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Invertibility of random matrices: norm of the inverse

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Abstract

Let A be an $n \times n$ matrix, whose entries are independent copies of a centered random variable satisfying the subgaussian tail estimate. We prove that the operator norm of A^{-1} does not exceed $Cn^{3/2}$ with probability close to 1.

1. Introduction

Let A be an $n \times n$ matrix, whose entries are independent, identically distributed random variables. The spectral properties of such matrices, in particular invertibility, have been extensively studied (see, e.g. [M] and the survey [DS]). While A is almost surely invertible whenever its entries are absolutely continuous, the case of discrete entries is highly nontrivial. Even in the case, when the entries of A are independent random variables taking values ± 1 with probability $1/2$, the precise order of probability that A is singular is unknown. Komlós [K1], [K2] proved that this probability is $o(1)$ as $n \to \infty$. This result was improved by Kahn, Komlós and Szemerédi [KKS], who showed that this probability is bounded above by θ^n for some absolute constant $\theta < 1$. The value of θ has been recently improved in a series of papers by Tao and Vu [TV1], [TV2] to $\theta = 3/4 + o(1)$ (the conjectured value is $\theta = 1/2 + o(1)$).

However, these papers do not address the quantitative characterization of invertibility, namely the norm of the inverse matrix, considered as an operator from $\mathbb{R}^n$ to $\mathbb{R}^n$. Random matrices are one of the standard tools in geometric functional analysis. They are used, in particular, to estimate the Banach-Mazur distance between finite-dimensional Banach spaces and to construct sections of convex bodies possessing certain properties. In all these questions condition number or the distortion $\|A\| \cdot \left\|A^{-1}\right\|$ plays the crucial role. Since the norm of A is usually highly concentrated, the distortion is determined by the norm of A^{-1}. The estimate of the norm of A^{-1} is known only in the case

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when A is a matrix with independent $N(0,1)$ Gaussian entries. In this case Edelman [Ed] and Szarek [Sz2] proved that $\left\|A^{-1}\right\| \leq c\sqrt{n}$ with probability close to 1 (see also [Sz1] where the spectral properties of a Gaussian matrix are applied to an important question from geometry of Banach spaces). For other random matrices, including a random ± 1 matrix, even a polynomial bound was unknown. Proving such a polynomial estimate is the main aim of this paper.

More results are known about rectangular random matrices. Let Γ be an $N \times n$ matrix, whose entries are independent random variables. If $N > n$, then such a matrix can be considered as a linear operator $\Gamma : \mathbb{R}^n \to Y$, where $Y = \Gamma\mathbb{R}^n$. If we consider a family Γ_n of such matrices with $n/N \to \alpha$ for a fixed constant $\alpha > 1$, then the norms of $(N^{-1/2} \cdot \Gamma_n|_Y)^{-1}$ converge a.s. to $(1-\sqrt{\alpha})^{-1}$, provided that the fourth moments of the entries are uniformly bounded [BY]. The random matrices for which $n/N = 1 - o(1)$ are considered in [LPRT]. If the entries of such a matrix satisfy certain moment conditions and $n/N > 1 - c/\log n$, then $\left\|(\Gamma|_Y)^{-1}\right\| \leq C(n/N) \cdot n^{-1/2}$ with probability exponentially close to 1.

The proof of the last result is based on the ε-net argument. To describe it we have to introduce some notation. For $p \geq 1$ let B_p^n denote the unit ball of the Banach space ℓ_p^n. Let $E \subset \mathbb{R}^n$ and let $B \subset \mathbb{R}^n$ be a convex symmetric body. Let $\varepsilon > 0$. We say that a set $F \subset \mathbb{R}^n$ is an ε-net for E with respect to B if

$$E \subset \bigcup_{x \in F} (x + \varepsilon B).$$

The smallest cardinality of an ε-net will be denoted by $N(E, B, \varepsilon)$. For a point $x \in \mathbb{R}^n$, $\|x\|$ stands for the standard Euclidean norm, and for a linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$, $\|T\|$ denotes the operator norm of $T : \ell_2^n \to \ell_2^m$. Let $E \subset S^{n-1}$ be a set such that for any fixed $x \in E$ there is a good bound for the probability that $\|\Gamma x\|$ is small. We shall call such a bound the small ball probability estimate. If $N(E, B_2^n, \varepsilon)$ is small, this bound implies that with high probability $\|\Gamma x\|$ is large for all x from an ε-net for E. Then the approximation is used to derive that in this case $\|\Gamma x\|$ is large for all $x \in E$. Finally, the sphere S^{n-1} is partitioned in two sets for which the above method works. This argument is applicable because the small ball probability is controlled by a function of N, while the size of an ε-net depends on $n < N$.

The case of a square random matrix is more delicate. Indeed, in this case the small ball probability estimate is too weak to produce a nontrivial estimate for the probability that $\|\Gamma x\|$ is large for all points of an ε-net. To overcome this difficulty, we use the ε-net argument for one part of the sphere and work with conditional probability on the other part. Also, we will need more elaborate small ball probability estimates than those employed in [LPRT]. To obtain such estimates we use the method of Halász, which lies in the foundation of the arguments of [KKS], [TV1], [TV2].

Let $\mathbb{P}(\Omega)$ denote the probability of the event Ω, and let $\mathbb{E}\xi$ denote the expectation of the random variable ξ. A random variable β is called subgaussian if for any $t > 0$

$$\mathbb{P}(|\beta| > t) \le C\exp(-ct^2). \tag{1.1}$$

The class of subgaussian variables includes many natural types of random variables, in particular, normal and bounded ones. It is well-known that the tail decay condition (1.1) is equivalent to the moment condition $\left(\mathbb{E}|\beta|^p\right)^{1/p} \le C'\sqrt{p}$ for all $p \ge 1$.

The letters c, C, C' etc. denote unimportant absolute constants, whose value may change from line to line. Besides these constants, the paper contains many absolute constants which are used throughout the proof. For the reader's convenience we use a standard notation for such important absolute constants. Namely, if a constant appears in the formulation of Lemma or Theorem x.y, we denote it $C_{\text{x.y}}$ or $c_{\text{x.y}}$.

The main result of this paper is the polynomial bound for the norm of A^{-1}. We shall formulate it in terms of the smallest singular number of A:

$$s_n(A) = \min_{x \in S^{n-1}} \|Ax\|.$$

Note that if the matrix A is invertible, then $\left\|A^{-1}\right\| = 1/s_n(A)$.

Theorem 1.1. *Let β be a centered subgaussian random variable of variance* 1. *Let A be an $n \times n$ matrix whose entries are independent copies of β. Then for any $\varepsilon > c_{1.1}/\sqrt{n}$*

$$\mathbb{P}\left(\exists\, x \in S^{n-1} \mid \|Ax\| < \frac{\varepsilon}{C_{1.1} \cdot n^{3/2}}\right) < \varepsilon$$

if n is large enough.

More precisely, we prove that the probability above is bounded by $\varepsilon/2 + 4\exp(-cn)$ for all $n \in \mathbb{N}$.

The inequality in Theorem 1.1 means that $\left\|A^{-1}\right\| \le C_{1.1} \cdot n^{3/2}/\varepsilon$ with probability greater than $1 - \varepsilon$. Equivalently, the smallest singular number of A is at least $\varepsilon/(C_{1.1} \cdot n^{3/2})$.

An important feature of Theorem 1.1 is its universality. Namely, the probability estimate holds for all subgaussian random variables, regardless of their nature. Moreover, the only place where we use the assumption that β is subgaussian is Lemma 3.3 below.

2. Overview of the proof

The strategy of the proof of Theorem 1.1 is based on the step-by-step exclusion of the points with singular small ball probability behavior. Since all coordinates of the vector Ax are identically distributed, it will be enough to consider the distribution of the first coordinate, which we shall denote by Y. If the entries of A have absolutely continuous distribution with a bounded density function, then for any $t > 0, \mathbb{P}(|Y| < t) \le Ct$. However, for a general random matrix, in particular, for a random ± 1 matrix, this estimate holds only for $t > t(x)$, where the cut-off level $t(x)$ is determined by the distribution of the coordinates of x.

We shall divide the sphere S^{n-1} into several parts according to the values of $t(x)$. For each part, except for the last one, we use the small ball probability estimate combined with the ε-net argument. However, the balance between the bound for the probability and the size of the net will be different at each case. More regular distribution of the coordinates of the vector x will imply bounds for the small ball probability $\mathbb{P}(\|Ax\| < \rho)$ for smaller values of ρ. To apply this result to a set of vectors, we shall need a finer ε-net.

Proceeding this way, we establish a uniform lower bound for $\|Ax\|$ for the set of vectors x whose coordinates are distributed irregularly. This leaves the set of vectors $x \in S^{n-1}$ with very regularly distributed coordinates. This set contains most of the points of the sphere, so the ε-net argument cannot be applied here. However, for such vectors x the value of $t(x)$ will be exceptionally small, so that their small ball probability behavior will be close to that of an absolutely continuous random variable. This, together with the conditional probability argument, will allow us to conclude the proof.

