Annals of Mathematics, **167** (2008), 601–641

Multi-critical unitary random matrix ensembles and the general Painlevé II equation

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Abstract

We study unitary random matrix ensembles of the form

$$Z_{n,N}^{-1} |\det M|^{2\alpha} e^{-N \operatorname{Tr} V(M)} dM,$$

where $\alpha > -1/2$ and V is such that the limiting mean eigenvalue density for $n, N \to \infty$ and $n/N \to 1$ vanishes quadratically at the origin. In order to compute the double scaling limits of the eigenvalue correlation kernel near the origin, we use the Deift/Zhou steepest descent method applied to the Riemann-Hilbert problem for orthogonal polynomials on the real line with respect to the weight $|x|^{2\alpha} e^{-NV(x)}$. Here the main focus is on the construction of a local parametrix near the origin with ψ-functions associated with a special solution q_α of the Painlevé II equation $q'' = sq + 2q^3 - \alpha$. We show that q_α has no real poles for $\alpha > -1/2$, by proving the solvability of the corresponding Riemann-Hilbert problem. We also show that the asymptotics of the recurrence coefficients of the orthogonal polynomials can be expressed in terms of q_α in the double scaling limit.

1. Introduction and statement of results

1.1. *Unitary random matrix ensembles.* For $n \in \mathbb{N}$, $N > 0$, and $\alpha > -1/2$, we consider the unitary random matrix ensemble

$$Z_{n,N}^{-1} |\det M|^{2\alpha} e^{-N \operatorname{Tr} V(M)}\, dM, \tag{1.1}$$

on the space of $n \times n$ Hermitian matrices M, where $V : \mathbb{R} \to \mathbb{R}$ is a real analytic function satisfying

$$\lim_{x \to \pm\infty} \frac{V(x)}{\log(x^2+1)} = +\infty. \tag{1.2}$$

Because of (1.2) and $\alpha > -1/2$, the integral

$$Z_{n,N} = \int |\det M|^{2\alpha} e^{-N \operatorname{Tr} V(M)}\, dM \tag{1.3}$$

converges and the matrix ensemble (1.1) is well- defined. It is well known, see for example [11], [36], that the eigenvalues of M are distributed according to a determinantal point process with a correlation kernel given by

$$K_{n,N}(x,y) = |x|^\alpha e^{-\frac{N}{2}V(x)}|y|^\alpha e^{-\frac{N}{2}V(y)} \sum_{k=0}^{n-1} p_{k,N}(x)p_{k,N}(y), \tag{1.4}$$

where $p_{k,N} = \kappa_{k,N}x^k + \cdots$, $\kappa_{k,N} > 0$, denotes the k-th degree orthonormal polynomial with respect to the weight $|x|^{2\alpha}e^{-NV(x)}$ on $\mathbb{R}$.

Scaling limits of the kernel (1.4) as $n, N \to \infty$, $n/N \to 1$, show a remarkable universal behavior which is determined to a large extent by the limiting mean density of eigenvalues

$$\psi_V(x) = \lim_{n\to\infty} \frac{1}{n} K_{n,n}(x,x). \tag{1.5}$$

Indeed, for the case $\alpha = 0$, Bleher and Its [5] (for quartic V) and Deift et al. [16] (for general real analytic V) showed that the sine kernel is universal in the bulk of the spectrum, i.e.,

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x_0)} K_{n,n}\left(x_0 + \frac{u}{n\psi_V(x_0)}, x_0 + \frac{v}{n\psi_V(x_0)}\right) = \frac{\sin \pi(u-v)}{\pi(u-v)}$$

whenever $\psi_V(x_0) > 0$. In addition, the Airy kernel appears generically at endpoints of the spectrum. If x_0 is a right endpoint and $\psi_V(x) \sim (x_0 - x)^{1/2}$ as $x \to x_0-$, then there exists a constant $c > 0$ such that

$$\lim_{n\to\infty} \frac{1}{cn^{2/3}} K_{n,n}\left(x_0 + \frac{u}{cn^{2/3}}, x_0 + \frac{v}{cn^{2/3}}\right) = \frac{\mathrm{Ai}\,(u)\mathrm{Ai}\,'(v) - \mathrm{Ai}\,'(u)\mathrm{Ai}\,(v)}{u-v},$$

where Ai denotes the Airy function; see also [13].

The extra factor $|\det M|^{2\alpha}$ in (1.1) introduces singular behavior at 0 if $\alpha \neq 0$. The pointwise limit (1.5) does not hold if $\psi_V(0) > 0$, since $K_{n,n}(0,0) = 0$ if $\alpha > 0$ and $K_{n,n}(0,0) = +\infty$ if $\alpha < 0$, due to the factor $|x|^\alpha|y|^\alpha$ in (1.4). However (1.5) continues to hold for $x \neq 0$ and also in the sense of weak* convergence of probability measures

$$\frac{1}{n} K_{n,n}(x,x)dx \xrightarrow{*} \psi_V(x)dx, \text{ as } n \to \infty.$$

Therefore we can still call ψ_V the limiting mean density of eigenvalues. Observe that ψ_V does not depend on α.

However, at a microscopic level the introduction of the factor $|\det M|^{2\alpha}$ changes the eigenvalue correlations near the origin. Indeed, for the case of a noncritical V for which $\psi_V(0) > 0$, it was shown in [35] that

$$\lim_{n\to\infty}\frac{1}{n\psi_V(0)}K_{n,n}\left(\frac{u}{n\psi_V(0)},\frac{v}{n\psi_V(0)}\right) = \pi\sqrt{u}\sqrt{v}\frac{J_{\alpha+\frac12}(\pi u)J_{\alpha-\frac12}(\pi v)-J_{\alpha-\frac12}(\pi u)J_{\alpha+\frac12}(\pi v)}{2(u-v)}, \tag{1.6}$$

where J_ν denotes the usual Bessel function of order ν.

We notice that universality results for orthogonal and symplectic ensembles of random matrices have been obtained only very recently, see [12], [13], [14].

1.2. *The multi-critical case.* It is the goal of this paper to study (1.1) in a critical case where ψ_V vanishes quadratically at 0, i.e.,

$$\psi_V(0)=\psi_V'(0)=0, \qquad \text{and} \qquad \psi_V''(0)>0. \tag{1.7}$$

The behavior (1.7) is among the possible singular behaviors that were classified in [15]. The classification depends on the characterization of the measure $\psi_V(x)dx$ as the unique minimizer of the logarithmic energy

$$I_V(\mu)=\iint\log\frac{1}{|x-y|}d\mu(x)d\mu(y)+\int V(x)d\mu(x) \tag{1.8}$$

among all probability measures μ on $\mathbb{R}$. The corresponding Euler-Lagrange variational conditions give that for some constant $\ell\in\mathbb{R}$,

$$2\int\log|x-y|\psi_V(y)dy-V(x)+\ell=0, \qquad \text{for } x\in\operatorname{supp}(\psi_V), \tag{1.9}$$

$$2\int\log|x-y|\psi_V(y)dy-V(x)+\ell\le 0, \qquad \text{for } x\in\mathbb{R}. \tag{1.10}$$

In addition one has that ψ_V is supported on a finite union of disjoint intervals, and

$$\psi_V(x)=\frac{1}{\pi}\sqrt{Q_V^-(x)}, \tag{1.11}$$

where Q_V is a real analytic function, and Q_V^- denotes its negative part. Note that the endpoints of the support correspond to zeros of Q_V with odd multiplicity.

The possible singular behaviors are as follows, see [15], [32].

Singular case I. Equality holds in the variational inequality (1.10) for some $x\in\mathbb{R}\setminus\operatorname{supp}(\psi_V)$.

Singular case II. ψ_V vanishes at an interior point of $\operatorname{supp}(\psi_V)$, which corresponds to a zero of Q_V in the interior of the support. The multiplicity of such a zero is necessarily a multiple of 4.

Singular case III. ψ_V vanishes at an endpoint to higher order than a square root. This corresponds to a zero of the function Q_V in (1.11) of odd multiplicity $4k+1$ with $k\ge 1$. (The multipicity $4k+3$ cannot occur in these matrix models.)

In each of the above cases, V is called singular, or, otherwise, regular. The above conditions correspond to a singular exterior point, a singular endpoint, and a singular interior point, respectively.

In each of the singular cases one expects a family of possible limiting kernels in a double scaling limit as $n, N \to \infty$ and $n/N \to 1$ at some critical rate [4]. As said before we consider the case (1.7) which corresponds to the singular case II with $k = 1$ at the singular point $x = 0$. For technical reasons we assume that there are no other singular points besides 0. Setting $t = n/N$, and letting $n, N \to \infty$ such that $t \to 1$, we have that the parameter t describes the transition from the case where $\psi_V(0) > 0$ (for $t > 1$) through the multi-critical case ($t = 1$) to the case where 0 lies in a gap between two intervals of the spectrum ($t < 1$). The appropriate double scaling limit will be such that the limit $\lim_{n,N\to\infty} n^{2/3}(t-1)$ exists.

The double scaling limit for $\alpha = 0$ was considered in [2], [6], [7] for certain special cases, and in [9] in general. The limiting kernel is built out of ψ-functions associated with the Hastings-McLeod solution [25] of the Painlevé II equation $q'' = sq + 2q^3$.

For general $\alpha > -1/2$, we are led to the general Painlevé II equation

$$q'' = sq + 2q^3 - \alpha. \tag{1.12}$$

The Painlevé II equation for general α has been suggested by the physics papers [1], [40]. The limiting kernels in the double scaling limit are associated with a special distinguished solution of (1.12), which we describe first. We assume from now on that $\alpha \neq 0$.

1.3. *The general Painlevé* II *equation.* Balancing sq and α in the differential equation (1.12), we find that there exist solutions such that

$$q(s) \sim \frac{\alpha}{s}, \qquad \text{as } s \to +\infty, \tag{1.13}$$

and balancing sq and $2q^3$, we see that there also exist solutions of (1.12) such that

$$q(s) \sim \sqrt{\frac{-s}{2}}, \qquad \text{as } s \to -\infty. \tag{1.14}$$

There is exactly one solution of (1.12) that satisfies both (1.13) and (1.14) (see [26], [27], [30]) and we denote it by q_α. This is the special solution that we need. It corresponds to the choice of Stokes multipliers

$$s_1 = e^{-\pi i\alpha}, \qquad s_2 = 0, \qquad s_3 = -e^{\pi i\alpha};$$

see Section 2 below. We call q_α the Hastings-McLeod solution of the general Painlevé II equation (1.12), since it seems to be the natural analogue of the Hastings-McLeod solution for $\alpha = 0$.

The Hastings-McLeod solution is meromorphic in s (as are all solutions of (1.12)) with an infinite number of poles. We need that it has no poles on the real line. From the asymptotic behavior (1.13) and (1.14) we know that there are no real poles for $|s|$ large enough, but that does not exclude the possibility of a finite number of real poles. While there is a a substantial literature on Painlevé equations and Painlevé transcendents, see e.g. the recent monograph [22], we have not been able to find the following result.

THEOREM 1.1. *Let q_α be the Hastings-McLeod solution of the general Painlevé* II *equation* (1.12) *with $\alpha > -1/2$. Then q_α is a meromorphic function with no poles on the real line.*

1.4. *Main result.* To describe our main result, we recall the notion of ψ-functions associated with the Painlevé II equation; see [20]. The Painlevé II equation (1.12) is the compatibility condition for the following system of linear differential equations for $\Psi = \Psi_\alpha(\zeta; s)$.

$$\frac{\partial \Psi}{\partial \zeta} = A\Psi, \qquad \frac{\partial \Psi}{\partial s} = B\Psi, \tag{1.15}$$

where

$$A = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir + \alpha/\zeta \\ 4\zeta q - 2ir + \alpha/\zeta & 4i\zeta^2 + i(s+2q) \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} -i\zeta & q \\ q & i\zeta \end{pmatrix}. \tag{1.16}$$

That is, (1.15) has a solution where $q = q(s)$ and $r = r(s)$ depend on s but not on ζ, if and only if q satisfies Painlev é II and $r = q'$.

Given s, q and r, the solutions of

$$\frac{\partial}{\partial \zeta}\begin{pmatrix} \Phi_1(\zeta) \\ \Phi_2(\zeta) \end{pmatrix} = A \begin{pmatrix} \Phi_1(\zeta) \\ \Phi_2(\zeta) \end{pmatrix} \tag{1.17}$$

are analytic with branch point at $\zeta = 0$. For $\alpha > -1/2$ and $s \in \mathbb{R}$, we take $q = q_\alpha(s)$ and $r = q_\alpha'(s)$ where q_α is the Hastings-McLeod solution of the Painlevé II equation, and we define $\begin{pmatrix} \Phi_{\alpha,1}(\zeta; s) \\ \Phi_{\alpha,2}(\zeta; s) \end{pmatrix}$ as the unique solution of (1.17) with asymptotics

$$e^{i(\frac{4}{3}\zeta^3 + s\zeta)} \begin{pmatrix} \Phi_{\alpha,1}(\zeta; s) \\ \Phi_{\alpha,2}(\zeta; s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(\zeta^{-1}), \tag{1.18}$$

uniformly as $\zeta \to \infty$ in the sector $\varepsilon < \arg \zeta < \pi - \varepsilon$ for any $\varepsilon > 0$. Note that this is well-defined for every $s \in \mathbb{R}$ because of Theorem 1.1.

The functions $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ extend to analytic functions on $\mathbb{C} \setminus (-i\infty, 0]$, which we also denote by $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$; see also Remark 2.33 below. Their values on the real line appear in the limiting kernel. The following is the main result of this paper.

