Annals of Mathematics, **168** (2008), 221–246

The derivation problem for group algebras

By Viktor Losert

Abstract

If G is a locally compact group, then for each derivation D from $L^1(G)$ into $L^1(G)$ there is a bounded measure $\mu \in M(G)$ with $D(a) = a * \mu - \mu * a$ for $a \in L^1(G)$ ("derivation problem" of B. E. Johnson).

Introduction

Let $\mathcal{A}$ be a Banach algebra, E an $\mathcal{A}$-bimodule. A linear mapping $D : \mathcal{A} \to E$ is called a *derivation*, if $D(a\,b) = a\,D(b) + D(a)\,b$ for all $a, b \in \mathcal{A}$ ([D, Def. 1.8.1]). For $x \in E$, we define the inner derivation $\mathrm{ad}_x : \mathcal{A} \to E$ by $\mathrm{ad}_x(a) = x\,a - a\,x$ (as in [GRW]; $\mathrm{ad}_x = -\delta_x$ in the notation of [D, (1.8.2)]).

If G is a locally compact group, we consider the group algebra $\mathcal{A} = L^1(G)$ and $E = M(G)$, with convolution (note that by Wendel's theorem [D, Th. 3.3.40], $M(G)$ is isomorphic to the multiplier algebra of $L^1(G)$ and also to the left multiplier algebra). The *derivation problem* asks whether all derivations are inner in this case ([D, Question 5.6.B, p. 746]). The question goes back to J. H. Williamson around 1965 (personal communication by H. G. Dales). The corresponding problem when $\mathcal{A} = E$ is a von Neumann algebra was settled affirmatively by Sakai [Sa], using earlier work of Kadison (see [D, p. 761] for further references). The derivation problem for the group algebra is linked to the name of B. E. Johnson, who pursued it over the years as a pertinent example in his theory of cohomology in Banach algebras. He developed various techniques and gave affirmative answers in a number of important special cases.

As an immediate consequence of the factorization theorem, the image of a derivation from $L^1(G)$ to $M(G)$ is always contained in $L^1(G)$. In [JS] (with A. Sinclair), it was shown that derivations on $L^1(G)$ are automatically continuous. In [JR] (with J. R. Ringrose), the case of discrete groups G was settled affirmatively. In [J1, Prop. 4.1], this was extended to SIN-groups and amenable groups (serving also as a starting point to the theory of amenable Banach algebras). In addition, some cases of semi-simple groups were considered in [J1] and this was completed in [J2], covering all connected locally compact groups.

A number of further results on the derivation problem were obtained in [GRW] (some of them will be discussed in later sections).

These problems were brought to my attention by A. Lau.

1. The main result

We use a setting similar to [J2, Def. 3.1]. Ω shall be a locally compact space, G a discrete group acting on Ω by homeomorphisms, denoted as a left *action* (or a left G*-module*), i.e., we have a continuous mapping $(x,\omega)\mapsto x\circ\omega$ from $G\times\Omega$ to Ω such that $x\circ(y\circ\omega)=(xy)\circ\omega$, $e\circ\omega=\omega$ for $x,y\in G$, $\omega\in\Omega$. Then $C_0(\Omega)$, the space of continuous (real- or complex-valued) functions on Ω vanishing at infinity becomes a right Banach G-module by $(h\circ x)(\omega)=h(x\circ\omega)$ for $h\in C_0(\Omega)$, $x\in G$, $\omega\in\Omega$. The space $M(\Omega)$ of finite Radon measures on the Borel sets $\mathcal{B}$ of Ω will be identified with the dual space $C_0(\Omega)'$ in the usual way and it becomes a left Banach G-module by $\langle x\circ\mu,\, h\,\rangle=\langle\,\mu\,,h\circ x\rangle$ for $\mu\in M(\Omega)$, $h\in C_0(\Omega)$, $x\in G$ (in particular, $x\circ\delta_\omega=\delta_{x\circ\omega}$ when $\mu=\delta_\omega$ is a point measure with $\omega\in\Omega$; see also [D, §3.3] and [J2, Prop. 3.2]).

A mapping $\Phi\colon G\to M(\Omega)$ (or more generally, $\Phi\colon G\to X$, where X is a left Banach G-module) is called a *crossed homomorphism* if $\Phi(xy)=\Phi(x)+x\circ\Phi(y)$ for all $x,y\in G$ ([J2, Def. 3.3]; in the terminology of [D, Def. 5.6.35], this is a G-derivation, if we consider the trivial right action of G on $M(\Omega)$). Now, Φ is called bounded if $\|\Phi\|=\sup_{x\in G}\|\Phi(x)\|<\infty$. For $\mu\in M(\Omega)$, the special example $\Phi_\mu(x)=\mu-x\circ\mu$ is called a *principal* crossed homomorphism (this follows [GRW]; the sign is taken opposite to [J2]).

Theorem 1.1. *Let Ω be a locally compact space, G a discrete group with a left action of G on Ω by homeomorphisms. Then any bounded crossed homomorphism Φ from G to $M(\Omega)$ is principal. There exists $\mu\in M(\Omega)$ with $\|\mu\|\le 2\,\|\Phi\|$ such that $\Phi=\Phi_\mu$.*

Corollary 1.2. *Let G denote a locally compact group. Then any derivation $D\colon L^1(G)\to M(G)$ is inner.*

Using [D, Th. 5.6.34 (ii)], one obtains the same conclusion for all derivations $D\colon M(G)\to M(G)$.

Proof. As mentioned in the introduction, we have $D(L^1(G))\subseteq L^1(G)$ and then D is bounded by a result of Johnson and Sinclair (see also [D, Th. 5.2.28]). Then by further results of Johnson, D defines a bounded crossed homomorphism Φ from G to $M(G)$ with respect to the action $x\circ\omega=x\,\omega\,x^{-1}$ of G on G ([D, Th. 5.6.39]) and (applying our Theorem 1.1) $\Phi=\Phi_\mu$ implies $D=\mathrm{ad}_\mu$. □

Corollary 1.3. *Let G denote a locally compact group, H a closed subgroup. Then any bounded derivation $D: M(H)\to M(G)$ is inner.*

Again, the same conclusion applies to bounded derivations $D\colon L^1(H) \to M(G)$.

Proof. $M(H)$ is identified with the subalgebra of $M(G)$ consisting of those measures that are supported by H (this gives also the structure of an $M(H)$-module considered in this corollary). As above, D defines a bounded crossed homomorphism Φ from H to $M(G)$ (for the restriction to H of the action considered in the proof of 1.2) and our claim follows. □

Corollary 1.4. *For any locally compact group G, the first continuous cohomology group $\mathcal{H}^1(L^1(G), M(G))$ is trivial.*

Note that

$$\mathcal{H}^1(M(G), M(G)) = \mathcal{H}^1(L^1(G), M(G))$$

holds by [D, Th. 5.6.34 (iii)].

Proof. Again, this is contained in [D, Th. 5.6.39]. □

Corollary 1.5. *Let G be a locally compact group and assume that $T \in \mathrm{VN}(G)$ satisfies $T * u - u * T \in M(G)$ for all $u \in L^1(G)$. Then there exists $\mu \in M(G)$ such that $T - \mu$ belongs to the centre of* $\mathrm{VN}(G)$.

Proof. This is Question 8.3 of [GRW]. With $\mathrm{VN}(G)$ denoting the von Neumann algebra of G (see [GRW, §1]), $M(G)$ is identified with the corresponding set of left convolution operators on $L^2(G)$ (see [D, Th. 3.3.19]) and is thus considered as a subalgebra of $\mathrm{VN}(G)$. By analogy, we also use the notation $S*T$ for multiplication in $\mathrm{VN}(G)$. Then $\mathrm{ad}_T(u) = T*u - u*T$ defines a derivation from $L^1(G)$ to $M(G)$ and (from Corollary 1.2) $\mathrm{ad}_T = \mathrm{ad}_\mu$ implies that $T - \mu$ centralizes $L^1(G)$. Since $L^1(G)$ is dense in $\mathrm{VN}(G)$ for the weak operator topology, it follows that $T - \mu$ is central. □

Remark 1.6. If G is a locally compact group with a continuous action on Ω (i.e., the mapping $G \times \Omega \to \Omega$ is jointly continuous; by the theorem of Ellis, this results from separate continuity), then Theorem 1.1 implies that bounded crossed homomorphisms from G to $M(\Omega)$ are automatically continuous for the w*-topology on $M(\Omega)$, i.e., for $\sigma(M(\Omega), C_0(\Omega))$ (since in this case the right action of G on $C_0(\Omega)$ is continuous for the norm topology). This is a counterpart to [D, Th. 5.6.34(ii)] which implies that bounded derivations from $M(G)$ to a dual module E' are automatically continuous for the strong operator topology on $M(G)$ and the w*- topology on E'. See also the end of Remark 5.6.

