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A counterexample to the strong version of Freedman's conjecture

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Abstract

A long-standing conjecture due to Michael Freedman asserts that the 4-dimensional topological surgery conjecture fails for non-abelian free groups, or equivalently that a family of canonical examples of links (the generalized Borromean rings) are not $A-B$ slice. A stronger version of the conjecture, that the Borromean rings are not even weakly $A-B$ slice, where one drops the equivariant aspect of the problem, has been the main focus in the search for an obstruction to surgery. We show that the Borromean rings, and more generally all links with trivial linking numbers, are in fact weakly $A-B$ slice. This result shows the lack of a non-abelian extension of Alexander duality in dimension 4, and of an analogue of Milnor's theory of link homotopy for general decompositions of the 4-ball.

1. Introduction

Surgery and the s-cobordism conjecture, central ingredients of the geometric classification theory of topological 4-manifolds, were established in the simply-connected case and more generally for elementary amenable groups by Freedman [1], [7]. Their validity has been extended to the groups of subexponential growth [8], [13]. A long-standing conjecture of Freedman [2] asserts that surgery fails in general, in particular for free fundamental groups. This is the central open question, since surgery for free groups would imply the general case, cf. [7].

There is a reformulation of surgery in terms of the slicing problem for a special collection of links, the untwisted Whitehead doubles of the Borromean rings and of a certain family of their generalizations; see Figure 2. (We work in the topological category, and a link in $S^3=\partial D^4$ is *slice* if its components bound disjoint, embedded, locally flat disks in D^4.) An "undoubling" construction [3] allows one to work with a more robust link, the Borromean rings, but the slicing

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condition is replaced in this formulation by a more general A–B *slice problem*. Freedman's conjecture pinpoints the failure of surgery in a specific example and states that the Borromean rings are not $A-B$ slice. This approach to surgery has been particularly attractive since it is amenable to the tools of link-homotopy theory and nilpotent invariants of links, and partial obstructions are known in restricted cases, cf [6], [10], [11]. At the same time it is an equivalent reformulation of the surgery conjecture, and if surgery holds there must exist specific $A-B$ decompositions solving the problem.

The $A-B$ slice conjecture is a problem at the intersection of 4-manifold topology and Milnor's theory of link homotopy [14]. It concerns codimension zero decompositions of the 4-ball. Here a *decomposition* of D^4, $D^4 = A \cup B$, is an extension of the standard genus one Heegaard decomposition of $\partial D^4 = S^3$. Each part A, B of a decomposition has an attaching circle (a distinguished curve in the boundary: $\alpha \subset \partial A, \beta \subset \partial B$) which is the core of the solid torus forming the Heegaard decomposition of ∂D^4. The two curves α, β form the Hopf link in S^3.

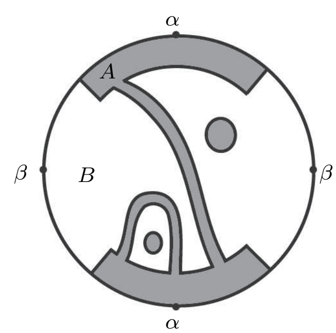

Figure 1: A 2-dimensional example of a decomposition (A,α), (B,β): $D^2 = A \cup B$, A is shaded; $(\alpha,\ \beta)$ are linked 0-spheres in ∂D^2.

Figure 1 is a schematic illustration of a decomposition: an example drawn in two dimensions. While the topology of decompositions in dimension 2 is quite simple, they illustrate important basic properties. In this dimension the attaching regions α, β are 0-spheres, and $(\alpha,\ \beta)$ form a “Hopf link” (two linked 0-spheres) in ∂D^2. Alexander duality implies that exactly one of the two possibilities holds: either α vanishes as a rational homology class in A, or β does in B. In dimension 2, this means that either α bounds an arc in A, as in the example in Figure 1, or β bounds an arc in B. (See Figure 12 in §5 for additional examples in 2 dimensions.)

Algebraic and geometric properties of the two parts A, B of a decomposition of D^4 are tightly correlated. The geometric implication of Alexander duality in dimension 4 is that either (an integer multiple of) α bounds an orientable *surface* in A or a multiple of β bounds a surface in B.

Alexander duality does not hold for homotopy groups, and this difference between being trivial homologically (bounding a surface) as opposed to *homotopically* (bounding a disk) is an algebraic reason for the complexity of decompositions of D^4.

A geometric refinement of Alexander duality is given by handle structures: under a mild condition on the handle decompositions which can be assumed without loss of generality, there is a one-to-one correspondence between 1-handles of each side and 2-handles of its complement. In general the interplay between the topologies of the two sides is rather subtle. Decompositions of D^4 are considered in more detail in Sections 2 and 4 of this paper.

We now turn to the main subject of the paper, the $A-B$ slice reformulation of the surgery conjecture. An n-component link L in S^3 is $A - B$ *slice* if there exist n decompositions (A_i, B_i) of D^4 and disjoint embeddings of all $2n$ manifolds $A_1, B_1, \dots, A_n, B_n$ into D^4 so that the attaching curves $\alpha_1, \dots, \alpha_n$ form the link L and the curves $\beta_1, \dots, \beta_n$ form an untwisted parallel copy of L. Moreover, the re-embeddings of A_i, B_i are required to be *standard* – topologically equivalent to the ones coming from the original decompositions of D^4. The connection of the $A - B$ slice problem for the Borromean rings to the surgery conjecture is provided by consideration of the universal cover of a hypothetical solution to a canonical surgery problem [3], [4]. The action of the free group by covering transformations is precisely encoded by the fact that the re-embeddings of A_i, B_i are standard. A formal definition and a more detailed discussion of the $A-B$ slice problem are given in Section 2. The following is the statement of Freedman's conjecture [2], [4] concerning the failure of surgery.

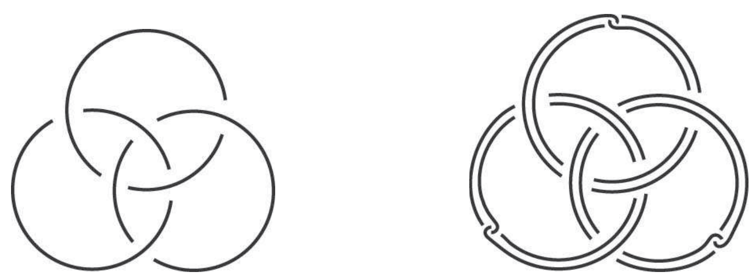

Figure 2: The Borromean rings and their untwisted Whitehead double.