THEOREM 1.2. *Let V be real analytic on $\mathbb{R}$ such that* (1.2) *holds. Suppose that ψ_V vanishes quadratically in the origin, i.e., $\psi_V(0) = \psi_V'(0) = 0$, and $\psi_V''(0) > 0$, and that there are no other singular points besides 0. Let $n, N \to \infty$ such that*

$$\lim_{n,N\to\infty} n^{2/3}(n/N-1) = L \in \mathbb{R}$$

exists. Define constants

$$c = \left(\frac{\pi\psi_V''(0)}{8}\right)^{1/3}, \tag{1.19}$$

and

$$s = 2\pi^{2/3}L\left[\psi_V''(0)\right]^{-1/3} w_{S_V}(0), \tag{1.20}$$

where w_{S_V} is the equilibrium density of the support of ψ_V (see Remark 1.3 *below). Then*

$$\lim_{n,N\to\infty} \frac{1}{cn^{1/3}} K_{n,N}\left(\frac{u}{cn^{1/3}}, \frac{v}{cn^{1/3}}\right) = K^{\mathrm{crit},\alpha}(u,v;s), \tag{1.21}$$

uniformly for u, v in compact subsets of $\mathbb{R}\setminus\{0\}$, where

$$K^{\mathrm{crit},\alpha}(u,v;s) = -e^{\frac{1}{2}\pi i\alpha[\mathrm{sgn}(u)+\mathrm{sgn}(v)]}\frac{\Phi_{\alpha,1}(u;s)\Phi_{\alpha,2}(v;s)-\Phi_{\alpha,1}(v;s)\Phi_{\alpha,2}(u;s)}{2\pi i(u-v)}. \tag{1.22}$$

Remark 1.3. The equilibrium measure of $S_V = \mathrm{supp}(\psi_V)$ is the unique probability measure ω_{S_V} on S_V that minimizes the logarithmic energy

$$I(\mu) = \iint \log\frac{1}{|x-y|}d\mu(x)d\mu(y)$$

among all probability measures on S_V. Since S_V consists of a finite union of intervals, and since 0 is an interior point of one of these intervals, ω_{S_V} has a density w_{S_V} with respect to Lebesgue measure, and $w_{S_V}(0) > 0$. This number is used in (1.20).

Remark 1.4. One can refine the calculations of Section 4 to obtain the following stronger result:

$$\frac{1}{cn^{1/3}} K_{n,N}\left(\frac{u}{cn^{1/3}}, \frac{v}{cn^{1/3}}\right) = K^{\mathrm{crit},\alpha}(u,v;s) + \mathcal{O}\left(\frac{|u|^\alpha|v|^\alpha}{n^{1/3}}\right), \tag{1.23}$$

uniformly for u, v in bounded subsets of $\mathbb{R}\setminus\{0\}$.

Remark 1.5. It is not immediate from the expression (1.22) that $K^{\mathrm{crit},\alpha}$ is real. This property follows from the symmetry

$$e^{\frac{1}{2}\pi i\alpha\mathrm{sgn}(u)}\Phi_{\alpha,2}(u;s) = \overline{e^{\frac{1}{2}\pi i\alpha\mathrm{sgn}(u)}\Phi_{\alpha,1}(u;s)}, \qquad \text{for } u \in \mathbb{R}\setminus\{0\},$$

which leads to the "real formula"

$$K^{\mathrm{crit},\alpha}(u,v;s) = -\frac{1}{\pi(u-v)}\mathrm{Im}\left(e^{\frac{1}{2}\pi i\alpha(\mathrm{sgn}(u)-\mathrm{sgn}(v))}\Phi_{\alpha,1}(u;s)\overline{\Phi_{\alpha,1}(v;s)}\right);$$

see Remark 2.11 below.

Remark 1.6. For $\alpha = 0$, the theorem is proven in [9]. The proof for the general case follows along similar lines, but we need the information about the existence of $q_\alpha(s)$ for real s, as guaranteed by Theorem 1.1.

1.5. *Recurrence coefficients for orthogonal polynomials.* In order to prove Theorem 1.2, we will study the Riemann-Hilbert problem for orthogonal polynomials with respect to the weight $|x|^{2\alpha}e^{-NV(x)}$. This analysis leads to asymptotics for the kernel $K_{n,N}$, but also provides the ingredients to derive asymptotics for the orthogonal polynomials and for the coefficients in the recurrence relation that is satisfied by them.

To state these results we introduce measures ν_t in the following way; see also [9] and Section 3.2. Take $\delta_0 > 0$ sufficiently small and let ν_t be the minimizer of $I_{V/t}(\nu)$ (see (1.8) for the definition of I_V) among all measures $\nu = \nu^+ - \nu^-$, where $\nu^\pm$ are nonnegative measures on $\mathbb{R}$ such that $\nu(\mathbb{R}) = 1$ and $\mathrm{supp}(\nu^-) \subset [-\delta_0, \delta_0]$. We use ψ_t to denote the density of ν_t.

We restrict ourselves to the one-interval case without singular points except for 0. Then $\mathrm{supp}(\psi_V) = [a,b]$ and $\mathrm{supp}(\psi_t) = [a_t, b_t]$ for t close to 1, where a_t and b_t are real analytic functions of t.

We write $\pi_{n,N}$ for the monic orthogonal polynomial of degree n with respect to the weight $|x|^{2\alpha}e^{-NV(x)}$. Those polynomials satisfy a three-term recurrence relation

$$\pi_{n+1,N} = (z - b_{n,N})\pi_{n,N} - a_{n,N}^2\pi_{n-1,N}, \tag{1.24}$$

with recurrence coefficients $a_{n,N}$ and $b_{n,N}$. In the large n expansion of $a_{n,N}$ and $b_{n,N}$, we observe oscillations in the $\mathcal{O}(n^{-1/3})$-term. The amplitude of the oscillations is proportional to $q_\alpha(s)$, while in general the frequency of the oscillations slowly varies with $t = n/N$.

THEOREM 1.7. *Let the conditions of Theorem* 1.2 *be satisfied and assume that* $\mathrm{supp}(\psi_V) = [a,b]$ *consists of one single interval. Consider the three-term recurrence relation* (1.24) *for the monic orthogonal polynomials* $\pi_{k,N}$ *with respect to the weight* $|x|^{2\alpha}e^{-NV(x)}$. *Then as* $n, N \to \infty$ *such that* $n/N - 1 = \mathcal{O}(n^{-2/3})$,

$$a_{n,N} = \frac{b-a}{4} - \frac{q_\alpha(s_{t,n})\cos(2\pi n\omega_t + 2\alpha\theta)}{2c}n^{-1/3} + \mathcal{O}(n^{-2/3}), \tag{1.25}$$

$$b_{n,N} = \frac{b+a}{2} + \frac{q_\alpha(s_{t,n})\sin(2\pi n\omega_t + (2\alpha+1)\theta)}{c}n^{-1/3} + \mathcal{O}(n^{-2/3}), \tag{1.26}$$

where $t = n/N$, c *is given by* (1.19),

$$s_{t,n} = n^{2/3}\frac{\pi}{c}\psi_t(0), \tag{1.27}$$

$$\theta = \arcsin\frac{b+a}{b-a}, \tag{1.28}$$

and

$$\omega_t = \int_0^{b_t}\psi_t(x)dx. \tag{1.29}$$

Remark 1.8. It was shown in [9] that $\frac{d}{dt}\psi_t(0)\big|_{t=1} = w_{S_V}(0)$, which in the situation of Theorem 1.7 implies that (since $S_V = [a,b]$ and $\psi_t(0)$ is real analytic as a function of t near $t=1$),

$$\psi_t(0) = (t-1)\frac{1}{\pi\sqrt{-ab}} + \mathcal{O}((t-1)^2), \qquad \text{as } t \to 1.$$

Then it follows from (1.27) that $s_{t,n} = n^{2/3}(t-1)\frac{1}{c\sqrt{-ab}} + \mathcal{O}(n^{-2/3})$ and we could in fact replace $s_{t,n}$ in (1.25) and (1.26) by

$$s_{t,n}^{*} = n^{2/3}(t-1)\frac{1}{c\sqrt{-ab}}.$$

We prefer to use $s_{t,n}$ since it appears more naturally from our analysis.

Remark 1.9. In [6], Bleher and Its derived (1.25) in the case where $\alpha = 0$ and where V is a critical even quartic polynomial. They also computed the $\mathcal{O}(n^{-2/3})$-term in the large n expansion for $a_{n,N}$. For even V we have that $a = -b$, $\theta = 0$, $\omega_t = 1/2$ and thus $\cos(2\pi n\omega_t + 2\alpha\theta) = (-1)^n$, so that (1.25) reduces to

$$a_{n,N} = \frac{b}{2} - \frac{q_\alpha(s_{t,n})(-1)^n}{2c}n^{-1/3} + \mathcal{O}(n^{-2/3}),$$

which is in agreement with the result of [6]. Also for even V the recurrence coefficient $b_{n,N}$ vanishes which is in agreement with (1.26).

Remark 1.10. In [4] an ansatz was made about the recurrence coefficients associated with a general (not necessarily even) critical quartic polynomial V in the case $\alpha = 0$. For fixed large N, the ansatz agrees with (1.25) and (1.26) up to an N- dependent phase shift in the trigonometric functions.

Remark 1.11. Since the submission of this manuscript several new results were obtained leading to a more complete description of the singular cases for the random matrix ensemble (1.1). See the discussion in section 1.2 for the singular cases I, II, and III.

The singular case I with $\alpha = 0$ was treated in [19] and later in [8], [37], [3]. For the singular case III with $\alpha = 0$, see [10], where a connection with the Painlevé I hierarchy was found.

The non-singular case III with $\alpha \neq 0$ is described by the Painlevé XXXIV equation in [28].

1.6. *Outline of the rest of the paper.* In Section 2, we comment on the Riemann-Hilbert problem associated with the Painlevé II equation. We also prove the existence of a solution to this RH problem for real values of the parameter s, and this existence provides the proof of Theorem 1.1. In Section 3, we state the RH problem for orthogonal polynomials and apply the Deift/Zhou steepest descent method. Our main focus will be the construction of a local parametrix near the origin. For this construction, we will use the RH problem from Section 2. In Section 4 and Section 5 finally, we use the results obtained in Section 3 to prove Theorem 1.2 and Theorem 1.7.

2. The RH problem for Painlevé II and the proof of Theorem 1.1

As before, we assume $\alpha > -1/2$.

2.1. *Statement of the* RH *problem.* Let $\Sigma = \bigcup_j \Gamma_j$ be the contour consisting of four straight rays oriented to infinity,

$$\Gamma_1 : \arg\zeta = \frac{\pi}{6}, \qquad \Gamma_2 : \arg\zeta = \frac{5\pi}{6}, \qquad \Gamma_3 : \arg\zeta = -\frac{5\pi}{6}, \qquad \Gamma_4 : \arg\zeta = -\frac{\pi}{6}.$$

The contour Σ divides the complex plane into four regions $S_1, \dots, S_4$ as shown in Figure 1. For $\alpha > -1/2$ and $s \in \mathbb{C}$, we seek a 2×2 matrix-valued function $\Psi_\alpha(\zeta; s) = \Psi_\alpha(\zeta)$ (we suppress notation of s for brevity) satisfying the following.

The RH *problem for* Ψ_α. (a) Ψ_α is analytic in $\mathbb{C} \setminus \Sigma$.

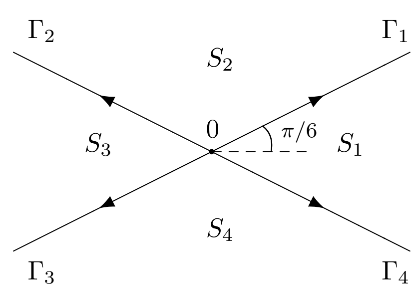

Figure 1: The contour Σ consisting of four straight rays oriented to infinity.

(b) Ψ_α satisfies the following jump relations on $\Sigma \setminus \{0\}$,

$$\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta)\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{2.1}$$

$$\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta)\begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2, \tag{2.2}$$

$$\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta)\begin{pmatrix} 1 & e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3, \tag{2.3}$$

$$\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta)\begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_4. \tag{2.4}$$

(c) Ψ_α has the following behavior at infinity,

$$\Psi_\alpha(\zeta) = (I + \mathcal{O}(1/\zeta))e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}, \qquad \text{as } \zeta \to \infty. \tag{2.5}$$

Here $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ denotes the third Pauli matrix.

(d) Ψ_α has the following behavior near the origin. If $\alpha < 0$,

$$\Psi_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^\alpha & |\zeta|^\alpha \\ |\zeta|^\alpha & |\zeta|^\alpha \end{pmatrix}, \qquad \text{as } \zeta \to 0, \tag{2.6}$$

and if $\alpha \geq 0$,

$$\Psi_\alpha(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_1 \cup S_3, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{\alpha} & |\zeta|^{-\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_2, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_4. \end{cases} \tag{2.7}$$

Note that Ψ_α depends on s only through the asymptotic condition (2.5).

Remark 2.1. This RH problem is a generalization of the RH problem for the case where $\alpha = 0$, used in [2], [9].

Remark 2.2. By standard arguments based on Liouville's theorem, see e.g. [11], [33], it can be verified that the solution of this RH problem, if it exists, is unique. Here it is important that $\alpha > -1/2$.

In the following we need more information on the behavior of solutions of the RH problem near 0. To this end, we make use of the following proposition, cf. [27]. We use G_j to denote the jump matrix of Ψ_α on Γ_j as given by (2.1)–(2.4).

PROPOSITION 2.3. *Let* Ψ *satisfy conditions* (a), (b), *and* (d) *of the* RH *problem for* Ψ_α.

(1) *If* $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, *then there exists an analytic matrix-valued function* E *and constant matrices* A_j *such that*

$$\Psi(\zeta) = E(\zeta) \begin{pmatrix} \zeta^{-\alpha} & 0 \\ 0 & \zeta^{\alpha} \end{pmatrix} A_j, \qquad \text{for } \zeta \in S_j, \tag{2.8}$$

where the branch cut of ζ^α *is chosen along* Γ_4. *The matrices* A_j *satisfy*

$$A_{j+1} = A_j G_j, \qquad \text{for } j = 1, 2, 3, \tag{2.9}$$

and

$$A_2 = \begin{pmatrix} 0 & -p^{-1} \\ p & \frac{p}{2\cos(\pi\alpha)} \end{pmatrix}, \qquad \text{for some } p \in \mathbb{C} \setminus \{0\}. \tag{2.10}$$

(2) *If* $\alpha - \frac{1}{2} \in \mathbb{N}_0$, *then there is logarithmic behavior of* Ψ *at the origin. There exist an analytic matrix-valued function* E *and constant matrices* A_j *such that*

$$\Psi(\zeta) = E(\zeta) \begin{pmatrix} \zeta^{-\alpha} & 0 \\ \frac{1}{\pi}\zeta^{\alpha} \ln \zeta & \zeta^{\alpha} \end{pmatrix} A_j, \qquad \text{for } \zeta \in S_j, \tag{2.11}$$

where again the branch cuts of ζ^α *and* $\ln \zeta$ *are chosen along* Γ_4. *The matrices* A_j *satisfy*

$$A_{j+1} = A_j G_j, \qquad \text{for } j = 1, 2, 3, \tag{2.12}$$

and for some $p \in \mathbb{C}$,

$$A_2 = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix}, & \text{if } \alpha - \frac{1}{2} \text{ is even}, \\[2ex] \begin{pmatrix} 0 & i \\ i & p \end{pmatrix}, & \text{if } \alpha - \frac{1}{2} \text{ is odd}. \end{cases} \tag{2.13}$$

Proof. (1) Define E by equation (2.8) with matrices A_j satisfying (2.9) and (2.10). Then E is analytic across Γ_1, Γ_2, and Γ_3 because of (2.9). For $\zeta \in \Gamma_4$ there is a jump

$$E_+(\zeta) = E_-(\zeta) \begin{pmatrix} \zeta^{-\alpha} & 0 \\ 0 & \zeta^{\alpha} \end{pmatrix}_- A_4 G_4 A_1^{-1} \begin{pmatrix} \zeta^{\alpha} & 0 \\ 0 & \zeta^{-\alpha} \end{pmatrix}_+. \tag{2.14}$$

Using $\zeta_-^\alpha = e^{2\pi i \alpha} \zeta_+^\alpha$ and the explicit expressions for the matrices G_j and A_j, we get from (2.14) that E is analytic across Γ_4 as well.