2. Decomposition of $M(\Omega)$

Let Ω be a left G-module as in Theorem 1.1. For $\mu, \lambda \in M(\Omega)$, singularity is denoted by $\mu \perp \lambda$, absolute continuity by $\mu \ll \lambda$, equivalence by $\mu \sim \lambda$ ($\Leftrightarrow$ $\mu \ll \lambda$ and $\lambda \ll \mu$). The measure λ is called G-invariant if $x \circ \lambda = \lambda$ for all $x \in G$. It is easy to see that the G-invariant elements form a norm-closed sublattice $M(\Omega)_{\text{inv}}$ in $M(\Omega)$ (which may be trivial). We introduce the following notation:

$$M(\Omega)_{\text{inf}} = \{\mu \in M(\Omega) : \ \mu \perp \lambda \ \text{ for all } \ \lambda \in M(\Omega)_{\text{inv}}\},$$

$$M(\Omega)_{\text{fin}} = \{\mu \in M(\Omega) : \mu \ll \lambda \ \text{ for some } \lambda \in M(\Omega)_{\text{inv}}\} \ .$$

Sometimes, we will also write $M(\Omega)_{\text{inf},G}$ and $M(\Omega)_{\text{fin},G}$ to indicate dependence on G. In the terminology of ordered vector spaces (see e.g., [Sch, §V.1.2]), $M(\Omega)_{\text{fin}}$ is the band generated by $M(\Omega)_{\text{inv}}$, and $M(\Omega)_{\text{inf}}$ is the orthogonal band to $M(\Omega)_{\text{fin}}$ (and also to $M(\Omega)_{\text{inv}}$). For spaces of measures, bands are also called L-subspaces. Since the action of G respects order and the absolute value, it follows that $M(\Omega)_{\text{inf}}$ and $M(\Omega)_{\text{fin}}$ are G-invariant. Furthermore,

$$M(\Omega) = M(\Omega)_{\text{inf}} \oplus M(\Omega)_{\text{fin}}$$

and the norm is additive with respect to this decomposition.

This gives contractive, G-invariant projections to the two parts of the sum. It follows that it will be enough to prove Theorem 1.1 separately for crossed homomorphisms with values in one of the two components.

The proof of Theorem 1.1 will be organized as follows: In Section 3, we recall some classical results. Sections 4–6 are devoted to $M(\Omega)_{\text{inf}}$ ("*infinite type*"). First (§§4, 5), we consider measures that are absolutely continuous with respect to some (finite) quasi-invariant measure. We will work with the extension of the action of G to the Stone-Čech compactification βG and in Section 5, we describe an approximation procedure which will produce the measure μ representing the crossed homomorphism (see Proposition 5.1). Then in Section 6 the general case for $M(\Omega)_{\text{inf}}$ is treated (Proposition 6.2). Finally, Section 7 covers the case $M(\Omega)_{\text{fin}}$ ("*finite type*", see Proposition 7.1). Here the behaviour of crossed homomorphisms is different and we will use weak compactness and the fixed point theorem of Section 3. As explained above, Propositions 6.2 and 7.1 will give a complete proof of Theorem 1.1. □

Remark 2.1. A similar decomposition technique has been applied in [Lo, proof of the proposition]. The distinction between finite and infinite types is related to corresponding notions for von Neumann algebras (see e.g., [T, §V.7]) and the states on these algebras ([KS]). Some proofs for Sakai's theorem (e.g., [JR]) also treat these cases separately.

In [GRW, §§5, 6], another sort of distinction was considered: for $\Omega = G$ a locally compact group with the action $x \circ y = x\,y\,x^{-1}$ (see the proof of

Corollary 1.2), they write N for the closure of the elements of G belonging to relatively compact conjugacy classes. Then Cond. 6.2 of [GRW] (which is satisfied e.g. for IN-groups or connected groups), implies that $M(G \setminus N)$ contains no nonzero G-invariant measures ($G \setminus N$ denoting the set-theoretical difference); thus $M(G \setminus N) \subseteq M(G)_{inf}$. Then ([GRW, Th. 6.8]), they showed that bounded crossed homomorphisms with values in $M(G \setminus N)$ are principal. But, as Example 2.2 below demonstrates, $M(G)_{\mathrm{inf}}$ is in general strictly larger and in Sections 4 - 6 we will extend the method of [GRW] to $M(\Omega)_{\mathrm{inf}}$.

Example 2.2. Put $\Omega = \mathbb{T}^2$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the one-dimensional torus group, $H = \mathrm{SL}(2, \mathbb{Z})$ with the action induced by the standard left action of H on $\mathbb{R}^2$. This is related to the example $G = \mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{T}^2$ discussed in [GRW], since for G (in the notation of Remark 2.1 above, putting $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$), we have $N = \{\pm I\} \ltimes \mathbb{T}^2$ (this is the maximal compact normal subgroup of G) and then $M(\Omega) \subseteq M(N)$ was a typical case left open in [GRW].

One can show (using disintegration and then unique ergodicity of irrational rotations on $\mathbb{T}$) that the extreme points of the set of H-invariant probability measures on Ω can be described as follows: put $K_0 = (0)$, $K_n = (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^2$, $K_\infty = \Omega$ (these are all the closed H-invariant subgroups of $\mathbb{T}^2$). Then the extreme points are just the normalized Haar measures of the compact groups K_n ($n = 0, 1, \ldots, \infty$) and $M(\Omega)_{\mathrm{inv}}$ is the norm-closed subspace generated by them. It follows that $\mu \in M(\Omega)_{\mathrm{fin}}$ if and only if $\mu = u + \nu$, where $u \in L^1(\mathbb{T}^2)$ (i.e., u is absolutely continuous with respect to Haar measure) and ν is an atomic measure concentrated on $(\mathbb{Q}/\mathbb{Z})^2 = \bigcup_{n \in \mathbb{N}} K_n$. Now, $\mu \in M(\Omega)_{\mathrm{inf}}$ if and only if $\mu \perp L^1(\mathbb{T}^2)$ and μ gives zero weight to all points of $(\mathbb{Q}/\mathbb{Z})^2$.

Example 2.3. Put $\Omega = \mathbb{T}$ which is now identified with the unit circle $\{v \in \mathbb{R}^2 : \|v\| = 1\}$. For $G = SL(2, \mathbb{R})$, we consider the action $A \circ v = \dfrac{Av}{\|Av\|}$. Here, although Ω is compact, there are no nonzero G-invariant measures (we consider first the orthogonal matrices in G; uniqueness of Haar measure makes the standard Lebesgue measure of $\mathbb{T}$ the only candidate, but this is not invariant under matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}$ with $\alpha \neq \pm 1$). Thus $M(\Omega) = M(\Omega)_{\mathrm{inf}}$ in this example. In [GRW] after their L. 6.3, a generalized version of their Condition 6.2 is formulated (this is slightly hidden on p. 382: "Suppose now ... "). It implies also the nonexistence of G-invariant measures, but it is applicable only for noncompact spaces Ω. The present example shows that the condition of [GRW] does not cover all actions without invariant measures. Of course (using the Iwasawa decomposition), Ω can be identified with the (left) coset space of G by the subgroup $\left\{\begin{pmatrix} \alpha & \beta \\ 0 & \frac{1}{\alpha} \end{pmatrix} : \alpha > 0,\, \beta \in \mathbb{R}\right\}$, with the action induced by left translation. Hence this is related to the semi-simple Lie group case and the methods of [J1, Prop. 4.3] (which were developed further in [J2]) apply. This amounts to consideration first of the restricted action on an appropriate subgroup, for example $\left\{\begin{pmatrix}\alpha & 0\\ 0 & \frac{1}{\alpha}\end{pmatrix} : \alpha > 0\right\}$ (see also the Remarks 4.3(a) and 5.6).

Further notation. Note that e will always mean the unit element of a group G. If G is a locally compact group, $L^1(G)$, $L^\infty(G)$ are defined with respect to a fixed left Haar measure on G. Duality between Banach spaces is denoted by $\langle\ \rangle$; thus for $f \in L^\infty(G)$, $u \in L^1(G)$, we have $\langle f, u\rangle = \int_G f(x)\,u(x)\,dx$. We write $\mathbf{1}$ for the constant function of value one.

3. Some classical results

For completeness, we collect here some results (and fix notation) for Banach spaces of measures and describe a fixed point theorem that will be used in the following sections.

All the elements of $M(\Omega)$ are countably additive set functions on $\mathcal{B}$ (the Borel sets of Ω). For a nonnegative $\lambda \in M(\Omega)$ (we write $\lambda \geq 0$), $L^1(\Omega, \lambda)$ is considered as a subset of $M(\Omega)$ in the usual way (see e.g., [D, App. A]).

Result 3.1 (Dunford-Pettis criterion). *Assume that $\lambda \in M(\Omega)$, $\lambda \geq 0$. A subset K of $L^1(\Omega, \lambda)$ is weakly relatively compact* (*i.e., for* $\sigma(L^1, L^\infty)$) *if and only if K is bounded and the measures in K are uniformly λ-continuous*; *this means explicitly*:

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 : \ A \in \mathcal{B},\ \lambda(A) < \delta \text{ implies } |\mu(A)| < \varepsilon \text{ for all } \mu \in K.$$

Be aware that weak topologies are always meant in the functional analytic sense ([DS, Def. A.3.15]). This is different from probabilistic terminology (where "weak convergence of measures" usually refers to $\sigma(M(\Omega), C_b(\Omega))$ and "vague convergence" to $\sigma(M(\Omega), C_0(\Omega))$, i.e., to the w*-topology). Recall that weak topologies are hereditary for subspaces (an easy consequence of the Hahn-Banach theorem; see e.g. [Sch, IV.4.1, Cor. 2]), thus $\sigma(M(\Omega), M(\Omega)')$ induces $\sigma(L^1, L^\infty)$ on $L^1(\Omega, \lambda)$. By [DS, Th. IV.9.2] this characterizes, also, weakly relatively compact subsets in $M(\Omega)$. Furthermore, by standard topological results ([D, Prop. A.1.7]), if K is as above, the weak closure $\overline{K}$ of such a set is w*-compact as well, i.e., for $\sigma(M(\Omega), C_0(\Omega))$.

