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On the dimensions of conformal repellers. Randomness and parameter dependency

By Hans Henrik Rugh

Abstract

Bowen's formula relates the Hausdorff dimension of a conformal repeller to the zero of a 'pressure' function. We present an elementary, self-contained proof to show that Bowen's formula holds for C^1 conformal repellers. We consider time-dependent conformal repellers obtained as invariant subsets for sequences of conformally expanding maps within a suitable class. We show that Bowen's formula generalizes to such a repeller and that if the sequence is picked at random then the Hausdorff dimension of the repeller almost surely agrees with its upper and lower box dimensions and is given by a natural generalization of Bowen's formula. For a random uniformly hyperbolic Julia set on the Riemann sphere we show that if the family of maps and the probability law depend real-analytically on parameters then so does its almost sure Hausdorff dimension.

1. Random Julia sets and their dimensions

Let (U, d_U) be an open, connected subset of the Riemann sphere avoiding at least three points and equipped with a hyperbolic metric. Let $K \subset U$ be a compact subset. We denote by $\mathcal{E}(K, U)$ the space of unramified conformal covering maps $f : \mathcal{D}_f \to U$ with the requirement that the covering domain $\mathcal{D}_f \subset K$. Denote by $Df : \mathcal{D}_f \to \mathbb{R}_+$ the conformal derivative of f, see equation (2.4), and by $\|Df\| = \sup_{f^{-1}K} Df$ the maximal value of this derivative over the set $f^{-1}K$. Let $\mathcal{F} = (f_n) \subset \mathcal{E}(K, U)$ be a sequence of such maps. The intersection

$$J(\mathcal{F}) = \bigcap_{n \geq 1} f_1^{-1} \circ \cdots \circ f_n^{-1}(U) \tag{1.1}$$

defines a uniformly hyperbolic Julia set for the sequence $\mathcal{F}$. Let (Υ, ν) be a probability space and let $\omega \in \Upsilon \to f_\omega \in \mathcal{E}(K, U)$ be a ν-measurable map. Suppose that the elements in the sequence $\mathcal{F}$ are picked independently, each according to the law ν. Then $J(\mathcal{F})$ becomes a random 'variable'. Our main objective is to establish the following

THEOREM 1.1. I. *Suppose that* $\mathbb{E}(\log \|Df_\omega\|) < \infty$. *Then almost surely, the Hausdorff dimension of* $J(\mathcal{F})$ *is constant and equals its upper and lower box dimensions. The common value is given by a generalization of Bowen's formula.*

II. *Suppose in addition that there is a real parameter* t *having a complex extension so that*: (a) *The family of maps* $(f_{t,\omega})_{\omega\in\Upsilon}$ *depends analytically upon* t. (b) *The probability measure* ν_t *depends real-analytically on* t. (c) *Given any local inverse,* $f_{t,\omega}^{-1}$, *the log-derivative* $\log Df_{t,\omega}\circ f_{t,\omega}^{-1}$ *is* (*uniformly in* $\omega \in \Upsilon$) *Lipschitz with respect to* t. (d) *For each* t *the condition number* $\|Df_{t,\omega}\|\cdot\|1/Df_{t,\omega}\|$ *is uniformly bounded in* $\omega \in \Upsilon$.

Then the almost sure Hausdorff dimension obtained in part I *depends real-analytically on* t. (*For a precise definition of the parameter* t *we refer to Section* 6.3, *for conditions* (a), (c) *and* (d) *see Definition* 6.8 *and Assumption* 6.13, *and for* (b) *see Definition* 7.1 *and Assumption* 7.3. *We prove Theorem* 1.1 *in Section* 7).

Example 1.2. Let $a \in \mathbb{C}$ and $r \geq 0$ be such that $|a| + r < \frac{1}{4}$. Suppose that $c_n \in \mathbb{C}$, $n \in \mathbb{N}$ are i.i.d. random variables uniformly distributed in the closed disk $\overline{B}(a,r)$ and that N_n, $n \in \mathbb{N}$ are i.i.d. random variables distributed according to a Poisson law of parameter $\lambda \geq 0$. We consider the sequence of maps $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ given by

$$f_n(z) = z^{N_n+2} + c_n. \tag{1.2}$$

An associated 'random' Julia set may be defined through

$$J(\mathcal{F}) = \partial\,\{z \in \mathbb{C} : f_n \circ \cdots \circ f_1(z) \to \infty\}. \tag{1.3}$$

We show in Section 6 that the family verifies all conditions of Theorem 1.1, parts I and II with a 4-dimensional real parameter $t = (\operatorname{re} a, \operatorname{im} a, r, \lambda)$ in the domain determined by $|a| + r < 1/4$, $r \geq 0$, $\lambda \geq 0$. For a given parameter the Hausdorff dimension of the random Julia set is almost surely constant and equals the upper/lower box dimensions. The common value $d(a, r, \lambda)$ depends real-analytically upon $\operatorname{re} a$, $\operatorname{im} a$, r and λ. Note that the sequence of degrees $(N_n)_{n\in\mathbb{N}}$ almost surely is unbounded when $\lambda > 0$.

Rufus Bowen, one of the founders of the Thermodynamic Formalism (henceforth abbreviated TF), saw more than twenty years ago [Bow79] a natural connection between the geometric properties of a conformal repeller and the TF for the map(s) generating this repeller. The Hausdorff dimension $\dim_H(\Lambda)$ of a smooth and compact conformal repeller (Λ, f) is precisely the unique zero s_{crit} of a 'pressure' function $P(s, \Lambda, f)$ having its origin in the TF. This relationship is now known as 'Bowen's formula'. The original proof by Bowen [Bow79] was in the context of Kleinian groups and involved a finite Markov partition and uniformly expanding conformal maps. Using TF he constructed a finite

Gibbs measure of zero 'conformal pressure' and showed that this measure is equivalent to the $s_{\rm crit}$-dimensional Hausdorff measure of Λ. The conclusion then follows.

Bowen's formula applies in many other cases. For example, when dealing with expanding 'Markov maps', the Markov partition need not be finite and one may eventually have a neutral fixed point in the repeller [Urb96], [SSU01]. One may also relax on smoothness of the maps involved, C^1 being sufficient. McCluskey and Manning in [MM83], were the first to note this for horse-shoe type maps. Barreira [Bar96] and also Gatzouras and Peres [GP97] were also able to demonstrate that Bowen's formula holds for classes of C^1 repellers. *A priori*, the classical TF does not apply in this setup. McCluskey and Manning used nonunique Gibbs states to show this. Gatzouras and Peres circumvene the problem by using an approximation argument and then apply the classical theory. Barreira, following the approach of Pesin [Pes88], defines the Hausdorff dimension as a Caratheodory dimension characteristic. By extending the TF itself, Barreira gets closer to the core of the problem and may also consider maps somewhat beyond the C^1 case mentioned. The proofs are, however, fairly involved and do not generalize easily either to a random set-up or to a study of parameter-dependency.

In [Rue82], Ruelle showed that the Hausdorff dimension of the Julia set of a uniformly hyperbolic rational map depends real-analytically on parameters. The original approach of Ruelle was indirect, using dynamical zeta-functions, [Rue76]. Other later proofs are based on holomorphic motions, (see [Zin99] as well as [UZ01] and [UZ04]). In another context, Furstenberg and Kesten, [FK60], had shown, under a condition of log-integrability, that a random product of matrices has a unique almost sure characteristic exponent. Ruelle, in [Rue79], required in addition that the matrices contracted uniformly a positive cone and satisfied a compactness and continuity condition with respect to the underlying probability space. He showed that under these conditions if the family of postive random matrices depends real-analytically on parameters then so does the almost sure characteristic exponent of their product. He did not, however, allow the probability law to depend on parameters. We note here that if the matrices contract uniformly a positive cone, the topological conditions in [Rue79] may be replaced by the weaker condition of measurablity + log-integrability. We also mention the more recent paper, [Rue97], of Ruelle. It is in spirit close to [Rue79] (not so obvious at first sight) but provides a more global and far more elegant point of view to the question of parameter-dependency. It has been an invaluable source of inspiration to our work.

In this article we depart from the traditional path pointed out by TF. In Part I we present a proof of Bowen's formula, Theorem 2.1, for a C^1 conformal repeller which bypasses measure theory and most of the TF. Measure theory can be avoided essentially because Λ is compact and the only element remaining from TF is a family of transfer operators which encodes geometric facts into analytic ones. Our proof is short and elementary and releases us from some of the smoothness conditions imposed by TF.

An elementary proof of Bowen's formula should be of interest on its own, at least in the author's opinion. It generalizes, however, also to situations where a 'standard' approach either fails or manages only with great difficulties. We consider classes of time-dependent conformal repellers. By picking a sequence of maps within a suitable equi-conformal class one may study the associated time-dependent repeller. Under the assumption of uniform equi-expansion and equi-mixing and a technical assumption of sub-exponential 'growth' of the involved sequences we show, Theorem 3.7, that the Hausdorff and box dimensions are bounded within the unique zeros of a lower and an upper conformal pressure. Similar results were found by Barreira [Bar96, Ths. 2.1 and 3.8]. When it comes to random conformal repellers, however, the approach of Pesin and Barreira seems difficult to generalize. Kifer [Kif96] and later, Crauel and Flandoni [CF98] and also Bogenschütz and Ochs [BO99], using time-dependent TF and Martingale arguments, considered random conformal repellers for certain classes of transformations, but under the smoothness restriction imposed by TF. In Theorem 4.4, a straight-forward application of Kingman's sub-ergodic theorem, [King68], allows us to deal with this case without such restrictions. In addition we obtain very general formulae for the parameter-dependency of the Hausdorff dimension.

Part II is devoted to random Julia sets on hyperbolic subsets of the Riemann sphere. Here statements and hypotheses attain much more elegant forms; cf. Theorem 1.1 and Example 1.2 above. Straight-forward Koebe estimates enables us to apply Theorem 4.4 to deduce Theorem 5.3 which in turn yields Theorem 1.1, part (I).[1] The parameter dependency is, however, more subtle. The central ideas are then the following:

(1) We introduce a 'mirror embedding' of our hyperbolic subset and then a related family of transfer operators and cones having a natural (real-) analytic structure.

(2) We compute the pressure function using a hyperbolic fixed point of a holomorphic map acting upon cone-sections. When the family of maps depends real-analytically on parameters, then the real-analytical dependency of the dimensions, Theorem 6.20, follows from an implicit function theorem.

[1] Within the framework of TF, methods of [Kif96], [PW96], [CF98] or [BO99] can also be used to prove this part.

(3) The above mentioned fixed point is hyperbolic. This implies an exponential decay with respect to 'time' and allows us in Section 7.1 to treat a real-analytic parameter dependency with respect to the underlying probability law. This concludes the proof of Theorem 1.1. □

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2. Part I: C^1 conformal repellers and Bowen's formula

Let (Λ, d) be a nonempty compact metric space without isolated points and let $f : \Lambda \to \Lambda$ be a continuous surjective map. Throughout Part I we will write interchangeably f_x or $f(x)$ for the map f applied to a point x. We say that f is C^1 *conformal* at $x \in \Lambda$ if and only if the following double limit exists:

$$Df(x) = \lim_{u \neq v \to x} \frac{d(f_u, f_v)}{d(u, v)}. \tag{2.4}$$

The limit is called the conformal derivative of f at x. The map f is said to be C^1 conformal on Λ if it is so at every point of Λ. A point $x \in \Lambda$ is said to be *critical* if and only if $Df(x) = 0$.

The product $Df^n(x) = Df(f^{n-1}(x)) \cdots Df(x)$ along the orbit of x is the conformal derivative for the n'th iterate of f. The map is said to be uniformly expanding if there are constants $C > 0$, $\beta > 1$ for which $Df^n(x) \geq C\beta^n$ for all $x \in \Lambda$ and $n \in \mathbb{N}$. We say that (Λ, f) is a C^1 conformal repeller if

(C1) f is C^1 conformal on Λ,

(C2) f is uniformly expanding,

(C3) f is an open mapping.

For $s \in \mathbb{R}$ we define the dynamical pressure of the s-th power of the conformal derivative by the formula:

$$P(s, \Lambda, f) = \liminf_{n} \frac{1}{n} \log \sup_{y \in \Lambda} \sum_{x \in \Lambda : f_x^n = y} (Df^n(x))^{-s}. \tag{2.5}$$

We then have the following:

Theorem 2.1 (Bowen's formula). *Let (Λ, f) be a C^1 conformal repeller. Then, the Hausdorff dimension of Λ coincides with its upper and lower box dimensions and is given as the unique zero of the pressure function $P(s, \Lambda, f)$.*

Many similar results, proved under various restrictions, appear in the literature, see e.g. [Bow79], [Rue82], [Fal89], [Bar96], [GP97] and our introduction. It seems to be the first time that it is stated in the above generality. For clarity of the proof we will here impose the additional assumption of strong mixing. We have delegated to Appendix A a sketch of how to remove this restriction. We have chosen to do so because (1) the proof is really much more elegant and (2) there seems to be no natural generalisation when dealing with the time-dependent case (apart from trivialities).

More precisely, to any given $\delta > 0$ we assume that there is an $n_0 = n_0(\delta) \in \mathbb{N}$ (denoted the δ-covering time for the repeller) such that for every $x \in \Lambda$:

(C4) $f^{n_0} B(x, \delta) = \Lambda$.

For the rest of this section (Λ, f) will be assumed to be a strongly mixing C^1 conformal repeller, thus verifying (C1)–(C4).

Recall that a countable family $\{U_n\}_{n\in\mathbb{N}}$ of open sets is a δ-cover(Λ) if diam $U_n < \delta$ for all n and their union contains (here equals) Λ. For $s \geq 0$ we set

$$M_\delta(s, \Lambda) = \inf \left\{ \sum_n (\operatorname{diam} U_n)^s : \{U_n\}_{n\in\mathbb{N}} \text{ is a } \delta-\text{cover}(\Lambda) \right\} \in [0, +\infty].$$

Then $M(s, \Lambda) = \lim_{\delta\to 0} M_\delta(s, \Lambda) \in [0, +\infty]$ exists and is called the s-dimensional Hausdorff measure of Λ. The Hausdorff dimension is the unique critical value $s_{\text{crit}} = \dim_H \Lambda \in [0, \infty]$ such that $M(s, \Lambda) = 0$ for $s > s_{\text{crit}}$ and $M(s, \Lambda) = \infty$ for $s < s_{\text{crit}}$. The Hausdorff measure is said to be finite if $0 < M(s_{\text{crit}}, \Lambda) < \infty$.

Alternatively we may replace the condition on the covering sets by considering finite covers by open balls $B(x, \delta)$ of fixed radii, $\delta > 0$. Then the limit as $\delta \to 0$ of $M_\delta(s, \Lambda)$ need not exist so we replace it by taking lim sup and lim inf. We then obtain the upper, respectively the lower s-dimensional box 'measure'. The upper and lower box dimensions, $\dim^B \Lambda$ and $\dim_B \Lambda$, are the corresponding critical values. It is immediate that

$$0 \leq \dim_H \Lambda \leq \dim_B \Lambda \leq \dim^B \Lambda \leq +\infty.$$

Remark 2.2. Let $J(f)$ denote the Julia set of a uniformly hyperbolic rational map f of the Riemann sphere. There is an open (hyperbolic) neighborhood U of $J(f)$ such that $V = f^{-1}U$ is compactly contained in U and such that f has no critical points in V. When d is the hyperbolic metric on U, $(J(f), d_{|J(f)})$ is a compact metric space and one verifies that $(J(f), f)$ is a C^1 conformal repeller.

Remark 2.3. Let X be a C^1 Riemannian manifold without boundaries and let $f : X \to X$ be a C^1 map. It is an exercise in Riemannian geometry to see that f is uniformly conformal at $x \in X$ if and only if $f_{*x} : T_x X \to T_{fx} X$ is a conformal map of tangent spaces and in that case, $Df(x) = \|f_{*x}\|$. When dim $X < \infty$, condition (C3) follows from (C1)–(C2). We note also that being C^1 (the double limit in equation 2.4) rather than just differentiable is important.

2.1. *Geometric bounds.* We will first establish sub-exponential geometric bounds for iterates of the map f. In the following we say that a sequence $(b_n)_{n\in\mathbb{N}}$ of positive real numbers is sub-exponential or of sub-exponential growth if and only if $\lim_n \sqrt[n]{b_n} = 1$. For notational convenience we will also assume that $Df(x) \geq \beta > 1$ for all $x \in \Lambda$. This can always be achieved in the present set-up by considering a high enough iterate of the map f possibly redefining β.

Define the *divided difference*,

$$f[u,v] = \begin{cases} \frac{d(f_u,f_v)}{d(u,v)} & u \neq v \in \Lambda, \\ Df(u) & u = v \in \Lambda. \end{cases} \tag{2.6}$$

Our hypothesis on f implies that $f[\cdot,\cdot]$ is continuous on the compact set $\Lambda\times\Lambda$, and not smaller than $\beta > 1$ on the diagonal of the product set. We let $\|Df\| = \sup_{u\in\Lambda} Df(u) < +\infty$ denote the maximal conformal derivative on the repeller.

Choose $1 < \lambda_0 < \beta$. Uniform continuity of $f[\cdot,\cdot]$ and (uniform) openness of the map f show that we may find $\delta_f > 0$ and then $\lambda_1 = \lambda_1(f) < +\infty$ such that

(C2′) $\lambda_0 \leq f[u,v] \leq \lambda_1$ whenever $u, v \in \Lambda$ and $d(u,v) < \delta_f$,

(C3′) $B(f_x, \delta_f) \subset fB(x, \delta_f)$ for all $x \in \Lambda$.

The constant δ_f gives a scale below which f is injective, uniformly expanding and (locally) onto. We note that $\Lambda \not\subset B(x,\delta_f)$ for any $x \in \Lambda$ (or else Λ would be reduced to a point). In the following we will assume that values of $\delta_f > 0$, $\lambda_0 > 1$ and $\lambda_1 < +\infty$ have been found so as to satisfy conditions (C2') and (C3').

We define the *distortion* of f at $x \in \Lambda$ and for $r > 0$ as follows:

$$\varepsilon_f(x,r) = \sup\{\, \log \frac{f[u_1,u_2]}{f[u_3,u_4]} : \text{all } u_i \in B(x,\delta_f) \cap f^{-1}B(f_x,r)\}. \tag{2.7}$$

This quantity tends to zero as $r \to 0^+$ uniformly in $x \in \Lambda$ (with the same compactness and continuity as before). Thus,

$$\varepsilon(r) = \sup_{x\in\Lambda} \varepsilon_f(x,r)$$

tends to zero as $r \to 0^+$. When $x \in \Lambda$ and the u_i's are as in (2.7) then also:

$$\left|\log \frac{f[u_1,u_2]}{Df(u_3)}\right| \leq \varepsilon(r) \quad \text{and} \quad \left|\log \frac{Df(u_1)}{Df(u_2)}\right| \leq \varepsilon(r). \tag{2.8}$$

For $n \in \mathbb{N}\cup\{0\}$ we define the n-th 'Bowen ball' around $x \in \Lambda$

$$B_n(x) \equiv B_n(x,\delta_f,f) = \{u \in \Lambda : d(f^k_x, f^k_u) < \delta_f,\ 0 \leq k \leq n\}.$$

We say that u is *n-close* to $x \in \Lambda$ if $u \in B_n(x)$. The Bowen balls act as 'reference' balls, getting uniformly smaller with increasing n. In particular, diam $B_n(x) \leq 2\ \delta_f\ \lambda_0^{-n}$, i.e. tends to zero exponentially fast with n. We also see that for each $x \in \Lambda$ and $n \geq 0$ the map

$$f\ :\ B_{n+1}(x) \to B_n(f_x)$$

is a uniformly expanding homeomorphism.

Expansiveness of f means that closeby points may follow very different future trajectories. Our assumptions assure, however, that closeby points have very similar backwards histories. The following two lemmas emphasize this point:

LEMMA 2.4 (Pairing). *For each $y, w \in \Lambda$ with $d(y,w) < \delta_f$ and for every $n \in \mathbb{N}$ the sets $f^{-n}\{y\}$ and $f^{-n}\{z\}$ may be paired uniquely into pairs of n-close points.*

Proof. Take $x \in f^{-n}\{y\}$. The map $f^n : B_n(x) \to B_0(f_x^n) = B(y, \delta_f)$ is a homeomorphism. Thus there is a unique point $u \in f^{-n}\{z\} \cap B_n(x)$. By construction, $x \in B_n(u)$ if and only if $u \in B_n(x)$. Therefore $x \in f^{-n}\{y\} \cap B_n(u)$ is the unique pre-image of y in the n-th Bowen ball around u and we obtain the desired pairing. □

LEMMA 2.5 (Sub-exponential distortion). *There is a sub-exponential sequence $(c_n)_{n\in\mathbb{N}}$ such that given any two points z and u which are n-close to $x \in \Lambda$ ($x \neq u$) one has*

$$\frac{1}{c_n} \leq \frac{d(f_u^n, f_x^n)}{d(u,x)\ Df^n(z)} \leq c_n \quad \text{and} \quad \frac{1}{c_n} \leq \frac{Df^n(x)}{Df^n(z)} \leq c_n.$$

Proof. For all $1 \leq k \leq n$ we have that $f_u^k \in B_{n-k}(f_x^k)$. Therefore, $d(f_u^k, f_x^k) < \delta_f \lambda_0^{k-n}$ and the distortion bound (2.8) implies that

$$|\log \frac{d(f_u^n, f_x^n)}{d(u,x)\ Df^n(z)}| \leq \varepsilon(\delta_f) + \varepsilon(\delta_f \lambda_0^{-1}) + \cdots + \varepsilon(\delta_f \lambda_0^{1-n}) \equiv \log c_n.$$

Since $\lim_{r\to 0} \varepsilon(r) = 0$ it follows that $\frac{1}{n} \log c_n \to 0$, whence that the sequence $(c_n)_{n\in\mathbb{N}}$ is of sub-exponential growth. This yields the first inequality and the second is proved e.g. by taking the limit $u \to x$. □

Remark 2.6. When $\varepsilon(t)/t$ is integrable at $t = 0^+$ one verifies that the distortion stays uniformly bounded, i.e. that $c_n \leq \varepsilon(\delta_f) + \int_0^{\delta_f} \frac{\varepsilon(t)}{t}\ \frac{dt}{\log\lambda} < \infty$ uniformly in n. This is the case, e.g. when ε is Hölder continuous at zero.

2.2. *Transfer operators.* Let $\mathcal{M}(\Lambda)$ denote the Banach space of bounded, real valued functions on Λ equipped with the sup-norm. We denote by χ_U the characteristic function of a subset $U \subset \Lambda$ and we write $\mathbf{1} = \chi_\Lambda$ for the constant function $\mathbf{1}(x) = 1$, $\forall x \in \Lambda$. For $\phi \in \mathcal{M}(\Lambda)$ and $s \geq 0$ we define the positive linear transfer[2] operator,

$$(L_s \phi)_y \equiv (L_{s,f}\phi)_y \equiv \sum_{x \in \Lambda : f_x = y} (Df(x))^{-s} \, \phi_x, \quad y \in \Lambda.$$

Since Λ has a finite δ_f-cover and Df is bounded these operators are necessarily bounded. The n'th iterate of the operator L_s is given by

$$(L_s^n \phi)_y = \sum_{x \in \Lambda : f_x^n = y} (Df^n(x))^{-s} \, \phi_x.$$

It is of importance to obtain bounds for the action upon the constant function. More precisely, for $s \geq 0$ and $n \in \mathbb{N}$, we denote

$$M_n(s) \equiv \sup_{y \in \Lambda} L_s^n \mathbf{1}(y) \quad \text{and} \quad m_n(s) \equiv \inf_{y \in \Lambda} L_s^n \mathbf{1}(y). \tag{2.9}$$

We then define the lower, respectively, the upper pressure through

$$-\infty \leq \underline{P}(s) \equiv \liminf_n \frac{1}{n} \log m_n(s) \quad \leq \quad \overline{P}(s) \equiv \limsup_n \frac{1}{n} \log M_n(s) \leq +\infty.$$

Lemma 2.7 (Operator bounds). *For each $s \geq 0$ the upper and lower pressures agree and are finite. We write $P(s) \equiv \underline{P}(s) = \overline{P}(s) \in \mathbb{R}$ for the common value. The function $P(s)$ is continuous, strictly decreasing and has a unique zero, $s_{\mathrm{crit}} \geq 0$.*

Proof. Fix $s \geq 0$. Since the operator is positive, the sequences $M_n = M_n(s)$ and $m_n = m_n(s)$, $n \in \mathbb{N}$ are sub-multiplicative and super-multiplicative, respectively. Thus,

$$m_k m_{n-k} \leq m_n \leq M_n \leq M_k M_{n-k}, \quad \forall\, 0 < k < n. \tag{2.10}$$

This implies convergence of both $\sqrt[n]{M_n}$ and $\sqrt[n]{m_n}$, the limit of the former sequence being the spectral radius of L_s acting upon $\mathcal{M}(\Lambda)$. Let us sketch a standard proof for the first sequence: Fixing $k \geq 1$ we write $n = pk + r$ with $0 \leq r < k$. Since k is fixed, $\limsup_n \max_{0<r<k} \sqrt[n]{M_r} = 1$. But then $\limsup_n \sqrt[n]{M_n} = \limsup_p \sqrt[pk]{M_{pk}} \leq \sqrt[k]{M_k}$. Taking lim inf (with respect to k) on the right-hand side we conclude that the limit exists. A similar proof works for the sequence $(m_n)_{n \in \mathbb{N}}$. Both limits are nonzero ($\geq m_1 > 0$) and finite ($\leq M_1 < \infty$). We need to show that the ratio M_n/m_n is of sub-exponential growth.

