Annals of Mathematics, **168** (2008), 643–674

Diffusion and mixing in fluid flow

By P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatoš

Abstract

We study enhancement of diffusive mixing on a compact Riemannian manifold by a fast incompressible flow. Our main result is a sharp description of the class of flows that make the deviation of the solution from its average arbitrarily small in an arbitrarily short time, provided that the flow amplitude is large enough. The necessary and sufficient condition on such flows is expressed naturally in terms of the spectral properties of the dynamical system associated with the flow. In particular, we find that weakly mixing flows always enhance dissipation in this sense. The proofs are based on a general criterion for the decay of the semigroup generated by an operator of the form $\Gamma + iAL$ with a negative unbounded self-adjoint operator Γ, a self-adjoint operator L, and parameter $A \gg 1$. In particular, they employ the RAGE theorem describing evolution of a quantum state belonging to the continuous spectral subspace of the hamiltonian (related to a classical theorem of Wiener on Fourier transforms of measures). Applications to quenching in reaction-diffusion equations are also considered.

1. Introduction

Let M be a smooth compact d-dimensional Riemannian manifold. The main objective of this paper is the study of the effect of a strong incompressible flow on diffusion on M. Namely, we consider solutions of the passive scalar equation

$$\phi_t^A(x,t) + Au \cdot \nabla \phi^A(x,t) - \Delta \phi^A(x,t) = 0, \quad \phi^A(x,0) = \phi_0(x). \tag{1.1}$$

Here Δ is the Laplace-Beltrami operator on M, u is a divergence free vector field, ∇ is the covariant derivative, and $A \in \mathbb{R}$ is a parameter regulating the strength of the flow. We are interested in the behavior of solutions of (1.1) for $A \gg 1$ at a fixed time $\tau > 0$.

It is well known that as time tends to infinity, the solution $\phi^A(x,t)$ will tend to its average,

$$\overline{\phi} \equiv \frac{1}{|M|}\int\limits_M \phi^A(x,t)\,d\mu = \frac{1}{|M|}\int\limits_M \phi_0(x)\,d\mu,$$

with $|M|$ being the volume of M. We would like to understand how the speed of convergence to the average depends on the properties of the flow and determine which flows are efficient in enhancing the relaxation process.

The question of the influence of advection on diffusion is very natural and physically relevant, and the subject has a long history. The passive scalar model is one of the most studied PDEs in both mathematical and physical literature. One important direction of research focused on homogenization, where in a long time–large propagation distance limit the solution of a passive advection-diffusion equation converges to a solution of an effective diffusion equation. Then one is interested in the dependence of the diffusion coefficient on the strength of the fluid flow. We refer to [29] for more details and references. The main difference in the present work is that here we are interested in the flow effect in a finite time without the long time limit.

On the other hand, the Freidlin-Wentzell theory [16], [17], [18], [19] studies (1.1) in $\mathbb{R}^2$ and, for a class of Hamiltonian flows, proves the convergence of solutions as $A \to \infty$ to solutions of an effective diffusion equation on the Reeb graph of the hamiltonian. The graph, essentially, is obtained by identifying all points on any streamline. The conditions on the flows for which the procedure can be carried out are given in terms of certain non-degeneracy and growth assumptions on the stream function. The Freidlin-Wentzell method does not apply, in particular, to ergodic flows or in odd dimensions.

Perhaps the closest to our setting is the work of Kifer and more recently a result of Berestycki, Hamel and Nadirashvili. Kifer's work (see [21], [22], [23], [24] where further references can be found) employs probabilistic methods and is focused, in particular, on the estimates of the principal eigenvalue (and, in some special situations, other eigenvalues) of the operator $-\epsilon\Delta + u\cdot\nabla$ when ϵ is small, mainly in the case of the Dirichlet boundary conditions. In particular, the asymptotic behavior of the principal eigenvalue λ_0^ϵ and the corresponding positive eigenfunction ϕ_0^ϵ for small ϵ has been described in the case where the operator $u\cdot\nabla$ has a discrete spectrum and sufficiently smooth eigenfunctions. It is well known that the principal eigenvalue determines the asymptotic rate of decay of the solutions of the initial value problem, namely

$$\lim_{t\to\infty} t^{-1}\log\|\phi^\epsilon(x,t)\|_{L^2} = -\lambda_0^\epsilon \tag{1.2}$$

(see e.g. [22]). In a related recent work [2], Berestycki, Hamel and Nadirashvili utilize PDE methods to prove a sharp result on the behavior of the principal eigenvalue λ_A of the operator $-\Delta + Au \cdot \nabla$ defined on a bounded domain $\Omega \subset \mathbb{R}^d$ with the Dirichlet boundary conditions.

The main conclusion is that λ_A stays bounded as $A \to \infty$ if and only if u has a first integral w in $H_0^1(\Omega)$ (that is, $u \cdot \nabla w = 0$). An elegant variational principle determining the limit of λ_A as $A \to \infty$ is also proved. In addition, [2] provides a direct link between the behavior of the principal eigenvalue and the dynamics which is more robust than (1.2): it is shown that $\|\phi^A(\cdot, 1)\|_{L^2(\Omega)}$ can be made arbitrarily small for any initial datum by increasing A if and only if $\lambda_A \to \infty$ as $A \to \infty$ (and, therefore, if and only if the flow u does not have a first integral in $H_0^1(\Omega)$). We should mention that there are many earlier works providing variational characterization of the principal eigenvalues, and refer to [2], [24] for more references.

Many of the studies mentioned above also apply in the case of a compact manifold without boundary or Neumann boundary conditions, which are the primary focus of this paper. However, in this case the principal eigenvalue is simply zero and corresponds to the constant eigenfunction. Instead one is interested in the speed of convergence of the solution to its average, the relaxation speed. A recent work of Franke [15] provides estimates on the heat kernels corresponding to the incompressible drift and diffusion on manifolds, but these estimates lead to upper bounds on $\|\phi^A(1) - \overline{\phi}\|$ which essentially do not improve as $A \to \infty$. One way to study the convergence speed is to estimate the spectral gap – the difference between the principal eigenvalue and the real part of the next eigenvalue. To the best of our knowledge, there is very little known about such estimates in the context of (1.1); see [22] p. 251 for a discussion. Neither probabilistic methods nor PDE methods of [2] seem to apply in this situation, in particular because the eigenfunction corresponding to the eigenvalue(s) with the second smallest real part is no longer positive and the eigenvalue itself does not need to be real.

Moreover, even if the spectral gap estimate were available, generally it only yields a limited asymptotic in time dynamical information of type (1.2), and how fast the long time limit is achieved may depend on A. Part of our motivation for studying the advection-enhanced diffusion comes from the applications to quenching in reaction-diffusion equations (see e.g. [4], [12], [27], [34], citeZ), which we discuss in Section 7. For these applications, one needs estimates on the A-dependent L^∞ norm decay at a fixed positive time, the type of information the bound like (1.2) does not provide. We are aware of only one case where enhanced relaxation estimates of this kind are available. It is the recent work of Fannjiang, Nonnemacher and Wolowski [10], [11], where such estimates are provided in the discrete setting (see also [22] for some related earlier references). In these papers a unitary evolution step (a certain measure-preserving map on the torus) alternates with a dissipation step, which, for example, acts simply by multiplying the Fourier coefficients by damping factors. The absence of sufficiently regular eigenfunctions appears as a key for the lack of enhanced relaxation in this particular class of dynamical systems. In [10], [11], the authors also provide finer estimates of the dissipation time for particular classes of toral automorphisms (that is, they estimate how many steps are needed to reduce the L^2 norm of the solution by a factor of two if the diffusion strength is ϵ).

Our main goal in this paper is to provide a sharp characterization of incompressible flows that are relaxation enhancing, in a quite general setup. We work directly with dynamical estimates, and do not discuss the spectral gap. The following natural definition will be used in this paper as a measure of the flow efficiency in improving the solution relaxation.

Definition 1.1. Let M be a smooth compact Riemannian manifold. The incompressible flow u on M is called *relaxation enhancing* if for every $\tau > 0$ and $\delta > 0$, there exist $A(\tau, \delta)$ such that for any $A > A(\tau, \delta)$ and any $\phi_0 \in L^2(M)$ with $\|\phi_0\|_{L^2(M)} = 1$,

$$\|\phi^A(\cdot, \tau) - \overline{\phi}\|_{L^2(M)} < \delta, \tag{1.3}$$

where $\phi^A(x, t)$ is the solution of (1.1) and $\overline{\phi}$ the average of ϕ_0.

Remarks. 1. In Theorem 5.5 we show that the choice of the L^2 norm in the definition is not essential and can be replaced by any L^p-norm with $1 \leq p \leq \infty$.

2. It follows from the proofs of our main results that the relaxation-enhancing class is not changed even when we allow the flow strength that ensures (1.3) to depend on ϕ_0, that is, if we require (1.3) to hold for all $\phi_0 \in L^2(M)$ with $\|\phi_0\|_{L^2(M)} = 1$ and all $A > A(\tau, \delta, \phi_0)$.

Our first result is as follows.

THEOREM 1.2. *Let M be a smooth compact Riemannian manifold. A Lipschitz continuous incompressible flow $u \in \mathrm{Lip}(M)$ is relaxation-enhancing if and only if the operator $u \cdot \nabla$ has no eigenfunctions in $H^1(M)$, other than the constant function.*

Any incompressible flow $u \in \mathrm{Lip}(M)$ generates a unitary evolution group U^t on $L^2(M)$, defined by $U^t f(x) = f(\Phi_{-t}(x))$. Here $\Phi_t(x)$ is a measure-preserving transformation associated with the flow, defined by $\frac{d}{dt}\Phi_t(x) = u(\Phi_t(x))$, $\Phi_0(x) = x$. Recall that a flow u is called weakly mixing if the corresponding operator U has only continuous spectrum. The weakly mixing flows are ergodic, but not necessarily mixing (see e.g. [5]). There exist fairly explicit examples of weakly mixing flows [1], [13], [14], [28], [35],u [33], some of which we will discuss in Section 6. A direct consequence of Theorem 1.2 is the following corollary.

Corollary 1.3. *Any weakly mixing incompressible flow* $u \in \mathrm{Lip}(M)$ *is relaxation enhancing.*

Theorem 1.2, as we will see in Section 5, in its turn follows from a quite general abstract criterion, which we are now going to describe. Let Γ be a self-adjoint, positive, unbounded operator with a discrete spectrum on a separable Hilbert space H. Let $0 < \lambda_1 \le \lambda_2 \le \dots$ be the eigenvalues of Γ, and e_j the corresponding orthonormal eigenvectors forming a basis in H. The (homogeneous) Sobolev space $H^m(\Gamma)$ associated with Γ is formed by all vectors $\psi = \sum_j c_j e_j$ such that

$$\|\psi\|^2_{H^m(\Gamma)} \equiv \sum_j \lambda_j^m |c_j|^2 < \infty.$$

Note that $H^2(\Gamma)$ is the domain $D(\Gamma)$ of Γ. Let L be a self-adjoint operator such that, for any $\psi \in H^1(\Gamma)$ and $t > 0$,

$$\|L\psi\|_H \le C\|\psi\|_{H^1(\Gamma)} \text{ and } \|e^{iLt}\psi\|_{H^1(\Gamma)} \le B(t)\|\psi\|_{H^1(\Gamma)} \tag{1.4}$$

with both the constant C and the function $B(t) < \infty$ independent of ψ and $B(t) \in L^2_{\mathrm{loc}}(0,\infty)$. Here e^{iLt} is the unitary evolution group generated by the self-adjoint operator L. One might ask whether one of the two conditions in (1.4) does not imply the other. We show at the end of Section 2, by means of an example, that this is not the case in general.

Consider a solution $\phi^A(t)$ of the Bochner differential equation

$$\frac{d}{dt}\phi^A(t) = iAL\phi^A(t) - \Gamma\phi^A(t), \quad \phi^A(0) = \phi_0. \tag{1.5}$$

Theorem 1.4. *Let* Γ *be a self-adjoint, positive, unbounded operator with a discrete spectrum and let a self-adjoint operator* L *satisfy conditions* (1.4). *Then the following two statements are equivalent*:

- *For any* $\tau, \delta > 0$ *there exists* $A(\tau,\delta)$ *such that for any* $A > A(\tau,\delta)$ *and any* $\phi_0 \in H$ *with* $\|\phi_0\|_H = 1$, *the solution* $\phi^A(t)$ *of the equation* (1.5) *satisfies* $\|\phi^A(\tau)\|^2_H < \delta$.
- *The operator* L *has no eigenvectors lying in* $H^1(\Gamma)$.