Proof [DS, p. 387] (Dieudonné's version). Observe that if $\lambda(\{\omega\}) = 0$ for all ω, then (since λ is finite) uniform λ-continuity implies that K is bounded. □

In addition, we will consider *finitely additive* measures. Let $ba(\Omega, \mathcal{B}, \lambda)$ denote the space of finitely additive (real- or complex-valued) measures μ on $\mathcal{B}$ such that for $A \in \mathcal{B}$, $\lambda(A) = 0$ implies $\mu(A) = 0$. These spaces investigated in

[DS, III.7], are Banach lattices; in particular, the expressions $|\mu|,\ \mu \geq 0,\ \mu_1 \perp \mu_2$ are meaningful for finitely additive measures as well. (Using abstract representation theorems for Boolean algebras, we see that all this could be reduced to countably additive measures on certain "big" compact spaces, but for our purpose, the classical viewpoint appears to be more suitable; some authors use the term "charge" to distinguish from countably additive measures; see [BB]).

RESULT 3.2. *For $\lambda \in M(\Omega)$ with $\lambda \geq 0$,*

$$L^1(\Omega,\lambda)'' \cong L^\infty(\Omega,\lambda)' \cong ba(\Omega,\mathcal{B},\lambda)\,.$$

For an indicator function c_A ($A \in \mathcal{B}$), the duality is given by $\langle \mu, c_A\rangle = \mu(A)(\,\mu \in ba(\Omega,\mathcal{B},\lambda)\,)$.

Proof [DS, Th. IV.8.16]. The result goes essentially back to Hildebrandt, Fichtenholz and Kantorovitch. In addition, it follows that the canonical embedding of $L^1(\Omega,\lambda)$ into its bidual is given by the usual correspondence between classes of integrable functions and measures. □

RESULT 3.3 (Yosida-Hewitt decomposition). *We have*

$$ba(\Omega,\mathcal{B},\lambda) \cong L^1(\Omega,\lambda) \oplus L^1(\Omega,\lambda)^\perp,$$

where $L^1(\Omega,\lambda)^\perp$ consists of the purely finitely additive measures in $ba(\Omega,\mathcal{B},\lambda)$. More explicitly, every $\mu \in ba(\Omega,\mathcal{B},\lambda)$ has a unique decomposition $\mu = \mu_a + \mu_s$ with $\mu_a \ll \lambda,\ \mu_s \perp \lambda$. Furthermore, $\|\mu\| = \|\mu_a\| + \|\mu_s\|$.

Proof. [DS, Th. III.7.8]. □

Defining $P_\lambda(\mu) = \mu_a$, gives a projection $P_\lambda \colon L^1(\Omega,\lambda)'' \to L^1(\Omega,\lambda)$ that is a left inverse to the canonical embedding.

RESULT 3.4. *For $\nu \in ba(\Omega,\mathcal{B},\lambda)$, we have $\nu \perp \lambda$ ("ν is purely finitely additive") if and only if for every $\varepsilon > 0$ there exists $A \in \mathcal{B}$ such that $\lambda(A) < \varepsilon$ and ν is concentrated on A (this means that $\nu(B) = 0$ for all $B \in \mathcal{B}$ with $B \subseteq \Omega \setminus A$; for $\nu \geq 0$, this is equivalent to $\nu(A) = \nu(\Omega)$).*

Proof. For the sake of completeness, we sketch the argument. It is rather obvious that the condition above implies singularity of ν and λ. For the converse, recall the formula for the infimum of two real measures (see e.g., [Se, Prop. 17.2.4] or [BB, Th. 2.2.1]): $(\lambda \wedge \nu)(C) = \inf\{\lambda(C_1) + \nu(C \setminus C_1) : C_1 \in \mathcal{B},\ C_1 \subseteq C\}$. We can assume that ν is real and then (using the Jordan decomposition [DS, III.1.8]) that $\nu \geq 0$. If $\lambda \wedge \nu = 0$ and $\varepsilon > 0$ is given, it follows (with $C = \Omega$) that there exist sets $A_n \in \mathcal{B}$ such that $\lambda(A_n) < \dfrac{\varepsilon}{2^n}$ and $\nu(\Omega \setminus A_n) < \dfrac{\varepsilon}{2^n}$. Put $A = \bigcup_{n=1}^\infty A_n$. Then σ-additivity of λ implies $\lambda(A) < \varepsilon$ and positivity of ν implies $\nu(\Omega \setminus A) = 0$. □

Lemma 3.5. *Let $(\mu_n)_{n=1}^{\infty}$ be a sequence in $ba(\Omega,\mathcal{B},\lambda) = L^1(\Omega,\lambda)''$ with $\mu_n \ge 0$ for all n. Assume that for some $c \ge 0$ there exist $A_n \in \mathcal{B}$ $(n = 1, 2, \dots)$ such that* $\liminf \mu_n(A_n) \ge c$ *and* $\sum_{n=1}^{\infty} \lambda(A_n) < \infty$. *Let μ be any* w*-*cluster point of the sequence (μ_n) (i.e., for $\sigma(ba(\Omega,\mathcal{B},\lambda), L^{\infty}(\Omega,\lambda))$). Then*

$$\|\mu - P_\lambda(\mu)\| \;=\; \|\mu_s\| \;\ge\; c\,.$$

Proof. Put $B_n = \bigcup_{m\ge n} A_m$. Then $\lambda(B_n) \to 0$ for $n \to \infty$ and for $m \ge n$, we have $\mu_m(B_n) \ge \mu_m(A_m)$. Since by Result 3.2, $\mu_m(B_n) = \langle \mu_m, c_{B_n}\rangle$ and c_{B_n} defines a w*-continuous functional on $ba(\Omega,\mathcal{B},\lambda)$, we conclude that $\mu(B_n) \ge c$ for all n. Since for $n \to \infty$ absolute continuity implies that $\langle P_\lambda(\mu), c_{B_n}\rangle \to 0$, we arrive at $\liminf \mu_s(B_n) \ge c$. □

Corollary 3.6. *$L^1(\Omega,\lambda)^{\perp}$ is "countably closed" for the* w*-*topology in $L^1(\Omega,\lambda)''$. This says that if C is a countable subset of $L^1(\Omega,\lambda)^{\perp}$, then its* w*-*closure $\overline{C}$ is still contained in $L^1(\Omega,\lambda)^{\perp}$.*

Proof. This is a special case of [T, Prop. III.5.8] (which is formulated for general von Neumann algebras); see also [A, Th. III.5]. If C consists of nonnegative elements, the result follows easily from Lemma 3.5. In the general case, a direct argument can be given as follows. Put $C = \{\mu_1, \mu_2, \dots\}$ (we may assume that C is infinite). By Result 3.4, there exists $A_n \in \mathcal{B}$ with $\lambda(A_n) < \frac{1}{2^n}$ such that μ_n is concentrated on A_n. As before, put $B_n = \bigcup_{m\ge n} A_m$. Then, if μ is any cluster point of the sequence (μ_n), it easily follows that μ is concentrated on B_n for all n. By Result 3.4, we obtain that $\mu \in L^1(\Omega,\lambda)^{\perp}$. □

Remark 3.7. We have chosen the term "countably closed" to distinguish from the classical notion "sequentially closed". Corollary 3.6 applies also to nets that are concentrated on a countable subset of $L^1(\Omega,\lambda)^{\perp}$, whereas the sequential closure usually restricts to convergent sequences.

It is not hard to see that $L^1(\Omega,\lambda)^{\perp}$ is w*-dense in $L^1(\Omega,\lambda)''$, unless the support $\operatorname{supp}\lambda$ has an isolated point. This demonstrates again that the w*-topology on $L^1(\Omega,\lambda)''$ is highly nonmetrizable.

Result 3.8 (Fixed point theorem). *Let X be a normed space, K a nonempty weakly compact convex subset. Assume that a group G acts by affine transformations $A(x)$ on X (i.e., $A(x)\,v = L(x)\,v + \phi(x)$ for $x \in G$, $v \in X$, where $L(x)\colon X \to X$ is linear, $\phi(x) \in X$) and that K is G-invariant. Furthermore, assume that* $\sup_{x\in G}\|L(x)\| < \infty$. *Then there exists a fixed point $v \in K$ for the action of G.*

Proof. This follows from [La, Th. p. 123] "on the property (F_2)", where the result is formulated for general locally convex spaces. For completeness, we include a direct proof, similar to that of Day's fixed point theorem (compare [Gr, p. 50]). It is enough to show the result for linear transformations $A(x)$

(otherwise, we pass to $\tilde{X} = X \times \mathbb{C}$, $\tilde{K} = K \times \{1\}$ and the usual linear extensions $\tilde{A}(x)$ of $A(x)$). For $v' \in X'$, we get a bounded linear mapping $T_{v'} : X \to l^\infty(G)$ by $T_{v'}(v)\,(x) = \langle\, v'\,,\, A(x)\, v\rangle$ for $v \in X$, $x \in G$. Then $T_{v'}(K)$ is weakly compact and $T_{v'}(v)\,(xy) = T_{v'}\big(A(y)\,v\big)\,(x)$. It follows that $T_{v'}(v)$ is a weakly, almost periodic, function on G ($T_{v'}(v) \in \mathrm{WAP}(G)$) for all $v \in K$. Let m be the invariant mean on $\mathrm{WAP}(G)$ (compare [Gr, § 3.1]). We fix $v \in K$ and define $v_0 \in X''$ by $\langle v_0, v'\rangle = m\big(T_{v'}(v)\big)$. Then $v_0 \in K$, since otherwise, the separation theorem for convex sets would give some $v' \in X'$ and $\alpha \in \mathbb{R}$ such that $\mathrm{Re}\,\langle v', w\rangle \le \alpha$ for all $w \in K$ and $\mathrm{Re}\,\langle v_0, v'\rangle > \alpha$ which contradicts the definition of v_0. Then invariance of m easily implies that $A(y)\,v_0 = v_0$ for all $y \in G$. □

Remark 3.9. This is related to Ryll-Nardzewski's fixed point theorem ([Gr, Th. A.2.2, p. 98]; in fact, the proof of the existence of an invariant mean on $\mathrm{WAP}(G)$ uses this result). Ryll-Nardzewski's fixed point theorem does not need our uniform boundedness assumption on the transformations, but it requires that the action of G be distal. Of course, as soon as one knows that a fixed point exists, one can use a translation so that the origin becomes a fixed point. Then uniform boundedness of the group of transformations $\{A(x)\}$ implies that the action has to be distal. But the assumptions above make it possible to show the existence of a fixed point without having to verify distality in advance (which appears to be a rather difficult task for the action that we consider in §7).

