(\pi, Z) \in X_\delta$, *then* $\|(\pi, Z)\| \lesssim \delta$. *For any fixed* $\delta > 0$ *the space* X_δ *is a complete metric space in the norm* (135).

Proof. Suppose $\|(\pi_n, Z_n)-(\pi_m, Z_m)\| \to 0$ as $n, m \to \infty$ where $(\pi_n, Z_n) \in X_\delta$. Recall that we are requiring that $\pi_n(0) = (\alpha_0, 0, 0, 0)$. Thus,

$$\begin{aligned}
&\sup_{t\geq 0}[|\alpha_n(t) - \alpha_m(t)| + |v_n(t) - v_m(t)| + |D_n(t) - D_m(t)|] \\
&\qquad \leq \|(\pi_n, Z_n) - (\pi_m, Z_m)\| \int_0^\infty (s^{-\frac{3}{4}}\chi_{[0<s<1]} + s^{-2}\chi_{[s>1]})\, ds \\
&\qquad \leq C\, \|(\pi_n, Z_n) - (\pi_m, Z_m)\|.
\end{aligned}$$

Define $(\alpha, D, v) := \lim_{n\to\infty}(\alpha_n, D_n, v_n)$ in the uniform sense. Then by (65)

$$\begin{aligned}
|(\alpha, D, v)(t_1) - (\alpha, D, v)(t_2)| &= \lim_{n\to\infty} |(\alpha_n, D_n, v_n)(t_1) - (\alpha_n, D_n, v_n)(t_2)| \\
&\leq \int_{t_1}^{t_2} \delta^2 \langle s\rangle^{-3}\, ds
\end{aligned}$$

for all $0 \leq t_1 < t_2$. This implies that $(\alpha, D, v) \in \mathrm{Lip}([0,\infty), \mathbb{R}^5)$ and that (65) also holds for (α, D, v). Let $\dot\gamma$ be the limit of $\dot\gamma_n = \dot{\tilde\gamma}_n - \dot v_n \cdot y_n$. Then define $\dot{\tilde\gamma} := \dot\gamma + \dot v \cdot y = \lim_{n\to\infty} \dot{\tilde\gamma}_n$. Since each $\dot{\tilde\gamma}_n$ satisfies (65), the same argument as before shows that $\dot{\tilde\gamma}$ does, too. Since

$$|\dot\gamma(t)| \leq |\dot{\tilde\gamma}(t)| + |\dot v(t)| C(1+t) \lesssim \langle t\rangle^{-2},$$

it follows that γ is also Lipschitz and hence $\pi \in \mathrm{Lip}([0,\infty), \mathbb{R}^8)$, as required in Definition 19.

For a.e. $t > 0$ let $Z(t) := \lim_{n\to\infty} Z_n(t)$, where the convergence takes place in $L^4 + L^\infty$. Since Z_n satisfy (66), for any Schwartz function ψ

$$\begin{aligned}
|\langle Z(t), \psi\rangle| + |\langle Z(t), \nabla\psi\rangle| &= \lim_{n\to\infty} (|\langle Z_n(t), \psi\rangle| + |\langle Z_n(t), \nabla\psi\rangle|) \\
&\leq \lim_{n\to\infty} (|\langle Z_n(t), \psi\rangle| + |\langle \nabla Z_n(t), \psi\rangle|) \\
&\leq c_0 \delta \|\psi\|_2.
\end{aligned}$$

It follows by the usual Hahn-Banach, Riesz-Fischer argument that (66) holds for $Z(t)$ and a.e. $t > 0$. For the same reason, the other estimates (67), (68) also persist in the limit. Finally, the $\mathcal{J}$ invariance clearly survives in the limit, and we are done. □

Next, we prove a simple technical lemma which will control the variation of various quantities in the path.

Lemma 27. *Define* $\mathcal{P} := \{\pi \in C^1([0,\infty),\mathbb{R}^8) \mid \pi(0) = (\alpha_0,0,0,0),$ $\|\pi\| \le 1\}$, *where* $\|\pi\|$ *is the* π-*part of* (135); *i.e.*,

$$\|\pi\| = \sup_{0<t\le 1} t^{\frac{3}{4}}|\dot{\tilde{\pi}}(t)| + \sup_{t\ge 1} t^2|\dot{\tilde{\pi}}(t)|. \tag{136}$$

Then, with $y^{(0)} = y(\pi^{(0)})$ *and* $\theta^{(0)} = \theta(\pi^{(0)})$ *as in* (5), (4), *and similarly for* $y^{(1)}, \theta^{(1)}$,

$$|\alpha_\infty^{(0)} - \alpha_\infty^{(1)}| + |v_\infty^{(0)} - v_\infty^{(1)}| \lesssim \|\pi^{(0)} - \pi^{(1)}\|, \tag{137}$$

$$\|e^{i\theta^{(0)}(t)}\phi(\cdot - y^{(0)}(t), \alpha^{(0)}(t)) - e^{i\theta^{(1)}(t)}\phi(\cdot - y^{(1)}(t), \alpha^{(1)}(t))\|_{L^1\cap L^\infty} \lesssim \langle t\rangle \|\pi^{(1)} - \pi^{(0)}\|, \tag{138}$$

$$\|\tilde{\xi}_j(\pi^{(0)})(t) - \tilde{\xi}_j(\pi^{(1)})(t)\|_{L^1\cap L^\infty} \lesssim \langle t\rangle \|\pi^{(0)} - \pi^{(1)}\|, \tag{139}$$

$$\|\tilde{\eta}_j(\pi^{(0)})(t) - \tilde{\eta}_j(\pi^{(1)})(t)\|_{L^1\cap L^\infty} \lesssim \langle t\rangle \|\pi^{(0)} - \pi^{(1)}\|, \tag{140}$$

$$\|\mathcal{S}_j(\pi^{(0)})(t) - \mathcal{S}_j(\pi^{(1)})(t)\|_{L^1\cap L^\infty} \lesssim \langle t\rangle \|\pi^{(0)} - \pi^{(1)}\|, \tag{141}$$

for all $\pi^{(0)}, \pi^{(1)} \in \mathcal{P}$. *For the definitions of the various quantities on the left-hand sides see Lemma* 12. *The implicit constants here depend on* α_0 *but are otherwise absolute.*

Proof. In view of the definitions of $\pi_\infty^{(0)}, \pi_\infty^{(1)}$ in Definition 3 we have the following bounds:

$$\begin{aligned}
&\sup_{t\ge 0}[|\alpha^{(0)}(t) - \alpha^{(1)}(t)| + |v^{(0)}(t) - v^{(1)}(t)| + |D^{(0)}(t) - D^{(1)}(t)|] \\
&\le \int_0^\infty (|\dot{\alpha}^{(0)}(s) - \dot{\alpha}^{(1)}(s)| + |\dot{v}^{(0)}(s) - \dot{v}^{(1)}(s)| + |\dot{D}^{(0)}(s) - \dot{D}^{(1)}(s)|)\, ds \\
&\lesssim \|\pi^{(0)} - \pi^{(1)}\| \int_0^\infty (s^{-\frac{3}{4}}\chi_{[0<s<1]} + s^{-2}\chi_{[s>1]})\, ds \lesssim \|\pi^{(0)} - \pi^{(1)}\|.
\end{aligned}$$

In particular,

$$|\alpha_\infty^{(0)} - \alpha_\infty^{(1)}| + |v_\infty^{(0)} - v_\infty^{(1)}| \lesssim \|\pi^{(1)} - \pi^{(0)}\|,$$

which is (137). Moreover (recall that $\pi^{(0)}(0) = \pi^{(1)}(0) = (\alpha_0, 0, 0, 0)$),

$$\begin{aligned}
|y^{(0)}(t) - y^{(1)}(t)| &\lesssim \int_0^t |v^{(0)}(s) - v^{(1)}(s)|\, ds + |D^{(0)}(t) - D^{(1)}(t)| \\
&\lesssim \Big[\int_0^t \int_0^s \langle\sigma\rangle^{-2}\, d\sigma ds + \int_0^t \langle s\rangle^{-2}\, ds\Big] \|\pi^{(0)} - \pi^{(1)}\| \\
&\lesssim \langle t\rangle \|\pi^{(0)} - \pi^{(1)}\|.
\end{aligned}$$

and

$$\begin{aligned}
|\gamma^{(0)}(t) - \gamma^{(1)}(t)| \le{}& \int_0^t |\dot{\tilde{\gamma}}^{(0)}(s) - \dot{\tilde{\gamma}}^{(1)}(s)|\, ds \\
&+ \int_0^t |\dot{v}^{(0)}(s)\cdot y^{(0)}(s) - \dot{v}^{(1)}(s)\cdot y^{(1)}(s)|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \langle s\rangle^{-2}\,ds\,\|\pi^{(0)}-\pi^{(1)}\| + \int_0^t |\dot v^{(0)}(s)-\dot v^{(1)}(s)|\langle s\rangle\,ds \\
&\quad + \int_0^t |\dot v^{(0)}(s)\langle s\rangle\,ds\,\|\pi^{(0)}-\pi^{(1)}\| \\
&\lesssim \log(2+t)\|\pi^{(0)}-\pi^{(1)}\| \lesssim \langle t\rangle\|\pi^{(0)}-\pi^{(1)}\|.
\end{aligned}
$$

Let θ be as in (4). Then

$$
\begin{aligned}
|\theta^{(0)}(t,x)-\theta^{(1)}(t,x)| &\lesssim |v^{(0)}(t)-v^{(1)}(t)||x| \\
&\quad + \int_0^t (|v^{(0)}(s)-v^{(1)}(s)| + |\alpha^{(0)}(s)-\alpha^{(1)}(s)|)\,ds \\
&\quad + |\gamma^{(0)}(t)-\gamma^{(1)}(t)| \lesssim (|x|+\langle t\rangle)\|\pi^{(0)}-\pi^{(1)}\|.
\end{aligned}
$$

The estimate (138) now follows easily. Indeed, observe that $|x|$ behaves like t in this context. The other estimates (139), (140), and (141) are easily deduced from (138). □

We will use the following simple extension of the contraction principle. Of course it is well-known, but we still record it here.

LEMMA 28. *Let $S \subset X$ be a closed subset of a Banach space X and $T \subset Y$ an arbitrary subset of some normed space Y. Suppose that $A : S \times T \to S$ so that with some $0<\gamma<1$*

$$
\begin{aligned}
\sup_{t\in T}\|A(x,t)-A(y,t)\|_X &\le \gamma\|x-y\|_X \quad \text{for all} \quad x,y\in S,\\
\sup_{x\in S}\|A(x,t_1)-A(x,t_2)\| &\le C_0\|t_1-t_2\|_Y \quad \text{for all} \quad t_1,t_2\in T.
\end{aligned}
$$

Then for every $t \in T$ there exists a unique fixed-point $x(t) \in S$ such that $A(x(t),t) = x(t)$. Moreover, these points satisfy the bounds

$$
\|x(t_1)-x(t_2)\|_X \le \frac{C_0}{1-\gamma}\|t_1-t_2\|_Y
$$

for all $t_1, t_2 \in T$.

Proof. Clearly, $x(t) = \lim_{n\to\infty} A(x_n(t),t)$ where for some fixed (i.e., independent of t) x_0

$$
x_0(t) := x_0, \quad x_{n+1}(t) = A(x_n(t),t).
$$

Then inductively,

$$
\begin{aligned}
\|x_{n+1}(t_1)-x_{n+1}(t_2)\|_X &\le \|A(x_n(t_1),t_1)-A(x_n(t_2),t_1)\|_X \\
&\quad + \|A(x_n(t_2),t_1)-A(x_n(t_2),t_2)\|_X \\
&\le \gamma\|x_n(t_1)-x_n(t_2)\|_X + C_0\|t_1-t_2\|_Y \\
&\le C_0\sum_{k=0}^n \gamma^k\,\|t_1-t_2\|_Y
\end{aligned}
$$

for all $n \ge 0$. Passing to the limit $n\to\infty$ proves the lemma. □

We are now ready to state the contraction property of Ψ.

LEMMA 29. *Under the hypotheses of Theorem* 1 *the map* $\Psi : X_\delta \to X_\delta$ *is a contraction in the norm* (135). *Thus* Ψ *has a fixed point* $(\pi, Z) \in X_\delta$, *which is completely determined by* R_0. *Hence, the function* $h(R_0, \pi^{(0)}, Z^{(0)})$ *now becomes a function* $h = h(R_0)$ *of* R_0 *alone. It satisfies* (95) *as well as the Lipschitz bound*

$$|h(R_0) - h(R_1)| \lesssim \delta \|R_0 - R_1\| \tag{142}$$

for any R_0, R_1 *satisfying* $P_u^+(\alpha_0)\binom{R_j}{\bar{R}_j} = 0$, *and* $\|R_j\| \le c_1\delta$, $j = 0, 1$.

Proof. Let $(\pi^{(0)}, Z^{(0)}), (\pi^{(1)}, Z^{(1)}) \in X_\delta$ and set

$$(\pi^{(2)}, Z^{(2)}) := \Psi(\pi^{(0)}, Z^{(0)}), \quad (\pi^{(3)}, Z^{(3)}) := \Psi(\pi^{(1)}, Z^{(1)}),$$

as well as

$$\begin{aligned} U^{(0)}(t) &:= M(\pi^{(0)})(t)\mathcal{G}_\infty(\pi^{(0)})(t)Z^{(0)}(t), \\ U^{(2)}(t) &:= M(\pi^{(0)})(t)\mathcal{G}_\infty(\pi^{(0)})(t)Z^{(2)}(t), \\ U^{(1)}(t) &:= M(\pi^{(1)})(t)\mathcal{G}_\infty(\pi^{(1)})(t)Z^{(1)}(t), \\ U^{(3)}(t) &:= M(\pi^{(1)})(t)\mathcal{G}_\infty(\pi^{(1)})(t)Z^{(3)}(t). \end{aligned}$$

Hence, by definition of Ψ we have the linear problems

$$\begin{aligned} i\partial_t U^{(2)} - \mathcal{H}(\alpha_\infty^{(0)})U^{(2)} &= -i\dot{\pi}^{(2)}\partial_\pi \widetilde{W}(\pi^{(0)}) + N(U^{(0)}, \pi^{(0)}) + V(\pi^{(0)})U^{(0)}, \\ \langle \dot{\pi}^{(2)}\partial_\pi \widetilde{W}(\pi^{(0)}), \xi_j(\pi^{(0)})\rangle &= i\langle U^{(2)}, \dot{\xi}_j(\pi^{(0)})\rangle + \langle U^{(2)}, E(\pi^{(0)})\xi_j(\pi^{(0)})\rangle \\ &\quad + \langle N(U^{(0)}, \pi^{(0)}), \xi_j(\pi^{(0)})\rangle \end{aligned} \tag{143}$$

for $1 \le j \le 8$ and

$$\begin{aligned} i\partial_t U^{(3)} - \mathcal{H}(\alpha_\infty^{(1)})U^{(3)} &= -i\dot{\pi}^{(3)}\partial_\pi \widetilde{W}(\pi^{(1)}) + N(U^{(1)}, \pi^{(1)}) + V(\pi^{(1)})U^{(1)}, \\ i\langle \dot{\pi}^{(3)}\partial_\pi \widetilde{W}(\pi^{(1)}), \xi_j(\pi^{(1)})\rangle &= i\langle U^{(3)}, \dot{\xi}_j(\pi^{(1)})\rangle + \langle U^{(3)}, E(\pi^{(1)})\xi_j(\pi^{(1)})\rangle \\ &\quad + \langle N(U^{(1)}, \pi^{(1)}), \xi_j(\pi^{(1)})\rangle \end{aligned} \tag{144}$$

for $1 \le j \le 8$. The initial conditions are

$$U^{(2)}(0) = \mathcal{G}_\infty(\pi^{(0)})(0)\Big[\begin{pmatrix} R_0 \\ \bar{R}_0 \end{pmatrix} + h^{(0)} f^+(\alpha_\infty^{(0)}) + \sum_{j=1}^{8} a_j^{(0)} \eta_j(\alpha_\infty^{(0)})\Big] \tag{145}$$

$$U^{(3)}(0) = \mathcal{G}_\infty(\pi^{(1)})(0)\Big[\begin{pmatrix} R_0 \\ \bar{R}_0 \end{pmatrix} + h^{(1)} f^+(\alpha_\infty^{(1)}) + \sum_{j=1}^{8} a_j^{(1)} \eta_j(\alpha_\infty^{(1)})\Big] \tag{146}$$

$$\pi^{(2)}(0) = \pi^{(3)}(0) = (\alpha_0, 0, 0, 0), \tag{147}$$

where we have set

$$h^{(0)} := h(R_0, \pi^{(0)}, Z^{(0)}), \quad h^{(1)} := h(R_0, \pi^{(1)}, Z^{(1)})$$