What remains to be shown is that the possible isolated singularity of E at the origin is removable. If $\alpha < 0$ it follows from (2.6) and (2.8) that

$$E(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{2\alpha} & 1 \\ |\zeta|^{2\alpha} & 1 \end{pmatrix}, \qquad \text{as } \zeta \to 0,$$

so that (since $2\alpha > -1$) the isolated singularity at the origin is indeed removable. If $\alpha > 0$ we have in sector S_2 by (2.7), (2.8), and (2.10) that

$$\begin{aligned} E(\zeta) &= \Psi(\zeta) A_2^{-1} \begin{pmatrix} \zeta^{\alpha} & 0 \\ 0 & \zeta^{-\alpha} \end{pmatrix} \\ &= \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{\alpha} & |\zeta|^{-\alpha} \end{pmatrix} \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \begin{pmatrix} \zeta^{\alpha} & 0 \\ 0 & \zeta^{-\alpha} \end{pmatrix} = \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \end{aligned}$$

as $\zeta \to 0$, $\zeta \in S_2$, where $*$ denotes an unimportant constant. Hence the singularity at the origin is not a pole. Moreover, from (2.7) and (2.8) it is also easy to check that E does not have an essential singularity at the origin either. Therefore the singularity is removable for the case $\alpha > 0$ as well, and the proof of part (1) is complete.

(2) The proof of part (2) is similar. □

Remark 2.4. The matrix A_2 in Proposition 2.3 is called the connection matrix, cf. [20, 24]. In all cases we have $\det A_2 = 1$ and the $(1,1)$-entry of A_2 is zero.

2.2. *Solvability of the* RH *Problem for* Ψ_α. We shall prove that the RH problem for Ψ_α is solvable for every $s \in \mathbb{R}$. We do that, as in [16], [24], [43], by showing that every solution of the homogeneous RH problem is identically zero. Such a result is known as a vanishing lemma [23], [24].

We briefly indicate why the vanishing lemma is enough to establish the solvability of the RH problem for Ψ_α. The RH problem is equivalent to a singular integral equation on the contour Σ. The singular integral equation can be stated in operator theoretic terms, and the operator is a Fredholm operator of zero index. The vanishing lemma yields that the kernel is trivial, and so the operator is onto which implies that the singular integral equation is solvable, and therefore the RH problem is solvable. For more details and other examples of this procedure see [16], [24], [43] and [29, App. A].

PROPOSITION 2.5 (the vanishing lemma). *Let $\alpha > -1/2$ and $s \in \mathbb{R}$. Suppose that $\widehat{\Psi}$ satisfies conditions* (a), (b), *and* (d) *of the* RH *problem for Ψ_α with the following asymptotic condition* (*instead of condition* (c))

$$\widehat{\Psi}(\zeta) e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty. \tag{2.15}$$

Then $\widehat{\Psi} \equiv 0$.

Proof. As before, we use G_j to denote the jump matrix of Γ_j, given by (2.1)–(2.4). Introduce an auxiliary matrix-valued function H with jumps only on $\mathbb{R}$, as follows.

$$H(\zeta) = \begin{cases} \widehat{\Psi}(\zeta)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_2 \cup S_4, \\ \widehat{\Psi}(\zeta)G_1 e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_1 \cap \mathbb{C}_+, \\ \widehat{\Psi}(\zeta)G_2^{-1} e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_3 \cap \mathbb{C}_+, \\ \widehat{\Psi}(\zeta)G_3 e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_3 \cap \mathbb{C}_-, \\ \widehat{\Psi}(\zeta)G_4^{-1} e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_1 \cap \mathbb{C}_-. \end{cases} \tag{2.16}$$

Then H satisfies the following RH problem.

The RH *problem for* H.

(a) $H : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic and satisfies the following jump relations on $\mathbb{R} \setminus \{0\}$,

(2.17)
$$H_+(\zeta) = H_-(\zeta)e^{-i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}\begin{pmatrix} 0 & -e^{-\pi i\alpha} \\ e^{\pi i\alpha} & 1 \end{pmatrix} e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, \quad \text{for } \zeta \in (-\infty, 0),$$

(2.18)
$$H_+(\zeta) = H_-(\zeta)e^{-i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}\begin{pmatrix} 0 & -e^{\pi i\alpha} \\ e^{-\pi i\alpha} & 1 \end{pmatrix} e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, \quad \text{for } \zeta \in (0, \infty).$$

(b) $H(\zeta) = \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty.$

(c) H has the following behavior near the origin: If $\alpha < 0$,

$$H(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^\alpha & |\zeta|^\alpha \\ |\zeta|^\alpha & |\zeta|^\alpha \end{pmatrix}, \qquad \text{as } \zeta \to 0, \tag{2.19}$$

and if $\alpha > 0$,

$$H(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^\alpha & |\zeta|^{-\alpha} \\ |\zeta|^\alpha & |\zeta|^{-\alpha} \end{pmatrix}, & \text{as } \zeta \to 0, \operatorname{Im}\zeta > 0, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^\alpha \\ |\zeta|^{-\alpha} & |\zeta|^\alpha \end{pmatrix}, & \text{as } \zeta \to 0, \operatorname{Im}\zeta < 0. \end{cases} \tag{2.20}$$

The jumps in (a) follow from straightforward calculation. The vanishing behavior (b) of H at infinity (in all sectors) follows from the triangular shape of the jump matrices G_j, see (2.1)–(2.4). For example, for $\zeta \in S_1 \cap \mathbb{C}_+$ we have

$\operatorname{Re} i(\frac{4}{3}\zeta^3 + s\zeta) < 0$ so that by (2.15) and (2.16)

$$H(\zeta) = \mathcal{O}(1/\zeta) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} e^{2i(\frac{4}{3}\zeta^3 + s\zeta)} & 1 \end{pmatrix} = \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty.$$

The behavior near the origin in (c) follows from Proposition 2.3. This is immediate for (2.19), while for $\alpha > 0$, $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, we have by (2.8), (2.9), (2.10), and (2.16),

$$H(\zeta)e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = \begin{cases} E(\zeta)\zeta^{-\alpha\sigma_3} A_2 = E(\zeta)\zeta^{-\alpha\sigma_3} \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}, & \text{if } \operatorname{Im}\zeta > 0, \\ E(\zeta)\zeta^{-\alpha\sigma_3} A_4 = E(\zeta)\zeta^{-\alpha\sigma_3} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, & \text{if } \operatorname{Im}\zeta < 0, \end{cases}$$

which yields (2.20) in case $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, since E is analytic. Using (2.13) instead of (2.10), we will see that the same argument works if $\alpha - \frac{1}{2} \in \mathbb{N}_0$.

Next we define (cf. [16], [24], [43])

$$M(\zeta) = H(\zeta)H(\bar{\zeta})^*, \qquad \text{for } \zeta \in \mathbb{C} \setminus \mathbb{R}, \tag{2.21}$$

where H^* denotes the Hermitian conjugate of H. From condition (c) of the RH problem for H it follows that M has the following behavior near the origin:

$$M(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^{2\alpha} & |\zeta|^{2\alpha} \\ |\zeta|^{2\alpha} & |\zeta|^{2\alpha} \end{pmatrix}, & \text{as } \zeta \to 0, \text{ in case } \alpha < 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } \zeta \to 0, \text{ in case } \alpha > 0. \end{cases}$$

Since $\alpha > -1/2$, it follows that each entry of M has an integrable singularity at the origin. Because $M(\zeta) = \mathcal{O}(1/\zeta^2)$ as $\zeta \to \infty$, and M is analytic in the upper half plane, it then follows by Cauchy's theorem that $\int_{\mathbb{R}} M_+(\zeta) d\zeta = 0$, and hence by (2.21)

$$\int_{\mathbb{R}} H_+(\zeta)H_-(\zeta)^* d\zeta = 0.$$

Adding this equation to its Hermitian conjugate, we find

$$\int_{\mathbb{R}} \left[H_+(\zeta)H_-(\zeta)^* + H_-(\zeta)H_+(\zeta)^*\right] d\zeta = 0. \tag{2.22}$$

Using (2.17), (2.18) and the fact that $(e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3})^* = e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}$ for $\zeta, s \in \mathbb{R}$ (here we use the fact that s is real!), we obtain from (2.22),

$$0 = \int_{\mathbb{R}} H_-(\zeta) \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} H_-(\zeta)^* d\zeta = 2\int_{\mathbb{R}} \left[|(H_-)_{12}(\zeta)|^2 + |(H_-)_{22}(\zeta)|^2\right] d\zeta.$$

This implies that the second column of H_- is identically zero. The jump relations (2.17) and (2.18) of H then imply that the first column of H_+ is identically zero as well.

To show that the second column of H_+ and the first column of H_- are also identically zero, we use an idea of Deift et al. [16, Proof of Th. 5.3, Step 3]. Since the second column of H_- is identically zero, the jump relations (2.17) and (2.18) for H yield for $j = 1, 2$,

$$(H_{j2})_+(\zeta) = -e^{\operatorname{sgn}(\zeta)\pi i\alpha} e^{-2i(\frac{4}{3}\zeta^3 + s\zeta)} (H_{j1})_-(\zeta), \qquad \text{for } \zeta \in \mathbb{R} \setminus \{0\}.$$

Thus if we define for $j = 1, 2$,

$$h_j(\zeta) = \begin{cases} H_{j2}(\zeta), & \text{if } \operatorname{Im}\zeta > 0, \\ H_{j1}(\zeta), & \text{if } \operatorname{Im}\zeta < 0, \end{cases} \tag{2.23}$$

then both h_1 and h_2 satisfy the following RH problem for a scalar function h.

The RH *problem for h.*

(a) h is analytic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies the following jump relation

$$h_+(\zeta) = -e^{\operatorname{sgn}(\zeta)\pi i\alpha} e^{-2i(\frac{4}{3}\zeta^3 + s\zeta)} h_-(\zeta), \qquad \text{for } \zeta \in \mathbb{R} \setminus \{0\},$$

(b) $h(\zeta) = \mathcal{O}(1/\zeta)$ as $\zeta \to \infty$.

(c) $h(\zeta) = \begin{cases} \mathcal{O}(|\zeta|^{\alpha}), & \text{as } \zeta \to 0, \text{ in case } \alpha < 0, \\ \mathcal{O}(|\zeta|^{-\alpha}), & \text{as } \zeta \to 0, \text{ in case } \alpha > 0. \end{cases}$

Take ζ_0 with $\operatorname{Im}\zeta_0 < -1$ and define

$$\widehat{h}(\zeta) = \begin{cases} \frac{\zeta^{\alpha}}{(\zeta-\zeta_0)^{\alpha}} h(\zeta), & \text{if } \operatorname{Im}\zeta > 0, \\ \frac{\zeta^{\alpha}}{(\zeta-\zeta_0)^{\alpha}} \left(-e^{\pi i\alpha} e^{-2i(\frac{4}{3}\zeta^3 + s\zeta)}\right) h(\zeta), & \text{if } -1 < \operatorname{Im}\zeta < 0, \end{cases} \tag{2.24}$$

where we use principal branches of the powers, so that ζ^{α} is defined with a branch cut along the negative real axis. Then it is easy to check that $\widehat{h}$ is analytic in $\operatorname{Im}\zeta > -1$, continuous and uniformly bounded in $\operatorname{Im}\zeta \geq -1$, and

$$\widehat{h}(\zeta) = \mathcal{O}(e^{-3|\operatorname{Re}\zeta|^2}), \qquad \text{as } \zeta \to \infty \text{ on the horizontal line } \operatorname{Im}\zeta = -1.$$

By Carlson's theorem, see e.g. [38], this implies that $\widehat{h} \equiv 0$, so that $h \equiv 0$, as well. This in turn implies that $h_1 \equiv 0$ and $h_2 \equiv 0$, so that $H \equiv 0$. Then also $\widehat{\Psi} \equiv 0$ and the proposition is proven. □

As noted before, Proposition 2.5 has the following consequence.

Corollary 2.6. *The* RH *problem for* Ψ_α, *see Section* 2.1, *has a unique solution for every* $s \in \mathbb{R}$ *and* $\alpha > -1/2$.

2.3. *Proof of Theorem* 1.1. Theorem 1.1 follows from the connection of the RH problem for Ψ_α of Section 2.1 with the RH problem associated with the general Painlevé II equation (1.12) as first described by Flaschka and Newell [20, §3D].

Proof of Theorem 1.1. Consider the matrix differential equation

$$\frac{\partial \Psi}{\partial \zeta} = A\Psi, \tag{2.25}$$

where A is as in (1.16) and s, q, and r are constants. For every $k = 0, 1, \dots, 5$, there is a unique solution Ψ_k of (2.25) such that

$$\Psi_k(\zeta) = (I + \mathcal{O}(1/\zeta))e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}$$

as $\zeta \to \infty$ in the sector $(2k-1)\frac{\pi}{6} < \arg\zeta < (2k+1)\frac{\pi}{6}$. The function

$$\Psi(\zeta) = \Psi_k(\zeta), \qquad \text{for } (2k-1)\tfrac{\pi}{6} < \arg\zeta < (2k+1)\tfrac{\pi}{6}, \tag{2.26}$$

is then defined on $\mathbb{C} \setminus (\Sigma \cup i\mathbb{R})$ and satisfies the following conditions.