More generally, the proof given above works if X is any (Hausdorff) locally convex space, K is a compact convex subset of X and a group G acts on K by continuous affine transformations $A(x)$ such that the functions $T_{v'}(v)$ (defined as above) are weakly almost periodic for all $v \in K$, $v' \in X'$.

Corollary 3.10. *A measure $\mu \in M(\Omega)$ belongs to $M(\Omega)_{\mathrm{fin}}$ if and only if the orbit $\{x \circ \mu : x \in G\}$ is weakly relatively compact. Thus $M(\Omega)_{\mathrm{fin}}$ consists exactly of the* WAP*-vectors for the action of G on $M(\Omega)$.*

Proof. Assume that $\mu \ll \lambda$ for some $\lambda \in M(\Omega)_{\mathrm{inv}}$. In addition, we may suppose that $\lambda \ge 0$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $A \in \mathcal{B}$, $\lambda(A) < \delta$ implies $|\mu(A)| < \varepsilon$. Since $\lambda(A) < \delta$ implies (see also the beginning of §4)

$$\lambda(x^{-1} \circ A) = \langle c_{x^{-1}\circ A}\,,\, \lambda\,\rangle = \langle\, c_A\,, x \circ \lambda\rangle = \lambda(A) \,<\, \delta\,,$$

it follows that for all $x \in G$,

$$|x \circ \mu(A)| = |\langle\, c_A, x \circ \mu\rangle| = |\langle c_A \circ x, \mu\rangle| = |\langle c_{x^{-1}\circ A}, \mu\,\rangle| = |\mu(x^{-1} \circ A)| < \varepsilon\,.$$

Thus, by the Dunford-Pettis criterion (Result 3.1), $\{x \circ \mu : x \in G\}$ is weakly relatively compact.

For the converse, recall that $|x \circ \mu| = x \circ |\mu|$; thus (using the existence of a "control measure" for weakly compact subsets of $M(\Omega)$ – see [DS, Th. IV.9.2];

and again Result 3.1) we may assume that $\mu \geq 0$ and (using the decomposition of §2 and the part already proved) that $\mu \in M(\Omega)_{\mathrm{inf}}$. Let K be the (norm- or weakly-) closed convex hull of $\{x \circ \mu : x \in G\}$. This is convex, G-invariant and, by classical results, it is weakly compact. Thus, by the fixed point theorem (Result 3.8), there exists $\lambda \in M(\Omega)_{\mathrm{inv}}$ with $\lambda \in K$. If $\lambda \neq 0$, then since $\{\nu \in M(\Omega) : \nu \perp \lambda\}$ is norm closed, it would follow that $x \circ \mu$ is not singular to λ for some $x \in G$. But this entails that μ is not singular to λ, contradicting $\mu \in M(\Omega)_{\mathrm{inf}}$. Thus $\lambda = 0$. But by elementary arguments, $\nu(\Omega) = \mu(\Omega)$ for all $\nu \in K$ and this gives $\mu = 0$. □

4. Quasi-invariant measures

A probability measure $\lambda \in M(\Omega)$ is called *quasi-invariant*, if $x \circ \lambda \sim \lambda$ for all $x \in G$. Then $L^1(\Omega, \lambda)$ is a G-invariant L-subspace of $M(\Omega)$. Abstractly, if X is a left Banach G-module (i.e., X is a Banach space and the transformations $v \mapsto x \circ v$ are linear and bounded for each $x \in G$), then its dual X' becomes a right G-module (as in [D, (2.6.4), p. 240]). By an easy computation, it follows that the right G-action on $L^\infty(\Omega, \lambda)$ ($\cong L^1(\Omega, \lambda)'$) is given by the same formula as that on $C_0(\Omega)$ (see the beginning of §1). In a similar way, the space of bounded Borel measurable functions on Ω can be embedded into $M(\Omega)'$ (see [D, Prop. 4.2.30]) and on this subspace the formula for the dual action of G is the same (this was used in the proof of Corollary 3.10).

Recall that βG (the Stone-Čech compactification of the discrete group G) can be made into a right topological semigroup (extending the multiplication of G; see [HS, Ch. 4]).

Lemma 4.1. *Let X be a left Banach G-module for which the action of G is uniformly bounded.*

(a) *The bidual X'' becomes a left βG-module, extending the action of G on X and such that for every fixed $x \in G$ the mapping $v \mapsto x \circ v$ is* w**-continuous on X'' and for every fixed $v \in X''$, the mapping $p \mapsto p \circ v$ is continuous from βG to X'' (with* w**-topology $\sigma(X'', X')$).*

(b) *Any bounded crossed homomorphism $\Phi \colon G \to X$ extends (uniquely) to a continuous crossed homomorphism from βG to X'' (with* w**-topology).*

This extension will be denoted by the same letter, Φ.

Proof. (a) can be proved as in [D, Th. 2.6.15] (see also [HS, Th. 4.8]). In fact, as an alternative definition, the product on βG can be obtained by restriction of the first Arens product $\Box$ on $l^1(G)''$. Similarly for (b), crossed homomorphisms on semigroups can be defined by the same functional equation as in the group case. □

LEMMA 4.2. *Assume that $\lambda \in M(\Omega)_{\mathrm{inf}}$ is a quasi-invariant probability measure. Then there exists $p \in \beta G$ such that $p \circ f \in L^1(\Omega,\lambda)^\perp$ for all $f \in L^1(\Omega,\lambda)$.*

Proof. It is easy to see that $f \geq 0$ implies $p \circ f \geq 0$; consequently, it will be enough to verify the property of p for a single $f \in L^1(\Omega,\lambda)$ such that $f(\omega) > 0$, λ- a.e. (indeed, if $p \circ f \in L^1(\Omega,\lambda)^\perp$, then by positivity, $p \circ (h\, f) \in L^1(\Omega,\lambda)^\perp$ for $h \in L^\infty$ with $0 \leq h \leq \mathbf{1}$ and by elementary measure theory, the set of these products $h\,f$ generates a norm dense subspace of $L^1(\Omega,\lambda)$). We take the constant function $f = \mathbf{1}$.

We argue by contradiction and assume that $P_\lambda(p \circ \mathbf{1}) \neq 0$ for all $p \in \beta G$ (P_λ denoting the projection to $L^1(\Omega,\lambda)$ defined after Result 3.3). Put $c = \inf_{p\in\beta G}\|P_\lambda(p \circ \mathbf{1})\|$. The first step is to show that the infimum is actually attained at some point $p_0 \in \beta G$ (in particular, our assumption then implies that $c > 0$).

Choose a sequence $(p_n)_{n\geq 1}$ in βG such that $\|P_\lambda(p_n \circ \mathbf{1})\|$ tends to c. Let $p_0 = \lim p_{n_i} \in \beta G$ be a cluster point, obtained as limit of a net refining the sequence. By Lemma 4.1(a), we have $p_0 \circ \mathbf{1} = \text{w*-}\lim p_{n_i} \circ \mathbf{1}$. Then let $w \in L^1(\Omega,\lambda)''$ be a w*-cluster point of the bounded net $\big(P_\lambda(p_{n_i} \circ \mathbf{1})\big)$. By Corollary 3.6, $p_0 \circ \mathbf{1} - w$ (being the w*-limit of a further refinement of the net $\big(p_{n_i} \circ \mathbf{1} - P_\lambda(p_{n_i} \circ \mathbf{1})\big)$ which is concentrated on a countable subset of $L^1(\Omega,\lambda)^\perp$) belongs to $L^1(\Omega,\lambda)^\perp$. Thus $P_\lambda(p_0 \circ \mathbf{1}) = P_\lambda(w)$. Lower semicontinuity of the norm implies $\|w\| \leq c$, from which we get $\|P_\lambda(p_0 \circ \mathbf{1})\| = c$.

Put $g = P_\lambda(p_0 \circ \mathbf{1})$. We claim that $\{x \circ g : x \in G\}$ should be relatively weakly compact (then by Corollary 3.10, this will imply $g \in M(\Omega)_{\mathrm{fin}}$, resulting in a contradiction to $\lambda \in M(\Omega)_{\mathrm{inf}}$ and $c > 0$).