for simplicity, and similarly for $a_j = a_j(h)$. We *cannot* compare $U^{(2)}$ and $U^{(3)}$ because they are given in terms of reference Hamiltonians which involve the vectors $\pi_\infty^{(0)}$ and $\pi_\infty^{(1)}$ and the latter cannot be compared. Indeed, since we only know that $|\dot\pi^{(0)}(t) - \dot\pi^{(1)}(t)| \le \langle t\rangle^{-2}\|\pi^{(0)} - \pi^{(1)}\|$, the best estimate on the "terminal translation" D_∞ here would be

$$|D_\infty^{(0)} - D_\infty^{(1)}| \lesssim \|\pi^{(0)} - \pi^{(1)}\| \log \|\pi^{(0)} - \pi^{(1)}\|^{-1},$$

which is too weak for the contraction. Therefore, we return to the system (21), (70). More precisely, with $Z^{(0)} = \binom{R^{(0)}}{\bar R^{(0)}}$ and $Z^{(1)} = \binom{R^{(1)}}{\bar R^{(1)}}$ one has the systems

$$\begin{aligned}
&i\partial_t Z^{(2)}(t) + \begin{pmatrix} \triangle + 2|W(\pi^{(0)})|^2 & W^2(\pi^{(0)}) \\ -\bar W^2(\pi^{(0)}) & -\triangle - 2|W(\pi^{(0)})|^2 \end{pmatrix} Z^{(2)}(t) \\
&\quad = \dot v^{(2)} \begin{pmatrix} -(x - y(\pi^{(0)})(t)) e^{i\theta(\pi^{(0)})(t)} \phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \\ (x - y(\pi^{(0)})(t)) e^{-i\theta(\pi^{(0)})(t)} \phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \end{pmatrix} \\
&\qquad + \dot{\tilde\gamma}^{(2)} \begin{pmatrix} -e^{i\theta(\pi^{(0)})(t)} \phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \\ e^{-i\theta(\pi^{(0)})(t)} \phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \end{pmatrix} \\
&\qquad + i\dot\alpha^{(2)} \begin{pmatrix} e^{i\theta(\pi^{(0)})(t)} \partial_\alpha \phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \\ e^{-i\theta(\pi^{(0)})(t)} \partial_\alpha \phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \end{pmatrix} \\
&\qquad + i\dot D^{(2)} \begin{pmatrix} -e^{i\theta(\pi^{(0)})(t)} \nabla \phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \\ -e^{-i\theta(\pi^{(0)})(t)} \nabla \phi(\cdot - y(\pi^{(0)})(t), \alpha^{(0)}(t)) \end{pmatrix} \\
&\qquad + \begin{pmatrix} -2|R^{(0)}|^2 W(\pi^{(0)})(t) - \bar W(\pi^{(0)})(t)(R^{(0)})^2 - |R^{(0)}|^2 R^{(0)} \\ 2|R^{(0)}|^2 \bar W(\pi^{(0)})(t) + W(\pi^{(0)})(t)(\bar R^{(0)})^2 + |R^{(0)}|^2 \bar R^{(0)} \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&i\partial_t Z^{(3)}(t) + \begin{pmatrix} \triangle + 2|W(\pi^{(1)})|^2 & W^2(\pi^{(1)}) \\ -\bar W^2(\pi^{(1)}) & -\triangle - 2|W(\pi^{(1)})|^2 \end{pmatrix} Z^{(3)}(t) \\
&\quad = \dot v^{(3)} \begin{pmatrix} -(x - y(\pi^{(1)})(t)) e^{i\theta(\pi^{(1)})(t)} \phi(\cdot - y(\pi^{(1)})(t), \alpha^{(1)}(t)) \\ (x - y(\pi^{(1)})(t)) e^{-i\theta(\pi^{(1)})(t)} \phi(\cdot - y(\pi^{(1)})(t), \alpha^{(1)}(t)) \end{pmatrix} \\
&\qquad + \dot{\tilde\gamma}^{(3)} \begin{pmatrix} -e^{i\theta(\pi^{(1)})(t)} \phi(\cdot - y(\pi^{(1)})(t), \alpha^{(1)}(t)) \\ e^{-i\theta(\pi^{(1)})(t)} \phi(\cdot - y(\pi^{(1)})(t), \alpha^{(1)}(t)) \end{pmatrix} \\
&\qquad + i\dot\alpha^{(3)} \begin{pmatrix} e^{i\theta(\pi^{(1)})(t)} \partial_\alpha \phi(\cdot - y(\pi^{(1)})(t), \alpha^{(1)}(t)) \\ e^{-i\theta(\pi^{(1)})(t)} \partial_\alpha \phi(\cdot - y(\pi^{(1)})(t), \alpha^{(1)}(t)) \end{pmatrix} \\
&\qquad + i\dot D^{(3)} \begin{pmatrix} -e^{i\theta(\pi^{(1)})(t)} \nabla \phi(\cdot - y(\pi^{(1)})(t), \alpha^{(1)}(t)) \\ -e^{-i\theta(\pi^{(1)})(t)} \nabla \phi(\cdot - y(\pi^{(1)})(t), \alpha^{(1)}(t)) \end{pmatrix} \\
&\qquad + \begin{pmatrix} -2|R^{(1)}|^2 W(\pi^{(1)})(t) - \bar W(\pi^{(1)})(t)(R^{(1)})^2 - |R^{(1)}|^2 R^{(1)} \\ 2|R^{(1)}|^2 \bar W(\pi^{(1)})(t) + W(\pi^{(1)})(t)(\bar R^{(1)})^2 + |R^{(1)}|^2 \bar R^{(1)} \end{pmatrix}.
\end{aligned}$$

Using the notations from Lemma 12 we rewrite these systems in the form

$$
\begin{aligned}
(148)\qquad & i\partial_t Z^{(2)} - \mathcal{H}(\pi^{(0)}(t))Z^{(2)} \\
&= i\Big[\sum_{\ell=1}^{3}(\dot D_\ell^{(2)}\tilde\eta_{5+\ell}^{(0)} - \dot v_\ell^{(2)}\tilde\eta_{2+\ell}^{(0)}) + \dot\alpha^{(2)}\tilde\eta_2^{(0)} - \dot{\tilde\gamma}^{(2)}\tilde\eta_1^{(0)}\Big] + N_*(Z^{(0)},\pi^{(0)}), \\
&=: -i\dot{\tilde\pi}^{(2)}\partial_\pi W(\pi^{(0)}) + N_*(Z^{(0)},\pi^{(0)}),
\end{aligned}
$$

$$
\begin{aligned}
(149)\qquad & i\partial_t Z^{(3)} - \mathcal{H}(\pi^{(1)}(t))Z^{(3)} \\
&= i\Big[\sum_{\ell=1}^{3}(\dot D_\ell^{(3)}\tilde\eta_{5+\ell}^{(1)} - \dot v_\ell^{(3)}\tilde\eta_{2+\ell}^{(1)}) + \dot\alpha^{(3)}\tilde\eta_2^{(1)} - \dot{\tilde\gamma}^{(3)}\tilde\eta_1^{(1)}\Big] + N_*(Z^{(1)},\pi^{(1)}) \\
&=: -i\dot{\tilde\pi}^{(3)}\partial_\pi W(\pi^{(1)}) + N_*(Z^{(1)},\pi^{(1)}),
\end{aligned}
$$

where $\tilde\eta_j^{(0)} := \tilde\eta_j(\pi^{(0)})$, $\tilde\eta_j^{(1)} := \tilde\eta_j(\pi^{(1)})$, and N_* are defined in the obvious way. By construction, $U^{(3)}, U^{(2)}$ satisfy the orthogonality conditions

$$\langle U^{(2)}(t), \xi_j(\pi^{(0)})(t)\rangle = \langle U^{(3)}, \xi_j(\pi^{(1)})(t)\rangle = 0$$

for all $1 \le j \le 8$ and $t \ge 0$. By Lemma 12 these are equivalent to

$$\langle Z^{(2)}(t), \tilde\xi_j(\pi^{(0)})(t)\rangle = \langle Z^{(3)}, \tilde\xi_j(\pi^{(1)})(t)\rangle = 0.$$

Taking the scalar products of (148) and (149) with $\tilde\xi_j(\pi^{(0)})$ and $\tilde\xi_j(\pi^{(1)})$, respectively, leads to the following modulation equations on the level of $Z(t)$:

$$
\begin{aligned}
&(150) \\
&-i\langle\dot{\tilde\pi}^{(2)}\partial_\pi W(\pi^{(0)}), \tilde\xi_j(\pi^{(0)})\rangle = \langle Z^{(2)}, \dot{\tilde\pi}^{(0)}\mathcal{S}_j(\pi^{(0)})\rangle - \langle N_*(Z^{(0)},\pi^{(0)}), \tilde\xi_j(\pi^{(0)})\rangle, \\
&(151) \\
&-i\langle\dot{\tilde\pi}^{(3)}\partial_\pi W(\pi^{(1)}), \tilde\xi_j(\pi^{(1)})\rangle = \langle Z^{(3)}, \dot{\tilde\pi}^{(1)}\mathcal{S}_j(\pi^{(1)})\rangle - \langle N_*(Z^{(1)},\pi^{(1)}), \tilde\xi_j(\pi^{(1)})\rangle.
\end{aligned}
$$

Here we used the notation from (48). Subtracting (148), (149), and (150), (151), respectively, we obtain the equations that will provide the estimates for the contraction step:

$$
\begin{aligned}
(152)\quad & i\partial_t(Z^{(3)} - Z^{(2)}) - \mathcal{H}(\pi^{(0)}(t))(Z^{(3)} - Z^{(2)}) \\
&= -i(\dot{\tilde\pi}^{(3)} - \dot{\tilde\pi}^{(2)})\partial_\pi W(\pi^{(0)}) + V(\pi^{(0)},\pi^{(1)})Z^{(3)} \\
&\quad + N_*(Z^{(1)},\pi^{(1)}) - N_*(Z^{(0)},\pi^{(0)}) + \dot{\tilde\pi}^{(3)}(\partial_\pi W(\pi^{(1)}) - \partial_\pi W(\pi^{(0)}));
\end{aligned}
$$

$$
\begin{aligned}
&(153) \\
&(Z^{(3)} - Z^{(2)})(0) = h^{(1)}f^+(\alpha_\infty^{(1)}) \\
&\qquad + \sum_{j=1}^{8} a_j^{(1)}\eta_j(\alpha_\infty^{(1)}) - [h^{(0)}f^+(\alpha_\infty^{(0)}) + \sum_{j=1}^{8} a_j^{(0)}\eta_j(\alpha_\infty^{(0)})] \\
&\qquad -i\langle(\dot{\tilde\pi}^{(3)} - \dot{\tilde\pi}^{(2)})\partial_\pi W(\pi^{(0)}), \tilde\xi_j(\pi^{(0)})\rangle
\end{aligned}
$$

$$
\begin{aligned}
(154) \qquad &= \langle Z^{(3)} - Z^{(2)}, \dot{\tilde{\pi}}^{(0)} \mathcal{S}_j(\pi^{(0)}) \rangle \\
&\quad + \langle Z^{(3)}, \dot{\tilde{\pi}}^{(1)} \mathcal{S}_j(\pi^{(1)}) - \dot{\tilde{\pi}}^{(0)} \mathcal{S}_j(\pi^{(0)}) \rangle \\
&\quad + \langle N_*(Z^{(1)}, \pi^{(1)}) - N_*(Z^{(0)}, \pi^{(0)}), \tilde{\xi}_j(\pi^{(1)}) \rangle \\
&\quad + \langle N_*(Z^{(0)}, \pi^{(0)}), \tilde{\xi}_j(\pi^{(1)}) - \tilde{\xi}_j(\pi^{(0)}) \rangle \\
&\quad + \langle \dot{\tilde{\pi}}^{(3)} (\partial_\pi W(\pi^{(0)}) - \partial_\pi W(\pi^{(1)})), \tilde{\xi}_j(\pi^{(0)}) \rangle \\
&\quad + \langle \dot{\tilde{\pi}}^{(3)} \partial_\pi W(\pi^{(0)}), \tilde{\xi}_j(\pi^{(0)}) - \tilde{\xi}_j(\pi^{(1)}) \rangle.
\end{aligned}
$$

Here

$$
\begin{aligned}
&\tilde{V}(\pi^{(0)}, \pi^{(1)}) := \mathcal{H}(\pi^{(1)}) - \mathcal{H}(\pi^{(0)}) \\
&= \begin{pmatrix} 2(\phi^2(\cdot - y^{(0)}, \alpha^{(0)}) - \phi^2(\cdot - y^{(1)}, \alpha^{(1)})) & e^{2i\theta^{(0)}} \phi^2(\cdot - y^{(0)}, \alpha^{(0)}) - e^{2i\theta^{(1)}} \phi^2(\cdot - y^{(1)}, \alpha^{(1)}) \\ -e^{-2i\theta^{(0)}} \phi^2(\cdot - y^{(0)}, \alpha^{(0)}) + e^{-2i\theta^{(1)}} \phi^2(\cdot - y^{(1)}, \alpha^{(1)}) & -2(\phi^2(\cdot - y^{(0)}, \alpha^{(0)}) - \phi^2(\cdot - y^{(1)}, \alpha^{(1)})) \end{pmatrix}.
\end{aligned}
$$

In view of (138),

$$
(155) \qquad \| \tilde{V}(\pi^{(0)}, \pi^{(1)}) \|_{L^1 \cap L^\infty} \lesssim \langle t \rangle \| \pi^{(0)} - \pi^{(1)} \|.
$$

Set $\mathcal{T}^{(0)}(t) := M(\pi^{(0)})(t) \mathcal{G}_\infty(\pi^{(0)})(t)$ and define $\tilde{U}^{(3)} := \mathcal{T}^{(0)}(t) Z^{(3)}(t)$. Hence,

$$
(\tilde{U}^{(3)} - U^{(2)})(t) = \mathcal{T}^{(0)}(t)(Z^{(3)}(t) - Z^{(2)}(t))
$$

satisfies the transformed equation

$$
\begin{aligned}
(156) \qquad & i\partial_t(\tilde{U}^{(3)} - U^{(2)}) - \mathcal{H}(\alpha_\infty^{(0)})(\tilde{U}^{(3)} - U^{(2)}) \\
&= \mathcal{T}^{(0)}(t)[-i(\dot{\tilde{\pi}}^{(3)} - \dot{\tilde{\pi}}^{(2)}) \partial_\pi W(\pi^{(0)}) + \tilde{V}(\pi^{(0)}, \pi^{(1)}) Z^{(3)}] \\
&\quad + \mathcal{T}^{(0)}(t)[N_*(Z^{(1)}, \pi^{(1)}) - N_*(Z^{(0)}, \pi^{(0)}) \\
&\quad - i\dot{\tilde{\pi}}^{(3)}(\partial_\pi W(\pi^{(1)}) - \partial_\pi W(\pi^{(0)}))] + V(\pi^{(0)})(\tilde{U}^{(3)} - U^{(2)}).
\end{aligned}
$$

As before, we need to split the evolution into the three pieces

$$
\begin{aligned}
\tilde{U}^{(3)} - U^{(2)} &= P_{\mathrm{s}}(\alpha_\infty^{(0)})(\tilde{U}^{(3)} - U^{(2)}) + P_{\mathrm{root}}(\alpha_\infty^{(0)})(\tilde{U}^{(3)} - U^{(2)}) \\
&\quad + P_{\mathrm{im}}(\alpha_\infty^{(0)})(\tilde{U}^{(3)} - U^{(2)}).
\end{aligned}
$$

In view of Lemma 12 and Lemma 27, for all $1 \le j \le 8$,

$$
\begin{aligned}
|\langle \tilde{U}^{(3)} - U^{(2)}, \xi_j(\pi^{(0)}) \rangle| &= |\langle Z^{(3)} - Z^{(2)}, \tilde{\xi}_j(\pi^{(0)}) \rangle| = |\langle Z^{(3)}, \tilde{\xi}_j(\pi^{(0)}) - \tilde{\xi}_j(\pi^{(1)}) \rangle| \\
&\le \| Z^{(3)} \|_{2+\infty} \langle t \rangle \, \| \pi^{(0)} - \pi^{(1)} \| \lesssim \delta \langle t \rangle^{-\frac{1}{2}} \| \pi^{(0)} - \pi^{(1)} \|.
\end{aligned}
$$

The conclusion is that

$$
\begin{aligned}
(157) \qquad & \| P_{\mathrm{root}}(\alpha_\infty^{(0)})(\tilde{U}^{(3)} - U^{(2)})(t) \|_{4+\infty} \\
&\lesssim \delta \langle t \rangle^{-\frac{1}{2}} \| \pi^{(0)} - \pi^{(1)} \| + \delta^2 \| \tilde{U}^{(3)}(t) - U^{(2)}(t) \|_{4+\infty} \langle t \rangle^{-1}.
\end{aligned}
$$