Now we describe the exclusion procedure in more detail. First, we consider the *peaked* vectors, namely the vectors x, for which a substantial part of the norm is concentrated in a few coordinates. For such vectors $t(x)$ is a constant. Translating this into the small ball probability estimate for the vector Ax, we obtain $\mathbb{P}(\|Ax\| < C\sqrt{n}) \le c^n$ for some $c < 1$. Since any peaked vector is close to some coordinate subspace of a small dimension, we can construct a small ε-net for the set of peaked vectors. Applying the union bound we show that $\|Ax\| > C\sqrt{n}$ for any vector x from the ε-net, and extend it by approximation to all peaked vectors.

For the set of *spread* vectors, which is the complement of the set of peaked vectors, we can lower the cut-off level $t(x)$ to $c/\sqrt{n}$. This in turn implies the small ball probability estimate $\mathbb{P}(\|Ax\| < C) \le (c/\sqrt{n})^n$. This better estimate allows us to construct a finer ε-net for the set of the spread vectors. However, an ε-net for the whole set of the spread vectors will be too large to guarantee that the inequality $\|Ax\| \ge C$ holds for all of its vectors with high probability.

Therefore, we shall further divide the set of the spread vectors into two subsets and apply the ε-net argument to the smaller one.

To this end we consider only the coordinates of the vector x whose absolute values lie in the interval $[r/\sqrt{n}, R/\sqrt{n}]$ for some absolute constants $0 < r < 1 < R$. We divide this interval into subintervals of the length Δ. If a substantial part of the coordinates of x lie in a few such intervals, we call x a vector of a Δ-*singular profile*. Otherwise, x is called a vector of a Δ-*regular profile*. At the first step we set $\Delta = c/n$. For such Δ the set of vectors of a Δ-singular profile admits an ε net of cardinality smaller than $(c\sqrt{n})^n$. Therefore, combining the small ball probability estimate for the spread vectors with the ε-net argument, we prove that the estimate $\|Ax\| \geq C$ holds for all vectors of a Δ-singular profile with probability exponentially close to 1.

Now it remains to treat the vectors of a Δ-regular profile. For such vectors we prove a new small ball probability estimate. Namely, we show that for any such vector x, the cut-off level $t(x) = \Delta$, which implies that $\mathbb{P}(\|Ax\| < C\Delta\sqrt{n}) \leq (c\Delta)^n$. The proof of this result is much more involved than the previous small ball probability estimates. It is based on the method of Halász which uses the estimates of the characteristic functions of random variables. To take advantage of this estimate we split the set of vectors of a c/n-regular profile into the set of vectors of Δ-singular and Δ-regular profiles for $\Delta = \varepsilon/n$. For the first set we repeat the ε-net argument with a different ε. This finally leads us to the vectors of ε/n-regular profile.

For such vectors we employ a different argument. Assume that $\|A^{-1}\|$ is large. This means that the rows $a_1, \dots, a_n$ of A are almost linearly dependent. In other words, one of the rows, say the last, is close to the linear combination of the other. Fixing on the first $n-1$ rows, we choose a vector x of an ε/n-regular profile for which $\|A'x\|$ is small, where A' is the matrix consisting of the first $n-1$ rows of A. Such a vector depends only on $a_1, \dots, a_{n-1}$. The almost linear dependence implies that the random variable $Z = \sum_{j=1}^{n} a_{n,j} x_j$ belongs to a small interval $I \subset \mathbb{R}$, which is defined by $a_1, \dots, a_{n-1}$. Since x has an ε/n-regular profile, the small ball probability estimate implies that the probability that $Z \in I$, and therefore the probability that $\|A^{-1}\|$ is large, will be small.

3. Preliminary results

Assume that l balls are randomly placed in k urns. Let $V \in \{1, \dots, k\}^l$ be a random vector whose i-th coordinate is the number of balls contained in the i-th urn. The distribution of V, called random allocation, has been extensively studied, and many deep results are available (see [KSC]). We need only a simple combinatorial lemma.

Lemma 3.1. *Let $k \le l$ and let $X(1),\dots,X(l)$ be i.i.d. random variables uniformly distributed on the set $\{1,\dots,k\}$. Let $\eta < 1/2$. Then with probability greater than $1-\eta^l$ there exists a set $J \subset \{1,\dots,l\}$ containing at least $l/2$ elements such that*

$$\sum_{i=1}^{k} |\{j \in J \mid X(j)=i\}|^2 \le C(\eta)\frac{l^2}{k}. \tag{3.1}$$

Remark 3.2. The proof yields $C(\eta)=\eta^{-16}$. This estimate is by no means exact.

Proof. Let $X=(X(1),\dots,X(l))$. For $i=1,\dots,k$ denote

$$P_i(X) = |\{j \mid X(j)=i\}|.$$

Let $2<\alpha<k/2$ be a number to be chosen later. Denote

$$I(X)=\{i \mid P_i(X) \ge \alpha\,\frac{l}{k}\}.$$

For any X we have $\sum_{i=1}^k P_i(X)=l$, so that $|I(X)|\le k/\alpha$. Set

$$J(X)=\{j \mid X(j)\in I(X)\}.$$

Assume that $|J(X)|\le l/2$. Then for the set $J'(X)=\{1,\dots,l\}\setminus J(X)$ we have $|J'(X)|\ge l/2$ and

$$\sum_{i=1}^{k} |\{j \in J'(X) \mid X(j)=i\}|^2 = \sum_{i\notin I(X)} P_i^2(X) \le k\cdot\left(\alpha\,\frac{l}{k}\right)^2 = \frac{\alpha^2 l^2}{k}.$$

Now we have to estimate the probability that $|J(X)|\ge l/2$. To this end we estimate the probability that $J(X)=J$ and $I(X)=I$ for fixed subsets $J\subset\{1,\dots,l\}$ and $I\subset\{1,\dots,k\}$ and sum over all relevant choices of J and I. We have

$$\begin{aligned}
\mathbb{P}\left(|J(X)|\ge l/2\right) &\le \sum_{|J|\ge l/2}\ \sum_{|I|\le k/\alpha} \mathbb{P}\left(J(X)=J,\ I(X)=I\right)\\
&\le \sum_{|J|\ge l/2}\ \sum_{|I|\le k/\alpha} \mathbb{P}\left(X(j)\in I \text{ for all } j\in J\right)\\
&\le 2^l (k/\alpha)\cdot\binom{k}{k/\alpha}\cdot(1/\alpha)^{l/2}\\
&\le k\cdot(e\alpha)^{k/\alpha}\cdot(4/\alpha)^{l/2},
\end{aligned}$$

since the random variables $X(1),\dots,X(l)$ are independent. If $k\le l$ and $\alpha>100$, the last expression does not exceed $\alpha^{-l/8}$. To complete the proof, set $\alpha=\eta^{-8}$ and $C(\eta)=\alpha^2$. If $\eta>(2/k)^{1/8}$, then the assumption $\alpha<k/2$ is satisfied. Otherwise, we can set $C(\eta)>(k/2)^2$, for which the inequality (3.1) becomes trivial. $\square$

The following result is a standard large deviation estimate (see e.g. [DS] or [LPRT], where a more general result is proved).

LEMMA 3.3. *Let $A = (a_{i,j})$ be an $n \times n$ matrix whose entries are i.i.d. centered subgaussian random variables of variance 1. Then*

$$\mathbb{P}\left(\|A : B_2^n \to B_2^n\| \geq C_{3.3}\sqrt{n}\right) \leq \exp(-n).$$

We will also need the volumetric estimate of the covering numbers $N(K, D, t)$ (see e.g. [P]). Denote by $|K|$ the volume of $K \subset \mathbb{R}^n$.

LEMMA 3.4. *Let $t > 0$ and let $K, D \subset \mathbb{R}^n$ be convex symmetric bodies. If $tD \subset K$, then*

$$N(K, D, t) \leq \frac{3^n |K|}{|tD|}.$$

4. Halász type lemma

Let $\xi_1, \dots, \xi_n$ be independent centered random variables. To obtain the small ball probability estimates below, we must bound the probability that $\sum_{j=1}^n \xi_j$ is concentrated in a small interval. One standard method of obtaining such bounds is based on the Berry-Esséen theorem (see, e.g. [LPRT]). However, this method has certain limitations. In particular, if $\xi_j = t_j \varepsilon_j$, where $t_j \in [1, 2]$ and ε_j are ± 1 random variables, then the Berry-Esséen theorem does not "feel" the distribution of the coefficients t_j, and thus does not yield bounds better than $c/\sqrt{n}$ for the small ball probability. To obtain better bounds we use the approach developed by Halász [Ha1], [Ha2].