Remark. Here L corresponds to $iu \cdot \nabla$ (or, to be precise, a self-adjoint operator generating the unitary evolution group U^t which is equal to $iu \cdot \nabla$ on $H^1(M)$), and Γ to $-\Delta$ in Theorem 1.2, with $H \subset L^2(M)$ the subspace of mean zero functions.

Theorem 1.4 provides a sharp answer to the general question of when a combination of fast unitary evolution and dissipation produces a significantly stronger dissipative effect than dissipation alone. It can be useful in any model describing a physical situation which involves fast unitary dynamics with dissipation (or, equivalently, unitary dynamics with weak dissipation). We prove Theorem 1.4 in Section 3. The proof uses ideas from quantum dynamics, in particularly the RAGE theorem (see e.g., [6]) describing evolution of a quantum state belonging to the continuous spectral subspace of a self-adjoint operator.

A natural concern is the consistency of the existence of rough eigenvectors of L and condition (1.4) which says that the dynamics corresponding to L preserves $H^1(\Gamma)$. In Section 4 we establish this consistency by providing examples where rough eigenfunctions exist yet (1.4) holds. One of them involves a discrete version of the celebrated Wigner-von Neumann construction of an imbedded eigenvalue of a Schrödinger operator [32]. Moreover, in Section 6 we describe an example of a smooth flow on the two dimensional torus $\mathbb{T}^2$ with discrete spectrum and rough (not $H^1(\mathbb{T}^2)$) eigenfunctions – this example essentially goes back to Kolmogorov [28]. Thus, the result of Theorem 1.4 is precise.

In Section 7, we discuss the application of Theorem 1.2 to quenching for reaction-diffusion equations on compact manifolds and domains. This corresponds to adding a non-negative reaction term $f(T)$ on the right-hand side of (1.1), with $f(0) = f(1) = 0$. Then the long-term dynamics can lead to two outcomes: $\phi^A \to 1$ at every point (complete combustion), or $\phi^A \to c < 1$ (quenching). The latter case is only possible if f is of the ignition type; that is, there exists θ_0 such that $f(T) = 0$ for $T \leq \theta_0$, and $c \leq \theta_0$. The question is then how the presence of strong fluid flow may aid the quenching process. We note that quenching/front propagation in infinite domains is also of considerable interest. Theorem 1.2 has applications in that setting as well, but they will be considered elsewhere.

2. Preliminaries

In this section we collect some elementary facts and estimates for the equation (1.5). Henceforth we are going to denote the standard norm in the Hilbert space H by $\|\cdot\|$, the inner product in H by $\langle\cdot,\cdot\rangle$, the Sobolev spaces $H^m(\Gamma)$ simply by H^m and norms in these Sobolev spaces by $\|\cdot\|_m$. We have the following existence and uniqueness theorem.

THEOREM 2.1. *Assume that for any* $\psi \in H^1$,

$$\|L\psi\| \leq C\|\psi\|_1. \tag{2.1}$$

Then for any $T > 0$, *there exists a unique solution* $\phi(t)$ *of the equation*

$$\phi'(t) = (iL - \Gamma)\phi(t), \quad \phi(0) = \phi_0 \in H^1.$$

This solution satisfies

$$\phi(t) \in L^2([0,T], H^2) \cap C([0,T], H^1), \;\; \phi'(t) \in L^2([0,T], H). \tag{2.2}$$

Remarks. 1. The proof of Theorem 2.1 is standard, and can proceed by construction of a weak solution using Galerkin approximations and then establishing uniqueness and regularity. We refer, for example, to Evans [8] where the construction is carried out for parabolic PDEs but, given the assumption (2.1), can be applied verbatim in the general case.

2. The existence theorem is also valid for initial data $\phi_0 \in H$, but the solution has rougher properties at intervals containing $t = 0$, namely

$$\phi(t) \in L^2([0,T], H^1) \cap C([0,T], H), \quad \phi'(t) \in L^2([0,T], H^{-1}). \tag{2.3}$$

The existence of a rougher solution can also be derived from the general semigroup theory, by checking that $iL-\Gamma$ satisfies the conditions of the Hille-Yosida theorem and thus generates a strongly continuous contraction semigroup in H (see, e.g. [7]).

Next we establish a few properties that are more specific to our particular problem. It will be more convenient for us, in terms of notation, to work with an equivalent reformulation of (1.5), by setting $\epsilon = A^{-1}$ and rescaling time by the factor ϵ^{-1}, thus arriving at the equation

$$(\phi^\epsilon)'(t) = (iL - \epsilon\Gamma)\phi^\epsilon(t), \quad \phi^\epsilon(0) = \phi_0. \tag{2.4}$$

Lemma 2.2. *Assume* (2.1); *then for any initial data* $\phi_0 \in H$, $\|\phi_0\| = 1$, *the solution* $\phi^\epsilon(t)$ *of* (2.4) *satisfies*

$$\epsilon \int_0^\infty \|\phi^\epsilon(t)\|_1^2 dt \le \frac{1}{2}. \tag{2.5}$$

Proof. Recall that if $\phi \in H^1(\Gamma)$, then $\Gamma\phi \in H^{-1}(\Gamma)$ and $\langle \Gamma\phi, \phi\rangle = \|\phi\|_1^2$. The regularity conditions (2.2)-(2.3) and the fact that L is self-adjoint allow us to compute

$$\frac{d}{dt}\|\phi^\epsilon\|^2 = \langle \phi^\epsilon, \phi^\epsilon_t\rangle + \langle \phi^\epsilon_t, \phi^\epsilon\rangle = -2\epsilon\|\phi^\epsilon\|_1^2. \tag{2.6}$$

Integrating in time and taking into account the normalization of ϕ_0, we obtain (2.5). □

An immediate consequence of (2.6) is the following result, that we state here as a separate lemma for convenience.

Lemma 2.3. *Suppose that for all times* $t \in (a,b)$ *we have* $\|\phi^\epsilon(t)\|_1^2 \ge N\|\phi^\epsilon(t)\|^2$. *Then the following decay estimate holds*:

$$\|\phi^\epsilon(b)\|^2 \le e^{-2\epsilon N(b-a)}\|\phi^\epsilon(a)\|^2.$$

Next we need an estimate on the growth of the difference between solutions corresponding to $\epsilon > 0$ and $\epsilon = 0$ in the H-norm.

Lemma 2.4. *Assume, in addition to* (2.1), *that for any* $\psi \in H^1$ *and* $t > 0$,

$$\|e^{iLt}\psi\|_1 \le B(t)\|\psi\|_1 \tag{2.7}$$

for some $B(t) \in L^2_{\mathrm{loc}}[0,\infty)$. *Let* $\phi^0(t), \phi^\epsilon(t)$ *be solutions of*

$$(\phi^0)'(t) = iL\phi^0(t), \quad (\phi^\epsilon)'(t) = (iL - \epsilon\Gamma)\phi^\epsilon(t),$$

satisfying $\phi^0(0) = \phi^\epsilon(0) = \phi_0 \in H^1$. *Then*

$$\frac{d}{dt}\|\phi^\epsilon(t) - \phi^0(t)\|^2 \le \frac{1}{2}\epsilon\|\phi^0(t)\|_1^2 \le \frac{1}{2}\epsilon B^2(t)\|\phi_0\|_1^2. \tag{2.8}$$

Remark. Note that $\phi^0(t) = e^{iLt}\phi_0$ by definition. Assumption (2.7) says that this unitary evolution is bounded in the $H^1(\Gamma)$ norm.

Proof. The regularity guaranteed by conditions (2.1), (2.7) and Theorem 2.1 allows us to multiply the equation

$$(\phi^\epsilon - \phi^0)' = iL(\phi^\epsilon - \phi^0) - \epsilon\Gamma\phi^\epsilon$$

by $\phi^\epsilon - \phi^0$. We obtain

$$\frac{d}{dt}\|\phi^\epsilon - \phi^0\|^2 \le 2\epsilon(\|\phi^\epsilon\|_1\|\phi^0\|_1 - \|\phi^\epsilon\|_1^2) \le \frac{1}{2}\epsilon\|\phi^0\|_1^2,$$

which is the first inequality in (2.8). The second inequality follows simply from assumption (2.7). □

The following corollary is immediate.

Corollary 2.5. *Assume that* (2.1) *and* (2.7) *are satisfied, and the initial data* $\phi_0 \in H^1$. *Then the solutions* $\phi^\epsilon(t)$ *and* $\phi^0(t)$ *defined in Lemma* 2.4 *satisfy*

$$\|\phi^\epsilon(t) - \phi^0(t)\|^2 \le \frac{1}{2}\epsilon\|\phi_0\|_1^2 \int_0^\tau B^2(t)\,dt$$

for any time $t \le \tau$.

Finally, we observe that conditions (2.1) and (2.7) are independent. Taking $L = \Gamma$ shows that (2.7) does not imply (2.1), because in this case the evolution e^{iLt} is unitary on H^1 but the domain of L is $H^2 \subsetneq H^1$. On the other hand, (2.1) does not imply (2.7), as is the case in the following example. Let $H \equiv L^2(0,1)$, define the operator Γ by $\Gamma f(x) \equiv \sum_n e^{n^2}\hat{f}(n)e^{2\pi inx}$ for all $f \in H$ such that $e^{n^2}\hat{f}(n) \in \ell^2(\mathbb{Z})$, and take $Lf(x) \equiv xf(x)$. Then L is bounded on H and so (2.1) holds automatically, but

$$\left[e^{itL}f\right](x) = f(x)e^{itx}$$

so that $e^{2\pi iL}e^{2\pi inx} = e^{2\pi i(n+1)x}$. It follows that $e^{2\pi iL}$ is not bounded on H^1 (and neither is e^{iLt} for any $t \neq 0$).

3. The abstract criterion

One direction in the proof of Theorem 1.4 is much easier. We start by proving this easy direction: that existence of $H^1(\Gamma)$ eigenvectors of L ensures existence of $\tau, \delta > 0$ and ϕ_0 with $\|\phi_0\| = 1$ such that $\|\phi^A(\tau)\| > \delta$ for all A – that is, if such eigenvectors exist, then the operator L is not relaxation enhancing.

Proof of the first part of Theorem 1.4. Assume that the initial datum $\phi_0 \in H^1$ for (1.5) is an eigenvector of L corresponding to an eigenvalue E, normalized so that $\|\phi_0\| = 1$. Take the inner product of (1.5) with ϕ_0. We arrive at

$$\frac{d}{dt}\langle \phi^A(t), \phi_0\rangle = iAE\langle \phi^A(t), \phi_0\rangle - \langle \Gamma\phi^A(t), \phi_0\rangle.$$

This and the assumption $\phi_0 \in H^1$ lead to

$$\left|\frac{d}{dt}\left(e^{-iAEt}\langle \phi^A(t), \phi_0\rangle\right)\right| \leq \frac{1}{2}\left(\|\phi^A(t)\|_1^2 + \|\phi_0\|_1^2\right).$$

Note that the value of the expression being differentiated on the left-hand side is equal to one at $t = 0$. By Lemma 2.2 (with a simple time rescaling) we have $\int_0^\infty \|\phi^A(t)\|_1^2\,dt \leq 1/2$. Therefore, for $t \leq \tau = (2\|\phi_0\|_1^2)^{-1}$ we have $|\langle \phi^A(t), \phi_0\rangle| \geq 1/2$. Thus, $\|\phi^A(\tau)\| \geq 1/2$, uniformly in A. □

Note also that we have proved that in the presence of an H^1 eigenvector of L, enhanced relaxation does not happen for some ϕ_0 even if we allow $A(\tau, \delta)$ to be ϕ_0-dependent as well. This explains Remark 2 after Definition 1.1.

The proof of the converse is more subtle, and will require some preparation. We switch to the equivalent formulation (2.4). We need to show that if L has no H^1 eigenvectors, then for all $\tau, \delta > 0$ there exists $\epsilon_0(\tau, \delta) > 0$ such that if $\epsilon < \epsilon_0$, then $\|\phi^\epsilon(\tau/\epsilon)\| < \delta$ whenever $\|\phi_0\| = 1$. The main idea of the proof can be naively described as follows. If the operator L has purely continuous spectrum or its eigenfunctions are rough then the H^1-norm of the free evolution (with $\epsilon = 0$) is large most of the time. However, the mechanism of this effect is quite different for the continuous and point spectra. On the other hand, we will show that for small ϵ the full evolution is close to the free evolution for a sufficiently long time. This clearly leads to dissipation enhancement.