Next, we turn to the dispersive piece $P_{\mathrm{s}}(\alpha_\infty^{(0)})(\tilde{U}^{(3)} - U^{(2)})$. This requires estimating each of the expressions on the right-hand side of (156) in the appropriate norms. It will be convenient to use the notations

$$
\mu_Z(t) := t^{\frac{1}{2}} \chi_{[t \ge 1]} + t^{\frac{3}{4}} \chi_{[0<t<1]}, \; \mu_\pi(t) := t^2 \chi_{[t \ge 1]} + t^{\frac{3}{4}} \chi_{[0<t<1]}.
$$

Then,

$$\|\mathcal{T}^{(0)}(t)[-i(\dot{\tilde\pi}^{(3)}-\dot{\tilde\pi}^{(2)})\partial_\pi W(\pi^{(0)})+V(\pi^{(0)},\pi^{(1)})Z^{(3)}]\|_{1\cap 2} \lesssim \|\pi^{(3)}-\pi^{(2)}\|\mu_\pi(t)^{-1}+\delta\langle t\rangle^{-\frac12}\|\pi^{(0)}-\pi^{(1)}\|. \tag{158}$$

Moreover, by (155),

$$\begin{aligned}\|V(\pi^{(0)})(\tilde U^{(3)}(t)-U^{(2)}(t))\|_{1\cap 2} &\lesssim \|V(\pi^{(0)})\|_{1\cap 4}\|\tilde U^{(3)}(t)-U^{(2)}(t)\|_{4+\infty}\\ &\lesssim \delta^2\|\tilde U^{(3)}(t)-U^{(2)}(t)\|_{4+\infty}.\end{aligned} \tag{159}$$

Another easy term is

$$\|\mathcal{T}^{(0)}(t)\dot{\tilde\pi}^{(3)}(\partial_\pi W(\pi^{(1)})-\partial_\pi W(\pi^{(0)}))\|_{1\cap\infty} \lesssim \delta^2\langle t\rangle^{-2}\|\pi^{(0)}-\pi^{(1)}\|. \tag{160}$$

Next, we turn to the nonlinear terms $N_*(Z^{(1)},\pi^{(1)})-N_*(Z^{(0)},\pi^{(0)})$. Recall that

$$N_*(Z^{(1)},\pi^{(1)})=\begin{pmatrix}-2|R^{(0)}|^2W(\pi^{(0)})(t)-\bar W(\pi^{(0)})(t)(R^{(0)})^2-|R^{(0)}|^2R^{(0)}\\ 2|R^{(0)}|^2\bar W(\pi^{(0)})(t)+W(\pi^{(0)})(t)(\bar R^{(0)})^2+|R^{(0)}|^2\bar R^{(0)}\end{pmatrix}.$$

The right-hand side here naturally divides into three columns, which we formally write as

$$|Z|^2\,W,\ Z^2\,\bar W,\ |Z|^2Z,$$

respectively. Let us start with the third column (we suppress t for the most part):

$$\begin{aligned}&\||Z^{(0)}|^2Z^{(0)}-|Z^{(1)}|^2Z^{(1)}\|_{1\cap\frac43}\\ &\quad\lesssim \|Z^{(0)}-Z^{(1)}\|_{4+\infty}(\|\,|Z^{(0)}|^2+|Z^{(1)}|^2\|_{\frac43\cap 1}+\|\,|Z^{(0)}|^2+|Z^{(1)}|^2\|_{2\cap\frac43})\\ &\quad\lesssim \delta^2\|Z^{(0)}-Z^{(1)}\|_{4+\infty}.\end{aligned}$$

This estimate is the reason we do not work on L^2+L^∞. Indeed, in the latter case we would be faced with $\|U(t)\|_\infty^2$, which we can only bound by $t^{-\frac32}$ for small t; see (68). This bound is nonintegrable at $t=0$. The first column satisfies

$$\begin{aligned}&\||Z^{(0)}|^2W(\pi^{(0)})-|Z^{(1)}|^2W(\pi^{(1)})\|_{1\cap\frac43}\\ &\lesssim \|Z^{(0)}-Z^{(1)}\|_{4+\infty}\Big(\|\,|Z^{(0)}W(\pi^{(0)})|+|Z^{(1)}W(\pi^{(1)})|\,\|_{\frac43\cap 1}\\ &\qquad\qquad+\|\,|Z^{(0)}W(\pi^{(0)})|+|Z^{(1)}W(\pi^{(1)})|\,\|_{2\cap\frac43}\Big)\\ &\quad+\|W(\pi^{(0)})-W(\pi^{(1)})\|_{1\cap\infty}\|Z^{(1)}\|_{2+\infty}^2\\ &\lesssim \delta\|Z^{(0)}-Z^{(1)}\|_{4+\infty}+\delta^2\langle t\rangle^{-2}\|\pi^{(0)}-\pi^{(1)}\|.\end{aligned}$$

An analogous bound holds for the second column. Collecting these bounds yields

$$\|\mathcal{T}^{(0)}[N_*(Z^{(1)},\pi^{(1)})-N_*(Z^{(0)},\pi^{(0)})]\|_{1\cap\frac43} \lesssim \delta\|Z^{(0)}-Z^{(1)}\|_{4+\infty}+\delta^2\langle t\rangle^{-2}\|\pi^{(0)}-\pi^{(1)}\|. \tag{161}$$

Combining (158), (159), (160), and (161) leads to

$$\begin{aligned}(162)\qquad &\|\text{right-hand side of (156)}\|_{1\cap\frac43}\\ &\lesssim \delta\langle t\rangle^{-\frac12}\|\pi^{(1)}-\pi^{(0)}\| + \delta\|Z^{(1)}(t)-Z^{(0)}(t)\|_{4+\infty}\\ &\quad + \delta^2\|\tilde U^{(3)}(t)-U^{(2)}(t)\|_{4+\infty} + \|\pi^{(3)}-\pi^{(2)}\|\mu_\pi(t)^{-1}.\end{aligned}$$

Denote the right-hand side of (156) by F. Estimating the Duhamel version of (156) therefore leads to the conclusion that

$$\begin{aligned}(163)\qquad &\|P_{\rm s}(\alpha_\infty^{(0)})(\tilde U^{(3)}-U^{(2)})(t)\|_{4+\infty}\\ &\le \|e^{-it\mathcal H(\alpha_\infty^{(0)})}P_{\rm s}(\alpha_\infty^{(0)})(\tilde U^{(3)}(0)-U^{(2)}(0))\|_{4+\infty}\\ &\quad + \int_0^{(t-1)_+}\|e^{-i(t-s)\mathcal H(\alpha_\infty^{(0)})}P_{\rm s}(\alpha_\infty^{(0)})F(s)\|_\infty\,ds\\ &\quad + \int_{(t-1)_+}^t\|e^{-i(t-s)\mathcal H(\alpha_\infty^{(0)})}P_{\rm s}(\alpha_\infty^{(0)})F(s)\|_4\,ds\\ &\lesssim t^{-\frac32}\|P_{\rm s}(\alpha_\infty^{(0)})[\tilde U^{(3)}(0)-U^{(2)}(0)]\|_1\chi_{[t\ge1]}\\ &\quad + t^{-\frac34}\|P_{\rm s}(\alpha_\infty^{(0)})[\tilde U^{(3)}(0)-U^{(2)}(0)]\|_{\frac43}\chi_{[0<t<1]}\\ &\quad + \int_0^t\big(\langle t-s\rangle^{-\frac32}+(t-s)^{-\frac34}\chi_{[(t-1)_+<s<t]}\big)\\ &\quad \cdot\Big(\delta\langle s\rangle^{-\frac12}\|\pi^{(1)}-\pi^{(0)}\| + \delta^2\|(Z^{(1)}-Z^{(0)})(s)\|_{4+\infty}\\ &\qquad + \delta^2\|\tilde U^{(3)}(s)-U^{(2)}(s)\|_{4+\infty} + \|\pi^{(3)}-\pi^{(2)}\|\mu_\pi(s)^{-1}\Big)\,ds.\end{aligned}$$

As far as the initial conditions are concerned, we infer from (145), (146), as well as (137) that

(164)

$$\begin{aligned}\|P_{\rm s}(\alpha_\infty^{(0)})[U^{(3)}(0)-U^{(2)}(0)]\|_{1\cap\frac43} &\lesssim |h^{(1)}|\|[P_{\rm s}(\alpha_\infty^{(0)})-P_{\rm s}(\alpha_\infty^{(1)})]f^+(\alpha_\infty^{(0)})\|_2\\ &+\sum_{k=1}^8|a_j^{(1)}|\|[P_{\rm s}(\alpha_\infty^{(0)})-P_{\rm s}(\alpha_\infty^{(1)})]\eta_k(\alpha_\infty^{(0)})\|_2 \lesssim \delta^2\|\pi^{(0)}-\pi^{(1)}\|.\end{aligned}$$

Further simplification of (163) therefore leads to

$$\begin{aligned}(165)\quad &\|P_{\rm s}(\alpha_\infty^{(0)})(\tilde U^{(3)}-U^{(2)})(t)\|_{4+\infty}\\ &\lesssim \delta\mu_Z(t)^{-1}\,\|\pi^{(0)}-\pi^{(1)}\| + \delta^2t^{-\frac12}\sup_{s\ge0}\mu_Z(s)\|\tilde U^{(3)}(s)-U^{(2)}(s)\|_{4+\infty}\\ &\quad +\delta\,t^{-\frac12}\big(\|\pi^{(0)}-\pi^{(1)}\| + \sup_{s\ge0}\mu_Z(s)\|Z^{(0)}(s)-Z^{(1)}(s)\|_{4+\infty}\big)\\ &\quad +t^{-\frac12}\|\pi^{(3)}-\pi^{(2)}\|,\end{aligned}$$

where we used the elementary estimate $\int_0^t\langle t-s\rangle^{-\frac32}\langle s\rangle^{-\frac12}\,ds\lesssim\langle t\rangle^{-\frac12}$. It remains to bound

$$\|P_{\rm im}(\alpha_\infty^{(0)})[\tilde U^{(3)}(t)-U^{(2)}(t)]\|_{4+\infty}.$$

To this end write

$$P_{\mathrm{im}}(\alpha_\infty^{(0)})[\tilde U^{(3)}(t) - U^{(2)}(t)] = b^+(t) f^+(\alpha_\infty^{(0)}) + b^-(t) f^-(\alpha_\infty^{(0)}),$$

with coefficients that are governed by the hyperbolic ODE

$$\frac{d}{dt}\begin{pmatrix} b^+(t) \\ b^-(t)\end{pmatrix} - \begin{pmatrix} \sigma(\alpha_\infty^{(0)}) & 0 \\ 0 & -\sigma(\alpha_\infty^{(0)})\end{pmatrix}\begin{pmatrix} b^+(t) \\ b^-(t)\end{pmatrix} = \begin{pmatrix} g^+ \\ g^-\end{pmatrix}. \tag{166}$$

Here

$$P_{\mathrm{im}}(\alpha_\infty^{(0)}) F(t) = g^+(t) f^+(\alpha_\infty^{(0)}) + g^-(t) f^-(\alpha_\infty^{(0)})$$

where F stands for the right-hand side of (156). Clearly, $g^\pm(t)$ satisfy the bound from (162). We need to find $b^\pm(0)$. To this end compute

$$\begin{aligned} P_{\mathrm{im}}(\alpha_\infty^{(0)})[U^{(3)}(0) - U^{(2)}(0)] &= b^+(0) f^+(\alpha_\infty^{(0)}) + b^-(0) f^-(\alpha_\infty^{(0)}) \\ &= P_{\mathrm{im}}(\alpha_\infty^{(0)}) \mathcal{G}_\infty(\pi^{(0)})(0)\big[(h^{(1)} - h^{(0)}) f^+(\alpha_\infty^{(0)}) - h^{(1)}[f^+(\alpha_\infty^{(0)}) - f^+(\alpha_\infty^{(1)})] \\ &\quad + \sum_{j=1}^{8} [a_j^{(1)} \eta_j(\alpha_\infty^{(1)}) - a_j^{(1)} \eta_j(\alpha_\infty^{(1)})]\big]. \end{aligned}$$

Thus,

$$\begin{aligned} |b^+(0) - (h^{(1)} - h^{(0)})| &\lesssim \delta^2 |h^{(1)} - h^{(0)}| + \delta^2 \|\pi^{(1)} - \pi^{(0)}\| + \delta^2 \sum_{j=1}^{8} |a_j^{(1)} - a_j^{(0)}| \\ &\lesssim \delta^2 |h^{(1)} - h^{(0)}| + \delta^2 \|\pi^{(1)} - \pi^{(0)}\|, \end{aligned}$$

where we used (82) in the final inequality. Moreover,

$$|b^-(0)| \lesssim |h^{(1)} - h^{(0)}| + \delta^2 \|\pi^{(1)} - \pi^{(0)}\|.$$

Since $b^\pm(t)$ is a bounded solution of the ODE (166), it follows from Lemma 23 and (162) that

$$\begin{aligned} |b^+(0)| &\lesssim \int_0^\infty e^{-\sigma(\alpha_\infty^{(0)})t} (|g^+(t)| + |g^-(t)|)\, dt \\ &\lesssim \int_0^\infty e^{-\sigma(\alpha_\infty^{(0)})t} \Big[\delta \langle t\rangle^{-\frac12} \|\pi^{(1)} - \pi^{(0)}\| + \delta \|Z^{(1)}(t) - Z^{(0)}(t)\|_{4+\infty} \\ &\qquad + \delta^2 \|\tilde U^{(3)}(t) - U^{(2)}(t)\|_{4+\infty} + \|\pi^{(3)} - \pi^{(2)}\| \mu_\pi(t)^{-1}\Big]\, dt \\ &\lesssim \delta \|(\pi^{(0)} - \pi^{(1)}, Z^{(0)} - Z^{(1)})\| \\ &\qquad + \delta^2 \|(\pi^{(3)} - \pi^{(2)}, Z^{(3)} - Z^{(2)})\| + \|\pi^{(3)} - \pi^{(2)}\| \end{aligned}$$

and thus also

$$\begin{aligned} |h^{(1)} - h^{(0)}| + |b^-(0)| \lesssim\, &\delta \|(\pi^{(0)} - \pi^{(1)}, Z^{(0)} - Z^{(1)})\| \\ &+ \delta^2 \|(\pi^{(3)} - \pi^{(2)}, Z^{(3)} - Z^{(2)})\| + \|\pi^{(3)} - \pi^{(2)}\|. \end{aligned} \tag{167}$$

Furthermore, in view of (93),

$$
\begin{aligned}
(168)\quad & \|P_{\mathrm{im}}(\alpha_\infty^{(0)})[\tilde U^{(3)}(t)-U^{(2)}(t)]\|_{4+\infty}\lesssim |b^+(t)|+|b^-(t)|\\
&\lesssim \int_t^\infty e^{-\sigma(\alpha_\infty^{(0)})(s-t)}\Big[\delta\langle s\rangle^{-\frac12}\|\pi^{(1)}-\pi^{(0)}\|+\delta\|Z^{(1)}(s)-Z^{(0)}(s)\|_{4+\infty}\\
&\quad+\delta^2\|\tilde U^{(3)}(s)-U^{(2)}(s)\|_{4+\infty}+\|\pi^{(3)}-\pi^{(2)}\|\mu_\pi(s)^{-1}\Big]\,ds\\
&\quad+e^{-\sigma(\alpha_\infty^{(0)})t}(\delta\|(\pi^{(0)}-\pi^{(1)},Z^{(0)}-Z^{(1)})\|\\
&\quad+\delta^2\|(\pi^{(3)}-\pi^{(2)},Z^{(3)}-Z^{(2)})\|+\|\pi^{(3)}-\pi^{(2)}\|)\\
&\quad+\int_0^t e^{-\sigma(\alpha_\infty^{(0)})(t-s)}\Big[\delta\langle s\rangle^{-\frac12}\|\pi^{(1)}-\pi^{(0)}\|+\delta\|Z^{(1)}(s)-Z^{(0)}(s)\|_{4+\infty}\\
&\quad+\delta^2\|\tilde U^{(3)}(s)-U^{(2)}(s)\|_{4+\infty}+\|\pi^{(3)}-\pi^{(2)}\|\mu_\pi(s)^{-1}\Big]\,ds\\
&\lesssim \langle t\rangle^{-\frac12}\big[\delta\|(\pi^{(0)}-\pi^{(1)},Z^{(0)}-Z^{(1)})\|\\
&\quad+\delta^2\|(\pi^{(3)}-\pi^{(2)},Z^{(3)}-Z^{(2)})\|+\|\pi^{(3)}-\pi^{(2)}\|\big].
\end{aligned}
$$

Now set

$$\varepsilon_0:=\|(\pi^{(0)}-\pi^{(1)},Z^{(0)}-Z^{(1)})\|,\quad \varepsilon_2:=\|(\pi^{(3)}-\pi^{(2)},Z^{(3)}-Z^{(2)})\|.$$

Combining (157), (165), and (168) leads to the bound

$$\sup_{t\ge0}\mu_Z(t)\|Z^{(3)}(t)-Z^{(2)}(t)\|_{4+\infty}\lesssim \delta\varepsilon_0+\delta^2\varepsilon_2+\|\pi^{(3)}-\pi^{(2)}\| \tag{169}$$

and thus also, in view of (167),

$$|h^{(1)}-h^{(0)}|\lesssim \delta\varepsilon_0+\delta^2\varepsilon_2+\|\pi^{(3)}-\pi^{(2)}\|. \tag{170}$$

We now turn to estimating the difference of the paths $\pi^{(3)},\pi^{(2)}$. Indeed, inserting some of the bounds we derived in (154) yields

$$
\begin{aligned}
|\dot\pi^{(3)}(t)-\dot\pi^{(2)}(t)| &\lesssim \delta^2\langle t\rangle^{-3}\|(Z^{(3)}-Z^{(2)})(t)\|_{4+\infty}\\
&\quad+\delta\langle t\rangle^{-\frac32}(\langle t\rangle^{-\frac12}\|\pi^{(0)}-\pi^{(1)}\|+\delta^2\langle t\rangle^{-2}\|\pi^{(1)}-\pi^{(0)}\|)\\
&\quad+\delta\langle t\rangle^{-\frac32}\|Z^{(0)}(t)-Z^{(1)}(t)\|_{4+\infty}\\
&\quad+\delta^2\langle t\rangle^{-2}\|\pi^{(1)}-\pi^{(0)}\|+\delta^2\langle t\rangle^{-2}\|\pi^{(1)}-\pi^{(0)}\|\\
&\lesssim \delta^2\mu_\pi(t)^{-1}\varepsilon_2+\delta\mu_\pi(t)^{-1}\varepsilon_0,
\end{aligned}
$$

which implies that

$$\|\pi^{(3)}-\pi^{(2)}\|=\sup_{t\ge0}\mu_\pi(t)|\dot\pi^{(3)}(t)-\dot\pi^{(2)}(t)|\lesssim \delta^2\varepsilon_2+\delta\varepsilon_0.$$

Combining this bound with (169) yields that $\varepsilon_2\lesssim\delta\varepsilon_0$, which is the same as

$$\|\Psi(R_0,\pi^{(0)},Z^{(0)})-\Psi(R_1,\pi^{(1)},Z^{(1)})\|\lesssim\delta\|(\pi^{(0)},Z^{(0)})-(\pi^{(1)},Z^{(1)})\|,$$

where we have included the initial condition R_0 in the notation. We have shown that Ψ is a contraction in X_δ.