Conjecture 1. *The untwisted Whitehead double of the Borromean rings* (*Figure* 2) *is not a freely slice link. Equivalently, the Borromean rings are not* $A - B$ *slice.*

Here a link is *freely* slice if it is slice, and in addition the fundamental group of the slice complement in the 4-ball is freely generated by meridians to the components of the link. An affirmative solution to this conjecture would exhibit the failure of surgery, since surgery predicts the existence of the free-slice complement of the link above.

A stronger version of Freedman's conjecture, that the Borromean rings are not even weakly $A-B$ slice, has been the main focus in the search for an obstruction to surgery. Here a link L is *weakly* $A-B$ slice if the re-embeddings of A_i, B_i are required to be disjoint but not necessarily standard in the definition above. To understand the context of this conjecture, consider the simplest example of a decomposition $D^4 = A \cup B$ where (A, α) is the 2-handle $(D^2 \times D^2, \partial D^2 \times \{0\})$ and B is just the collar on its attaching curve β. This decomposition is trivial in the sense that all topology is contained in one side, A. It is easy to see that a link L is weakly $A-B$ slice with this particular choice of a decomposition if and only if L is *slice*. The Borromean rings is not a slice link (cf [14]), so it is not weakly $A-B$ slice with the trivial decomposition. However to find an obstruction to surgery, one needs to find an obstruction for the Borromean rings to be weakly $A-B$ slice for all possible decompositions.

Freedman's program in the $A-B$ slice approach to surgery could be roughly summarized as follows. First consider *model* decompositions, defined using Alexander duality and introduced in [6] (see also Section 4). The main step is then to show that any decomposition is algebraically approximated, in some sense, by the models – in this case a suitable algebraic analogue of the partial obstruction for model decompositions should give rise to an obstruction to surgery. The first step, formulating an obstruction for model decompositions, was carried out in [11], [12]. We now state the main result of this paper which shows that the second step is substantially more subtle than previously thought, involving not just the submanifolds but also their embedding information.

THEOREM 1. *Let L be the Borromean rings or more generally any link in S^3 with trivial linking numbers. Then L is weakly $A-B$ slice.*

The linking numbers provide an obstruction to being weakly $A-B$ slice (see §3), so in fact Theorem 1 asserts that a link is weakly $A-B$ slice if and only if it has trivial linking numbers.

To formulate the main ingredient in the proof of this result in the geometric context of link homotopy, it is convenient to introduce the notion of a robust 4-manifold. Recall that a link L in S^3 is *homotopically trivial* if its components bound disjoint maps of disks in D^4. Otherwise, L is called homotopically essential. (The Borromean rings is a homotopically essential link [14] with trivial linking numbers.) Let (M, γ) be a pair (4-manifold, attaching curve in ∂M). The pair (M, γ) is *robust* if whenever several copies (M_i, γ_i) are properly disjointly embedded in (D^4, S^3), the link formed by the curves $\{\gamma_i\}$ in S^3 is homotopically trivial. The following question relates this notion to the $A-B$ slice problem: *Given a decomposition* $(A, \alpha), (B, \beta)$ *of* D^4, *is one of the two* *pairs* $(A, \alpha), (B, \beta)$ *necessarily robust*? The answer has been affirmative for all previously known examples, including the model decompositions [11], [12]. In contrast, we prove

Lemma 2. *There exist decompositions* $D^4 = A \cup B$ *where neither of the two sides* A, B *is robust.*

This result suggests an intriguing possibility that there are 4-manifolds which are not robust, but which admit *robust embeddings* into D^4. (The definition of a robust embedding $e\colon (M, \gamma) \hookrightarrow (D^4, S^3)$ is analogous to the definition of a robust pair above, with the additional requirement that each of the embeddings $(M_i, \gamma_i) \subset (D^4, S^3)$ is equivalent to e.) Then the question relevant for the surgery conjecture is: *given a decomposition* $D^4 = A \cup B$*, is one of the given* embeddings $A \hookrightarrow D^4$, $B \hookrightarrow D^4$ *necessarily robust?*

Theorem 1 has a consequence in the context of *topological arbiters*, introduced in [5]. Roughly speaking, it points out a substantial difference in the structure of the invariants of submanifolds of D^4, depending on whether they are endowed with a specific embedding or not. We refer the reader to that paper for the details on this application.

Section 2 reviews the background material on surgery and the $A - B$ slice problem which, for two-component links, is considered in Section 3; it is shown that Alexander duality provides an obstruction for links with non-trivial linking numbers. The proof of Theorem 1 starts in Section 4 with a construction of the relevant decompositions of D^4. The final section completes the proof of the theorem.

Acknowledgements. This paper concerns the program on the surgery conjecture developed by Michael Freedman. I would like to thank him for sharing his insight into the subject on numerous occasions.

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2. 4-dimensional surgery and the the $A - B$ slice problem

The surgery conjecture asserts that given a 4-dimensional Poincaré pair (X, N), the sequence

$$\mathcal{S}^h_{\mathrm{TOP}}(X, N) \longrightarrow \mathcal{N}_{\mathrm{TOP}}(X, N) \longrightarrow L^h_4(\pi_1 X)$$

is exact (cf. [7, Ch. 11]). This result, as well as the 5-dimensional topological s-cobordism theorem, is known to hold for a class of *good* fundamental groups. The simply-connected case followed from Freedman's disk embedding theorem [1] allowing one to represent hyperbolic pairs in $\pi_2(M^4)$ by embedded spheres. Currently the class of good groups is known to include the groups of subexponential growth [8], [13] and it is closed under extensions and direct limits. There is a specific conjecture for the failure of surgery for free groups [2]:

CONJECTURE 2.1. *There does not exist a topological* 4*-manifold* M*, homotopy equivalent to* $\vee^3 S^1$ *and with* ∂M *homeomorphic to* $\mathcal{S}^0(\mathrm{Wh}(\mathrm{Bor}))$*, the zero-framed surgery on the Whitehead double of the Borromean rings.*

This statement is seen to be equivalent to Conjecture 1 in the introduction by consideration of the complement in D^4 of the slices for Wh(Bor). This is one of a collection of canonical surgery problems with free fundamental groups, and solving them is equivalent to the surgery theorem without restrictions on the fundamental group. The $A-B$ slice problem, introduced in [3], is a reformulation of the surgery conjecture, and it may be roughly summarized as follows. Assuming on the contrary that the manifold M in the conjecture above exists, consider its universal cover $\widetilde{M}$. It is shown in [3] that the end point compactification of $\widetilde{M}$ is homeomorphic to the 4-ball. The group of covering transformations (the free group on three generators) acts on D^4 with a prescribed action on the boundary, and roughly speaking the $A-B$ slice problem is a program for finding an obstruction to the existence of such actions. To state a precise definition, consider decompositions of the 4-ball:

Definition 2.2. A *decomposition* of D^4 is a pair of compact codimension zero submanifolds with boundary $A, B \subset D^4$, satisfying conditions (1) – (3) below. Denote

$$\partial^+ A = \partial A \cap \partial D^4,\ \ \partial^+ B = \partial B \cap \partial D^4,\ \ \partial A = \partial^+ A \cup \partial^- A,\ \ \partial B = \partial^+ B \cup \partial^- B.$$

(1) $A \cup B = D^4$,
(2) $A \cap B = \partial^- A = \partial^- B$,
(3) $S^3 = \partial^+ A \cup \partial^+ B$ is the standard genus 1 Heegaard decomposition of S^3.