The first ingredient that we need to recall is the so-called RAGE theorem.

THEOREM 3.1 (RAGE). *Let L be a self-adjoint operator in a Hilbert space H. Let P_c be the spectral projection on its continuous spectral subspace. Let C be any compact operator. Then for any $\phi_0 \in H$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|Ce^{iLt}P_c\phi_0\|^2\,dt = 0.$$

Clearly, the result can be equivalently stated for a unitary operator U, replacing e^{iLt} with U^t. The proof of the RAGE theorem can be found, for example, in [6].

A direct consequence of the RAGE theorem is the following lemma. Recall that we denote by $0 < \lambda_1 \leq \lambda_2 \leq \dots$ the eigenvalues of the operator Γ and by $e_1, e_2, \dots$ the corresponding orthonormal eigenvectors. Let us also denote by P_N the orthogonal projection on the subspace spanned by the first N eigenvectors $e_1, \dots, e_N$ and by $S = \{\phi \in H : \ \|\phi\| = 1\}$ the unit sphere in H. The following lemma shows that if the initial data lie in the continuous spectrum of L then the L-evolution will spend most of its time in the higher modes of Γ.

LEMMA 3.2. *Let $K \subset S$ be a compact set. For any $N, \sigma > 0$, there exists $T_c(N, \sigma, K)$ such that for all $T \geq T_c(N, \sigma, K)$ and any $\phi \in K$,*

$$\frac{1}{T}\int_0^T \|P_N e^{iLt} P_c \phi\|^2 \, dt \leq \sigma. \tag{3.1}$$

Remark. The key observation of Lemma 3.2 is that the time $T_c(N, \sigma, K)$ is uniform for all $\phi \in K$.

Proof. Since P_N is compact, we see that for any vector $\phi \in S$, there exists a time $T_c(N, \sigma, \phi)$ that depends on the function ϕ such that (3.1) holds for $T > T_c(N, \sigma, \phi)$ – this is assured by Theorem 3.1. To prove the uniformity in ϕ, note that the function

$$f(T, \phi) = \frac{1}{T}\int_0^T \|P_N e^{iLt} P_c \phi\|^2 \, dt$$

is uniformly continuous on S for all T (with constants independent of T):

$$\begin{aligned}
&|f(T,\phi) - f(T,\psi)| \\
&\quad \leq \frac{1}{T}\int_0^T \big| \|P_N e^{iLt} P_c \phi\| - \|P_N e^{iLt} P_c \psi\| \big| \left(\|P_N e^{iLt} P_c \phi\| + \|P_N e^{iLt} P_c \psi\| \right) dt \\
&\quad \leq (\|\phi\| + \|\psi\|) \frac{1}{T}\int_0^T \|P_N e^{iLt} P_c(\phi - \psi)\| dt \leq 2\|\phi - \psi\|.
\end{aligned}$$

Now, existence of a uniform $T_c(N, \sigma, K)$ follows from compactness of K by standard arguments. □

We also need a lemma which controls from below the growth of the H^1 norm of free solutions corresponding to rough eigenfunctions. We denote by P_p the spectral projection on the pure point spectrum of the operator L.

LEMMA 3.3. *Assume that not a single eigenvector of the operator L belongs to $H^1(\Gamma)$. Let $K \subset S$ be a compact set. Consider the set $K_1 \equiv \{\phi \in K \,|\, \|P_p\phi\| \geq 1/2\}$. Then for any $B > 0$ we can find $N_p(B,K)$ and $T_p(B,K)$ such that for any $N \geq N_p(B,K)$, any $T \geq T_p(B,K)$ and any $\phi \in K_1$,*

$$\frac{1}{T}\int_0^T \|P_N e^{iLt} P_p\phi\|_1^2\,dt \geq B. \tag{3.2}$$

Remark. Note that unlike (3.1), we have the H^1 norm in (3.2).

Proof. The set K_1 may be empty, in which case there is nothing to prove. Otherwise, denote by E_j the eigenvalues of L (distinct, without repetitions) and by Q_j the orthogonal projection on the space spanned by the eigenfunctions corresponding to E_j. First, let us show that for any $B > 0$ there is $N(B,K)$ such that for any $\phi \in K_1$,

$$\sum_j \|P_N Q_j\phi\|_1^2 \geq 2B \tag{3.3}$$

if $N \geq N(B,K)$. It is clear that for each fixed ϕ with $P_p\phi \neq 0$ we can find $N(B,\phi)$ so that (3.3) holds, since by assumption $Q_j\phi$ does not belong to H^1 whenever $Q_j\phi \neq 0$. Assume that $N(B,K)$ cannot be chosen uniformly for $\phi \in K_1$. This means that for any n, there exists $\phi_n \in K_1$ such that

$$\sum_j \|P_n Q_j \phi_n\|_1^2 < 2B.$$

Since K_1 is compact, we can find a subsequence n_l such that ϕ_{n_l} converges to $\tilde\phi \in K_1$ in H as $n_l \to \infty$. For any N and any $n_{l_1} > N$ we have

$$\sum_j \|P_N Q_j\tilde\phi\|_1^2 \leq \sum_j \|P_{n_{l_1}} Q_j\tilde\phi\|_1^2 \leq \liminf_{l\to\infty} \sum_j \|P_{n_{l_1}} Q_j \phi_{n_l}\|_1^2.$$

The last inequality follows by Fatou's lemma from the convergence of ϕ_{n_l} to $\tilde\phi$ in H and the fact that $\|P_{n_{l_1}} Q_j\psi\|_1 \leq \lambda_{n_{l_1}}^{1/2}\|\psi\|$ for any n_{l_1}. But now the expression on the right-hand side is less than or equal to

$$\liminf_{l\to\infty} \sum_j \|P_{n_l} Q_j\phi_{n_l}\|_1^2 \leq 2B.$$

Thus $\sum_j \|P_N Q_j\tilde\phi\|_1^2 \leq 2B$ for any N, a contradiction since $\tilde\phi \in K_1$.

Next, take $\phi \in K_1$ and consider

$$\frac{1}{T}\int_0^T \|P_N e^{iLt}P_p\phi\|_1^2\,dt = \sum_{j,l} \frac{e^{i(E_j-E_l)T}-1}{i(E_j-E_l)T}\langle \Gamma P_N Q_j\phi, P_N Q_l\phi\rangle. \tag{3.4}$$

In (3.4), we set $(e^{i(E_j-E_l)T}-1)/i(E_j-E_l)T \equiv 1$ if $j=l$. Notice that the sum above converges absolutely. Indeed, $P_N Q_j\phi = \sum_{i=1}^N \langle Q_j\phi, e_i\rangle e_i$, and $\langle \Gamma e_i, e_k\rangle = \lambda_i \delta_{ik}$; therefore

$$\langle \Gamma P_N Q_j\phi, P_N Q_l\phi\rangle = \sum_{i=1}^N \lambda_i \langle Q_j\phi, e_i\rangle \overline{\langle Q_l\phi, e_i\rangle}$$

and further, the sum on the right-hand side of (3.4) does not exceed

$$\begin{aligned} (3.5)\qquad \sum_{i=1}^N \lambda_i \sum_{j,l} &|\langle Q_j\phi, e_i\rangle\langle Q_l\phi, e_i\rangle| \\ &\le \lambda_N \sum_{i=1}^N \sum_{j,l} \|Q_j\phi\|\|Q_l\phi\| |\langle Q_j\phi/\|Q_j\phi\|, e_i\rangle \langle Q_l\phi/\|Q_l\phi\|, e_i\rangle| \\ &\le \lambda_N \sum_{i=1}^N \sum_{j,l} \|Q_l\phi\|^2 |\langle Q_j\phi/\|Q_j\phi\|, e_i\rangle|^2 \le \lambda_N N, \end{aligned}$$

with the second step obtained from the Cauchy-Schwartz inequality, and the third by $\|\phi\| = \|e_i\| = 1$. Then for each fixed N, we have by the dominated convergence theorem that the expression in (3.4) converges to $\sum_j \|\Gamma^{1/2} P_N Q_j\phi\|^2 = \sum_j \|P_N Q_j\phi\|_1^2$ as $T \to \infty$. Now assume $N \ge N_p(B,K) \equiv N(B,K)$, so that (3.3) holds. We claim that we can choose $T_p(B,K)$ so that for any $T \ge T_p(B,K)$ we have

$$\begin{aligned} (3.6)\qquad &\left| \frac{1}{T}\int_0^T \|P_N e^{iLt} P_p\phi\|_1^2\,dt - \sum_j \|P_N Q_j\phi\|_1^2 \right| \\ &\qquad = \left| \sum_{l\ne j} \frac{e^{i(E_j-E_l)T}-1}{i(E_j-E_l)T} \langle \Gamma P_N Q_j\phi, P_N Q_l\phi\rangle \right| \le B \end{aligned}$$

for all $\phi \in K_1$. Indeed, this follows from convergence to zero for each individual ϕ as $T \to \infty$, compactness of K_1, and uniform continuity of the expression in the middle of (3.6) in ϕ for each T (with constants independent of T). The latter is proved by estimating the difference of these expressions for $\phi, \psi \in K_1$ and any T by

$$\sum_{l\ne j} |\langle \Gamma P_N Q_j\phi, P_N Q_l(\phi-\psi)\rangle| + |\langle \Gamma P_N Q_j(\phi-\psi), P_N Q_l\psi\rangle|,$$

which is then bounded by $2\lambda_N N\|\phi-\psi\|$ when we use the trick from (3.5). Combining (3.3) and (3.6) proves the lemma. □

We can now complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Recall that given any $\tau, \delta > 0$, we need to show the existence of $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, then $\|\phi^\epsilon(\tau/\epsilon)\| < \delta$ for any initial datum $\phi_0 \in H$, $\|\phi_0\| = 1$. Here $\phi^\epsilon(t)$ is the solution of (2.4). Let us outline the idea of the proof. Lemma 2.3 tells us that if the H^1 norm of the solution $\phi^\epsilon(t)$ is large, relaxation is happening quickly. If, on the other hand, $\|\phi^\epsilon(\tau_0)\|_1^2 \leq \lambda_M \|\phi^\epsilon(\tau_0)\|^2$, where M is to be chosen depending on τ, δ, then the set of all unit vectors satisfying this inequality is compact, and so we can apply Lemma 3.2 and Lemma 3.3. Using these lemmas, we will show that even if the H^1 norm is small at some moment of time τ_0, it will be large on the average in some time interval after τ_0. Enhanced relaxation will follow.

We now provide the details. Since Γ is an unbounded positive operator with a discrete spectrum, we know that its eigenvalues $\lambda_n \to \infty$ as $n \to \infty$. Let us choose M large enough, so that $e^{-\lambda_M \tau/80} < \delta$. Define the sets $K \equiv \{\phi \in S \,|\, \|\phi\|_1^2 \leq \lambda_M\} \subset S$ and as before, $K_1 \equiv \{\phi \in K \,|\, \|P_p\phi\| \geq 1/2\}$. It is easy to see that K is compact. Choose N so that $N \geq M$ and $N \geq N_p(5\lambda_M, K)$ from Lemma 3.3. Define

$$\tau_1 \equiv \max\left\{T_p(5\lambda_M, K), T_c(N, \tfrac{\lambda_M}{20\lambda_N}, K)\right\},$$

where T_p is from Lemma 3.3, and T_c from Lemma 3.2. Finally, choose $\epsilon_0 > 0$ so that $\tau_1 < \tau/2\epsilon_0$, and

$$\epsilon_0 \int_0^{\tau_1} B^2(t)\,dt \leq \frac{1}{20\lambda_N}, \tag{3.7}$$

where $B(t)$ is the function from condition (2.7).