Denote the unique fixed-point in X_δ by $(\pi(R_0), Z(R_0))$. We claim that this fixed-point is Lipschitz in R_0 in the following sense:

$$\|(\pi(R_0), Z(R_0)) - (\pi(R_1), Z(R_1))\| \lesssim \|\!|R_0 - R_1|\!\|. \tag{171}$$

In view of Lemma 28 it suffices to show that

$$\|\Psi(R_0, \pi^{(0)}, Z^{(0)}) - \Psi(R_1, \pi^{(0)}, Z^{(0)})\| \lesssim \|\!|R_0 - R_1|\!\|. \tag{172}$$

To prove this, set

$$(\pi^{(3)}, Z^{(3)}) = \Psi(R_1, \pi^{(0)}, Z^{(0)}),\ (\pi^{(2)}, Z^{(2)}) = \Psi(R_0, \pi^{(0)}, Z^{(0)}).$$

The difference of these functions is controlled by the equations (152), (154) with $\pi^{(0)} = \pi^{(1)}, Z^{(0)} = Z^{(1)}$. Hence,

$$\begin{aligned} i\partial_t(Z^{(3)} - Z^{(2)}) - \mathcal{H}(\pi^{(0)}(t))(Z^{(3)} - Z^{(2)}) &= -i(\dot{\tilde{\pi}}^{(3)} - \dot{\tilde{\pi}}^{(2)})\partial_\pi W(\pi^{(0)}),\\ -i\langle(\dot{\tilde{\pi}}^{(3)} - \dot{\tilde{\pi}}^{(2)})\partial_\pi W(\pi^{(0)}), \tilde{\xi}_j(\pi^{(0)})\rangle &= \langle Z^{(3)} - Z^{(2)}, \dot{\tilde{\pi}}^{(0)}\mathcal{S}_j(\pi^{(0)})\rangle, \end{aligned}$$

with initial conditions

$$(Z^{(3)} - Z^{(2)})(0) = \begin{pmatrix} R_1\\ \bar{R}_1\end{pmatrix} - \begin{pmatrix} R_0\\ \bar{R}_0\end{pmatrix} + (h^{(1)} - h^{(0)})f^+(\alpha_\infty^{(0)}) + \sum_{j=1}^{8}(a_j^{(1)} - a_j^{(0)})\eta_j(\alpha_\infty^{(0)});$$

cf. (153). The orthogonality conditions

$$\langle Z^{(3)}(t), \tilde{\xi}_j(\pi^{(0)})(t)\rangle = \langle Z^{(2)}(t), \tilde{\xi}_j(\pi^{(0)})(t)\rangle = 0$$

hold for all $t \geq 0$ by construction. Setting

$$\tilde{U}^{(3)} := \mathcal{T}^{(0)}Z^{(3)},\ U^{(2)} := \mathcal{T}^{(0)}Z^{(2)}$$

as before, we obtain the transformed equations

$$\begin{aligned} i\partial_t(\tilde{U}^{(3)} - U^{(2)}) &- \mathcal{H}(\alpha_\infty^{(0)})(\tilde{U}^{(3)} - U^{(2)})\\ &= \mathcal{T}^{(0)}[-i(\dot{\tilde{\pi}}^{(3)} - \dot{\tilde{\pi}}^{(2)})\partial_\pi W(\pi^{(0)})] + V(\pi^{(0)})(\tilde{U}^{(3)} - U^{(2)}),\\ (\tilde{U}^{(3)} - U^{(2)})(0) &= \mathcal{G}_\infty(0)\Big[\begin{pmatrix} R_1\\ \bar{R}_1\end{pmatrix} - \begin{pmatrix} R_0\\ \bar{R}_0\end{pmatrix} + (h^{(1)} - h^{(0)})f^+(\alpha_\infty^{(0)})\\ &\qquad + \sum_{j=1}^{8}(a_j^{(1)} - a_j^{(0)})\eta_j(\alpha_\infty^{(0)})\Big]. \end{aligned}$$

The orthogonality conditions are $\langle \tilde{U}^{(3)} - U^{(2)}, \xi_j(\pi^{(0)})\rangle = 0$. The estimate (172) now follows by using the same techniques we have employed repeatedly in order to control the solution $\tilde{U}^{(3)} - U^{(2)}$. We skip the details.

Combining (171) with (170) leads to the statement that

$$|h(R_0) - h(R_1)| \lesssim \delta\|\!|R_0 - R_1|\!\|,$$

as claimed. □

Proof of Theorem 1. Given R_0, the previous lemma establishes the existence of $h = h(R_0) \in \mathbb{R}$ as well as $(\pi, Z) = (\pi(R_0), Z(R_0)) \in X_\delta$ where $\delta = C_0 \|\!|R_0|\!\|$, which solve

$$\begin{aligned} &i\partial_t Z(t) + \begin{pmatrix} \triangle + 2|W(\pi)|^2 & W^2(\pi) \\ -\bar{W}^2(\pi) & -\triangle - 2|W(\pi)|^2 \end{pmatrix} Z(t) \\ &= \dot{v}\begin{pmatrix} -xe^{i\theta(\pi)(t)}\phi(\cdot - y(\pi)(t), \alpha(t)) \\ xe^{-i\theta(\pi)(t)}\phi(\cdot - y(\pi)(t), \alpha(t)) \end{pmatrix} + \dot{\gamma}\begin{pmatrix} -e^{i\theta(\pi)(t)}\phi(\cdot - y(\pi)(t), \alpha(t)) \\ e^{-i\theta(\pi)(t)}\phi(\cdot - y(\pi)(t), \alpha(t)) \end{pmatrix} \\ &\quad + i\dot{\alpha}\begin{pmatrix} e^{i\theta(\pi)(t)}\partial_\alpha\phi(\cdot - y(\pi)(t), \alpha(t)) \\ e^{-i\theta(\pi)(t)}\partial_\alpha\phi(\cdot - y(\pi)(t), \alpha(t)) \end{pmatrix} + i\dot{D}\begin{pmatrix} -e^{i\theta(\pi)(t)}\nabla\phi(\cdot - y(\pi)(t), \alpha(t)) \\ -e^{-i\theta(\pi)(t)}\nabla\phi(\cdot - y(\pi)(t), \alpha(t)) \end{pmatrix} \\ &\quad + \begin{pmatrix} -2|R|^2 W(\pi)(t) - \bar{W}(\pi)(t)R^2 - |R|^2 R \\ 2|R|^2\bar{W}(\pi)(t) + W(\pi)(t)\bar{R}^2 + |R|^2\bar{R} \end{pmatrix} \end{aligned}$$

with initial conditions

$$\begin{aligned} Z(0) &= \begin{pmatrix} R_0 \\ \bar{R}_0 \end{pmatrix} + h(R_0)f^+(\alpha_\infty) + \sum_{j=1}^{8} a_j(R_0)\eta_j(\alpha_\infty), \\ \pi(0) &= (\alpha_0, 0, 0, 0). \end{aligned}$$

Here $a_j = a_j(h(R_0)) \in \mathbb{R}$. Define

$$\Phi(R_0) := h(R_0)f^+(\alpha_\infty) + \sum_{j=1}^{8} a_j(R_0)\eta_j(\alpha_\infty).$$

Since

$$|h(R_0)| + \sum_{j=1}^{8} |a_j(h(R_0))| \lesssim \delta^2 \lesssim \|\!|R_0|\!\|^2,$$

the estimate (8) follows. Moreover, (9) follows from (142). Since $Z = \binom{R(t)}{\bar{R}(t)}$ is $\mathcal{J}$-invariant, it follows from Lemma 5 that

$$\psi(t) := W(\pi(t)) + R(t)$$

is an H^1-solution of the nonlinear Schrödinger equation (1). Finally,

$$\|R(t)\|_{W^{1,2}} \lesssim \delta, \quad \|R(t)\|_\infty \lesssim \delta t^{-\frac{3}{2}}$$

follows from (66) and (67), whereas (65) ensures that the path is admissible and therefore converges to $\pi(\infty)$ with

$$\sup_{t\geq 0} |\pi(t) - \pi(\infty)| \lesssim \delta^2.$$

Finally, we turn to the scattering statement. According to Lemma 6,

$$\begin{aligned} i\partial_t U - \mathcal{H}(\alpha_\infty)U &= -i\dot{\pi}\partial_\pi\widetilde{W}(\pi) + N(U, \pi) + V(\pi)U \\ U(0) &= \mathcal{G}_\infty(\pi)(0)\Big[\begin{pmatrix} R_0 \\ \bar{R}_0 \end{pmatrix} + \begin{pmatrix} \Phi(R_0) \\ \overline{\Phi(R_0)} \end{pmatrix}\Big]. \end{aligned} \tag{173}$$

Denoting the right-hand side (173) by $F(t)$, we have

$$U(t) = e^{-it\mathcal{H}(\alpha_\infty)}U(0) - i\int_0^t e^{-i(t-s)\mathcal{H}(\alpha_\infty)}F(s)\,ds.$$

The estimates (65)–(67) imply that

$$\|F(s)\|_{\frac{3}{2}\cap 2} \lesssim \langle s\rangle^{-\frac{5}{2}}, \quad \|F(s)\|_{1\cap 2} \lesssim \langle s\rangle^{-\frac{3}{2}}, \quad \int_0^\infty \|F(s)\|_2\,ds < \infty.$$

This allows us to define

$$U_1 := PU(0) - i\int_0^\infty e^{is\mathcal{H}(\alpha_\infty)}PF(s)\,ds \in L^2(\mathbb{R}^3)$$

where we have set $P := P_{\mathrm{s}}(\alpha_\infty) + P_{\mathrm{root}}(\alpha_\infty) = 1 - P_{\mathrm{im}}(\alpha_\infty)$. We are using here that

$$\|e^{is\mathcal{H}(\alpha_\infty)}PF(s)\|_2 \lesssim \langle s\rangle^{-\frac{3}{2}},$$

which follows from the fact that growth of $e^{is\mathcal{H}(\alpha_\infty)}$ on the root-space can be at most s. Clearly, U_1 was defined so that

$$PU(t) - e^{-it\mathcal{H}(\alpha_\infty)}U_1 = i\int_t^\infty e^{-i(t-s)\mathcal{H}(\alpha_\infty)}PF(s)\,ds$$

which implies that

$$\|PU(t) - e^{-it\mathcal{H}(\alpha_\infty)}U_1\|_2 \lesssim \langle t\rangle^{-\frac{1}{2}} \to 0$$

as $t \to \infty$. As far as the hyperbolic part is concerned, we define

$$U_2 := P^-_{\mathrm{im}}(\alpha_\infty)U(0) - i\int_0^\infty e^{-s\sigma(\alpha_\infty)}P^-_{\mathrm{im}}(\alpha_\infty)F(s)\,ds.$$

Because of Lemma 23,

$$P_{\mathrm{im}}(\alpha_\infty)U(t) - e^{-it\mathcal{H}(\alpha_\infty)}U_2 = i\int_t^\infty e^{(t-s)\sigma(\alpha_\infty)}P_{\mathrm{im}}(\alpha_\infty)F(s)\,ds.$$

In conjunction with the P-part this shows that

$$U(t) - e^{-it\mathcal{H}(\alpha_\infty)}(U_1+U_2) = i\int_t^\infty e^{-i(t-s)\mathcal{H}(\alpha_\infty)}PF(s)\,ds + i\int_t^\infty e^{(t-s)\sigma(\alpha_\infty)}P_{\mathrm{im}}(\alpha_\infty)F(s)\,ds. \tag{174}$$

Therefore, as $t \to \infty$,

$$U(t) = e^{-it\mathcal{H}(\alpha_\infty)}(U_1+U_2) + o_{L^2}(1). \tag{175}$$

Another consequence of (174) is the estimate

$$\|e^{-it\mathcal{H}(\alpha_\infty)}(U_1+U_2)\|_3 \lesssim \|U(t)\|_3 + \int_t^\infty (t-s)^{-\frac{1}{2}}\|F(s)\|_{\frac{3}{2}}\,ds \lesssim \langle t\rangle^{-\frac{1}{2}}. \tag{176}$$

This implies that in fact $P_{\rm root}(U_1+U_2)=0$. Seeing this requires some care, as we do not know that $U_1+U_2\in L^1(\mathbb{R}^3)$. However, (176) implies that

$$\|e^{-it\mathcal{H}(\alpha_\infty)}(U_1+U_2)\|_{L^4_t(L^3_x(\mathbb{R}^3))}<\infty.$$

On the other hand, by the Strichartz estimate (216),

$$\|e^{-it\mathcal{H}(\alpha_\infty)}P_{\rm s}(U_1+U_2)\|_{L^4_t(L^3_x(\mathbb{R}^3))}\lesssim \|P_{\rm s}(U_1+U_2)\|_2<\infty.$$

Hence, also

$$\|e^{-it\mathcal{H}(\alpha_\infty)}P_{\rm root}(U_1+U_2)\|_{L^4_t(L^3_x(\mathbb{R}^3))}<\infty.$$

However, this is only possible if in fact $P_{\rm root}(U_1+U_2)=0$, as claimed. Therefore,

$$U_1:=P_{\rm s}U(0)-i\int_0^\infty e^{is\mathcal{H}(\alpha_\infty)}P_{\rm s}F(s)\,ds$$

which in particular implies the dispersive bound

(177)

$$\|e^{-it\mathcal{H}(\alpha_\infty)}(U_1+U_2)\|_{2+\infty}\lesssim \|U(t)\|_{2+\infty}+\int_t^\infty \langle t-s\rangle^{-\frac32}\|F(s)\|_{1\cap 2}\,ds\lesssim \langle t\rangle^{-\frac32}.$$