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[2]The ‘transfer’-terminology, inherited from statistical mechanics, refers here to the ‘transfer’ of the encoded geometric information at a small scale to a larger scale, using the dynamics of the map, f.

Consider $w, z \in \Lambda$ with $d(w, z) < \delta_f$ and $n > 0$. The Pairing Lemma shows that we may pair the pre-images $f^{-n}\{w\}$ and $f^{-n}\{z\}$ into pairs of n-close points, say $(w_\alpha, z_\alpha)_{\alpha \in I_n}$, over a finite index set I_n which possibly depends on the pair (w, z). Applying the second distortion bound in Lemma 2.5 to each pair yields

$$L_s^n \, \mathbf{1}(z) \geq \left(\frac{1}{c_n}\right)^s L_s^n \, \mathbf{1}(w). \tag{2.11}$$

Choose $w \in \Lambda$ such that $L_s^n \, \mathbf{1}(w) \geq M_n/2$. Given an arbitrary $y \in \Lambda$ our strong mixing assumption (C4), with $n_0 = n_0(\delta_f)$, implies that the set $B(w, \delta_f) \cap f^{-n_0}\{y\}$ contains at least one point. Using (2.11) we obtain

$$L_s^{n+n_0} \, \mathbf{1}(y) = \sum_{z : f_z^{n_0} = y} (Df^{n_0}(z))^{-s} \, L_s^n \, \mathbf{1}(z) \geq (\|Df\|^{n_0} c_n)^{-s} \, \frac{M_n}{2}.$$

Thus,

$$m_{n+n_0} \geq (\|Df\|^{n_0} c_n)^{-s} M_n/2 \tag{2.12}$$

and since c_n is of sub-exponential growth then so is M_n/m_{n+n_0} and therefore also $M_{n+n_0}/m_{n+n_0} \leq M_{n_0} M_n/m_{n+n_0}$.

The functions, $s \log \beta + P(s)$ and $s \log \|Df\| + P(s)$, are nonincreasing and nondecreasing, respectively. Also $0 \leq P(0) < +\infty$. It follows that $s \mapsto P(s)$ is continuous and that P has a unique zero $s_{\text{crit}} \geq 0$. □

Remark 2.8. Super- and sub-multiplicativity (2.10) imply the bounds[3]

$$m_n(s) \leq e^{nP(s)} \leq M_n(s), \quad n \in \mathbb{N}.$$

Clearly, if the distortion constants c_n are uniformly bounded then so is the ratio $M_n(s)/m_n(s) \leq K(s) < \infty$.

In order to prove Theorem 2.1 it suffices to show that $s_{\text{crit}} \leq \dim_H(\Lambda)$ and $\dim^B(\Lambda) \leq s_{\text{crit}}$.

2.3. *$\dim_H(\Lambda) \geq s_{\text{crit}}$*. Let $U \subset \Lambda$ be an open nonempty subset of diameter not exceeding δ_f. We will iterate U by f until the size of $f^k U$ becomes 'large' compared to δ_f. As long as f^k stays injective on U the set $\{z \in U : f_z^k = y\}$ contains at most one element for any $y \in \Lambda$. Therefore, for such k-values

$$(L_s^k \chi_U)(y) \leq \sup_{z \in U} \left(Df^k(z)\right)^{-s}, \quad \forall \, y \in \Lambda. \tag{2.13}$$

Choose $x = x(U) \in U$ and let $k = k(U) \geq 0$ be the largest positive integer for which $U \subset B_k(x)$. In other words:

[3] Such bounds are useful in applications as they imply computable rigorous bounds for the dimensions.

(a) $d(f^l_x, f^l_u) < \delta_f$ for $0 \le l \le k$ and all $u \in U$ and

(b) $d(f^{k+1}_x, f^{k+1}_u) \ge \delta_f$ for some $u \in U$.

Note that $k(U)$ is finite because the open set U contains at least two distinct points which are going to be separated when iterating. Because of (a) f^k is injective on U so that (2.13) applies. On the other hand, (a) and (b) imply that there is $u \in U$ for which $\delta_f \le d(f^{k+1}_x, f^{k+1}_u) \le \lambda_1(f) d(f^k_u, f^k_x)$ where $\lambda_1(f)$ was the maximal dilation of f on δ_f-separated points. Our sub-exponential distortion estimate shows that for any $z \in U$

$$\frac{\delta_f/\lambda_1(f)}{\operatorname{diam} U}\ \frac{1}{Df^k(z)} \le \frac{d(f^k_u, f^k_x)}{d(u,x)}\ \frac{1}{Df^k(z)} \le c_k.$$

Inserting this in (2.13) and using the definition of $m_n(s)$ we see that for any $y \in \Lambda$,

$$(L^k_s \chi_U)(y) \le\ (\operatorname{diam} U)^s\ (\frac{\lambda_1(f) c_k}{\delta_f})^s\, \mathbf{1} \le (\operatorname{diam} U)^s \left[(\frac{\lambda_1(f) c_k}{\delta_f})^s\ \frac{1}{m_k(s)}\right]\ L^k_s \mathbf{1}.$$

Choosing now $0 < s < s_{\text{crit}}$, the sequence $m_k(s)$ tends exponentially fast to infinity (when $s_{\text{crit}} = 0$ there is nothing to show). Since the sequence $((c_k)^s)_{k\in\mathbb{N}}$ is sub-exponential the factor in square-brackets is uniformly bounded in k, say by $\gamma_1(s) < \infty$ (independent of U). Positivity of the operator implies that for any $n \ge k(U)$ we have

$$L^n_s \chi_U \le \gamma_1(s)\ (\operatorname{diam} U)^s\ L^n_s \mathbf{1}.$$

If $(U_\alpha)_{\alpha\in\mathbb{N}}$ is an open δ_f-cover of the compact set Λ then it has a finite sub-cover, say $\Lambda \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_m}$. Taking now $n = \max\{k(U_{\alpha_1}), \ldots, k(U_{\alpha_m})\}$ we obtain

$$L^n_s \mathbf{1} \le \sum_{i=1}^m L^n_s \chi_{U_{\alpha_i}} \le \gamma_1(s) \sum_{i=1}^m (\operatorname{diam} U_{\alpha_i})^s\, L^n_s \mathbf{1} \le \gamma_1(s) \sum_\alpha (\operatorname{diam} U_\alpha)^s\, L^n_s \mathbf{1}.$$

This equation shows that $\sum_\alpha (\operatorname{diam} U_\alpha)^s$ is bounded uniformly from below by $1/\gamma_1(s) > 0$. The Hausdorff dimension of Λ is then not smaller than s, whence not smaller than s_{crit}.

2.4. $\dim^B \Lambda \le s_{\text{crit}}$. Fix $0 < r < r_0 \equiv \frac{\delta_f}{\lambda_1(f)^{n_0}}$ and let $x \in \Lambda$. This time we wish to iterate a ball $U = B(x,r)$ until it has a 'large' interior and contains a ball of size δ_f. This may, however, not be good enough (cf. Figure 1). We also need to control the distortion. Again these two goals combine nicely when considering the sequence of Bowen balls $B_k \equiv B_k(x)$, $k \ge 0$. It forms a sequence of neighborhoods of x, shrinking to $\{x\}$. Hence, there is a smallest integer $k = k(x,r) \ge 1$ such that $B_k \subset U$. Note that k must be strictly positive, or else $\Lambda = f^{n_0} B_0 \subset f^{n_0} B(x, r_0) \subset B(f^{n_0}(x), \delta_f)$ which is not possible. Now,

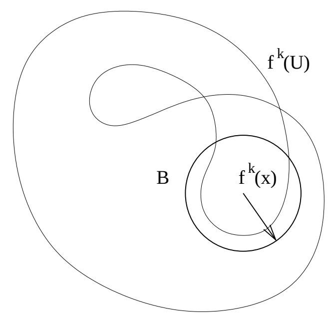

Figure 1: An iterate $f^k(U)$ which covers $B = B(f^k(x), \delta_f)$ but not in the 'right' way.

f^k maps B_k homeomorphically onto $B_0(f_x^k) = B(f_x^k, \delta_f)$ and positivity of L_s shows that

$$L_s^k \chi_U \geq L_s^k \chi_{B_k} \geq \inf_{z \in B_k} \Big(Df^k(z) \Big)^{-s} \chi_{B(f_x^k, \delta_f)}.$$

By assumption $B_{k-1} \not\subset U$ and so there must be a point $y \in B_{k-1}$ with $d(y,x) \geq r$. As y is $(k-1)$-close to x our distortion estimate shows that for any $z \in B_k \subset B_{k-1}$,

$$\frac{\delta_f}{r} \frac{\|Df\|}{Df^k(z)} > \frac{d(f_y^{k-1}, f_x^{k-1})}{d(y,x)} \frac{1}{Df^{k-1}(z)} \geq \frac{1}{c_{k-1}}.$$

Therefore,

$$L_s^k \chi_U \geq r^s (\delta_f c_{k-1} \|Df\|)^{-s} \chi_{B(f_x^k, \delta_f)}.$$

If we iterate another $n_0 = n_0(\delta_f)$ times then $f^{n_0} B(f_x^k, \delta_f)$ covers all of Λ due to mixing and using the definition of $M_n(s)$ we have

$$L_s^{k+n_0} \chi_U \geq r^s (\delta_f c_{k-1} \|Df\|^{1+n_0})^{-s} \mathbf{1} \geq (4r)^s \left[\frac{(4\|Df\|^{1+n_0} \delta_f c_{k-1})^{-s}}{M_{k+n_0}(s)} \right] L_s^{k+n_0} \mathbf{1}.$$

When $s > s_{\rm crit}$, $M_{k+n_0}(s)$ tends expontially fast to zero. As the rest is sub-exponential, the quantity in the square brackets is uniformly bounded from below by some $\gamma_2(s) > 0$. Using the positivity of the operator we see that

$$L_s^n \chi_U \geq \gamma_2(s) (4r)^s L_s^n \mathbf{1}, \tag{2.14}$$

whenever $n \geq k(x,r) + n_0$.

Now, let $x_1, \ldots, x_N$ be a finite maximal $2r$ separated set in Λ. Thus, the balls $\{B(x_i, 2r)\}_{i=1,\ldots,N}$ cover Λ whereas the balls $\{B(x_i, r)\}_{i=1,\ldots,N}$ are mutually disjoint. For $n \geq \max_i k(x_i, r) + n_0$,

$$L_s^n \mathbf{1} \geq \sum_i L_s^n \chi_{B(x_i, r)} \geq \gamma_2(s) \; N (4r)^s L_s^n \mathbf{1}.$$

We have deduced the bound,

$$\sum_{i=1}^{N} (\text{diam } B(x_i, 2r))^s \leq 1/\gamma_2(s).$$

This shows that $\dim^B \Lambda$ does not exceed s, whence not s_{crit}. We have proven Theorem 2.1 in the case of a strongly mixing repeller and refer to Appendix A for the extension to the general case. □

Corollary 2.9. *If $\frac{\varepsilon(t)}{t}$ is integrable at $t = 0^+$ and the repeller is strongly mixing (cf. Remark* A.1) *then the Hausdorff measure is finite and between $1/\gamma_1(s_{\text{crit}}) > 0$ and $1/\gamma_2(s_{\text{crit}}) < +\infty$.*

Proof. The hypothesis implies that for fixed s the sequences $(c_n(s))_n$ and $M_n(s)/m_n(s)$ in the sub-exponential distortion and operator bounds, respectively, are both uniformly bounded in n (Remarks 2.6 and 2.8). All the (finite) estimates may then be carried out at $s = s_{\text{crit}}$ and the conclusion follows. (Note that no measure theory was used to reach this conclusion). □

3. Time dependent conformal repellers

Let (K, d) denote a complete metric space without isolated points and let $\Delta > 0$ be such that K is covered by a finite number, say N_Δ balls of size Δ. To avoid certain pathologies we will also assume that (K, d) is Δ-*homogeneous*, i.e. that there is a constant $0 < \delta < \Delta$ such that for any $y \in K$

$$B(y, \Delta) \setminus B(y, \delta) \neq \emptyset. \tag{3.15}$$

For example, if K is connected or consists of a finite number of connected components then K is Δ-homogeneous.

Let $\beta > 1$ and let $\varepsilon : [0, \Delta] \to [0, +\infty[$ be an ε-function, i.e. a continuous function with $\varepsilon(0) = 0$. In the following we will consider C^1-conformal unramified covering maps of the form

$$f : \Omega_f \to K$$

from a nonempty (not necessarily connected) domain $\Omega_f \subset K$ onto K and of finite maximal degree $d^o_{\max}(f) = \max_{y \in K} \deg(f; y) \in \mathbb{N}$. More precisely, we will consider the class $\mathcal{E} = \mathcal{E}(\Delta, \beta, \varepsilon)$ of such maps that in addition verify the following 'equi-uniform' requirements:

Assumption 3.1. *There are constants $0 < \delta(f) \leq \Delta$ and $\lambda_1(f) < +\infty$, and a function $\delta_f : x \in \Omega_f \mapsto [\delta(f), \Delta]$ such that:*

(T0) *For all distinct $x, x' \in f^{-1}\{y\}$ (with $y \in K$) the balls $B(x, 2\delta_f(x))$ and $B(x', 2\delta_f(x'))$ are disjoint (local injectivity).*

(T1) *For all* $x \in \Omega_f$: $B(f(x),\Delta) \subset f\,(B(x,\delta_f(x)) \cap \Omega_f)$ (*openness*).

(T2) *For all* $u, x \in \Omega_f$ *with* $d(u,x) < \delta_f(x)$: $\beta \leq f[u,x] \leq \lambda_1(f)$ (*dilation*).

(T3) *For all* $x \in \Omega_f$: $\varepsilon_f(x,r) \leq \varepsilon(r)$, $\forall\ 0 < r \leq \Delta$ (*distortion*).

Here, $f[\cdot,\cdot]$ is the divided difference from equation (2.6) and the distortion, a restricted version of equation (2.7), for $x \in \Omega_f$ and $r > 0$ is given by

$$\varepsilon_f(x,r) = \sup\left\{ \left|\log \frac{f[u_1,x]}{Df(u_2)}\right| : u_1, u_2 \in B(x,\delta_f(x)) \cap f^{-1}B(f(x),r) \right\}.$$

We tacitly understand by writing $f^{-1}\{y\}$ that we are looking at the pre-images of $y \in K$ within Ω_f, i.e. where the map is defined. We also write $\|Df\|$ for the supremum of the conformal derivative of f over its domain of definition Ω_f. By (T2) and by setting $u = x$ we also see that

$$\beta \leq \|Df\| \leq \lambda_1(f). \tag{3.16}$$

When $f \in \mathcal{E}(\Delta,\beta,\varepsilon)$ and $f(x) = y \in K$ then by Δ-homogeneity (3.15) and property (T1), there must be $u \in B(x,\delta_f(x))$ with $f(u) \in B(y,\Delta)\setminus B(y,\delta)$ and δ as in (3.15). By the above definition of the distortion, $\varepsilon_f(x,r)$, it follows that

$$0 < \kappa \equiv \delta e^{-\varepsilon(\Delta)} \leq \delta_f(x) Df(x), \quad \forall x \in \Omega_f. \tag{3.17}$$

In the following let $\mathcal{F} = (f_k)_{k\in\mathbb{N}} \subset \mathcal{E}(\Delta,\beta,\varepsilon)$ be a fixed sequence of such mappings and let us fix $\delta_{f_k}(x)$, $\delta(f_k) = \inf_{x\in\Omega_{f_k}} \delta_{f_k}(x) > 0$ and $\lambda_1(f_k)$ so as to satisfy conditions (T0)–(T3). Let $\Omega_0(\mathcal{F}) = K$ and for $n \geq 1$ define

$$\Omega_n(\mathcal{F}) = f_1^{-1} \circ \cdots \circ f_n^{-1}(K)$$

and then

$$\Lambda(\mathcal{F}) = \bigcap_{n\geq 0} \Omega_n(\mathcal{F}).$$

Letting $\sigma(\mathcal{F}) = (f_{k+1})_{k\in\mathbb{N}}$ denote the shift of the sequence we set $\Lambda_t = \Lambda(\sigma^t(\mathcal{F}))$, $t \geq 0$. Recall that K was assumed complete (though not necessarily compact) and each $\delta(f_k)$ is strictly positive. It follows then that each Λ_t is closed, whence complete. Each Λ_t also has finite open covers of arbitrarily small diameters (obtained by pulling back a finite Δ-cover of K), whence each Λ_t is compact and nonempty. Also $f_t(\Lambda_{t-1}) = \Lambda_t$ so we have obtained a time-dependent sequence of compact conformal repellers,

$$\Lambda_0 \xrightarrow{f_1} \Lambda_1 \xrightarrow{f_2} \Lambda_2 \longrightarrow \cdots .$$

For $t \geq 0$, $k \geq 0$ we denote by $f_t^{(k)} = f_{t+k} \circ \cdots \circ f_{t+1}$ the k'th iterated map from $\Omega_k(\sigma^t(\mathcal{F}))$ onto K ($f_t^{(0)}$ is the identity map on K). We write simply $f^{(k)} \equiv f_0^{(k)} : \Omega_k(\mathcal{F}) \to K$ for the iterated map starting at time zero and

$Df^{(k)}(x)$ for the conformal derivative of this iterated map.

For $n \geq 0$, $x \in \Omega_n(\mathcal{F})$ (and similarly for $u \in \Omega_n(\mathcal{F})$) we write $x_j = f^{(j)}(x)$, $0 \leq j \leq n$ for its iterates. Using this notation we define for $n \geq 0$ the n'th Bowen ball around x:

$$B_n(x) = \{u \in \Omega_n(\mathcal{F}) : d(x_j, u_j) < \delta_{f_{j+1}}(x_j), 0 \leq j \leq n\}$$

and then for $n \geq 1$ also the $(n-1, \Delta)$-Bowen ball around $x \in \Omega_n(\mathcal{F})$:

$$B_{n-1,\Delta}(x) = \{u \in B_{n-1}(x) \cap \Omega_n(\mathcal{F}) : d(x_n, u_n) < \Delta\}.$$

Then $f^{(n)} : B_{n-1,\Delta}(x) \to B(f^{(n)}(x), \Delta)$, $n \geq 1$, is a uniformly expanding homeomorphism for all $x \in \Omega_n(\mathcal{F})$. When $u \in B_{n-1,\Delta}(x)$ we say that u and x are $(n-1, \Delta)$-close. Our hypotheses imply that being $(n-1, \Delta)$-close is a reflexive relation (perhaps not so obvious since $\delta_f(x)$ depends on x) as is shown in the proof of the following:

Lemma 3.2 (Pairing). *For $n \in \mathbb{N}$, $y, w \in K$ with $d(y, w) < \Delta$, the sets $(f^{(n)})^{-1}\{y\}$ and $(f^{(n)})^{-1}\{w\}$ may be paired uniquely into pairs of $(n-1, \Delta)$-close points.*

Proof. Fix $f = f_n$ and let $x \in f^{-1}\{y\}$. By (T1), $f(B(x, \delta_f(x)) \cap \Omega_f)$ contains $B(f(x), \Delta) \ni w$. Let $z \in f^{-1}\{w\} \cap B(x, \delta_f(x))$ be at a distance $d(x, z) < \delta_f(x) \leq \Delta$ to x. We claim that then also $x \in B(z, \delta_f(z))$. If not so, there must be $x' \in B(z, \delta_f(z)) \cap f^{-1}\{y\}$ for which $d(x', z) < \delta_f(z) \leq d(x, z) < \delta_f(x)$ so that $d(x, x') < 2\delta_f(x)$ and this contradicts $(T0)$. But then also the point z must be unique: If $z, z' \in f^{-1}\{w\} \cap B(x, \delta_f(x))$ then $x \in B(z, \delta_f(z)) \cap B(z', \delta_f(z'))$ implies $z = z'$ by (T0). Returning to the sequence of mappings we obtain by recursion in n the unique pairing. □

Lemma 3.3 (Sub-exponential distortion). *There is a sub-exponential sequence $(c_n)_{n \in \mathbb{N}}$ (depending on the equi-distortion function ε but not on the actual sequence of maps) such that the following holds: Given $n \geq 1$ and points z and u that are $(n-1, \Delta)$-close to $x \in \Omega_n(\mathcal{F})$, $x \neq u$ we have*

$$\frac{1}{c_n} \leq \frac{d(f^{(n)}(u), f^{(n)}(x))}{d(u, x)\, Df^{(n)}(z)} \leq c_n \quad \textit{and} \quad \frac{1}{c_n} \leq \frac{Df^{(n)}(x)}{Df^{(n)}(z)} \leq c_n.$$

Proof. As in Lemma 2.5, but more precisely, we have $\log c_n = \varepsilon(\Delta) + \varepsilon(\Delta/\beta) + \cdots + \varepsilon(\Delta/\beta^{n-1})$. □

For $s \geq 0$, $f \in \mathcal{E}(\Delta, \beta, \varepsilon)$ we define as before a transfer operator $L_{s,f} : \mathcal{M}(K) \to \mathcal{M}(K)$ by setting:

$$(L_{s,f}\phi)(y) \equiv \sum_{x \in f^{-1}\{y\}} (Df(x))^{-s}\, \phi_x, \quad y \in K, \quad \phi \in \mathcal{M}(K). \tag{3.18}$$

We write $L_s^{(n)} = L_{s,f_n} \circ \cdots \circ L_{s,f_1}$ for the n'th iterated operator from $\mathcal{M}(K)$ to $\mathcal{M}(K)$, $n \in \mathbb{N}$. We denote by $\mathbf{1} = \chi_{_K}$ the constant function which equals one on K. As in (2.9) we define for $n \in \mathbb{N}$ (omitting the dependency on $\mathcal{F}$ in the notation):

$$M_n(s) \equiv \sup_{y \in \Lambda_n} L_s^{(n)}\mathbf{1}(y) \quad \text{and} \quad m_n(s) \equiv \inf_{y \in \Lambda_n} L_s^{(n)}\mathbf{1}(y)$$

and then the lower and upper s-conformal pressures:

$$-\infty \leq \underline{P}(s) \equiv \liminf_n \frac{1}{n} \log m_n(s) \quad \leq \quad \overline{P}(s) \equiv \limsup_n \frac{1}{n} \log M_n(s) \leq +\infty.$$

These limits need not be equal nor finite. As in Lemma 2.7 one shows that both $s \log \beta + \underline{P}(s)$ and $s \log \beta + \overline{P}(s)$ are nonincreasing so that the functions $\underline{P}(s)$ and $\overline{P}(s)$ are strictly decreasing (when finite). Regarding explicit formulae, we have e.g. for the lower pressure, similar to (2.5):

$$\underline{P}(s) = \liminf_n \ \frac{1}{n} \ \log \inf_{y \in \Lambda_n} \sum_{x \in (f^{(n)})^{-1}\{y\}} \left(Df^{(n)}(x) \right)^{-s}.$$

We define the following *lower and upper critical exponents* with values in $[0, +\infty]$:

$$\underline{s}_{\rm crit} = \sup\{s \geq 0 : \underline{P}(s) > 0\} \quad \text{and} \quad \overline{s}_{\rm crit} = \inf\{s \geq 0 : \overline{P}(s) \leq 0\}.$$

It will be necessary to make some additional assumptions on mixing and growth rates. For our purposes the following suffices:

ASSUMPTION 3.4. (T4) *There is $n_0 \in \mathbb{N}$ such that the sequence $(f_k)_{k \in \mathbb{N}}$ is (n_0, Δ)-mixing, i.e. for any $t \in \mathbb{N} \cup \{0\}$ and $x \in \Lambda_t$:*

$$f_t^{(n_0)}(B(x, \Delta) \cap \Lambda_t) = \Lambda_{t+n_0}.$$

(T5) *The sequence $(\lambda_1(f_k))_{k \in \mathbb{N}}$ is sub-exponential, i.e.*

$$\lim_k \frac{1}{k} \log \lambda_1(f_k) = 0.$$

LEMMA 3.5. *Assuming* (T0)–(T5) *we have* (*the limits need not be finite*):

$$\overline{P}(s) = \limsup_n \frac{1}{n} \log m_n(s) = \limsup_n \frac{1}{n} \log M_n(s),$$
$$\underline{P}(s) = \liminf_n \frac{1}{n} \log m_n(s) = \liminf_n \frac{1}{n} \log M_n(s).$$

Proof. We proceed as in the last half of the proof of the operator bounds, Lemma 2.7. Through a small modification, notably replacing δ_f by Δ, and making use of mixing, condition (T4), and the distortion bounds in Lemma 3.3 we deduce similarly to (2.12) that

$$m_{n+n_0}(s) \geq (\|Df_{n+1}\| \cdots \|Df_{n+n_0}\| c_n)^{-s} M_n(s)/2,$$

in which the sequence c_n is sub-exponential. Due to (3.16), (T5) and as n_0 is fixed the sequence $M_n(s)/m_{n+n_0}(s)$ is of sub-exponential growth. Whether finite or not, the above lim inf's and lim sup's agree. $\square$

LEMMA 3.6. *Assuming* (T0)–(T5) *we have the following dichotomy*: *Either* Λ_0 *is a finite set or* Λ_0 *is a perfect set.*

Proof. Suppose that Λ_k is a singleton for some $k \in \mathbb{N}$. Then also Λ_n is a singleton for all $n \geq k$ and Λ_0 is a finite set because all the (preceeding) maps are of finite degree. Suppose instead that no Λ_k is reduced to a singleton and let us take $x \in \Lambda_0$ as well as $n \geq 0$. By (T4) there is $z \in \Lambda_n \cap B(f^{(n)}(x), \Delta)$, $z \neq f^{(n)}(x)$. Because of Lemma 3.2, z must have an n'th pre-image in Λ_0 distinct from x and at a distance less than $\beta^{-n}\Delta$ to x. Thus, x is a point of accumulation of other points in Λ_0. $\square$

We have the following (see [Bar96, Ths. 2.1 and 3.8] for similar results):

THEOREM 3.7. *Let* Λ_0 *denote the time-zero conformal repeller for a sequence of* $\mathcal{E}(\Delta, \beta, \varepsilon)$*-maps*, $(f_k)_{k \in \mathbb{N}}$, *verifying conditions* (T0)–(T5). *Then there exist the following inequalities* (*note that the first is actually an equality*), *regarding dimensions of* $\Lambda_0 = \Lambda(\mathcal{F})$*:*

$$\underline{s}_{\mathrm{crit}} = \dim_H \Lambda_0 \leq \dim_B \Lambda_0 \leq \dim^B \Lambda_0 \leq \overline{s}_{\mathrm{crit}}.$$

If, in addition, $\lim \frac{1}{n} \log m_n(\underline{s}_{\mathrm{crit}}) = 0$ *then* $\underline{s}_{\mathrm{crit}} = \overline{s}_{\mathrm{crit}}$ *and all the above dimensions agree.*

Proof. When Λ_0 is a finite set it is easily seen that $\overline{P}(0) = 0$ and then that $\underline{s}_{\mathrm{crit}} = \overline{s}_{\mathrm{crit}} = 0$ in agreement with our claim. In the following we assume that Λ_0 has no isolated points.