Take any $\epsilon < \epsilon_0$. If we have $\|\phi^\epsilon(s)\|_1^2 \geq \lambda_M \|\phi^\epsilon(s)\|^2$ for all $s \in [0, \tau]$ then Lemma 2.3 implies that $\|\phi^\epsilon(\tau/\epsilon)\| \leq e^{-2\lambda_M \tau} \leq \delta$ by the choice of M and we are done. Otherwise, let τ_0 be the first time in the interval $[0, \tau/\epsilon]$ such that $\|\phi^\epsilon(\tau_0)\|_1^2 \leq \lambda_M \|\phi^\epsilon(\tau_0)\|^2$ (it may be that $\tau_0 = 0$, of course). We claim that the following estimate holds for the decay of $\|\phi^\epsilon(t)\|$ on the interval $[\tau_0, \tau_0 + \tau_1]$:

$$\|\phi^\epsilon(\tau_0 + \tau_1)\|^2 \leq e^{-\lambda_M \epsilon \tau_1/20} \|\phi^\epsilon(\tau_0)\|^2. \tag{3.8}$$

For the sake of transparency, henceforth we will denote $\phi^\epsilon(\tau_0) = \phi_0$. On the interval $[\tau_0, \tau_0 + \tau_1]$, consider the function $\phi^0(t)$ satisfying $\frac{d}{dt}\phi^0(t) = iL\phi^0(t)$, $\phi^0(\tau_0) = \phi_0$. Note that by the choice of ϵ_0, (3.7) and Corollary 2.5, we have

$$\|\phi^\epsilon(t) - \phi^0(t)\|^2 \leq \frac{\lambda_M}{40\lambda_N} \|\phi_0\|^2 \tag{3.9}$$

for all $t \in [\tau_0, \tau_0 + \tau_1]$. Split $\phi^0(t) = \phi_c(t) + \phi_p(t)$, where $\phi_{c,p}$ also solve the free equation $\frac{d}{dt}\phi_{c,p}(t) = iL\phi_{c,p}(t)$, but with initial data $P_c\phi_0$ and $P_p\phi_0$ at $t = \tau_0$, respectively. We will now consider two cases.

Case I. Assume that $\|P_c\phi_0\|^2 \geq \frac{3}{4}\|\phi_0\|^2$, or, equivalently, $\|P_p\phi_0\|^2 \leq \frac{1}{4}\|\phi_0\|^2$. Note that since $\phi_0/\|\phi_0\| \in K$ by the hypothesis, we can apply

Lemma 3.2. Our choice of τ_1 implies

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|P_N\phi_c(t)\|^2\,dt \le \frac{\lambda_M}{20\lambda_N}\|\phi_0\|^2. \tag{3.10}$$

By elementary considerations,

$$\begin{aligned}\|(I-P_N)\phi^0(t)\|^2 &\ge \frac{1}{2}\|(I-P_N)\phi_c(t)\|^2 - \|(I-P_N)\phi_p(t)\|^2\\ &\ge \frac{1}{2}\|\phi_c(t)\|^2 - \frac{1}{2}\|P_N\phi_c(t)\|^2 - \|\phi_p(t)\|^2.\end{aligned}$$

Taking into account the fact that the free evolution e^{iLt} is unitary, $\lambda_N \ge \lambda_M$, and our assumptions on $\|P_{c,p}\phi_0\|$ and (3.10), we obtain

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|(I-P_N)\phi^0(t)\|^2\,dt \ge \frac{1}{10}\|\phi_0\|^2. \tag{3.11}$$

Using (3.9), we conclude that

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|(I-P_N)\phi^\epsilon(t)\|^2\,dt \ge \frac{1}{40}\|\phi_0\|^2. \tag{3.12}$$

This estimate implies

$$\int_{\tau_0}^{\tau_0+\tau_1} \|\phi^\epsilon(t)\|_1^2\,dt \ge \frac{\lambda_N\tau_1}{40}\|\phi_0\|^2. \tag{3.13}$$

Combining (3.13) with (2.6) yields

$$\|\phi^\epsilon(\tau_0+\tau_1)\|^2 \le \left(1-\frac{\lambda_N\epsilon\tau_1}{20}\right)\|\phi^\epsilon(\tau_0)\|^2 \le e^{-\lambda_N\epsilon\tau_1/20}\|\phi^\epsilon(\tau_0)\|^2. \tag{3.14}$$

This finishes the proof of (3.8) in the first case since $\lambda_N \ge \lambda_M$.

Case II. Now suppose that $\|P_p\phi_0\|^2 \ge \frac{1}{4}\|\phi_0\|^2$. In this case $\phi_0/\|\phi_0\| \in K_1$, and we can apply Lemma 3.3. In particular, by the choice of N and τ_1,

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|P_N\phi_p(t)\|_1^2\,dt \ge 5\lambda_M\|\phi_0\|^2. \tag{3.15}$$

Since (3.10) still holds because of our choice of τ_0 and τ_1, it follows that

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|P_N\phi_c(t)\|_1^2\,dt \le \frac{\lambda_M}{20}\|\phi_0\|^2. \tag{3.16}$$

Note that the H-norm in (3.10) has been replaced in (3.16) by the H^1-norm at the expense of the factor of λ_N. Together, (3.15) and (3.16) imply

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1}\|P_N\phi^0(t)\|_1^2\,dt \geq 2\lambda_M\|\phi_0\|^2. \tag{3.17}$$

Finally, (3.17) and (3.9) give

$$\int_{\tau_0}^{\tau_0+\tau_1}\|P_N\phi^\epsilon(t)\|_1^2\,dt \geq \frac{\lambda_M\tau_1}{2}\|\phi_0\|^2 \tag{3.18}$$

since $\|P_N\phi^\epsilon - P_N\phi^0\|_1^2 \leq \lambda_N\|\phi^\epsilon - \phi^0\|^2$. As before, (3.18) implies

$$\|\phi^\epsilon(\tau_0+\tau_1)\|^2 \leq e^{-\lambda_M\epsilon\tau_1}\|\phi^\epsilon(\tau_0)\|^2, \tag{3.19}$$

which finishes the proof of (3.8) in the second case.

Summarizing, we see that if $\|\phi^\epsilon(\tau_0)\|_1^2 \leq \lambda_M\|\phi^\epsilon(\tau_0)\|^2$, then

$$\|\phi^\epsilon(\tau_0+\tau_1)\|^2 \leq e^{-\lambda_M\epsilon\tau_1/20}\|\phi^\epsilon(\tau_0)\|^2. \tag{3.20}$$

On the other hand, for any interval $I=[a,b]$ such that $\|\phi^\epsilon(t)\|_1^2 \geq \lambda_M\|\phi^\epsilon(t)\|^2$ on I, we have by Lemma 2.3 that

$$\|\phi^\epsilon(b)\|^2 \leq e^{-2\lambda_M\epsilon(b-a)}\|\phi^\epsilon(a)\|^2. \tag{3.21}$$

Combining all the decay factors gained from (3.20) and (3.21), and using $\tau_1 < \tau/2\epsilon$, we find that there is $\tau_2 \in [\tau/2\epsilon, \tau/\epsilon]$ such that

$$\|\phi^\epsilon(\tau_2)\|^2 \leq e^{-\lambda_M\epsilon\tau_2/20} \leq e^{-\lambda_M\tau/40} < \delta^2$$

by our choice of M. Then (2.6) gives $\|\phi^\epsilon(\tau/\epsilon)\| \leq \|\phi^\epsilon(\tau_2)\| < \delta$, which finishes the proof of Theorem 1.4. □

4. Examples with rough eigenvectors

It is not immediately obvious that condition (2.7), $\|e^{iLt}\phi\|_1 \leq B(t)\|\phi\|_1$ for any $\phi_0 \in H^1$, is consistent with the existence of eigenvectors of L which are not in H^1. The purpose of this section is to show that, in general, rough eigenvectors may indeed be present under the conditions of Theorem 1.4. We provide here two simple examples of operators Γ and L in which (2.7) is satisfied and L has only rough eigenfunctions. In both cases L will be a discrete Schrödinger operator on $\mathbb{Z}^+$ resp., more generally, a Jacobi matrix, and Γ a multiplication operator. One more example with rough eigenfunctions will deal with an actual fluid flow and will be discussed in Section 6.

The first is an explicit example with one rough eigenvector that is a discrete version of the celebrated Wigner-von Neumann construction [32] of an imbedded eigenvalue of a Schrödinger operator with a decaying potential. The second example is implicit, its existence being guaranteed by a result of Killip and Simon [25], and demonstrates that all eigenvectors of L can be rough while at the same time the eigenvalues can be dense in the spectrum of L.

Example 1. Let $\tilde{\Gamma}$ be the operator of multiplication by n on $l^2(\mathbb{Z}^+)$, $\mathbb{Z}^+ = \{1, 2, \ldots, \}$. Furthermore, let $\tilde{L}$ be the discrete Schrödinger operator on $l^2(\mathbb{Z}^+)$:

$$\tilde{L}u_n = u_{n+1} + u_{n-1} + v_n u_n$$

for $n \geq 1$, with the potential

$$v_n \equiv \begin{cases} -\frac{2}{n+2} & n \text{ even}, \\ \frac{2}{n-1} & n > 1 \text{ odd}, \\ -1 & n = 1, \end{cases}$$

and the self-adjoint boundary condition $u_0 \equiv 0$. Then $\tilde{L}$ has eigenvalue zero with eigenfunction u given by

$$u_{2n-1} = u_{2n} = \tfrac{(-1)^n}{n}$$

for $n \geq 1$, because then $\tilde{L}u \equiv 0$ and $u \in \ell^2(\mathbb{Z}^+)$. Note that u does not belong to $H^1(\tilde{\Gamma})$.

It is not difficult to show that $\tilde{L}$ has no more eigenvalues in its essential spectrum $[-2, 2]$ (for example, using the so-called EFGP transform, see [26] for more details). The eigenvalue zero is a consequence of a resonant structure of the potential which is tuned to this energy. There may be (and there are) other eigenvalues outside $[-2, 2]$, with eigenfunctions that are exponentially decaying and so do belong to $H^1(\tilde{\Gamma})$. It is also known that $\tilde{L}$ has no singular continuous spectrum and that it has absolutely continuous spectrum that fills $[-2, 2]$. More precisely, the absolutely continuous part of the spectral measure gives positive weight to any set of positive Lebesgue measure lying in $[-2, 2]$ (see, e.g., [25]).

To get an example where we have only rough eigenfunctions, we will project away the eigenfunctions lying in H^1. Namely, denote by D the subspace spanned by all eigenfunctions of $\tilde{L}$, with the exception of u. Denote P the projection on the orthogonal complement of D, and set $\Gamma = P\tilde{\Gamma}P$, $L = P\tilde{L}P$. Then Γ, L are self-adjoint on the infinite dimensional Hilbert space $H = Pl^2(\mathbb{Z}^+)$, and by construction L has absolutely continuous spectrum filling $[-2, 2]$ as well as a single eigenvalue equal to zero. The corresponding eigenfunction is u and it does not belong to $H^1(\Gamma)$ because

$$\langle \Gamma u, u \rangle = \langle P\tilde{\Gamma}Pu, u \rangle = \langle \tilde{\Gamma}u, u \rangle \geq \sum_n |n|(n^{-1})^2 = \infty.$$

Let us check the conditions of Theorem 1.4. First, Γ is positive because $\tilde{\Gamma}$ is. It is also unbounded and has a discrete spectrum. Indeed, let $H_R \subset H$ be the subspace of all vectors $\phi \in H$ such that

$$\langle \Gamma\phi, \phi\rangle \le R\langle \phi, \phi\rangle. \tag{4.1}$$

Then for each such ϕ we also have (4.1) with $\tilde{\Gamma}$ instead of Γ. By the minimax principle for self-adjoint operators this implies that $\lambda_n \ge \tilde{\lambda}_n = n$, where λ_n, $\tilde{\lambda}_n$ are the n-th eigenvalues of Γ and $\tilde{\Gamma}$, respectively (counting multiplicities).

Also, L is a bounded operator on H (since $\tilde{L}$ is) and so (2.1) is satisfied automatically. Finally, observe that for any $\phi \in H^1(\Gamma)$,

$$|\langle \Gamma L\phi, \phi\rangle| = |\langle P\tilde{\Gamma}P\tilde{L}P\phi, \phi\rangle| = |\langle \tilde{\Gamma}\tilde{L}\phi, \phi\rangle| \le C\|\phi\|_1^2. \tag{4.2}$$

The second equality in (4.2) follows from the fact that $\tilde{L}$ and P commute by construction and $P\phi = \phi$ for $\phi \in H$. The inequality in (4.2) holds since $\|\tilde{L}\phi\|_1 \le C\|\phi\|_1$, which follows from the fact that $\tilde{L}$ is tridiagonal and both $\tilde{\lambda}_{n+1}/\tilde{\lambda}_n$ and v_n are bounded. Now given $\phi \in H^1(\Gamma)$, set $\phi(t) = e^{iLt}\phi$. Then

$$\left|\frac{d}{dt}\|\phi(t)\|_1^2\right| \le 2|\langle \Gamma L\phi(t), \phi(t)\rangle| \le C\|\phi(t)\|_1^2$$

by (4.2). This *a priori* estimate and Gronwall's lemma allow one to conclude that (2.7) holds with $B(t) = e^{Ct/2}$. This concludes our first example.