Recall the definition of an $A-B$ slice link [4], [6]:

Definition 2.3. Given an n-component link $L = (l_1, \dots, l_n) \subset S^3$, let $D(L) = (l_1, l_1', \dots, l_n, l_n')$ denote the $2n$-component link obtained by adding an untwisted parallel copy L' to L. The link L is $A-B$ *slice* if there exist decompositions $(A_i, B_i), i = 1, \dots, n$ of D^4 and self-homeomorphisms ϕ_i, ψ_i of D^4, $i = 1, \dots, n$ such that all sets in the collection $\phi_1 A_1, \dots, \phi_n A_n, \psi_1 B_1, \dots, \psi_n B_n$ are disjoint and satisfy the boundary data: $\phi_i(\partial^+ A_i)$ is a tubular neighborhood of l_i and $\psi_i(\partial^+ B_i)$ is a tubular neighborhood of l_i', for each i.

The surgery conjecture is equivalent to the statement that the Borromean rings (and a family of their generalizations) are $A-B$ slice. The idea of the proof of one implication is sketched above; the converse is also true: if the generalized Borromean rings were $A-B$ slice, consider the complement of the entire collection $\phi_i(A_i), \psi_i(B_i)$. Gluing the boundary according to the homeomorphisms, one gets solutions to the canonical surgery problems (see the proof of Theorem 2 in [3].)

The restrictions $\phi_i|_{A_i}, \psi_i|_{B_i}$ in the definition above provide disjoint embeddings into D^4 of the entire collection of $2n$ manifolds $\{A_i, B_i\}$. Moreover, these re-embeddings are *standard*: they are restrictions of self-homeomorphisms of D^4, so in particular the complement $D^4 \smallsetminus \phi_i(A_i)$ is homeomorphic to B_i, and $D^4 \smallsetminus \psi_i(B_i) \cong A_i$. This requirement that the re-embeddings are standard is removed in the following definition:

Definition 2.4. A link $L = (l_1, \ldots, l_n)$ in S^3 is *weakly* $A - B$ *slice* if there exist decompositions $((A_1, \alpha_1),\ (B_1, \beta_1)), \ldots, ((A_n, \alpha_n), (B_n, \beta_n))$ of D^4 and disjoint embeddings of all manifolds A_i, B_i into D^4 so that the attaching curves $\alpha_1, \ldots, \alpha_n$ form the link L and the curves $\beta_1, \ldots, \beta_n$ form an untwisted parallel copy of L.

3. Abelian versus non-abelian Alexander duality

This section uses Alexander duality to show that the vanishing of the linking numbers is a necessary condition in Theorem 1. Specifically, we prove

PROPOSITION 3.1. *Let L be a link with a non-trivial linking number. Then L is not weakly $A - B$ slice.*

Proof. It suffices to consider 2-component links, since any sub-link of a weakly $A - B$ slice link is also weakly $A - B$ slice. Let $L = (l_1, l_2)$ with $\mathrm{lk}(l_1, l_2) \neq 0$, and consider any two decompositions $D^4 = A_1 \cup B_1 = A_2 \cup B_2$.

Consider the long exact sequences of the pairs $(A_i, \partial^+ A_i), (B_i, \partial^+ B_i)$, where the homology groups are taken with rational coefficients:

$$0 \longrightarrow H_2 A_i \longrightarrow H_2(A_i, \partial^+ A_i) \longrightarrow H_1 \partial^+ A_i \longrightarrow H_1 A_i \longrightarrow H_1(A_i, \partial^+ A_i) \longrightarrow 0,$$
$$0 \longrightarrow H_2 B_i \longrightarrow H_2(B_i, \partial^+ B_i) \longrightarrow H_1 \partial^+ B_i \longrightarrow H_1 B_i \longrightarrow H_1(B_i, \partial^+ B_i) \longrightarrow 0.$$

Recall that $\partial^+ A_i, \partial^+ B_i$ are solid tori (regular neighborhoods of the attaching curves α_i, β_i). The claim is that for each i, the attaching curve on exactly one side vanishes in its first rational homology group. Both of them can't vanish simultaneously, since the linking number is 1. Suppose neither of them vanishes. Then the boundary map in each sequence above is trivial, and $rk\, H_2(A_i) = rk\, H_2(A_i, \partial^+ A_i)$. On the other hand, by Alexander duality $rk\, H_2(A_i) = rk\, H_1(B_i, \partial^+ B_i)$, $\ rk\, H_2(A_i, \partial^+ A_i) = rk\, H_1(B_i)$. This is a contradiction, since $H_1 \partial^+ B_i \cong \mathbb{Q}$ is in the kernel of $H_1 B_i \longrightarrow H_1(B_i, \partial^+ B_i)$.

Now to show that the link $L = (l_1, l_2)$ is not weakly $A - B$ slice, set $(C_i, \gamma_i) = (A_i, \alpha_i)$ if $\alpha_i = 0 \in H_1(A_i; \mathbb{Q})$ or $(C_i, \gamma_i) = (B_i, \beta_i)$ otherwise. If L were weakly $A - B$ slice, there would exist disjoint embeddings $(C_1, \gamma_1) \subset$

(D^4, S^3), $(C_2, \gamma_2) \subset (D^4, S^3)$ so that γ_1 is either l_1 or its parallel copy, and γ_2 is l_2 or its parallel copy. Then $\mathrm{lk}(\gamma_1, \gamma_2) \neq 0$, a contradiction. □

Proposition 3.1 should be contrasted with Theorem 1. Milnor's link-homotopy invariant of the Borromean rings, $\overline{\mu}_{123}(\mathrm{Bor})$, equals 1 [14]. Also, $\overline{\mu}_{123}$, defined using the quotient $\pi_1/(\pi_1)^3$ of the fundamental group by the third term of the lower central series, is a non-abelian analogue of the linking number of a link. Our result, Theorem 1, shows the lack of a non-abelian extension of Alexander duality in dimension 4.