($\underline{s}_{\mathrm{crit}} \leq \dim_H \Lambda_0$): Let $U \subset \Lambda_0$ be a nonempty open subset (for the induced topology on Λ_0) of diameter not exceeding $\delta(f_1)$. Choose $x = x(U) \in U$ and let $k = k(U) \geq 0$ be the largest integer (finite as Λ_0 was without isolated points) such that $U \subset B_k(x)$. Then there is $u \in U \setminus B_{k+1}(x) \subset B_k(x) \setminus B_{k+1}(x)$ for which we must have $\delta(f_{k+2}) \leq d(x_{k+1}, u_{k+1}) \leq \lambda_1(f_{k+1}) d(x_k, u_k)$. Proceeding as in section 2.3 we obtain the bound

$$L_s^{(k)} \chi_U \leq (\mathrm{diam}\, U)^s \left[\left(\frac{\lambda_1(f_{k+1}) c_k}{\delta(f_{k+2})} \right)^s \frac{1}{m_k(s)} \right] L_s^{(k)} \chi_{\Lambda_0}.$$

By hypothesis (T5), $\lambda_1(f_k)$ is a sub-exponential sequence. Because of Δ-homogeneity, or more precisely (3.17) and (T2), we see that $\delta(f_k) \geq \kappa / \lambda_1(f_k)$ is also sub-exponential. If $\underline{s}_{\mathrm{crit}} = 0$ there is nothing to show. If $0 \leq s < \underline{s}_{\mathrm{crit}}$ then $m_k(s)$ tends to infinity exponentially fast (recall that $\underline{P}(s)$ is strictly decreasing in s) and the factor in the square bracket is uniformly bounded from above by a constant $\gamma_1(s) < \infty$. We thus arrive at

$$L_s^{(k)}\chi_U \le \gamma_1(s)\,(\text{diam } U)^s\, L_s^{(k)}\chi_{\Lambda_0}.$$

We may proceed as in Section 2.3 to conclude that $\dim_H\Lambda_0 \ge \underline{s}_{\text{crit}}$.

($\underline{s}_{\text{crit}} \ge \dim_H\Lambda_0$): To obtain this converse inequality we will use a standard trick which amounts to constructing explicit covers of small diameter and giving bounds for their Hausdorff measure.

Let $n \ge 1$. By our initial assumption we may find a finite Δ-cover $\{V_1, \ldots, V_{N_\Delta}\}$ of Λ_n (because K has this property). Let $i \in \{1, \ldots, N_\Delta\}$ and pick $x_i \in V_i \cap \Lambda_n$ and write $(f^{(n)})^{-1}\{x_i\} = \bigcup_{\alpha \in I_i}\{x_{i,\alpha}\}$ over a finite index set I_i. By Lemma 3.2 we see that to each $x_{i,\alpha}$ there corresponds a pre-image $V_{i,\alpha} = (f^{(n)})^{-1}V_i \cap B_{n-1,\Delta}(x_{i,\alpha})$ (the union over α yields a partition of $(f^{(n)})^{-1}V_i$). Whence, by Lemma 3.3,

$$\text{diam } V_{i,\alpha} \le \frac{2c_n\Delta}{Df^{(n)}(x_{i,\alpha})}.$$

Then

$$\sum_\alpha (\text{diam } V_{i,\alpha})^s \le (2c_n\Delta)^s (L_s^n\chi_{\Lambda_0})(x_i)$$

and consequently

$$\sum_{i,\alpha} (\text{diam } V_{i,\alpha})^s \le [N_\Delta (2c_n\Delta)^s M_n(s)].$$

Let $s > \underline{s}_{\text{crit}}$. Then $\underline{P}(s) < 0$ and there is a sub-sequence n_k, $k \in \mathbb{N}$, for which $m_{n_k}(s)$ and, by Lemma 3.5, also $M_{n_k}(s)$ tend exponentially fast to zero. For that sub-sequence the expression in the square brackets is uniformly bounded in n_k. Since diam $V_{i,\alpha} \le 2c_n\Delta\ \beta^{-n}$ which tends to zero with n the family $\{V_{i,\alpha}\}_{n_k}$ exhibits covers of Λ_0 of arbitrarily small diameter. This implies that $\dim_H(\Lambda)$ does not exceed s nor $\underline{s}_{\text{crit}}$.

($\dim^B\Lambda_0 \le \overline{s}_{\text{crit}}$): For the upper bound on the box dimensions, consider for $0 < r \le \frac{\delta}{\lambda_1(f_1)}$ (with $\delta > 0$ as in (3.15)) and $x \in \Lambda_0$ the ball $U = B(x, r)$. Let $k = k(x, r) \ge 2$ be the smallest integer such that $B_{k-1,\Delta}(x) \subset U$. Note that Δ-homogeneity (3.15) shows that $B(f_1(x), \Delta) \not\subset B(f_1(x), \delta)$. By (T1) and (T2), $B_{0,\Delta}(x) \not\subset B(x, \frac{\delta}{\lambda_1(f_1)})$, so that *a fortiori*, $k \ge 2$. We then have

$$L_s^{(k)}\chi_U \ge L_s^{(k)}\chi_{B_{k-1,\Delta}(x)} \ge \inf_{z \in B_{k-1,\Delta}(x)} \left(Df^{(k)}(z)\right)^{-s} \chi_{B(f^{(k)}(x),\Delta)}.$$

By definition of k there is $y \in B_{k-2,\Delta}(x) \setminus U$, so that in particular, $d(y, x) \ge r$. When $z \in B_{k-1,\Delta}(x)$, Lemma 3.3 shows that

$$\frac{\Delta}{r}\,\frac{\|Df_k\|}{Df^{(k)}(z)} \ge \frac{d(f^{(k-1)}(y), f^{(k-1)}(x))}{d(y,x)\ Df^{(k-1)}(z)} \ge \frac{1}{c_{k-1}}$$

and we deduce that

$$L_s^{(k)}\chi_U \ge r^s (c_{k-1}\Delta\|Df_k\|)^{-s}\chi_{B(f^{(k)}(x),\Delta)}.$$

Iterating another n_0 times we will by hypothesis (T4) cover all of Λ_{k+n_0}. Reasoning as in Section 2.4, we see that

$$L_s^{(k+n_0)}\chi_U \geq (4r)^s \left[(4c_{k-1}\Delta \prod_{j=0}^{n_0} \|Df_{k+j}\|)^{-s} \frac{1}{M_{k+n_0}(s)} \right] L_s^{(k+n_0)}\chi_{\Lambda_0}.$$

If $s > \overline{s}_{\rm crit}$ the sequence, $M_k(s)$, tends to zero exponentially fast (recall that $\underline{P}(s)$ is strictly decreasing at $\overline{s}_{\rm crit}$). The sub-exponential bounds in hypothesis (T5) imply that the factor in the brackets remains uniformly bounded from below. We may proceed to conclude that $\dim^B\Lambda$ does not exceed s, whence not $\overline{s}_{\rm crit}$.

Finally, for the last assertion suppose that $\frac{1}{n}\log m_n(\underline{s}_{\rm crit}) = 0$, i.e. the limit exists and equals zero (cf. the remark below). Lemma 3.5 shows that the lower and upper pressures agree so that $\underline{P}(\underline{s}_{\rm crit}) = \overline{P}(\underline{s}_{\rm crit}) = 0$. Now, both pressure functions are strictly decreasing (because $\beta > 1$). Therefore, $\overline{s}_{\rm crit} = \underline{s}_{\rm crit}$ and the conclusion follows. $\square$

Remark 3.8. A Hölder inequality (for fixed n) shows that $s \mapsto \frac{1}{n}\log M_n$ is convex in s. Convexity is preserved when taking limsup (but in general not when taking liminf) so that $s \mapsto \overline{P}(s)$ is convex. Even if $\frac{1}{n}\log\, M_n(\overline{s}_{\rm crit})$ converges, however, it can happen that $\limsup \frac{1}{n}\log\, M_n(s) = +\infty$ for $s < \overline{s}_{\rm crit}$. In that case convergence of $\frac{1}{n}\log M_n(\overline{s}_{\rm crit})$ could be towards a strictly negative number and $\underline{s}_{\rm crit}$ could turn out to be strictly smaller than $\overline{s}_{\rm crit}$.

4. Random conformal maps and parameter-dependency

The distortion function ε gives rise to a natural metric on $\mathcal{E} \equiv \mathcal{E}(\Delta, \beta, \varepsilon)$. We assume in the following that ε is extended to all of $\mathbb{R}_+$ and is a strictly increasing concave function (or else replace it by an extension of its concave 'hull' and make it strictly increasing). For $f, \widetilde{f} \in \mathcal{E}$ we set $d_{\mathcal{E}}(f, \widetilde{f}) = +\infty$ if there is $y \in K$ for which $\deg(f; y) \neq \deg(\widetilde{f}; y)$. Note that by pairing, $\deg(f; y)$ is locally constant. When the local degrees coincide everywhere we proceed as follows: For $y \in K$, we let Π_y denote the family of bijections, $\pi : f^{-1}y \xrightarrow{1:1} \widetilde{f}^{-1}y$, and for $x \in f^{-1}y$ we set

$$\rho_{\pi,x}(f, \widetilde{f}) = \varepsilon\left(\frac{\beta}{\beta - 1} d(x, \pi(x))\right) + \left|\log \frac{D\widetilde{f} \circ \pi(x)}{Df(x)}\right|. \tag{4.19}$$

The distance between f and $\widetilde{f}$ is then defined as:

$$d_{\mathcal{E}}(f, \widetilde{f}) = \sup_{y\in K} \inf_{\pi\in\Pi_y} \sup_{x\in f^{-1}(y)} \rho_{\pi,x}(f, \widetilde{f}). \tag{4.20}$$

Let f_1, f_2, f_3 be maps at a finite 'distance'. Fixing $y \in K$ we pick corresponding bijections, $\pi_1 : f_1^{-1}y \xrightarrow{1:1} f_2^{-1}y$ and $\pi_2 : f_2^{-1}y \xrightarrow{1:1} f_3^{-1}y$. For $x \in f_1^{-1}y$ our hypotheses on ε imply that $\rho_{\pi_2\circ\pi_1,x}(f_1,f_3) \leq \rho_{\pi_1,x}(f_1,f_2)+\rho_{\pi_2,\pi_1(x)}(f_2,f_3)$ from which we deduce that $d_{\mathcal{E}}$ fulfills a triangular inequality. It is then checked that indeed, $d_{\mathcal{E}}$ defines a metric on $\mathcal{E}$.

Lemma 4.1. *Let $u \leq \Delta$ and $d_{\mathcal{E}}(f,\widetilde{f}) \leq \varepsilon(u)$. Then for all $y,\widetilde{y} \in K$ with $d(y,\widetilde{y}) < u$ there exists a pairing $(x_\alpha,\widetilde{x}_\alpha)_{\alpha\in J}$ (for some finite index set J) of $f^{-1}(y)$ and $\widetilde{f}^{-1}(\widetilde{y})$ for which $\forall\alpha \in J$,*

$$d(x_\alpha,\widetilde{x}_\alpha) < u \quad \text{and} \quad \left|\log \frac{Df(x_\alpha)}{D\widetilde{f}(\widetilde{x}_\alpha)}\right| \leq 2\varepsilon(u).$$

Proof. Let $y \in K$ and choose a bijection $\pi : f^{-1}(y) \xrightarrow{1:1} \widetilde{f}^{-1}(y)$ for which $\rho_{\pi,x}(f,\widetilde{f}) \leq \varepsilon(u)$, $\forall x \in f^{-1}y$. For any fixed $x \in f^{-1}y$ we then have: (a) $\varepsilon\left(\frac{\beta}{\beta-1}d(x,\pi(x))\right) \leq \varepsilon(u)$ and (b) $|\log \frac{D\widetilde{f}(\pi(x))}{Df(x)}| \leq \varepsilon(u)$. As ε is strictly increasing (a) implies $d(x,\pi(x)) \leq (1-\frac{1}{\beta})u$. Since $d(y,\widetilde{y}) < u \leq \Delta$ Lemma 3.2 gives a (unique) pairing $j : \widetilde{f}^{-1}y \xrightarrow{1:1} \widetilde{f}^{-1}\widetilde{y}$ for which $j(x') \in B(x',\delta_{\widetilde{f}}(x'))$ and $d(x',j(x')) \leq d(y,\widetilde{y})/\beta < u/\beta$, $x' \in \widetilde{f}^{-1}y$. We then obtain a pairing $j\circ\pi : f^{-1}y \xrightarrow{1:1} \widetilde{f}^{-1}y \xrightarrow{1:1} \widetilde{f}^{-1}\widetilde{y}$ (in general not unique) for which $d(x,\widetilde{x}) < u(1-\frac{1}{\beta}) + \frac{u}{\beta} = u$, $\widetilde{x} = j\circ\pi(x)$ as wanted. By definition of the distortion we also have $|\log \frac{D\widetilde{f}(x')}{D\widetilde{f}(j(x'))}| \leq \varepsilon(d(y,\widetilde{y})) \leq \varepsilon(u)$. Setting $x' = \pi(x)$, $\widetilde{x} = j(\pi(x))$ and combining this with the bound from (b), we see that the last claim follows. □

Given two sequences, $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$ and $\widetilde{\mathcal{F}} = (\widetilde{f}_n)_{n\in\mathbb{N}}$, in $\mathcal{E}$, we define their distance (with some further caution one could replace sup by lim-sup),

$$d_\infty(\mathcal{F},\widetilde{\mathcal{F}}) = \sup_n d_{\mathcal{E}}(f_n,\widetilde{f}_n).$$

For compact sets, A and B, we write $\mathrm{dist}_H(A,B)$ for their Hausdorff distance.

Proposition 4.2. *When $d_\infty(\mathcal{F},\widetilde{\mathcal{F}}) \leq r = \varepsilon(u) \leq \varepsilon(\Delta)$ then:*

$$\mathrm{dist}_H(\Lambda(\mathcal{F}),\Lambda(\widetilde{\mathcal{F}})) \leq u,$$

$$\left|\underline{P}(s,\mathcal{F}) - \underline{P}(s,\widetilde{\mathcal{F}})\right| \leq 2rs, \quad s \geq 0 \quad \text{and}$$

$$\left(1+\frac{2r}{\log\beta}\right)^{-1} \leq \frac{\underline{s}_{\mathrm{crit}}(\mathcal{F})}{\underline{s}_{\mathrm{crit}}(\widetilde{\mathcal{F}})} \leq 1+\frac{2r}{\log\beta}.$$

(If $\underline{P}(s,\mathcal{F})$ equals $\pm\infty$ then so does $\underline{P}(s,\widetilde{\mathcal{F}})$. If $\underline{s}_{\mathrm{crit}}(\mathcal{F})$ equals $+\infty$ then so does $\underline{s}_{\mathrm{crit}}(\widetilde{\mathcal{F}})$). Now, the upper pressures $\overline{P}$ and upper critical value $\overline{s}_{\mathrm{crit}}$ have the same bounds.

Proof. Let $x \in \Lambda(\mathcal{F})$. By the pairing in Lemma 4.1, the decreasing sequence $A_n = \{\widetilde{x} \in \Omega_n(\widetilde{\mathcal{F}}) : d(f^{(j)}(x),\widetilde{f}^{(j)}(\widetilde{x})) \leq u,\ 0 \leq j \leq n\}$ has a nonempty intersection (which could contain more than one point): $\emptyset \neq \bigcap_{n\geq 0} A_n \subset \Lambda(\widetilde{\mathcal{F}})$. A point in this intersection is at a distance not exceeding u to $x \in \Lambda(\mathcal{F})$. Interchanging the roles of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ we conclude that $\mathrm{dist}_H(\Lambda(\mathcal{F}), \Lambda(\widetilde{\mathcal{F}})) \leq u$.

Given $y \in \Lambda_n(\mathcal{F})$ we may thus find $\widetilde{y} \in \Lambda_n(\widetilde{\mathcal{F}})$ at a distance not exceeding $u \leq \Delta$. We perform a recursive pairing of their pre-images at distances less than u and with $\varepsilon(u) \leq r$. Using Lemma 4.1 for the bounds on the derivatives we obtain

$$\frac{1}{k}\left|\log \frac{L^{(k)}_{s,\mathcal{F}}\mathbf{1}(y)}{L^{(k)}_{s,\widetilde{\mathcal{F}}}\mathbf{1}(\widetilde{y})}\right| \leq 2rs.$$

The second claim then follows by taking suitable limits. For the last claim suppose e.g. that $s_c = \underline{s}_{\mathrm{crit}}(\mathcal{F}) < \widetilde{s}_c = \underline{s}_{\mathrm{crit}}(\widetilde{\mathcal{F}}) < +\infty$ and that $\underline{P}(s_c, \mathcal{F}) = \underline{P}(\widetilde{s}_c, \widetilde{\mathcal{F}}) = 0$. Now, $s \mapsto \underline{P}(s, \widetilde{\mathcal{F}}) + s\log\beta$ is nonincreasing so $(\widetilde{s}_c - s_c)\log\beta \leq \underline{P}(s_c, \widetilde{\mathcal{F}}) - \underline{P}(\widetilde{s}_c, \widetilde{\mathcal{F}}) = \underline{P}(s_c, \widetilde{\mathcal{F}}) - \underline{P}(s_c, \mathcal{F}) \leq 2rs_c$. Thus, $\widetilde{s}_c/s_c \leq 1 + \frac{2r}{\log\beta}$ and the last bound follows. $\square$

Associated to the metric space $(\mathcal{E}, d_{\mathcal{E}})$ there is a corresponding Borel σ-algebra and this allows us to construct measurable maps into $\mathcal{E}$. In the following let (Ω, μ) be a probability space and let $\tau : \Omega \to \Omega$ be a μ-ergodic transformation. We use $\mathbb{E}$ to denote an average with respect to μ.

Definition 4.3. We write $\mathcal{E}_\Omega \equiv \mathcal{E}_\Omega(\Delta, \beta, \varepsilon)$ for the *space of measurable maps*, $\mathbf{f} : \omega \in (\Omega, \mu) \mapsto \mathbf{f}_\omega \in (\mathcal{E}, d_{\mathcal{E}})$, whose image is almost surely separable (i.e. the image of a subset of full measure contains a countable dense set). Following standard conventions we say that the map is *Bochner-measurable*.

We write $\mathcal{F}_\omega = (\mathbf{f}_{\tau^{n-1}\omega})_{n\in\mathbb{N}}$ for the sequence of maps fibered at the orbit of $\omega \in \Omega$. Denote by $\mathbf{f}^{(n)}_\omega = \mathbf{f}_{\tau^{n-1}(\omega)} \circ \cdots \circ \mathbf{f}_\omega$, $n \in \mathbb{N}$ (and $\mathbf{f}^{(0)}_\omega = \mathrm{id}$) the iterated map defined on the domain, $\Omega_n(\mathcal{F}_\omega) = \mathbf{f}^{-1}_\omega \circ \mathbf{f}^{-1}_{\tau(\omega)} \circ \cdots \circ \mathbf{f}^{-1}_{\tau^{n-1}(\omega)}(K)$ (and $\Omega_0(\mathcal{F}_\omega) = K$). The 'random' Julia set is then the compact, nonempty intersection

$$J(\mathbf{f})_\omega \equiv \Lambda(\mathcal{F}_\omega) = \bigcap_{n\geq 0} \Omega_n(\mathcal{F}_\omega). \tag{4.21}$$

Lemma 4.1 implies that $(f_1, \ldots, f_n) \in \mathcal{E}^n \mapsto f_1^{-1} \circ \cdots \circ f_n^{-1}(K) \subset K$ is continuous when K is equipped with the Hausdorff topology for its nonempty subsets. It follows that $\omega \mapsto \Omega_n(\mathcal{F}_\omega)$ is measurable. Uniform contraction implies that $\Omega_n(\mathcal{F}_\omega)$ converge exponentially fast to $\Lambda(\mathcal{F}_\omega)$ in the Hausdorff topology, whence the 'random' conformal repeller, $\Lambda(\mathcal{F}_\omega)$, is measurable for the Hausdorff σ-algebra.

Using the estimates from the previous proposition, the function, $(f_1, \ldots$ $\ldots, f_n) \in \mathcal{E}^n \mapsto M_n(s, (f_1, \ldots, f_n))$, is continuous. Almost sure separability of $\{\mathbf{f}_\omega : \omega \in \Omega\} \subset \mathcal{E}$ implies then that $\omega \mapsto M_n(s, \mathcal{F}_\omega)$ is measurable (with the standard Borel σ-algebra on the reals). For example, if V_1, V_2 are open subsets of $\mathcal{E}$, the pre-image of $V_1 \times V_2$ by $\omega \mapsto (\mathbf{f}_\omega, \mathbf{f}_{\tau\omega})$ is $\mathbf{f}^{-1}(V_1) \cap \tau^{-1}\mathbf{f}^{-1}(V_2)$ which is measurable. The function, $\overline{P}(s, \mathcal{F}_\omega)$, being a lim sup of measurable functions, is then also measurable (and the same is true for m_n and $\underline{P}$). We define the distance between $\mathbf{f}, \widetilde{\mathbf{f}} \in \mathcal{E}_\Omega$ to be

$$d_{\mathcal{E},\Omega}(\mathbf{f}, \widetilde{\mathbf{f}}) = \mu\text{-ess}\sup_{\omega} d_{\mathcal{E}}(\mathbf{f}_\omega, \widetilde{\mathbf{f}}_\omega) \in [0, +\infty]. \tag{4.22}$$

THEOREM 4.4. *Let τ be an ergodic transformation on (Ω, μ) and let $\mathbf{f} = (\mathbf{f}_\omega)_{\omega\in\Omega} \in \mathcal{E}_\Omega$ be Bochner-measurable* (*Definition* 4.3). *We suppose that there is $n_0 < \infty$ such that (a.s.) the sequence $\mathcal{F}_\omega = (\mathbf{f}_{\tau^{n-1}\omega})_{n\in\mathbb{N}}$ is (n_0, Δ)-mixing* (*Condition* (T4) *in Assumption* 3.4).