Example 2. We let $H \equiv \ell^2(\mathbb{Z}^+)$ and define Γ to be the multiplication by e^n. In order to provide an example with a much richer set of rough eigenfunctions, we will now consider L to be a Jacobi matrix

$$Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + v_n u_n,$$

with $a_n > 0$, $v_n \in \mathbb{R}$ and boundary condition $u_0 \equiv 0$. We choose ν to be a pure point measure of total mass $\frac{1}{2}$, whose mass points are contained and dense in $(-2, 2)$, and define the probability measure $d\mu(x) \equiv d\nu(x) + \frac{1}{8}\chi_{[-2,2]}(x)dx$. By the Killip-Simon [25] characterization of spectral measures of Jacobi matrices that are Hilbert-Schmidt perturbations of the free half-line Schrödinger operator (with $a_n = 1$, $v_n = 0$), there is a unique Jacobi matrix L such that $a_n - 1, v_n \in \ell^2(\mathbb{Z}^+)$, and its spectral measure is μ. In particular, the eigenvalues of L are dense in its spectrum $\sigma(L) = [-2, 2]$.

The conditions of Theorem 1.4 are again satisfied, with the key estimate $\|L\phi\|_1 \le C\|\phi\|_1$ holding because $\lambda_{n+1}/\lambda_n, a_n, v_n$ are bounded. Moreover, it is easy to show (see below) that the fact that eigenvalues of L are inside $(-2, 2)$ and $a_n - 1, v_n \in \ell^2$ imply that eigenfunctions of L decay slower than $e^{-C\sqrt{n}}$ for some C. More precisely, if u is an eigenfunction of L, then

$$\lim_n (u_n^2 + u_{n-1}^2)e^{C\sqrt{n}} = \infty,$$

and so obviously $u \notin H^1(\Gamma)$ (actually, $u \notin H^s(\Gamma)$ for any $s > 0$).

To obtain the well-known bound on the eigenfunction decay, let u be an eigenfunction of L corresponding to eigenvalue $E \in (-2, 2)$; that is,

$$Eu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + v_n u_n \tag{4.3}$$

for $n \geq 1$. Define the square of the Prüfer amplitude of u by

$$R_n \equiv u_n^2 + u_{n-1}^2 - Eu_n u_{n-1} = \frac{2-|E|}{2}(u_n^2 + u_{n-1}^2) + \frac{|E|}{2}(u_n - u_{n-1})^2 > 0$$

and $c_n \equiv |a_n - 1| + |a_{n-1} - 1| + |v_n| \in \ell^2$. After expressing u_{n+1} in terms of u_n and u_{n-1} using (4.3), one obtains (with each $|O(c_n)| \leq C_E c_n$)

$$\frac{R_{n+1}}{R_n} = (1 + O(c_n)) \cdot \frac{\frac{2-|E|}{2}((1+O(c_n))u_n^2 + (1+O(c_n))u_{n-1}^2) + \frac{|E|}{2}(1+O(c_n))(u_n - u_{n-1})^2}{\frac{2-|E|}{2}(u_n^2 + u_{n-1}^2) + \frac{|E|}{2}(u_n - u_{n-1})^2}$$

if $E \neq 0$ and

$$\frac{R_{n+1}}{R_n} = (1 + O(c_n)) \frac{(1+O(c_n))u_n^2 + (1+O(c_n))u_{n-1}^2 + O(c_n)u_n u_{n-1}}{u_n^2 + u_{n-1}^2}$$

if $E = 0$. In either case, $R_{n+1}/R_n = 1 + O(c_n)$, which means that

$$\begin{aligned} R_n \geq R_{n_0} \prod_{k=n_0+1}^{n} (1 - C_E c_k) &\geq R_{n_0} \exp(-2C_E \sum_{k=n_0+1}^{n} c_k) \\ &\geq R_{n_0} \exp(-2C_E \|c_k\|_2 \sqrt{n}) \end{aligned}$$

if n_0 is chosen so that $C_E c_k < \frac{1}{2}$ for $k > n_0$. But then the definition of R_n shows that $\lim_n (u_n^2 + u_{n-1}^2) e^{C\sqrt{n}} = \infty$ for some $C < \infty$. This concludes the example.

We have thus proved

THEOREM 4.1. *There exist a self-adjoint, positive, unbounded operator* Γ *with a discrete spectrum and a self-adjoint operator* L *such that the following conditions are satisfied.*

- $\|L\phi\| \leq C\|\phi\|_1$ *and* $\|e^{iLt}\phi\|_1 \leq B(t)\|\phi\|_1$ *for some* $C < \infty$, $B(t) \in L^1_{\mathrm{loc}}[0, \infty)$ *and any* $\phi \in H^1(\Gamma)$;
- L *has eigenvectors but not a single one belongs to* $H^s(\Gamma)$ *for any* $s > 0$.

Later we will discuss examples of relaxation enhancing flows on manifolds. One of our examples is derived from a construction going back to Kolmogorov [28], and yields a smooth flow with discrete spectrum and rough eigenfunctions. This example is even more striking than the ones we discussed here since the spectrum is discrete. However, the construction is more technical and is postponed till Section 6.

5. The fluid flow theorem

In this section we discuss applications of the general criterion to various situations involving diffusion in a fluid flow. First, we are going to prove Theorem 1.2. Most of the results we need regarding the evolution generated by incompressible flows are well-known and can be found, for example, in [30] in the Euclidean space case. There are no essential changes in the more general manifold setting.

Proof of Theorem 1.2. It is well known that the Laplace-Beltrami operator Δ on a compact smooth Riemannian manifold is self-adjoint, nonpositive, unbounded, and has a discrete spectrum (see e.g. [3]). Moreover, it is negative when considered on the invariant subspace of mean zero L^2 functions. Henceforth, this will be our Hilbert space: $H \equiv L^2(M) \ominus \mathbf{1}$. Obviously it is sufficient to prove Theorem 1.2 for $\phi_0 \in H$ (i.e., when $\overline{\phi} = 0$). The Lipschitz class divergence-free vector field u generates a volume measure-preserving transformation $\Phi_t(x)$, defined by

$$\frac{d}{dt}\Phi_t(x) = u(\Phi_t(x)), \;\; \Phi_0(x) = x \tag{5.1}$$

(see, e.g. [30]). The existence and uniqueness of solutions to the system (5.1) follows from the well-known theorems on existence and uniqueness of solutions to first order systems of ODEs involving Lipschitz class functions. With this transformation we can associate a unitary evolution group U^t in $L^2(M)$ where $U^t f(x) = f(\Phi_{-t}(x))$. It is easy to see that H is an invariant subspace for this group. The group U^t corresponds to e^{iLt} in the abstract setting of Section 3. Since $\frac{d}{dt}(U^t f) = -u \cdot \nabla(U^t f)$ for all $f \in H^1(M)$ (the usual Sobolev space on M), we see that the group's self-adjoint generator, L, is defined by $L = iu \cdot \nabla$ on functions from $H^1(M)$. It is clear that condition (2.1) is satisfied, since $\|u \cdot \nabla f\| \leq C\|f\|_1$ for all $f \in H^1$. It remains to check that the condition (2.7) is satisfied, that is, $\|e^{iLt}f\|_1 \leq B(t)\|f\|_1$. Notice that if $u(x)$ is Lipschitz, so is $\Phi_t(x)$ for any t. This follows from the estimate (in the local coordinates and for a sufficiently small time t)

$$|\Phi_t(x) - \Phi_t(y)| \leq |x-y| + \int_0^t |u(\Phi_s(x)) - u(\Phi_s(y)|\, ds.$$

Applying Gronwall's lemma, we get

$$|\Phi_t(x) - \Phi_t(y)| \leq |x-y| e^{\|u\|_{\mathrm{Lip}} t}$$

for any x, y. Now by the well-known results on change of variables in Sobolev functions (see e.g. [37]) and by the fact that Φ_t is measure-preserving, we have that

$$\|U^t f\|_1 \leq C\|\Phi_t\|_{\mathrm{Lip}}\|f\|_1.$$

This is exactly (2.7), and the application of Theorem 1.4 finishes the proof. □

The criterion of Theorem 1.4 can be applied to boundary value problems as well. For the sake of simplicity, consider a bounded domain $\Omega \subset \mathbb{R}^d$ with a C^2 boundary $\partial\Omega$. Let $u \in \mathrm{Lip}(\Omega)$ be a Lipschitz incompressible flow such that its normal component is zero on the boundary: $u(x) \cdot \hat{n}(x) = 0$ for $x \in \partial\Omega$, with $\hat{n}(x)$ the outer normal at x. Let $\phi^A(x,t)$ be the solution of

$$\partial_t \phi^A(x,t) + Au \cdot \nabla \phi^A(x,t) - \Delta \phi^A(x,t) = 0, \qquad (5.2)$$
$$\phi^A(x,0) = \phi_0(x), \;\; \frac{\partial \phi^A}{\partial n} = 0 \text{ if } x \in \partial\Omega,$$

where the Neumann boundary condition is satisfied in the trace sense for almost every $t > 0$. The existence of a solution to (5.2) can be proved similarly to Theorem 2.1.

THEOREM 5.1. *In the Neumann boundary conditions setting, the flow $u \in$ $\mathrm{Lip}(\Omega)$ is relaxation-enhancing according to the Definition* 1.1 *if and only if the operator $u \cdot \nabla$ has no eigenfunctions in $H^1(\Omega)$ other than the constant function.*

Proof. The proof is essentially identical to that of Theorem 1.2. The Laplacian operator with Neumann boundary conditions restricted to mean zero functions plays a role of the self-adjoint operator Γ. The condition $u \cdot \hat{n} = 0$ ensures that the vector field u generates a measure-preserving flow $\Phi_t(x)$ via (5.1), and thus the corresponding evolution group is unitary. The estimates necessary for Theorem 1.4 to apply are verified in the same way as in the proof of Theorem 1.2. □

To treat other types of boundary conditions, such as Dirichlet, one needs to modify the relaxation enhancement definition. This is due to the fact that in this case the solution of (1.1) always tends to zero, rather than to the average of the initial datum.

Definition 5.2. Let $\phi^A(x,t)$ solve evolution equation (5.2), but with Dirichlet or more general heat loss type boundary conditions

$$\frac{\partial \phi^A}{\partial n}(x,t) + \sigma(x)\phi^A(x,t) = 0, \;\; x \in \partial\Omega, \;\; \sigma(x) \in C(\partial\Omega), \;\; \sigma(x) > 0 \qquad (5.3)$$

where n is the outer normal to $\partial\Omega$. Then we call the *divergence-free flow $u \in \mathrm{Lip}(\Omega)$ relaxation enhancing* if for every τ and δ there exists $A(\tau,\delta)$ such that for $A > A(\tau,\delta)$ and $\|\phi_0\|_{L^2(\Omega)} = 1$ we have $\|\phi^A(x,\tau)\|_{L^2(\Omega)} < \delta$.

Remarks. 1. Note that $\sigma(x) = \infty$ is not excluded and may lead to the Dirichlet boundary conditions on a part of the boundary.

2. The more general definition encompassing both Definitions 1.1 and 5.2 would assume that the solution tends to a certain limit and would define relaxation enhancement in terms of speed-up in reaching this limit.

It is well known that the Laplace operator with boundary conditions (5.3) is self-adjoint on the domain of $H^2(\Omega)$ functions satisfying (5.3) in the trace sense in $L^2(\partial\Omega)$. We denote this operator Δ_σ. The corresponding $H^1_\sigma(\Omega)$ space is the domain of the quadratic form of Δ_σ, consisting of all functions $\phi \in H^1(\Omega)$ such that $\int_{\partial\Omega} \sigma(x)|\phi(x)|^2 ds$ is finite. In the Dirichlet boundary condition case, formally corresponding to $\sigma(x) \equiv \infty$, we obtain the standard space $H^1_0(\Omega)$. Then we have

THEOREM 5.3. *In the case of the heat loss boundary condition* (5.3), *the flow* $u \in \mathrm{Lip}(\Omega)$ *satisfying* $u \cdot \hat{n} = 0$ *on the boundary is relaxation enhancing according to Definition* 5.2 *if and only if the operator* $u\cdot\nabla$ *has no eigenfunctions in* $H^1_\sigma(\Omega)$.