LEMMA 4.1. *Let $c > 0$, $0 < \Delta < a/(2\pi)$ and let $\xi_1, \dots, \xi_n$ be independent random variables such that $\mathbb{E}\xi_i = 0$, $\mathbb{P}(\xi_i > a) \geq c$ and $\mathbb{P}(\xi_i < -a) \geq c$. For $y \in \mathbb{R}$ set*

$$S_\Delta(y) = \sum_{j=1}^n \mathbb{P}\left(\xi_j - \xi_j' \in [y - \pi\Delta, y + \pi\Delta]\right),$$

where ξ_j' is an independent copy of ξ_j. Then for any $v \in \mathbb{R}$,

$$\mathbb{P}\left(\left|\sum_{j=1}^n \xi_j - v\right| < \Delta\right) \leq \frac{C}{n^{5/2}\Delta} \int_{3a/2}^{\infty} S_\Delta^2(y)\, dy + ce^{-c'n}.$$

Proof. For $t \in \mathbb{R}$ define

$$\varphi_k(t) = \mathbb{E}\exp(i\xi_k t)$$

and set

$$\varphi(t) = \mathbb{E} \exp\left(it \sum_{k=1}^{n} \xi_k \right) = \prod_{k=1}^{n} \varphi_k(t).$$

Then by a lemma of Esséen [E], for any $v \in \mathbb{R}$,

$$Q = \mathbb{P}\left(\left| \sum_{j=1}^{n} \xi_j - v \right| < \Delta \right) \le c \int_{[-\pi/2,\pi/2]} |\varphi(t/\Delta)|\, dt.$$

Let ξ_k' be an independent copy of ξ_k and let $\nu_k = \xi_k - \xi_k'$. Then $\mathbb{P}(|\nu_k| > 2a) \ge 2c^2 = \bar{c}$. We have

$$|\varphi_k(t)|^2 = \mathbb{E} \cos \nu_k t \tag{4.1}$$

and since $|x|^2 \le \exp(-(1-|x|^2))$ for any $x \in \mathbb{C}$,

$$|\varphi(t)| \le \left(\prod_{k=1}^{n} \exp\left(-1 + |\varphi_k(t)|^2\right) \right)^{1/2} = \exp\left(-\frac{1}{2} \sum_{k=1}^{n} (1 - |\varphi_k(t)|^2) \right).$$

Define a new random variable τ_k by conditioning on $|\nu_k| > 2a$. For a Borel set $A \subset \mathbb{R}$ put

$$\mathbb{P}\,(\tau_k \in A) = \frac{\mathbb{P}\,(\nu_k \in A \setminus [-2a, 2a])}{\mathbb{P}\,(|\nu_k| > 2a)}.$$

Then by (4.1),

$$1 - |\varphi_k(t)|^2 \ge \mathbb{E}(1 - \cos \tau_k t) \cdot \mathbb{P}\,(|\nu_k| > 2a) \ge \bar{c} \cdot \mathbb{E}(1 - \cos \tau_k t),$$

so that

$$|\varphi(t)| \le \exp(-c' f(t)),$$

where

$$f(t) = \mathbb{E} \sum_{k=1}^{n} (1 - \cos \tau_k t).$$

Let $T(m, r) = \{t \mid f(t/\Delta) \le m,\ |t| \le r\}$ and let

$$M = \max_{|t| \le \pi/2} f(t/\Delta).$$

Then, obviously, $M \le n$. To estimate M from below, notice that

$$\begin{aligned} M = \max_{|t| \le \pi/2} f(t/\Delta) &\ge \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \mathbb{E} \sum_{k=1}^{n} (1 - \cos(\tau_k/\Delta)t)\, dt \\ &= \mathbb{E} \sum_{k=1}^{n} \left(1 - \frac{2}{\pi} \cdot \frac{\sin(\tau_k/\Delta)\pi/2}{\tau_k/\Delta} \right) \ge cn, \end{aligned}$$

since $|\tau_k|/\Delta > 2a/\Delta > 4\pi$.

To estimate the measure of $T(m, \pi/2)$ we use the argument of [Ha1]. For reader's convenience we present a complete proof.

LEMMA 4.2. *Let* $0 < m < M/4$. *Then*

$$|T(m, \pi/2)| \leq c\sqrt{\frac{m}{M}} \cdot |T(M/4, \pi)|.$$

Proof. Let $l = \sqrt{M/4m}$. Taking the integer part if necessary, we may assume that l is an integer. For $k \in \mathbb{N}$ set

$$S_k = \{\sum_{j=1}^{k} t_j \mid t_j \in T(m, \pi/2)\}.$$

Note that $S_1 = T(m, \pi/2)$. Since

$$1 - \cos\alpha = 2\sin^2(\alpha/2)$$

and

$$\sin^2\left(\sum_{j=1}^{k} \alpha_j\right) \leq \left(\sum_{j=1}^{k} |\sin\alpha_j|\right)^2 \leq k\sum_{j=1}^{k} \sin^2\alpha_j,$$

we conclude that

$$S_k \subset T(k^2 m, k\pi/2). \tag{4.2}$$

For $k \leq l$ we have $k^2 m < M$, so that $(-\pi/2, \pi/2) \setminus T(k^2 m, k\pi/2) \neq \emptyset$. For a Borel set A let $\mu(A) = |A \cap [-\pi, \pi]|$, where $|B|$ denotes the Lebesgue measure of B. Now we shall prove by induction that for all $k \leq l$

$$\mu(S_k) \geq (k/2) \cdot \mu(S_1).$$

Obviously, $\mu(S_2) = |S_2| \geq 2 \cdot |S_1|$, and so this inequality holds for $k = 2$. Assume that $\mu(S_{k-1}) \geq (k-1)/2 \cdot \mu(S_1)$. Note that the sets S_k are closed. Let $v \in (-\pi/2, \pi/2)$ be a boundary point of S_k. Such point exists since $S_k \subset T(k^2 m, \pi/2)$, and so $(-\pi/2, \pi/2) \setminus S_k \neq \emptyset$. Let $\{v_j\}_{j=1}^{\infty}$ be a sequence of points in $(-\pi/2, \pi/2) \setminus S_k$ converging to v. Then $(v_j - S_1) \cap S_{k-1} = \emptyset$, so by continuity we have

$$\mu((v - S_1) \cap S_{k-1}) = 0.$$

Since the set S_1 is symmetric, this implies

$$\mu((v + S_1) \cup S_{k-1}) = \mu(v + S_1) + \mu(S_{k-1}).$$

Both sets on the right-hand side are contained in S_{k+1} (to see this for S_{k-1} note that $0 \in S_2$). Since $v + S_1 \subset [-\pi, \pi]$, the induction hypothesis implies

$$\mu(S_{k+1}) \geq \mu(v + S_1) + \mu(S_{k-1}) \geq \mu(S_1) + \frac{k-1}{2} \cdot \mu(S_1) = \frac{k+1}{2} \cdot \mu(S_1).$$

Finally, by (4.2), $S_l \cap [-\pi, \pi] \subset T(l^2 m, \pi)$, and so

$$|T(l^2 m, \pi)| \geq \mu(S_l) \geq \frac{l}{2} \cdot \mu(S_1) = \frac{l}{2} \cdot |T(m, \pi/2)|. \qquad \square$$

We continue to prove Lemma 4.1. Recall that

$$Q \le C \int_{[-\pi/2,\pi/2]} |\varphi(t/\Delta)|\,dt \le C \int_{[-\pi/2,\pi/2]} \exp(-c' f(t/\Delta))\,dt$$
$$\le \bar{C} \int_0^n |T(m,\pi/2)| e^{-c'm}\,dm.$$

Applying Lemma 4.2 for $0 \le m \le M/4$ and using the trivial bound $|T(m,\pi/2)| \le \pi$ for $m > M/4$, we obtain

$$Q \le \frac{C'}{\sqrt{M}} \cdot |T(\frac{M}{4},\pi)| + ce^{-C'M/16} \le \frac{C'}{\sqrt{M}} \cdot |T(\frac{M}{4},\pi)| + ce^{-c'n}. \tag{4.3}$$

To complete the proof we have to estimate the measure of $T = T(M/4,\pi)$ from above. For any $t \in T$ we have

$$g(t) = \sum_{k=1}^n \mathbb{E}\cos(\tau_k t/\Delta) \ge n - M/4 \ge n/2.$$

Let $w(x) = (1-|x|/\pi)\cdot \chi_{[-\pi,\pi]}(x)$ and put $W = \hat{w}$. Then $W \ge 0$ and $W(t) \ge c$ for $|t| \le \pi$. Hence by Parceval's equality,

$$|T| \le \left(\frac{n}{2}\right)^{-2} \int_T |g(t)|^2 \le C\left(\frac{n}{2}\right)^{-2} \int_{\mathbb{R}} W^2(t)|g(t)|^2\,dt$$
$$= \frac{C}{n^2} \int_{\mathbb{R}} \left|\mathbb{E}\sum_{k=1}^n w(\tau_k/\Delta - y)\right|^2 dy.$$

Since $w \le \chi_{[-\pi,\pi]}$, the last expression does not exceed

$$\frac{C}{n^2}\int_{\mathbb{R}} \left(\sum_{k=1}^n \mathbb{P}\left(\frac{\tau_k}{\Delta} \in [y-\pi, y+\pi]\right)\right)^2 dy$$
$$\le \frac{C}{n^2\Delta}\int_{\mathbb{R}} \left(\sum_{k=1}^n \mathbb{P}\left(\tau_k \in [z-\pi\Delta, z+\pi\Delta]\right)\right)^2 dz.$$

Since $\tau_k \notin [-2a,2a]$ and $\pi\Delta < a/2$, we can integrate only over $\mathbb{R}\setminus[-3a/2,3a/2]$.