(a) Ψ is analytic in $\mathbb{C} \setminus (\Sigma \cup i\mathbb{R})$.

(b) There exist constants $s_1, s_2, s_3 \in \mathbb{C}$ (Stokes multipliers) satisfying

$$s_1 + s_2 + s_3 + s_1 s_2 s_3 = -2i \sin \pi\alpha \tag{2.27}$$

such that the following jump conditions hold, where all rays are oriented to infinity,

$$\Psi_+ = \begin{cases} \Psi_- \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, & \text{on } \Gamma_1, \\ \Psi_- \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}, & \text{on } i\mathbb{R}^+, \\ \Psi_- \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix}, & \text{on } \Gamma_2, \end{cases} \qquad \Psi_+ = \begin{cases} \Psi_- \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, & \text{on } \Gamma_3, \\ \Psi_- \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, & \text{on } i\mathbb{R}^-, \\ \Psi_- \begin{pmatrix} 1 & s_3 \\ 0 & 1 \end{pmatrix}, & \text{on } \Gamma_4. \end{cases}$$

(c) $\Psi(\zeta) = (I + \mathcal{O}(1/\zeta))e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}, \qquad \text{as } \zeta \to \infty.$

The Stokes multipliers s_1, s_2, s_3 depend on s, q and r. However, if $q = q(s)$ satisfies the second Painlevé equation $q'' = sq + 2q^3 - \alpha$, and if $r = q'(s)$, then the Stokes multipliers are constant. In this way there is a one-to-one correspondence between solutions of the Painlevé II equation and Stokes multipliers s_1, s_2, s_3 satisfying (2.27). This also means that there exists a solution of the above RH problem which is built out of solutions of (2.25) if and only if s is not a pole of the Painlevé II function that corresponds to the Stokes multipliers s_1, s_2, s_3. The Painlevé II function itself may then be recovered from the RH problem by the formula [20]

$$q(s) = \lim_{\zeta\to\infty} 2i\zeta\Psi_{12}(\zeta)e^{-i(\frac{4}{3}\zeta^3+s\zeta)},$$

with Ψ_{12} the $(1,2)$-entry of Ψ. In particular, condition (c) of the RH problem can be strengthened to

(2.28)

$$\Psi(\zeta) = \left(I + \frac{1}{2i\zeta}\begin{pmatrix} u(s) & q(s) \\ -q(s) & -u(s)\end{pmatrix} + \mathcal{O}(1/\zeta^2)\right)e^{-i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, \qquad \text{as } \zeta\to\infty,$$

where $u = (q')^2 - sq^2 - q^4 + 2\alpha q$.

The RH problem for Ψ_α in Section 2.1 corresponds to

$$s_1 = e^{-\pi i\alpha}, \qquad s_2 = 0, \qquad s_3 = -e^{\pi i\alpha}. \tag{2.29}$$

These Stokes multipliers are very special in two respects [26], [30]. First, since $s_2 = 0$, the corresponding solution of the Painlevé II equation decays as $s \to +\infty$, i.e.,

$$q(s) \sim \frac{\alpha}{s}, \qquad \text{as } s \to +\infty. \tag{2.30}$$

Secondly, since $s_1 s_3 = -1$ the Painlevé II solution increases as $s \to -\infty$, i.e.,

$$q(s) \sim \pm\sqrt{-\frac{s}{2}}, \qquad \text{as } s\to -\infty, \tag{2.31}$$

where the choice $s_1 = e^{-\pi i\alpha}$, $s_3 = -e^{\pi i\alpha}$ corresponds to the $+$ sign, while the interchange of s_1 and s_3 corresponds to the $-$ sign in (2.31). Thus the special choice (2.29) corresponds to q_α, the Hastings-McLeod solution of the general Painlevé II equation; see (1.13) and (1.14).

Then as a consequence of the fact that the RH problem for Ψ_α stated in Section 2.1 is solvable for every real s by Corollary 2.6, we conclude that q_α has no poles on the real line, which proves Theorem 1.1. □

Remark 2.7. Its and Kapaev [26] use a slightly modified, but equivalent, version of the RH problem for Ψ_α. The solutions are connected by the transformation

$$\Psi_\alpha \leftrightarrow e^{\frac{\pi i}{4}\sigma_3}\Psi_\alpha e^{-\frac{\pi i}{4}\sigma_3}, \tag{2.32}$$

which results in a transformation of the Stokes multipliers $s_j \leftrightarrow (-1)^j i s_j$.

For later use, we record the following corollary.

COROLLARY 2.8. *For every fixed $s_0 \in \mathbb{R}$, there exists an open neighborhood U of s_0 such that the* RH *problem for Ψ_α is solvable for every $s \in U$.*

Proof. Since q_α is meromorphic in $\mathbb{C}$, there is an open neighborhood of s_0 without poles. This implies [20] that the RH problem for Ψ_α is solvable for every s in that open neighborhood of s_0, as well. $\square$

Remark 2.9. The function $\Psi_\alpha(\zeta; s)$ is analytic as a function of both $\zeta \in \mathbb{C} \setminus \Sigma$ and $s \in \mathbb{C} \setminus \mathcal{P}_\alpha$, where $\mathcal{P}_\alpha$ denotes the set of poles of q_α; see [20]. As a consequence, one can check that (2.5), (2.6) and (2.7) hold uniformly for s in compact subsets of $\mathbb{C} \setminus \mathcal{P}_\alpha$.

Remark 2.10. The functions $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ defined by (1.15) and (1.18) are connected with Ψ_α as follows. Define

(2.33)
$$\Phi_\alpha(\zeta; s) = \begin{cases} \Psi_\alpha(\zeta; s) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \zeta \in S_1, \\ \Psi_\alpha(\zeta; s), & \text{for } \zeta \in S_2, \\ \Psi_\alpha(\zeta; s) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \zeta \in S_3, \\ \Psi_\alpha(\zeta; s) \begin{pmatrix} 1 & -e^{\pi i \alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \zeta \in S_4, \operatorname{Re} \zeta > 0, \\ \Psi_\alpha(\zeta; s) \begin{pmatrix} 1 & -e^{-\pi i \alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \zeta \in S_4, \operatorname{Re} \zeta < 0. \end{cases}$$

Then it follows from the RH problem for Ψ_α that Φ_α is analytic on $\mathbb{C} \setminus (-i\infty, 0]$. Moreover, we also see from (1.15) and (1.18) that

$$\Phi_\alpha = \begin{pmatrix} \Phi_{\alpha,1} & * \\ \Phi_{\alpha,2} & * \end{pmatrix}, \tag{2.34}$$

where $*$ denotes an unspecified unimportant entry. It also follows that $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ have analytic continuations to $\mathbb{C} \setminus (-i\infty, 0]$.

Remark 2.11. We show that the kern $K^{\mathrm{crit},\alpha}(u, v; s)$ is real. This will follow from the identity

(2.35)
$$e^{\frac{1}{2}\pi i \alpha \operatorname{sgn}(u)} \Phi_{\alpha,2}(u; s) = \overline{e^{\frac{1}{2}\pi i \alpha \operatorname{sgn}(u)} \Phi_{\alpha,1}(u; s)}, \qquad \text{for } u \in \mathbb{R} \setminus \{0\} \text{ and } s \in \mathbb{R},$$

since obviously (2.35) implies that

$$K^{\mathrm{crit},\alpha}(u, v; s) = -\frac{1}{\pi(u - v)} \operatorname{Im} \left(e^{\frac{1}{2}\pi i \alpha (\operatorname{sgn}(u) - \operatorname{sgn}(v))} \Phi_{\alpha,1}(u; s) \overline{\Phi_{\alpha,1}(v; s)} \right).$$

The identity (2.35) will follow from the RH problem. It is easy to check that $\sigma_1 \overline{\Psi_\alpha(\bar{\zeta}; s)} \sigma_1$, with $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, also satisfies the RH conditions for Ψ_α.

Because of the uniqueness of the solution of the RH problem, this implies

$$\Psi_\alpha(\zeta; s) = \sigma_1 \overline{\Psi_\alpha(\overline{\zeta}; s)} \sigma_1. \tag{2.36}$$

For $\zeta \in S_4$, the equality of the $(2,1)$ entries of (2.36) yields by (2.33) and (2.34)

$$e^{\pi i\alpha}\Phi_{\alpha,2}(\zeta; s) = \overline{\Phi_{\alpha,1}(\overline{\zeta}; s)}, \qquad \text{for } \zeta \in S_4,\ \operatorname{Re}\zeta > 0, \tag{2.37}$$

and

$$e^{-\pi i\alpha}\Phi_{\alpha,2}(\zeta; s) = \overline{\Phi_{\alpha,1}(\overline{\zeta}; s)}, \qquad \text{for } \zeta \in S_4,\ \operatorname{Re}\zeta < 0. \tag{2.38}$$

Since both sides of (2.37) are analytic in the right half-plane we find the identity (2.35) for $u > 0$, and similarly since both sides of (2.38) are analytic in the left half-plane, we obtain (2.35) for $u < 0$.

3. Steepest descent analysis of the RH problem

In this section we write the kernel $K_{n,N}$ in terms of the solution Y of the RH problem for orthogonal polynomials (due to Fokas, Its and Kitaev [21]) and apply the Deift/Zhou steepest descent method [18] to the RH problem for Y to get the asymptotics for Y. These asymptotics will be used in the next sections to prove Theorems 1.2 and 1.7.

We will restrict ourselves to the one-interval case, which means that ψ_V is supported on one interval, although the RH analysis can be done in general. We comment below in Remark 3.1 (see the end of this section) on the modifications that have to be made in the multi-interval case.

As in Theorems 1.2 and 1.7 we also assume that besides 0 there are no other singular points.

3.1. *The* RH *problem for orthogonal polynomials.* The starting point is the RH problem that characterizes the orthogonal polynomials associated with the weight $|x|^{2\alpha}e^{-NV(x)}$. The 2×2 matrix-valued function $Y = Y_{n,N}$ satisfies the following conditions.

The RH *problem for* Y.

(a) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & |x|^{2\alpha}e^{-NV(x)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } x \in \mathbb{R}.$

(c) $Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad \text{as } z \to \infty.$

(d) Y has the following behavior near the origin,

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z|^{2\alpha} \\ 1 & |z|^{2\alpha} \end{pmatrix}, & \text{as } z \to 0, \text{ if } \alpha < 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to 0, \text{ if } \alpha \geq 0. \end{cases} \tag{3.1}$$

Here we have oriented the real axis from the left to the right and $Y_+(x)$ ($Y_-(x)$) in part (b) denotes the limit as we approach $x \in \mathbb{R}$ from the upper (lower) half-plane. This RH problem possesses a unique solution given by [21] (see [33], [35] for the condition (d)),

$$Y(z) = \begin{pmatrix} \frac{1}{\kappa_{n,N}} p_{n,N}(z) & \frac{1}{2\pi i \kappa_{n,N}} \int_{\mathbb{R}} \frac{p_{n,N}(y)|y|^{2\alpha} e^{-NV(y)}}{y - z} dy \\ -2\pi i \kappa_{n-1,N} p_{n-1,N}(z) & -\kappa_{n-1,N} \int_{\mathbb{R}} \frac{p_{n-1,N}(y)|y|^{2\alpha} e^{-NV(y)}}{y - z} dy \end{pmatrix}, \tag{3.2}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where $p_{n,N}(z) = \kappa_{n,N} z^n + \cdots$, is the n-th degree orthonormal polynomial with respect to the weight $|x|^{2\alpha} e^{-NV(x)}$, and $\kappa_{n,N}$ is the leading coefficient of $p_{n,N}$.

The correlation kernel $K_{n,N}$ can be expressed in terms of the solution of this RH problem. Indeed, using the Christoffel-Darboux formula for orthogonal polynomials, we get from (1.4), (3.2), and the fact that $\det Y \equiv 1$,

$$\begin{aligned} K_{n,N}(x,y) &= |x|^{\alpha} e^{-\frac{1}{2}NV(x)} |y|^{\alpha} e^{-\frac{1}{2}NV(y)} \frac{\kappa_{n-1,N}}{\kappa_{n,N}} \\ &\quad \times \frac{p_{n,N}(x) p_{n-1,N}(y) - p_{n-1,N}(x) p_{n,N}(y)}{x - y} \\ &= |x|^{\alpha} e^{-\frac{1}{2}NV(x)} |y|^{\alpha} e^{-\frac{1}{2}NV(y)} \frac{1}{2\pi i (x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_{\pm}^{-1}(y) Y_{\pm}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

The asymptotics of $K_{n,N}$ follows from a steepest descent analysis of the RH problem for Y, see [9], [16], [17], [34], [35], [42]. The Deift/Zhou steepest descent analysis consists of a series of explicit transformations $Y \mapsto T \mapsto S \mapsto R$ so that it leads to an RH problem for R which is normalized at infinity and which has jumps uniformly close to the identity matrix I. Then R itself is uniformly close to I. By going back in the series of transformations we then have the asymptotics for Y from which the asymptotics of $K_{n,N}$ in different scaling regimes can be deduced.

The main issue of the present situation is the construction of a local parametrix near 0 with the aid of the RH problem for Ψ_α introduced in Section 2. For the case $\alpha = 0$ this was done in [9] and we use the ideas introduced in that paper.

Throughout the rest of the paper we use the notation

$$t = n/N, \qquad \text{and} \qquad V_t = \frac{1}{t}V. \tag{3.4}$$

3.2. *First transformation* $Y \mapsto T$. In the first transformation we normalize the RH problem at infinity. The standard approach would be to use the equilibrium measure in the external field V_t, see [11], [39]. This is the probability measure that minimizes

$$I_{V_t}(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V_t(x) d\mu(x)$$

among all Borel probability measures μ on $\mathbb{R}$. The minimizer for $t = 1$ has density ψ_V which by assumption vanishes at the origin. For $t < 1$, the origin is outside of the support and for t slightly less than 1, there is a gap in the support around 0. An annoying consequence is that the equality in the variational conditions is not valid near the origin. Therefore, a modified measure ν_t was introduced in [9] to overcome this problem.