The claim will again be proved by contradiction. An equivalent condition to weak relative compactness of the set $\{x \circ g : x \in G\}$ is that the w*-closure of this set in the bidual $L^1(\Omega,\lambda)''$ is contained in $L^1(\Omega,\lambda)$. Thus we assume that this set has a w*-cluster point $w \in L^1(\Omega,\lambda)''$ with $w \notin L^1(\Omega,\lambda)$. Put $w_0 = w - P_\lambda(w)$, $c_0 = \|w_0\|$. Then $w_0 \perp L^1(\Omega,\lambda)$, $c_0 > 0$. Observe that $g, w, P_\lambda(w), w_0 \geq 0$. By Result 3.4, there exists $A_n \in \mathcal{B}$ with $\lambda(A_n) < \frac{1}{2^n}$, $\langle w_0, c_{A_n}\rangle = c_0$. Then $P_\lambda(w) \geq 0$ implies $\langle w, c_{A_n}\rangle \geq c_0$, consequently, there exists $x_n \in G$ such that

$$\big\langle x_n \circ g,\, c_{A_n} \big\rangle \;>\; c_0 - \frac{1}{n} \qquad (n = 1, 2, \dots)\,.$$

Let $q \in \beta G$ be a cluster point of the sequence (x_n) and put $w' = q \circ g$. Then Lemma 3.5 implies $\|w' - P_\lambda(w')\| \geq c_0$ (put $\mu_n = x_n \circ g$, considered as a countably additive measure on Ω; then by Lemma 4.1(a), w' is a w*-cluster point of (μ_n)). By Result 3.3, we have $\|w'\| = \|P_\lambda(w')\| + \|w' - P_\lambda(w')\|$ and this gives $\|P_\lambda(w')\| \leq \|w'\| - c_0$. Note that $x_n \circ (p_0 \circ \mathbf{1}) = x_n \circ g + x_n \circ (p_0 \circ \mathbf{1} - g)$ and the second part of this sum belongs to $L^1(\Omega,\lambda)^\perp$. As before, it follows that $P_\lambda\big(q \circ (p_0 \circ \mathbf{1})\big) = P_\lambda(q \circ g) = P_\lambda(w')$ and this would imply (making use of the semigroup structure of βG)

$$\|P_\lambda(\, qp_0 \circ \mathbf{1}\,)\| \;=\; \|P_\lambda(w')\| \;\le\; \|w'\| - c_0 \;\le\; c - c_0\,,$$

contradicting the definition of c. This proves our claim and, as explained above, completes the proof of Lemma 4.2. □

Remark 4.3. (a) There are numerous examples of transformation groups that admit a quasi-invariant probability measure but no finite invariant measure (see also §6). An easy example is $\Omega = \mathbb{R}$ with $G = \mathbb{R}_d$ (i.e., $\mathbb{R}$ with discrete topology) acting by $x \circ y = x + y$. Then any measure λ that is equivalent to standard Lebesgue measure will be quasi-invariant. $\beta\mathbb{R}_d$ maps continuously to the compactification $[-\infty, \infty]$ of $\mathbb{R}$. It is not hard to see that any $p \in \beta\mathbb{R}_d$ lying above $\pm\infty$ has the property that $p \circ L^1(\Omega, \lambda) \subseteq L^1(\Omega, \lambda)^\perp$ (intuitively speaking: functions are "shifted out to infinity").

In Example 2.3, the standard Lebesgue measure λ is quasi-invariant (but not invariant) for the action of G. Put $H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} : \alpha > 0 \right\}$ $(\cong]0, \infty[\,)$. Note that βH_d maps continuously to the compactification $[0, \infty]$ of $]0, \infty[$. It is not hard to see that any $p \in \beta H_d$ lying above $0, \infty$ has the property that $p \circ L^1(\Omega, \lambda) \subseteq L^1(\Omega, \lambda)^\perp$. If p lies above ∞, we obtain for $p \circ \mathbf{1}$ a finitely additive measure on Ω that projects (by restricting the functional to continuous functions) to $\frac{1}{2}(\delta_{\binom{1}{0}} + \delta_{\binom{-1}{0}})$ (which is an H-invariant measure). Hence this Example shows another interpretation of "infinity".

(b) The case of quasi-invariant measures is used as an intermediate step in the proof of the infinite case (Proposition 6.2). Quasi-invariance of λ is a necessary condition for G- invariance of $L^1(\Omega, \lambda)$. Of course, there are always the actions of G on $M(\Omega)$ and that of βG on $M(\Omega)''$ defined by Lemma 4.1. But without quasi-invariance, one cannot guarantee that for $p \in \beta G$ and $f \in L^1(\Omega, \lambda)$ the element $p \circ f$ belongs to the subspace $L^1(\Omega, \lambda)''$ of $M(\Omega)''$. Working with general elements of $M(\Omega)''$ (rather than $ba(\Omega, \mathcal{B}, \lambda)$) would make the argument considerably more abstract. In the examples of (a), it is possible to choose $p \in \beta G$ so that $p \circ M(\Omega) \subseteq M(\Omega)^\perp$, but it is not clear if this can be done in general (for the infinite part of the action; see also Remark 5.6).

(c) If G is a locally compact group and G_d denotes the group with discrete topology, then βG_d maps continuously to βG. If the action of G on X is uniformly bounded and continuous (i.e., $x \mapsto x \circ v$ is continuous for each $v \in X$), then it is easy to see that $p \circ v$ depends for $v \in X$ only on the image of $p \in \beta G_d$ in βG. Thus $p \circ v$ is well defined for $p \in \beta G$. This applies in particular to the action of G on $L^1(\Omega, \lambda)$ when we have a continuous action of G on Ω as in Remark 1.6. Thus, in the two examples above, we might have said as well that $p \circ L^1(\Omega, \lambda) \subseteq L^1(\Omega, \lambda)^\perp$ for $p \in \beta\mathbb{R} \setminus \mathbb{R}$ (resp., $p \in \beta H \setminus H$).

The technical problem is that in general βG cannot be made into a semigroup in a reasonable way (see [HS, Th. 21.47]); furthermore, $p \circ v$ cannot be defined in the same way for $v \in X''$, i.e., one cannot speak of an "action" of βG on X''. Therefore we are restricted to the discrete case.

5. The approximation procedure

We will generalize now the approach developed by G. Willis in Section 6 of [GRW] for bounded crossed homomorphisms with values in $M(G \setminus N)$ (see Remark 2.1); similar ideas were used in [J2] and earlier in [J1, p. 51ff]. The main result is Proposition 5.1 which extends Theorem 6.8 of [GRW]. Technically, the main difference is to replace convergence to the "ideal point ∞" as defined in [GRW, p. 380], by consideration instead of the extended crossed homomorphism (Lemma 4.1(b)) at some point $p \in \beta G$ satisfying the property of Lemma 4.2.

PROPOSITION 5.1. *Assume that $\lambda \in M(\Omega)_{\text{inf}}$ is a quasi-invariant probability measure and that $p \in \beta G$ satisfies $p \circ L^1(\Omega, \lambda) \subseteq L^1(\Omega, \lambda)^{\perp}$. For a bounded crossed homomorphism $\Phi\colon G \to L^1(\Omega, \lambda)$ put $u = P_\lambda\big(\Phi(p)\big)$. Then*

$$u \in L^1(\Omega, \lambda)\,, \quad \|u\| \,=\, \frac{1}{2}\,\|\Phi\| \,=\, \frac{1}{2}\lim_{x \to p} \|\Phi(x)\| \qquad \textit{and}$$

$$\Phi(x) = u - x \circ u \qquad \textit{for all } x \in G \qquad (\textit{thus } \Phi \textit{ is principal}).$$

Let P_λ denote the projection $L^1(\Omega, \lambda)'' \to L^1(\Omega, \lambda)$ defined after Result 3.3. The proof will be given at the end of the section after several lemmas. The structure follows closely that of [GRW, §6]. The basic strategy is to study Φ at those points x where $\|\Phi(x)\|$ comes close to $\|\Phi\|$. Throughout this section, we fix $p \in \beta G$ given by Lemma 4.2 and we make the convention that in expressions of the type $\lim\limits_{x \to p} F(x)$, where F is some function, x shall always be restricted to elements of G (e.g., in Proposition 5.1 and Lemma 5.2, we do not claim that $\|\Phi\| = \|\Phi(p)\|$).

LEMMA 5.2 (see [GRW, L. 6.4]). $\|\Phi\| \,=\, \lim\limits_{y \to p} \|\Phi(y)\|$.