It remains to show that one has scattering for the evolution of $\mathcal{H}(\alpha_\infty)$. This is a standard Cook's method argument. Indeed, write

$$\mathcal{H}(\alpha_\infty)=\begin{pmatrix} -\triangle+\alpha_\infty^2 & 0\\ 0 & \triangle-\alpha_\infty^2\end{pmatrix}+\begin{pmatrix} -2\phi_\infty^2 & -\phi_\infty^2\\ \phi_\infty^2 & 2\phi_\infty^2\end{pmatrix}=:\mathcal{H}_0(\alpha_\infty)+V,$$

where $\phi_\infty:=\phi(\cdot,\alpha_\infty)$. Then

$$\begin{aligned} e^{-it\mathcal{H}(\alpha_\infty)}(U_1+U_2) &= e^{-it\mathcal{H}_0(\alpha_\infty)}(U_1+U_2)\\ &\quad -i\int_0^t e^{-i(t-s)\mathcal{H}_0(\alpha_\infty)}Ve^{-is\mathcal{H}(\alpha_\infty)}(U_1+U_2)\,ds\end{aligned}$$

and thus

$$U_3:=\lim_{t\to\infty}e^{it\mathcal{H}_0(\alpha_\infty)}e^{-it\mathcal{H}(\alpha_\infty)}(U_1+U_2) \tag{178}$$

exists as a strong L^2 limit. Indeed, this follows from

$$\begin{aligned}&\int_0^\infty \|e^{is\mathcal{H}_0(\alpha_\infty)}Ve^{-is\mathcal{H}(\alpha_\infty)}(U_1+U_2)\|_2\,ds\\ &\qquad\lesssim \int_0^\infty \|e^{-is\mathcal{H}(\alpha_\infty)}(U_1+U_2)\|_{2+\infty}\,ds<\infty;\end{aligned}$$

see (177). It follows from (175) and (178) that

$$U(t)=e^{-it\mathcal{H}_0(\alpha_\infty)}U_3+o_{L^2}(1).$$

Finally,

$$Z(t)=\mathcal{G}_\infty(t)^{-1}M(t)^{-1}U(t)=e^{-it\mathcal{H}_0}\mathcal{G}_\infty^{-1}(0)U_3+o_{L^2}(1),$$

where $\mathcal{H}_0 = \begin{pmatrix} -\triangle & 0 \\ 0 & \triangle \end{pmatrix}$. Setting $\mathcal{G}_\infty^{-1}(0)U_3 = \binom{f_0}{\bar{f}_0}$ and $Z(t) = \binom{R(t)}{\bar{R}(t)}$, we obtain

$$R(t) = e^{it\triangle} f_0 + o_{L^2}(1),$$

and the theorem is proved. □

Proof of Theorem 2. The idea is as follows: Given α_0, consider the nonlinear Schrödinger equation (1) with initial data $\phi(\cdot, \alpha_0) + R_0$. Applying the usual eight-parameter family of symmetries (Galilei giving six parameters, modulation one, and scaling also one — scaling here is the same as the parameter α), we transform this to $W(0, \cdot) + R_1$ where $W(0, x)$ is a soliton with a general parameter vector π_0 which is close to $(0, 0, 0, \alpha_0)$. Hence, we can apply Theorem 1 to conclude that these initial data will give rise to global solutions with the desired properties as long as $W(0, x) + R_1$ lies on the stable manifold associated with $W(0, x)$. To prove that we obtain eight dimensions back in this fashion requires checking that the derivatives of $W(0, x)$ in its parameters are transverse to the linear space $\mathcal{S}$ of Theorem 1. However, these derivatives are basically the elements of the root space $\mathcal{N}$ of $\mathcal{H}(\alpha_0)$, whereas we know that $\mathcal{S}$ is perpendicular to the root space $\mathcal{N}^*$ of $\mathcal{H}(\alpha_0)^*$. But Lemma 10 implies that no nonzero vector in $\mathcal{N}$ is perpendicular to $\mathcal{N}^*$, which proves that $\mathcal{N}$ is transverse to $\mathcal{S}$, as desired. □

7. The linear analysis: Dispersive theory

We now consider the estimates on the linear evolution $e^{it\mathcal{H}}$ where $\mathcal{H} = \mathcal{H}(\alpha^2)$ is as in (6). It is for the estimates in this section that we will need to assume the absence of imbedded eigenvalues in the essential spectrum of $\mathcal{H}$. The reader should consult Section 4 for the spectral properties of $\mathcal{H}(\alpha)$. Although the results of this section are abstract and refer to general matrix-valued Schrödinger operators, see [17], we will use some facts about $\mathcal{H}$ established in that section.

In what follows, we need to bound the resolvents on weighted L^2 spaces. For that purpose, let $L^{2,\sigma} = \langle x \rangle^{-\sigma} L^2$ and

$$X_\sigma := L^{2,\sigma}(\mathbb{R}^3) \times L^{2,\sigma}(\mathbb{R}^3).$$

Clearly, $X_\sigma^* = X_{-\sigma}$. Recall that we have shown above that the edges $\pm\alpha^2$ of the essential spectrum of $\mathcal{H}$ are regular. This means that

$$I + (\mathcal{H}_0 - (\lambda \pm i0))^{-1} V : X_{-1-} \to X_{-1-} \tag{179}$$

is invertible for $\lambda = \pm\mu$.

The following result, which is proved in [17], see also [15], is a version of the usual limiting absorption principle for Schrödinger operators, but for the non-selfadjoint case.

Proposition 30. *For any* $0 < \mu < \alpha^2$

$$\sup_{|\lambda|\ge\mu,\,0<\varepsilon} |\lambda|^{\frac12}\,\|(\mathcal{H}-(\lambda\pm i\varepsilon))^{-1}\| < \infty \tag{180}$$

where the norm is the one from $X_{1+} \to X_{-1-}$. *If the supremum in* (180) *is only taken over* $|\lambda| \ge \lambda_0$ *where* $\lambda_0 > \alpha^2$, *then* (180) *also holds in the norms of* $X_{\frac12+} \to X_{-\frac12-}$.

Proof. See [17]. □

Furthermore, the resolvents attain boundary values that satisfy the same bounds as in Proposition 30.

Corollary 31. *It is possible to define*

$$(\mathcal{H}-(\lambda\pm i0))^{-1} := (I+(\mathcal{H}_0-(\lambda\pm i0))^{-1}V)^{-1}(\mathcal{H}_0-(\lambda\pm i0))^{-1} \tag{181}$$

for all $|\lambda| > \mu$ *where* μ *is as in* (180). *Then as* $\varepsilon \to 0+$,

$$\|(\mathcal{H}-(\lambda\pm i\varepsilon))^{-1} - (\mathcal{H}-(\lambda\pm i0))^{-1}\| \to 0$$

in the norm of $X_{1+} \to X_{-1-}$ *and one can extend* (180) *to* $\varepsilon \ge 0$. *The same statements hold with* $X_{\frac12+} \to X_{-\frac12-}$ *provided* $|\lambda| \ge \lambda_0 > \alpha^2$.

Proof. See [17]. □

Finally, we will need to differentiate the resolvents in the energy.

Corollary 32. *With the same notation as in the previous two results, for every* $\lambda_0 > \alpha^2$,

$$\sup_{|\lambda|\ge\lambda_0} \left\|\partial_\lambda(\mathcal{H}-(\lambda\pm i0))^{-1}\right\|_{X_{\frac32+}\to X_{-\frac32-}} \lesssim 1,$$
$$\sup_{|\lambda|\ge\lambda_0} \left\|\partial_\lambda^2(\mathcal{H}-(\lambda\pm i0))^{-1}\right\|_{X_{\frac52+}\to X_{-\frac52-}} \lesssim 1.$$

Proof. See [17]. □

Finally, there is the following representation formula for the evolution; see [17].

Lemma 33. *Assume that there are no embedded eigenvalues in the essential spectrum. Then there is the representation*

$$e^{it\mathcal{H}} = \frac{1}{2\pi i}\int_{|\lambda|\ge\mu} e^{it\lambda}\,[(\mathcal{H}-(\lambda+i0))^{-1}-(\mathcal{H}-(\lambda-i0))^{-1}]\,d\lambda + \sum_j e^{it\mathcal{H}}P_{\zeta_j}, \tag{182}$$

where the sum runs over the entire discrete spectrum $\{\zeta_j\}_j$ *and* P_{ζ_j} *is the Riesz projection corresponding to the eigenvalue* ζ_j. *The formula* (182) *and the* *convergence of the integral are to be understood in the following weak sense: If* ϕ, ψ *belong to* $[W^{2,2} \times W^{2,2}(\mathbb{R}^3)] \cap X_{1+}$, *then*

$$\langle e^{it\mathcal{H}}\phi, \psi\rangle = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{R\geq|\lambda|\geq\mu} e^{it\lambda} \big\langle [(\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1}]\phi, \psi \big\rangle \, d\lambda + \sum_j \langle e^{it\mathcal{H}} P_{\zeta_j}\phi, \psi\rangle$$

for all t, *where the integrand is well-defined by the limiting absorption principle from above.*

Proof. See [17]. □

As an easy consequence, one can bound the evolution uniformly on L^2 provided the discrete spectrum is removed. As ususal, we denote by P_{s} the projection $I - P_d$ where P_d is the Riesz projection onto the discrete spectrum of $\mathcal{H}$.

THEOREM 34. *Assume as before that there are no embedded eigenvalues. Then the following* stability *bound holds*:

$$\sup_{t\in\mathbb{R}} \|e^{it\mathcal{H}} P_{\mathrm{s}}\|_{2\to 2} \leq C. \tag{183}$$

Proof. See [17]. We remark that this bound holds irrespective of $\pm\alpha^2$ being regular. □

We now turn to the dispersive bound. The proof applies to all matrix potentials V with the decay $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for all $x \in \mathbb{R}^3$ with some $\beta > 3$. In addition we again need to assume that there are no embedded eigenvalues in the essential spectrum and that the thresholds $\pm\alpha^2$ are regular.

THEOREM 35. *There is the dispersive bound*

$$\|e^{it\mathcal{H}} P_{\mathrm{s}}\|_{1\to\infty} \lesssim |t|^{-\frac{3}{2}}. \tag{184}$$

Proof. We will use the method of proof from [21]. We start from Lemma 33; i.e.,

$$e^{it\mathcal{H}} P_{\mathrm{s}} = \frac{1}{2\pi i} \int_{|\lambda|\geq\mu} e^{it\lambda} [(\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1}] \, d\lambda. \tag{185}$$

We distinguish between energies close to the thresholds $\pm\mu$ and those separated from these points. Thus let $\chi_+(\lambda) = 1$ if $\lambda - \mu > 2\lambda_1$ and $= 0$ if $\lambda - \mu \leq \lambda_1$ where $\lambda_1 > 0$ is some small number. Similarly, $\chi_-(\lambda) = 1$ if $\lambda + \mu < -2\lambda_1$ and $= 0$ if $\lambda + \mu \geq -\lambda_1$. We will use the notation $\chi_+(\mathcal{H})$ and $\chi_-(\mathcal{H})$ formally with the obvious meaning. Let $R_0^{\pm}(\lambda^2)$ and $R_V^{\pm}(\lambda^2)$ be the resolvents of $\mathcal{H}_0$, and

$\mathcal{H} = \mathcal{H}_0 + V$, respectively. Then, by a finite resolvent expansion and a change of variables $\lambda \to \lambda^2 + \mu$,

(186)

$$\langle e^{it\mathcal{H}}\chi_+(\mathcal{H})P_{\rm s}f, g\rangle = \frac{e^{it\mu}}{\pi i}\int_0^\infty \lambda e^{it\lambda^2}\chi_+(\lambda^2+\mu)\big\langle [R_V^+(\lambda^2+\mu) - R_V^-(\lambda^2+\mu)]f, g\big\rangle\, d\lambda$$

(187)

$$= e^{it\mu}\sum_{\ell=0}^{2m-1}\frac{(-1)^\ell}{\pi i}\int_0^\infty \lambda e^{it\lambda^2}\chi_+(\lambda^2+\mu)\big\langle [R_0^+(\lambda^2+\mu)(VR_0^+(\lambda^2+\mu))^\ell - R_0^-(\lambda^2+\mu)(VR_0^-(\lambda^2+\mu))^\ell]f, g\big\rangle\, d\lambda$$
$$+ \frac{e^{it\mu}}{\pi i}\int_0^\infty \lambda e^{it\lambda^2}\chi_+(\lambda^2+\mu)\big\langle [(R_0^+(\lambda^2+\mu)V)^m R_V^+(\lambda^2+\mu)(VR_0^+(\lambda^2+\mu))^m - (R_0^-(\lambda^2+\mu)V)^m R_V^-(\lambda^2+\mu)(VR_0^-(\lambda^2+\mu))^m]f, g\big\rangle\, d\lambda.$$

We need to show that each of the $2m$ terms in the finite (Born) sum is in absolute value $\le C(\ell, V)\,|t|^{-\frac{3}{2}}\|f\|_1\|g\|_1$, and similarly for the remaining term containing R_V.

Each of the first $2m$ terms of the Born series is written out explicitly using the free scalar resolvent ($\Im z > 0, \Im\sqrt{z} > 0$)

$$(-\triangle - z)^{-1}(x,y) = \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|},$$

which implies for the matrix case

$$R_0^\pm(\lambda^2+\mu)(x,y) = \begin{pmatrix} \frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|} & 0 \\ 0 & \frac{e^{-\sqrt{2\mu+\lambda^2}|x-y|}}{4\pi|x-y|} \end{pmatrix}. \tag{188}$$

Consider the case $\ell = 0$ in (186). Upon recombining the two $\pm$ parts the lower right-hand corner of (188) drops out, and one is lead to proving an oscillatory integral bound of the form

$$\Big|\int_0^\infty e^{it\lambda^2}\lambda\chi_+(\lambda^2+\mu)\sin(\lambda|x-y|)\, d\lambda\Big| \lesssim t^{-\frac{3}{2}}|x-y|, \tag{189}$$

To prove (189), we argue as follows:

$$\begin{aligned}\Big|\int_0^\infty e^{it\lambda^2}\lambda\chi_+(\lambda^2+\mu)\sin(\lambda|x-y|)\, d\lambda\Big| &= \frac{1}{2}\Big|\int_{-\infty}^\infty e^{it\lambda^2}\lambda\chi_+(\lambda^2+\mu)\sin(\lambda|x-y|)\, d\lambda\Big| \\ &\lesssim t^{-1}|x-y|\,\Big|\int_{-\infty}^\infty e^{it\lambda^2}\chi_+(\lambda^2+\mu)\cos(\lambda|x-y|)\, d\lambda\Big|\end{aligned} \tag{190}$$

$$
\begin{aligned}
&+ t^{-1}\left|\int_{-\infty}^{\infty} e^{it\lambda^2}\lambda\chi_+'(\lambda^2+\mu)\sin(\lambda|x-y|)\,d\lambda\right| \\
&\lesssim t^{-\frac{3}{2}}|x-y|\big\|[\chi_+(\lambda^2+\mu)\cos(\lambda|x-y|)]^\vee\big\|_{\mathcal{M}} \\
&\quad + t^{-\frac{3}{2}}\big\|[\lambda\chi_+'(\lambda^2+\mu)\sin(\lambda|x-y|)]^\vee\big\|_{\mathcal{M}} \\
&\lesssim t^{-\frac{3}{2}}|x-y|.
\end{aligned}
$$

Here we used the $L^1 \to L^\infty$ estimate for the one-dimensional Schrödinger equation, as well as the elementary facts

$$
\sup_{a\in\mathbb{R}}\big\|[\chi_+(\lambda^2+\mu)\cos(\lambda a)]^\vee\big\|_{\mathcal{M}} \le C,
$$

$$
\sup_{a\in\mathbb{R}}|a|^{-1}\big\|[\lambda\chi_+'(\lambda^2+\mu)\sin(\lambda a)]^\vee\big\|_{\mathcal{M}} \le C,
$$

where $\|\cdot\|_{\mathcal{M}}$ stands for the total variation norm of measures. The first is proved by writing it as the convolution of two measures of mass $\lesssim 1$ uniformly in a. The second is done similarly, but first write

$$
\sin(\lambda a) = \lambda\int_0^a \cos(\lambda\alpha)\,d\alpha. \tag{191}
$$

This yields that the $\ell = 0$ in (187) contributes $\lesssim t^{-\frac{3}{2}}\|f\|_1\|g\|_1$, as desired. Next, we sketch the argument for the case $\ell = 1$. The argument for larger ℓ is similar, and we will discuss it later. Writing $f = \binom{f_1}{f_2}, g = \binom{g_1}{g_2}$ this term becomes (we ignore the factor $e^{it\mu}$ as well as other constants and write $dx = dx_0dx_1dx_2$ for simplicity)

$$
\int_{\mathbb{R}^9}\int_0^\infty e^{it\lambda^2}\lambda\chi_+(\lambda^2+\mu)\sin(\lambda(|x_0-x_1|+|x_1-x_2|))\,d\lambda\,\frac{U(x_1)f_1(x_0)\bar g_1(x_2)}{|x_0-x_1||x_1-x_2|}\,dx \tag{192}
$$

$$
+\int_{\mathbb{R}^9}\int_0^\infty e^{it\lambda^2}\lambda\chi_+(\lambda^2+\mu)\sin(\lambda|x_0-x_1|)e^{-\sqrt{2\mu+\lambda^2}|x_2-x_1|}\,d\lambda\,\frac{W(x_1)f_1(x_0)\bar g_2(x_2)}{|x_0-x_1||x_1-x_2|}\,dx \tag{193}
$$

$$
-\int_{\mathbb{R}^9}\int_0^\infty e^{it\lambda^2}\lambda\chi_+(\lambda^2+\mu)\sin(\lambda|x_2-x_1|)e^{-\sqrt{2\mu+\lambda^2}|x_1-x_0|}\,d\lambda\,\frac{W(x_1)f_2(x_0)\bar g_1(x_2)}{|x_0-x_1||x_1-x_2|}\,dx. \tag{194}
$$