We suppose also that $\mathbb{E}\log\|D\mathbf{f}\| < +\infty$. (*We say that the family is of* bounded average logarithmic dilation). *Then*

(a) *For any $s \geq 0$ and μ-almost surely, the pressure functions, $\underline{P}(s, \mathcal{F}_\omega)$ and $\overline{P}(s, \mathcal{F}_\omega)$, are independent of ω and equal in value. We write $P(s, \mathbf{f})$ for this almost sure common value. The various dimensions of the random conformal repeller $\Lambda(\mathcal{F}_\omega)$ agree* (a.s.) *in value. Their common value is* (a.s.) *constant and given by*

$$s_c(\mathbf{f}) \equiv \sup\{s \geq 0 : P(s, \mathbf{f}) > 0\} \in [0, +\infty].$$

(b) $s_c(\mathbf{f})$ *is finite if and only if $P(0, \mathbf{f}) < +\infty$* (*this is the case, e.g. if* $\mathbb{E}\log d^o_{\max}(\mathbf{f}) < \infty$) *and*

$$\frac{\mathbb{E}\log d^o_{\min}(\mathbf{f})}{\mathbb{E}\log\|D\mathbf{f}\|} \leq s_c(\mathbf{f}) \leq \frac{\mathbb{E}\log d^o_{\max}(\mathbf{f})}{-\mathbb{E}\log\|1/D\mathbf{f}\|}.$$

(c) *If $P(0, \mathbf{f}) < +\infty$ the map $\mathbf{f} \in (\mathcal{E}_\Omega, d_{\mathcal{E},\Omega}) \mapsto \log s_c(\mathbf{f})$ is $\frac{2}{\log\beta}$-Lipschitz at distances $\leq \varepsilon(\Delta)$.*

Proof. Write $\phi(\omega) = \log\|D\mathbf{f}_\omega\| \geq \log\beta > 0$ and similarly $\phi^{(n)}(\omega) = \log\|D\mathbf{f}^{(n)}_\omega\|$. Then $\phi^{(n)}(\omega) \leq \phi^{(k)}(\omega) + \phi^{(n-k)} \circ \tau^k(\omega),\ \ 0 < k < n$ and since ϕ is integrable we get by Kingman's subergodic theorem [King68] that the limit

$$\log\beta \ \leq\ \lim_n \frac{1}{n}\phi^{(n)}(\omega) \ <\ +\infty$$

exists (and is constant) μ-almost surely. As a consequence,

$$\lim_n \frac{1}{n}\phi \circ \tau^n(\omega) = \lim \frac{n+1}{n}\ \frac{1}{n+1}\phi^{(n+1)}(\omega) - \frac{1}{n}\phi^{(n)}(\omega) = 0$$

μ-almost surely. Thus the sequence of maximal dilations is almost surely subexponential (Condition (T5) of Assumption 3.4). Condition (T4) of that assumption is a.s. verified by the hypotheses stated in our Theorem. It follows by Theorem 3.7 that the Hausdorff dimension of the random repeller $\Lambda(\mathcal{F}_\omega)$

a.s. is given by $\underline{s}_{\rm crit}(\mathcal{F}_\omega)$. In order to prove (a) we must show that (a.s.) the value is constant and equals $\overline{s}_{\rm crit}(\mathcal{F}_\omega)$.

The family $m_n(s,\mathcal{F}_\omega)$ is super-multiplicative, i.e.

$$m_n(s,\mathcal{F}_\omega) \geq m_{n-k}(s,\mathcal{F}_{\tau^k\omega}) m_k(s,\mathcal{F}_\omega),$$

for $0<k<n$ and $\omega \in \Omega$. Writing $\log_+ x = \max\{0, \log x\}$, $x>0$, we have

$$\mathbb{E} \log_+ \frac{1}{m_1(s,\mathcal{F}_\omega)} \leq s\, \mathbb{E} \log \|D\mathbf{f}\|.$$

As the latter quantity is assumed finite we may for s fixed apply Kingman's super-ergodic theorem to m_n, i.e. the sub-ergodic theorem to the sequence $1/m_n$ to deduce that the limit

$$\lim_n \frac{1}{n} \log m_n(s,\mathcal{F}_\omega) \in (-\infty,+\infty]$$

exists (and is constant) μ-almost surely. In view of Lemma 3.5 this implies (a.s.) that $\underline{P}(s,\mathcal{F}_\omega) = \overline{P}(s,\mathcal{F}_\omega) = \mathrm{const}(s)$. We write $P(s,\mathbf{f})$ for this a.s. limit.

From the expression for the operator and for fixed n and $\omega \in \Omega$, we see that the sequence $\|D\mathbf{f}_\omega^{(n)}\|^s m_n(s,\mathcal{F}_\omega)$ is a nondecreasing function of s. The same is then true for

$$s\frac{1}{n}\log \|D\mathbf{f}_\omega^{(n)}\| + \frac{1}{n}\log m_n(s,\mathcal{F}_\omega).$$

Apply now Kingman's sub-ergodic, respectively super-ergodic, theorem to these two terms. We are allowed to do so because the first (a.s.) has a finite limit, bounded by $s\,\mathbb{E}\,\log\|D\mathbf{f}\|$. It follows that

$$s\,\mathbb{E}\,\log\|D\mathbf{f}\| + P(s,\mathbf{f}) \in (-\infty,+\infty]$$

is a nondecreasing function of s. Similarly, we see that

$$s\,\log\beta + P(s,\mathbf{f}) \in (-\infty,+\infty]$$

is nonincreasing. The latter bound shows that $P(s,\mathbf{f})$ is strictly decreasing in s which implies that $s_c(\mathbf{f}) \equiv \overline{s}_{\rm crit} = \underline{s}_{\rm crit} \in [0,+\infty]$. From the two bounds we also obtain the following dichotomy: Either (1) $P(0,\mathbf{f})=\infty$, $P(s,\mathbf{f})$ is infinite for all $s\geq 0$ and $\underline{s}_{\rm crit} = \overline{s}_{\rm crit} = +\infty$, or (2) $P(0,\mathbf{f}) < +\infty$ in which case the function $s\mapsto P(s,\mathbf{f})$ is continuous, strictly decreasing and has a unique zero $\underline{s}_{\rm crit} = \overline{s}_{\rm crit} \in [0,+\infty)$. In either case, Theorem 3.7 shows that the common value (a.s.) equals all of the various dimensions. This proves (a) and also the first part of (b).

We have the following bounds for the action of the transfer operator $L_{s,f}$ upon a positive function $\phi>0$:

$$\frac{d^o_{\min}(f)}{\|Df\|^s}\ \min\phi \leq L_{s,f}\ \phi \leq d^o_{\max}(f)\ \|\frac{1}{Df}\|^s\ \max\phi. \tag{4.23}$$

Here, $d^o_{\max}(f)$ and $d^o_{\min}(f)$ denotes the maximal, respectively, the minimal pointwise degree of the map, f. The estimate in (b) for the dimensions is obtained then by taking averages as above. Finally, (c) is a consequence of Proposition 4.2 and the fact that s_{crit} a.s. equals the dimensions. □

Example 4.5. Let $K = \{\phi \in \ell^2(\mathbb{N}) : \|\phi\| \leq 1\}$ and denote by e_n, $n \in \mathbb{N}$, the canonical basis for $\ell^2(\mathbb{N})$. The domains $D_n = \text{Cl } B(\frac{2}{3}e_n, \frac{1}{6})$, $n \in \mathbb{N}$, maps conformally onto K by $x \mapsto 6(x - \frac{2}{3}e_n)$. For each $n \in \mathbb{N}$ we consider the conformal map f_n of degree n which maps $D_1 \cup \ldots \cup D_n$ onto K by the above mappings. Finally let ν be a probability measure on $\mathbb{N}$. Picking an i.i.d. sequence of the mappings f_n according to the distribution ν we obtain a conformal repeller for which all dimensions almost surely agree. In this case we have equality for the estimates in Theorem 4.4 (b) so that the a.s. common value for the dimensions is given by

$$\frac{\sum_n n\, \nu(n)}{\log 6}.$$

Finiteness of the dimension thus depends on n having finite average or not (cf. also [DT01, Ex. 2.1]).

The Lipschitz continuity of the dimensions with respect to parameters is somewhat misleading because it is with respect to our particular metric on $\mathcal{E}$. In practice, when constructing parametrized families of mappings it is really the modulus of continuity of Df, i.e. the ε-function in $\mathcal{E}(K, \Delta, \varepsilon)$, that comes into play:

Example 4.6. We consider here just the case of one stationary map $f \in \mathcal{E}$. Let T_t, $t \geq 0$, be a Lipschitz motion of (Ω_f, f) in $\mathcal{E}(K, \Delta, \varepsilon)$. By this we mean that $T_t^{-1} : \Omega_f \to K$, $t \geq 0$, is a family of conformal injective mappings with $T_0(x) = x$, $d(x, T_t^{-1}x) \leq b\,t$, $|\log DT_t^{-1}(x)| \leq c\,t$ (for $t \geq 0$) and such that $f \circ T_t : T_t^{-1}\Omega_f \to K$ belongs to $\mathcal{E}(K, \Delta, \varepsilon)$ for $t \geq 0$. One checks that $d_{\mathcal{E}}(f \circ T_t, f) \leq \varepsilon\left(\frac{\beta}{\beta-1} b\,t\right) + c\,t$ (use $\widetilde{f} = f \circ T_t$ and $\pi = T_t^{-1}$ in (4.19)). By Theorem 4.4 (c), the map $t \mapsto d(t) = \dim_H \Lambda(f \circ T_t)$ for t small verifies:

$$|\log \frac{d(t)}{d(0)}| \leq \frac{2}{\log \beta} \left(\varepsilon \left(\frac{\beta}{\beta - 1} b\,t \right) + c\,t \right).$$

When Thermodynamic Formalism applies, in particular when a bit more smoothness is imposed, a similar result could be deduced within the framework (and restrictions) of TF. I am not aware, however, of any results published on this.

5. Part II: Random Julia sets and parameter dependency

Let $U \subset \widehat{\mathbb{C}}$ be an open nonempty connected subset of the Riemann sphere omitting at least three points. We denote by (U, d_U) the set U equipped with a hyperbolic metric d_U. As U will be fixed throughout we will usually write $d = d_U$ for this hyperbolic metric. As normalisation we use $ds = 2|dz|/(1-|z|^2)$ on the unit disk $\mathbb{D}$ and the hereby induced metric for the hyperbolic Riemann surface U (cf. Remark 5.1 below). In particular, for the unit disk and $z \in \mathbb{D}$ we have

$$d_{\mathbb{D}}(0,z) = \log \frac{1+|z|}{1-|z|}, \qquad |z| = \tanh \frac{d_{\mathbb{D}}(0,z)}{2}.$$

We write $B(u,r) \equiv B_U(u,r)$ for the hyperbolic ball of radius $r > 0$ centered at $u \in (U,d)$, $B_{\mathbb{D}}(u,r)$ for the similar hyperbolic ball in $(\mathbb{D}, d_{\mathbb{D}})$, $u \in \mathbb{D}$ and $B_{\mathbb{C}}(u,r) = \{z \in \mathbb{C} : |z-u| < r\}$ for a standard Euclidean ball in $\mathbb{C}$.

Recall that when $K \subset U$ is a compact subset the inclusion mapping $(\mathrm{Int}K, d_{\mathrm{Int}K}) \hookrightarrow (\mathrm{Int}K, d_U)$ is a strict contraction [CG93, Th. 4.2, p.13] by some factor $\beta = \beta(K,U) > 1$ depending on K and U only. We consider the family $\mathcal{E}(K,U)$ of finite degree unramified conformal covering maps

$$f : \mathcal{D}_f \to U$$

for which the domain $\mathcal{D}_f$ is a subset of the compact set K. We may assume without loss of generality that K is the closure of its own interior. Our first goal is to show that such maps *a fortiori* verify conditions (T0)–(T3) from the previous section, in which the set K is the same as here and the metric d on K is the restriction of the hyperbolic metric d_U to K.

Let $\ell = \ell(K,U) > 0$ be the infimum length of closed noncontractible curves (sometimes called essential loops) intersecting K and let $\alpha = \tanh(\ell/4)$ (we set $\ell = +\infty$ and $\alpha = \tanh(+\infty) \equiv 1$ when U is simply connected). We define the constant

$$\Delta = \Delta(K,U) = \log \frac{1+\alpha/10}{1-\alpha/10} \tag{5.24}$$

and for $0 \le r < \ell/2$ the ε-function

$$\varepsilon_\ell(r) = -6 \log \left(1 - \frac{\tanh(r/2)}{\tanh(\ell/4)}\right). \tag{5.25}$$

One has: $\tanh \frac{\Delta}{2} = \frac{\alpha}{10}$, $\Delta < \ell/20$ and $\varepsilon_\ell(\Delta) < 1$.

Remark 5.1. We recall some facts about universal covering maps of Riemann surfaces: Let $\phi : \mathbb{D} \to U$ be a universal conformal covering map of U. For $x, y \in U$ their hyperbolic distance is defined as $d_U(x,y) = \min\{d_{\mathbb{D}}(\widehat{x}, \widehat{y})\}$ where the minimum is taken over lifts $\widehat{x} \in \phi^{-1}\{x\}$ and $\widehat{y} \in \phi^{-1}\{y\}$ of x and y, respectively. If p, $p' \in \phi^{-1}\{y\}$ are two disctinct lifts of a point $y \in K$ then

$d_{\mathbb{D}}(p,p') \geq \ell$. Otherwise the geodesic connecting p and p' projects to a closed noncontractible curve in U intersecting K and of length $< \ell$, contradicting our definition of ℓ. For the same reason, the map $\phi : B_{\mathbb{D}}(p,\ell/2) \to B(y,\ell/2)$ must be a conformal bijection which preserves distances to y; i.e., if $z \in B_{\mathbb{D}}(p,\ell/2)$ then $d_{\mathbb{D}}(z,p) = d_U(\phi(z),y)$. Note, however, that ϕ need not be an isometry on the full disc, since two points in $B(y,\ell/2) \setminus K$ may have lifts closer than their lifts in $B_{\mathbb{D}}(p,\ell/2)$.

We have the following

Lemma 5.2. *Let $f \in \mathcal{E}(K,U)$. Write $\|Df\| = \|Df\|_{f^{-1}K}$ for the maximal conformal derivative of f on the set $f^{-1}K$. Define $\lambda_1(f) = \frac{3}{2}\|Df\|$. Let $x \in \mathcal{D}_f \cap f^{-1}K$ and set*

$$\delta_f(x) = \min\{\log \frac{8\ Df(x) + \alpha}{8\ Df(x) - \alpha}, \Delta\}. \tag{5.26}$$

Let also $\delta(f) = \min\{\log \frac{8\|Df\|+\alpha}{8\|Df\|-\alpha}, \Delta\} > 0$ be the minimum value of $\delta_f(x)$ over the compact set $\mathcal{D}_f \cap f^{-1}K$. Then $B(x,\delta_f(x)) \subset \mathcal{D}_f$ and we have the following properties:

(0) *If $x' \neq x$ is another pre-image of $f(x)$, then $B(x, 2\delta_f(x))$ and $B(x', 2\delta_f(x'))$ are disjoint.*

(1) *f is univalent on the hyperbolic disk $B(x,\delta_f(x))$ and $B(f_x,\Delta) \subset fB(x,\delta_f(x))$.*

(2) *$\beta \leq f[u,x] \leq \lambda_1(f)$ for $u \in B(x,\delta_f(x))$.*

(3) *If $u,v \in B(x,\delta_f(x))$ and $f_u, f_v \in B(f_x, r)$ with $0 < r \leq \Delta$ then*

$$\left| \log \left(\frac{d(f_x,f_u)}{d(x,u)Df(v)} \right) \right| \leq \varepsilon_\ell(r). \tag{5.27}$$

Proof. Let C be a connected component of $\mathcal{D}_f \subset K$ and fix an $x \in C$ for which $y = f(x) \in K \subset U$. Pick universal conformal covering maps, $\phi_x : \mathbb{D} \to U$ and $\phi_y : \mathbb{D} \to U$, for which $\phi_x(0) = x$ and $\phi_y(0) = y$. Let $\widehat{C} = \phi_x^{-1} C \subset \mathbb{D}$ be the lift of the connected component C containing x. The composed map, $f \circ \phi_x : \widehat{C} \to U$ is a conformal covering map of U. Since $\phi_y : \mathbb{D} \to U$ is a universal covering there is a unique (conformal) map, $\psi = \psi_{x,y} : \mathbb{D} \to \widehat{C}$, such that $\psi_{x,y}(0) = 0$ and (cf. Figure 2)

$$f \circ \phi_x \circ \psi_{x,y} \equiv \phi_y : \mathbb{D} \to U.$$

By definition of the hyperbolic metric the conformal derivative of f at x is given by

$$\lambda \equiv Df(x) = 1/|\psi'(0)|.$$

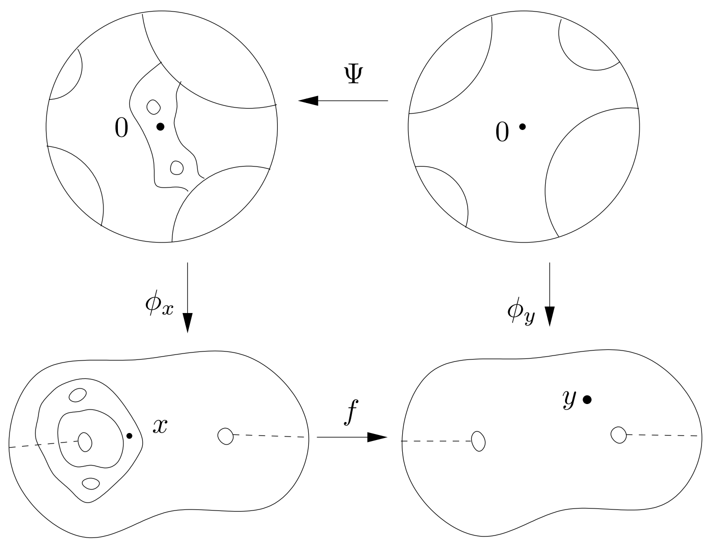

Figure 2: An illustration of a covering map of degree 2 and its 'inverse' in the universal cover. Cuts along the dotted lines become arcs in the lift. One fundamental domain is sketched in each cover.

More generally, if $u = \phi_x(\psi(z)) \in C$, $z \in \mathbb{D}$ then

$$Df(u) = 1/D\psi(z) = \frac{1}{|\psi'(z)|} \frac{1 - |\psi(z)|^2}{1 - |z|^2}.$$

The value does not depend on the choices of covering maps because the conformal line element $ds = 2|dz|/(1 - |z|^2)$ is invariant under conformal automorphisms of the unit disk (both in the source and in the image).

The map $\psi : (\mathbb{D}, d_{\mathbb{D}}) \to (\widehat{C}, d_{\widehat{C}})$ is nonexpanding [CG93, Th. 4.1, p. 12]. As $C \subset K$ the inclusion $(C, d_C) \hookrightarrow (U, d_U)$ is β^{-1}-Lipschitz. In the covering space also $(\widehat{C}, d_{\widehat{C}}) \hookrightarrow (\mathbb{D}, d_{\mathbb{D}})$ is β^{-1}-Lipschitz so the composed map $(\mathbb{D}, d_{\mathbb{D}}) \xrightarrow{\psi} (C, d_C) \hookrightarrow (U, d_U)$ must be β^{-1}-Lipschitz. The map ψ need not, however, be univalent on all of $\mathbb{D}$ because a noncontractible loop in C may be contractible in U (as is the case in Figure 2). On the other hand, the map $\phi_y : B_{\mathbb{D}}(0, \ell/2) \to B(y, \ell/2) = B(f(x), \ell/2)$ is a conformal bijection (Remark 5.1). Writing j_y for the inverse of this map we see that

$$h_x = \phi_x \circ \psi \circ j_y : B(f(x), \ell/2) \to B(x, \ell/(2\beta)) \tag{5.28}$$

defines a local inverse of f. The map ψ is univalent on the disk $B_{\mathbb{D}}(0, \ell/2) = B_{\mathbb{C}}(0, \alpha)$ so that

$$g : t \in \mathbb{D} \mapsto \frac{\psi(t\alpha)}{\alpha\psi'(0)} \in \mathbb{C}$$

is univalent and verifies $g(0)=0$ and $g'(0)=1$. The Koebe distortion theorem [CG93, Th. 1.6, p 3] applied to g shows that if $|z|<\alpha$ (recall that $\lambda = Df(x) = 1/|\psi'(0)|$) then

$$\begin{aligned} \frac{1}{(1+|z|/\alpha)^2} \quad &\leq \quad \left|\frac{\lambda\psi(z)}{z}\right| \quad \leq \quad \frac{1}{(1-|z|/\alpha)^2}, \\ \frac{1-|z|/\alpha}{(1+|z|/\alpha)^3} \quad &\leq \quad \left|\lambda\psi'(z)\right| \quad \leq \quad \frac{1+|z|/\alpha}{(1-|z|/\alpha)^3}. \end{aligned} \tag{5.29}$$

With our definition of $\delta_f(x)$ we have that $B_{\mathbb{D}}(0,\delta_f(x)) \subset B_{\mathbb{C}}(0,\alpha/8\lambda)$. Using the first bound in (5.29) for $|z|=\alpha$ we see that

$$B_{\mathbb{D}}(0,2\delta_f(x)) \subset B_{\mathbb{C}}(0,\alpha/4\lambda) \subset \psi B_{\mathbb{C}}(0,\alpha) = \psi B_{\mathbb{D}}(0,\ell/2). \tag{5.30}$$

Since $\frac{1}{10}\frac{1}{(1-\frac{1}{10})^2} = \frac{10}{9\cdot 9} \leq 1/8$ and $\frac{1}{5}\frac{1}{(1+\frac{1}{5})^2} = \frac{5}{6\cdot 6} \geq 1/8$ we also have:

$$\psi B_{\mathbb{C}}(0,\frac{\alpha}{10}) \subset B_{\mathbb{C}}(0,\frac{\alpha}{8\lambda}) \subset \psi B_{\mathbb{C}}(0,\frac{\alpha}{5}). \tag{5.31}$$

Going back to hyperbolic distances and U, noting that also $\psi B_{\mathbb{C}}(0,\alpha/10) \subset B_{\mathbb{C}}(0,\alpha/10)$ (we need this here because we want $\delta_f(x) \leq \Delta$; cf. Assumption 3.1 above), we obtain

$$h_x B(y,\Delta) \subset B(x,\delta_f(x)) \subset B(x,2\delta_f(x)) \subset h_x B(y,\ell/2) \tag{5.32}$$

with Δ and $\delta_f(x)$ as in (5.24) and (5.26). In particular, $B(x,\delta_f(x)) \subset C \subset \mathcal{D}_f$.

Property (0): Let x' be another pre-image of y distinct from x. Suppose that $B(x,2\delta_f(x)) \cap B(x',2\delta_f(x')) \neq \emptyset$. By the last inclusion in (5.32) we may find a path γ from x to x' whose image $f(\gamma)$ is a loop of length $< \ell/2+\ell/2 = \ell$, contains y and is noncontractible in U. This contradicts the definition of ℓ.

Property (1): The first inclusion in (5.32) implies that $fB(x,\delta_f(x)) \supset B(f(x),\Delta)$. Concerning the local inverse we have $h_x B(y,\ell/2) \supset B(x,\delta_f(x))$ so that the map f is univalent on $B(x,\delta_f(x))$.

Property (2), $f[u,x] \geq \beta$: For

$$u \in B(x,\delta_f(x)) \subset B(x,2\delta_f(x)) \subset h_x B(f(x),\ell/2)$$

we have: $d(u,x) = d(h_x(f(u)),h_x(f(x))) \leq \beta^{-1} d(f(u),f(x))$.

Property (2)$'$, $f[u,x] \leq \lambda_1(f)$: By Schwarz' Lemma, $|\psi(z)| \leq |z|$, $z \in \mathbb{D}$. Also, from the expression $d_{\mathbb{D}}(z,0) = \log\frac{1+|z|}{1-|z|} = 2|z|\int_0^1 \frac{dt}{1-|z|^2t^2}$, it follows that

$$1 \leq \frac{d_{\mathbb{D}}(z,0)}{d_{\mathbb{D}}(\psi(z),0)}\frac{|\psi(z)|}{|z|} \leq \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Now, let $|z| < \alpha$ and set $u = \phi_x(\psi(z))$. Using also the first bound in (5.29) we obtain

$$(1 - |z|/\alpha)^2 \le \frac{f[u,x]}{Df(x)} = \frac{d_{\mathbb{D}}(z,0)}{d_{\mathbb{D}}(\psi(z),0)\lambda} \le \frac{(1+|z|/\alpha)^2}{1-|z|^2}. \tag{5.33}$$

When $u \in B(x, \delta_f(x))$ then $|\psi(z)| \le \frac{\alpha}{8\lambda}$ and by (5.31) we must have $|z| \le \alpha/5 \le 1/5$. Thus,

$$f[x,u] \le \frac{(1+1/5)^2}{1-(1/5)^2}\, Df(x) = \frac{3}{2} Df(x) \le \lambda_1(f), \quad u \in B(x, \delta_f(x)).$$

Property (3): Let $|z|, |w| \le r < \alpha$ and set $u = \phi_x(\psi(z))$, $v = \phi_x(\psi(w))$. By (5.29), the second inequality,

$$\frac{1 - r/\alpha}{(1+r/\alpha)^3}(1-r^2) \le \frac{Df(x)}{Df(v)} = \lambda |\psi'(w)| \frac{1-|w|^2}{1-|\psi(w)|^2} \le \frac{1+r/\alpha}{(1-r/\alpha)^3}.$$

Multiplying this and the inequality in (5.33) we obtain

$$\left| \log \frac{f[u,x]}{Df(v)} \right| \le \log \frac{(1+r/\alpha)^3}{(1-r/\alpha)^3(1-r^2)^2} \le 6 \log \frac{1}{1-r/\alpha},$$

i.e. (5.27) with the ε_ℓ function as defined in (5.25). □

The above lemma implies that conditions (T0)–(T3) of the previous section are verified for our class of maps $\mathcal{E}(K,U)$ when setting $\Omega_f = \mathcal{D}_f \cap f^{-1}K$ and looking at the metric space (K, d_U), the ε-function ε_ℓ and finally β, Δ and $\delta_f(x)$ as defined above. We may also proceed as in Section 4 and consider $\mathcal{E}(K,U)$ as a metric space, whence also as a Borel measurable space.