4. Decompositions of D^4

This section starts the proof of Theorem 1 by constructing the relevant decompositions of D^4. The simplest decomposition $D^4 = A \cup B$ where A is the 2-handle $D^2 \times D^2$ and B is just the collar on its attaching curve, was discussed in the introduction. Now consider the genus one surface S with a single boundary component α, and set $A_0 = S \times D^2$. Moreover, one has to specify its embedding into D^4 to determine the complementary side, B. Consider the standard embedding (take an embedding of the surface in S^3, push it into the 4-ball and take a regular neighborhood). Note that given any decomposition, by Alexander duality the attaching curve of exactly one of the two sides vanishes in it homologically, at least rationally. Therefore the decomposition under consideration now may be viewed as the first level of an "algebraic approximation" to an arbitrary decomposition.

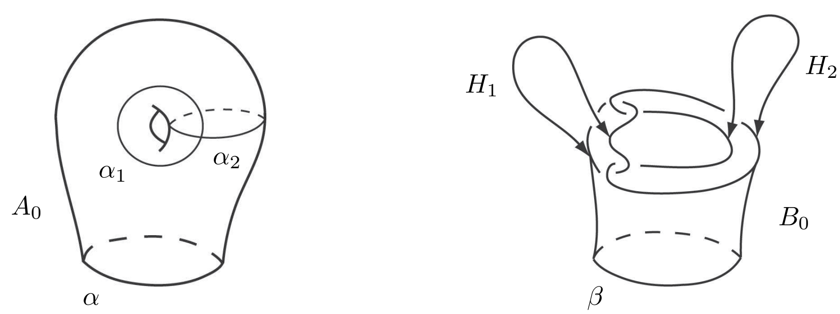

Figure 3

Proposition 4.1. *Let $A_0 = S \times D^2$, where S is the genus-one surface with a single boundary component α. Consider the standard embedding $(A_0, \alpha \times \{0\}) \subset (D^4, S^3)$. Then the complement B_0 is obtained from the collar on its attaching curve, $S^1 \times D^2 \times I$, by attaching a pair of zero-framed 2-handles to the Bing double of the core of the solid torus $S^1 \times D^2 \times \{1\}$, Figure 3.*

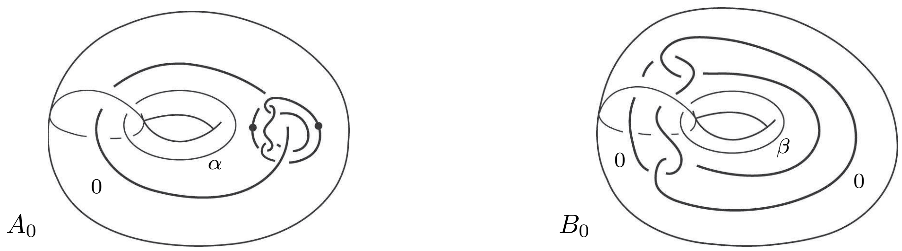

Figure 4

The proof is a standard exercise in Kirby calculus; see for example [6]. A precise description of these 4-manifolds is given in terms of Kirby diagrams in Figure 4. Rather than considering handle diagrams in the 3-sphere, we find it convenient to draw them in the solid torus, so that the 4-manifolds are obtained from $S^1 \times D^2 \times I$ by attaching the 1- and 2-handles as shown in the diagrams. To make sense of the "zero framing" of curves which are not null-homologous in the solid torus, recall that the solid torus is embedded into $S^3 = \partial D^4$ as the attaching region of a 4-manifold, and the 2-handle framings are defined using this embedding.

This example illustrates the general principle that (in all examples considered in this paper) the 1-handles of each side are in one-to-one correspondence with the 2-handles of the complement. This is true since the embeddings in D^4 considered here are all standard, and in particular each 2-handle is unknotted in D^4. The statement follows from the fact that 1-handles may be viewed as standard 2-handles removed from a collar, a standard technique in Kirby calculus (see Chapter 1 in [9]). Moreover, in each of our examples the attaching curve α on the A-side bounds a surface in A, so that it has a zero framed 2-handle attached to the core of the solid torus. On the 3-manifold level, the zero surgery on this core transforms the solid torus corresponding to A into the solid torus corresponding to B. The Kirby diagram for B is obtained by taking the diagram for A, performing the surgery as above, and replacing all zeroes with dots, and conversely all dots with zeroes. (The 2-handles in all our examples are zero-framed.)

Note that a distinguished pair of curves α_1, α_2, forming a symplectic basis in the surface S, is determined as the meridians (linking circles) to the cores of the 2-handles H_1, H_2 of B_0 in D^4. In other words, α_1, α_2 are fibers of the circle normal bundles over the cores of H_1, H_2 in D^4.

An important observation [6] is that this construction may be iterated: consider the 2-handle H_1 in place of the original 4-ball. The pair of curves (α_1, the attaching circle β_1 of H_1) forms the Hopf link in the boundary of H_1. In H_1 consider the standard genus-one surfaces bounded by β_1. As discussed above, its complement is given by two zero-framed 2-handles attached to the

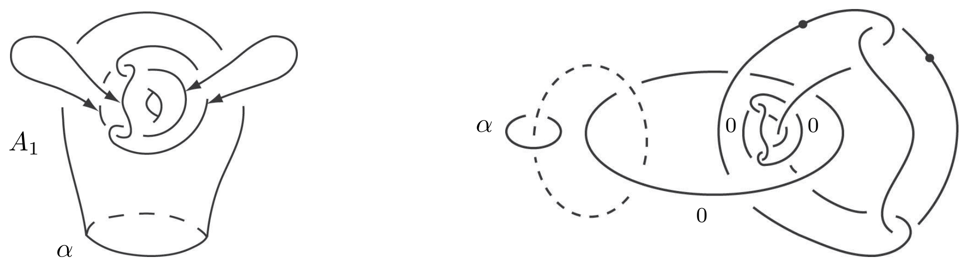

Figure 5

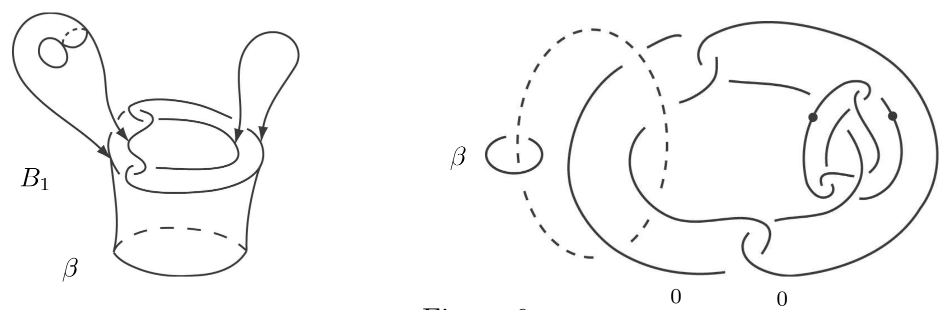

Figure 6

Bing double of α_1. Assembling the data, consider the new decomposition $D^4 = A_1 \cup B_1$, Figures 5, 6. As above, the diagrams are drawn in solid tori (complements in S^3 of unknotted circles drawn dashed in the figures). The handlebodies A_1, B_1 are examples of *model decompositions* [6] obtained by iterated applications of the construction above. It is shown in [11], [12] that such model handlebodies are robust, or in other words the Borromean rings are not weakly $A - B$ slice when restricted to the class of model decompositions.