Here, we follow [9, §3]. We take a small $\delta_0 > 0$ so that $\psi_V(x) > 0$ for $x \in [-\delta_0, \delta_0] \setminus \{0\}$, and we consider the problem to minimize $I_{V_t}(\nu)$ among all signed measures $\nu = \nu^+ - \nu^-$ where $\nu^\pm$ are nonnegative measures such that $\int d\nu = 1$ and $\operatorname{supp}(\nu^-) \subset [-\delta_0, \delta_0]$. There is a unique minimizer which we denote by ν_t. This signed measure is absolutely continuous with density ψ_t and its support $S_t = [a_t, b_t]$ is an interval if t is sufficiently close to 1. The following variational conditions are satisfied: there exists a constant $\ell_t \in \mathbb{R}$ such that

$$2\int \log|x-y| \psi_t(y) dy - V_t(x) + \ell_t = 0, \qquad \text{for } x \in [a_t, b_t], \tag{3.5}$$

$$2\int \log|x-y| \psi_t(y) dy - V_t(x) + \ell_t \leq 0, \qquad \text{for } x \in \mathbb{R}. \tag{3.6}$$

In addition, it was shown in [9] that for t sufficiently close to 1,

$$\psi_t(x) = \frac{1}{\pi}(-Q_t(x))^{1/2}, \qquad \text{for } x \in [a_t, b_t], \tag{3.7}$$

where

$$Q_t(z) = \left(\frac{V'(z)}{2t}\right)^2 - \frac{1}{t}\int \frac{V'(z) - V'(y)}{z-y} \psi_t(y) dy. \tag{3.8}$$

For $t > 1$, we take the square root in (3.7) which is positive for $x = 0$, while for $t < 1$ we take the square root which is negative for $x = 0$.

For the first transformation, we introduce the following 'g-function' associated with ν_t,

$$g_t(z) = \int \log(z-y) d\nu_t(y) = \int \log(z-y)\psi_t(y) dy, \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}, \tag{3.9}$$

where we take the branch cut of the logarithm along the negative real axis. We define

$$T(z) = e^{\frac{1}{2}n\ell_t\sigma_3} Y(z) e^{-ng_t(z)\sigma_3} e^{-\frac{1}{2}n\ell_t\sigma_3}, \qquad \text{for } z \in \mathbb{C}\setminus\mathbb{R}. \tag{3.10}$$

We also use the functions

$$\varphi_t(z) = \int_{b_t}^{z} (Q_t(s))^{1/2} ds, \tag{3.11}$$

$$\tilde{\varphi}_t(z) = \int_{a_t}^{z} (Q_t(s))^{1/2} ds, \tag{3.12}$$

where the path of integration does not cross the real axis. The relations that exist between g_t, φ_t and $\tilde{\varphi}_t$ are described in [9, §5.2]. Using these, we find that T is the unique solution of the following RH problem.

The RH *problem for* T.

(a) $T : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $T_+(x) = T_-(x) v_T(x)$ for $x \in \mathbb{R}$, with

$$v_T(x) = \begin{cases} \begin{pmatrix} e^{2n\varphi_{t,+}(x)} & |x|^{2\alpha} \\ 0 & e^{2n\varphi_{t,-}(x)} \end{pmatrix}, & \text{for } x \in (a_t, b_t), \\ \begin{pmatrix} 1 & |x|^{2\alpha} e^{-2n\varphi_t(x)} \\ 0 & 1 \end{pmatrix}, & \text{for } x \in (b_t, \infty), \\ \text{[3ex]} \begin{pmatrix} 1 & |x|^{2\alpha} e^{-2n\tilde{\varphi}_t(x)} \\ 0 & 1 \end{pmatrix}, & \text{for } x \in (-\infty, a_t). \end{cases}$$

(c) $T(z) = I + \mathcal{O}(1/z), \qquad$ as $z \to \infty$.

(d) T has the same behavior as Y near the origin, given by (3.1).

3.3. *Second transformation* $T \mapsto S$. In this subsection, we open the lens as in Figure 2. The opening of the lens is based on the factorization of the jump matrix v_T for $x \in (a_t, b_t)$, which is

$$v_T(x) = \begin{pmatrix} e^{2n\varphi_{t,+}(x)} & |x|^{2\alpha} \\ 0 & e^{2n\varphi_{t,-}(x)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ |x|^{-2\alpha} e^{2n\varphi_{t,-}(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ |x|^{-2\alpha} e^{2n\varphi_{t,+}(x)} & 1 \end{pmatrix}. \tag{3.13}$$

We deform the RH problem for T into an RH problem for S by opening a lens around $[a_t, b_t]$ going through the origin, as shown in Figure 2. The precise form

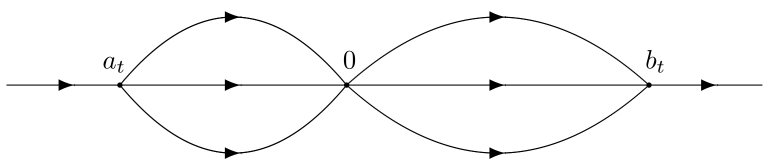

Figure 2: The lens shaped contour Σ_S going through the origin.

of the lens is not yet specified but for now we choose the lens to be contained in the region of analyticity of V and we can do it in such a way that for any given $\delta > 0$, there exists $\gamma > 0$ so that, for every t sufficiently close to 1, we have that

$$\operatorname{Re} \varphi_t(z) < -\gamma, \tag{3.14}$$

for z on the upper and lower lips of the lens with the exception of δ-neighborhoods of 0, a, and b. See also [9, §5.3].

Let ω be the analytic continuation of $x \mapsto |x|^{2\alpha}$ to $\mathbb{C} \setminus (i\mathbb{R})$; i.e.,

$$\omega(z) = \begin{cases} z^{2\alpha}, & \text{if } \operatorname{Re} z > 0, \\ (-z)^{2\alpha}, & \text{if } \operatorname{Re} z < 0. \end{cases} \tag{3.15}$$

The second transformation is then defined by

(3.16)

$$S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lens,} \\ T(z) \begin{pmatrix} 1 & 0 \\ -\omega(z)^{-1} e^{2n\varphi_t(z)} & 1 \end{pmatrix}, & \text{for } z \text{ in the upper parts of the lens,} \\ T(z) \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1} e^{2n\varphi_t(z)} & 1 \end{pmatrix}, & \text{for } z \text{ in the lower parts of the lens.} \end{cases}$$

Then S is the unique solution of the following RH problem posed on the contour Σ_S which is the union of $\mathbb{R}$ with the upper and lower lips of the lens.

The RH *problem for* S.

(a) $S : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $S_+ = S_- v_S$ on Σ_S, where

$$v_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1} e^{2n\varphi_t(z)} & 1 \end{pmatrix}, & \text{for } z \in \Sigma_S \setminus \mathbb{R}, \\ \begin{pmatrix} 0 & |z|^{2\alpha} \\ -|z|^{-2\alpha} & 0 \end{pmatrix}, & \text{for } z \in (a_t, b_t), \\ \begin{pmatrix} 1 & |z|^{2\alpha} e^{-2n\varphi_t(z)} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in (b_t, \infty), \\ \begin{pmatrix} 1 & |z|^{2\alpha} e^{-2n\tilde{\varphi}_t(z)} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in (-\infty, a_t). \end{cases}$$

(c) $S(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

(d) S has the following behavior near the origin. If $\alpha < 0$,

$$S(z) = \mathcal{O}\begin{pmatrix} 1 & |z|^{2\alpha} \\ 1 & |z|^{2\alpha} \end{pmatrix}, \qquad \text{as } z \to 0, z \in \mathbb{C} \setminus \Sigma_S, \tag{3.17}$$

and if $\alpha \geq 0$,

$$S(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ from outside the lens}, \\ \mathcal{O}\begin{pmatrix} |z|^{-2\alpha} & 1 \\ |z|^{-2\alpha} & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ from inside the lens}. \end{cases} \tag{3.18}$$

3.4. *Parametrix away from special points.* On the lips of the lens and on $(-\infty, a_t) \cup (b_t, \infty)$, the jump matrix for S is close to the identity matrix if n is large and t is close to 1. This follows from the inequality (3.14) and the fact that $\varphi_t(x) > 0$ for $x > b_t$ and $\tilde{\varphi}_t(x) > 0$ for $x < a_t$. Ignoring these jumps we are led to the following RH problem.

The RH *problem for* $P^{(\infty)}$.

(a) $P^{(\infty)} : \mathbb{C} \setminus [a_t, b_t] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P_+^{(\infty)}(x) = P_-^{(\infty)}(x) \begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix}, \qquad \text{for } x \in (a_t, b_t) \setminus \{0\}.$

(c) $P^{(\infty)}(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

Note that $P^{(\infty)}$ depends on n and N through the parameter t. As in [31], [33], [35] we construct $P^{(\infty)}$ in terms of the Szegő function D associated with $|x|^{2\alpha}$ on (a_t, b_t). This is an analytic function in $\mathbb{C} \setminus [a_t, b_t]$, satisfying $D_+(x)D_-(x) = |x|^{2\alpha}$ for $x \in (a_t, b_t) \setminus \{0\}$, which does not vanish anywhere in $\overline{\mathbb{C}} \setminus [a_t, b_t]$. It is easy to check that D is given by

$$D(z) = z^\alpha \phi\left(\frac{2z - a_t - b_t}{b_t - a_t}\right)^{-\alpha}, \qquad \text{for } z \in \mathbb{C} \setminus [a_t, b_t], \tag{3.19}$$

where $\phi(z) = z + (z-1)^{1/2}(z+1)^{1/2}$ is the conformal map from $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit circle. Since $\phi(z) = 2z + \mathcal{O}(1/z)$ as $z \to \infty$ we have

$$\lim_{z\to\infty} D(z) = \left(\frac{4}{b_t - a_t}\right)^{-\alpha} \equiv D_\infty.$$

Now the transformed matrix-valued function

$$\widehat{P}^{(\infty)} = D_\infty^{-\sigma_3} P^{(\infty)} D^{\sigma_3} \tag{3.20}$$

satisfies conditions (a) and (c) of the RH problem and it has the jump matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on (a_t, b_t). The construction of $\widehat{P}^{(\infty)}$ has been done in [11], [16], [17], and leads us to the solution of the RH problem for $P^{(\infty)}$:

$$P^{(\infty)}(z) = D_\infty^{\sigma_3} \begin{pmatrix} \frac{\beta(z)+\beta(z)^{-1}}{2} & \frac{\beta(z)-\beta(z)^{-1}}{2i} \\ \frac{\beta(z)-\beta(z)^{-1}}{-2i} & \frac{\beta(z)+\beta(z)^{-1}}{2} \end{pmatrix} D(z)^{-\sigma_3}, \qquad \text{for } z \in \mathbb{C} \setminus [a_t, b_t], \tag{3.21}$$

where

$$\beta(z) = \frac{(z-b_t)^{1/4}}{(z-a_t)^{1/4}}, \qquad \text{for } z \in \mathbb{C} \setminus [a_t, b_t]. \tag{3.22}$$

3.5. *Parametrix near endpoints.* The jump matrices of S and $P^{(\infty)}$ are not uniformly close to each other near the origin and near the endpoints of $[a_t, b_t]$. We surround a_1 and b_1 (the endpoints of S_V) with small disks $U_\delta(a)$ and $U_\delta(b)$ of radius δ. For t sufficiently close to 1, the endpoints a_t and b_t are in these disks, and then local parametrices $P^{(a_t)}$ and $P^{(b_t)}$ can be constructed with Airy functions as in [11], [16], [17].

3.6. *Parametrix near the origin.* Near the origin a local parametrix will be constructed with the aid of the RH problem for Ψ_α of Section 2. Let U_δ be a small disk with center at 0 and radius $\delta > 0$. We seek a 2×2 matrix-valued function P in U_δ, with the same jumps as S, with the same behavior as S near the origin, which matches with $P^{(\infty)}$ on the boundary ∂U_δ of the disk. We thus seek a 2×2 matrix-valued function that satisfies the following RH problem.

The RH *problem for* P.

(a) P is defined and analytic in $U_{\delta'} \setminus \Sigma_S$ for some $\delta' > \delta$.

(b) On $\Sigma_S \cap U_\delta$, P satisfies the jump relations

$$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1} e^{2n\varphi_t(z)} & 1 \end{pmatrix}, \qquad \text{for } z \in (\Sigma_S \setminus \mathbb{R}) \cap U_\delta, \tag{3.23}$$

$$P_+(x) = P_-(x) \begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix}, \qquad \text{for } x \in (-\delta, \delta) \setminus \{0\}. \tag{3.24}$$

(c) P satisfies the matching condition

$$P(z) = (I + \mathcal{O}(n^{-1/3}))P^{(\infty)}(z), \tag{3.25}$$

as $n, N \to \infty$ such that $n^{2/3}(n/N - 1) \to L$, uniformly for $z \in \partial U_\delta \setminus \Sigma_S$.

(d) P has the same behavior near the origin as S, given by (3.17) and (3.18).

In order to solve the RH problem for P we work as follows. First, we seek P such that it satisfies conditions (a), (b), and (d). To do this, we transform (in the first step) the RH problem into an RH problem for $\widehat{P}$ with constant jump matrices. In the second step we solve the RH problem for $\widehat{P}$ explicitly by using the RH problem for Ψ_α. In the final step we take also the matching condition (c) into account.

Step 1: *Transformation to constant jump matrices.* In the first step we transform the RH problem for P into an RH problem for $\widehat{P}$ with constant jump matrices. We seek P in the form

$$P(z) = E(z)\widehat{P}(z)e^{n\varphi_t(z)\sigma_3}e^{\frac{1}{2}\pi i\alpha\sigma_3}z^{-\alpha\sigma_3}, \qquad \text{if } \operatorname{Im} z > 0, \tag{3.26}$$

$$P(z) = E(z)\widehat{P}(z)e^{-n\varphi_t(z)\sigma_3}e^{\frac{1}{2}\pi i\alpha\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}z^{-\alpha\sigma_3}, \qquad \text{if } \operatorname{Im} z < 0, \tag{3.27}$$

where the invertible matrix-valued function $E = E_{n,N}$ (we suppress notation of the indices) is analytic in $U_{\delta'}$ and where the branch cut of z^α is chosen along the negative real axis.