THEOREM 5.3. *Let τ be an ergodic transformation on (Ω, μ). Let $(\mathbf{f}_\omega)_{\omega\in\Omega} \in \mathcal{E}_\Omega(K,U)$ be a measurable family satisfying $\mathbb{E}(\log \|D\mathbf{f}_\omega\|) < +\infty$. Then μ-almost surely the various dimensions of the random Julia set $J(\mathbf{f})_\omega$, equation* (4.21), *agree and are given as the unique zero $s_c(\mathbf{f})$ of the pressure function $P(s,\mathbf{f})$ from Theorem* 4.4 (a).

Proof. We will apply Theorem 4.4. The assumption of *bounded average logarithmic dilation* is included in our hypothesis. We need to show that (n_0, Δ)-mixing holds for some n_0. This follows, however, directly from connectivity of U and the properties of our conformal maps. The diameter of K is finite within U. Given two points y and z in K, choose a geodesic between them of length not greater than the diameter of K. By taking preimages we obtain paths of exponentially shrinking lengths. It suffices to take n_0 such that diam $K/\beta^{n_0} \le \Delta$ to assure that (T4) of Assumption 3.4 holds. An area estimate for $f \in \mathcal{E}(K,U)$ yields: $d^o(f)\text{Area}(K) = \int_{f^{-1}K} |Df|^2 d\text{Area} \le \|Df\|^2 \text{Area}(K)$, whence the degree of f verifies:

$$d^o(f) \le \|Df\|^2. \tag{5.34}$$

Therefore, $\mathbb{E}(\log d^o(\mathbf{f}_\omega)) < +\infty$ and we may apply Theorem 4.4 (b) to obtain the desired conclusion. For $\phi \geq 0$ we also have by change of variables,

$$\int_K L_{s=2}\phi \, dA = \int_{f^{-1}K} \phi \, dA \leq \int_K \phi \, dA,$$

which incidently shows that $\overline{s}_{\rm crit} \leq 2$ (as it should be !). $\square$

6. Mirror embedding and real-analyticity of the Hausdorff dimension

The dependence of the Hausdorff dimension on parameters may be studied through the dependence of the pressure function on those parameters. A complication arises, namely that our transfer operators do not depend analytically on the expanding map. In [Rue82], Ruelle circumvented this problem in the case of a (nonrandom) hyperbolic Julia set by instead looking at an associated dynamical zeta-function. Urbański and Zdunik [UZ04] considered a holomorphic motion and constructed a family of operators that depends analytically on parameters, through a conjugation with the holomorphic motion. Here, we tackle the problem differently and introduce a *mirror embedding* : We embed our function space into a larger space and semi-conjugate our family of transfer operators to operators with an explicit real-analytic dependency on parameters and mappings. We establish a Perron-Frobenius theorem through the contraction of cones of 'real-analytic' functions. The pressure function may then be calculated as the averaged action of the operator on a hyperbolic fixed point (cf. [Rue79], [Rue97]) which has the wanted dependence on parameters. Finally as the pressure function cuts the horisontal axis transversally the result will follow from another implicit function theorem.

6.1. *Mirror extension and mirror embedding.* Possibly after conjugation with a Möbius transformation we may assume that U is a hyperbolic subset of $\mathbb{C}$, i.e. that U does not contain the point at infinity as well as (at least) two other points. The compact subset $K \subset U$ is then a bounded subset of the complex plane. We write $\overline{U} = \{\overline{z} : z \in U\}$ for the complex conjugated domain (not the closure) and we define the mirror extension of U as the product $\widehat{U} = U \times \overline{U} \subset \mathbb{C}^2$. Given two points $\xi = (\xi_1, \xi_2)$ and $\zeta = (\zeta_1, \zeta_2)$ in $\widehat{U}$ we define their $\widehat{U}$-distance to be $d_{\widehat{U}}(\xi, \zeta) = \max\{d_U(\xi_1, \zeta_1), d_{\overline{U}}(\xi_2, \zeta_2)\}$. The map $z \in U \mapsto (z, \overline{z}) \in \widehat{U}$ is then an isometric embedding of U onto the mirror diagonal,

$$\operatorname{diag} U = \{(z, \overline{z}) : z \in U\} \subset \widehat{U}.$$

The 'exchange-conjugation'

$$c(u, v) = (\overline{v}, \overline{u}), \quad (u, v) \in U \times \overline{U}$$

defines an involution on the mirror extension leaving invariant the mirror diagonal. Let $X \subset \widehat{U}$ be an open subset. We call X mirror symmetric if and only if $c(X) = X$. We say that X is connected to the diagonal if any connected component of X has a nonempty intersection with diag U. We write $A(X) = C^0(\mathrm{Cl}\ X) \cap C^\omega(X)$ for the space of holomorphic functions on X, having a continuous extension to Cl X.

LEMMA 6.1. *Let $X \subset \widehat{U}$ be an open, mirror symmetric subset, connected to the diagonal and let $A = A(X)$. Then*

(1) *A is a unital Banach algebra* (*in the sup-norm*) *with a complex involution*,

$$\phi^*(u,v) = \overline{\phi(\overline{v},\overline{u})} \equiv \overline{\phi}(v,u), \quad (u,v) \in X,\ \phi \in A.$$

(2) *Denote $A_{\mathbb{R}} = \{\phi \in A : \phi^* = \phi\}$, the space of self-adjoint elements in A. Such functions are real-valued on the mirror diagonal and $A = A_{\mathbb{R}} \oplus iA_{\mathbb{R}}$.*

(3) *A function $\phi \in A$ is uniquely determined by its restriction to* (diag U) $\cap X$. (*For this*, *mirror-symmetry of X is not needed*).

Proof. (1) and (2) are clear. Suppose now that ϕ vanishes on the mirror diagonal. Because any point in X is path-connected to the diagonal it suffices to show that ϕ vanishes on an open neighborhood of a diagonal point $(y,\overline{y}) \in X \cap \mathrm{diag}\ U$. For u, v small enough we have a convergent power series expansion,

$$\phi(y+u, \overline{y}+v) = \sum_{k,l \geq 0} a_{k,l} u^k v^l.$$

Setting $u = r\,e^{i\theta}$, $v = \overline{u}$, we obtain for r small enough:

$$0 = \phi(y+u, y+\overline{u}) = \sum_{m \geq 0} r^m \sum_{k=0}^{m} a_{k,m-k} e^{i(2k-m)\theta},$$

which vanishes if and only if $a_{k,l} = 0$ for all $k, l \geq 0$. □

Let $G(K)$ denote the geodesic closure of K. This set is constructed as follows: For $x, u \in K$ denote by $\{z \in \gamma \mid \gamma : x \to u\}$ the set of points belonging to the shortest geodesic(s) connecting x with u in U. Then $G(K) = \mathrm{Cl} \bigcup \{z \in \gamma \mid \gamma : x \to u,\ x, u \in K\}$. With the constant Δ as in (5.24) we write $K_\Delta \subset U$ for the open Δ neighborhood of the compact set K and G_Δ for the closed Δ-neighborhood of $G(K)$. The set G_Δ is contained in the diam K_Δ-neighborhood of K, whence it is compact and by construction connected in (U, d_U). Since $\infty \notin U$, G_Δ is also a compact subset of U in $(\mathbb{C}, d_{\mathbb{C}})$ so there is a constant $r_c > 0$ such that

$$B_{\mathbb{C}}(y, r_c) \subset U, \quad \forall y \in G_\Delta. \tag{6.35}$$

The set G_Δ is compact both for the hyperbolic and the Euclidean metric so that these two metrics must be smoothly (in fact, real-analytically) equivalent when restricted to G_Δ. Writing $ds = \mu_U(z)|dz|$ for the hyperbolic metric it follows that there is $C < \infty$ for which

$$\frac{1}{C} \le \mu_U(z) \le C, \quad \forall z \in G_\Delta. \tag{6.36}$$

The constants r_c and C will be used frequently in the following.

6.2. *Mirror extended transfer operators and cone contractions.* Let $\widehat{K}_\Delta$ denote the open Δ-neighborhood of diag K in $(\widehat{U}, d_{\widehat{U}})$. Let $f \in \mathcal{E}(K,U)$ and let $\widehat{f} = (f, \overline{f})$ be the mirror extended map. For $\widehat{y} = (y, \overline{y}) \in \text{diag}\,(K)$, we write for its mirror-preimages:

$$P_{\widehat{f}}(\widehat{y}) \equiv \widehat{f}^{-1}\{\widehat{y}\} \cap \text{diag } K \equiv \{(x_i, \overline{x}_i)\}_{i \in J}$$

where $f^{-1}\{y\} = \{x_i\}_{i\in J}$ for some index set J. We wish to define an analytic continuation of this ensemble to preimages of points in Cl $\widehat{K}_\Delta$. Given $\xi \in$ Cl $\widehat{K}_\Delta$ we pick a point $\widehat{y} \in$ diag K Δ-close to ξ, i.e. such that $d_{\widehat{U}}(\widehat{y}, \xi) = \max\{d(y,\xi_1), d(\overline{y},\xi_2)\} \le \Delta$. For each $i \in J$, $x_i \in f^{-1}\{y\}$, we write $h_{x_i} : B(y, \ell/2) \to B(x_i, \ell/2\beta)$ for the local inverse of f as defined in (5.28). We write $\overline{h}_{\overline{x}_i}(z) = \overline{h_{x_i}(\overline{z})}$ for the conjugated map and claim that

$$P_{\widehat{f}}(\xi) \equiv \left\{\left(h_{x_i}(\xi_1), \overline{h}_{\overline{x}_i}(\xi_2)\right)\right\}_{i\in J} \subset \widehat{K}_{\Delta/\beta}$$

yields the desired analytic continuation and that this continuation is unique (up to a permutation of J). Local analyticity is clear. To see that it is well-defined (and unique), suppose that $\widehat{w} = (w, \overline{w}) \in$ diag K is another point for which $d(\widehat{w}, \xi) \le \Delta$. Then $d(y,w) \le d(y,\xi_1) + d(w,\xi_1) \le 2\Delta < \ell/10$ so that $B(w,\Delta) \subset B(y, \ell/2)$. Setting $z_i = h_{x_i}(w)$ we see that

$$(h_{x_i})_{|B(w,\Delta)} = (h_{z_i})_{|B(w,\Delta)}. \tag{6.37}$$

But then $h_{z_i}(\xi_1) = h_{x_i}(\xi_1)$ and $\overline{h}_{\overline{z}_i}(\xi_2) = \overline{h}_{\overline{x}_i}(\xi_2)$. Interchanging the roles of y and w we see that $P_{\widehat{f}}(\xi)$ is independent of the choices made (up to a permutation of J). In the following we will denote by

$$D_{\widehat{f}} = \{v \in P_{\widehat{f}}(\xi) : \xi \in \widehat{K}_\Delta\} \tag{6.38}$$

the set of all pre-images of points in $\widehat{K}_\Delta$ obtained this way. We have the inclusions,

$$\widehat{f}^{-1}\text{diag } K \cap \text{diag } K \subset D_{\widehat{f}} \subset \widehat{f}^{-1}\widehat{K}_\Delta \cap \widehat{K}_{\Delta/\beta}.$$

By construction, $D_{\widehat{f}}$ is connected to diag K. It turns out to be convenient to define a metric on the subset Cl $\widehat{K}_\Delta \subset \widehat{U}$, better tailored to suit our purposes than $d_{\widehat{U}}$.

Definition 6.2. For $\xi, \zeta \in \mathrm{Cl}\, \widehat{K}_\Delta$,

$$d_\Delta(\xi,\zeta) = \inf\{d_{\widehat{U}}(\xi,\widehat{y}) + d_{\widehat{U}}(\zeta,\widehat{w}) + d_{\widehat{U}}(\widehat{y},\widehat{w})\} \tag{6.39}$$

where the infimum is taken over all $\widehat{y}, \widehat{w} \in \mathrm{diag}\, K$ for which $d_{\widehat{U}}(\xi,\widehat{y}) \le \Delta$ and $d_{\widehat{U}}(\zeta,\widehat{w}) \le \Delta$.

Remark 6.3. Thus, when measuring the distance $d_\Delta(\xi,\zeta)$ between points that are off-diagonal, we first find Δ-close points on the diagonal and then consider the total length of the concatenated path $\gamma : \xi \to \widehat{y} \to \widehat{w} \to \zeta$. Compactness of diag K ensures that the infimum is actually realized for some such admissible path γ. Note that a shortest geodesic from $\widehat{y}$ to $\widehat{w}$ may be realized by $(\alpha,\overline{\alpha})$ where α is a shortest geodesic from y to w. In particular, and this is our motivation for making the above definition, the path γ may be chosen so that it stays within $\widehat{G}_\Delta$, the closed Δ-neighborhood of diag $G(K)$ (the mirror diagonal of the geodesic closure of K).

Lemma 6.4. *Define* $g_f(x,\overline{x}) = \log|f'(x)|,\ x \in f^{-1}K$. *Then*

(1) g_f *has a unique holomorphic extension*, $g_{\widehat{f}} \in A(D_{\widehat{f}})$.

(2) *For* $\xi, \zeta \in \mathrm{Cl}\, \widehat{K}_\Delta$, *let* γ *be a minimal admissible path as in the previous remark. Let* $v \in P_{\widehat{f}}(\xi)$ *and* $w \in P_{\widehat{f}}(\zeta)$ *be connected by an* $\widehat{f}$*-preimage of* γ. *Then there exists the following estimate* (*with the constants* r_c *and* C *from* (6.35) *and* (6.36)):

$$\left| g_{\widehat{f}}(v) - g_{\widehat{f}}(u) \right| \le \frac{4C}{r_c}\ d_\Delta(\xi,\zeta).$$

Proof. (1) We define $g_{\widehat{f}}$ locally as $(v_1,v_2) \in \mathrm{Cl}\, D_{\widehat{f}} \mapsto \frac{1}{2}\log(f'(v_1)\overline{f'}(v_2))$ for a suitable choice of logarithmic branch. Analyticity, and by Lemma 6.1 then also uniqueness, follow if we can show how to choose the branch globally. First, take $\widehat{y} = (y,\overline{y}) \in \mathrm{diag}\, K$, $x_i \in f^{-1}\{y\}$, $i \in J$ and let $h_{x_i} : B(y,\ell/2) \to B(x_i,\ell/2\beta)$ be a local inverse of f as above. We define

$$g_{x_i} : (z_1,z_2) \in B(y,\ell/2) \times B(\overline{y},\ell/2) \mapsto g_{x_1}(z_1,z_2) = -\frac{1}{2}\log h'_{x_i}(z_1)\overline{h'_{\overline{x}_i}}(z_2)$$

with the normalization $g_{x_i}(z,\overline{z}) = -\log|h'_{x_i}(z)|$, $z \in B(y,\ell/2)$. This is well-defined because of simply connectedness of the product and because h_{x_i} has no critical points.

For $\xi = (\xi_1,\xi_2) \in \mathrm{Cl}\, \widehat{K}_\Delta$, let $\widehat{y} = (y,\overline{y}) \in \mathrm{diag}\, K$ be Δ-close to ξ. We claim that $g_{x_i}(\xi_1,\xi_2)$ is independent of the choices made (i.e., y and x_i). Suppose that $\widehat{w} = (w,\overline{w}) \in \mathrm{diag}\, K$ is another point, Δ-close to ξ, and set $z_i = h_{x_i}(w)$. Since $d(y,w) \le 2\Delta < \ell/2 - \Delta$ we have

$$(h_{x_i})_{|B(w,\Delta)} = (h_{z_i})_{|B(w,\Delta)}$$

so that $h'_{x_i}(u)\overline{h}'_{\overline{x}_i}(\overline{u}) = h'_{z_i}(u)\overline{h}'_{\overline{z}_i}(\overline{u}) > 0$ for all $u \in B(w, \Delta)$. By our choice of branches their logarithms then agree (are real-valued) and for $(\xi_1, \xi_2) \in B(w, \Delta) \times B(\overline{w}, \Delta)$ we then have $\log h'_{x_i}(\xi_1)\overline{h}'_{\overline{x}_i}(\xi_2) = \log h'_{z_i}(\xi_1)\overline{h}'_{\overline{z}_i}(\xi_2) \in \mathbb{C}$, as claimed. (In fact, our proof shows that g_f extends to a holomorphic function on a neighborhood of Cl $D_{\widehat{f}}$).

(2) Given a point, (z_1, z_2), on the path γ there is a local inverse h with complex conjugate $\overline{h}$ defined in a neighborhood of z_1 and z_2, respectively. Calculating the differential we have

$$-2\, dg_{\widehat{f}}(z_1, z_2) = \frac{h''(z_1)}{h'(z_1} \, dz_1 + \frac{\overline{h}''(z_2)}{\overline{h}'(z_2} \, dz_2.$$

As $z_1, \overline{z}_2 \in G_\Delta$ we see by (6.35) that h extends to a univalent function on $B(z_1, r_c)$ so the fundamental Köebe estimate shows that $|h''(z_1)|/|h'(z_1)| \leq \frac{4}{r_c}$. The same holds for $\overline{h}$ at z_2 so that by integrating the above bound along γ and using (6.36) we get

$$|g_{\widehat{f}}(v) - g_{\widehat{f}}(w)| \leq \frac{2}{r_c} \int_{\gamma:\xi\to\zeta} (|dz_1| + |dz_2|) \leq \frac{4C}{r_c} \, d_\Delta(\xi, \zeta). \qquad \square$$

Recall that $A(\widehat{K}_\Delta) = C^0(\text{Cl } \widehat{K}_\Delta) \cap C^\omega(\widehat{K}_\Delta)$ denotes the space of holomorphic functions on $\widehat{K}_\Delta$ having a continuous extension to the boundary. We define for $s \in \mathbb{C}$ the (bounded and linear) operator, $\mathcal{L}_{s,\widehat{f}} : A(\widehat{K}_\Delta) \to A(\widehat{K}_\Delta)$ (with $g_{\widehat{f}}$ from Lemma 6.4):

$$\text{(6.40)} \qquad \mathcal{L}_{s,\widehat{f}}\, \phi(\xi) = \sum_{v \in P_{\widehat{f}}(\xi)} e^{-s g_{\widehat{f}}(v)} \phi(v), \quad \phi \in A(\widehat{K}_\Delta), \xi \in \text{Cl } \widehat{K}_\Delta.$$

Remark 6.5. When restricted to a diagonal point $\widehat{y} = (y, \overline{y}) \in \text{diag } K$ we have $\mathcal{L}_{s,\widehat{f}}\, \phi(y, \overline{y}) = \sum_{x \in f^{-1}(y)} |f'(x)|^{-s} \phi(x, \overline{x})$ (cf. Lemma 6.4). This restriced operator thus acts in the same way as our usual transfer operator (3.18), $L_{s,f}$, in the nonanalytic setting except that we use here the Euclidean derivate $|f'(x)|$ instead of the hyperbolic derivative $Df(x)$. Because of (6.36) this is, however, irrelevant for the spectral properties, whence the way that we determine the Hausdorff dimension (the two operators are conjugated through a multiplication by $\mu_U(z)$ which is smooth and bounded from above and below on G_Δ).

For the moment let us fix a real value of $s \geq 0$. Then $\mathcal{L}_{s,\widehat{f}}$ preserves $A_\mathbb{R}(\widehat{K}_\Delta)$, the space of self-adjoint elements. We define for $\sigma > 0$ a closed proper convex cone:

$$\text{(6.41)} \quad \mathcal{C}_\sigma = \{\phi \in A_\mathbb{R}(\widehat{K}_\Delta) : |\phi(\xi) - \phi(\widehat{\eta})| \leq \phi(\widehat{\eta})(e^{\sigma d_\Delta(\xi, \widehat{\eta})} - 1),$$
$$\xi \in \text{Cl } \widehat{K}_\Delta, \ \widehat{\eta} \in \text{diag } K\}.$$

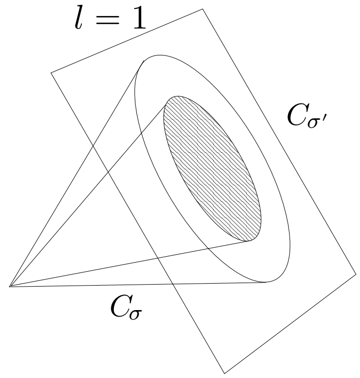

Figure 3: The cone contraction. The sliced cone $\mathcal{C}_{\sigma',\ell=1}$ has an R-neighborhood which is contained in $\mathcal{C}_\sigma$.