Proof. Consider $\varepsilon > 0$ and take some $x_0 \in G$ with

$$\|\Phi(x_0)\| > \|\Phi\| - \varepsilon\,. \tag{1}$$

Put $f = |\Phi(x_0)|$. Then $x_0^{-1} \circ f \in L^1(\Omega, \lambda)\,,\ \|x_0^{-1} \circ f\| = \|\Phi(x_0)\|$ and $p \circ f \in L^1(\Omega, \lambda)^{\perp}$. By Result 3.4, there exists $B \in \mathcal{B}$ such that

$$\langle\, p \circ f\,,\, c_B \,\rangle \,=\, 0 \tag{2}$$

and

$$\langle\, x_0^{-1} \circ f \,,\, c_B \,\rangle \;>\; \|\Phi(x_0)\| - \varepsilon \underset{(1)}{>} \|\Phi\| - 2\varepsilon\,. \tag{3}$$

Thus,

$$\langle\, x_0^{-1} \circ f \,,\, c_{\Omega\setminus B} \,\rangle \;<\; 2\varepsilon\,. \tag{4}$$

The defining equation for crossed homomorphisms implies that for all $y \in G$ we have

$$\Phi(x_0 y x_0) \;=\; \Phi(x_0) + x_0 \circ \Phi(y) + x_0 y \circ \Phi(x_0)\,.$$

This gives (since G acts isometrically on $L^1(\Omega,\lambda)$)

$$\|\Phi\| \;\geq\; \|\Phi(x_0 y x_0)\| \;\geq\; \|\, x_0^{-1} \circ \Phi(x_0) + y \circ \Phi(x_0)\,\| - \|\Phi(y)\|\,. \tag{5}$$

Observe that by Lemma 4.1(a) and (2),

$$\lim_{y\to p} \langle\, y \circ f \,,\, c_B \,\rangle \;=\; \langle\, p \circ f \,,\, c_B \,\rangle \;=\; 0\,.$$

Consequently, there exists a neighbourhood U of p such that

$$\langle\, y \circ |\Phi(x_0)| \,,\, c_B \,\rangle \;<\; \varepsilon \qquad \text{for all } y \in U\,. \tag{6}$$

This implies that for $y \in U \cap G$, we have

$$\langle\, y \circ |\Phi(x_0)| \,,\, c_{\Omega\setminus B} \,\rangle \;=\; \|\Phi(x_0)\| - \langle\, y \circ |\Phi(x_0)| \,,\, c_B \,\rangle \underset{(1)}{>} \|\Phi\| - 2\varepsilon\,. \tag{7}$$

Decomposition of the integral defining the L^1-norm into the domains B and $\Omega \setminus B$ gives

$$\begin{aligned}
\|\, x_0^{-1} \circ \Phi(x_0) \;+\; y \circ \Phi(x_0)\,\| \quad &\geq \quad \langle\, x_0^{-1} \circ |\Phi(x_0)| - y \circ |\Phi(x_0)|, c_B \,\rangle \\
&\qquad + \langle\, y \circ |\Phi(x_0)| - x_0^{-1} \circ |\Phi(x_0)|, c_{\Omega\setminus B} \,\rangle \\
&\underset{(3),(6),(7),(4)}{\geq} \|\Phi\| - 2\varepsilon - \varepsilon + \|\Phi\| - 2\varepsilon - 2\varepsilon \\
&= \quad 2\,\|\Phi\| - 7\varepsilon.
\end{aligned}$$

Combined with (5), this yields $\|\Phi(y)\| > \|\Phi\| - 8\varepsilon$ for all $y \in U \cap G$. □

LEMMA 5.3. *Take $B \in \mathcal{B}$ and $\varepsilon > 0$.*

(a) *Assume that $x, z \in G$ satisfy the conditions*

$$\langle\, |\Phi(x)| \,,\, c_B \,\rangle > \|\Phi\| - \varepsilon \quad \textit{and} \quad \|\Phi(z)\| > \|\Phi\| - \varepsilon\,.$$

Then $\quad \langle\, |\Phi(z)| \,,\, c_B \,\rangle > \dfrac{\|\Phi\|}{2} - 2\varepsilon\,.$

(b) *In addition to* (a), *assume that the condition $\langle\, z \circ |\Phi(x)| \,,\, c_B \,\rangle < \varepsilon$ holds. Then $\langle\, |\Phi(z)| \,,\, c_B \,\rangle < \dfrac{\|\Phi\|}{2} + 2\varepsilon$.*

Proof (*compare* [GRW, L. 6.5]). For (a), assume that $\langle\, |\Phi(z)|\,,\, c_B\,\rangle \leq \dfrac{\|\Phi\|}{2} - 2\varepsilon$. Then, by the conditions of (a),

$$\langle\, |\Phi(x) - \Phi(z)|, c_B\rangle > \frac{\|\Phi\|}{2} + \varepsilon, \qquad \langle\, |\Phi(z)|, c_{\Omega\setminus B}\rangle > \frac{\|\Phi\|}{2} + \varepsilon\,,$$

and furthermore $\langle\, |\Phi(x)|, c_{\Omega\setminus B}\rangle < \varepsilon$.

This implies $\langle\, |\Phi(x) - \Phi(z)|, c_{\Omega\setminus B}\,\rangle > \dfrac{\|\Phi\|}{2}$ and then $\|\Phi(x) - \Phi(z)\| > \|\Phi\| + \varepsilon$. But, since $\Phi(x) - \Phi(z) = z \circ \Phi(z^{-1}x)$, this is a contradiction.

For (b), assume that $\langle\, |\Phi(z)|\,,\, c_B\,\rangle \geq \dfrac{\|\Phi\|}{2} + 2\varepsilon$. Using $\Phi(zx) = \Phi(z) + z \circ \Phi(x)$, the condition of (b) implies $\langle\, |\Phi(zx)|\,,\, c_B\,\rangle > \dfrac{\|\Phi\|}{2} + \varepsilon$. Furthermore, the assumption gives $\langle\, |\Phi(z)|\,,\, c_{\Omega\setminus B}\,\rangle \leq \dfrac{\|\Phi\|}{2} - 2\varepsilon$ and (since the condition of (a) implies $\|\Phi(x)\| > \|\Phi\| - \varepsilon\,$), we have $\langle\, z \circ |\Phi(x)|\,,\, c_{\Omega\setminus B}\,\rangle > \|\Phi\| - 2\varepsilon$. This entails $\langle\, |\Phi(zx)|\,,\, c_{\Omega\setminus B}\,\rangle > \dfrac{\|\Phi\|}{2}$ and combined, we get $\|\Phi(zx)\| > \|\Phi\| + \varepsilon$, a contradiction. ☐

Corollary 5.4 (compare[GRW, L. 6.5]). *Assume that $x \in G$, $B \in \mathcal{B}$, $\varepsilon > 0$ are given such that*

$$\|\,\Phi(x)\, c_B\| > \|\Phi\| - \varepsilon \quad \text{and} \quad \langle\, p \circ |\,\Phi(x)\,|\,,\, c_B\,\rangle = 0\,.$$

Then there exists a neighbourhood U of p such that

$$\frac{\|\Phi\|}{2} - 2\varepsilon \,<\, \|\,\Phi(z)\, c_B\| \,<\, \frac{\|\Phi\|}{2} + 2\varepsilon \qquad \text{for all} \quad z \in U \cap G\,.$$

Proof. By Lemma 5.2 and Lemma 4.1 (a), the conditions of Lemma 5.3 are satisfied when $z \in G$ is sufficiently close to p. ☐

Lemma 5.5 (compare [GRW, L. 6.6]). *Assume that $B \in \mathcal{B}$ satisfies the condition $\langle\, p \circ \mathbf{1}\,,\, c_B\,\rangle = 0$. Then $\big(\Phi(x)\, c_B\big)$ is a Cauchy-net in $L^1(\Omega, \lambda)$ for $x \to p$. More explicitly*: $\forall\, \varepsilon > 0$, $\exists\, U$ *a neighbourhood of p such that*

$$\big\|\big(\Phi(x) - \Phi(y)\big)\, c_B\big\| < \varepsilon \;\forall\, x, y \in U \cap G.$$

Proof. Fix $\varepsilon > 0$ and take $x_0 \in G$ such that $\|\Phi(x_0)\| > \|\Phi\| - \dfrac{\varepsilon}{24}$. By Result 3.4, there exists $B_1 \in \mathcal{B}$ with $B_1 \supseteq B$, satisfying

$$\|\,\Phi(x_0)\, c_{B_1}\| > \|\Phi\| - \frac{\varepsilon}{24}\,, \qquad \langle\, p \circ \mathbf{1}\,,\, c_{B_1}\,\rangle = 0\,.$$

Note that this implies $\langle\, p \circ |\Phi(x_0)|\,,\, c_{B_1}\,\rangle = 0$ (see the beginning of the proof of Lemma 4.2). By Corollary 5.4 and Lemma 5.2 there exists a neighbourhood

U_1 of p such that

$$\frac{\|\Phi\|}{2}-\frac{\varepsilon}{12} < \|\,\Phi(z)\,c_{B_1}\| < \frac{\|\Phi\|}{2}+\frac{\varepsilon}{12} \qquad \text{and} \tag{8}$$

$$\|\Phi(z)\| > \|\Phi\|-\frac{\varepsilon}{24} \qquad \text{for all } z\in U_1\cap G\,. \tag{9}$$

Fix some $z\in U_1\cap G$. Then (repeating the argument with z, B_1 instead of x_0, B) there exists $B_2\in\mathcal{B}$ with $B_2\supseteq B_1$, satisfying

$$\|\,\Phi(z)\,c_{B_2}\| > \|\Phi\|-\frac{\varepsilon}{24} \quad \text{and} \quad \langle\, p\circ f\,,\, c_{B_2}\,\rangle=0\,. \tag{10}$$

Finally, we get a neighbourhood U_2 of p, contained in U_1 and such that

$$\frac{\|\Phi\|}{2}-\frac{\varepsilon}{12} < \|\,\Phi(x)\,c_{B_2}\| < \frac{\|\Phi\|}{2}+\frac{\varepsilon}{12} \qquad \text{for all} \quad x\in U_2\cap G\,. \tag{11}$$

Note that in combination with (8), this implies

$$\|\,\Phi(x)\,c_{B_2\setminus B_1}\| < 2\cdot\frac{\varepsilon}{12} = \frac{\varepsilon}{6}\,. \tag{12}$$