The term (192) can be treated by means of (189). Indeed, using this bound it reduces to

$$
\lesssim t^{-\frac{3}{2}}\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|U(y)|}{|x-y|}\,dy\|f\|_1\|g\|_1.
$$

Hence it is enough to assume that the so-called *Kato norm*

$$
\|U\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|U(y)|}{|x-y|}\,dy < \infty
$$

in order to obtain the desired decay for that term. Since we are assuming the pointwise bound $|U(x)| \lesssim \langle x\rangle^{-3-}$, the Kato norm is indeed finite. Now consider the λ-integral in (193). Extending the integral to $(-\infty,\infty)$ and integrating by parts yields

(195)
$$2it\int_{-\infty}^{\infty} e^{it\lambda^2}\lambda\chi_+(\lambda^2+\mu)\sin(\lambda|x_0-x_1|)e^{-\sqrt{2\mu+\lambda^2}|x_2-x_1|}\,d\lambda$$
$$= -\int_{-\infty}^{\infty} e^{it\lambda^2}2\lambda\chi_+'(\lambda^2+\mu)\sin(\lambda|x_0-x_1|)e^{-\sqrt{2\mu+\lambda^2}|x_2-x_1|}\,d\lambda$$

(196)
$$-\int_{-\infty}^{\infty} e^{it\lambda^2}\chi_+(\lambda^2+\mu)\cos(\lambda|x_0-x_1|)e^{-\sqrt{2\mu+\lambda^2}|x_2-x_1|}\,d\lambda\;|x_0-x_1|$$

(197)
$$+\int_{-\infty}^{\infty} e^{it\lambda^2}\chi_+(\lambda^2+\mu)\sin(\lambda|x_0-x_1|)e^{-\sqrt{2\mu+\lambda^2}|x_2-x_1|}\frac{2\lambda}{\sqrt{\mu+\lambda^2}}\,d\lambda\;|x_1-x_2|.$$

The integrals in (195) and (196) can be treated by the same type of arguments which lead up to (190) provided we show that

(198)
$$\sup_{b\ge 0}\Big\|\int_{-\infty}^{\infty} e^{-b\sqrt{2\mu+\lambda^2}}e^{-i\lambda u}\,d\lambda\Big\|_{\mathcal{M}_u} = \sup_{\mu\ge 0}\Big\|\int_{-\infty}^{\infty} e^{-\sqrt{\mu+\lambda^2}}e^{-i\lambda u}\,d\lambda\Big\|_{\mathcal{M}_u}<\infty.$$

Now

$$\partial_\lambda e^{-\sqrt{\mu+\lambda^2}} = -\frac{\lambda}{\sqrt{\mu+\lambda^2}}e^{-\sqrt{\mu+\lambda^2}}, \tag{199}$$
$$\partial_\lambda^2 e^{-\sqrt{\mu+\lambda^2}} = \Big(-\frac{\mu}{(\mu+\lambda^2)^{\frac32}}+\frac{\lambda^2}{\mu+\lambda^2}\Big)e^{-\sqrt{\mu+\lambda^2}}$$

are both in $L^1(\mathbb{R})$, and their L^1 norms are uniformly bounded in $\mu>0$. It follows that

$$\sup_{\mu\ge 0}(1+u^2)\Big|\int_{-\infty}^{\infty} e^{-\sqrt{\mu+\lambda^2}}e^{-i\lambda u}\,d\lambda\Big|\lesssim 1$$

and (198) holds. Therefore, arguing as in (190), we have

$$|(195)|+|(196)|\lesssim t^{-\frac12}|x_0-x_1|.$$

To deal with (197), note that because of (191), the same type of argument as before will yield

$$|(197)|\lesssim t^{-\frac12}|x_0-x_1|$$

provided we can show that

$$\sup_{b>0}\Big\|\int_{-\infty}^{\infty} e^{-i\lambda u}\lambda\partial_\lambda e^{-b\sqrt{2\mu+\lambda^2}}\,d\lambda\Big\|_{\mathcal{M}_u} = \sup_{\mu>0}\Big\|\int_{-\infty}^{\infty} e^{-i\lambda u}\lambda\partial_\lambda e^{-\sqrt{\mu+\lambda^2}}\,d\lambda\Big\|_{\mathcal{M}_u}<\infty. \tag{200}$$

We leave it to reader to check that

$$\sup_{\mu>0}\Big[\|\lambda\partial_\lambda e^{-\sqrt{\mu+\lambda^2}}\|_1+\|\partial_\lambda\lambda\partial_\lambda e^{-\sqrt{\mu+\lambda^2}}\|_1+\|\partial_\lambda^2\lambda\partial_\lambda e^{-\sqrt{\mu+\lambda^2}}\|_1\Big]<\infty, \tag{201}$$

which implies that

$$\sup_{\mu\geq 0}(1+u^2)\Big|\int_{-\infty}^{\infty} e^{-i\lambda u}\lambda\partial_\lambda e^{-\sqrt{\mu+\lambda^2}}\,d\lambda\Big|\lesssim 1$$

and (200) holds. As a side remark, we note the difference between (199) and (201). If $\mu=0$, then the former holds because $\partial_\lambda^2 e^{-|\lambda|}$ contains a δ-measure at the origin. Hence it is not possible to increase this to three derivatives. On the other hand, $\partial_\lambda^3\lambda e^{-|\lambda|}$ is again a measure, which makes (201) hold. Hence, we conclude that for all $t>0$

$$|(193)|+|(194)|\lesssim t^{-\frac32}\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|W(y)|}{|x-y|}\,dy\|f\|_1\|g\|_1\lesssim t^{-\frac32}\|f\|_1\|g\|_1.$$

Recall that this leads to the desired dispersive bound for the term $\ell=1$ in (187). The cases $\ell>1$ are similar. Indeed, the reader will easily check that in the general case one arrives at oscillatory integrals of the form, cf. (192), (193), (194),

$$\int_{-\infty}^{\infty} e^{it\lambda^2}\lambda\chi_+(\lambda^2+\mu)\sin\big(\lambda\sum_{j\in\mathcal{J}}|x_{j+1}-x_j|\big)\exp\big(-\sqrt{2\mu+\lambda^2}\sum_{k\in\mathcal{J}^*}|x_{k+1}-x_k|\big)\,d\lambda$$

where $\mathcal{J}\cup\mathcal{J}^*=\{0,1,\ldots,\ell\}$ is a disjoint partition with $\mathcal{J}\neq\emptyset$. This integral is exactly of the type that we have just dealt with. Therefore, it is bounded by

$$\lesssim t^{-\frac32}\sum_{j\in\mathcal{J}}|x_{j+1}-x_j|.$$

Combining the oscillatory integral with the potentials that accompany it, we are lead to estimating

$$\int_{\mathbb{R}^{3(\ell+2)}}\sum_{j\in\mathcal{J}}|x_{j+1}-x_j|\prod_{k=1}^{n}\frac{|V(x_k)|}{|x_{k+1}-x_k|}\,\frac{|f(x_0)||g(x_{\ell+1})|}{|x_0-x_1|}dx \lesssim(\ell+1)\|V\|_{\mathcal{K}}^{\ell}\|f\|_1\|g\|_1.$$

To pass to the final inequality we invoke a simple lemma from [37] which says that for any positive integer ℓ

$$\sup_{x_0,x_{\ell+1}\in\mathbb{R}^3}\int_{\mathbb{R}^{3\ell}}\frac{\prod_{j=1}^{\ell}|V(x_j)|}{\prod_{j=0}^{\ell}|x_j-x_{j+1}|}\sum_{\ell=0}^{\ell}|x_\ell-x_{\ell+1}|\,dx_1\ldots dx_\ell\leq(\ell+1)\|V\|_{\mathcal{K}}^{\ell}.$$

See Section 2 of [37] for the proof of this. It follows that each of the first $2m$ terms in (187) satisfies the desired dispersive bound.

In order to bound the "remainder" in (186), which is the final summand containing the perturbed resolvents $R_V^{\pm}(\lambda^2+\mu)$, we need to regard the resolvents as operators $L^{2,\sigma}\to L^{2,-\sigma}$ with $\sigma>\frac12$ (this is the limiting absorption principle from Proposition 30. Note that we only need $\sigma>\frac12$ rather than $\sigma>1$ since the energies are separated from the thresholds, although this is not too important). Moreover, not only are the resolvents bounded $L^{2,\frac12+}\to L^{2,-\frac12-}$, but their operator norms decay like $\lambda^{-\frac12}$. Note that this makes the composition of resolvents and V, which appears in (186), well-defined provided $|V(x)|\lesssim(1+|x|)^{-1-}$ (recall that we are assuming $-3-$ decay). Set

$$\begin{aligned} G_{\pm,x}(\lambda^2)(x_1) &:= \begin{pmatrix} e^{\mp i\lambda|x|} & 0 \\ 0 & 1 \end{pmatrix} R_0^{\pm}(\lambda^2+\mu)(x_1,x) \\ &= \begin{pmatrix} \frac{e^{\pm i\lambda(|x_1-x|-|x|)}}{4\pi|x_1-x|} & 0 \\ 0 & \frac{e^{-\sqrt{2\mu+\lambda^2}|x-x_1|}}{4\pi|x-x_1|} \end{pmatrix}. \end{aligned}$$

Let $e_1=\binom{1}{0}$ and $e_2=\binom{0}{1}$. Removing f,g from (186), we are led to proving that

$$\begin{aligned} \Big|\int_0^\infty e^{it\lambda^2}e^{\pm i\lambda(|x|+|y|)}\,\chi(\lambda)\lambda\Big\langle VR_V^{\pm}(\lambda^2)V(R_0^{\pm}(\lambda^2)V)^m G_{\pm,y}(\lambda^2)e_1, \\ (R_0^{\mp}(\lambda^2)V^*)^m G_{\pm,x}^*(\lambda^2)e_1\Big\rangle\,d\lambda\Big| \lesssim |t|^{-\frac32}, \end{aligned} \tag{202}$$

uniformly in $x,y\in\mathbb{R}^3$ as well as

$$\begin{aligned} &\Big|\int_0^\infty e^{it\lambda^2}e^{\pm i\lambda|x|}\,\chi(\lambda)\lambda\Big\langle VR_V^{\pm}(\lambda^2)V(R_0^{\pm}(\lambda^2)V)^m G_{\pm,y}(\lambda^2)e_2, \\ &\qquad (R_0^{\mp}(\lambda^2)V^*)^m G_{\pm,x}^*(\lambda^2)e_1\Big\rangle\,d\lambda\Big| \\ &+\Big|\int_0^\infty e^{it\lambda^2}e^{\pm i\lambda|y|}\,\chi(\lambda)\lambda\Big\langle VR_V^{\pm}(\lambda^2)V(R_0^{\pm}(\lambda^2)V)^m G_{\pm,y}(\lambda^2)e_1 \\ &\qquad (R_0^{\mp}(\lambda^2)V^*)^m G_{\pm,x}^*(\lambda^2)e_2\Big\rangle\,d\lambda\Big| \\ &+\Big|\int_0^\infty e^{it\lambda^2}\,\chi(\lambda)\lambda\Big\langle VR_V^{\pm}(\lambda^2)V(R_0^{\pm}(\lambda^2)V)^m G_{\pm,y}(\lambda^2)e_2, \\ &\qquad (R_0^{\mp}(\lambda^2)V^*)^m G_{\pm,x}^*(\lambda^2)e_2\Big\rangle\,d\lambda\Big| \lesssim |t|^{-\frac32}, \end{aligned}$$

uniformly in $x,y\in\mathbb{R}^3$. We first verify (202). It is a simple matter to check that the derivatives of $G_{+,x}(\lambda^2)$ satisfy the estimates

$$\begin{aligned} &\sup_{x\in\mathbb{R}^3}\Big\|\frac{d^j}{d\lambda^j}G_{+,x}(\lambda^2)e_k\Big\|_{L^{2,-\sigma}} \lesssim \langle x\rangle^{-1} \quad\text{provided}\quad \sigma>\frac32+j, \\ &\sup_{x\in\mathbb{R}^3}\Big\|\frac{d^j}{d\lambda^j}G_{+,x}(\lambda^2)e_k\Big\|_{L^{2,-\sigma}} \lesssim 1 \quad\text{provided}\quad \sigma>\frac12+j \end{aligned} \tag{203}$$

for all $j \geq 0$ and $k = 1, 2$. Rewrite the integral in (202) in the form

$$I^{\pm}(t,x,y) := \int_0^\infty e^{it\lambda^2 \pm i\lambda(|x|+|y|)} a^{\pm}_{x,y}(\lambda)\,d\lambda.$$

Then in view of the limiting absorption principle of Corollaries 31, 32 and the estimate (203) one concludes that $a^{\pm}_{x,y}(\lambda)$ has two derivatives in λ and

$$(204)\qquad \Big|\frac{d^j}{d\lambda^j} a^{\pm}_{x,y}(\lambda)\Big| \lesssim (1+\lambda)^{-2+}(\langle x\rangle\langle y\rangle)^{-1} \quad \text{for} \quad j=0,1, \quad \text{and all} \quad \lambda > 1$$
$$\Big|\frac{d^2}{d\lambda^2} a^{\pm}_{x,y}(\lambda)\Big| \lesssim (1+\lambda)^{-2+} \quad \text{for all} \quad \lambda > 1$$

which in particular shows that the integral in (202) is absolutely convergent. This requires that one take m sufficiently large and that $|V(x)| \lesssim (1+|x|)^{-\beta}$ for some $\beta > 3$. The latter condition arises as follows: Consider, for example, the case where two derivatives fall one of the G-terms at the ends. Then V has to compensate for $\frac{5}{2}+$ powers because of (203), and also a $\frac{1}{2}+$ power from

$$\|R_0^{\pm}(\lambda^2)f\|_{X_{-\frac{1}{2}-}} \lesssim \lambda^{-1+}\|f\|_{X_{\frac{1}{2}+}}.$$

Similarly with the other terms.

As far as estimating $I^+(t,x,y)$ is concerned, note that on the support of $a^{\pm}_{x,y}(\lambda)$ the phase $t\lambda^2 + \lambda(|x|+|y|)$ has no critical point. Two integrations by parts yield the bound $|I^+(t,x,y)| \lesssim t^{-2}$. In the case of $I^-(t,x,y)$ the phase $t\lambda^2 - \lambda(|x|+|y|)$ has a unique critical point at $\lambda_0 = (|x|+|y|)/(2t)$. If $\lambda_0 \ll \lambda_1$, then two integration by parts again yield a bound of t^{-2}. If $\lambda_0 \gtrsim \lambda_1$ then the bound $\max(|x|,|y|) \gtrsim t$ is also true, and stationary phase contributes $t^{-\frac{1}{2}}(\langle x\rangle\langle y\rangle)^{-1} \lesssim t^{-\frac{3}{2}}$, as desired. Strictly speaking, these estimates are only useful when $t > 1$. On the other hand, when $0 < t < 1$ there is nothing to prove since $I^{\pm}(t,x,y) \lesssim 1$ by (204).

Now consider the other three terms following (202) which involve one or more e_2. The two integrals involving exactly one e_2 can be handled by the exact same argument as (202), the only difference being that the critical point is at $\frac{|x|}{2t}$ or $\frac{|y|}{2t}$. But since (203) takes the same form for e_2 (actually a better estimate holds here, but we ignore that since it is of no use), no other changes are needed. Finally, concerning the integral involving two e_2's: It is estimated by two integrations by parts if $t > 1$, and by putting absolute values inside it if $0 < t < 1$. Indeed, in this case the critical point is at $\lambda = 0$, which falls outside the support of the integrand. Hence, two integrations by parts give a decay of t^{-2}.

The conclusion of the preceding is that (187) and (186) satisfy the desired dispersive bounds. Therefore,

$$|\langle e^{it\mathcal{H}}\chi_+(\mathcal{H})P_{\mathrm{s}}f, g\rangle| \lesssim t^{-\frac{3}{2}}\|f\|_1\|g\|_1,$$

and the same bound holds for $e^{it\mathcal{H}}\chi_-(\mathcal{H})P_{\mathrm{s}}$.