Given $\phi_1, \phi_2 \in \mathcal{C}_\sigma - \{0\}$, we define $\beta(\phi_1, \phi_2) = \inf\{\lambda > 0 : \lambda\phi_1 - \phi_2 \in \mathcal{C}_\sigma\}$ and write $d_\mathcal{C} = \log(\beta(\phi_1, \phi_2)\beta(\phi_2, \phi_1)) \in [0, +\infty]$ for the corresponding projective Hilbert metric (cf. [Bir67], [Liv95], [Rugh02]). Fix a point $\widehat{y}_0 = (y_0, \overline{y}_0) \in$ diag (K) and denote by $\ell \in A(\widehat{K}_\Delta)'$ the (real-analytic) linear functional

$$\ell(\phi) = \phi(\widehat{y}_0), \quad \phi \in A(\widehat{K}_\Delta).$$

We use this to introduce the *sliced cone*

$$\mathcal{C}_{\sigma,\ell=1} \equiv \{\phi \in \mathcal{C}_\sigma : \ell(\phi) = 1\}.$$

Lemma 6.6 (Cone contraction). *Let $s \geq 0$ and choose $\sigma = \sigma(s) > 0$ large enough so that*

$$\sigma' = \sigma'(s) \equiv \frac{4C}{r_c} s + \frac{1}{\beta}\sigma < \sigma.$$

Then there is $\theta = \theta(K, U, \sigma, s) < 1$ such that for every $f \in \mathcal{E}(K, U)$ the operator $\mathcal{L} = \mathcal{L}_{s,\widehat{f}}$ maps $\mathcal{C}_\sigma$ into $\mathcal{C}_{\sigma'}$ and is a θ-Lipschitz contraction for the Hilbert metric $d_{\mathcal{C}_\sigma}$. Furthermore,

(a) *There is $k = k(K, U, \sigma) > 0$ such that $\forall \phi \in \mathcal{C}_\sigma$: $\ell(\phi) \geq k\|\phi\|$*

(b) *There is $R = R(K, U, \sigma, s) > 0$ such that $\forall \phi \in \mathcal{C}_{\sigma',\ell=1}$: $B(\phi, R) \subset \mathcal{C}_\sigma$.*

Proof. Fix $\xi = (\xi_1, \xi_2) \in \mathrm{Cl}\, \widehat{K}_\Delta$, $\widehat{\eta} = (\eta, \overline{\eta}) \in$ diag (K) and let $\gamma = (\gamma_1, \gamma_2)$ be an admissible path as in Remark 6.3 that minimizes $d \equiv d_\Delta(\xi, \widehat{\eta})$ and stays within $\widehat{G}_\Delta$. Taking pre-images by $\widehat{f}$ of the path γ we obtain a pairing ($i \in J$) of $v_i \in P_{\widehat{f}}(\xi)$ and $u_i \in P_{\widehat{f}}(\widehat{\eta}) \subset$ diag K (and also with $x_i \in P_{\widehat{f}}(\widehat{y})$). We then have

$$d_\Delta(u_i, v_i) \leq \beta^{-1} d \quad \text{and} \quad d_\Delta(x_i, v_i) \leq \beta^{-1}\Delta \tag{6.42}$$

because of contraction of the local inverse maps and because a path pre-image is again admissible in the sense of Remark 6.3. By Lemma 6.4, we have $|g_{\widehat{f}}(u_i) - g_{\widehat{f}}(v_i)| \leq \frac{4C}{r_c} d$. Combining this with the first bound in (6.42) and the definition of the cone we obtain:

$$\begin{aligned}
|\mathcal{L}\phi(\xi) - \mathcal{L}\phi(\widehat{\eta})| &\leq \sum_{i\in J} \left| e^{-sg_{\widehat{f}}(v_i)}\phi(v_i) - e^{-sg_{\widehat{f}}(u_i)}\phi(u_i) \right| \\
&\leq \sum_{i\in J} |e^{-sg_{\widehat{f}}(v_i)}|\ |\phi(v_i) - \phi(u_i)| + |e^{-sg_{\widehat{f}}(v_i)} - e^{-sg_{\widehat{f}}(u_i)}|\ \phi(u_i) \\
&\leq (e^{\frac{4C}{r_c}sd}(e^{\beta^{-1}\sigma d} - 1) + (e^{\frac{4C}{r_c}sd} - 1))\mathcal{L}\phi(\widehat{\eta}) \\
&\leq (e^{\sigma' d} - 1)\mathcal{L}\phi(\widehat{\eta}),
\end{aligned}$$

with $\sigma' = \frac{4C}{r_c} s + \beta^{-1}\sigma$. Thus, $\mathcal{L} : \mathcal{C}_\sigma \to \mathcal{C}_{\sigma'}$ and we get for the projective diameter (for this standard calculation we refer to e.g. [Liv95] or [Rugh02, App. A, eq. (A.148)]):

$$\text{(6.43)} \qquad \operatorname{diam}_{\mathcal{C}_\sigma} \mathcal{C}_{\sigma'} \leq \rho \equiv 2 \log \frac{\sigma + \sigma'}{\sigma - \sigma'} + \sigma' D_\Delta < \infty,$$

with $D_\Delta = \operatorname{diam} K + \Delta$ being an upper bound for the d_Δ distance of points $\xi \in \widehat{K}_\Delta$ and $\widehat{\eta} \in \operatorname{diag} K$. By Birkhoff's theorem (see [Bir67], [Liv95] or [Rugh02, Lemma A.4]), this implies a uniform contraction for the hyperbolic metric on $\mathcal{C}_\sigma$. Writing $\theta = \tanh(\rho/4) < 1$ we have for $\phi_1, \phi_2 \in \mathcal{C}_\sigma$:

$$d_{\mathcal{C}}(\mathcal{L}\phi_1, \mathcal{L}\phi_2) \leq \theta\ d_{\mathcal{C}}(\phi_1, \phi_2).$$

Property (a) is clear from the definition of the cone which shows that

$$|\phi(\xi)| \leq \phi(\widehat{y}_0)\ e^{\sigma d_\Delta(\xi, \widehat{y}_0)} \leq \ell(\phi)\ e^{\sigma D_\Delta} \equiv \frac{1}{k}\ell(\phi), \quad \phi \in \mathcal{C}_\sigma,\ \xi \in \widehat{K}_\Delta,$$

with $k = \exp(-\sigma D_\Delta) > 0$. To see (b) we set $\kappa = \frac{2}{\tanh(\Delta/2)}$ and let $\phi \in A(\widehat{K}_\Delta)$. We claim that for $\widehat{\eta} = (\eta, \overline{\eta}) \in \operatorname{diag}(K)$ and $\xi \in \operatorname{Cl} \widehat{K}_\Delta$:

$$\text{(6.44)} \qquad |\phi(\xi) - \phi(\widehat{\eta})| \leq \|\phi\|\, \kappa\, d_\Delta(\widehat{\eta}, \xi).$$

For $d_\Delta(\xi, \widehat{\eta}) \geq \Delta/2$ this inequality is clear; so assume that $0 < d \equiv d_\Delta(\xi, \widehat{\eta}) < \Delta/2$. Let $p : \mathbb{D} \to U$ be a universal covering map with $p(0) = \eta$, $p(z_1) = \xi_1$ and $p(z_2) = \overline{\xi_2}$. Denoting $B = B_{\mathbb{D}}(0, \Delta) = B_{\mathbb{C}}(0, \tanh \frac{\Delta}{2})$ and $\widehat{B} = B \times B$, we see that $\phi \circ \widehat{p}$ is analytic on $\widehat{B}$. Then also $t \in \mathbb{D} \mapsto a(t) \equiv \phi \circ \widehat{p}(\frac{t}{d} \tanh \frac{\Delta}{2}\ \xi)$ is analytic so that by Schwarz' lemma, $|a(t) - a(0)| \leq 2|t|\ \|\phi\|$. Setting $t = \frac{d}{\tanh(\Delta/2)}$ we obtain:

$$|\phi(\xi) - \phi(\widehat{\eta})| \leq \frac{2\|\phi\|}{\tanh(\Delta/2)}\, d_\Delta(\xi, \widehat{\eta}).$$

Consider $h \in \mathcal{C}_{\sigma',\ell=1}$ and $\phi \in A_{\mathbb{R}}(\widehat{K}_\Delta)$. In order for $h+\phi$ to belong to $\mathcal{C}_\sigma$ we need that

$$\left|\frac{h(\xi)+\phi(\xi)}{h(\widehat{\eta})+\phi(\widehat{\eta})}-1\right| \leq \exp(\sigma d_\Delta(\xi,\widehat{\eta}))-1$$

is verified for all $\xi \in \mathrm{Cl}\ \widehat{K}_\Delta$ and $\widehat{\eta} \in \mathrm{diag}\ (K)$. With $d = d_\Delta(\xi,\widehat{\eta})$ this is the case provided

$$|h(\xi)-h(\widehat{\eta})| + |\phi(\xi)-\phi(\widehat{\eta})| \leq (h(\widehat{\eta}) - \|\phi\|)(e^{\sigma d}-1).$$

Using that $h \in \mathcal{C}_{\sigma'}$, $\ell(h) = 1 \leq \frac{1}{k}\ h(\widehat{\eta})$ and the above distortion estimate (6.44) for ϕ we obtain as a sufficient condition that for all $d > 0$,

$$\|\phi\| \leq k\,\frac{e^{\sigma d}-e^{\sigma' d}}{\kappa d + e^{\sigma d}-1}.$$

As $d \to 0$ the right-hand side tends to $k(\sigma-\sigma')/(\kappa+\sigma) > 0$ and in the $d \to \infty$ limit it tends to $k > 0$. It follows that it has a minimum $R > 0$ and we have shown property (b). □

Consider now a sequence $\mathcal{L}_1, \mathcal{L}_2, \ldots$ of operators as in the above lemma. We write $\mathcal{L}^{(n)} = \mathcal{L}_n \circ \mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_1$ for the n'th iterated operator.

Lemma 6.7. *There are constants $c_1, c_2 < \infty$ (and $\theta < 1$ from Lemma 6.6) such that for $h, h' \in \mathbb{C}_{\sigma',\ell=1}$, $\phi \in A(\widehat{K}_\Delta)$ and $n \geq 0$:*

(1) $$\left\|\frac{\mathcal{L}^{(n)}h}{\ell(\mathcal{L}^{(n)}h)} - \frac{\mathcal{L}^{(n)}h'}{\ell(\mathcal{L}^{(n)}h')}\right\| \leq c_1\theta^n,$$

(2) $$\left\|\frac{\mathcal{L}^{(n)}\phi}{\ell(\mathcal{L}^{(n)}h)} - \frac{\mathcal{L}^{(n)}h}{\ell(\mathcal{L}^{(n)}h)}\ \frac{\ell(\mathcal{L}^{(n)}\phi)}{\ell(\mathcal{L}^{(n)}h)}\right\| \leq c_2\theta^n\|\phi\|.$$

Proof. Outer regularity, i.e. Property (a) of the above lemma, and a computation[4] show that (with k from Lemma 6.6 (a)):

$$\|\phi_1-\phi_2\| \leq \frac{2}{k}(e^{d(\phi_1,\phi_2)}-1), \quad \forall\ \phi_1,\phi_2 \in \mathcal{C}_{\sigma,\ell=1}.$$

When $\phi_1, \phi_2 \in \mathcal{L}^{(n)}\mathcal{C}_{\sigma'}$, $\ell(\phi_1) = \ell(\phi_2) = 1$ and $n \geq 0$ we have that $d_{\mathcal{C}_\sigma}(\phi_1,\phi_2) \leq \theta^n\rho$, with ρ from eq. (6.43). Therefore,

$$\|\phi_1-\phi_2\| \leq \frac{2}{k}(e^{\theta^n\rho}-1)$$

which is smaller that $c_1\theta^n$ for a suitable choice of c_1. This yields the first bound.

[4]See e.g. [Rugh02, App. A, Lemma A.3] for this computation.

For the second bound note that $B(h,R) \subset \mathcal{C}_\sigma$. For $\phi \in A_{\mathbb{R}}(\widehat{K}_\Delta)$ (small), one has $d_{\mathcal{C}}(h+\phi,h) \leq \frac{2}{R}\|\phi\| + o(\|\phi\|)$ and therefore

$$\left\| \frac{\mathcal{L}^{(n)}(h+\phi)}{\ell(\mathcal{L}^{(n)}(h+\phi))} - \frac{\mathcal{L}^{(n)}(h)}{\ell(\mathcal{L}^{(n)}(h))} \right\| \leq \frac{2}{k}\theta^n \frac{2}{R}\|\phi\| + o(\|\phi\|)$$

By linearizing this bound (and losing a factor of at most $\sqrt{2}$) we may extend this bound to any complex $\phi \in A(\widehat{K}_\Delta)$ to obtain the second inequality with $c_2 = \sqrt{2}\frac{4}{kR}$. □

6.3. *Analytic conformal families and mirror extensions.* Let $\mathcal{O}_{\mathbb{C}} \subset \mathbb{C}^n$ be an open nonempty ball, invariant under complex conjugation and denote by $\mathcal{O}_{\mathbb{R}} = \mathcal{O}_{\mathbb{C}} \cap \mathbb{R}^n$ its real (nonempty) section. Recall that $U \subset \mathbb{C}$ so that $\mathrm{Cl}\ K_\Delta \subset U$ is a compact subset of $\mathbb{C}$. As in the previous section we will work with the Euclidean derivative, $f' = \frac{df}{dz}$, of f rather than the hyperbolic derivative Df.

Definition 6.8. Let $t \in \mathcal{O}_{\mathbb{C}} \subset \mathbb{C}^n \to f_t \in \mathcal{E}(K,U)$ be a continuous map.

(1) $(f_t)_{t\in\mathcal{O}_{\mathbb{C}}}$ is an *analytic family* if $\{(t,z) : t \in \mathcal{O}_{\mathbb{C}}, z \in \mathcal{D}_{f_t}\} \mapsto f_t(z) \in \mathbb{C}$ is analytic.

We say that the family f_t verifies an *L-Lipschitz condition* $(0 < L < +\infty)$ if for any $z \in K_\Delta$ and any local inverse $f_t^{-1}(z)$ (see Remarks 6.9 and 6.10 below) the map $t \in \mathcal{O}_{\mathbb{C}} \mapsto \log \frac{\partial f_t}{\partial z} \circ f_t^{-1}(z) \in \mathbb{C}$ is L-Lipschitz (for some, whence any, local choice of logarithmic branch).

(2) We define the *condition number* of $f \in \mathcal{E}(K,U)$ to be

$$\Gamma(f) = \|f'\|_{f^{-1}K}\, \|1/f'\|_{f^{-1}K}.$$

It is no lack of generality to assume that the parameters are one-dimensionial $(n = 1)$. We may also assume that $\mathcal{O}_{\mathbb{C}} = \mathbb{D}$, i.e. is the unit-disk so that $\mathcal{O}_{\mathbb{R}} = \mathbb{D} \cap \mathbb{R} =]-1,1[$ is its real section. In the following let $t \in \mathbb{D} \mapsto f_t \in \mathcal{E}(K,U)$ be an analytic family, verifying an L-Lipschitz condition.

We obtain a conjugated analytic family if we set $\mathcal{D}_{\overline{f}_t} \equiv \overline{(\mathcal{D}_{f_{\bar{t}}})} \subset \overline{U}$ and for $x' \in \mathcal{D}_{\overline{f}_t}$, $\overline{f}_t(x') \equiv \overline{f_{\bar{t}}(\overline{x}')}$. Then $\overline{f}_t(x')$ is analytic in t and x' on the domain $\{(t,z) : t \in \mathbb{D}, z \in \mathcal{D}_{\overline{f}_t}\}$. We also define for $t \in \mathbb{D}$ the product map $F_t : (x,x') \in \mathcal{D}_{f_t} \times \mathcal{D}_{\overline{f}_t} \mapsto (f_t(x), \overline{f}_t(x')) \in U \times \overline{U}$. Again, this map is analytic in x, x' and t on its domain of definition. Note that when $t \in \mathbb{D} \cap \mathbb{R}$ then $F_t = \widehat{f}_t = (f_t, \overline{f}_t)$.

Let $y \in \mathrm{Cl}\ K_\Delta$ and fix $x_0 \in f_0^{-1}\{y\}$. Let h_{0,x_0} denote a local inverse of f_0, equation (5.28), for which $h_{0,x_0}(y) = x_0$. By the implicit function theorem we may find an analytic continuation $x_t \in f_t^{-1}\{y\}$ along any path in $\mathbb{D}$ emanating

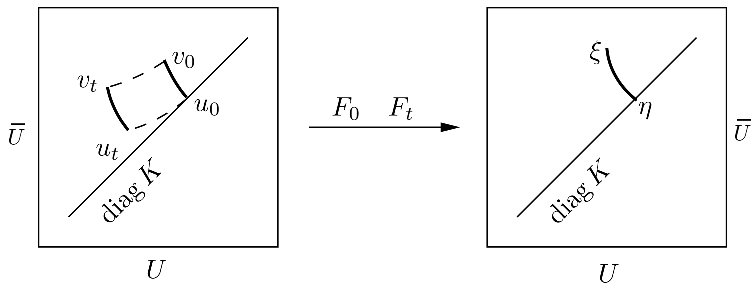

Figure 4: An inverse $v_t \in F_t^{-1}(\xi)$ obtained by lift and analytic continuation.

from $t = 0$. Simple connectness of $\mathbb{D}$ and of $B(y, \ell/2)$ show that h_{0,x_0} extends to an analytic map

$$t \in \mathbb{D},\ z \in B(y, \ell/2) \mapsto h_{t,x_0}(z) \in f_t^{-1}\{z\} \subset U. \tag{6.45}$$

In particular, for fixed $z \in B(y, \ell/2)$, the map $t \in (\mathbb{D}, d_{\mathbb{D}}) \mapsto z_t = h_{t,x_0}(z) \in (U, d) \equiv (U, d_U)$ is a contraction, i.e.

$$d(h_{t,x_0}(z), z) = d(z_t, z_0) \leq d_{\mathbb{D}}(t, 0), \quad z \in B(y, \ell/2). \tag{6.46}$$

Similarly, we see that the map

$$t \in \mathbb{D},\ w \in B(\overline{y}, \ell/2) \mapsto \overline{h}_{t,\overline{x}_0}(w) \in \overline{f}_t^{-1}\{w\} \subset U \tag{6.47}$$

is well-defined, analytic and a contraction with respect to t.

Remark 6.9. We may now give a more precise statement of the Lipschitz condition in 6.8 (2): If $h_{t,x_0}(z)$ is a local inverse of f as in (6.45) then (again for any branch of log)

$$\left| \frac{\partial}{\partial t} \log \frac{\partial h_{t,x_0}(z)}{\partial z} \right| \leq L, \quad \forall t \in \mathbb{D},\ z \in B(f_0(x_0), \ell/2). \tag{6.48}$$

Remark 6.10. Suppose that $\phi \in C^\omega(\mathbb{D})$ with $\phi(0) = 1$ and $-L \leq \log|\phi(t)| \leq L$, $t \in \mathbb{D}$. If we choose a logarithmic branch so that $\log(\phi(0)) = 0$ then If we choose a logarithmic branch so that $\log(\phi(0)) = 0$ then

$$|\log \phi(t)| \leq \frac{2L}{\pi} \log\ \frac{1 + |t|}{1 - |t|}.$$

(Apply Schwarz' inequality to the map $t \in \mathbb{D} \mapsto \tan \frac{\pi \log \phi(t)}{4L} \in \mathbb{D}$). As a consequence, a Lipschitz condition on $\log f'$, $\log|f'|$ or $\log(Df)$ are all equivalent if we shrink the domain in t and allow for a larger Lipschitz constant L (cf. Remark 6.5).

Let $\xi \in \mathrm{Cl}\,\widehat{K}_\Delta$ be Δ-close to $\widehat{y}=(y,\overline{y}) \in \mathrm{diag}\,K$. As above we associate to each $x_0^i \in f_0^{-1}(y)$, $i \in J$ local inverses, h_{t,x_0^i} and $\overline{h}_{t,\overline{x_0^i}}$. Then

$$P_{F_t}(\xi) = \{v_t^i\}_{i\in J} \equiv \{(h_{t,x_0^i}(\xi_1), \overline{h}_{t,\overline{x_0^i}}(\xi_2)\}_{i\in J}, \quad t \in \mathbb{D} \tag{6.49}$$

yields a unique analytic continuation (a holomorphic motion) of the pre-images in $P_{\widehat{f}_0}(\xi)$ (see Figure 4). Using (6.46) as well as its conjugated version we see for any $i \in J$ that

$$d_{\widehat{U}}(v_t^i, v_0^i) \le d_{\mathbb{D}}(t,0). \tag{6.50}$$

For t real the second bound in (6.42) implies that $v_t^i \in \mathrm{Cl}\,\widehat{K}_{\Delta/\beta}$. When making t complex we want to remain within $\widehat{K}_\Delta$ and by the above it suffices to have $\Delta/\beta + d_{\mathbb{D}}(t,0) < \Delta$ or, equivalently,

$$|t| < \tanh\left(\frac{\Delta}{2}(1-\frac{1}{\beta})\right). \tag{6.51}$$

When this condition is fulfilled we may analytically continue the transfer operator in t. For $t \in \mathbb{D}$ the domain of F_t is $D_{F_t} = \bigcup\{P_{F_t}(\xi) : \xi \in \widehat{K}_\Delta\}$. We note that $g_{\widehat{f}_0}$ from Lemma 6.4 extends uniquely (same proof) to an analytic function g_{F_t} on $\{(z,t) : z \in D_{F_t}, t \in \mathbb{D}\}$, with a continuous extension to the domain $\{(z,t) : z \in \mathrm{Cl}\,D_{F_t}, t \in \mathbb{D}\}$ (locally this is defined as $-\frac{1}{2}\log(h'_{t,x_0}(\xi_1)\overline{h}'_{t,\overline{x}_0}(\xi_2))$ for a suitable choice of x_0 and logarithmic branch). For $s \in \mathbb{C}$ and t verifying (6.51) we define an operator $\mathcal{L}_{s,F_t} \in L(A(\widehat{K}_\Delta))$ by setting:

$$\mathcal{L}_{s,F_t}\phi(\xi) = \sum_{v \in P_{F_t}(\xi)} e^{-s\,g_{F_t}(v)}\phi(v), \quad \phi \in A(\widehat{K}_\Delta), \quad \xi \in \mathrm{Cl}\,\widehat{K}_\Delta. \tag{6.52}$$

LEMMA 6.11. *Let $h \in \mathcal{C}_{\sigma'}$. Choose $x_0 \in f_0^{-1}K$ and set $\lambda = \log|f'(x_0)| > 0$. Let $\xi \in \mathrm{Cl}\,\widehat{K}_\Delta$, $\widehat{\eta} \in \mathrm{diag}\,K$ and let γ be a minimizing path for $d_\Delta(\xi,\widehat{\eta})$* (*cf. Remark* 6.3). *For $d_{\mathbb{D}}(t,0) < \Delta(1-\frac{1}{\beta})$, let $v_t \in P_{F_t}(\xi)$ and $u_t \in P_{F_t}(\widehat{\eta})$ be pre-images* (*analytic in t*) *that join a pre-image by F_t^{-1} of the path γ and for which $u_0 \in \mathrm{diag}\,K$. Then*

$$\left|\frac{h(v_t)e^{-s(g_{F_t}(v_t)-\lambda)}}{h(u_0)e^{-s_0(g_{F_0}(u_0)-\lambda)}} - 1\right| \le e^{q(\xi,\widehat{\eta},s,t)} - 1$$

with

$$q(\xi,\widehat{\eta},s,t) = |s-s_0|\log\Gamma(f_0) + \left(\sigma + |s|\frac{L}{2}\right)d_{\mathbb{D}}(t,0) + \left(\frac{\sigma}{\beta} + |s|\frac{4C}{r_c}\right)d_\Delta(\xi,\widehat{\eta}).$$

Proof. The defining equation for the cone, equation (6.41), shows that $h \in \mathcal{C}_{\sigma'}$ verifies

$$\left|\frac{h(v_t)}{h(u_0)} - 1\right| \le e^{\sigma d_\Delta(v_t,u_0)} - 1.$$

The distance in the exponent may be bounded as follows, cf. (6.50) and (6.42):

$$d_\Delta(v_t, u_0) \le d_\Delta(v_t, v_0) + d_\Delta(v_0, u_0) \le d_{\mathbb{D}}(t, 0) + \frac{1}{\beta} d_\Delta(\xi, \widehat{\eta}).$$

By Lemma 6.4,

$$|g_{F_0}(v_0) - g_{F_0}(u_0)| \le \frac{4C}{r_c}\, d_\Delta(\xi, \widehat{\eta}).$$

A small modification of the proof of that lemma, with a t-derivative rather than a z-derivative and by (6.48), also shows that

$$\begin{aligned} |g_{F_t}(v_t) - g_{F_0}(v_0)| &\le \frac{1}{2}\int_0^t \left| \frac{\partial}{\partial t} \log \left(\frac{\partial h_{t,x(t)}}{\partial z}(\xi_1) \frac{\partial \overline{h}_{t,\overline{x}(t)}}{\partial z}(\xi_2) \right) \right| |dt| \\ &\le L\,|t| \le \frac{L}{2}\, d_{\mathbb{D}}(t, 0), \end{aligned} \tag{6.53}$$

with $h_{t,x(t)}$ being a suitable inverse of f_t. By definition 6.8(3) of the condition number for f' and writing $u_0 = (z_0, \overline{z}_0) \in \operatorname{diag} K$ we also obtain

$$|g_{F_0}(u_0) - \lambda| = \big|\log|f'(z_0)| - \log|f'(x_0)|\big| \le \log \Gamma(f).$$

The inequality

$$\left|\prod e^{a_i} - 1\right| \le e^{\sum |a_i|} - 1$$

is valid for any finite set of complex numbers, $a_1, \ldots, a_n$. Now, insert the four estimates above to obtain the claimed inequality. □

We define

$$\pi_{s,F_t}(\phi) = \frac{\mathcal{L}_{s,F_t}\phi}{\ell(\mathcal{L}_{s,F_t}\phi)}, \quad \phi \in A(\widehat{K}_\Delta), \tag{6.54}$$

for all ϕ for which the denominator does not vanish.

LEMMA 6.12. *Let $f_t \in \mathcal{E}(K, U)$, $t \in \mathbb{D}$ verify an L-Lipschitz condition. Choose $x_0 \in f_0^{-1}K$ and set $\lambda = \log|f'(x_0)| > 0$. For $s_0 \ge 0$ we let W^{s_0} denote the open neighborhood of $(s_0, 0) \subset \mathbb{C} \times \mathbb{C}$ consisting of all (s, t) that verify*

$$|s - s_0| \log \Gamma(f_0) + \left(\sigma + |s|\frac{L}{2}\right) d_{\mathbb{D}}(t, 0) < \log\frac{4}{3} \text{ and } d_{\mathbb{D}}(t, 0) < \Delta\left(1 - \frac{1}{\beta}\right).$$

Define $\rho(s) = \frac{k}{4}\exp(-\sigma'(|s|) D_\Delta) > 0$, with k and $\sigma'(|s|) = \frac{\sigma}{\beta} + \frac{4C}{r_c}|s|$ from Lemma 6.6 and $D_\Delta = \operatorname{diam} K + \Delta$. Let $h \in \mathcal{C}_{\sigma',\ell=1}$ and denote $B(r) = \{\phi \in A(\widehat{K}_\Delta) : \|\phi\| \le r\}$.