Proof. In the case of heat loss boundary conditions, it is well-known that the principal eigenvalue of Δ_σ is positive, and so we can set $\Gamma = -\Delta_\sigma$. Our space H is now equal to $L^2(\Omega)$. The rest of the proof remains the same as in Theorem 5.1. $\square$

We note that the case of Dirichlet boundary conditions has been treated in [2] in a more general setting $u \in L^\infty(\Omega)$ and without the assumption $u \cdot \hat{n} = 0$. The methods of [2] are completely different from ours, and rely on the estimates of the principal eigenvalue of $-\Delta + Au \cdot \nabla$ and positivity of the corresponding eigenfunction. In particular, as described in the introduction, these methods do not seem to be directly applicable to the study of the enhanced relaxation in the case of a compact manifold without boundary or Neumann boundary conditions, where the principal eigenvalue is always zero. The results of [2] show that in the Dirichlet boundary condition case, the flow u is relaxation enhancing in the sense of Definition 5.2 if and only if u does not have a first integral in $H^1_0(\Omega)$. In other words, if and only if the operator $u\cdot\nabla$ does not have an $H^1_0(\Omega)$ eigenfunction corresponding to the eigenvalue zero. The discrepancy between this result and Theorem 5.3 may seem surprising, but in fact the explanation is simple.

PROPOSITION 5.4. *Let* $u \in \mathrm{Lip}(\Omega)$. *If* $\phi \in H^1(\Omega)$ $(H^1_\sigma(\Omega))$ *is an eigenfunction of the operator* $u \cdot \nabla$ *corresponding to the eigenvalue* $i\lambda$, *then* $|\phi| \in H^1(\Omega)$ $(H^1_\sigma(\Omega))$ *and it is the first integral of* u, *that is*, $u \cdot \nabla|\phi| = 0$.

Proof. The fact that $|\phi| \in H^1$ follows from the well-known properties of Sobolev functions (see e.g. [8]). A direct computation using $u \cdot \nabla\phi = i\lambda\phi$ then verifies that $u \cdot \nabla|\phi| = 0$. $\square$

As a consequence, when $\sigma \not\equiv 0$, the condition of no H^1_σ eigenfunctions in the statement of Theorem 5.3 can be replaced by the condition of no first integrals in H^1_σ. In the settings of Theorems 1.2 and 5.1, the above argument still applies but does not allow change of their statements. Indeed — on one hand the operator $u \cdot \nabla$ always has eigenvalue zero with an eigenfunction that is smooth, namely a constant. Existence of this first integral, however, tells us nothing about relaxation enhancement. On the other hand, existence of mean zero H^1 eigenfunctions need not guarantee the existence of a mean zero first integral, as can be seen in the following well-known example.

Example. Let $M \equiv T^d$ be the flat d-dimensional torus with period one. Let α be a d dimensional constant vector generating irrational rotation on the torus (that is, we assume that components of α are independent over the field of rationals). It is well-known that the flow generated by the constant vector field α is ergodic but not weakly mixing. The self-adjoint operator $L = i\alpha \cdot \nabla$ has eigenvalues $2\pi\alpha \cdot k$, where k are all possible vectors with integer components. The corresponding eigenfunctions are $e^{-2\pi i k \cdot x}$, $x \in T^d$. Their absolute value is 1, which is a first integral of α, but there are no other first integrals. In particular, every non-constant eigenfunction of L corresponds to a non-zero eigenvalue. Thus, this flow is not relaxation enhancing even though it has no first integrals other than a constant function.

Finally, we show that the L^2 norm in the Definitions 1.1, 5.2 can be replaced by other L^p norms with $1 \le p \le \infty$ without any change to the statements of Theorems 1.2, 5.1, 5.3. This result is important for applications to quenching in reaction-diffusion equations.

THEOREM 5.5. *Theorems* 1.2, 5.1, 5.3 *remain true if*, *in Definitions* 1.1, 5.2, "$\|\phi_0\|_{L^2} = 1$" *is replaced by* "$\|\phi_0\|_{L^p} = 1$" *and* "$\|\phi^A(\tau) - \bar{\phi}\|_{L^2} < \delta$" (*resp.* "$\|\phi^A(\tau)\|_{L^2} < \delta$") *by* "$\|\phi^A(\tau) - \bar{\phi}\|_{L^q} < \delta$" (*resp.* "$\|\phi^A(\tau)\|_{L^q} < \delta$") *for any* $p, q \in [1, \infty]$.

For the sake of consistency of notation, we will consider the compact manifold case. The case of a domain Ω with Dirichlet or heat-loss boundary conditions is handled similarly (see below).

We start with the proof of a general $L^1 \to L^\infty$ estimate for solutions of

$$\psi_t + v \cdot \nabla \psi - \Delta \psi = 0 \tag{5.4}$$

on a compact manifold M. The point is that this estimate will be independent of the incompressible flow v and so, in particular, of the amplitude A in (1.1). It appeared, for example, in [12], where the domain was a strip in $\mathbb{R}^2$. The crucial ingredient of the proof was a Nash inequality. In the general case, we follow a part of the argument, but our proof of the corresponding inequality (5.6) is different.

LEMMA 5.6. *For any smooth Riemannian manifold M of dimension d and any $\varepsilon \ge 0$ (resp. $\varepsilon > 0$) if $d \ge 3$ (resp. $d = 2$), there exists $C = C(M, \varepsilon) > 0$*

such that for any incompressible flow $v \in \mathrm{Lip}(M)$ *and any mean zero* $\phi_0 \in L^2(M)$, *the solution of* (5.4) *and* $\phi(x,0) = \phi_0(x)$ *satisfies*

$$\|\phi(x,t)\|_{L^\infty(M)} \leq C t^{-d/2-\varepsilon} \|\phi_0\|_{L^1(M)}. \tag{5.5}$$

Proof. First note that by Hölder and Poincaré inequalities we have for any mean zero $\psi \in H^1(M)$, any $p \geq (d+2)/4$ (if $d = 2$, then for any $p > (d+2)/4$), and some C_p,

$$\|\psi\|_{L^2}^2 \leq \|\psi\|_{L^1}^{1/p} \|\psi\|_{L^{(2p-1)/(p-1)}}^{(2p-1)/p} \leq C_p \|\psi\|_{L^1}^{1/p} \|\nabla\psi\|_{L^2}^{(2p-1)/p}.$$

That is,

$$\|\nabla\psi\|_{L^2}^2 \geq C_q \|\psi\|_{L^2}^{2+q} \|\psi\|_{L^1}^{-q} \tag{5.6}$$

for $q \equiv 2/(2p-1)$ so that $q \leq 4/d$ if $d \geq 3$ and $q < 2$ if $d = 2$.

After multiplying (5.4) by ϕ and integrating over M we obtain for $t > 0$

$$\frac{d}{dt}\|\phi\|_{L^2}^2 = -2\|\nabla\phi\|_{L^2}^2 \leq -2C_q \|\phi\|_{L^2}^{2+q} \|\phi\|_{L^1}^{-q} \leq -2C_q \|\phi\|_{L^2}^{2+q} \|\phi_0\|_{L^1}^{-q}. \tag{5.7}$$

The last inequality follows from the positivity of and the preservation of L^1 norms of solutions of (5.4) with initial conditions $\phi_{0,\pm} \equiv \max\{\pm\phi_0, 0\}$, which shows that $\|\phi\|_{L^1}$ is non-increasing.

Next we divide (5.7) by $-\|\phi\|_{L^2}^{2+q}$ and integrate in time to obtain $\|\phi(x,t)\|_{L^2}^{-q} \geq qC_q t \|\phi_0\|_{L^1}^{-q}$. This in turn gives (with a new C_q),

$$\|\phi(x,t)\|_{L^2} \leq C_q t^{-1/q} \|\phi_0\|_{L^1}. \tag{5.8}$$

Hence we have shown that $\|\mathcal{P}_t(v)\|_{L^1 \to L^2} \leq C_q t^{-1/q}$ where $\mathcal{P}_t(v)$ is the solution operator for (5.4). But since $\mathcal{P}_t(v) = (\mathcal{P}_t(-v))^*$ is the adjoint of the operator $\mathcal{P}_t(-v)$, which satisfies the same bound, we obtain

$$\begin{aligned}\|\mathcal{P}_{2t}(v)\|_{L^1 \to L^\infty} &\leq \|\mathcal{P}_t(v)\|_{L^1 \to L^2} \|\mathcal{P}_t(v)\|_{L^2 \to L^\infty} \\ &= \|\mathcal{P}_t(v)\|_{L^1 \to L^2} \|\mathcal{P}_t(-v)\|_{L^1 \to L^2} \leq C_q^2 t^{-2/q}\end{aligned}$$

which is (5.5). □

Proof of Theorem 5.5. Assume for simplicity that the total volume of M is equal to one. Then it is clear that Lemma 5.6 also holds with

$$\|\phi(x,t)\|_{L^p(M)} \leq C t^{-d/2-\varepsilon} \|\phi_0\|_{L^q(M)} \tag{5.9}$$

in place of (5.5), with any $p, q \in [1, \infty]$ and the same C.

Assume now that we know that for some u and $p, q \geq 1$, given any $\tau, \delta > 0$, we can find $A_{p,q}(\tau,\delta)$ such that $\|\phi^A(x,\tau)\|_{L^p} < \delta$ for any $A > A_{p,q}(\tau,\delta)$ and any mean zero $\phi_0 \in L^2(M)$ with $\|\phi_0\|_{L^q} = 1$. Take any other $p', q' \geq 1$. Then for any $A > A_{p,q}(\tau,\delta)$,

$$\begin{aligned}\|\phi^A(x,3\tau)\|_{L^{p'}} &\leq C\tau^{-d/2-\varepsilon} \|\phi^A(x,2\tau)\|_{L^p} \\ &\leq \delta C\tau^{-d/2-\varepsilon} \|\phi^A(x,\tau)\|_{L^q} \leq \delta (C\tau^{-d/2-\varepsilon})^2 \|\phi_0\|_{L^{q'}}.\end{aligned}$$

This shows that when $A_{p,q}(\tau,\delta)$ exists for some p,q and all τ,δ, for any p',q',τ,δ we have $A_{p',q'}(\tau,\delta) = A_{p,q}(\tau/3, \delta C^{-2}(\tau/3)^{d+2\varepsilon})$ and so $A_{p',q'}(\tau,\delta)$ exists for all τ,δ. That is, Definition 1.1 describes the same class of flows regardless of which $L^p \to L^q$ decay it addresses. This finishes the proof. $\square$

We note that in the case of a bounded domain Ω with Dirichlet boundary conditions, the proof is identical. When we have heat-loss boundary conditions, the only change is that the equality in (5.7) reads

$$\frac{d}{dt}\|\phi\|_{L^2}^2 = -2(\|\nabla\phi\|_{L^2}^2 + \|\sigma^{1/2}\phi\|_{L^2(\partial\Omega)}^2)$$

and the Poincaré inequality is replaced by

$$\begin{aligned}\|\psi\|_{L^{(2p-1)/(p-1)}} &\le C_p(\|\psi\|_{L^2} + \|\nabla\psi\|_{L^2}) \\ &\le (C_p + \lambda_0^{-1/2})(\|\nabla\psi\|_{L^2} + \|\sigma^{1/2}\psi\|_{L^2(\partial\Omega)})\end{aligned}$$

which is due to the Sobolev inequality and the fact that the principal eigenvalue λ_0 of the Laplacian on Ω with heat-loss boundary conditions is positive.

6. Examples of relaxation enhancing flows

Here we discuss examples of flows that are relaxation enhancing. Most of the results in this section are not new and are provided for illustration purposes. According to Theorem 1.2 a flow $u \in \mathrm{Lip}(M)$ is relaxation enhancing if all of its eigenfunctions are not in $H^1(M)$. One natural class satisfying this condition is weakly mixing flows – for which the spectrum is purely continuous. Examples of weakly mixing flows on $\mathbb{T}^2$ go back to von Neumann [33] and Kolmogorov [28]. The flow in von Neumann's example is continuous; in the construction suggested by Kolmogorov the flow is smooth. The technical details of the construction were carried out in [35]; see also [20] for a review. Recently, Fayad [13] generalized this example to show that weakly mixing flows are generic in a certain sense. For more results on weakly mixing flows, see for example [14], [20]. To describe the result in [13] in more detail, we recall that a vector α in $\mathbb{R}^d$ is called β-Diophantine if there exists a constant C such that for each $k \in \mathbb{Z}^d \setminus \{0\}$ we have

$$\inf_{p\in\mathbb{Z}} |\langle\alpha,k\rangle + p| \ge \frac{C}{|k|^{d+\beta}}.$$

The vector α is Liouvillean if it is not Diophantine for any $\beta > 0$. The Liouvillean numbers (and vectors) are the ones which can be very well approximated by rationals.