Substituting this estimate into (4.3), we get

$$Q \le \frac{C}{n^{5/2}\Delta}\int_{\mathbb{R}\setminus[-3a/2,3a/2]} \left(\sum_{k=1}^n \mathbb{P}\left(\tau_k \in [z-\pi\Delta, z+\pi\Delta]\right)\right)^2 dz + ce^{-c'n}.$$

To finish the proof, recall that the variables τ_k are symmetric. This allows us to change the integration set in the previous inequality to $(3a/2,\infty)$. Moreover, if $z \in (3a/2,\infty)$, then

$$\mathbb{P}\left(\tau_k \in [z-\pi\Delta, z+\pi\Delta]\right) \le \frac{1}{\bar{c}} \cdot \mathbb{P}\left(\nu_k \in [z-\pi\Delta, z+\pi\Delta]\right),$$

so that the random variables τ_k can be replaced by $\nu_k = \xi_k - \xi_k'$. $\square$

Remark 4.3. A more delicate analysis shows that the term $ce^{-c'n}$ in the formulation of Lemma 4.1 can always be eliminated. However, we shall not prove it since this term does not affect the results below.

We shall apply Lemma 4.1 to weighted copies of the same random variable. To formulate the result we have to introduce a new notion.

Definition 4.4. Let $x \in \mathbb{R}^m$. For $\Delta > 0$ define the Δ-profile of the vector x as a sequence $\{P_k(x,\Delta)\}_{k=1}^{\infty}$ such that

$$P_k(x,\Delta) = |\{j \mid |x_j| \in (k\Delta,(k+1)\Delta]\}.$$

THEOREM 4.5. *Let β be a random variable such that $\mathbb{E}\beta = 0$ and $\mathbb{P}(\beta > c) \geq c'$, $\mathbb{P}(\beta < -c) \geq c'$ for some $c, c' > 0$. Let $\beta_1 \dots \beta_m$ be independent copies of β. Let $\Delta > 0$ and let $(x_1 \dots x_m) \in \mathbb{R}^m$ be a vector such that $a < |x_j| < \lambda a$ for some $a > 0$ and $\lambda > 1$. Then for any $\Delta < a/(2\pi)$ and for any $v \in \mathbb{R}$*

$$\mathbb{P}\left(\left|\sum_{j=1}^{m} \beta_j x_j - v\right| < \Delta\right) \leq \frac{C_{4.5}}{m^{5/2}} \sum_{k=1}^{\infty} P_k^2(x,\Delta).$$

Here $C_{4.5}$ depends only on λ.

Proof. We shall apply Lemma 4.1 to the random variables $\xi_j = x_j\beta_j$.

Let $\mathcal{M}(\mathbb{R})$ be the set of all probability measures on $\mathbb{R}$. Consider the function $F : \mathcal{M}(\mathbb{R}) \to \mathbb{R}_+$ defined by

$$F(\mu) = \int_{3a/2}^{\infty} \tilde{S}_{\Delta}^2(y)\, dy,$$

where

$$\tilde{S}_{\Delta}(y) = \sum_{j=1}^{m} \mu\left(\frac{1}{|x_j|} \cdot [y - \pi\Delta, y + \pi\Delta]\right).$$

Since F is a convex function on $\mathcal{M}(\mathbb{R})$, it attains the maximal value at an extreme point of this set, i.e. at some delta-measure δ_t, $t \in \mathbb{R}$. Note that in this case

$$\tilde{S}_{\Delta}(y) = |\{j \mid t|x_j| \in [y - \pi\Delta, y + \pi\Delta]\} = \sum_{j=1}^{m} \chi(t|x_j| - y),$$

where $\chi = \chi_{[-\pi\Delta,\pi\Delta]}$ is the indicator function of $[-\pi\Delta, \pi\Delta]$. For $t < \frac{1}{2\lambda}$ we have $t|x_j| < a/2$, so $\tilde{S}_{\Delta}(y) = 0$ for any $y \geq 3a/2$, and thus $F(\delta_t) = 0$. If $t \geq \frac{1}{2\lambda}$, then

$$\begin{aligned} F(\delta_t) &= \sum_{j=1}^{m}\sum_{l=1}^{m} \int_{3a/2}^{\infty} \chi(t|x_j| - y)\chi(t|x_l| - y)\, dy \\ &\leq 2\pi\Delta|\{(j,l) \mid t\big||x_j| - |x_l|\big| \leq \pi\Delta\}| = g(t). \end{aligned}$$

Since the function g is decreasing,

$$F(\delta_t) \le g(\frac{1}{2\lambda}) \le 2\pi\Delta \sum_{l=1}^{\infty} |\{j \mid \left||x_j| - l\Delta\right| \le 2\pi\Delta \cdot \lambda\}|^2$$
$$\le \bar{C}\Delta \sum_{k=1}^{\infty} |\{j \mid |x_j| \in (k\Delta, (k+1)\Delta]\}|^2.$$

The last inequality holds since we can cover each interval $[l\Delta - 2\pi\Delta\lambda, l\Delta + 2\pi\Delta\lambda]$ by at most $2\pi\lambda + 2$ intervals $(k\Delta, (k+1)\Delta]$.

Let μ be the distribution of the random variable $\beta - \beta'$, where β' is an independent copy of β. Applying Lemma 4.1 to the random variables $\xi_j = x_j \cdot \beta_j$, we have

$$\mathbb{P}\left(\left|\sum_{j=1}^{m} \beta_j x_j - v\right| < \Delta\right) \le \frac{C}{m^{5/2}\Delta} F(\mu) + ce^{-c'm}$$
$$\le \frac{C'}{m^{5/2}} \sum_{k=1}^{\infty} |\{j \mid |x_j| \in (k\Delta, (k+1)\Delta]\}|^2 + ce^{-c'm}.$$

Since the sum on the right-hand side is at least m, the second term is negligible compare to the first one. Thus,

$$\mathbb{P}\left(\left|\sum_{j=1}^{m} \beta_j x_j - v\right| < \Delta\right) \le \frac{2C'}{m^{5/2}} \sum_{k=1}^{\infty} |\{j \mid |x_j| \in (k\Delta, (k+1)\Delta]\}|^2. \qquad \square$$

5. Small ball probability estimates

Let G be an $n \times n$ Gaussian matrix. If $x \in S^{n-1}$ is any unit vector, then $y = Gx$ is the standard Gaussian vector in $\mathbb{R}^n$. Hence for any $t > 0$ we have $\mathbb{P}(|y_j| < t) \le t \cdot \sqrt{2/\pi}$ for any coordinate. Moreover,

$$\mathbb{P}(\|y\| \le t \cdot \sqrt{n}) \le (2\pi)^{-n/2} \mathrm{vol}(t\sqrt{n}B_2^n) \le (Ct)^n.$$

We would like to have the same small ball probability estimates for the random vector $y = Ax$. However, it is easy to see that it is impossible to achieve such estimate for *all* directions $x \in S^{n-1}$. Indeed, if A is a random ± 1 matrix and $x = (1/\sqrt{2}, 1/\sqrt{2}, 0 \dots 0)$, then $\mathbb{P}(y_j = 0) = 1/2$ and $\mathbb{P}(y = 0) = 2^{-n}$. Analyzing this example, we see that the reason that the small ball estimate fails is the concentration of the Euclidean norm of x on a few coordinates. If the vector x is "spread", we can expect a more regular behavior of the small ball probability.

Although we cannot prove the Gaussian type estimates for all directions and all $t > 0$, it is possible to obtain such estimates for *most* directions provided that t is sufficiently large ($t > t_0$). Moreover, the more we assume about the regularity of distribution of the coordinates of x, the smaller value of t_0 we can take. This general statement is illustrated by the series of results below.

The first result is valid for any direction. The following lemma is a particular case of [LPRT, Prop. 3.4].

LEMMA 5.1. *Let A be an $n \times n$ matrix with i.i.d. subgaussian entries. Then for every $x \in S^{n-1}$*

$$\mathbb{P}\left(\|Ax\| \leq C_{5.1}\sqrt{n}\right) \leq \exp(-c_{5.1}n).$$

The example considered at the beginning of this section shows that this estimate cannot be improved for a general random matrix.

If we assume that all coordinates of the vector x are comparable, then we have the following lemma, which is a particular case of Proposition 3.4 [LPRTV2] (see also Proposition 3.2 [LPRT]).

LEMMA 5.2. *Let β be a random variable such that $\mathbb{E}\beta = 0$, $\mathbb{E}\beta^2 = 1$ and let $\beta_1, \ldots, \beta_m$ be independent copies of β. Let $0 < r < R$ and let $x_1, \ldots, x_m \in \mathbb{R}$ be such that $r/\sqrt{m} \leq |x_j| \leq R/\sqrt{m}$ for any j. Then for any $t \geq c_{5.2}/\sqrt{m}$ and for any $v \in \mathbb{R}$*

$$\mathbb{P}\left(\left|\sum_{j=1}^{m} \beta_j x_j - v\right| < t\right) \leq C_{5.2}t.$$

Here $c_{5.2}$ and $C_{5.2}$ depend only on r and R.

Proof. The proof is based on the Berry-Esséen theorem (cf., e.g., [St, §2.1]).