We are now in a position to define the decomposition $D^4 = A \cup B$ used in the proof of Theorem 1.

Definition 4.2. Consider $B = (B_1 \cup$ zero-framed 2-handle) attached as shown in the Kirby diagram in Figure 7. The effect of this 2-handle on the complement $A = D^4 \setminus B$ is shown in Figure 8: it adds a 1-handle to the diagram of A_1. Figure 9 shows a handle diagram of A after a handle slide. Note that a $(1-, 2-)$ handle pair may be canceled, the result is given on the left in Figure 12. This fact will be used in the proof of Theorem 1.

Imprecisely (up to homotopy, on the level of spines) B may be viewed as a $B_1 \cup$ 2-cell attached along the attaching circle β of B_1, followed by a

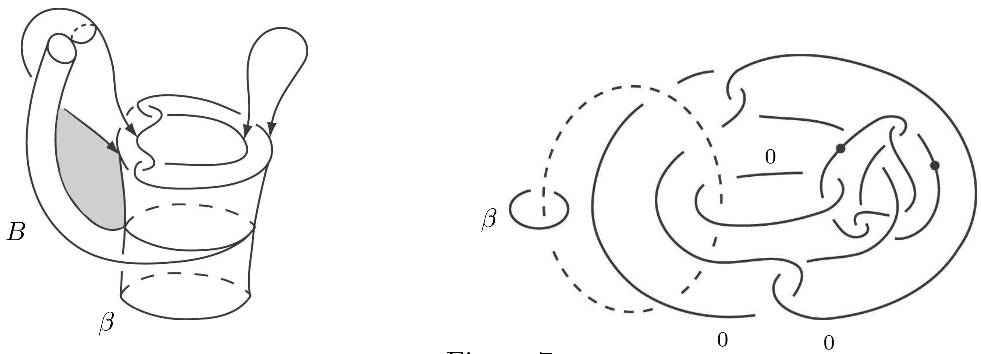

Figure 7

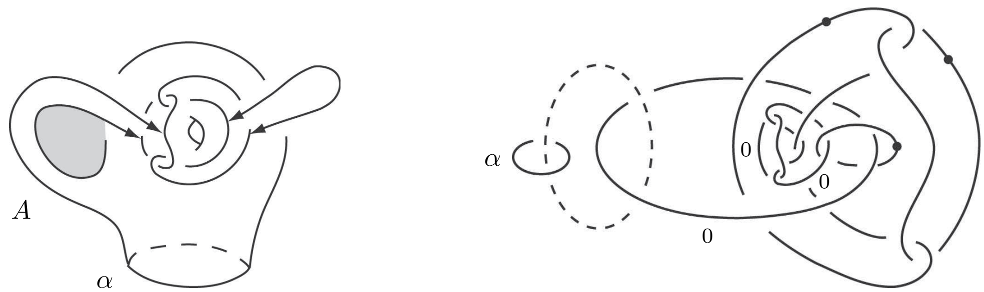

Figure 8

curve representing a generator of H_1 of the second stage surface of B_1. This 2-cell is schematically shown in the spine picture of B in the first part of Figure 7 as a cylinder connecting the two curves. The shading indicates that the new generator of π_1 created by adding the cylinder is filled in with a disk. Similarly, one checks that the effect of this operation on the A-side is that one of the 2-handles at the second stage is connected-summed with the first stage surface, Figure 8. (This is seen in the handle diagram by canceling a 1-, 2-handle pair, as shown in Figure 12.) Again, the shading indicates that no new generators of π_1 are created. The figures showing the spines are provided only as a motivation for the construction; a precise description of A, B is of course given by their handle diagrams. While the proof of Theorem 1 below is given in terms of Kirby diagrams, it can easily be followed at the level of spines.

5. Proof of Theorem 1: a relative slice problem

We start this section by recalling the technique which will be useful in completing the proof of Theorem 1, the *relative slice problem*, introduced in

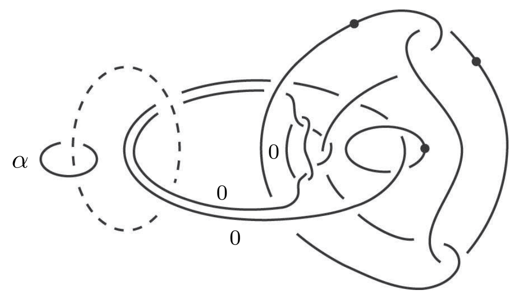

Figure 9: A handle diagram for A after a handle slide.

[6]. The setup in our context is as follows: suppose two codimension zero submanifolds M, N of D^4 are given; each one has an attaching circle $\gamma \subset \partial M$, $\delta \subset \partial N$. The submanifolds are proper in the sense that one has embeddings of pairs $(M, \gamma) \subset (D^4, S^3)$, $(N, \delta) \subset (D^4, S^3)$, where each circle γ, δ is unknotted in the 3-sphere.

The problem that has to be analyzed is: can (M, γ), (N, δ) be embedded *disjointly* into (D^4, S^3) so that the curves γ, δ form the Hopf link in the 3-sphere? Assume that M, N have handle decompositions, relative to the attaching regions $S^1 \times D^2$, with only 1- and 2-handles. Let γ, δ form the Hopf link in ∂D^4, and consider the 4-ball $D' = D^4 \smallsetminus (\text{collar on } \partial D^4)$. To be precise, denote the 1-handles of M, N by $\mathcal{H}_1$, $\mathcal{H}'_1$, and their 2-handles by $\mathcal{H}_2$, $\mathcal{H}'_2$. As usual, we view the 1-handles of M, N as standard slices removed from their collars. Denote these slices by $\mathcal{H}^*_1$, $\mathcal{H}'^*_1$. Then M, N embed disjointly into D^4 if and only if there are disjoint embeddings of the 2-handles $\mathcal{H}_2 \cup \mathcal{H}'_2$, attached to the collars, in the handlebody $D' \cup \mathcal{H}^*_1 \cup \mathcal{H}'^*_1$.