Example 1. Consider the flow on a torus $\mathbb{T}^{d+1}$ that is a time change of a linear translation flow:

$$\frac{dx}{dt} = \frac{\alpha}{F(x,y)}, \quad \frac{dy}{dt} = \frac{1}{F(x,y)}, \quad (x,y) \in \mathbb{T}^{d+1} \tag{6.1}$$

with a smooth positive function $F(x,y)$. Such flows have a unique invariant measure $d\mu = F(x,y)dxdy$. Let us denote by $C^r(\mathbb{T}^d,\mathbb{R}^+)$ the set of C^r functions on the torus that are positive. We have

Proposition 6.1 ([13]). *Assume the irrational vector $\alpha \in \mathbb{R}^d$ is not β-Diophantine, for some $\beta > 0$. Then, for a dense G_δ of functions F in $C^{\beta+d}(\mathbb{T}^d,\mathbb{R}^+)$, the flow* (6.1) *is weakly mixing* (*for the unique invariant measure $F(x,y)dxdy$*).

To obtain examples of relaxation enhancing flows, we can now consider the generic flows of Proposition 6.1 on the torus with a metric such that the volume element is $F(x,y)dxdy$. Alternatively, we can just view a problem in a weighted space and consider the operator $\frac{1}{F}\nabla(F\nabla)$, which is self-adjoint on $L^2(\mathbb{T}^{d+1}, Fdxdy)$, instead of Δ. It is also straightforward to obtain more physical examples of the relaxation enhancing flows that are incompressible with respect to the usual flat metric. Indeed, assume for the sake of simplicity that we are working with a unit torus and that the total integral of F is also one. Then it is not difficult to construct an explicit measure preserving invertible transformation Z from $\mathbb{T}^{d+1}(F(x,y)dxdy)$ to $\mathbb{T}^{d+1}(dpdq)$, as smooth as the function F (for general results on existence of such maps see [31]). If we denote $w(x,y) = (\alpha/F(x,y), 1/F(x,y))$ the vector field in (6.1), then the vector field $u(p,q) = Z \circ w \circ Z^{-1}$ will be incompressible. Moreover, the unitary evolutions generated by w and u in $L^2(F(x,y)dxdy)$ and $L^2(dpdq)$ respectively are unitary equivalent and so have the same spectra.

We now describe an example of a different class of flows to which Theorem 1.2 applies. Namely, we will sketch a construction of a smooth incompressible flow $u(p,q)$, $\nabla\cdot u = 0$, on a torus $\mathbb{T}^2$ such that it has a purely discrete spectrum but none of the eigenfunctions are in $H^1(\mathbb{T}^2)$. We could not find an exact statement regarding the existence of such flows in the literature, although the idea of the construction appears in [28] and the result is presumably well-known in the dynamical systems community. In particular, it follows in a fairly direct way for example from considerations in [1], [20]. We briefly sketch the construction, without presenting well-known technical details.

Example 2. Let us denote by Φ_t^u the flow on the torus generated by u and by U^t the flow on $L^2(\mathbb{T}^2)$ generated by Φ_t^u: $(U^t f)(x) = f(\Phi_{-t}^u(x))$.

Proposition 6.2. *There exists a smooth incompressible* (*with respect to the Lebesgue measure*) *flow $u(x,y)$ on a two-dimensional torus $\mathbb{T}^2$ so that the corresponding unitary evolution U^t has a discrete spectrum on $L^2(\mathbb{T}^2)$ but none of the eigenfunctions of U are in $H^1(\mathbb{T}^2)$.*

Proof. The example will be given by a flow of type (6.1), with $d = 2$ and appropriately chosen α and $F(x,y)$. We remark that while the form (6.1) may seem quite special, in fact any analytic flow in two dimensions with an integral invariant can be mapped analytically to the linear translation flow (6.1) with some α, F (see e.g. [28]). The idea of the construction is to find a smooth flow (6.1) which can be mapped to a constant flow $(\alpha, 1)$ by a measure preserving map S with very low regularity properties. Since the eigenfunctions of the constant flow are explicitly computable, we can compute the eigenfunctions of the original flow. Due to the roughness of S, these will prove highly irregular. To obtain an incompressible flow, we will then proceed as in the first example.

In order to find such a flow w, we start with a smooth periodic function $Q \in C^\infty(\mathbb{S}^1)$ and an irrational number $\alpha \in \mathbb{R}$ so that the homology equation,

$$R(\xi + \alpha) - R(\xi) = Q(\xi) - 1, \quad \xi \in \mathbb{S}^1, \tag{6.2}$$

has a solution $R(\xi)$ that is very rough. Note that for (6.2) to have a measurable solution the function $Q(\xi)$ should satisfy the normalization [1]

$$\int_0^1 Q(\xi) d\xi = 1.$$

The following proposition is a particular case of Theorem 4.5 of [20].

Proposition 6.3. *Let α be a Liouvillean irrational number. There exists a $C^\infty(\mathbb{S}^1)$ function $Q(\xi)$ so that the homology equation* (6.2) *has a unique* (*up to an additive constant*) *measurable solution $R(\xi) : \mathbb{S}^1 \to \mathbb{R}$ such that for any $\lambda \in \mathbb{R}_* = \mathbb{R}\backslash\{0\}$, the function $R_\lambda(\xi) = e^{i\lambda R(\xi)}$ is discontinuous everywhere.*

Note that without loss of generality we may assume that $Q(\xi)$ is positive – otherwise we choose M so that $Q(\xi) + M > 1$ and consider a rescaled function $Q_M(\xi) = (M + Q(\xi))/(M + 1)$. Then the function $R_M(\xi) = R(\xi)/(M + 1)$ is the solution of (6.2) with Q_M on the right side and, of course, $R_M(\xi)$ has the same properties as $R(\xi)$.

Given a Liouvillean irrational number α and a function $Q(\xi)$ that satisfies the conclusion of Proposition 6.3 we define a function $F(x, y)$ on the torus $\mathbb{T}^2$ as follows. Choose $m > 0$ so that $m < \min Q(s)$ and a smooth function $\psi(y) \geq 0$ such that

$$\int_0^1 \psi(y) dy = 1. \tag{6.3}$$

In addition,

$$\psi(y) = 0 \text{ for } 0 \leq y \leq y_0 \text{ and } y_1 \leq y \leq 1 \quad \text{with } y_0 \text{ close to zero and } y_1 \text{ close to one.} \tag{6.4}$$

The choice of m ensures that the function

$$F(x, y) = m + \psi(y)(Q(x - \alpha y) - m), \quad 0 \leq x, y \leq 1 \tag{6.5}$$

is positive – then we extend $F(x, y)$ periodically in both variables to the whole plane $\mathbb{R}^2$. The resulting function is smooth because of (6.4) and, in addition, it has total mass equal to one. The normalization (6.3) implies that the functions F and Q are related by

$$Q(\xi) = \int_0^1 F(\xi + \alpha z, z)dz. \tag{6.6}$$

Now, the required transformation $S : (x, y) \to (X, Y)$ is defined by [28], [36]

$$X(x, y) = x + \alpha(Y(x, y) - y), \quad Y(x, y) = T(x - \alpha y, y) + R(x - \alpha y) \tag{6.7}$$

with the function $R(x)$ that satisfies (6.2), and $T(x, y)$ defined by

$$T(x, y) = \int_0^y F(x + \alpha z, z)dz.$$

Note that the transformation (6.7) implies that $x - \alpha y = X - \alpha Y$, and so it preserves the flow trajectories. The homology equation (6.2) together with the definition (6.5) of the function $F(x, y)$ imply that S is well-defined as a mapping $\mathbb{T}^2 \to \mathbb{T}^2$. It is also straightforward to check that S maps the flow w onto the uniform flow $w_{\text{unif}} = (\alpha, 1)$. One can also verify that S is invertible with measurable inverse, and is measure preserving:

$$\int [S^* f](x, y)F(x, y)dxdy = \int f(S(x, y))F(x, y)dxdy = \int f(X, Y)dXdY \tag{6.8}$$

for any function $f \in C(\mathbb{T}^2)$. Hence, S^* may be extended as an operator $L^2(dxdy) \to L^2(d\mu)$ with the preservation of the corresponding norms. We conclude that the unitary evolutions U_w^t and U_{unif}^t generated by the flow w given by (6.1) and the uniform flow w_{unif}, respectively, are conjugated by means of a unitary transformation $S^* :\ L^2(\mathbb{T}^2, dXdY) \to L^2(\mathbb{T}^2, d\mu)$ and, now we have $U_{\text{unif}}^t = [S^*]^{-1} U_w^t S^*$. It follows that U_w^t and U_{unif}^t have the same spectrum: $\lambda_{nl} = 2\pi i n\alpha + 2\pi i l, \quad l, n \in \mathbb{Z}$. It also follows that the eigenfunctions of the operator U_w may be written as

$$\begin{aligned} \psi_{nl}^w(x, y) &= e^{2\pi i n X(x,y) + 2\pi i l Y(x,y)} = e^{2\pi i n(x - \alpha y + \alpha Y(x,y)) + 2\pi i l Y(x,y)} \\ &= e^{2\pi i n(x-\alpha y)} e^{(2\pi i n\alpha + 2\pi i l)(T(x-\alpha y, y) + R(x-\alpha y))} \\ &= \zeta(x, y) e^{(2\pi i n\alpha + 2\pi i l)R(x-\alpha y)} \end{aligned} \tag{6.9}$$

with a smooth function $\zeta(x, y) \in C^\infty([0, 1]^2)$ (note that the function $\zeta(x, y)$ is not periodic in y). In order to verify that ψ_{nl}^w are not in $H^1(\mathbb{T}^2)$ it suffices to check that the function

$$\Theta_\lambda(x, y) = e^{i\lambda R(x-\alpha y)} = R_\lambda(x - \alpha y)$$

is not in $H^1([0, 1]^2)$ for any real $\lambda \neq 0$. The function $R_\lambda(s)$ is defined in Proposition 6.3 and is everywhere discontinuous. If $\Theta_\lambda(x, y)$ were in $H^1([0, 1]^2)$, it would force $R_\lambda(s)$ to be in $H^1(\mathbb{S}^1)$ and hence continuous; but this function is discontinuous everywhere. Therefore, the eigenfunctions ψ^w_{nl} cannot be in $H^1(\mathbb{T}^2)$ unless $n = l = 0$.

Finally, to obtain an incompressible flow, we introduce a smooth transformation $Z:\ (x,y) \to (p,q)$ by setting

$$p = \int_0^x \bar{F}(s)ds, \quad q = \frac{1}{\bar{F}(x)}\int_0^y F(x,z)dz, \text{ where } \bar{F}(x) = \int_0^1 F(x,z)dz.$$

It is immediate to verify that Z maps the measure $d\mu$ onto the Lebesgue measure $dpdq$. Hence, the evolution group generated by the image $u(p,q)$ of the flow $w(x,y)$ will have the same discrete spectrum as U_w. In addition, the eigenfunctions ψ^w_{nl} of U_w are the images of the eigenfunctions ψ^u_{nl} of u under Z^*: $\psi^w_{nl} = Z^*\psi^u_{nl} = \psi^u_{nl} \circ Z$. As the functions ψ^w_{nl} are not in $H^1(\mathbb{T}^2)$ and the map Z is smooth, it follows that all the eigenfunctions of the incompressible flow $u(p,q)$ are not in $H^1(\mathbb{T}^2)$. This finishes the proof of Proposition 6.2. □

7. Quenching in reaction-diffusion equations

In this section we describe the application of our results to questions of quenching in reaction-diffusion-advection equations. We will consider the problem

$$T^A_t(x,t) + Au\cdot\nabla T^A(x,t) - \Delta T^A(x,t) = f(T^A(x,t)), \quad T^A(x,0) = T_0(x) \tag{7.1}$$

on a smooth compact Riemannian manifold M with $T_0(x) \in [0,1]$. Here T is the (normalized) temperature of a premixed flammable gas that is advected by the incompressible flow $Au(x) \in \mathrm{Lip}(M)$. The nonlinear right-hand side term accounts for temperature increase due to burning and will be assumed to be of *ignition* type. That is,

(7.2) (i) $f(0) = f(1) = 0$ and $f(T)$ is Lipschitz continuous on $[0,1]$,

(ii) $\exists\theta_0 \in (0,1)$ such that $f(T) = 0$ for $T \in [0,\theta_0]$ and $f(T) > 0$ for $T \in (\theta_0, 1)$.