THEOREM 5.3. *Let $(\zeta_j)_{i=1}^m$ be a sequence of independent random variables with expectation 0 and finite third moments, and let $A^2 := \sum_{j=1}^m \mathbb{E}|\zeta_j|^2$. Then for every $\tau \in \mathbb{R}$,*

$$\left|\mathbb{P}\left(\sum_{j=1}^{m} \zeta_j < \tau A\right) - \mathbb{P}\left(g < \tau\right)\right| \leq (c/A^3)\sum_{j=1}^{m} \mathbb{E}|\zeta_j|^3,$$

where g is a Gaussian random variable with $N(0,1)$ distribution and $c \geq 1$ is a universal constant.

Let $\zeta_j = \beta_j x_j$. Then $A^2 := \sum_{j=1}^m \mathbb{E}\zeta_j^2 = \|x\|^2 \geq r^2$. Since the random variables β_j are copies of a subgaussian random variable β, $\mathbb{E}|\beta|^3 \leq C$ for some absolute constant C. Hence, $\mathbb{E}\sum_{j=1}^m |\zeta_j|^3 \leq C\sum_{j=1}^m |x_j|^3 \leq C'/\sqrt{m}$. By

Theorem 5.3 we get

$$\mathbb{P}\left(\left|\sum_{j=1}^{m}\beta_j x_j - v\right| < t\right) \le \mathbb{P}\left(\frac{v-t}{c} \le g < \frac{v+t}{c}\right) + \frac{c'}{\sqrt{m}}$$
$$\le C''t + \frac{c'}{\sqrt{m}} \le 2C''t,$$

provided $t \ge \frac{C''}{c'\sqrt{m}}$. $\square$

If $x = (1/\sqrt{m}, \dots, 1/\sqrt{m})$, then

$$\mathbb{P}\left(\left|\sum_{j=1}^{m}\beta_j x_j\right| = 0\right) \ge C/\sqrt{m}.$$

This shows that the bound $t \ge c_{5.2}/\sqrt{m}$ in Lemma 5.2 is necessary.

The proofs of Lemma 5.1 and Lemma 5.2 are based on Paley-Zygmund inequality and the Berry-Esséen theorem respectively. To obtain the linear decay of small ball probability for $t \le c_{5.2}/\sqrt{m}$, we use the third technique, namely Halász method. However, since the formulation of the result requires several technical assumptions on the vector x, we postpone it to Section 7, where these assumptions appear.

To translate the small ball probability estimate for a single coordinate to a similar estimate for the norm we use the Laplace transform technique, developed in [LPRT]. The following lemma improves the argument used in the proof of Theorem 3.1 [LPRT].

Lemma 5.4. *Let $\Delta > 0$ and let Y be a random variable such that for any $v \in \mathbb{R}$ and for any $t \ge \Delta$,*

$$\mathbb{P}(|Y - v| < t) \le Lt.$$

Let $y = (Y_1, \dots, Y_n)$ be a random vector, whose coordinates are independent copies of Y. Then for any $z \in \mathbb{R}^n$

$$\mathbb{P}\left(\|y - z\| \le \Delta\sqrt{n}\right) \le (C_{5.4}L\Delta)^n.$$

Proof. We have

$$\begin{aligned}\mathbb{P}\left(\|y-z\| \le \Delta\sqrt{n}\right) &= \mathbb{P}\left(\sum_{i=1}^n (Y_i - z_i)^2 \le \Delta^2 n\right) \\ &= \mathbb{P}\left(n - \frac{1}{\Delta^2}\sum_{i=1}^n (Y_i - z_i)^2 \ge 0\right) \\ &\le \mathbb{E}\exp\left(n - \frac{1}{\Delta^2}\sum_{i=1}^n (Y_i - z_i)^2\right) \\ &= e^n \cdot \prod_{i=1}^n \mathbb{E}\exp(-\frac{1}{\Delta^2}(Y_i - z_i)^2).\end{aligned}$$

To estimate the last expectation we use the small ball probability estimate for the random variable Y, assumed in the lemma. Note that if $t < \Delta$, then $\mathbb{P}(|Y-z| < t) \le L\Delta$ for any $z \in \mathbb{R}$. Hence,

$$\begin{aligned}\mathbb{E}\exp(-\frac{1}{\Delta^2}(Y_i - z_i)^2) &= \int_0^1 \mathbb{P}\left(\exp\Big(-\frac{1}{\Delta^2}(Y_i - z_i)^2\Big) > s\right) ds \\ &= \int_0^\infty 2ue^{-u^2}\mathbb{P}\left(|Y_i - z_i| < \Delta u\right) du \\ &\le \int_0^1 2ue^{-u^2} L\Delta\, du + \int_1^\infty 2ue^{-u^2} L\Delta u\, du \\ &\le \bar{C}L\Delta.\end{aligned}$$

Substituting this into the previous inequality, we get

$$\mathbb{P}\left(\|y-z\| \le \Delta\sqrt{n}\right) \le (e \cdot \bar{C}L\Delta)^n. \qquad \square$$

6. Partition of the sphere

To apply the small ball probability estimates proved in the previous section we have to decompose the sphere into different regions depending on the distribution of the coordinates of a point. We start by decomposing the sphere S^{n-1} in two parts following [LPRT], [LPRTV1], [LPRTV2]. We shall define two sets: V_P – the set of vectors, whose Euclidean norm is concentrated on a few coordinates, and V_S – the set of vectors whose coordinates are evenly spread. Let $r < 1 < R$ be the numbers to be chosen later. Given $x = (x_1, \dots, x_n) \in S^{n-1}$, set $\sigma(x) = \{i \mid |x_i| \le R/\sqrt{n}\}$. Let P_I be the coordinate projection on the set $I \subset \{, \dots, n\}$. Set

$$\begin{aligned}V_P &= \{x \in S^{n-1} \mid \left\|P_{\sigma(x)}x\right\| < r\}, \\ V_S &= \{x \in S^{n-1} \mid \left\|P_{\sigma(x)}x\right\| \ge r\}.\end{aligned}$$

First we shall show that with high probability $\|Ax\| \geq C\sqrt{n}$ for any $x \in V_P$.

For a single vector $x \in \mathbb{R}^n$ this probability was estimated in Lemma 5.1. We shall combine this estimate with an ε-net argument.

Lemma 6.1. *For any* $r < 1/2$

$$\log N(V_P, B_2^n, 2r) \leq \frac{n}{R} \cdot \log\left(\frac{3R}{r}\right).$$

Proof. If $x \in B_2^n$, then $|\{1, \dots, n\} \setminus \sigma(x)| \leq n/R$. Hence, the set V_P is contained in the sum of two sets: rB_2^n and

$$W_P = \{x \in B_2^n \mid |\text{supp}(x)| \leq n/R^2\}.$$

Since W_P is contained in the union of unit balls in all coordinate subspaces of dimension $l = n/R$, Lemma 3.4 implies

$$N(W_P, B_2^n, r) \leq \binom{n}{l} \cdot N(B_2^l, B_2^l, r) \leq \binom{n}{l} \cdot \left(\frac{3}{r}\right)^l.$$

Finally,

$$\log N(V_P, B_2^n, 2r) \leq \log N(W_P, B_2^n, r) \leq l \cdot \log\left(\frac{3n}{lr}\right) \leq \frac{n}{R} \cdot \log\left(\frac{3R}{r}\right). \quad \square$$

Recall that $C_{5.1} < C_{3.3}$. Set $r = C_{5.1}/2C_{3.3}$ and choose the number $R > 1$ so that

$$\frac{1}{R} \cdot \log\left(\frac{3R}{r}\right) < \frac{c_{5.1}}{2}.$$

For these parameters we prove that the norm of Ax is bounded below for all $x \in V_P$ with high probability.

Lemma 6.2.

$$\mathbb{P}\left(\exists x \in V_P \mid \|Ax\| \leq C_{5.1}\sqrt{n}/2\right) \leq 2\exp(-c_{5.1}n).$$

Proof. By Lemma 6.1 and the definition of r and R, the set V_P contains a $(C_{5.1}/2C_{3.3})$-net $\mathcal{N}$ in the ℓ_2-metric of cardinality at most $\exp(c_{5.1}n/2)$. Let

$$\Omega_0 = \{\omega \mid \|A\| > C_{3.3}\sqrt{n}\}$$

and let

$$\Omega_P = \{\omega \mid \exists x \in \mathcal{N} \ \ \|A(\omega)x\| \leq C_{5.1}\sqrt{n}\}.$$

Then Lemma 5.1 implies

$$\mathbb{P}(\Omega_0) + \mathbb{P}(\Omega_P) \leq \exp(-n) + \exp(-c_{5.1}n) \leq 2\exp(-c_{5.1}n).$$

Let $\omega \notin \Omega_P$. Pick any $x \in V_P$. There exists $y \in \mathcal{N}$ such that $\|x - y\|_2 \le C_{5.1}/2C_{3.3}$. Hence

$$\|Ax\| \ge \|Ay\| - \|A(x-y)\| \ge C_{5.1}\sqrt{n} - \|A : B_2^n \to B_2^n\| \cdot \|x-y\|_2 \ge \frac{C_{5.1}}{2}\sqrt{n}. \qquad \square$$