We now deal with the contribution by those λ which are close to $\pm\mu$. This requires showing that

$$
\begin{aligned}
(205)\quad &\langle e^{it\mathcal{H}}(1-\chi_+(\mathcal{H}))P_{\mathrm{s}}f,g\rangle \\
&= \frac{e^{it\mu}}{\pi i}\int_0^\infty \lambda e^{it\lambda^2}(1-\chi_+)(\lambda^2+\mu)\big\langle[R_V^+(\lambda^2+\mu)-R_V^-(\lambda^2+\mu)]f,g\big\rangle\,d\lambda
\end{aligned}
$$

is $\lesssim t^{-\frac{3}{2}}\|f\|_1\|g\|_1$ in absolute value, and similarly for χ_-. We use the resolvent identity in the form

$$
(206)\quad R_V^\pm(\lambda^2+\mu) = R_0^\pm(\lambda^2+\mu) - R_0^\pm(\lambda^2+\mu)V(I+R_0^\pm(\lambda^2+\mu)V)^{-1}R_0^\pm(\lambda^2+\mu)
$$

and write $R_0^\pm(\lambda^2+\mu) = R_0^\pm(\mu) + B^\pm(\lambda)$. Then

$$
(207)\qquad [I+R_0^\pm(\lambda^2+\mu)V]^{-1} = S_0^{-1}[I+B^\pm(\lambda)VS_0^{-1}]^{-1},
$$

where $S_0 = I + R_0^\pm(\mu)V$. In view of (188)

$$
R_0^\pm(\mu)(x,y) = \begin{pmatrix} \frac{1}{4\pi|x-y|} & 0 \\ 0 & \frac{e^{-\sqrt{2\mu}|x-y|}}{4\pi|x-y|} \end{pmatrix}.
$$

As far as the invertibility of S_0 is concerned, we note the following: First, if $\sigma,\alpha > \frac{1}{2}$, and $\sigma+\alpha > 2$, then one checks from the explicit form of the scalar, free resolvent that

$$
\sup_\lambda \|R_0^\pm(\lambda^2)\|_{HS(\sigma,-\alpha)} \le C_{\sigma,\alpha}
$$

where $HS(\sigma,-\alpha)$ refers to the Hilbert-Schmidt norm of $X_\sigma \to X_{-\alpha}$. Hence, if $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 3$, it follows that the operator $R_0^\pm(\lambda)V$ is compact on the weighted space $X_\sigma(\mathbb{R}^3)$ for all choices of $-\frac{5}{2} \le \sigma < -\frac{1}{2}$. Thus, the invertibility of S_0 depends only on whether a solution exists in X_σ to the equation $\psi = -R_0(\mu)V\psi$. However, if such a solution ψ satisfies $\psi \in X_\sigma$ for some $\sigma \ge -\frac{5}{2}$, then $\psi = -R_0(\mu)V\psi \in X_\alpha$ for any choice of $\alpha < -\frac{3}{2}$. Applying this bootstrapping process again, we see that the solution ψ must lie in X_α for all $\alpha < -\frac{1}{2}$. Evidently, this would contradict (179).

Returning to (207), a simple estimation of the explicit kernel

$$
(208)\qquad B^\pm(\lambda)(x,y) = \begin{pmatrix} \frac{e^{\pm i\lambda|x-y|}-1}{4\pi|x-y|} & 0 \\ 0 & \frac{e^{-\sqrt{2\mu+\lambda^2}|x-y|}-e^{-\sqrt{2\mu}|x-y|}}{4\pi|x-y|} \end{pmatrix}
$$

shows that if $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some choice of $\beta > 3$, then

$$
\lim_{\lambda\to 0} \|B^\pm(\lambda)VS_0^{-1}\|_{HS(\sigma,\sigma)} = 0
$$

for all $\sigma \in (-\frac{5}{2},-\frac{1}{2})$. For sufficiently small $\lambda^2 < \lambda_1$, it is then possible to expand

$$
\tilde{B}^\pm(\lambda) := [I+B^\pm(\lambda)VS_0^{-1}]^{-1}
$$

as a Neumann series in the norm $\|\cdot\|_{HS(\sigma,\sigma)}$ for all values $-\frac{5}{2} < \sigma < -\frac{1}{2}$. Moreover, the symmetry $\tilde{B}^-(\lambda) = \tilde{B}^+(-\lambda)$ holds. For ease of notation, define $\chi_0(\lambda) = (1-\chi_+)(\lambda^2+\mu)$ and extend it as an even function of λ. In view of (205) and (206) we wish to control the size of

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty e^{it\lambda^2}\lambda\chi_0(\lambda)\Big[\big[R_0^+(\lambda^2+\mu) - R_0^-(\lambda^2+\mu)\big] - \big[R_0^+(\lambda^2+\mu)VS_0^{-1}\tilde{B}^+(\lambda)R_0^+(\lambda^2+\mu) - R_0^-(\lambda^2+\mu)VS_0^{-1}\tilde{B}^-(\lambda)R_0^-(\lambda^2+\mu)\big]\Big](x,y)\,d\lambda\Big|$$

which is

$$\lesssim \sup_{x,y\in\mathbb{R}^3}\Big|\int_{-\infty}^\infty e^{it\lambda^2}\lambda\chi_0(\lambda)\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}d\lambda\Big| \tag{209}$$

$$+ \sup_{x,y\in\mathbb{R}^3}\Big|\int_{-\infty}^\infty e^{it\lambda^2}\lambda\iint_{\mathbb{R}^6}\frac{U(x_4)e^{i\lambda|y-x_4|}}{|y-x_4|} \cdot\langle e_1, \big(S_0^{-1}(\chi_0\tilde{B}^+)(\lambda)(x_4,x_1)\big)e_1\rangle\frac{e^{i\lambda|x-x_1|}}{|x-x_1|}\,dx_1dx_4d\lambda\Big| \tag{210}$$

$$+ \sup_{x,y\in\mathbb{R}^3}\Big|\int_{-\infty}^\infty e^{it\lambda^2}\lambda\iint_{\mathbb{R}^6}\frac{W(x_4)e^{i\lambda|y-x_4|}}{|y-x_4|} \cdot\langle e_1, \big(S_0^{-1}(\chi_0\tilde{B}^+)(\lambda)(x_4,x_1)\big)e_2\rangle\frac{e^{-\sqrt{2\mu+\lambda^2}|x-x_1|}}{|x-x_1|}\,dx_1dx_4d\lambda\Big| \tag{211}$$

$$+ \sup_{x,y\in\mathbb{R}^3}\Big|\int_{-\infty}^\infty e^{it\lambda^2}\lambda\iint_{\mathbb{R}^6}\frac{W(x_4)e^{-\sqrt{2\mu+\lambda^2}|y-x_4|}}{|y-x_4|} \cdot\langle e_2\big(S_0^{-1}(\chi_0\tilde{B}^+)(\lambda)(x_4,x_1)\big), e_1\rangle\frac{e^{i\lambda|x-x_1|}}{|x-x_1|}\,dx_1dx_4d\lambda\Big| \tag{212}$$

$$+ \sup_{x,y\in\mathbb{R}^3}\Big|\int_{-\infty}^\infty e^{it\lambda^2}\lambda\iint_{\mathbb{R}^6}\frac{U(x_4)e^{-\sqrt{2\mu+\lambda^2}|y-x_4|}}{|y-x_4|} \cdot\langle e_2, \big(S_0^{-1}(\chi_0\tilde{B}^+)(\lambda)(x_4,x_1)\big)e_2\rangle\frac{e^{-\sqrt{2\mu+\lambda^2}|x-x_1|}}{|x-x_1|}\,dx_1dx_4d\lambda\Big|. \tag{213}$$

The first term (209) is simply the low-energy part of the free Schrödinger evolution, which is known to be dispersive. The second term (210) can be integrated by parts once, leaving

$$\sup_{x,y\in\mathbb{R}^3}\frac{1}{2t}\Big|\int_{-\infty}^\infty e^{it\lambda^2}\iint_{\mathbb{R}^6}\frac{d}{d\lambda}\Big[\frac{U(x_4)e^{i\lambda|y-x_4|}}{|y-x_4|} \cdot\big(S_0^{-1}(\chi_0\tilde{B}^+)(\lambda)(x_4,x_1)\big)\frac{e^{i\lambda|x-x_1|}}{|x-x_1|}\Big]\,dx_1dx_4d\lambda\Big| \tag{214}$$

to be controlled. Note that we have dropped e_1 on both sides of the matrix operator in the middle. This does no harm, as long as the absolute value on the outside is interpreted entry-wise. The same comment is in effect for the remainder of the proof. The other terms (211), (212), and (213) are treated similarly to (210). In fact, we verified in (198) that for $a > 0$

$$\int_{-\infty}^{\infty} e^{i\tau\lambda} e^{-a\sqrt{2\mu+\lambda^2}}\, d\lambda =: \nu_a(d\tau)$$

is a measure with mass $\sup_{a>0} \|\nu_a\| < \infty$. This simple fact allows one to use the same argument which is sketched here for (210) in the other three cases as well, up to some obvious modifications. We now return to (214), which is essentially identical to the analogous term arising in the scalar case treated in [21]. Since we see no reason to repeat the details verbatim, we provide a sketch and refer the reader to [21] for more details. Consider the term where $\frac{d}{d\lambda}$ falls on $\tilde{B}^+(\lambda)$. The others will be similar. Using Parseval's identity, and the fact that $\|(e^{it(\cdot)^2})^\wedge(u)\|_{L^\infty(u)} = Ct^{-1/2}$, this is less than

$$\sup_{x,y\in\mathbb{R}^3} \frac{1}{t^{3/2}} \int_{-\infty}^{\infty} \Big| \iint_{\mathbb{R}^6} \frac{U(x_4)}{|y-x_4|} S_0^{-1}\big[\chi_0(\tilde{B}^+)'\big]^\vee \cdot \big(u + |y-x_4| + |x-x_1|\big)(x_4,x_1) \frac{1}{|x-x_1|}\, dx_1 dx_4 \Big|\, du.$$

If the absolute value is taken inside the inner integral, then Fubini's theorem may be used to exchange the order of integration to obtain

$$\begin{aligned}
&\sup_{x,y\in\mathbb{R}^3} \frac{1}{t^{3/2}} \iint_{\mathbb{R}^6} \int_{-\infty}^{\infty} \frac{|U(x_4)|}{|y-x_4|} \cdot \Big| S_0^{-1}\big[\chi_0(\tilde{B}^+)'\big]^\vee \big(u + |y-x_4| + |x-x_1|\big)(x_4,x_1) \Big| \frac{1}{|x-x_1|}\, du\, dx_1 dx_4 \\
&\le \sup_{x,y\in\mathbb{R}^3} \frac{1}{t^{3/2}} \Big\| \frac{|U(\cdot)|}{|y-\cdot|} \Big\|_{L^{2,2+}} \cdot \big\| \textstyle\int |S_0^{-1}[\chi_0(\tilde{B}^+)']^\vee(u)|\, du \big\|_{OP(-1-,-2-)} \ \big\| |x-\cdot|^{-1} \big\|_{L^{2,-1-}},
\end{aligned}$$

where $OP(-1-,-2-)$ stands for the operator norm from $X_{-1-} \to X_{-2-}$. The two norms at the ends of the last line are easily seen to be uniformly bounded in $x, y \in \mathbb{R}^3$. It therefore only remains to control the size of

$$\big\| \textstyle\int |S_0^{-1}[\chi_0(\tilde{B}^+)']^\vee(u)|\, du \big\|_{OP(-1-,-2-)}.$$

Minkowski's Inequality allows us to bring the norm inside the integral. Recall that S_0^{-1} is a bounded operator on $L^{2,-2-}$. Furthermore, it is an easy matter to check that the operator $|S_0^{-1}|$ whose kernel is the absolute value of the kernel of S_0^{-1}, is also a bounded operator on $L^{2,-2-}$. The problem then reduces to showing that

$$\int_{-\infty}^{\infty} \|[\chi_0(\tilde{B}^+)']^\vee(u)\|_{OP(-1-,-2-)}\, du < \infty \tag{215}$$

provided the support of χ_0 is sufficiently small. The operators $\tilde{B}^+(\lambda)$ are defined by the convergent Neuman series

$$\tilde{B}^+(\lambda) \;=\; [I + B^+(\lambda)VS_0^{-1}]^{-1} \;=\; \sum_{n=0}^{\infty} \big(- B^+(\lambda)VS_0^{-1}\big)^n.$$

Exploiting the explicit form of the kernel of B^+, see (208), it is possible to control the Fourier transform in λ of each term in this Neuman series in the appropriate weighted L^2 spaces, leading to (215) upon summation. For these details we refer the reader to the end of the paper [21]. □

Finally, we discuss Strichartz estimates. The usual derivation for Strichartz estimates involves TT^* arguments where $(Tf)(t,x) = (e^{-itH}f)(x)$. This relies on the unitarity of the evolution, since one wants

$$TT^*F(t,x) = \int_{-\infty}^{\infty} (e^{-i(t-s)H}F(s,\cdot))(x)\,ds.$$

In the system case, this cannot be done. We therefore rely on a different approach which is perturbative in nature. It uses Kato's notion of an $\mathcal{H}_0$-smooth and $\mathcal{H}$-smooth operator, and originates in [37]. In addition, we use the following lemma, which is due to Christ-Kiselev [9]. See also Sogge, Smith [42].

Lemma 36. *Let X, Y be Banach spaces and let $K(t,s)$ be the kernel of the operator*

$$K : L^p([0,T];X) \to L^q([0,T];Y).$$

Denote by $\|K\|$ the operator norm of K. Define the lower diagonal operator

$$\tilde{K} :\; L^p([0,T];X) \to L^q([0,T];Y)$$

to be

$$\tilde{K}f(t) = \int_0^t K(t,s)f(s)\,ds.$$

Then the operator $\tilde{K}$ is bounded from $L^p([0,T];X)$ to $L^q([0,T];Y)$ and its norm $\|\tilde{K}\| \le c\|K\|$, provided that $p < q$.

Now we can state the Strichartz estimates.

Corollary 37. *Under the same assumptions as in Theorem* 35, *one has the Strichartz estimates*

$$\|e^{-it\mathcal{H}}P_{\mathrm{s}}f\|_{L_t^r(L_x^p)} \le C\|f\|_{L^2}, \tag{216}$$

$$\Big\| \int_0^t e^{-i(t-s)\mathcal{H}}P_{\mathrm{s}}F(s)\,ds \Big\|_{L_t^r(L_x^p)} \le C\|F\|_{L_t^{a'}(L_x^{b'})}, \tag{217}$$

provided $(r,p), (a,b)$ are admissible; i.e., $2 < r \le \infty$ and $\frac{2}{r} + \frac{3}{p} = \frac{3}{2}$ and the same for (a,b).

Proof. Let ($\mathcal{S}$ for "Strichartz")

$$(\mathcal{S}F)(t,x) = \int_0^t (e^{-i(t-s)\mathcal{H}} P_{\mathrm{s}}\, F(s,\cdot))(x)\, ds.$$

In this proof it will be understood that all times are ≥ 0. Then by (183),

$$\|\mathcal{S}F\|_{L_t^\infty(L_x^2)} \lesssim \|F\|_{L_t^1(L_x^2)},$$

and more generally, by the usual fractional integration argument based on Theorem 35,

$$\|\mathcal{S}F\|_{L_t^r(L_x^p)} \lesssim \|F\|_{L_t^{r'}(L_x^{p'})} \tag{218}$$

for any admissible pair (r,p). In the unitary case this implies (216) via a TT^* argument, but this reasoning does not apply here. Instead, we rely on a Kato theory type approach as in [37, §4]. Since $\mathcal{H} = \mathcal{H}_0 + V$, Duhamel's formula yields

$$e^{-it\mathcal{H}} P_{\mathrm{s}} = e^{-it\mathcal{H}_0} P_{\mathrm{s}} - i\int_0^t e^{-i(t-s)\mathcal{H}_0} V e^{-is\mathcal{H}} P_{\mathrm{s}}\, ds. \tag{219}$$

Define $V = \tilde{M}\tilde{M}^{-1}V$, where $\tilde{M}$ is

$$\tilde{M} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$$

with $\rho(x) = \langle x\rangle^{-1-}$. Then observe that

$$\Big\| \int_0^\infty e^{-i(t-s)\mathcal{H}_0} \tilde{M} g(s)\, ds \Big\|_{L_t^r(L_x^p)} \lesssim \Big\| \int_0^\infty e^{is\mathcal{H}_0} \tilde{M} g(s)\, ds \Big\|_{L^2} \lesssim \|g\|_{L_s^2(L_x^2)},$$

where the last inequality is the dual of the smoothing bound

$$\int_0^\infty \Big\| \tilde{M} e^{-is\mathcal{H}_0^*} \psi \Big\|_2^2\, ds \lesssim \|\psi\|_2^2.$$

Now one applies the Christ-Kiselev lemma to conclude that

$$\Big\| \int_0^t e^{-i(t-s)\mathcal{H}_0} \tilde{M} g(s)\, ds \Big\|_{L_t^r(L_x^p)} \lesssim \|g\|_{L_s^2(L_x^2)}$$

for any admissible pair (r,p). Hence, continuing in (219), one obtains (using that $\|P_{\mathrm{s}} f\|_2 \lesssim \|f\|_2$)

$$\|e^{-it\mathcal{H}} P_{\mathrm{s}} f\|_{L_t^r(L_x^p)} \lesssim \|f\|_2 + \Big\| \tilde{M}^{-1} V e^{-is\mathcal{H}} P_{\mathrm{s}} f \Big\|_{L_s^2(L_x^2)}.$$