Using (3.24), (3.26) and (3.27), and keeping track of the branches of z^α, we can easily check that $\widehat{P}$ has no jumps on $(-\delta, \delta) \setminus \{0\}$. What remains are jumps on the contour $(\Sigma_S \setminus \mathbb{R}) \cap U_\delta = \bigcup_{j=1}^4 \Sigma_j$, which is shown in Figure 3. We have reversed the orientation of Σ_2 and Σ_3 towards infinity, so that now the orientation of the Σ_j's corresponds to the orientation of the Γ_j's in Figure 1. The contour divides U_δ into four regions $\mathrm{I}, \mathrm{II}, \mathrm{III}$ and IV, also shown in Figure 3.

We will now determine the jump relations for $\widehat{P}$. By (3.15), (3.23), and (3.26), $\widehat{P}$ should have the following jump matrix on Σ_1,

$$\begin{aligned} &\widehat{P}_-(z)^{-1}\widehat{P}_+(z) \\ &\quad = e^{n\varphi_t(z)\sigma_3}e^{\frac{1}{2}\pi i\alpha\sigma_3}z^{-\alpha\sigma_3}\begin{pmatrix} 1 & 0 \\ \omega(z)^{-1}e^{2n\varphi_t(z)} & 1 \end{pmatrix}z^{\alpha\sigma_3}e^{-\frac{1}{2}\pi i\alpha\sigma_3}e^{-n\varphi_t(z)\sigma_3} \\ &\quad = \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1}z^{2\alpha}e^{-\pi i\alpha} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}. \end{aligned} \tag{3.28}$$

For $z \in \Sigma_2$ we have, because of the reversal of the orientation, an extra minus sign in the $(2,1)$-entry of the jump matrix. The result is

$$\widehat{P}_-(z)^{-1}\widehat{P}_+(z) = \begin{pmatrix} 1 & 0 \\ -\omega(z)^{-1}z^{2\alpha}e^{-\pi i\alpha} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} & 1 \end{pmatrix}, \tag{3.29}$$

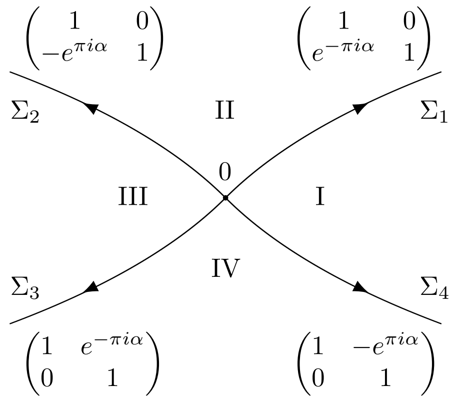

Figure 3: Contour and jumps for the RH problem for $\widehat{P}$.

where the latter equality follows from the fact that $\omega(z)^{-1}z^{2\alpha} = e^{2\pi i\alpha}$ by (3.15), since $\operatorname{Re} z < 0$ in this case. By equations (3.24) and (3.27), the jump matrices for $\widehat{P}$ on Σ_3 and Σ_4 can be determined similarly. The result is that

$$\widehat{P}_+(z) = \begin{cases} \widehat{P}_-(z) \begin{pmatrix} 1 & e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_3, \\ \widehat{P}_-(z) \begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_4. \end{cases} \tag{3.30}$$

We arrive at the following RH problem for $\widehat{P}$. If it is satisfied by $\widehat{P}$ then P defined by (3.26)–(3.27) satisfies the parts (a), (b), and (d) of the RH problem for P.

The RH *problem for* $\widehat{P}$.

(a) $\widehat{P}$ is defined and analytic in $U_{\delta'} \setminus \bigcup_j \Sigma_j$ for some $\delta' > \delta$.

(b) $\widehat{P}$ satisfies the following jump relations

$$\widehat{P}_+(z) = \begin{cases} \widehat{P}_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, & \text{for } z \in \Sigma_1, \\ \widehat{P}_-(z) \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} & 1 \end{pmatrix}, & \text{for } z \in \Sigma_2, \\ \widehat{P}_-(z) \begin{pmatrix} 1 & e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_3, \\ \widehat{P}_-(z) \begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_4. \end{cases} \tag{3.31}$$

(c) $\widehat{P}$ has the following behavior near the origin. If $\alpha < 0$,

$$\widehat{P}(z) = \mathcal{O}\begin{pmatrix} |z|^{\alpha} & |z|^{\alpha} \\ |z|^{\alpha} & |z|^{\alpha} \end{pmatrix}, \qquad \text{as } z \to 0, \tag{3.32}$$

and if $\alpha \geq 0$,

$$\widehat{P}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z|^{-\alpha} & |z|^{-\alpha} \\ |z|^{-\alpha} & |z|^{-\alpha} \end{pmatrix}, & \text{as } z \to 0,\ z \in \mathrm{I} \cup \mathrm{III}, \\ \mathcal{O}\begin{pmatrix} |z|^{\alpha} & |z|^{-\alpha} \\ |z|^{\alpha} & |z|^{-\alpha} \end{pmatrix}, & \text{as } z \to 0,\ z \in \mathrm{II}, \\ \mathcal{O}\begin{pmatrix} |z|^{-\alpha} & |z|^{\alpha} \\ |z|^{-\alpha} & |z|^{\alpha} \end{pmatrix}, & \text{as } z \to 0,\ z \in \mathrm{IV}. \end{cases} \tag{3.33}$$

Note that if $\widehat{P}$ has the behavior near the origin as described in part (c) of the RH problem, then P defined by (3.26) and (3.27) has the same behavior near the origin as S, as required by part (d) of the RH problem for P.

Step 2: *Construction of* $\widehat{P}$. Observe that the jump matrices and the behavior near the origin of the RH problem for $\widehat{P}$ correspond exactly to the jump matrices and the behavior near the origin of the RH problem for Ψ_α. We use the solution of the latter RH problem to solve the RH problem for $\widehat{P}$.

We seek $\widehat{P}$ in the form

$$\widehat{P}(z) = \Psi_\alpha\left(n^{1/3} f(z); n^{2/3} s_t(z)\right), \tag{3.34}$$

where f and s_t are analytic functions on U_δ which are real on $(-\delta, \delta)$, and s_t is such that

$$n^{2/3} s_t(z) \in \mathbb{C} \setminus \mathcal{P}_\alpha, \qquad \text{for } z \in U_\delta, \tag{3.35}$$

where $\mathcal{P}_\alpha$ is the set of poles of q_α. In addition, f is a conformal map from U_δ onto a convex neighborhood $f(U_\delta)$ of 0 such that $f(0) = 0$ and $f'(0) > 0$. Depending on f we open the lens around $[a_t, b_t]$ such that that $f(\Sigma_i) = \Gamma_i$ for $i = 1, 2, 3, 4$, where the Γ_i's are the jump contours for the RH problem for Ψ_α; see Figure 1. Recall that the lens was not fully specified and we still have the freedom to make this choice.

It remains to determine f and s_t so that the matching condition for P is also satisfied. Here we again follow [9]. As in [9, §5.6] we take

$$f(z) = \left[\frac{3}{4}\int_0^z (-Q_1(y))^{1/2} dy\right]^{1/3} = z\left(\frac{\pi \psi_V''(0)}{8}\right)^{1/3} + \mathcal{O}\left(z^2\right), \quad \text{as } z \to 0, \tag{3.36}$$

and

$$s_t(z) f(z) = \int_0^z \left((-Q_t(y))^{1/2} - (-Q_1(y))^{1/2}\right) dy. \tag{3.37}$$

Then f is analytic with $f(0)=0$ and $f'(0)>0$, it does not depend on t, and it is a conformal mapping on U_δ provided δ is small enough. Since the right-hand side of (3.37) is analytic and vanishes for $z=0$, we can divide by $f(z)$ and obtain an analytic function s_t. From [9, (5.26)], we get that there exists a constant $K>0$ such that

$$|s_t(z)-\pi c^{1/3}(t-1)w_{S_V}(0)|\le K(t-1)|z|+o(t-1) \qquad \text{as } t\to 1, \tag{3.38}$$

uniformly for z in a neighborhood of 0. Now assume that $|n^{2/3}(t-1)|\le M$ with n large enough. Then it easily follows from (3.38) and the fact that q_α has no real poles, that there exists a $\delta>0$, depending only on M, such that

$$|\operatorname{Im} n^{2/3}s_t(z)|<\min\{|\operatorname{Im} s| \mid s \text{ is a pole of } q_\alpha\} \qquad \text{for } |z|\le\delta. \tag{3.39}$$

Then (3.35) holds and (3.34) is well-defined and analytic since $\Psi_\alpha(\zeta;s)$ is jointly analytic in its two arguments, see Remark 2.9.

It follows from (3.36) and (3.37) that

$$-i\left[\frac{4}{3}f(z)^3+s_t(z)f(z)\right]=\begin{cases}\varphi_{t,+}(0)-\varphi_t(z), & \text{if } \operatorname{Im} z>0,\\ \varphi_{t,+}(0)+\varphi_t(z), & \text{if } \operatorname{Im} z<0,\end{cases} \tag{3.40}$$

see also [9, §5.6]. Hence by (2.5), which by Remark 2.9 holds uniformly for s in compact subsets of $\mathbb{C}\setminus\mathcal{P}_\alpha$, we have

$$\begin{aligned}\widehat{P}(z)&=\Psi_\alpha\left(n^{1/3}f(z);n^{2/3}s_t(z)\right)\left[I+\mathcal{O}(1/n^{1/3})\right]e^{n\varphi_{t,+}(0)\sigma_3}\\&\quad\times\begin{cases}e^{-n\varphi_t(z)\sigma_3}, & \text{if } \operatorname{Im} z>0,\\ e^{n\varphi_t(z)\sigma_3}, & \text{if } \operatorname{Im} z<0,\end{cases}\end{aligned} \tag{3.41}$$

as $n,N\to\infty$, uniformly for $z\in\partial U_\delta$.

Step 3: *Matching condition.* In the final step we determine E such that the matching condition (c) of the RH problem for P is satisfied. By (3.26), (3.27), and (3.41) we have for $z\in\partial U_\delta$,

$$P(z)=\begin{cases}E(z)\left[I+\mathcal{O}(1/n^{1/3})\right]e^{n\varphi_{t,+}(0)\sigma_3}e^{\frac12\pi i\alpha\sigma_3}z^{-\alpha\sigma_3}, & \text{if } \operatorname{Im} z>0,\\ E(z)\left[I+\mathcal{O}(1/n^{1/3})\right]e^{n\varphi_{t,+}(0)\sigma_3}e^{\frac12\pi i\alpha\sigma_3}\begin{pmatrix}0&-1\\1&0\end{pmatrix}z^{-\alpha\sigma_3}, & \text{if } \operatorname{Im} z<0,\end{cases}$$

as $n,N\to\infty$. This has to match the outside parametrix $P^{(\infty)}$, so that we are led to the following definition for the prefactor $E(z)$, for $z\in U_\delta$:

$$E(z)=\begin{cases}P^{(\infty)}(z)z^{\alpha\sigma_3}e^{-\frac12\pi i\alpha\sigma_3}e^{-n\varphi_{t,+}(0)\sigma_3}, & \text{if } \operatorname{Im} z>0,\\ [1ex]P^{(\infty)}(z)z^{\alpha\sigma_3}\begin{pmatrix}0&1\\-1&0\end{pmatrix}e^{-\frac12\pi i\alpha\sigma_3}e^{-n\varphi_{t,+}(0)\sigma_3}, & \text{if } \operatorname{Im} z<0.\end{cases} \tag{3.42}$$

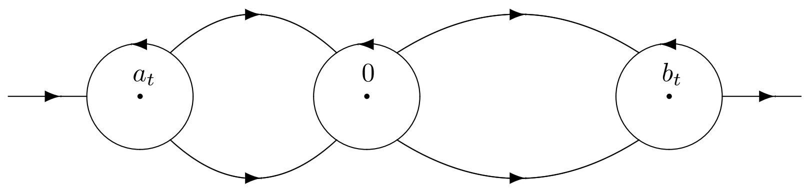

Figure 4: The contour Σ_R after the third and final transformation.

One can check as in [35], [41] that E is invertible and analytic in a full neighborhood of U_δ. In addition we have the matching condition (3.25). This completes the construction of the parametrix near the origin.

3.7. *Third transformation*: $S \mapsto R$. Having the parametrices $P^{(\infty)}$, $P^{(a_t)}$, $P^{(b_t)}$, and P, we now define

$$R(z) = \begin{cases} S(z)P^{-1}(z), & \text{for } z \in U_\delta, \\ S(z)\left(P^{(a_t)}\right)^{-1}(z), & \text{for } z \in U_\delta(a), \\ S(z)\left(P^{(b_t)}\right)^{-1}(z), & \text{for } z \in U_\delta(b), \\ S(z)\left(P^{(\infty)}\right)^{-1}(z), & \text{for } z \in \mathbb{C} \setminus (U_\delta \cup U_\delta(a) \cup U_\delta(b) \cup \Sigma_S). \end{cases} \tag{3.43}$$

Then R has only jumps on the reduced system of contours Σ_R shown in Figure 4, and R satisfies the following RH problem; cf. [9]. The circles around 0, a_t and b_t are oriented counterclockwise.

The RH *problem for* R.

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $R_+(z) = R_-(z)v_R(z)$ for $z \in \Sigma_R$, with

$$v_R = \begin{cases} P^{(\infty)}(P^{(a_t)})^{-1}, & \text{on } \partial U_\delta(a), \\ P^{(\infty)}(P^{(b_t)})^{-1}, & \text{on } \partial U_\delta(b), \\ P^{(\infty)}P^{-1}, & \text{on } \partial U_\delta, \\ P^{(\infty)}v_S(P^{(\infty)})^{-1}, & \text{on the rest of } \Sigma_R. \end{cases} \tag{3.44}$$

(c) $R(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty,$

(d) R remains bounded near the intersection points of Σ_R.

Now we let $n, N \to \infty$ such that $|n^{2/3}(n/N - 1)| \leq M$, so that δ does not depend on n. Then it follows from the construction of the parametrices that

$$v_R = \begin{cases} I + \mathcal{O}(1/n), & \text{on } \partial U_\delta(a) \cup \partial U_\delta(b), \\ I + \mathcal{O}(n^{-1/3}), & \text{on } \partial U_\delta, \\ {[1ex]}I + \mathcal{O}(e^{-\gamma n}), & \text{on the rest of } \Sigma_R, \end{cases} \tag{3.45}$$

where $\gamma > 0$ is some fixed constant. All $\mathcal{O}$-terms hold uniformly on their respective contours.