Then for $(s, t) \in W^{s_0}$ and $\phi, \phi_1 \in B(\rho(s))$:

$$1 \le \|\pi_{s,F_t}(h + \phi)\| \le \frac{1}{k\,\rho(s)} + 1, \tag{6.55}$$

$$\left| \frac{\ell(e^{s\lambda}\mathcal{L}_{s,F_t}(h + \phi))}{\ell(e^{s_0\lambda}\mathcal{L}_{s_0,F_0}(h))} - 1 \right| \le \frac{2}{3}. \tag{6.56}$$

and

$$\left|\log \frac{\ell(\mathcal{L}_{s,F_t}(h+\phi))}{\ell(\mathcal{L}_{s,F_t}(h+\phi_1))}\right| \leq \frac{\|\phi-\phi_1\|}{\rho(s)}. \tag{6.57}$$

Furthermore, the map, $(s,t,\phi) \in W^{s_0} \times B(\rho(s)) \mapsto \pi_{s,F_t}(h+\phi)$ *is real-analytic. Real-analyticity means it is analytic (in norm) and that for* $(t,s) \in W^{s_0} \cap \mathbb{R}^2$ *and* $\phi \in A_{\mathbb{R}}(\widehat{K}_\Delta) \cap B(\rho(s))$, *i.e. a selfadjoint element in* $B(\rho(s))$, *the image is also in* $A_{\mathbb{R}}(\widehat{K}_\Delta)$.

Proof. Let $\eta \in \operatorname{diag} K$. We first use our previous lemma for $\xi = \eta$. We let $q = q(\eta,\eta,s,t)$ and u_t, $v_t (= u_t)$ be as in that lemma. Our assumptions imply that $e^q - 1 < \frac{1}{3}$ and therefore

$$\left|h(u_t)e^{-s(g_{F_t}(u_t)-\lambda)} - h(u_0)e^{-s_0(g_{F_0}(u_0)-\lambda)}\right| \leq \frac{1}{3}h(u_0)e^{-s_0(g_{F_0}(u_0)-\lambda)}.$$

Summing this inequality over all pairs of pre-images and then dividing by the right hand side we obtain

$$\left|\frac{\ell(e^{s\lambda}\mathcal{L}_{s,F_t}h)}{\ell(e^{s_0\lambda}\mathcal{L}_{s_0,F_0}h)} - 1\right| \leq \frac{1}{3}. \tag{6.58}$$

In particular, $|\ell(e^{s\lambda}\mathcal{L}_{s,F_t}h)| \geq \frac{2}{3}e^{s_0\lambda}\ell(\mathcal{L}_{s_0,F_0}h)$, for all $(s,t) \in W^{s_0}$. Now, for general $\xi \in \operatorname{Cl} \widehat{K}_\Delta$ and when $(s,t) \in W^{s_0}$ we have that $e^q = e^{q(\xi,\eta,s,t)} \leq \frac{4}{3}\exp(\sigma'(|s|)D_\Delta) = \frac{k}{3\rho(s)}$. Thus,

$$|e^{-s(g_{F_t}(v_t)-\lambda)}| \leq \frac{k}{3\rho(s)}e^{-s_0(g_{F_0}(u_0)-\lambda)} \leq \frac{1}{3\rho(s)}e^{-s_0(g_{F_0}(u_0)-\lambda)}h(u_0).$$

Summing over pre-images, we obtain

$$\|e^{s\lambda}\mathcal{L}_{s,F_t}\| \leq \frac{1}{3\rho(s)}\ \ell(e^{s_0\lambda}\mathcal{L}_{s_0,F_0}h). \tag{6.59}$$

When $\|\phi\| \leq \rho(s)$,

$$\frac{|\ell(e^{s\lambda}\mathcal{L}_{s,F_t}\phi)|}{\ell(e^{s_0\lambda}\mathcal{L}_{s_0,F_0}h)} \leq \frac{1}{3}. \tag{6.60}$$

Since also $\|h\| \leq \frac{1}{k}$ we obtain the upper bound in (6.55):

$$\|\pi_{s,F_t}(h+\phi)\| = \frac{\|e^{s\lambda}\mathcal{L}_{s,F_t}(h+\phi)\|}{|\ell(e^{s\lambda}\mathcal{L}_{s,F_t}(h+\phi)|} \leq \frac{\frac{1}{3\rho(s)}\|h+\phi\|}{\frac{2}{3}-\frac{1}{3}} \leq \frac{1}{k\rho(s)} + 1.$$

The lower bound is clear. The bound in (6.56) follows from (6.58) and (6.60). From (6.58) and (6.59) we also obtain

$$\left|\frac{\ell\mathcal{L}_{s,F_t}\phi)}{\ell(\mathcal{L}_{s,F_t}h)}\right| \leq \frac{\|\phi\|}{3\rho(s)}\frac{3}{2} = \frac{\|\phi\|}{2\rho(s)} \leq \frac{1}{2}.$$

Using the inequality $|\log \frac{1+a}{1+b}| \leq 2|a-b|$, valid when $|a|,|b| \leq \frac{1}{2}$, we obtain (6.57). For the real-analyticity of our map, π, note that $\mathcal{L}_{s,F_t}(h+\phi)$ from equation (6.52) is real-analytic in the sense described in the lemma, because $(\mathcal{L}_{s,F_{t,\omega}}(h+\phi))^* = \mathcal{L}_{\overline{s},\overline{F}_{t,\omega}}(h+(\phi)^*)$, $\phi \in A(\widehat{K}_\Delta)$. The linear form $\ell : A(\widehat{K}_\Delta) \to \mathbb{C}$ is real-valued on $A_{\mathbb{R}}(\widehat{K}_\Delta)$ and by (6.56) uniformly bounded away from zero when acting upon $\mathcal{L}_{s,F_t}(h+\phi)$, $(s,t) \in W^{s_0}$, $\phi \in B(\rho(s))$. Therefore, $\pi_{s,F_t}(h+\phi)$ is real-analytic on the stated domain. □

6.4. *Analytic measurable sections.* Let us now return to the probability space (Ω, μ) and an invertible[5] μ-ergodic transformation $\tau : \Omega \to \Omega$.

We view the space $\Omega \times A(\widehat{K}_\Delta)$ as a (trivial) fiber bundle over Ω with each fiber being $A(\widehat{K}_\Delta)$. We denote by $\mathcal{A}$ the set of essentially bounded measurable sections of this fiber bundle and write $\|\Phi\|$ for the μ-essential sup of an element $\Phi \in \mathcal{A}$. Since $A(\widehat{K}_\Delta)$ is separable, measurability and Bochner-measurability are here the same. Also $\mathcal{A}$ is a unital Banach algebra when we define the analytic operations to be performed fiber-wise. To see this note that measurability is preserved under such operations and also by taking uniform limits. We write $\mathcal{A}_{\mathbb{R}}$ for the subspace of real-analytic sections. Let $\mathcal{C}_{\sigma,\ell=1}(\Omega)$ denote the space of 'sliced' measurable sliced cone-sections of $\Omega \times \mathcal{C}_{\sigma,\ell=1}$. This space is bounded in $\mathcal{A}_{\mathbb{R}}$. We define $\mathcal{O}_{\mathbb{C}} \subset \mathbb{C}^n$ and $\mathcal{O}_{\mathbb{R}} = \mathcal{O}_{\mathbb{C}} \cap \mathbb{R}^n$ as in the previous section and recall that $U \subset \mathbb{C}$ so that $\widehat{K}_\Delta$ is a compact subset of the complex plane.

ASSUMPTION 6.13. *Let $t \in \mathcal{O}_{\mathbb{C}} \to (\mathbf{f}_{t,\omega})_{\omega\in\Omega} \in \mathcal{E}_\Omega(K,U)$ be a continous map such that*:

1. *For each $\omega \in \Omega$ the map $t \in \mathcal{O}_{\mathbb{C}} \to \mathbf{f}_{t,\omega} \in \mathcal{E}(K,U)$ is analytic in the sense of Definition* 6.8. (*Note that we are implicitly assuming that for each fixed $t \in \mathcal{O}_{\mathbb{C}}$ the mapping $\omega \in \Omega \mapsto \mathbf{f}_{t,\omega} \in \mathcal{E}(K,U)$ is measurable as in Definition* 4.3).
2. *For $\omega \in \Omega$ the map $t \in \mathbb{D} \mapsto \mathbf{f}_{t,\omega}$ verifies an L-Lipschitz condition as in Definition* 6.8 *and for the same number* $0 < L < \infty$.
3. *The condition numbers $\Gamma(\mathbf{f}_{0,\omega})$, $\omega \in \Omega$ are uniformly bounded by some $\Gamma < +\infty$.*
4. $\mathbb{E}\,(\log \|\mathbf{f}'_{0,\omega}\|_{\mathbf{f}^{-1}_{0,\omega}K}) < +\infty$ (*cf. also Remark* 6.5).

As before, it is no restriction to assume that the parameter space is one-dimensional and that $\mathcal{O}_{\mathbb{C}} = \mathbb{D}$. In the following we will thus consider an analytic family $t \in \mathbb{D} \mapsto (\mathbf{f}_{t,\omega})_{\omega\in\Omega} \in \mathcal{E}_\Omega(K,U)$ verifying Assumption 6.13

[5]Invertibility of τ may be avoided. We have imposed it here in order to simplify the proofs.

above. Let $F_{t,\omega} = (\mathbf{f}_{t,\omega}, \overline{\mathbf{f}}_{t,\omega})$ denote the (holomorphic) mirror-extension of $\mathbf{f}_{t,\omega}$ and let $s_0 = \dim_H(J(\mathbf{f}_{0,\cdot})) \in [0,2]$ be the (a.s.) Hausdorff dimension of the random Julia set at $t = 0$ (Theorem 5.3). We choose $\sigma = \sigma(s_0)$ and $\sigma' = \sigma'(s_0)$ so as to verify the Cone contraction conditions in Lemma 6.6. Let $W^{s_0} \subset \mathbb{C}^2$ and $s \mapsto \rho(s)$ be chosen as in Lemma 6.12 and let $\mathbf{h} \in \mathcal{C}_{\sigma',\ell=1}(\Omega)$. By the very same lemma we obtain for $(s,t) \in W^{s_0}$ that

$$\pi_{s,t}(\Phi)_\omega \equiv \pi_{s,F_{t,\omega}}(\Phi_{\tau^{-1}\omega}) = \frac{\mathcal{L}_{s,F_{t,\omega}}\Phi_{\tau^{-1}\omega}}{\ell(\mathcal{L}_{s,F_{t,\omega}}\Phi_{\tau^{-1}\omega})}, \quad \Phi \in B(\mathbf{h}, \rho(s)),\ \omega \in \Omega$$

defines a map $\pi_{s,t} : B(\mathbf{h}, \rho(s)) \to \mathcal{A}$ whose image is bounded in norm by $\frac{1}{k\rho(s)} + 1$. It takes the value of Φ at the shifted fiber $\tau^{-1}\omega$, acts with the transfer operator, normalises according to ℓ and assigns it to the fiber at ω. Note that we use τ^{-1} here because we iterate by composing with the operator to the left. Measurability of the image is a consequence of the map $(s, F_t) \mapsto \mathcal{L}_{s,F_t}$ being continous and ℓ being nonzero on the image. We write $\pi^{(n)}_{s_0,0} : \mathcal{C}_{\sigma',\ell=1}(\Omega) \to \mathcal{C}_{\sigma',\ell=1}(\Omega)$ for the n'th iterated map of $\pi_{s_0,0}$ restricted to the sliced cone-section.

The reader may note that the (nonnormalised) family $(\mathcal{L}_{s,F_{t,\omega}})_{\omega\in\Omega}$ need not be uniformly norm-bounded, whence need not even define a bounded linear operator when acting upon sections of $\mathcal{A}$. This is the case e.g. for our example in the introduction.

LEMMA 6.14. *There are constants, $c_1, c_2 < +\infty$ such that*

(1) *For* $\mathbf{h}, \mathbf{h}' \in \mathcal{C}_{\sigma',\ell=1}(\Omega)$

$$\|\pi^{(n)}_{s_0,0}(\mathbf{h}) - \pi^{(n)}_{s_0,0}(\mathbf{h}')\| \le c_1\, \theta^n.$$

(2) *Taking the derivative in $\mathcal{A}$ of $\pi^{(n)}_{s_0,0}$ at the point $\mathbf{h} \in \mathcal{C}_{\sigma',\ell=1}(\Omega)$ gives*

$$\|D_{\mathbf{h}}\pi^{(n)}_{s_0,0}(\mathbf{h})\| \le c_2\, \theta^n.$$

(3) *For $\mathbf{h} \in \mathcal{C}_{\sigma',\ell=1}(\Omega)$ the map*

$$(s,t) \in W^{s_0},\ \Phi \in B(\mathbf{h}, \rho(s)) \mapsto \pi_{s,t}(\Phi) \in \mathcal{A}$$

is real-analytic.

Proof. (1) and (2) are reformulations of the bounds already given in Lemma 6.7 (with the constants from that lemma). By the last part of Lemma 6.12, each $\pi_{s,t}(\Phi)_\omega$ is real-analytic in $(s,t) \in W^{s_0}$ and $\Phi \in B(\mathbf{h}, \rho(s))$ (for fixed ω). The bound (6.55) shows that $\pi_{s,t}(\Phi)_\omega$ is uniformly bounded in $\omega \in \Omega$. A Cauchy formula (with $r > 0$ small enough)

$$t \mapsto \left(\oint_{|t'-t|=r} \frac{\pi_{s,t'}(\Phi)_\omega}{t'-t} \frac{dt'}{2\pi i} \right)_{\omega\in\Omega},$$

then enables us to recover a power series in the t-variable (similarly for s and Φ) within $\mathcal{A}$. The map is real-analytic in the sense that it maps $(s,t) \in W^{s_0} \cap \mathbb{R}^2$, $\Phi \in B(\mathbf{h}, \rho(s)) \cap \mathcal{A}_{\mathbb{R}}$ into $\mathcal{A}_{\mathbb{R}}$. □

First, we consider the real case fixing for the moment $(s,t) \in W^{s_0} \cap \mathbb{R}^2$ and letting $\mathbf{h}^{(0)} \equiv \mathbf{1} \in \mathcal{A}_{\mathbb{R}}$ be the unit section of our bundle. We define recursively iterates $\mathbf{h}^{(n+1)}_{s,t} = \pi_{s,t}(\mathbf{h}^{(n)}_{s,t}) \in \mathcal{C}_{\sigma',\ell=1}(\Omega)$, $n \geq 0$. Lemma 6.14 shows that $\|\mathbf{h}^{(n+m)}_{s,t} - \mathbf{h}^{(n)}_{s,t}\| \leq c_1 \theta^n$ which tends exponentially fast to zero. The sequence thus converges uniformly in $\mathcal{A}_{\mathbb{R}}$ towards a fixed point

$$\mathbf{h}^*_{s,t} = \pi_{s,t}(\mathbf{h}^*_{s,t}) \in \mathcal{C}_{\sigma',\ell=1}(\Omega), \quad (s,t) \in W^{s_0} \cap \mathbb{R}^2. \tag{6.61}$$

We are interested in the normalisation factor,

$$p_{s,t,\omega} = \ell(\mathcal{L}_{s,F_{t,\omega}} \mathbf{h}^*_{s,t,\tau^{-1}\omega}) \tag{6.62}$$

at the fixed point. This function is real and strictly positive (recall that for the moment s and t are real). By Theorems 5.3 and 4.4 (a) we know that for μ-a.s. the upper and lower pressures agree and are independent of ω. As in Theorem 4.4 we write $P(s, \mathbf{f}_{t,\cdot})$ for this (a.s.) common value of the pressure.

Lemma 6.15. *We have for $(s,t) \in W^{s_0} \cap \mathbb{R}^2$ the following formula for the pressure*:

$$P(s, \mathbf{f}_{t,\cdot}) = \int_\Omega \log p_{s,t,\omega}\, d\mu(\omega).$$

Proof. The embedding $j : z \in K \to (z, \overline{z}) \in \operatorname{diag} K \subset \widehat{U}$ induces a pull-back $j^* : \mathcal{C}_{\sigma'} \to \mathcal{M}(K)$. On $\mathcal{M}(K)$ (before the mirror embedding) we act with the operator $L_{s,\mathbf{f}_{t,\omega}}$ from equation (3.18) (except for the fact that we here use the Euclidean derivative; cf. Remark 6.5) and on the cone with the mirror extended operator $\mathcal{L}_{s,\widehat{\mathbf{f}}_{t,\omega}}$ (recall that t is real here so that $F_{t,\omega} = \widehat{\mathbf{f}}_{t,\omega}$). Consider $h \in \mathcal{C}_{\sigma',\ell=1}$. Then $L_{s,\mathbf{f}_{t,\omega}} j^* h = j^* \mathcal{L}_{s,\widehat{\mathbf{f}}_{t,\omega}} h$ and the cone properties show that $k \leq h_{|\operatorname{diag} K} \leq \frac{1}{k}$ (with k from Lemma 6.6 (a)). We write $L^{(n)}_{s,t,\omega} = L_{s,f_{t,\tau^{n-1}\omega}} \circ \cdots \circ L_{s,f_{t,\omega}}$ and similarly for $\mathcal{L}^{(n)}_{s,t,\omega}$. As stated above the pressure may be calculated μ-a.s. as an upper pressure and so for μ-almost every $\omega \in \Omega$ we have:

$$P(s, \mathbf{f}_{t,\cdot}) = \lim_n \frac{1}{n} \log \|L^{(n)}_{s,t,\omega}\|_{\mathcal{M}(K)}. \tag{6.63}$$

It follows that for μ-almost every $\omega \in \Omega$:

$$\begin{aligned} P(s, \mathbf{f}_{t,\cdot}) &= \lim_n \frac{1}{n} \log \|L^{(n)}_{s,t,\omega}\|_{\mathcal{M}(K)} \\ &= \lim_n \frac{1}{n} \log \ell(\mathcal{L}^{(n)}_{s,t,\omega} \mathbf{h}^*_{s,t,\tau^{-1}\omega}) = \lim_n \sum_{k=0}^{n-1} \frac{1}{n} \log p_{s,t,\tau^k\omega}. \end{aligned} \tag{6.64}$$

Use (5.34) and note that $\frac{1}{L} \leq \|f'_{t,\omega}/f'_{0,\omega}\| \leq L$ to see that for $0 \leq s \leq 2$:

$$k\left(L\|f'_{0,\omega}\|\right)^{-s} \leq p_{s,t,\omega} \leq \frac{1}{k}\left(\|f'_{0,\omega}\|L\right)^2. \tag{6.65}$$

By Assumption 6.13, $\mathbb{E}(\log\|f'_{0,\omega}\|) < +\infty$ and so also $|\log p_{s,t,\omega}|$ is μ-integrable. By Birkhoff's theorem, (6.64) converges μ-almost surely towards the integral of $\log p$ as desired. $\square$

Remark 6.16. The pressure does not depend on the choice of ℓ (of course, it should not). If one makes another choice $\tilde{\ell}$ for the normalisation this simply introduces a co-cycle that vanishes upon integration.

Remark 6.17. Equation (6.64) shows that the use of the fixed point, $\mathbf{h}$, transforms the sub-additive (with respect to n) quantity, $\log\|L^{(n)}_{s,t,\omega}\|$, into an additive quantity. This is really the crux of the matter of the proof and explains why we are so interested in the fixed point of π (see also [Rue97]).

We will use the following real-analytic version of the implicit function theorem:

THEOREM 6.18 (Implicit Function Theorem). *Let $\pi : \mathbb{C}^2 \times \mathcal{A} \to \mathcal{A}$ be a real-analytic map defined on a neighborhood of $(x_0, \phi_0) \in \mathbb{R}^2 \times \mathcal{A}_{\mathbb{R}}$. Let $T_0 = D_\phi\pi(x_0, \phi_0) \in L(\mathcal{A})$ denote the ϕ-derivative of this map at (x_0, ϕ_0). Suppose that $\phi_0 = \pi(x_0, \phi_0) \in \mathcal{A}_{\mathbb{R}}$ and that the spectral radius of the derivative, $\rho(T_0)$, is strictly smaller than* 1. *Then there exists a neighborhood $U \subset \mathbb{C}^2$ of x_0 and a real-analytic map* (*unique if U is small enough*) *$x \in U \mapsto \phi(x) \in \mathcal{A}$ for which $\phi_0 = \phi(x_0)$ and $\phi(x) = \pi(x, \phi(x))$ for all $x \in U$.*

Proof. The map

$$\Gamma(x,\phi) = (1-T_0)^{-1}(\pi(x,\phi) - \phi_0 - T_0(\phi-\phi_0)) + \phi_0$$

is real-analytic and verifies $\Gamma(x_0,\phi_0) = \phi_0$ and $D_\phi\Gamma(x_0,\phi_0) = 0$. We may therefore find a neighborhood U of x_0 and a closed neighborhood W of ϕ_0 such that Γ is a uniform contraction on the real-analytic sections, $U \to W$. The fixed point $\phi(x) = \Gamma(x,\phi(x))$, $x \in U$, is then itself a real-analytic section and has the desired properties. $\square$

LEMMA 6.19. *The fixed point $\mathbf{h}^*_{s,t} = \pi_{s,t}(\mathbf{h}^*_{s,t})$ from equation* (6.61) *and the logarithmic normalisation from equation* (6.62) *with ω fixed*, log $p_{s,t,\omega}$ *extends to real-analytic functions on an open neighborhood $U^{s_0} \subset \mathbb{C}^2$ of* $(s_0, 0)$.

Proof. Lemma 6.14 (2) shows that the spectral radius of $D_{\mathbf{h}}\pi_{s_0,0}(\mathbf{h}^*_{s_0,0})$ does not exceed $\theta < 1$. By the above Implicit Function Theorem there is a real-analytic map

$$(s,t) \in U^{s_0} \mapsto \mathbf{h}^*_{s,t} = \pi_{s,t}(\mathbf{h}^*_{s,t}) \in B(\mathbf{h}^*_{s_0,0}, \rho(s)) \subset \mathcal{A},$$

defined in a neighborhood $U^{s_0} \subset W^{s_0}$ of $(s_0, 0)$ and extending the fixed point found previously for real s and t. On this complex neighborhood we define as before the function $p_{s,t,\omega} = \ell(\mathcal{L}_{s,F_{t,\omega}} \mathbf{h}^*_{s,t,\tau^{-1}\omega}) \in \mathbb{C}$. For fixed ω this function is analytic in $(s,t) \in U^{s_0}$.

Set $\lambda_\omega = \log |\mathbf{f}'_{0,\omega}(x_{0,\omega})|$ with $x_{0,\omega} \in \mathbf{f}^{-1}_{0,\omega} K$. Lemma 6.12, equation (6.56), shows that the function

$$\gamma_{s,t,\omega} = \frac{e^{s\lambda_\omega} p_{s,t,\omega}}{e^{s_0\lambda_\omega} p_{s_0,0,\omega}}$$

is analytic in $(s,t) \in W^{s_0}$ and verifies $|\gamma_{s,t,\omega} - 1| \leq 2/3$. Therefore, $\gamma_{s,t,\omega}$ has a well-defined logarithm, $\log \gamma_{s,t,\omega}$, analytic in $(s,t) \in W^{s_0}$ and bounded by $\log 3$ in absolute value. We then obtain the following very explicit formula for analytic continuation of the logarithm of the normalization factor:

$$\log p_{s,t,\omega} = (s_0 - s)\lambda_\omega + \log p_{s_0,0,\omega} + \log \gamma_{s,t,\omega}. \tag{6.66}$$ □

In the case when the ergodic measure does not depend on parameters we obtain the following:

Theorem 6.20. *Let τ be an ergodic transformation on a fixed measure space (Ω, μ). Let $(\mathbf{f}_{t,\omega})_{\omega\in\Omega} \in \mathcal{E}_\Omega(K,U)$ be an analytic family verifying a uniform L-Lipschitz condition and with uniform bounded condition numbers, i.e. Assumption* 6.13 *above. Then* (a.s.) *the Hausdorff dimension of the random Julia set $J(\mathbf{f}_t)_\omega$, equation* (4.21), *is independent of ω and depends real-analytically upon t.*

Proof. Let U^{s_0} be as in the previous lemma. The pressure function in Lemma 6.15 then has an analytic extension to that neighborhood. To see this note that all terms in (6.66) are absolutely μ-integrable so that

$$\mathcal{P}(s,t) = \int_\Omega \log\, p_{s,t,\omega}\, \mu(d\omega), \quad (s,t) \in U^{s_0}$$

is well-defined and yields a real-analytic extension of the pressure. Consider now $t \in \mathbb{D} \cap \mathbb{R}$. Theorem 5.3 shows that (a.s.) $d(t) = \dim_H J(\mathbf{f}_{t,\cdot})_\omega$ is independent of ω and that $\mathcal{P}(d(t),t) = P(d(t), \mathbf{f}_{t,\cdot}) = 0$ whenever $(d(t),t) \in U^{s_0}$, $t \in \mathbb{R}$. Now $\mathcal{P}$ is real-analytic and $\frac{\partial \mathcal{P}}{\partial s}(d(t),t) = \frac{\partial}{\partial s} P(s, \mathbf{f}_{t,\cdot})_{|s=d(t)} \leq -\log\beta < 0$ for real t-values. We may then apply another implicit function theorem to $\mathcal{P}$ and conclude that there is an open neighborhood $V_0 \subset \mathbb{C}$ of 0 and a real-analytic function $t \in V_0 \mapsto (\widehat{d}(t), t) \in U^{s_0}$ such that $\mathcal{P}(\widehat{d}(t),t) = 0$ for all $t \in V_0$. The function $\widehat{d}(t)$ yields the desired real-analytic extension of the dimension. □

7. Proof of Theorem 1.1

We now consider the case when both the conformal maps and the ergodic measure depend on parameters. When the ergodic measure μ_t itself depends real-analytically on t (Definition 7.1 below) then Theorem 6.20 extends trivially to include this (we leave the details to the reader). This situation occurs rarely in practice and is subsequently not of particular interest. We shall here consider the case of i.i.d. random variables but refer to Remark 7.4 for an alternative approach using Gibbs measures.