This gives

$$\begin{aligned}\big\|\big(\Phi(x)-\Phi(z)\big)\,c_{\Omega\setminus B_2}\big\| &\geq \|\,\Phi(x)\,c_{\Omega\setminus B_2}\|-\|\,\Phi(z)c_{\Omega\setminus B_2}\|\\ &\underset{(9),(11),(10)}{\geq} \|\Phi\|-\frac{\varepsilon}{24}-\Big(\frac{\|\Phi\|}{2}+\frac{\varepsilon}{12}\Big)-\frac{\varepsilon}{24} = \frac{\|\Phi\|}{2}-\frac{\varepsilon}{6}\,,\end{aligned}$$

and

$$\begin{aligned}\big\|\big(\Phi(x)-\Phi(z)\big)\,c_{B_2\setminus B_1}\big\| &\geq \|\,\Phi(z)\,c_{B_2\setminus B_1}\|-\|\,\Phi(x)\,c_{B_2\setminus B_1}\|\\ &\underset{(10),(8),(12)}{\geq} \|\Phi\|-\frac{\varepsilon}{24}-\Big(\frac{\|\Phi\|}{2}+\frac{\varepsilon}{12}\Big)-\frac{\varepsilon}{6} = \frac{\|\Phi\|}{2}-\frac{7\varepsilon}{24}\,.\end{aligned}$$

Since (see the proof of Lemma 5.3 (a)), $\|\Phi(x)-\Phi(z)\|\leq\|\Phi\|$, we get in combination

$$\begin{aligned}\big\|\big(\Phi(x)-\Phi(z)\big)\,c_B\big\| &\leq \big\|\big(\Phi(x)-\Phi(z)\big)\,c_{B_1}\big\|\\ &\leq \|\Phi\|-\Big(\frac{\|\Phi\|}{2}+\frac{\varepsilon}{6}\Big)-\Big(\frac{\|\Phi\|}{2}+\frac{7\varepsilon}{24}\Big)\\ &=\frac{11\varepsilon}{24} < \frac{\varepsilon}{2} \qquad \text{for all} \quad x\in U_2\cap G\,.\end{aligned}$$

This leads to $\big\|\big(\Phi(x)-\Phi(y)\big)\,c_B\big\|<\varepsilon$ for all $x,y\in U_2\cap G$. □

Proof of Proposition 5.1. For $B\in\mathcal{B}$ with $\langle\, p\circ\mathbf{1}\,,\, c_B\,\rangle=0$ put

$$u_B=\lim_{x\to p}\,\Phi(x)\,c_B \qquad \text{(in the norm topology)} \tag{13}$$

which defines an element of $L^1(\Omega,\lambda)$ by Lemma 5.5. If $B_1\in\mathcal{B}$ is a subset of B with $\big\langle\,|\,\Phi(p)-P_\lambda(\Phi(p))\,|\,,\, c_{B_1}\,\big\rangle=0$, then

$$\langle u_B, c_{B_1}\rangle=\lim_{x\to p}\,\langle\Phi(x), c_{B_1}\rangle=\langle\Phi(p), c_{B_1}\rangle=\langle P_\lambda(\Phi(p))\,,\, c_{B_1}\,\rangle\,.$$

The set of all c_{B_1}, with B_1 as above, generates (by Result 3.4) a w*–dense subspace of $L^\infty(B)$ (i.e., for $\sigma(L^\infty, L^1)$). Thus, we conclude that

$$u_B = P_\lambda(\Phi(p))\, c_B \qquad \text{for all } B \in \mathcal{B} \text{ with } \langle p \circ \mathbf{1}, c_B \rangle = 0\,. \tag{14}$$

From Corollary 5.4 and (13), we get $\|u_B\| = \lim_{x\to p} \|\Phi(x)\, c_B\| \le \frac{\|\Phi\|}{2}$ for all B as above (the first condition of Corollary 5.4 can always be enforced by temporarily enlarging the set B). Furthermore (again by Corollary 5.4), for any $\varepsilon > 0$ there exists B as above such that $\lim_{x\to p} \|\Phi(x)c_B\| \ge \frac{\|\Phi\|}{2} - 2\varepsilon$, thus $\|u_B\| \ge \frac{\|\Phi\|}{2} - 2\varepsilon$. Combining this with (14), we get

$$\| P_\lambda(\Phi(p)) \| = \frac{\|\Phi\|}{2} \tag{15}$$

for any bounded crossed homomorphism $\Phi\colon G \to L^1(\Omega, \lambda)$.

Now, put $u = P_\lambda(\Phi(p))$, $\Phi_1(x) = u - x \circ u$, $\Phi_2(x) = \Phi(x) - \Phi_1(x)$ $(x \in G)$. It is easy to see that $\Phi_1, \Phi_2 : G \to L^1(\Omega, \lambda)$ are bounded crossed homomorphisms, $\Phi_1(p) = u - p \circ u$ (by Lemma 4.1); hence (since $p \circ u \in L^1(\Omega, \lambda)^\perp$) we get $P_\lambda(\Phi_1(p)) = u$, $\quad P_\lambda(\Phi_2(p)) = 0$. Applying (15) to Φ_2, we see that this implies $\|\Phi_2\| = 0$; thus $\Phi = \Phi_1$. $\square$

Remark 5.6. The element $u \in L^1(\Omega, \lambda)$ such that $\Phi(x) = u - x \circ u$ is uniquely determined ($\lambda \in M(\Omega)_{\text{inf}}$ implies that $L^1(\Omega, \lambda) \subseteq M(\Omega)_{\text{inf}}$; u' defines the same crossed homomorphism Φ if and only if $u - u' \in L^1(\Omega, \lambda) \cap M(\Omega)_{\text{inv}} = (0)$).

Note that p just depends on λ and not on the particular crossed homomorphism Φ. Put $W = \{h \in L^\infty(\Omega, \lambda) : \langle p \circ \mathbf{1}, |h| \rangle = 0\}$. The condition defining W is equivalent to w*-$\lim_{x\to p} |h| \circ x = 0$ (in the definition of W, one can replace the constant function $\mathbf{1}$ by any function $f \in L^1(\Omega, \lambda)$ such that $f(\omega) > 0$ λ-a.e.; see the beginning of the proof of Lemma 4.2). It is not hard to see that W is a (proper) norm-closed, w*-dense subspace of $L^\infty(\Omega, \lambda)$ and an ideal. It follows from the arguments in the proof of Proposition 5.1 that $u = \sigma(L^1, W)$-$\lim_{x\to p} \Phi(x)$ and for pointwise products, one has even $u\,h = \|\ \|_1$-$\lim_{x\to p} \Phi(x)h$ for all $h \in W$; in particular, convergence of $\Phi(x)$ holds in λ-measure as well ([DS, Def. III.2.6]). But observe that $\sigma((L^1)'', L^\infty)$-$\lim_{x\to p} \Phi(x) = \Phi(p)$; thus convergence of $\Phi(x)$ to $u = P_\lambda(\Phi(p))$ cannot take place in general for the weak topology (i.e., $\sigma(L^1, L^\infty)$; in particular, weak convergence is impossible if u is nonnegative and nonzero). Intuitively: half of the mass of $\Phi(x)$ drifts to infinity, the "location of infinity" being determined by W.

In the first example of Remark 4.3(a), W contains all compactly supported functions in $L^\infty(\mathbb{R}, \lambda)$. If W contains all the functions of compact support, one can say that $\Phi(x)$ converges to u in the sense of w*-convergence of measures (i.e., for $\sigma(M(\Omega), C_0(\Omega))$). But even this need not be true in general. Consider Example 2.2. Let Ω_0 be a (countable) $\mathrm{SL}(2,\mathbb{Z})$-orbit in $\mathbb{T}^2$ consisting of irrational points and choose an (atomic) probability measure λ on Ω_0 giving nonzero weight to each of its points. Clearly, λ is quasi-invariant and, by our discussion in Example 2.2, it belongs to $M(\Omega)_{\rm inf}$. Similarly as above, it follows from compactness of Ω that w*-convergence of $\Phi(x)$ to u is impossible whenever $u \in L^1(\Omega, \lambda)$ is nonnegative and nonzero (there is a canonical w*-continuous projection of $L^\infty(\Omega,\lambda)'$ to $M(\Omega)$, given by the dual of the embedding $C_0(\Omega) \to L^\infty(\Omega, \lambda)$. In this example the image of $p \circ u \in L^1(\Omega,\lambda)^\perp$ in $M(\Omega)$ is nonzero; thus $\sigma(M(\Omega), C_0(\Omega))\text{-}\lim_{x\to p} \Phi(x)$ exists, but it is different from u).

In the setting of [GRW, §6] (see our Remark 2.1), Condition 6.2 of [GRW] makes it possible always to choose p so that W contains the functions of compact support. One even gets a slightly stronger conclusion. Explicitly, if p is some cluster point of the filter base $\mathcal{W}$ defined as in [GRW, after L. 6.3], then their Condition 6.2 implies (considering now the G-module $M(G\setminus N)$) that $p\circ\mu$ belongs to $M(G\setminus N)^\perp$ ($\subseteq M(G\setminus N)''$) for each $\mu \in M(G\setminus N)$. It follows from Theorem 6.8 of [GRW] that for each bounded crossed homomorphism $\Phi\colon G \to M(G\setminus N)$, one has $\Phi = \Phi_\mu$, with $\mu = w^* - \lim_{x\to p} \Phi(x) (\in M(G\setminus N))$. Furthermore ([GRW, L. 6.7]), $\Phi(x)\, c_B$ converges in norm to $c_B\mu$ (when $x \to p$) for any relatively compact Borel set B, similarly under the generalized version of their Condition 6.2, described after L. 6.3 of [GRW]. This does not need a quasi-invariant measure controlling the range of Φ.