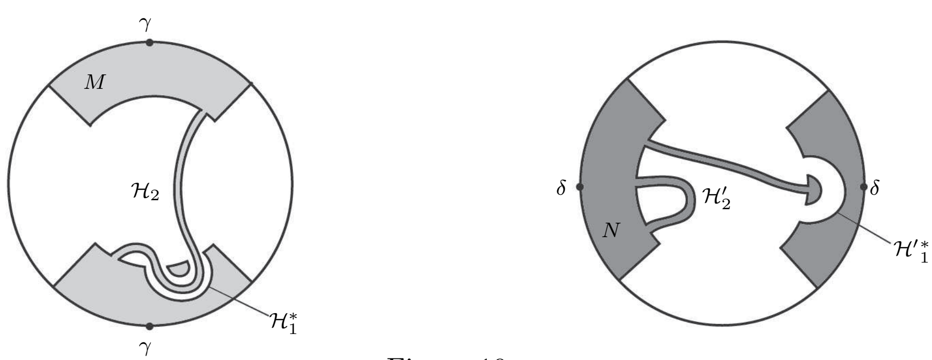

Figure 10

An example of M, N drawn in two dimensions is given in Figure 10, and a solution to this relative-slice problem – disjoint embeddings of M, N in D^4 with their attaching circles γ, δ forming a Hopf link in ∂D^4 – is shown in Figure 11.

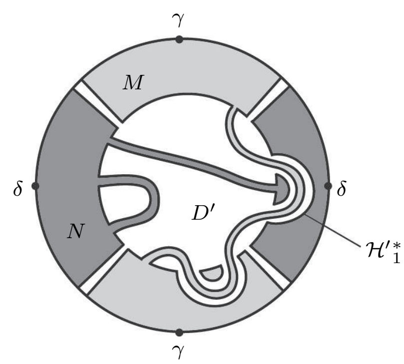

Figure 11: Disjoint embeddings of (M, γ), (N, δ) in Figure 10 into (D^4, S^3), where γ, δ form a Hopf link in S^3.

Note that the handle $\mathcal{H}_2$ of M in the solution goes over the "helping" handle $\mathcal{H}'^*_1$ attached to D'.

Consider the decomposition $D^4 = A \cup B$ constructed in Definition 4.2. The proof of Theorem 1 follows from Lemmas 5.1 and 5.2 below.

Lemma 5.1. *Let S denote the genus-one surface with one boundary component, $\gamma = \partial S$. Denote by S_0 its untwisted 4-dimensional thickening, $S_0 = S \times D^2$, and set $\gamma_0 = \gamma \times \{0\}$. Then there exists a proper embedding $(A, \alpha) \subset (S_0, \gamma_0)$.*

Proof. Kirby diagrams of A are given in Figures 8, 9. Observe that a (1-, 2-handle) pair in the diagram in Figure 9 may be canceled, the result is shown on the left in Figure 12.

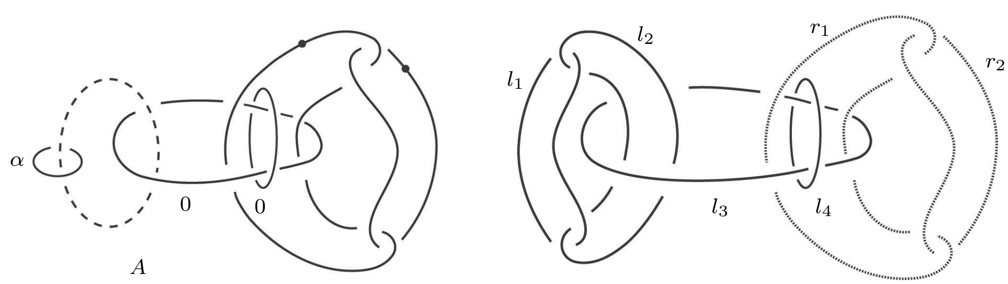

Figure 12

In light of Proposition 4.1, to prove that A embeds in S_0 it suffices to show that (A, α) embeds in the complement of a standard embedding of two zero-framed 2-handles attached to the Bing double of a meridian to α in S^3. This is an instance of the relative-slice problem discussed above, where $(M, \gamma) = (A, \alpha)$

and N is obtained from a collar on δ by attaching 2-handles to the Bing double of the core. (Note that (N, δ) equals (B_0, β) considered in Sections 4; see Figures 3 and 4.) This relative-slice problem is shown on the right in Figure 12. The link is considered in the 3-sphere boundary of the 4-ball D', and the link $l_1, \dots, l_4$ has to be sliced in the handlebody $D' \cup$ (2-handles $\mathcal{H}_1^*$) where the handles are attached with zero framings along r_1, r_2. Here l_1, l_2 are the attaching curves for the 2-handles of N and l_3, l_4 are the attaching curves for the 2-handles of M. Note that the slices for l_1, l_2 constructed in the proof are required to be standard in D^4, to make sure that their complement is the thickened surface S_0.

A solution to this relative-slice problem is given in Figures 13, 14. The slices are described in terms of the Morse function given by the radial coordinate in the 4-ball D'. Denote the 3-sphere at the radius R from the origin by S^3_R, $0 < R \leq 1$. The link on the right in Figure 12 lies in $\partial D' = S^3_1$. The link components move by an isotopy for $1 > R > 3/4$, and at $R = 3/4$ the component l_4 is connected-summed with a parallel copy of r_2. The result is denoted by l_4', Figure 13. Note that l_4' bounds a disk in $S^3_{3/4}$ in the complement of all other curves. To make the slice non-degenerate in terms of the Morse function, let l_4' bound a disk as R decreases from 3/4 to 1/2, while all other curves move by an isotopy. The link in $S^3_{1/2}$ is shown on the right in Figure 13.

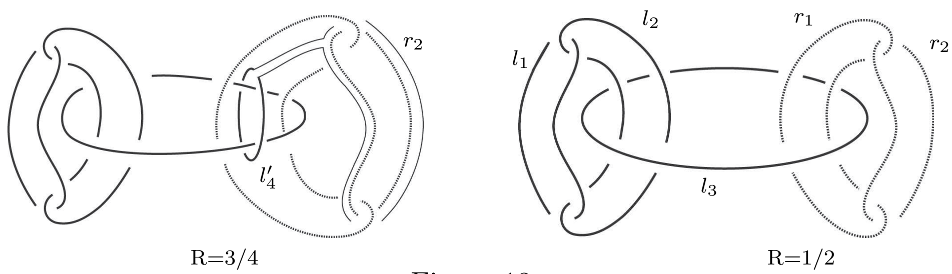

Figure 13

The curves $r_i \subset \partial D'$ bound disjoint embedded disks Δ_i: the cores of the zero-framed 2-handles $\mathcal{H}_1^*$ attached to D'. As the Morse function R changes from 1 to 0, it is important to note that the curves r_i move by an isotopy and no other curves intersect them. Therefore, r_1, r_2 at each radius R_0 bound disjoint disks: the disks Δ_i as above, union with the annuli corresponding to the isotopy of r_i for $1 > R > R_0$. Moreover, since the handles $\mathcal{H}_1^*$ attached to D' are zero-framed, untwisted parallel copies of r_i also bound disjoint embedded disks.