For large n, the jump matrix of R is close to the identity matrix, both in L^∞ and in the L^2-sense on Σ_R. Then arguments as in [11], [16], [17] (which are based on estimates on Cauchy operators as well as on contour deformations), guarantee that

$$R(z) = I + \mathcal{O}(n^{-1/3}), \qquad \text{uniformly for } z \in \mathbb{C} \setminus \Sigma_R, \tag{3.46}$$

as $n, N \to \infty$ such that $|n^{2/3}(n/N - 1)| \leq M$.

This completes the steepest descent analysis. Following the effect of the transformation on the correlation kernel $K_{n,N}$ and using (3.46) we will prove the main Theorem 1.2. This will be done in the next section. For the proof of Theorem 1.7 we need to expand $v_R(z)$ in (3.45) up to order $n^{-1/3}$, from which it follows that

$$R(z) = I + \frac{R^{(1)}(z)}{n^{1/3}} + \mathcal{O}(n^{-2/3}), \qquad \text{uniformly for } z \in \mathbb{C} \setminus \Sigma_R,$$

with an explicitly computable $R^{(1)}(z)$. The asymptotic behavior of the recurrence coefficients is expressed in terms of $R^{(1)}$ and this leads to the proof of Theorem 1.7 which will be given in Section 5.

Remark 3.1. The steepest descent analysis was done under the assumption that $\operatorname{supp}(\psi_V)$ consists of one interval. In the multi-interval case, the construction of the outside parametrix $P^{(\infty)}$ is more complicated, since it uses Θ-functions as in [16, Lemma 4.3] and the Szegő function for multiple intervals as in [35, §4]. With these modifications the asymptotic analysis can be carried through in the multi-interval case without any additional difficulty.

4. Proof of Theorem 1.2

As in the statement of Theorem 1.2, we assume that $n, N \to \infty$ with $n^{2/3}(t-1) \to L$, where $t = n/N$. Let $M > |L|$ and take n sufficiently large so that $|n^{2/3}(t-1)| \leq M$. Let $\delta > 0$ be such that (3.39) holds. We start by writing the kernel $K_{n,N}$ explicitly in terms of the matrix-valued function Φ_α defined in (2.33). For notational convenience we introduce

$$B(z) = R(z)E(z), \tag{4.1}$$

where E and R are given by (3.42) and (3.43), respectively.

PROPOSITION 4.1. *Let $x, y \in (-\delta, \delta) \setminus \{0\}$. Then*

(4.2)
$$K_{n,N}(x,y) = \frac{1}{2\pi i(x-y)} e^{\frac{1}{2}\pi i\alpha(\operatorname{sgn}(x)+\operatorname{sgn}(y))} \begin{pmatrix} 0 & 1 \end{pmatrix} \Phi_\alpha^{-1}\left(n^{1/3}f(y); n^{2/3}s_t(y)\right)$$
$$\times B^{-1}(y)B(x)\Phi_\alpha\left(n^{1/3}f(x); n^{2/3}s_t(x)\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where Φ_α is as given by (2.33).

Proof. From (3.3), (3.10), and the fact that $NV = nV_t$, the kernel $K_{n,N}$ can be written as

$$K_{n,N}(x,y) = |x|^\alpha e^{\frac{1}{2}n(2g_{t,+}(x)-V_t(x)+\ell_t)}|y|^\alpha e^{\frac{1}{2}n(2g_{t,+}(y)-V_t(y)+\ell_t)}$$
$$\times \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} T_+^{-1}(y)T_+(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using the relation

$$2g_{t,+} - V_t + \ell_t = -2\varphi_{t,+} \qquad \text{on } [a_t, b_t],$$

(see [9], and (3.16) to express T in terms of S), we find for x and y in $(a_t, b_t) \setminus \{0\}$,

(4.3)
$$K_{n,N}(x,y) = \frac{|x|^\alpha e^{-n\varphi_{t,+}(x)}|y|^\alpha e^{-n\varphi_{t,+}(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -|y|^{-2\alpha}e^{2n\varphi_{t,+}(y)} & 1 \end{pmatrix} S_+^{-1}(y)$$
$$\times S_+(x)\begin{pmatrix} 1 & 0 \\ |x|^{-2\alpha}e^{2n\varphi_{t,+}(x)} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 \end{pmatrix} |y|^{-\alpha\sigma_3}e^{n\varphi_{t,+}(y)\sigma_3}S_+^{-1}(y)$$
$$\times S_+(x)|x|^{\alpha\sigma_3}e^{-n\varphi_{t,+}(x)\sigma_3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We further simplify this expression by writing S in terms of R and the parametrix P near the origin. Consider the case that $x \in (0, \delta)$. Then, since $S_+(x) = R(x)P_+(x)$ by (3.43), we have by (3.26),

$$S_+(x) = B(x)\widehat{P}(x)e^{\frac{1}{2}\pi i\alpha\sigma_3}e^{n\varphi_{t,+}(x)\sigma_3}|x|^{-\alpha\sigma_3}, \qquad \text{for } x \in (0,\delta), \tag{4.4}$$

where B is given by (4.1). By (4.4), (3.34), and (2.33) we then find for $x \in (0,\delta)$,

$$S_+(x)|x|^{\alpha\sigma_3}e^{-n\varphi_{t,+}(x)\sigma_3}\begin{pmatrix}1\\1\end{pmatrix} \tag{4.5}$$
$$= B(x)\Phi_\alpha\left(n^{1/3}f(x);n^{2/3}s_t(x)\right)\begin{pmatrix}1 & 0\\ -e^{-\pi i\alpha} & 1\end{pmatrix}e^{\frac{1}{2}\pi i\alpha\sigma_3}\begin{pmatrix}1\\1\end{pmatrix}$$
$$= e^{\frac{1}{2}\pi i\alpha\,\mathrm{sgn}(x)}B(x)\Phi_\alpha\left(n^{1/3}f(x);n^{2/3}s_t(x)\right)\begin{pmatrix}1\\0\end{pmatrix}.$$

A similar calculation shows that (4.5) also holds for $x \in (-\delta,0)$. Similarly, we have

$$\begin{pmatrix}-1 & 1\end{pmatrix}|y|^{-\alpha\sigma_3}e^{n\varphi_{t,+}(y)\sigma_3}S_+^{-1}(y) \tag{4.6}$$
$$= e^{\frac{1}{2}\pi i\alpha\,\mathrm{sgn}(y)}\begin{pmatrix}0 & 1\end{pmatrix}\Phi_\alpha^{-1}\left(n^{1/3}f(y);n^{2/3}s_t(y)\right)B^{-1}(y),$$

for $y \in (-\delta,\delta)\setminus\{0\}$. Inserting (4.5) and (4.6) into (4.3), we arrive at (4.2), which proves the proposition. □

Proof of Theorem 1.2. Let $u,v \in \mathbb{R}\setminus\{0\}$, and put $u_n = u/(cn^{1/3})$ and $v_n = v/(cn^{1/3})$ with c given by (1.19). Note that, by (3.36),

$$\lim_{n\to\infty} n^{1/3}f(u_n) = u, \qquad \lim_{n\to\infty} n^{1/3}f(v_n) = v. \tag{4.7}$$

Furthermore, by (3.38), (1.19), and (1.20),

$$|n^{2/3}s_t(z) - s| \le Kn^{2/3}(t-1)|z| + n^{2/3}o(t-1) + |n^{2/3}(t-1) - L|\pi c^{1/3}w_{S_V}(0)$$

uniformly for z in a neighborhood of 0. Then it easily follows that, since $n^{2/3}(t-1) \to L$,

$$\lim_{n,N\to\infty} n^{2/3}s_t(u_n) = \lim_{n,N\to\infty} n^{2/3}s_t(v_n) = s. \tag{4.8}$$

Now, similarly, as in [35], we use the fact that the entries of B are analytic and uniformly bounded in U_δ, to obtain

$$\lim_{n,N\to\infty} B^{-1}(v_n)B(u_n) = I. \tag{4.9}$$

Inserting (4.7), (4.8), and (4.9) into (4.2), we find that

$$\lim_{n,N\to\infty}\frac{1}{cn^{1/3}}K_{n,N}(u_n,v_n)$$
$$= \frac{1}{2\pi i(u-v)}e^{\frac{1}{2}\pi i\alpha(\mathrm{sgn}(u)+\mathrm{sgn}(v))}\begin{pmatrix}0 & 1\end{pmatrix}\Phi_\alpha^{-1}(v;s)\Phi_\alpha(u;s)\begin{pmatrix}1\\0\end{pmatrix}$$
$$= -e^{\frac{1}{2}\pi i\alpha(\mathrm{sgn}(u)+\mathrm{sgn}(v))}\frac{\Phi_{\alpha,1}(u;s)\Phi_{\alpha,2}(v;s) - \Phi_{\alpha,1}(v;s)\Phi_{\alpha,2}(u;s)}{2\pi i(u-v)}.$$

This completes the proof of Theorem 1.2. □

5. Proof of Theorem 1.7

In this section we determine the asymptotic behavior of the recurrence coefficients $a_{n,N}$ and $b_{n,N}$ as $n,N\to\infty$ such that $|n^{2/3}(n/N-1)|\leq M$ for some $M>0$. As in Theorem 1.7 we assume that $S_V=[a,b]$ is an interval, and that there are no other singular points besides 0. Then it follows that $\operatorname{supp}(\psi_t)$ consists of one interval $[a_t,b_t]$ if t is sufficiently close to 1. In addition we have that the endpoints a_t and b_t are real analytic functions in t, see [32, Th. 1.3], so that

$$a_t=a+\mathcal{O}(n^{-2/3}),\qquad b_t=b+\mathcal{O}(n^{-2/3}), \tag{5.1}$$

since $t=n/N=1+\mathcal{O}(n^{-2/3})$.

We make use of the following result; see for example [11], [17]. Let Y be the unique solution of the RH problem for Y. There exist 2×2 constant (independent of z, but depending on n,N) matrices Y_1,Y_2 such that

$$Y(z)\begin{pmatrix} z^{-n} & 0\\ 0 & z^n\end{pmatrix}=I+\frac{Y_1}{z}+\frac{Y_2}{z^2}+\mathcal{O}(1/z^3),\qquad \text{as } z\to\infty,$$

and

$$a_{n,N}=\sqrt{(Y_1)_{12}(Y_1)_{21}},\qquad b_{n,N}=(Y_1)_{11}+\frac{(Y_2)_{12}}{(Y_1)_{12}}. \tag{5.2}$$

We need to determine the constant matrices Y_1 and Y_2. For large $|z|$ we have by (3.10), (3.16) and (3.43) that

$$Y(z)=e^{-\frac{1}{2}n\ell_t\sigma_3}R(z)P^{(\infty)}(z)e^{ng_t(z)\sigma_3}e^{\frac{1}{2}n\ell_t\sigma_3}. \tag{5.3}$$

So in order to compute Y_1 and Y_2 we need the asymptotic behavior of $P^{(\infty)}(z)$, $e^{ng_t(z)\sigma_3}$ and $R(z)$ as $z\to\infty$.

Asymptotic behavior of $P^{(\infty)}(z)$ as $z\to\infty$. ¿From (3.19) and (3.22) it is straightforward to determine the asymptotic behavior of the scalar functions $D(z)$ and $\beta(z)$ as $z\to\infty$. Indeed, as $z\to\infty$,

$$\begin{pmatrix} \frac{\beta(z)+\beta(z)^{-1}}{2} & \frac{\beta(z)-\beta(z)^{-1}}{2i}\\ \frac{\beta(z)-\beta(z)^{-1}}{-2i} & \frac{\beta(z)+\beta(z)^{-1}}{2}\end{pmatrix}$$

$$=I-\frac{1}{4}(b_t-a_t)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{z}+\frac{i}{8}(b_t^2-a_t^2)\begin{pmatrix} * & 1\\ -1 & *\end{pmatrix}\frac{1}{z^2}+\mathcal{O}(1/z^3),$$

and

$$D(z)^{-\sigma_3}=\left[I-\frac{\alpha}{2}(b_t+a_t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{z}+\begin{pmatrix} * & 0\\ 0 & *\end{pmatrix}\frac{1}{z^2}+\mathcal{O}(1/z^3)\right]D_\infty^{-\sigma_3},$$

where $*$ denotes an unspecified unimportant entry. Inserting these equations into (3.21) and using (5.1) gives us the asymptotic behavior of $P^{(\infty)}$ at infinity,

$$P^{(\infty)}(z) = I + \frac{P_1^{(\infty)}}{z} + \frac{P_2^{(\infty)}}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \tag{5.4}$$

with

$$\begin{aligned} P_1^{(\infty)} &= D_\infty^{\sigma_3} \begin{pmatrix} -\frac{\alpha}{2}(b_t + a_t) & \frac{i}{4}(b_t - a_t) \\ -\frac{i}{4}(b_t - a_t) & \frac{\alpha}{2}(b_t + a_t) \end{pmatrix} D_\infty^{-\sigma_3} \\ &= D_\infty^{\sigma_3} \begin{pmatrix} -\frac{\alpha}{2}(b + a) & \frac{i}{4}(b - a) \\ -\frac{i}{4}(b - a) & \frac{\alpha}{2}(b + a) \end{pmatrix} D_\infty^{-\sigma_3} + \mathcal{O}(n^{-2/3}), \end{aligned} \tag{5.5}$$

and

(5.6)
$$\begin{aligned} P_2^{(\infty)} &= D_\infty^{\sigma_3} \begin{pmatrix} * & \frac{i}{8}(\alpha + 1)(b_t^2 - a_t^2) \\ \frac{i}{8}(\alpha - 1)(b_t^2 - a_t^2) & * \end{pmatrix} D_\infty^{-\sigma_3} \\ &= D_\infty^{\sigma_3} \begin{pmatrix} * & \frac{i}{8}(\alpha + 1)(b^2 - a^2) \\ \frac{i}{8}(\alpha - 1)(b^2 - a^2) & * \end{pmatrix} D_\infty^{-\sigma_3} + \mathcal{O}(n^{-2/3}). \end{aligned}$$

Asymptotic behavior of $e^{ng_t(z)\sigma_3}$ *as* $z \to \infty$. By (3.9),

$$e^{ng_t(z)\sigma_3} \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{G_1}{z} + \frac{G_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \tag{5.7}$$

with

$$G_1 = -n \int_{a_t}^{b_t} y\psi_t(y)dy \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}. \tag{5.8}$$