This shows, in particular, that T remains in $[0,1]$. The main question will be under what conditions on u one can always choose A large enough so that for some time $\tau > 0$ we have $\|T^A(x,\tau)\|_{L^\infty(M)} \le \theta_0$; that is, quenching — extinction of flames — happens. Of course this question is meaningless for certain initial data T_0. Namely, if $\|T_0\|_{L^1} > \theta_0 \mathrm{vol}(M)$ or $\|T_0\|_{L^1} = \theta_0\mathrm{vol}(M)$ but $T_0 \not\equiv \theta_0$, then we show easily, using

$$\frac{d}{dt}\|T^A\|_{L^1} = \int f(T^A(x,t))dx \tag{7.3}$$

that $\|T^A\|_{L^1}$ must be strictly increasing with the limit equal to the volume of M. This motivates the following definition.

Definition 7.1. We say that u is *strongly quenching* if for any nonlinearity f as in (7.2), and any solution T^A of (7.1) with initial datum $T_0(x) \in [0,1]$ with $\|T_0\|_{L^1(M)} < \theta_0 \mathrm{vol}(M)$, there exists $A(T_0, f)$ such that if $A > A(T_0, f)$, then for some $\tau > 0$ one has $\|T^A(x,\tau)\|_{L^\infty(M)} \le \theta_0$.

Then we have

THEOREM 7.2. *An incompressible flow $u \in \mathrm{Lip}(M)$ is strongly quenching if and only if it is relaxation enhancing.*

Proof. Assume that the volume of M is one. First, u is strongly quenching when it is relaxation enhancing. Indeed, assume u is relaxation enhancing and let c be the Lipshitz constant for f so that $f(T) \le cT$. If ϕ^A solves (1.1) with $\phi_0 \equiv T_0$, then $T^A(x,t) \le e^{ct}\phi^A(x,t)$ by the comparison principle. But this means

$$\|T^A(x,\tau)\|_{L^\infty} \le e^{c\tau}\|\phi^A(x,\tau)\|_{L^\infty} \le e^{c\tau}(\bar{T}_0 + \|\phi^A(x,\tau) - \bar{\phi}\|_{L^\infty})$$

which can be made as close to $\bar{T}_0 = \|T_0\|_{L^1(M)} < \theta_0$ as we wish by taking small enough $\tau, \delta > 0$ and $A > A_{1,\infty}(\tau,\delta)$ from the proof of Theorem 5.5. Since f was arbitrary, it follows that u is strongly quenching.

Hence, we are left to prove that u being strongly quenching implies that u is relaxation enhancing. Assume this is not the case, that is, there exists an ignition nonlinearity f, a mean zero $\phi_0 \in L^2(M)$ and $\tau, \delta \in (0,1)$ such that for all $A < \infty$ and $t \le \tau$, the solution ϕ^A of (1.1) satisfies $\|\phi^A(x,t)\|_{L^1} > \delta$. We can assume without loss of generality that $\|\phi_0\|_{L^\infty} \le 1$.

Let $T_0 \equiv \theta_1 + \gamma\phi_0$ where $\gamma \equiv \frac{1}{3}\min\{\theta_0, 1-\theta_0\}$ and $\theta_1 \in (\theta_0 - \beta, \theta_0)$ with $\beta \equiv \frac{1}{8}\min\{\gamma\delta, \tau\delta\kappa, \theta_0\}$ and $\kappa \equiv \min\{f(T) \,|\, T \in [\theta_0 + \frac{\gamma\delta}{8}, \frac{1}{3}(2+\theta_0)]\} > 0$. Note that $T_0(x) \in \frac{1}{3}[\theta_0, 1+2\theta_0]$.

For each $t \le \tau$ let $B_t \equiv \{x \,|\, \phi^A(x,t) \ge \frac{\delta}{4}\}$. Since $\phi^A(x,t)$ is mean zero with L^1 norm more than δ, and $|\phi^A(x,t)| \le 1$, we must have $|B_t| \ge \frac{\delta}{4}$ for each $t \le \tau$ (recall that M has volume one). Thus

$$T^A(x,t) \ge \theta_1 + \tfrac{\gamma\delta}{4} \ge \theta_0 + \tfrac{\gamma\delta}{8}$$

for $x \in B_t$. If for some $t \le \tau$ there is a set $B_t' \subseteq B_t$ with $|B_t'| \ge \frac{\delta}{8}$ and $T^A(x,t) \ge \frac{1}{3}(2+\theta_0)$ for $x \in B_t'$, then since $\psi^A \equiv \theta_1 + \gamma\phi^A \le T^A$ by the comparison principle (because ψ^A satisfies (1.1) with $\psi^A(x,0) = T_0(x)$) and $0 < \inf\{T_0(x)\} \le \psi^A \le \|T_0\|_{L^\infty} \le \frac{1}{3}(1+2\theta_0)$ by the maximum principle,

$$\|T^A(x,t)\|_{L^1} \ge \tfrac{\delta}{8}[\tfrac{1}{3}(2+\theta_0) - \tfrac{1}{3}(1+2\theta_0)] + \|\psi^A(x,t)\|_{L^1} \ge \tfrac{\gamma\delta}{8} + \theta_1 > \theta_0.$$

This is a contradiction because then u cannot be strongly quenching by the argument before Definition 7.1.

Therefore for each $t \le \tau$ there must be a set $B_t'' \subseteq B_t$ such that $|B_t''| \ge \frac{\delta}{8}$ and $T^A(x,t) \in [\theta_0 + \frac{\gamma\delta}{8}, \frac{1}{3}(2+\theta_0)]$ for $x \in B_t''$. But then $f(T^A(x,t)) \ge \kappa$ for

$x \in B_t''$, and so (7.3) gives

$$\|T^A(x,\tau)\|_{L^1} \geq \theta_1 + \tau\tfrac{\delta\kappa}{8} > \theta_0$$

and we have a contradiction again. Hence u has to be relaxation enhancing. □

Applications of our results on relaxation enhancement to quenching on infinite domains (where front propagation can occur as well) will be considered elsewhere.

Acknowledgement. PC has been partially supported by NSF-DMS grant 0202531. AK and LR have been supported in part by Alfred P. Sloan Research Fellowships. In addition, AK and AZ has been partially supported by NSF-DMS grant 0314129, and AZ have been supported by NSF-DMS grant 0632442. AK and AZ thank the University of Chicago for its hospitality in the Spring of 2005. The authors thank Mark Freidlin, Alex Furman, Stas Molchanov and Laurent Saloff-Coste for interesting discussions and for pointing out some relevant literature.

University of Chicago, Chicago, IL
E-mail address: const@cs.uchicago.edu

University of Wisconsin, Madison, WI
E-mail address: kiselev@math.wisc.edu

University of Chicago, Chicago, IL
E-mail address: ryzhik@math.uchicago.edu

University of Chicago, Chicago, IL
E-mail address: zlatos@math.uchicago.edu

References

[1] D. V. Anosov, The additive functional homology equation that is connected with an ergodic rotation of the circles (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 1259–1274.

[2] H. Berestycki, F. Hamel, and N. Nadirashvili, Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena, *Comm. Pure Appl. Math.* **253** (2005), 451–480.

[3] M. Berger, *t A Panoramic View of Riemannian Geometry*. Springer-Verlag, New York, 2003.

[4] P. Constantin, A. Kiselev, and L. Ryzhik, Quenching of flames by fluid advection, *Comm. Pure Appl. Math.* **54** (2001), 1320–1342.

[5] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1982.

[6] H. Cycon, R. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators*, Springer-Verlag, New York, 1987.

[7] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[8] L. C. Evans, *Partial Differential Equations*, *Graduate Studies in Math*. **19**, A. M. S., Providence, RI, 1998.

[9] A. Fannjiang and G. Papanicolau, Convection enhanced diffusion for periodic flows, SIAM *J. Appl. Math*. **54** (1994), 333–408.

[10] A. Fannjiang and L. Wolowski, Noise induced dissipation in Lebesgue-measure preserving maps on d-dimensional torus, *J. Statist. Phys*. **113** (2003), 335–378.

[11] A. Fannjiang, S. Nonnenmacher, and L. Wolowski, Dissipation time and decay of correlations, *Nonlinearity* **17** (2004), 1481–1508.

[12] A. Fannjiang, A. Kiselev, and L. Ryzhik, Quenching of reaction by cellular flows, *Geom. Funct. Anal*. **16** (2006), 40–69.

[13] B. Fayad, Weak mixing for reparameterized linear flows on the torus, *Ergodic Theory Dynam. Systems* **22** (2002), 187–201.

[14] ———, Mixing diffeomorphisms and flows with purely singular spectra, preprint.

[15] B. Franke, Integral inequalities for the fundamental solutions of diffusions on manifolds with divergence-free drift, *Math. Z*. **246** (2004), 373–403.

[16] M. Freidlin, *Functional Integration and Partial Differential Equations*, *Princeton Math. Studies* **109**, Princeton Univ. Press, Princeton, NJ, 1985.

[17] ———, Reaction-diffusion in incompressible fluid: asymptotic problems, *J. Differential Equations* **179** (2002), 44–96.

[18] M. Freidlin and A. Wentzell, *Random Perturbations of Hamiltonian Systems*, *Mem. Amer. Math. Soc*., A. M. S., Providence, RI, 1994.

[19] ———, *Random Perturbations of Dynamical Systems*, 2nd ed., Springer-Verlag, New York, 1998.

[20] A. Katok, Cocycles, cohomology and combinatorial constructions in ergodic theory (in collaboration with E. A. Robinson, Jr.), *Proc. Sympos. Pure Math*. **69** in *Smooth Ergodic Theory and its Applications* (Seattle, WA, 1999), 107–173, A. M. S., Providence, RI, 2001.

[21] Y. Kifer, On the principal eigenvalue in a singular perturbation problem with hyperbolic limit points and circles, *J. Differential Equations* **37** (1980), 108–139.

[22] ———, *Random Perturbations of Dynamical Systems*, Birkhäuser Boston Inc., Boston, 1988.

[23] ———, Principal eigenvalues, topological pressure, and stochastic stability of equilibrium states, *Israel J. Math*. **70** (1990), 1–47.

[24] ———, *Random Perturbations of Dynamical Systems*: *a New Approach*, *Lectures in Appl. Math*. **27**, A. M. S., Providence, RI, 1991.

[25] R. Killip and B. Simon, Sum rules for Jacobi matrices and their applications to spectral theory, *Ann. of Math*. **158** (2003), 253–321.

[26] A. Kiselev, Y. Last, and B. Simon, Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators, *Comm. Math. Phys*. **194**, (1998), 1–45.

[27] A. Kiselev and A. Zlatoš, Quenching of combustion by shear flows, *Duke Math. J*. **132** (2006), 49–72.

[28] A. N. Kolmogorov, On dynamical systems with an integral invariant on the torus (Russian), *Dok. Akad. Nauk* SSSR **93** (1953), 763–766.

[29] A. Majda and P. Kramer, Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena, *Phys. Rep*. **314** (1999), 237–574.

[30] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Springer-Verlag, New York, 1994.

[31] J. Moser, On the volume elements on a manifold, *Trans. Amer. Math. Soc.* **120** (1965), 286–294.

[32] J. von Neumann and E. P. Wigner, Über merkwürdige diskrete Eigenwerte, *Z. Phys.* **30** (1929), 465–467.

[33] J. von Neuamnn, Zur operatorenmethode in der classichen mechanik, *Ann. of Math.* **33** (1932), 587–642.

[34] J.-M. Roquejoffre, Eventual monotonicity and convergence to travelling fronts for the solutions of parabolic equations in cylinders, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), 499–552.

[35] M. D. Šklover, Classical dynamical systems on the torus with continuous spectrum, *Izv. Vuzov* **10** (1967), 113–124.

[36] S. Sternberg, On differential equations on the torus, *Amer. J. Math.* **79** (1957), 397–402.

[37] W. P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York, 1989.

[38] A. Zlatoš, Quenching and propagation of combustion without ignition temperature cutoff, *Nonlinearity* **18** (2005), 1463–1475.

(Received September 28, 2005)