For $x = (x_1, \dots, x_n) \in V_S$ denote

$$J(x) = \left\{ j \mid \frac{r}{2\sqrt{n}} \le |x_j| \le \frac{R}{\sqrt{n}} \right\}. \tag{6.1}$$

Note that

$$\sum_{j \in J(X)} x_j^2 \ge \sum_{j \in \sigma(X)} x_j^2 - \frac{r^2}{2} \ge \frac{r^2}{2},$$

so that

$$|J(x)| \ge (r^2/2R^2) \cdot n =: m.$$

Let $0 < \Delta < r/2\sqrt{n}$ be a number to be chosen later. We shall cover the interval $[\frac{r}{2\sqrt{n}}, \frac{R}{\sqrt{n}}]$ by

$$k = \left\lceil \frac{R - r/2}{\sqrt{n}\Delta} \right\rceil$$

consecutive intervals $(j\Delta, (j+1)\Delta]$, where $j = k_0, (k_0+1), \dots, (k_0+k)$, and k_0 is the largest number such that $k_0\Delta < r/2\sqrt{n}$. Then we shall decompose the set V_S into two subsets: one containing the points whose coordinates are concentrated in a few such intervals, and the other containing points with evenly spread coordinates. This will be done using the Δ-profile, defined in 4.4. Note that if m coordinates of the vector x are evenly spread among k intervals, then

$$\sum_{i=1}^{\infty} P_i^2(x, \Delta) \sim \frac{m^2}{k} \sim m^{5/2}\Delta.$$

This observation leads to the following

Definition 6.3. Let $\Delta > 0$ and let $Q > 1$. We say that a vector $x \in V_S$ has a (Δ, Q)-regular profile if there exists a set $J \subset J(x)$ such that $|J| \ge m/2$ and

$$\sum_{i=1}^{\infty} P_i^2(x|_J, \Delta) \le Q m^{5/2} \Delta =: C_{6.3} Q \cdot \frac{m^2}{k}.$$

Here $x|_J$ is the restriction of x to the set J.

If such a set J does not exist, we call x a vector of (Δ, Q)-singular profile.

Note that $\sum_{i=1}^{\infty} P_i^2(x|_J, \Delta) \geq m/2$. Hence, if $\Delta < m^{-3/2}/2$, then every vector in V_S will be a vector of a (Δ, Q)-singular profile.

Vectors of regular and singular profile will be treated differently. Namely, in Section 7 we prove that vectors of regular profile satisfy the small ball probability estimate of the type Ct for $t \geq \Delta$. This allows us to use conditioning to estimate the probability that $\|Ax\|$ is small for some vector x of regular profile. In Section 8 we prove that the set of vectors of singular profile admits a small ε-net. This fact combined with Lemma 5.2 allows us to estimate the probability that there exists a vector x of singular profile such that $\|Ax\|$ is small using the standard ε-net argument.

7. Vectors of a regular profile

To estimate the small ball probability for a vector of a regular profile we apply Theorem 4.5.

LEMMA 7.1. *Let $\Delta \leq \frac{r}{4\pi\sqrt{n}}$. Let $x \in V_S$ be a vector of (Δ, Q)-regular profile. Then for any $t \geq \Delta$*

$$\mathbb{P}\left(\left|\sum_{j=1}^{n} \beta_j x_j - v\right| < t\right) \leq C_{7.1} Q \cdot t.$$

Proof. Let $J \subset \{1, \dots, n\}$, $|J| \geq m/2$ be the set from Definition 6.3. Denote by $\mathbb{E}_{J^c}$ the expectation with respect to the random variables β_j, where $j \in J^c = \{1, \dots, n\} \setminus J$. Then

$$\begin{aligned} &\mathbb{P}\left(\left|\sum_{j=1}^{n} \beta_j x_j - v\right| < t\right) \\ &= \mathbb{E}_{J^c}\mathbb{P}\left(\left|\sum_{j\in J} \beta_j x_j - (v + \sum_{j\in J^c} \beta_j x_j)\right| < t \mid \beta_j,\ j \in J^c\right). \end{aligned}$$

Hence, it is enough to estimate the conditional probability.

Recall that β is a centered subgaussian random variable of variance 1. It is well-known that such a variable satisfies $\mathbb{P}(\beta > c) \geq c'$, $\mathbb{P}(\beta < -c) \geq c'$ for some absolute constants c, c'. Moreover, a simple Paley-Zygmund type argument shows that these estimates hold if we assume only that $\mathbb{E}\beta = 0$ and the second and the fourth moment of β are comparable. Hence, for $t = \Delta$ the lemma follows from Theorem 4.5, where we set $a = r/\sqrt{n}$, $\lambda = R/r$.

To prove the lemma for other values of t, assume first that $t = \Delta_s = 2^s\Delta < \frac{r}{4\pi\sqrt{n}}$ for some $s \in \mathbb{N}$. Consider the Δ_s-profile of $x|_J$:

$$P_l(x|_J, \Delta_s) = |\{j \in J \mid |x_j| \in (l\Delta_s, (l+1)\Delta_s]\}|.$$

Notice that each interval $(l\Delta_s, (l+1)\Delta_s]$ is a union of 2^s intervals $(i\Delta, (i+1)\Delta]$. Hence

$$\sum_{l=1}^{\infty} P_l^2(x|_J, \Delta_s) \le 2^s \sum_{i=1}^{\infty} P_i^2(x|_J, \Delta) \le 2^s Q m^{5/2} \Delta = Q m^{5/2} t.$$

Applying Theorem 4.5 with Δ replaced by Δ_s and $v' = v + \sum_{j \in J^c} \beta_j x_j$, we obtain

$$\mathbb{P}\left(\left| \sum_{j \in J} \beta_j x_j - (v + \sum_{j \in J^c} \beta_j x_j) \right| < t \mid \beta_j,\ j \in J^c \right) \le C_{4.5} Q t.$$

For $2^s\Delta < t < 2^{s+1}\Delta$ the result follows from the previous inequality applied for $t = 2^s\Delta$. If $t \ge c_{5.2}/\sqrt{m} = \frac{\sqrt{2}\, c_{5.2} R}{r\sqrt{n}}$, Lemma 5.2 implies

$$\mathbb{P}\left(\left| \sum_{j \in J} \beta_j x_j - (v + \sum_{j \in J^c} \beta_j x_j) \right| < t \mid \beta_j,\ j \in J^c \right) \le C_{5.2} t \le C_{5.2} Q t.$$

Finally, if $\frac{r}{4\pi\sqrt{n}} < t < \frac{\sqrt{2}\, c_{5.2} R}{r\sqrt{n}}$, the previous inequality applied to $t_0 = \frac{\sqrt{2}\, c_{5.2} R}{r\sqrt{n}}$ implies

$$\mathbb{P}\left(\left| \sum_{j \in J} \beta_j x_j - (v + \sum_{j \in J^c} \beta_j x_j) \right| < t \mid \beta_j,\ j \in J^c \right) \le C_{5.2} Q t_0 \le C Q t,$$

where $C = C_{5.2} \cdot \frac{\sqrt{2}\, c_{5.2} R}{r} \cdot \frac{4\pi}{r}$. □

Now we estimate the probability that $\|A(\omega)x\|$ is small for some vector of a regular profile.

Theorem 7.2. *Let $\Delta > 0$ and let U be the set of vectors of (Δ, Q)-regular profile. Then*

$$\mathbb{P}\left(\exists\, x \in U \mid \|Ax\| \le \frac{\Delta}{2\sqrt{n}} \right) \le C_{7.1} Q \Delta n.$$

Proof. Set

$$s = \frac{\Delta}{2\sqrt{n}}.$$

Let Ω be the event described in Theorem 7.2. Denote the rows of A by $a_1, \dots, a_n$. Note that since

$$\min_{x \in S^{n-1}} \|Ax\| = \min_{u \in S^{n-1}} \left\| A^T u \right\|,$$

for any $\omega \in \Omega$ there exists a vector $u = (u_1, \dots, u_n) \in S^{n-1}$ such that

$$u_1 a_1 + \dots + u_n a_n = z,$$

where $\|z\| < s$. Then $\Omega = \cup_{k=1}^n \Omega_k$, where Ω_k is the event $|u_k| \ge 1/\sqrt{n}$. Since the events Ω_k have the same probability, it is enough to estimate $\mathbb{P}(\Omega_n)$.