Morse-theoretically the connected sum at $R = 3/4$ in Figure 13 corresponds to a saddle point of the slice for l_4. This slice is of the form shown in Figure 15 (disregard the labels in that figure, which are used for a later argument).

To finish the proof of the relative-slice problem, let the link in $S^3_{1/2}$ move by an isotopy for $1/2 > R > 1/4$, and at $R = 1/4$ the components l_1, l_2 are connected-summed with r_1, r_2 as shown on the left in Figure 14. Denote the resulting curves by l'_1, l'_2. The components l'_1, l'_2, l_3 form the unlink. This is seen by performing an isotopy (at $1/4 > R > 1/8$) to the link on the right in Figure 14. Now let all curves bound disks at $1/8 > R > 0$. The slices for l_1, l_2 again have the form shown in Figure 15; the slice for l_3 has just a single critical point.

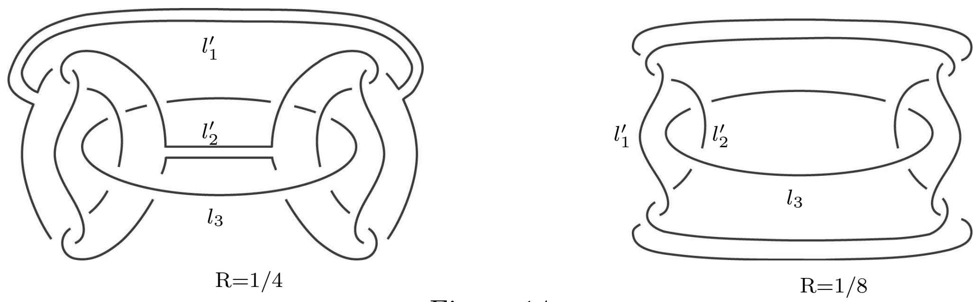

Figure 14

This concludes the proof of the relative-slice problem. It remains to show that the slices S_1, S_2 for l_1, l_2 constructed above are standard. We start by recording the data involved in their construction. (γ, δ) is a Hopf link in ∂D^4, $D' = D \setminus$(collar on ∂D^4). Since the slices were described in terms of the radial Morse function on D', to be specific consider D' as the 4-ball of radius 1 in D^4 of radius 2. The curves l_1, l_2 are in the boundary of D'; extending them by the product $l_i \times [1, 2]$ we will consider them as curves in ∂D^4.

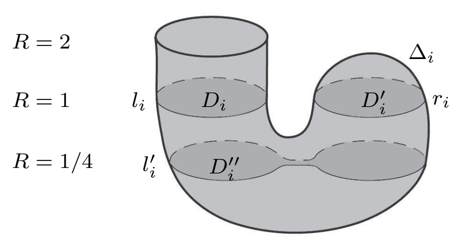

Figure 15: The 3-ball B_i.

For the rest of this argument, we only need to consider the curves l_1, l_2 and their slices; the slices for the other components are disregarded. Since l_1, l_2 form the unlink they therefore bound disjoint embedded disks D_1, D_2 in ∂D^4. We will show that the slices S_1, S_2 for l_1, l_2 are standard by constructing disjoint embedded 3-balls B_1, B_2 in D^4, with $\partial B_i = D_i \cup S_i$ for each i. The

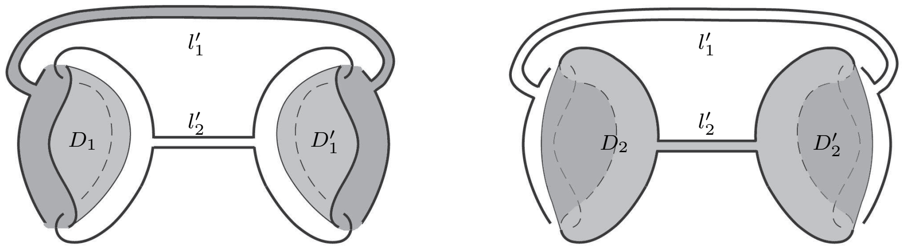

Figure 16: Disjoint disks D_1'', D_2'': on the left D_1'' is a band sum of D_1, D_1', on the right D_2'' is a band sum of D_2, D_2'.

existence of these 3-balls provides an isotopy in D^4 from S_1, S_2 to D_1, D_2 and shows that the slices are standard.

The construction of S_i is illustrated in Figure 15. The vertical axis in this figure corresponds to the radial component in D^4. There is a single maximum point given by the core Δ_i of the 2-handle attached to D' along r_i. Recall that Δ_1, Δ_2 are embedded in D^4 in a standard way, and so they are isotopic to disjoint embedded disks D_1', D_2' bounded by r_1, r_2 in the 3-sphere slice $\partial D' = S_1^3$. The curves l_i, r_i, and the disks bounded by them: D_i, D_i' move by an isotopy as R decreases from 1 to 1/4 until the index 1 critical points of the slices at $R = 1/4$ (shown in Figure 14).

The analysis of the disks at the level $R = 1/4$ is presented in Figure 16. At the level of these critical points, the disks D_i and D_i' are band-summed, and as a result, the disks D_1'', D_2'' are disjoint. The component l_1' on the left in Figure 16 bounds D_1'', the component l_2' on the right bounds D_2''. (Figure 16 has two copies of the link (l_1', l_2') just for convenience of visualization of the disks D_1'', D_2''.) Finally, at $R < 1/4$ the disks D_i'' move by an isotopy and shrink to points.

We summarize the construction of the disjoint 3-balls B_i, $i = 1, 2$: in the 3-sphere S_R^3, each component of $S_R^3 \cap S_i$ bounds a disk: $l_i = \partial D_i$, $r_i = \partial D_i'$, $l_i' = \partial D_i''$. These disks are the levels of the radial Morse function restricted to B_i. This concludes the proof that (A, α) embeds into (S_0, γ_0). $\square$

Lemma 5.2. *B embeds in a collar on its attaching curve. More precisely, there exists a proper embedding* $(B, \beta) \subset (S^1 \times D^2 \times [0, 1],\ S^1 \times \{0\} \times \{0\})$.

One needs to show that (B, β) embeds in the complement of a standard disk bounded by the meridian to β. The proof is again a relative-slice problem, shown in Figure 17. Here l_1 is the meridian which is required to bound a standard disk; l_2, l_3, l_4 are the attaching curves of the 2-handles of B, and r_1, r_2 are the attaching curves for the 2-handles attached to D'. Therefore the link $l_1, \ldots, l_4$ has to be sliced in $D' \cup_{r_1, r_2}$(zero-framed 2-handles), so that the slice for l_1 is standard in D^4.