7.1. *Parameter dependency of the measure.* Let Υ be a measure space and consider $\mathcal{M} \equiv \mathcal{M}(\Upsilon)$, the Banach space of complex measures on Υ in the variation norm, $\|\nu\|_1 = \int_\Upsilon d|\nu|$ (see e.g. [Lang93, VII, §3]). The set of probability measures $\mathcal{P} \equiv \{\nu \in \mathcal{M} : \nu \geq 0, \nu(\Upsilon) = 1\}$ forms a real affine subspace of $\mathcal{M}$.

Let $m : \Upsilon \to [1, +\infty[$ be measurable and not smaller than one. We define the stronger norm on $\mathcal{M}_1$,

$$\|\nu\|_m = \int_\Upsilon m \, d|\nu|, \tag{7.67}$$

and we denote by $\mathcal{M}_m$ the Banach space of complex measures on Υ for which $\|\nu\|_m$ is finite.

Definition 7.1. We say that a family of measures $\nu_t \in \mathcal{M}$, $t \in \mathbb{D}$ is *a real-analytic family of probability measures* with respect to the weight $m \geq 1$ if $\nu_t \in \mathcal{P}$ when $t \in \mathbb{D} \cap \mathbb{R} =]-1, 1[$ (i.e. is a genuine probability measure for t real) and if

$$t \in \mathbb{D} \mapsto \nu_t \in \mathcal{M}_m$$

is norm-analytic. Since $m \geq 1$, such a family is also real-analytic with respect to the weight 1.

Example 7.2. The Poission law $p_\lambda(k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k \in \mathbb{N}$, is real-analytic in $\lambda \geq 0$ with respect to the weight $1 + K$ (with $K(k) = k$). It has a complex extension to every $\lambda \in \mathbb{C}$ with variation norms

$$\|p_\lambda\|_1 = e^{|\lambda| - \mathrm{re}\lambda} \quad \text{and} \quad \|p_\lambda\|_{1+K} = (1 + |\lambda|) e^{|\lambda| - \mathrm{re}\lambda}.$$

Theorem 1.1 is formulated in terms of independent random variables. We reformulate this in terms of an ergodic transformation as follows: Let ν_t be a real-analytic family of probability measures on a measure space Υ (Definition 7.1) depending on the complex parameter $t \in \mathbb{D}$ (again it is no lack of generality to assume that the parameter is one-dimensional). When $t \in \mathbb{D} \cap \mathbb{R}$ is real ν_t is indeed a probability measure and we define

$$(\Omega, \mu_t) = (\prod_{\mathbb{Z}} \Upsilon, \otimes_{\mathbb{Z}} \nu_t)$$

to be the (Kolmogorov extension of the) direct product of probability spaces. We let τ be the shift on this space, i.e. $(\tau(\boldsymbol{\omega}))_n = (\boldsymbol{\omega})_{n+1}$, $n \in \mathbb{Z}$. Then τ is an invertible ergodic transformation of (Ω, μ_t).

Let the family $f_{t,\omega}$ (with $t \in \mathbb{D}$, $\omega \in \Upsilon$) be as in Theorem 1.1. We define $\mathbf{f}_{t,\boldsymbol{\omega}} = f_{t,(\boldsymbol{\omega})_0}$ on the extended space, Ω. Then $(f_n)_{n\geq 1} = (\mathbf{f}_{t,\boldsymbol{\omega}}, \mathbf{f}_{t,\tau\boldsymbol{\omega}}, \mathbf{f}_{t,\tau^2\boldsymbol{\omega}}, \ldots)$ corresponds to the sequence of random conformal maps described in Theorem 1.1. As in the previous section we assume that $U \subset \mathbb{C}$ so that K is compact in $\mathbb{C}$. By Lemma 6.15 we obtain for t real the pressure as an integral:

$$P(s, \mathbf{f}_{t,\cdot}) = \int_\Omega \log p_{s,t,\omega} d\mu_t(\boldsymbol{\omega}).$$

We encounter here the following problem: In general when t is complex μ_t does not make any sense (as a measure). Even when $t \neq t'$ are real, μ_t and $\mu_{t'}$ are in general mutually singular measures. In spite of these caveats we will show that, nevertheless, the pressure P extends to a real-analytic function in a neighborhood of $(s_0, 0)$. This suffices to prove Theorem 1.1 after repetition of the last part of the proof of Theorem 6.20.

ASSUMPTION 7.3. *We consider the weight* $m(\omega) = 1 + \left|\log \|f'_{0,\omega}\|\right|$ *on* Υ *and assume that the family* ν_t, $t \in \mathbb{D}$, *is real-analytic with respect to the weight* m.

Let $\mathbf{h}^*_{s,t}$, $(s,t) \in U^{s_0}$ be the fixed point obtained from Lemma 6.19. Recall that $D_{\mathbf{h}}\pi_{s_0,0}(\mathbf{h}^*_{s_0,0})$ has spectral radius not greater than $\theta < 1$. Possibly shrinking the neighborhood $U_1 \subset U^{s_0}$ of $(s_0, 0)$, continuity of the derivative, $D_{\mathbf{h}}\pi_{s,t}$, implies that we may find $\rho_1 > 0$, $C_1 < \infty$, $\theta < \theta_1 < 1$ such that for all $\Phi \in B(\mathbf{h}^*_{s_0,0}, \rho_1)$ and $n \geq 0$,

$$\|\pi^{(n)}_{s,t}(\Phi) - \mathbf{h}^*_{s,t}\| \leq C_1\theta_1^n \qquad \text{and} \qquad \pi^{(n)}_{s,t}(\Phi) \in B(\mathbf{h}^*_{s_0,0}, \rho(s)).$$

Shrinking the neighborhood further, $(s_0, 0) \in U_2 \subset U_1$, mere continuity of π shows that we may also find $k_1 \in \mathbb{N}$ such that ($\mathbf{1}$ being the constant unit-section of $\mathcal{A}$)

$$\pi^{(k_1)}_{s,t}(\mathbf{1}) \in B(\mathbf{h}^*_{s_0,0}, \rho_1), \qquad \forall (s,t) \in U_2.$$

Since $\|\nu_0\| = 1$ and ν_t is analytic,

$$\mathcal{D}_0 = \{(s,t) : (s,t) \in U_2, \|\nu_t\|_1 < 1/\theta_1\}$$

defines a complex neighborhood of $(s_0, 0)$. For $(s,t) \in U_2$ set $\mathbf{h}^{(k)}_{s,t} \equiv \pi^{(k)}_{s,t}(\mathbf{1})$, $k \in \mathbb{N}$. The operator $\mathcal{L}_{s,F_{t,\omega_0}}$ depends only on the first symbol $\omega_0 = (\boldsymbol{\omega})_0$ of $\boldsymbol{\omega}$ and since Ω is a direct product we see that $h^{(n)}_{s,t,\omega_0,\ldots,\omega_{-(n-1)}} = \mathbf{h}^{(n)}_{s,t,\boldsymbol{\omega}}$, $n \geq 1$ depends only on the n symbols $(\omega_0, \ldots, \omega_{-(n-1)})$ of $\boldsymbol{\omega}$. It follows that $p^{(n)}_{s,t,\boldsymbol{\omega}} \equiv \ell(\mathcal{L}_{s,F_{t,\omega_0}} \mathbf{h}^{(n)}_{s,t,\tau^{-1}\boldsymbol{\omega}}) = \ell(\mathcal{L}_{s,F_{t,\omega_0}} h^{(n)}_{s,t,\omega_{-1},\ldots,\omega_{-n}})$, $n \in \mathbb{N}$ only depends on $n+1$ symbols of $\boldsymbol{\omega}$. Now, by the choice of k_1 we may proceed as in Lemma 6.19 to see that $\log p^{(k_1)}_{s,t,\boldsymbol{\omega}}$ extends to a real-analytic function of $(s,t) \in \mathcal{D}_0$. Using (6.56) and (6.65) we see that

$$\left|\log p^{(k_1)}_{s,t,\boldsymbol{\omega}}\right| \leq \text{const}(s)(1 + |\log \|f'_{0,\omega_0}\| \,|)$$

and for $n \geq k_1$ we get because of (6.57):

$$\left|\log \frac{p^{(n+1)}_{s,t,\boldsymbol{\omega}}}{p^{(n)}_{s,t,\boldsymbol{\omega}}}\right| = \frac{\|\mathbf{h}^{(n+1)}_{s,t,\tau^{-1}\boldsymbol{\omega}} - \mathbf{h}^{(n)}_{s,t,\tau^{-1}\boldsymbol{\omega}}\|}{\rho(s)} \leq \mathrm{const}(s)\ \theta_1^n.$$

Integrating with respect to the analytic continuation of our probability measure we see that

$$\int \left|\log p^{(k_1)}_{s,t,\boldsymbol{\omega}}\right| d|\nu_t|(\omega_0)\cdots d|\nu_t|(\omega_{-k_1}) \leq \mathrm{const}(s)\ \|\nu_t\|_m\ \|\nu_t\|_1^{k_1} \tag{7.68}$$

and also that for $n \geq k_1$

$$\int \left|\log \frac{p^{(n+1)}_{s,t,\boldsymbol{\omega}}}{p^{(n)}_{s,t,\boldsymbol{\omega}}}\right| d|\nu_t|(\omega_0)\cdots d|\nu_t|(\omega_{-(n+1)}) \leq \mathrm{const}(s)\ \theta_1^n\ \|\nu_t\|_1^{n+2}. \tag{7.69}$$

The sum of all these terms (the latter for all $n \geq k_1$) is bounded by

$$\mathrm{const}(s)\left(\|\nu_t\|_m \|\nu_t\|_1^{k_1} + \sum_{n\geq k_1} \theta_1^n\ \|\nu_t\|_1^{n+2}\right)$$

which converges uniformly on compact subsets of $\mathcal{D}_0$. The following function:

$$\begin{aligned}\mathcal{P}(s,t) \equiv &\int \log p^{(k_1)}_{s,t,\boldsymbol{\omega}} d\nu_t(\omega_0)\cdots d\nu_t(\omega_{-k_1})\\ &+ \sum_{n\geq k_1} \int \log \frac{p^{(n+1)}_{s,t,\boldsymbol{\omega}}}{p^{(n)}_{s,t,\boldsymbol{\omega}}} d\nu_t(\omega_0)\cdots d\nu_t(\omega_{-(n+1)})\end{aligned}$$

therefore defines an analytic function of $(s,t) \in \mathcal{D}_0$. For t real we have

$$\int \log p^{(k_1)}_{s,t,\boldsymbol{\omega}} d\nu_t(\omega_0)\cdots d\nu_t(\omega_{-k_1}) = \int \log p^{(k_1)}_{s,t,\boldsymbol{\omega}} d\mu_t(\boldsymbol{\omega})$$

and similarly for the integral in the sum; so the function reduces in this case to

$$\begin{aligned}\mathcal{P}(s,t) &= \int \left(\log\ p^{(k_1)}_{s,t,\boldsymbol{\omega}} + \sum_{n\geq k_1} \log \frac{p^{(n+1)}_{s,t,\boldsymbol{\omega}}}{p^{(n)}_{s,t,\boldsymbol{\omega}}}\right) d\mu_t(\boldsymbol{\omega})\\ &= \int \log\ p_{s,t,\boldsymbol{\omega}} d\mu_t(\boldsymbol{\omega}) = P(s, \mathbf{f}_{t,\cdot}).\end{aligned}$$

In other words $\mathcal{P}(s,t)$, $(s,t) \in \mathcal{D}_0$ provides us with the desired real-analytic continuation of the pressure. As mentioned before we may proceed as in the proof of Theorem 6.20 to conclude the proof of Theorem 1.1. □

Remark 7.4. An alternative generalisation would be to pick the maps $\mathbf{f}_{t,\boldsymbol{\omega}}$ according to a Gibbs measure on a shift space over a finite alphabet. The Hausdorff dimension also here depends real-analytically (use the exponential decay of correlations to see this) upon the Hölder potential defining the Gibbs state. This result does not, however, cover example 1.2 in the introduction.

8. Proof of Example 1.2

We define for $0 \le \rho < 1$ the complex annulus $A_\rho = \{z \in \mathbb{C} : \rho < |z| < 1/\rho\}$ ($= \mathbb{C}^*$ for $\rho = 0$). The conditions on parameters may be written as

$$|a| + r \le \frac{k^2}{4},$$

where k is a constant $0 < k < 1$. We set $U = A_{k^2/2}$ and $K = \overline{A_{k/2}}$ (a compact subset of U).

The maps under consideration, $f = z^{N+2} + c$, then belongs to $\mathcal{E}(K, U)$. The neighborhood K_Δ may be written as $\overline{A_\kappa}$ for a suitable value of $\kappa \in]k^2/2, k/2[$. For $w = f(z) \in K_\Delta$ we have

$$f'(z) = (N+2)z^{N+1} = (N+2)\frac{w-c}{z},$$

which is comparable to N (because both w and z belong to K_Δ). The condition $\mathbb{E}(\log \|Df\|) < +\infty$ is then equivalent to $\mathbb{E}(N) < +\infty$ which is clearly verified for a Poisson distribution of N. Also, the condition numbers $\|f'\| \, \|1/f'\|$ are uniformly bounded (this is in fact true for the family of all maps $f \in \mathcal{E}(K, U)$ for which $f^{-1}U$ is connected). Writing $f(z) = z^{N+2} + a + r\xi$, with ξ a random variable uniformly distributed in $\mathbb{D}$, we obtain an explicit (real-) analytic parametrization of f in terms of a and r.

To see that a local inverse depends uniformly Lipshitz in parameters consider e.g.:

$$\frac{\partial f^{-1}}{\partial a} = -\frac{\partial f}{\partial a} / \frac{\partial f}{\partial z} = -\frac{1}{N+2} \, \frac{z}{w-c},$$

which is uniformly bounded on K_Δ. Similarly,

$$\frac{\partial}{\partial a} \log f' \circ f^{-1} = \frac{N+1}{z} \, \left(-\frac{w-c}{(N+2)z}\right) = \frac{N+1}{N+2} \, \frac{c-w}{z^2},$$

which is again uniformly bounded, independent of the value of N (but only just so!). Finally, the Poisson distribution is real-analytic in the parameter $\lambda \ge 0$ and with respect to the weight $1 + N$ (Example 7.2). We are in a position where we may apply Theorem 1.1, thus proving our claims in Example 1.2. □

Appendix A. Removing the mixing condition

Our mixing condition (C4) was convenient but not strictly necessary. For completeness we will show how to get rid of this condition. In the following we write $\delta = \delta_f$ for the injectivity constant associated with f (Section 2.1). Our first reduction replaces (C4) by topological transitivity. This amounts to saying that there is $n_0 = n_0(\delta)$ such that

$$\text{(C4}'\text{)} \qquad \bigcup_{k=0}^{n_0} f^k(B(x,\delta) \cap \Lambda) = \Lambda.$$

Repeating the previous steps we see that (2.12) is replaced by the inequality

$$\max_{0\le k\le n_0} m_{n+k} \ge (\|Df\|^{n_0} c_n)^{-s} M_n/2$$

from which the operator distortion bounds follow. The proof of the lower bounds for the Hausdorff dimension does not change and in the upper bounds for the Box dimension the left-hand side of the inequality (2.14) is replaced by $\sum_{0\le j\le n_0} L^{j+n_0}\chi_{B(x,r)}$ which leads to the bound

$$\sum_{i=1}^{N} (\text{diam } B(x_i,2r))^s \le 4^s \gamma_2(s) \sum_{0\le j\le n_0} \|L_s^j\|.$$

In the general situation we will replace Λ by a subset Λ' of the same dimensions but which is f-invariant and topologically transitive. First, define a local pressure at $x\in\Lambda$ within Λ:

$$\overline{P}_x(s,\Lambda) = \limsup_n \frac{1}{n} \log L_s^n \mathbf{1}_\Lambda(x).$$

From the very definition it is clear that

$$\overline{P}_x(s,\Lambda) \le \overline{P}_{fx}(s,\Lambda). \tag{A.70}$$

Also if $x\in\Lambda$ and $y\in\Lambda$ are at a distance less than δ, the ratio of $L_s^n\mathbf{1}(x)$ and $L_s^n\mathbf{1}(y)$ are sub-exponentially bounded in n. The local pressures at x and y are thus the same. Say that two points $x,y\in\Lambda$ are δ-connected if and only if there is a finite sequence of points

$$x_0 = x, x_1, \ldots, x_n, x_{n+1} = y \subset \Lambda$$

for which $d(x_i,x_{i+1})<\delta$ for all $0\le i\le n$. This partitions Λ into δ-connected components $\Lambda = \Lambda_1\cup\ldots\cup\Lambda_m$. Each Λ_i is δ-separated from its complement, whence open and compact within Λ. Thus there is a uniform bound on the number N_δ of intermediate points needed to connect any x and y within the same component.

The partition is not Markovian. For example, for a connected hyperbolic Julia set there is only one δ-connected component. It does, however, enjoy some Markov like properties: If $f\Lambda_i\cap\Lambda_j\neq\emptyset$ then $f\Lambda_i\supset\Lambda_j$. To see this note that if $x\in\Lambda_i$, $y=f(x_i)\in\Lambda_j$ and $v\in B(y,\delta)\subset\Lambda_j$ then there is (a unique) $u\in B(x,\delta)\subset\Lambda_i$ for which $f(u)=v$ and thus $v\in\Lambda_i$. This shows that $f\Lambda_i$ is δ-connected, whence contains Λ_j. We may introduce a transition matrix, writing $t_{ji}=1$ when $f\Lambda_i\supset\Lambda_j$ and zero otherwise. A partial ordering among the partition elements Λ_i is then given by

$$\Lambda_i \prec \Lambda_j \quad \text{if and only if} \quad \exists n = n(i,j) : t_{ji}^n \ge 1$$

and an equivalence relation,

$$\Lambda_i \sim \Lambda_j \quad \text{if and only if} \quad \Lambda_i\prec\Lambda_j \ \text{ and } \ \Lambda_j\prec\Lambda_i.$$

The equivalence classes provides a new partition of Λ:

$$\Lambda = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_k$$

which inherits the partial ordering from before. Each equivalence class is topologically transitive and the local pressures are constant on each class (because of (A.70)). Writing P_i for the pressure on class i we also see that $P_i \leq P_j$ for $i \prec j$.

Consider now the critical s-value s_{crit} and let $\mathcal{C}_{i_0}$ be a class which is minimal for the inherited partial ordering and such that the local pressure vanishes for some, whence every point in this class $\overline{P}_x(s_{\text{crit}}, \Lambda) = 0$, $x \in \mathcal{C}_{i_0}$. We denote by

$$\Lambda' = \cap_{j \geq 0} f^{-j} \mathcal{C}_{i_0}$$

the corresponding f invariant subset of the class. This subset is topologically transitive (clear) and we claim that this set has Hausdorff and box dimensions that agree and equal s_{crit}. For this it suffices to show that the pressure of that subset $P(s_{\text{crit}}, \Lambda', f)$ vanishes.

For $1 \leq i \leq k$, we define:

$$N_i \phi (= N_{s,i} \phi) = \chi_{\mathcal{C}_i} L_s \phi = L_s((\chi_{\mathcal{C}_i} \circ f)\phi).$$

If $\mathcal{C}_i \prec \mathcal{C}_j$ and they are not equal then $N_i N_j \equiv 0$ and the spectral radius, $r_{\text{sp}}(N_i)$, of N_i does not exceed that of N_j. Similarly, if $\mathcal{C}_i$ and $\mathcal{C}_j$ are not related then $N_i N_j = N_j N_i \equiv 0$. In either case we have $(N_i + N_j)^n = N_i^n + N_j N_i^{n-1} + \cdots N_j^{n-1} N_i + N_j^n$, which implies that $r_{\text{sp}}(N_i + N_j)$ does not exceed $\max\{r_{\text{sp}}(N_j), r_{\text{sp}}(N_i)\}$. When $L_s = \sum_{i : \mathcal{C}_i \subset \Lambda'} N_{s,i}$, it follows that $r_{\text{sp}}(L_{s_{\text{crit}}}) = r_{\text{sp}}(N_{i_0})$. But this implies precisely that $P(s_{\text{crit}}, \Lambda', f) = 0$.

Remark A.1. In this setting, even when distortions remain uniformly bounded the Hausdorff measure need not be finite (essentially because the powers of a matrix of spectral radius-one need not be bounded when the eigenvalue-one is not simple).

Department of Mathematics, University of Cergy-Pontoise,
CNRS UMR 8088, F-95000 Cergy-Pontoise, France
E-mail address: hhrugh@math.u-cergy.fr

References

[Bar96] L. M. Barreira, A nonadditive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems, *Ergodic Theory Dynam. Systems* **16** (1996), 871–927.

[BO99] T. Bogenschütz and G. Ochs, The Hausdorff dimension of conformal repellers under random perturbation, *Nonlinearity* **12** (1999), 1323–1338.

[Bow79] R. Bowen, Hausdorff dimension of quasi-circles, *IHES Publ.* **50** (1979), 259–273.

[Bir67] G. Birkhof, *Lattice Theory*, 3rd ed., Amer. Math. Soc., Providence, RI 1967.

[CG93] L. Carleson and T. Gamelin, *Complex Dynamics*, Springer-Verlag, New York, 1993.

[CF98] H. Crauel and F. Flandoli, Hausdorff dimension of invariant sets for random dynamical systems, *J. Dynam. Differential Equations* **10** (1998), 449-474.

[DT01] A. V. Dryakhlov and A. A. Tempelman, *New York J. Math.* **7** (2001), 99–115.

[Fal89] K. Falconer, Dimensions and measures of quasi self-similar sets, *Proc. Amer. Math. Soc.* **106** (1989), 543–554.

[FK60] H. Furstenberg and H. Kesten, Products of random matrices, *Ann. Math. Statist.* **31** (1960), 457–469.

[GP97] D. Gatzouras and Yu. Peres, Invariant measures of full dimension for some expanding maps, *Ergodic Theory Dynam. Systems* **17** (1997), 147–167.

[King68] J. F. C. Kingman, The ergodic theory of subadditive stochastic processes, *J. Royal Statist. Soc.* **B 30** (1968), 499–510.

[Kif96] Y. Kifer, Fractal dimensions and random transformations, *Trans. Amer. Math. Soc.* **348** (1996), 2003–2038.

[Lang93] S. Lang, *Real and Functional Analysis*, Springer-Verlag, New York (1993).

[Liv95] C. Liverani, Decay of correlations, *Ann. of Math.* **142** (1995), 239–301.

[MM83] H. McCluskey and A. Manning, Hausdorff dimension for horseshoes, *Ergodic Theory Dynam. Systems* **3** (1983), 251–260.

[Pes88] Ya. B. Pesin, Dimension-type characteristics for invariant sets of dynamical systems, *Russian Math. Surveys* **43** (1988), 111–151.

[PW96] Ya. B. Pesin and H.Ẇeiss, On the dimension of deterministic and random Cantor-like sets, symbolic dynamics, and the Eckmann-Ruelle conjecture, *Comm. Math. Phys.* **182** (1996), 105–153.

[Rue76] D. Ruelle, Zeta functions for expanding maps and Anosov flows, *Invent. Math.* **34** (1976), 231–242.

[Rue79] ———, Analytic properties of the characteristic exponents of random matrix products, *Adv. in Math.* **32** (1979), 68–80.

[Rue82] D. Ruelle, Repellers for real analytic maps, *Ergodic Theory Dynam. Systems* **2** (1982), 99–107.

[Rue97] ———, Differentiation of SRB states, *Comm. Math. Phys.* **187** (1997), 227–241.

[Rugh02] H. H . Rugh, Coupled maps and analytic function spaces, *Ann. Scient. École Norm. Sup.* **35** (2002), 489–535.

[SSU01] K. Simon, B. Solomyak, and M. Urbański, Hausdorff dimensions of limit sets for parabolic IFS with overlaps, *Pacific J. Math.* **201** (2001), 441–478.

[Urb96] M. Urbański, Parabolic Cantor sets, *Fund. Math.* **151** (1996), 241–277.

[UZ01] M. Urbański and M. Zinsmeister, Geometry of hyperbolic Julia-Lavaurs sets, *Indagationes Math.* **12** (2001), 273–292.

[UZ04] M. Urbański and A. Zdunik, Real analyticity of Hausdorff dimension of finer Julia sets of exponential family, *Ergodic Theory Dynam. Systems* **24** (2004), 279–315.

[Zin99] M. Zinsmeister, *Thermodynamic Formalism and Holomorphic Dynamical Systems*, Amer. Math. Soc., Providence, RI, 1999.

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