To this end we condition on the first $n-1$ rows of the matrix $A = A(\omega)$:

$$\mathbb{P}(\Omega_n) = \mathbb{E}_{a_1,\dots,a_{n-1}} \mathbb{P}(\Omega_n \mid a_1, \dots, a_{n-1}).$$

Here $\mathbb{E}_{a_1,\dots,a_{n-1}}$ is the expectation with respect to the first $n-1$ rows of the matrix A. Take *any* vector $y \in U$ such that

$$\sum_{j=1}^{n-1} \langle a_j, y\rangle^2 < s^2.$$

If such a vector does not exist, then $\|Ay\| \ge s$ for all $y \in U$, and so $\omega \notin \Omega$. Note that the vector y can be chosen using only $a_1, \dots, a_{n-1}$. We have

$$a_n = \frac{1}{u_n}(u_1 a_1 + \ldots + u_{n-1} a_{n-1} - z),$$

so that for $\omega \in \Omega_n$

$$\begin{aligned} |\langle a_n, y\rangle| &= \frac{1}{|u_n|} \left| \sum_{j=1}^{n-1} u_j \langle a_j, y\rangle - \langle z, y\rangle \right| \\ &\le \sqrt{n} \left(\left(\sum_{j=1}^{n-1} u_j^2 \right)^{1/2} \left(\sum_{j=1}^{n-1} \langle a_j, y\rangle^2 \right)^{1/2} + \|z\| \right) \le 2\sqrt{n} \cdot s = \Delta. \end{aligned}$$

The row a_n is independent of $a_1, \dots, a_{n-1}$. Hence, Lemma 7.1 implies

$$\begin{aligned} \mathbb{P}(\Omega_n \mid a_1, \dots, a_{n-1}) &\le \mathbb{P}(|\langle a_n, y\rangle| \le \Delta \mid a_1, \dots, a_{n-1}) \\ &= \mathbb{P}\left(\left| \sum_{j=1}^{n} \beta_{n,j} y_j \right| \le \Delta \mid a_1, \dots, a_{n-1} \right) \le C_{7.1} Q \Delta. \end{aligned}$$

Taking the expectation with respect to $a_1, \dots, a_{n-1}$, we obtain $\mathbb{P}(\Omega_n) \le C_{7.1} Q\Delta$, and so

$$\mathbb{P}(\Omega) \le n \cdot \mathbb{P}(\Omega_n) \le C_{7.1} Q \Delta n. \qquad \square$$

8. Vectors of a singular profile

We prove first that the set of vectors of singular profile admits a small Δ-net in the ℓ_∞-metric.

Lemma 8.1. *Let $\overline{C_{8.1}} n^{-3/2} \le \Delta \le n^{-1/2}$, where $\overline{C_{8.1}} = \frac{2R^3}{r^2}$ and let W_S be the set of vectors of (Δ, Q)-singular profile. Let $\eta < 1$ be such that*

$$C(\eta) < C_{6.3} Q,$$

where $C(\eta)$ is the function defined in Lemma 3.1. *Then there exists a Δ-net $\mathcal{N}$ in W_S in ℓ_∞-metric such that*

$$|\mathcal{N}| \le \left(\frac{C_{8.1}}{\Delta\sqrt{n}} \eta^{c_{8.1}} \right)^n .$$

Remark 8.2. Lemma 3.4 implies that there exists a Δ-net for S^{n-1} in the ℓ_∞-metric with less than $(C/\Delta\sqrt{n})^n$ points. Thus, considering only vectors of a singular profile, we gain the factor $\eta^{c_{8.1}\cdot n}$ in the estimate of the size of a Δ-net.

Proof. Let $J \subset \{1, \dots, n\}$ and denote $J' = \{1, \dots, n\} \setminus J$. Let $W_J \subset W_S$ be the set of all vectors x of a (Δ, Q)-singular profile for which $J(x) = J$. We shall construct Δ-nets in each W_J separately. To this end we shall use Lemma 3.1 to construct a Δ-net for the set $P_J W_J$, where P_J is the coordinate projection on $\mathbb{R}^J$. Then the product of this Δ-net and a Δ-net for the ball $B_2^{J'}$ will form a Δ-net for the whole W_J.

Assume that $J = \{1, \dots, l\}$, where $l \ge m$. Let $I_1, \dots, I_k$ be consecutive subintervals $(i\Delta, (i+1)\Delta],\ i = k_0, \dots, k_0 + k$, covering the interval $[\frac{r}{2\sqrt{n}}, \frac{R}{\sqrt{n}}]$, which appear in the definition of profile. Recall that

$$k = \left\lceil \frac{R - r/2}{\sqrt{n}\Delta} \right\rceil$$

The restriction on Δ implies that $k \le m$. Let d_i be the center of the interval I_i. Set

$$\mathcal{M}_J = \{x \in \mathbb{R}^J \mid |x_j| \in \{d_1, \dots, d_k\} \text{ for } j \in J\}.$$

Then $|\mathcal{M}_J| = (2k)^l$. Let $\mathcal{N}_J$ be the set of all $x \in \mathcal{M}_J$ for which there exists a vector $y \in W_J$ such that $-\Delta/2 < y_j - x_j \le \Delta/2$ for all $j \in J$. The set $\mathcal{N}_J$ forms a Δ-net for W_J in the ℓ_∞ metric. To estimate its cardinality we use the probabilistic method.

Let $X(1), \dots, X(l)$ be independent random variables uniformly distributed on the set $\{1, \dots, k\}$. Let $N \subset \{1, \dots, k\}^l$ be the set of all l-tuples $(v(1), \dots, v(l))$ such that $|x_j| = d_{v(j)},\ j = 1, \dots, l$ for some $x = (x_1, \dots, x_l) \in \mathcal{N}_J$. Since both $\mathcal{M}_J$ and $\mathcal{N}_J$ are invariant under changes of signs of the coordinates,

$$\mathbb{P}\left((X(1), \dots, X(l)) \in N\right) = \frac{|\mathcal{N}_J|}{|\mathcal{M}_J|}.$$

Let $(X(1), \dots, X(l)) \in N$ and let $x \in \mathbb{R}^l$ be such that $x_j = d_{X(j)}$. Let $y \in W_J$ be a vector such that $-\Delta/2 < y_j - x_j \le \Delta/2$ for all $j \in J$. Then for any $j \in J,\ y_j \in I_i$ implies that $X(j) = i$. Let $E \subset J$ be any set containing at least $m/2$ elements. Then

$$\sum_{i=1}^{\infty} P_i^2(y|_E, \Delta) = \sum_{i=1}^{k} |\{j \in E \mid X(j) = i\}|^2.$$

Since y is a vector of a singular profile, this implies

$$\sum_{i=1}^{k} |\{j \in E \mid X(j) = i\}|^2 \geq Qm^{5/2}\Delta = C_{6.3} \cdot Q\frac{m^2}{k} > C(\eta) \cdot \frac{m^2}{k}.$$

Now Lemma 3.1 implies that $\mathbb{P}\left((X(1), \dots, X(l)) \in N\right) \leq \eta^l$, and so

$$|\mathcal{N}_J| \leq (2k\eta)^l = \left(\frac{R-2r}{\Delta\sqrt{n}}\eta\right)^l.$$

To estimate the cardinality of the Δ-net for the whole W_J we use Lemma 3.4. Since $\Delta \leq 1/\sqrt{|J|}$, $\Delta B_\infty^J \subset B_2^J$, so that

$$N(P_{J'}W_J, B_\infty^{J'}, \Delta) \leq N(B_2^{J'}, B_\infty^{J'}, \Delta) \leq 3^{n-l}\frac{|B_2^{J'}|}{|\Delta B_\infty^{J'}|} \leq \left(\frac{c}{\Delta\sqrt{n-l}}\right)^{n-l}.$$

Because the function $f(t) = (a/t)^t$ is increasing for $0 < t < a/e$, the right-hand side of the previous inequality is bounded by $(c/\Delta\sqrt{n})^n$. Hence,

$$\begin{aligned} N(W_J, B_\infty^n, \Delta) &\leq N(P_J W_J, B_\infty^J, \Delta) \cdot N(P_{J'}W_J, B_\infty^{J'}, \Delta) \\ &\leq |\mathcal{N}_J| \cdot \left(\frac{c}{\Delta\sqrt{n}}\right)^n \leq \left(\frac{c'}{\Delta\sqrt{n}}\eta^{l/n}\right)^n \end{aligned}$$

Finally, set

$$\mathcal{N} = \bigcup_{|J|\geq m} \mathcal{N}_J.$$

Then

$$|\mathcal{N}| \leq \sum_{l=m}^{n}\sum_{|J|=l} |\mathcal{N}_J| \leq 2^n\left(\frac{c'}{\Delta\sqrt{n}}\eta^{m/n}\right)^n.$$

Thus, Lemma 8.1 holds with $c_{8.1} = m/n = \frac{r^2}{2R^2}$. □

Now we are ready to show that $\|Ax\| \geq c$ for *all* vectors of a (Δ, Q)-singular profile with probability exponentially close to 1.

THEOREM 8.3. *There exists an absolute constant Q_0 with the following property. Let $\Delta \geq C_{8.3}n^{-3/2}$, where $C_{8.3} = \max(c_{5.2}, \overline{C_{8.1}})$. Denote by Ω_Δ the event that there exists a vector $x \in V_S$ of (Δ, Q_0)-singular profile such that $\|Ax\| \leq \frac{\Delta}{2}n$. Then*

$$\mathbb{P}\left(\Omega_\Delta\right) \leq 3\exp(-n).$$

Proof. We consider two cases. First, we assume that $\Delta \geq \Delta_1 = c_{5.2}/n$. In this case we estimate the small ball probability using Lemma 5.2 and the size of the ε-net using Lemma 8.1. Note that only the second estimate uses the profile of the vectors. Then we conclude the proof with the standard approximation argument.