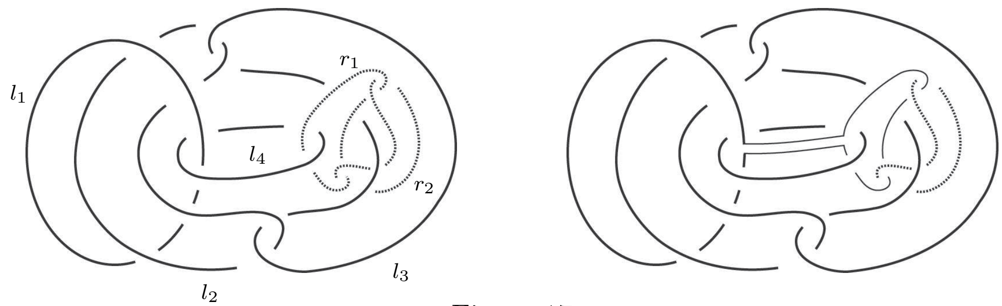

Figure 17

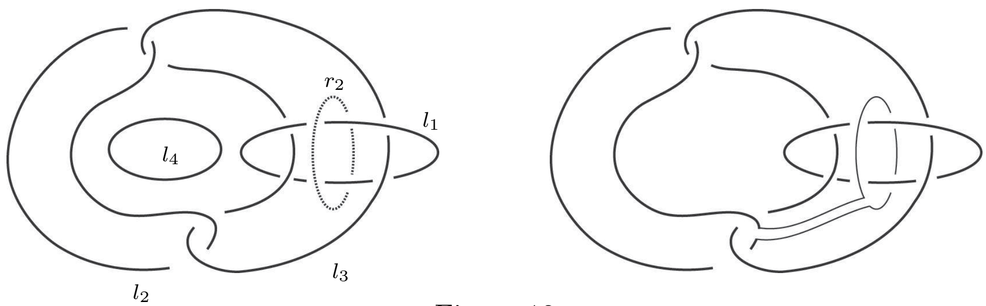

Figure 18

Taking a connected sum of l_1 and r_1 as shown in Figure 17, one gets the link on the left in Figure 18. Now taking a connected sum of l_2 and r_2 results in the trivial link, and the components are capped off with disjoint disks in D'. The proof that the slice for l_1 is standard is directly analogous to the corresponding proof in Lemma 5.1. □

Proof of Theorem 1 in the central case $L =$ Bor, the Borromean rings, follows from Lemmas 5.1, 5.2. The components l_i of Bor bound in D^4 disjoint embedded surfaces S_i; S_1 is a genus one surface, and S_2, S_3 are disks. Thinking of the radial coordinate of D^4 as time where ∂D^4 corresponds to time 0, we see that l_1 bounds a surface S_1 (shaded in Figure 19) at time $1/2$ and the other two

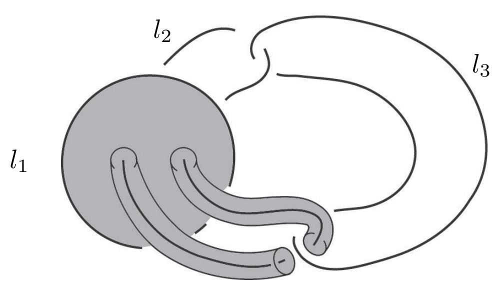

Figure 19

components bound disjoint disks at $t > 1/2$. Consider three decompositions of D^4: (A_1, B_1) is the decomposition constructed in Section 4. Define (A_2, B_2) and (A_3, B_3) to be the trivial decomposition: $A_2 = A_3 = D^2 \times D^2$, $B_2 = B_3$ are collars on their attaching curves. Lemmas 5.1, 5.2 imply that the Borromean rings are weakly $A - B$ slice with these decompositions.

To prove Theorem 1 for all links with trivial linking numbers, a variation of Lemmas 5.1, 5.2 is needed, for higher genus surfaces. That is, given any g there is a decomposition $D^4 = A_g \cup B_g$ such that A_g embeds in (surface of genus g) $\times D^2$ and B_g embeds in a collar. These are variations of the decompositions A, B in Definition 4.2; the case $g = 2$ is shown in Figure 20. The proof is analogous to the proof of Lemmas 5.1, 5.2. To complete the proof of Theorem 1, note that the components of any link with trivial linking numbers bound disjoint embedded surfaces in D^4. $\square$

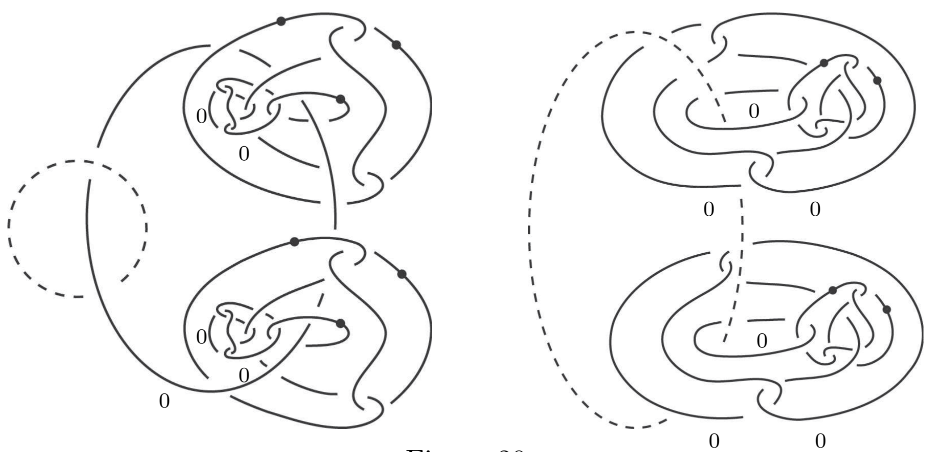

Figure 20

Remark. If the embeddings $(A, \alpha) \subset (S_0, \gamma_0) \subset (D^4, S^3)$ and $(B, \beta) \subset (M, \gamma) \subset (D^4, S^3)$, constructed in Lemmas 5.1, 5.2 were *standard*, then taking the complement of the six submanifolds (three copies of each of A and B) bounding the Borromean rings and their parallel copy in D^4 and gluing up the boundary, one would get a solution to a canonical surgery problem. Considering the generalized Borromean rings, one would get solutions to all canonical problems, and therefore a proof of the topological 4-dimensional surgery conjecture for all fundamental groups. However the embeddings constructed in the proof are not standard. This raises the question mentioned in the introduction: Given a decomposition $D^4 = A \cup B$, is one of the *embeddings* $A \hookrightarrow D^4$, $B \hookrightarrow D^4$ necessarily robust?

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