Asymptotic behavior of $R(z)$ *as* $z \to \infty$. The computation of R_1 and R_2 is more involved. For $z \in \partial U_\delta \cap \mathbb{C}_+$, we have by (3.44), (3.26), (3.34), and (3.42),

$$\begin{aligned} v_R(z) &= P^{(\infty)}(z) P^{-1}(z) \\ &= P^{(\infty)}(z) z^{\alpha\sigma_3} e^{-\frac{1}{2}\pi i \alpha \sigma_3} e^{-n\varphi_t(z)\sigma_3} \Psi_\alpha^{-1}(n^{1/3} f(z); n^{2/3} s_t(z)) \\ &\quad \times e^{n\varphi_{t,+}(0)\sigma_3} e^{\frac{1}{2}\pi i \alpha \sigma_3} z^{-\alpha\sigma_3} (P^{(\infty)})^{-1}(z). \end{aligned} \tag{5.9}$$

Using (2.28) and (3.40), we then find

$$v_R(z) = I + \frac{\Delta^{(1)}(z)}{n^{1/3}} + \mathcal{O}(n^{-2/3}), \tag{5.10}$$

where

$$\Delta^{(1)}(z) = -\frac{1}{2if(z)} P^{(\infty)}(z) z^{\alpha\sigma_3} e^{-\frac{1}{2}\pi i\alpha\sigma_3} e^{-n\varphi_{t,+}(0)\sigma_3} \tag{5.11}$$
$$\times \begin{pmatrix} u_\alpha(n^{2/3}s_t(z)) & q_\alpha(n^{2/3}s_t(z)) \\ -q_\alpha(n^{2/3}s_t(z)) & -u_\alpha(n^{2/3}s_t(z)) \end{pmatrix} e^{n\varphi_{t,+}(0)\sigma_3} e^{\frac{1}{2}\pi i\alpha\sigma_3} z^{-\alpha\sigma_3} (P^{(\infty)})^{-1}(z),$$

for $z \in \partial U_\delta \cap \mathbb{C}_+$. A similar calculation leads to an analogous formula for $z \in \partial U_\delta \cap \mathbb{C}_-$, which together with (5.11) shows that $\Delta^{(1)}$ has an extension to an analytic function in a punctured neighborhood of 0 with a simple pole at 0.

To calculate the residue at 0, we use (3.19) together with the fact that $\phi_+(x) = \exp(i \arccos x)$ for $x \in [-1, 1]$ to find

$$\lim_{z\to 0+i0} \frac{D(z)}{z^\alpha} = \exp\left(-i\alpha \arccos\left(-\frac{b_t + a_t}{b_t - a_t}\right)\right),$$

so that by (3.20)

$$\lim_{z\to 0+i0} P^{(\infty)}(z) z^{\alpha\sigma_3} e^{-\frac{1}{2}\pi i\alpha\sigma_3} = D_\infty^{\sigma_3} \widehat{P}_+^{(\infty)}(0) e^{i\alpha\theta_t\sigma_3}, \qquad \text{with } \theta_t = \arcsin \tfrac{b_t+a_t}{b_t-a_t}. \tag{5.12}$$

Also note that by (3.7), (3.11), and (1.29),

$$-\varphi_{t,+}(0) = \pi i \int_0^{b_t} \psi_t(x) dx = \pi i \omega_t. \tag{5.13}$$

Now use (3.36), (1.19), (5.12), and (5.13) in (5.11) to find

$$\operatorname{Res}(\Delta^{(1)}; 0) = -\frac{1}{2ic} D_\infty^{\sigma_3} \widehat{P}_+^{(\infty)}(0) e^{i(\pi n\omega_t + \alpha\theta_t)\sigma_3} \tag{5.14}$$
$$\times \begin{pmatrix} u_\alpha(n^{2/3}s_t(0)) & q_\alpha(n^{2/3}s_t(0)) \\ [1ex] - q_\alpha(n^{2/3}s_t(0)) & -u_\alpha(n^{2/3}s_t(0)) \end{pmatrix} e^{-i(\pi n\omega_t + \alpha\theta)\sigma_3} (\widehat{P}_+^{(\infty)})^{-1}(0) D_\infty^{-\sigma_3}.$$

Combining (3.37), (3.36), and (3.7) we see that $n^{2/3}s_t(0) = s_{t,n}$ as defined in (1.27). From (3.20), (3.21), and (3.22) it follows that

$$\widehat{P}_+^{(\infty)}(0) = \begin{pmatrix} \frac{\beta_+(0)+\beta_+(0)^{-1}}{2} & \frac{\beta_+(0)-\beta_+(0)^{-1}}{2i} \\ \frac{\beta_+(0)-\beta_+(0)^{-1}}{2i} & \frac{\beta_+(0)+\beta_+(0)^{-1}}{2} \end{pmatrix},$$

where $\beta_+(0) = e^{i\pi/4} (-b_t/a_t)^{1/4}$.

We insert this into (5.14) and after some straightforward calculations we find

$$-\operatorname{Res}(\Delta^{(1)}; 0) = D_\infty^{\sigma_3} (r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3) D_\infty^{-\sigma_3}, \tag{5.15}$$

where the Pauli matrices are $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and

(5.16)
$$\begin{aligned} r_1 &= -\frac{1}{2ic}\left(u_\alpha(s_{t,n})\frac{b_t-a_t}{2\sqrt{-a_tb_t}} + q_\alpha(s_{t,n})\frac{b_t+a_t}{2\sqrt{-a_tb_t}}\sin(2\pi n\omega_t + 2\alpha\theta_t)\right) \\ &= -\frac{1}{2ic}\left(u_\alpha(s_{t,n})\frac{b-a}{2\sqrt{-ab}} + q_\alpha(s_{t,n})\frac{b+a}{2\sqrt{-ab}}\sin(2\pi n\omega_t + 2\alpha\theta)\right) + \mathcal{O}(n^{-2/3}), \end{aligned}$$

$$\begin{aligned} r_2 &= \frac{q_\alpha(s_{t,n})}{2c}\cos(2\pi n\omega_t + 2\alpha\theta_t) \\ &= \frac{q_\alpha(s_{t,n})}{2c}\cos(2\pi n\omega_t + 2\alpha\theta) + \mathcal{O}(n^{-2/3}), \end{aligned} \tag{5.17}$$

(5.18)
$$\begin{aligned} r_3 &= \frac{1}{2c}\left(q_\alpha(s_{t,n})\frac{b_t-a_t}{2\sqrt{-a_tb_t}}\sin(2\pi n\omega_t + 2\alpha\theta_t) + u_\alpha(s_{t,n})\frac{b_t+a_t}{2\sqrt{-a_tb_t}}\right) \\ &= \frac{1}{2c}\left(q_\alpha(s_{t,n})\frac{b-a}{2\sqrt{-ab}}\sin(2\pi n\omega_t + 2\alpha\theta) + u_\alpha(s_{t,n})\frac{b+a}{2\sqrt{-ab}}\right) + \mathcal{O}(n^{-2/3}), \end{aligned}$$

where we used (5.1).

From (5.10),

$$R(z) = I + \frac{R^{(1)}(z)}{n^{1/3}} + \mathcal{O}(n^{-2/3}), \tag{5.19}$$

where $R^{(1)}_+ = R^{(1)}_- + \Delta^{(1)}$ on ∂U_δ and $R^{(1)}(z) \to 0$ as $z \to \infty$. Since $\Delta^{(1)}$ is analytic with a simple pole at $z = 0$, we can find explicitly

$$R^{(1)}(z) = \begin{cases} -\frac{1}{z}\mathrm{Res}(\Delta^{(1)};0) + \Delta^{(1)}(z), & \text{for } z \in U_\delta, \\ -\frac{1}{z}\mathrm{Res}(\Delta^{(1)};0), & \text{for } z \in \mathbb{C}\setminus \overline{U}_\delta. \end{cases} \tag{5.20}$$

As in [17] the matrix-valued function R has the following asymptotic behavior at infinity,

$$R(z) = I + \frac{R_1}{z} + \frac{R_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty. \tag{5.21}$$

The compatibility with (5.19) and (5.20) yields that

$$R_1 = -\mathrm{Res}(\Delta^{(1)};0)n^{-1/3} + \mathcal{O}(n^{-2/3}), \qquad R_2 = \mathcal{O}(n^{-2/3}). \tag{5.22}$$

Now, we are ready to determine the asymptotics of the recurrence coefficients.

Proof of Theorem 1.7. Note that by (5.3), (5.4), (5.7) and (5.21),

$$Y_1 = e^{-\frac{1}{2}n\ell_t\sigma_3}\left[P_1^{(\infty)} + G_1 + R_1\right]e^{\frac{1}{2}n\ell_t\sigma_3} \tag{5.23}$$

and

$$Y_2 = e^{-\frac{1}{2}n\ell_t\sigma_3}\left[P_2^{(\infty)} + G_2 + R_2 + R_1P_1^{(\infty)} + \left(P_1^{(\infty)} + R_1\right)G_1\right]e^{\frac{1}{2}n\ell_t\sigma_3}. \tag{5.24}$$

We start with the recurrence coefficient $a_{n,N}$. Inserting (5.23) into (5.2) and using (5.5) and the facts that $(G_1)_{12} = (G_1)_{21} = 0$ (by (5.8)), and $(R_1)_{12}(R_1)_{21} = \mathcal{O}(n^{-2/3})$ (by (5.22)), we obtain

$$\begin{aligned}
a_{n,N} &= \left[(P_1^{(\infty)})_{12}(P_1^{(\infty)})_{21} + (P_1^{(\infty)})_{12}(R_1)_{21} + (P_1^{(\infty)})_{21}(R_1)_{12} + \mathcal{O}(n^{-2/3})\right]^{1/2} \\
&= \left[\left(\frac{b-a}{4}\right)^2 + i\frac{b-a}{4}\left(D_\infty^2(R_1)_{21} - D_\infty^{-2}(R_1)_{12}\right) + \mathcal{O}(n^{-2/3})\right]^{1/2} \\
&= \frac{b-a}{4} + \frac{i}{2}\left(D_\infty^2(R_1)_{21} - D_\infty^{-2}(R_1)_{12}\right) + \mathcal{O}(n^{-2/3}).
\end{aligned}$$

From (5.22) and (5.15) we then arrive at,

$$\begin{aligned}
a_{n,N} &= \frac{b-a}{4} - r_2 n^{-1/3} + \mathcal{O}(n^{-2/3}) \\
&= \frac{b-a}{4} - \frac{q_\alpha(s_{t,n})\cos(2\pi n\omega_t + 2\alpha\theta)}{2c}n^{-1/3} + \mathcal{O}(n^{-2/3}).
\end{aligned} \tag{5.25}$$

Next, we consider the recurrence coefficient $b_{n,N}$. Inserting (5.23) and (5.24) into (5.2), and using the facts that $(G_1)_{11} + (G_1)_{22} = 0$ (by (5.8)), and $(R_2)_{12} = \mathcal{O}(n^{-2/3})$ (by (5.22)) we obtain

$$\begin{aligned}
b_{n,N} &= (P_1^{(\infty)})_{11} + (R_1)_{11} + \frac{(P_2^{(\infty)})_{12} + (R_1P_1^{(\infty)})_{12} + \mathcal{O}(n^{2/3})}{(P_1^{(\infty)} + R_1)_{12}} \\
&= (P_1^{(\infty)})_{11} + (R_1)_{11} + \left(1 - \frac{(R_1)_{12}}{(P_1^{(\infty)})_{12}} + \mathcal{O}(n^{-2/3})\right) \\
&\quad \times \left(\frac{(P_2^{(\infty)})_{12}}{(P_1^{(\infty)})_{12}} + (R_1)_{11} + \frac{(P_1^{(\infty)})_{22}}{(P_1^{(\infty)})_{12}}(R_1)_{12} + \mathcal{O}(n^{-2/3})\right).
\end{aligned}$$

From equations (5.5), (5.6), (5.22), and (5.15), we then arrive at

$$\begin{aligned}
b_{n,N} &= \frac{b+a}{2} + 2(R_1)_{11} + 2i\frac{b+a}{b-a}D_\infty^{-2}(R_1)_{12} + \mathcal{O}(n^{-2/3}) \\
&= \frac{b+a}{2} + 2\left(r_3 + i\frac{b+a}{b-a}(r_1 - ir_2)\right) + \mathcal{O}(n^{-2/3}).
\end{aligned} \tag{5.26}$$

Using (5.16), (5.17), and (5.18) in (5.26) we will see that the terms containing u_α cancel against each other. What remains are the terms containing q_α:

$$b_{n,N} = \frac{b+a}{2} + \frac{q_\alpha(s_{t,n})}{c}\left[\frac{b+a}{b-a}\cos(2\pi n\omega_t + 2\alpha\theta) + \frac{2\sqrt{-ab}}{b-a}\sin(2\pi n\omega_t + 2\alpha\theta)\right]n^{-1/3} + \mathcal{O}(n^{-2/3}). \tag{5.27}$$

Since $\frac{b+a}{b-a} = \sin\theta$ and $\frac{2\sqrt{-ab}}{b-a} = \cos\theta$, we can combine the two terms within square brackets and the result is

$$b_{n,N} = \frac{b+a}{2} + \frac{q_\alpha(s_{t,n})\sin(2\pi n\omega_t + (2\alpha+1)\theta)}{c}n^{-1/3} + \mathcal{O}(n^{-2/3}). \tag{5.28}$$

Theorem 1.24 is proven by (5.25) and (5.28). □

Acknowledgments. We are grateful to Pavel Bleher and Alexander Its for very useful and stimulating discussions.

The authors are supported by FWO research projects G.0176.02 and G.0455.04. The first author is Postdoctoral Fellow of the Fund for Scientific Research - Flanders (Belgium). The second author is also supported by K.U.Leuven research grant OT/04/21, by INTAS Research Network NeCCA 03-51-6637, by NATO Collaborative Linkage Grant PST.CLG.979738, by grant BFM2001-3878-C02-02 of the Ministry of Science and Technology of Spain and by the European Science Foundation Program Methods of Integrable Systems, Geometry, Applied Mathematics (MISGAM) and the European Network in Geometry, Mathematical Physics and Applications (ENIGMA). The third author was Postdoctoral Fellow of the Fund for Scientific Research - Flanders (Belgium). He is grateful to the Department of Mathematics of the Ruhr Universität Bochum where he has spent the academic year 2004–2005, for hospitality.

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(Received August 30, 2005)