The case $\Delta \le \Delta_1$ is more involved. From Case 1 we know that there exists Q_1 such that all vectors of (Δ_1, Q_1)-singular profile satisfy $\|Ax\| \ge \frac{\Delta_1}{2} n$ with probability at least $1 - e^{-n}$. Hence, it is enough to consider only vectors whose profile is regular on the scale Δ_1 and singular on the scale Δ. For these vectors we use the regular profile in Lemma 7.1 to estimate the small ball probability and singular profile in Lemma 8.1 to estimate the size of the ε-net. The same approximation argument finishes the proof.

Case 1. Assume first that $\Delta \ge \Delta_1 = c_{5.2}/n$. Let $Q_1 > 1$ be a number to be chosen later. Let $\mathcal{M}$ be the smallest $\frac{\Delta}{2C_{3.3}}$-net in the set of the vectors of (Δ, Q_1)-singular profile in ℓ_∞ metric.

Let $x \in V_S$ and let $J = J(x)$ be as defined in (6.1). Denote $J^c = \{1, \dots, n\} \setminus J$. Then Lemma 5.2 implies

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{j=1}^{n} \beta_j x_j\right| \le t\right) &= \mathbb{E}_{J^c}\mathbb{P}\left(\left|\sum_{j\in J} \beta_j x_j + \sum_{j\in J^c} \beta_j x_j\right| \le t \mid \beta_j,\ j \in J^c\right)\\ &\le C_{5.2} t\end{aligned}$$

for all $t \ge c_{5.2}/\sqrt{n}$. Since $\Delta\sqrt{n} \ge c_{5.2}/\sqrt{n}$, by Lemma 5.4,

$$\mathbb{P}\left(\|Ax\| \le \Delta n\right) \le (C_{5.4}\Delta\sqrt{n})^n$$

and so,

$$\mathbb{P}\left(\exists x \in \mathcal{M} \mid \|Ax\| \le \Delta n\right) \le |\mathcal{M}|(C_{5.4}\Delta\sqrt{n})^n. \tag{8.1}$$

We shall show that Q_1 can be chosen so that the last quantity will be less than e^{-n}. Recall that by Lemma 8.1, there exists a Δ-net $\mathcal{N}$ for the set of vectors of (Δ, Q_1)-singular profile satisfying

$$|\mathcal{N}| \le \left(\frac{C_{8.1}}{\Delta\sqrt{n}}\eta^{c_{8.1}}\right)^n,$$

provided

$$C(\eta) < C_{6.3}Q_1. \tag{8.2}$$

Covering each cube of size Δ with the center in $\mathcal{N}$ by the cubes of size $\frac{\Delta}{2C_{3.3}}$ and using Lemma 3.4, we obtain

$$|\mathcal{M}| \le |\mathcal{N}| \cdot N(\Delta B_\infty^n, \Delta B_\infty^n, \frac{1}{2C_{3.3}}) \le \left(\frac{6C_{8.1}\cdot C_{3.3}}{\Delta\sqrt{n}}\eta^{c_{8.1}}\right)^n.$$

Substitution of this estimate into (8.1) yields

$$\begin{aligned}\mathbb{P}\left(\exists x \in \mathcal{M} \mid \|Ax\| \le \Delta n\right) &\le \left(\frac{6C_{8.1}\cdot C_{3.3}}{\Delta\sqrt{n}}\eta^{c_{8.1}}\right)^n \cdot (C_{5.4}\Delta\sqrt{n})^n\\ &\le (C'\eta^{c_{8.1}})^n.\end{aligned}$$

Now choose η so that $C'\eta^{c_{8.1}} < 1/e$ and choose Q_1 satisfying (8.2). With this choice the probability above is smaller than e^{-n}. Combining this estimate with Lemma 3.3, we have that $\|A\| \le C_{3.3}\sqrt{n}$ and $\|Ax\| \ge \Delta n$ for all $x \in \mathcal{M}$ with probability at least $1 - 2e^{-n}$.

Let $y \in V_S$ be a vector of (Δ, Q_1)-singular profile. Choose $x \in \mathcal{M}$ such that $\|x - y\|_\infty \le \frac{\Delta}{2C_{3.3}}$. Then $\|x - y\| \le \frac{\Delta\sqrt{n}}{2C_{3.3}}$ and

$$\|Ay\| \ge \|Ax\| - \|A(x-y)\| \ge \Delta n - \|A\|\,\|x-y\| \ge \frac{\Delta}{2}n.$$

Case 2. Assume that $C_{8.3}n^{-3/2} \le \Delta < \Delta_1 = c_{5.2}/n$. Let Ω_1 be the event that $\|Ax\| < \frac{\Delta_1}{2}n = c_{5.2}/2$ for some vector of (Δ_1, Q_1)-singular profile. We proved in Case 1 that

$$\mathbb{P}(\Omega_1) < 2e^{-n}. \tag{8.3}$$

Let $Q_2 > 1$ be a number to be chosen later and let W be the set of all vectors of (Δ_1, Q_1)-regular and (Δ, Q_2)-singular profile. By Lemma 7.1 any vector $x \in W$ satisfies

$$\mathbb{P}\left(\left|\sum_{j=1}^n \beta_j x_j\right| \le t\right) \le C_{7.1}Q_1 t$$

for all $t \ge \Delta_1$.

Now we can finish the proof as in Case 1. Since $\Delta\sqrt{n} \ge \Delta_1$, Lemma 5.4 implies

$$\mathbb{P}\left(\|Ax\| \le \Delta n\right) \le (C'\Delta\sqrt{n})^n$$

for any $x \in W$. Here $C' = C_{5.4} \cdot C_{7.1}Q_1$.

Let $\mathcal{N}$ be the smallest $\frac{\Delta}{2C_{3.3}}$-net in W in ℓ_∞ metric. Note that $\Delta \ge C_{8.3}n^{-3/2} \ge \overline{C_{8.1}}n^{-3/2}$. Arguing as in the Case 1, we show that

$$|\mathcal{N}| \le \left(\frac{6C_{8.1} \cdot C_{3.3}}{\Delta\sqrt{n}}\eta^{c_{8.1}}\right)^n$$

for any η satisfying

$$C(\eta) < C_{6.3}Q_2. \tag{8.4}$$

Hence,

$$\mathbb{P}\left(\exists x \in \mathcal{N} \mid \|Ax\| \le \Delta n\right) \le |\mathcal{N}|(C'\Delta\sqrt{n})^n \le (C''\eta^{c_{8.1}})^n.$$

Choose η so that the last quantity is less than e^{-n} and choose Q_2 so that (8.4) holds. Then the approximation argument used in Case 1 shows that the inequality

$$\|Ay\| \ge \frac{\Delta}{2}n$$

holds for any $y \in W$ with probability greater than $1 - e^{-n}$. Combining it with (8.3), we complete the proof of Case 2. Finally, we unite two cases setting $Q_0 = \max(Q_1, Q_2)$. $\square$

9. Proof of Theorem 1.1

To prove Theorem 1.1 we combine the probability estimates of the previous sections. Let $\varepsilon > c_{1.1}/\sqrt{n}$, where the constant $c_{1.1}$ will be chosen later. Define the exceptional sets:

$$\Omega_0 = \{\omega \mid \|A\| > C_{3.3}\sqrt{n}\},$$
$$\Omega_P = \{\omega \mid \exists\, x \in V_P \ \ \|Ax\| < C_{5.1}\sqrt{n}\ \}.$$

Then Lemma 3.3 and Lemma 6.2 imply

$$\mathbb{P}\,(\Omega_0) + \mathbb{P}\,(\Omega_P) \le 3\exp(-c_{5.1}n).$$

Let Q_0 be the number defined in Theorem 8.3. Set

$$\Delta = \frac{\varepsilon}{2C_{7.1}Q_0 \cdot n}.$$

The assumption on ε implies $\Delta \ge C_{8.3}n^{-3/2}$ if we set $c_{1.1} = 2C_{7.1}Q_0 \cdot C_{8.3}$. Denote by W_S the set of vectors of (Δ, Q_0)-singular profile and by W_R the set of vectors of (Δ, Q_0)-regular profile. Set

$$\Omega_S = \{\omega \mid \exists\, x \in W_S \ \ \|Ax\| < \frac{\Delta}{2}n = \frac{1}{4C_{7.1}Q_0}\varepsilon\},$$
$$\Omega_R = \{\omega \mid \exists\, x \in W_R \ \ \|Ax\| < \frac{\Delta}{2\sqrt{n}} = \frac{1}{4C_{7.1}Q_0}\varepsilon \cdot n^{-3/2}\}.$$

By Theorem 8.3, $\mathbb{P}\,(\Omega_S) \le 3e^{-n}$, and by Theorem 7.2, $\mathbb{P}\,(\Omega_R) \le \varepsilon/2$. Since $S^{n-1} = V_P \cup W_S \cup W_R$, we conclude that

$$\mathbb{P}\,(\omega \mid \exists\, x \in S^{n-1} \ \ \|Ax\| < \frac{1}{2C_{7.1}Q_0}\varepsilon \cdot n^{-3/2}\} \le \varepsilon/2 + 4\exp(-c_{5.1}n) < \varepsilon$$

for large n. □

Remark 9.1. The proof shows that the set of vectors of a regular profile is critical. On the other sets the norm of Ax is much greater with probability exponentially close to 1.

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