It remains to show that $\tilde{M}^{-1}V$ is $\mathcal{H}P_{\mathrm{s}}$-smoothing; i.e.,

$$\Big\| \tilde{M}^{-1} V e^{-is\mathcal{H}} P_{\mathrm{s}} f \Big\|_{L_s^2(L_x^2)} \lesssim \|f\|_2. \tag{220}$$

Taking the Fourier transform in s shows that (220) is equivalent to

$$\int_{-\infty}^{\infty} \|\tilde{M}^{-1}V[P_{\rm s}(\mathcal{H}-\lambda-i0)P_{\rm s}]^{-1}P_{\rm s}f\|_2^2\,d\lambda \lesssim \|f\|_2^2. \tag{221}$$

This is an instance of a Kato smoothing estimate. It is standard that this holds for $\mathcal{H}_0$ instead of $\mathcal{H}=\mathcal{H}_0+V$ (without any projections), and follows for $\mathcal{H}$ via the resolvent identity

$$(\mathcal{H}-\lambda-i0)^{-1} = \big[1-(\mathcal{H}_0-\lambda-i0)^{-1}V\big]^{-1}(\mathcal{H}_0-\lambda-i0)^{-1}.$$

Indeed, since the thresholds $\pm\alpha^2$ are not resonances, we obtain that

$$\sup_{|\lambda|>\alpha^2/2}\Big\|\big[1-(\mathcal{H}_0-\lambda-i0)^{-1}V\big]^{-1}\Big\|_{L^2\to X^{-1-}} < \infty$$

and thus also

$$\sup_{|\lambda|>\alpha^2/2}\Big\|\tilde{M}^{-1}V\big[1+(\mathcal{H}_0-\lambda-i0)^{-1}V\big]^{-1}\Big\|_{L^2\to L^2} < \infty.$$

Therefore,

$$\int_{|\lambda|>\alpha^2/2} \|\tilde{M}^{-1}V(\mathcal{H}-\lambda-i0)^{-1}P_{\rm s}f\|_2^2\,d\lambda \lesssim \int_{|\lambda|>\alpha^2/2} \|(\mathcal{H}_0-\lambda-i0)^{-1}f\|_2^2\,d\lambda \lesssim \|f\|_2^2.$$

On the other hand,

$$\sup_{|\lambda|\le\alpha^2/2}\big\|[P_{\rm s}(\mathcal{H}-\lambda-i0)P_{\rm s}]^{-1}\big\|_{2\to2} < \infty$$

and (221) follows. The conclusion is that

$$\|e^{-it\mathcal{H}}P_{\rm s}f\|_{L_t^r(L_x^p)} \lesssim \|f\|_2$$

for any admissible (r,p), which is (216). The proof of (217) is now the usual interpolation argument. Indeed, in view of the preceding one has the following bounds on $\mathcal{S}$ for any admissible pair (r,p):

$$\mathcal{S}: L_t^1(L_x^2)\to L_t^r(L_x^p), \tag{222}$$

$$\mathcal{S}: L_t^{r'}(L_x^{p'})\to L_t^r(L_x^p), \tag{223}$$

$$\mathcal{S}: L_t^{r'}(L_x^{p'})\to L_t^\infty(L_x^2). \tag{224}$$

These estimates arise as follows: (223) is exactly (218), whereas (222) follows from (216) by means of Minkowski's inequality. Finally, (224) is dual to the bound

$$\Big\|\int_t^\infty e^{i(t-s)\mathcal{H}^*}\tilde{P}_cG(s)\,ds\Big\|_{L_t^r(L_x^p)} \lesssim \|G\|_{L_t^1(L_x^2)}. \tag{225}$$

Here $\tilde{P}_c$ corresponds to $\mathcal{H}^*$ in the same way that $P_{\rm s}$ does to $\mathcal{H}$. In particular, one has

$$\|e^{-it\mathcal{H}^*}\tilde{P}_c\|_{1\to\infty} \lesssim t^{-\frac{3}{2}}$$

and therefore, (225) is derived by the same methods as (222). It is important to notice that $P_{\rm s}^* = \tilde{P}_s$ which is essential for the duality argument here. This can be seen, for example, by writing the Riesz projections onto (generalized) eigenspaces as contour integrals around circles surrounding the eigenvalues. Since the (complex) eigenvalues always come in pairs, the adjoints have the desired property. Interpolating between (222) and (223) yields (217) for the range $a' \le r'$ or $a \ge r$, whereas interpolating between (222) and (223) yields (217) in the range $a \le r$. □

Finally, we introduce derivatives into the estimates of Theorems 34, 35 and Corollary 37.

COROLLARY 38. *Under the same assumptions as in Theorem* 35,

$$\big\|e^{it\mathcal{H}}P_{\rm s}f\big\|_{W^{k,p'}(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{p'})}\|f\|_{W^{k,p}(\mathbb{R}^3)}$$

for $0 \le k \le 2$ *and* $1 < p \le 2$.

Proof. The case $k=0$ is obtained by interpolating between Theorems 34 and 35 and holds for the entire range $1 \le p \le 2$. We need to require $p > 1$ only for the derivatives. If a is sufficiently large, then

$$(\mathcal{H} - ia)^{-1} : L^2 \times L^2 \to W^{2,2} \times W^{2,2}$$

is an isomorphism. More generally,

$$(\mathcal{H} - ia)^{-\frac{1}{2}} : L^p \times L^p \to W^{2,p} \times W^{2,p}$$

is an isomorphism for $1 < p < \infty$. This can be seen from the resolvent identity

$$(\mathcal{H} - ia)^{-1} = (\mathcal{H}_0 - ia)^{-1}[1 + V(\mathcal{H}_0 - ia)^{-1}]^{-1},$$

since $\|V\|_\infty < \infty$ implies that

$$\|V(\mathcal{H}_0 - ia)^{-1}\|_{p\to p} < \frac{1}{2}$$

if a is large enough, and because

$$(\mathcal{H}_0 - ia)^{-\frac{1}{2}} : L^p \times L^p \to W^{2,p} \times W^{2,p}$$

for any $a \neq 0$ as an isomorphism. Hence,

$$\begin{aligned}\|\triangle e^{it\mathcal{H}}P_{\rm s}f\|_{p'} &\lesssim \|(\mathcal{H} - ia)e^{it\mathcal{H}}f\|_{p'} = \|e^{it\mathcal{H}}(\mathcal{H} - ia)f\|_{p'} \\ &\lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{p'})}\|(\mathcal{H} - ia)f\|_p \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{p'})}\|f\|_{W^{2,p}(\mathbb{R}^3)}.\end{aligned}$$

This gives the case $k=2$ of the lemma, whereas $k=1$ follows by interpolation between $k=0$ and $k=2$. □

And now the same for the Strichartz estimates.

COROLLARY 39. *Under the same assumptions as in Corollary* 37, *one has the Strichartz estimates*

$$\|e^{-it\mathcal{H}} P_{\mathrm{s}} f\|_{L_t^r(W_x^{k,p})} \le C\|f\|_{W^{k,2}}, \tag{226}$$

$$\Big\| \int_0^t e^{-i(t-s)\mathcal{H}} P_{\mathrm{s}} F(s)\, ds \Big\|_{L_t^r(W_x^{k,p})} \le C\|F\|_{L_t^{a'}(W_x^{k,b'})}, \tag{227}$$

provided $(r,p),(a,b)$ *are admissible*; *i.e.*, $2 < r \le \infty$ *and* $\frac{2}{r}+\frac{3}{p}=\frac{3}{2}$ *and the same for* (a,b). *Here* k *is an integer*, $0 \le k \le 2$.

Proof. The case $k=0$ is just Corollary 37. As in the previous proof, we rely on the fact that (because $\|V\|_\infty < \infty$),

$$\|\triangle f\|_q \lesssim \|(\mathcal{H}-ia)f\|_q$$

for any $1<q<\infty$. Hence,

$$\begin{aligned}\|e^{-it\mathcal{H}} P_{\mathrm{s}} f\|_{L_t^r(W_x^{2,p})} &\lesssim \|(\mathcal{H}-ia)e^{-it\mathcal{H}} P_{\mathrm{s}} f\|_{L_t^r(L_x^p)} = \|e^{-it\mathcal{H}} P_{\mathrm{s}}(\mathcal{H}-ia) f\|_{L_t^r(L_x^p)}\\ &\lesssim \|(\mathcal{H}-ia)f\|_2 \lesssim \|f\|_{W^{2,2}},\end{aligned}$$

which is (226) for $k=2$. Similarly, one proves (227) for $k=2$. The case $k=1$ is then obtained by interpolation. □

UNIVERSITY OF CHICAGO, CHICAGO, IL
E-mail address: schlag@math.uchicago.edu

REFERENCES

[1] S. AGMON, Spectral properties of Schrödinger operators and scattering theory, *Ann. Scuola Norm. Sup. Pisa Cl. Sci*. **2** (1975), 151–218.

[2] ———, *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations*: *Bounds on Eigenfunctions of N-body Schrödinger Operators*, *Mathematical Notes*. **29**, Princeton Univ. Press, Princeton, NJ, 1982.

[3] P. W. BATES and C. K. R. T. JONES, Invariant manifolds for semilinear partial differential equations, in *Dynamics Reported*, Vol. 2, 1–38, *Dynam. Report. Ser. Dynam. Systems Appl*. **2**, Wiley, Chichester, 1989.

[4] H. BERESTYCKI and T. CAZENAVE, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires, *C. R. Acad. Sci. Paris Sér. I Math*. **293** (1981), 489–492.

[5] H. BERESTYCKI and P. L. LIONS, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.* **82** (1983), 313–345.

[6] V. S. BUSLAEV and G. S. PERELMAN, Scattering for the nonlinear Schrödinger equation: states that are close to a soliton (Russian), *Algebra i Analiz* **4** (1992), 63–102; translation in *St. Petersburg Math. J*. **4** (1993), 1111–1142.

[7] ———, On the stability of solitary waves for nonlinear Schrödinger equations. Nonlinear evolution equations, 75–98, *Amer. Math. Soc. Transl. Ser*. 2, **164**, Amer. Math. Soc., Providence, RI, 1995.

[8] T. CAZENAVE and P.-L. LIONS, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Comm. Math. Phys*. **85** (1982), 549–561.

[9] M. Christ and A. Kiselev, Maximal functions associated with filtrations, *J. Funct. Anal.* **179** (2001), 409-425.

[10] C. V. Coffman, Uniqueness of the ground state solution for $\triangle u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Rat. Mech. Anal.* **46** (1972), 81–95.

[11] A. Comech and D. Pelinovsky, Purely nonlinear instability of standing waves with minimal energy, *Comm. Pure Appl. Math.* **56** (2003), 1565–1607.

[12] O. Costin and A. Soffer, Resonance theory for Schrödinger operators, *Comm. Math. Phys.* **224** (2001), 133–152.

[13] S. Cuccagna, Stabilization of solutions to nonlinear Schrödinger equations, *Comm. Pure Appl. Math.* **54** (2001), 1110–1145.

[14] S. Cuccagna and D. Pelinovsky, Bifurcations from the endpoints of the essential spectrum in the linearized nonlinear Schrödinger problem, *J. Math. Phys.* **46** (2005), 15 pp.

[15] S. Cuccagna, D. Pelinovsky, and V. Vougalter, Spectra of positive and negative energies in the linearized NLS problem, *Comm. Pure Appl. Math.* **58** (2005), 1–29.

[16] L. Demanet and W. Schlag, Numerical verification of a gap condition for a linearized NLS equation, *Nonlinearity* **19** (2006), 829-852.

[17] M. B. Erdoğan and W. Schlag, Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: II, *J. d'Analyse* **99** (2006), 199–248.

[18] J. Fröhlich, S. Gustafson, B. L. G. Jonsson, and I. M. Sigal, Solitary wave dynamics in an external potential, *Comm. Math. Phys.* **250** (2004), 613–642.

[19] J. Fröhlich, T.-P. Tsai, and H.-T. Yau, On the point-particle (Newtonian) limit of the non-linear Hartree equation, *Comm. Math. Phys.* **225** (2002), 223–274.

[20] F. Gesztesy, C. K. R. T. Jones, Y. Latushkin, and M. Stanislavova, A spectral mapping theorem and invariant manifolds for nonlinear Schrödinger equations, *Indiana Univ. Math. J.* **49** (2000), 221–243.

[21] M. Goldberg and W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, *Comm. Math. Phys.* **251** (2004), 157–178.

[22] M. Grillakis, Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system, in *Integrable Systems and Applications* (Ile d'Oléron, 1988), 154–191, *Lecture Notes in Phys.* **342**, Springer-Verlag, New York, 1989.

[23] ———, Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system, *Comm. Pure Appl. Math.* **43** (1990), 299–333.

[24] M. Grillakis, J. Shatah, and W. Strauss, Stability theory of solitary waves in the presence of symmetry. I, *J. Funct. Anal.* **74** (1987), 160–197.

[25] ———, Stability theory of solitary waves in the presence of symmetry. II, *J. Funct. Anal.* **94** (1990), 308–348.

[26] P. D. Hislop and I. M. Sigal, *Introduction to Spectral Theory. With Applications to Schrödinger Operators*, *Applied Mathematical Sciences* **113**, Springer-Verlag, New York, 1996.

[27] D. Hundertmark and Y. R. Lee, Exponential decay of eigenfunctions and generalized eigenfunctions of non-self-adjoint matrix Schrödinger operators related to NLS, *Bull. London Math. Soc.* **39** (2007), 709–720.

[28] C. Kenig and F. Merle, Global well-posedness, scattering, and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, *Invent. Math.* **166** (2006), 645–675.

[29] J. Krieger and W. Schlag, Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension, *J. Amer. Math. Soc.* **19** (2006), 815–920.

[30] M. K. Kwong, Uniqueness of positive solutions of $\triangle u - u + u^p = 0$ in $\mathbb{R}^n$, *Arch. Rat. Mech. Anal.* **65** (1989), 243–266.

[31] C. Li and S. Wiggins, *Invariant Manifolds and Fibrations for Perturbed Nonlinear Schrödinger Equations*, *Applied Mathematical Sciences* **128**, Springer-Verlag, New York, 1997.

[32] G. Perelman, Some results on the scattering of weakly interacting solitons for nonlinear Schrödinger equations, in *Spectral Theory*, *Microlocal Analysis*, *Singular Manifolds*, Akad. Verlag, *Math. Top.* **14** (1997), 78–137.

[33] ———, On the formation of singularities in solutions of the critical nonlinear Schrödinger equation, *Ann. Henri Poincaré* **2** (2001), 605–673.

[34] ———, Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations, *Comm. Partial Differential Equations* **29** (2004), 1051–1095.

[35] C. A. Pillet and C. E. Wayne, Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations, *J. Differential Equations* **141** (1997), 310–326.

[36] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press, New York, 1978.

[37] I. Rodnianski and W. Schlag, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, *Invent. Math.* **155** (2004), 451–513.

[38] I. Rodnianski, W. Schlag, and A. Soffer, Dispersive analysis of charge transfer models, *Comm. Pure Appl. Math.* **58** (2005), 149–216.

[39] ———, Asymptotic stability of N-soliton states of NLS, preprint 2003.

[40] J. Shatah, Stable standing waves of nonlinear Klein-Gordon equations, *Comm. Math. Phys.* **91** (1983), 313–327.

[41] J. Shatah and W. Strauss, Instability of nonlinear bound states, *Comm. Math. Phys.* **100** (1985), 173–190.

[42] H. Smith and C. Sogge, Global Strichartz estimates for nontrapping perturbations of the Laplacian, *Comm. Partial Differential Equations* **25** (2000), 2171–2183.

[43] A. Soffer and M. Weinstein, Multichannel nonlinear scattering for nonintegrable equations, *Comm. Math. Phys.* **133** (1990), 119–146.

[44] ———, Multichannel nonlinear scattering for nonintegrable equations, II. The case of anisotropic potentials and data, *J. Differential Equations* **98** (1992), 376–390.

[45] W. A. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* **55** (1977), 149–162.

[46] ———, *Nonlinear Wave Equations*, CBMS *Regional Conference Series in Math.* **37**, A. M. S., Providence, RI, 1989.

[47] C. Sulem and P.-L. Sulem, The nonlinear Schrödinger equation. Self-focusing and wave collapse, *Applied Math. Sciences* **139**, Springer-Verlag, New York, 1999.

[48] T.-P. Tsai and H.- T. Yau, Stable directions for excited states of nonlinear Schrödinger equations, *Comm. Partial Differential Equations* **27** (2002), 2363–2402.

[49] M. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations. SIAM *J. Math. Anal.* **16** (1985), 472–491.

[50] ———, Lyapunov stability of ground states of nonlinear dispersive evolution equations, *Comm. Pure Appl. Math.* **39** (1986), 51–67.

(Received November 2, 2004)