In the presence of a quasi-invariant probability measure λ, one can also give a characterization of infiniteness of λ in the style of Condition 6.2 of [GRW]: $\lambda \in M(\Omega)_{\rm inf}$ if and only if there exists an ideal $\mathcal{K}$ of compact subsets of Ω such that $\sup_{K\in\mathcal{K}} \lambda(K) = 1$ and for each $K \in \mathcal{K}$ and each $\varepsilon > 0$ there exists $x \in G$ satisfying $\lambda(x\circ K) < \varepsilon$ (it is clear that this excludes the existence of an invariant measure that is absolutely continuous with respect to λ; for the converse, take $p \in \beta G$ as in Lemma 4.2, $\mathcal{K} = \{K : \langle p\circ \mathbf{1}, c_K\rangle = 0\}$). In examples, such a family $\mathcal{K}$ can often be obtained more directly, and then one can define a filter base $\mathcal{W}$ as in [GRW, after L. 6.3] so that any cluster point p of $\mathcal{W}$ satisfies the property of Lemma 4.2. In Example 2.2, when λ is concentrated on a (countable) $\mathrm{SL}(2,\mathbb{Z})$-orbit Ω_0 in $\mathbb{T}^2$ consisting of irrational points, one can take for $\mathcal{K}$ the finite subsets of Ω_0 (the condition in [GRW, after L. 6.3] amounts to the case where $\mathcal{K}$ consists of all compact subsets of Ω and $\mu(x\circ K) < \varepsilon$ is achievable for each probability measure $\mu \in M(\Omega)$).

In Example 2.3 (where Ω is again compact), when choosing p as described in Remark 4.3(a), lying above ∞, the space W contains all continuous functions h on Ω with $h\binom{\pm 1}{0} = 0$ (but no other continuous functions). Here one can take for $\mathcal{K}$ the compact subsets of Ω that do not contain $\binom{\pm 1}{0}$.

If G is a locally compact group with a continuous action on Ω, λ is a quasi-invariant probability measure on Ω, Φ is a bounded crossed homomorphism such that $\Phi(x) \ll \lambda$ for all x, then one can show (using Theorem 1.1) that

Φ is continuous for the norm-topology on $M(\Omega)$ (compare Remark 1.6). If in addition, G is σ-compact, the converse holds as well; i.e., there exists a quasi-invariant probability measure as above (compare the proof of Proposition 6.2).

6. The infinite case

In this section, Theorem 1.1 is proved for bounded crossed homomorphisms with values in $M(\Omega)_{\mathrm{inf}}$ (Proposition 6.2). The proof reduces the problem to the case where a quasi-invariant "control measure" exists (Proposition 5.1). A major step is separated in the following lemma. Note that if H is a subgroup of G, then $M(\Omega)_{\mathrm{inv},H} \supseteq M(\Omega)_{\mathrm{inv},G}$, $M(\Omega)_{\mathrm{inf},H} \subseteq M(\Omega)_{\mathrm{inf},G}$ and $M(\Omega)_{\mathrm{fin},H} \supseteq M(\Omega)_{\mathrm{fin},G}$ (see §2 for notation). $P_H \colon M(\Omega) \to M(\Omega)_{\mathrm{inf},H}$ denotes the corresponding projection with kernel $M(\Omega)_{\mathrm{fin},H}$.

Lemma 6.1. *Assume that $\rho \in M(\Omega)_{\mathrm{inf},G}$. Then there exists a countable subgroup H of G such that $\rho \in M(\Omega)_{\mathrm{inf},H}$.*

Proof. By Corollary 3.10, $\{x \circ \rho : x \in G\}$ is not weakly relatively compact. By Eberlein's theorem (see [Sch, Th. 11.1]), there exists a sequence (x_n) in G such that $\{x_n \circ \rho : n \in \mathbb{N}\}$ is not weakly relatively compact. Let H_0 be the subgroup of G generated by (x_n). Then $\rho \notin M(\Omega)_{\mathrm{fin},H_0}$; thus $P_{H_0}(\rho) \neq 0$. Observe that for $H_0 \subseteq H_1$, one has $P_{H_0} = P_{H_0} \circ P_{H_1} = P_{H_1} \circ P_{H_0}$. Hence, by an easy argument, we can choose a countable subgroup H_0 so that

$$\|P_{H_0}\rho\| = \sup\{\|P_H\rho\| : H \text{ is a countable subgroup of } G\}\,.$$

Assume that $P_{H_0}\rho \neq \rho$. Then (since $P_{H_0}\rho \in M(\Omega)_{\mathrm{inf},G}$) there exists a countable subgroup H_1 of G with $P_{H_1}(\rho - P_{H_0}\rho) \neq 0$ and we may assume that $H_1 \supseteq H_0$. Then $P_{H_1}\rho = P_{H_0}\rho + P_{H_1}(\rho - P_{H_0}\rho)$ and $P_{H_1}(\rho - P_{H_0}\rho) \ll \rho - P_{H_0}\rho \perp P_{H_0}\rho$. This would give $\|P_{H_1}\rho\| > \|P_{H_0}\rho\|$ resulting in a contradiction. It follows that $\rho = P_{H_0}\rho \in M(\Omega)_{\mathrm{inf},H_0}$. □

Proposition 6.2. *Let $\Phi \colon G \to M(\Omega)_{\mathrm{inf}}$ be a bounded crossed homomorphism. Then there exists $\mu \in M(\Omega)_{\mathrm{inf}}$ such that $\|\mu\| = \frac{1}{2}\|\Phi\|$ and $\Phi(x) = \mu - x \circ \mu$ for all $x \in G$.*

Proof. (a) First, we assume that $G = \{x_n : n = 1, 2, \dots\}$ is countable. Put

$$\lambda_0 = \sum_{n,m=1}^{\infty} \frac{1}{2^{n+m}}\, x_n \circ |\Phi(x_m)|\,, \qquad \lambda = \frac{\lambda_0}{\|\lambda_0\|}\,.$$

Then we have $\lambda \in M(\Omega)_{\mathrm{inf}}$ and it is a quasi-invariant probability measure such that $\Phi(x) \ll \lambda$ for all $x \in G$. Now Proposition 6.2 follows in this case from Lemma 4.2 and Proposition 5.1.

(b) In the general case, we consider a countable subgroup H_0 of G satisfying $\|\Phi\| = \sup_{x\in H_0}\|\Phi(x)\|$. By Lemma 6.1, there exists a countable subgroup H_1 of G such that $H_1 \supseteq H_0$ and $\Phi(x) \in M(\Omega)_{\mathrm{inf},H_1}$ for all $x \in H_0$. Put $\Phi_1(x) = P_{H_1}(\Phi(x))$. Then $\Phi_1 : H_1 \to M(\Omega)_{\mathrm{inf},H_1}$ is a crossed homomorphism satisfying $\|\,\Phi_1\| = \|\Phi\|$. By (a), there exists $\lambda_1 \in M(\Omega)_{\mathrm{inf},H_1}$ and $\mu \in L^1(\Omega,\lambda_1)$ such that λ_1 is an H_1-quasi-invariant probability measure, $\|\mu\| = \dfrac{\|\Phi\|}{2}$ and $\Phi_1(x) = \mu - x\circ\mu$ for $x \in H_1$. This implies that $\Phi(x) = \mu - x\circ\mu$ for $x \in H_0$.

Fix an arbitrary $y \in G$. Then by Lemma 6.1, there exists a countable subgroup H_2 of G such that $y \in H_2$, $\Phi(y) \in M(\Omega)_{\mathrm{inf},H_2}$ and we may assume $H_2 \supseteq H_1$. Put $\Phi_2(x) = P_{H_2}(\Phi(x))$. As above, there exists an H_2-quasi-invariant probability measure $\lambda_2 \in M(\Omega)_{\mathrm{inf},H_2}$ and $\mu_2 \in L^1(\Omega,\lambda_2)$ such that $\|\mu_2\| = \dfrac{\|\Phi\|}{2}\left(= \dfrac{\|\,\Phi_2\|}{2}\right)$ and $\Phi_2(x) = \mu_2 - x\circ\mu_2$ for all $x \in H_2$. We claim that $\mu_2 = \mu$; then $\Phi(y) = \Phi_2(y) = \mu - y\circ\mu$ and since this applies to an arbitrary $y \in G$, this will prove Proposition 6.2.

We can assume that $\lambda_1 \ll \lambda_2$ (by the uniqueness statement in Remark 5.6, μ_2 does not depend on λ_2). Using Lebesgue decomposition, let $\lambda_1^{\perp} \in M(\Omega)_{\mathrm{inf},H_2}$ be a probability measure such that $\lambda_1 + \lambda_1^{\perp} \sim \lambda_2$ and $\lambda_1 \perp \lambda_1^{\perp}$. Put $\lambda_1' = P_{H_1}(\lambda_1^{\perp})$ and $\lambda_1'' = \lambda_1^{\perp} - \lambda_1'$. Then $L^1(\Omega,\lambda_2) = L^1(\Omega,\lambda_1) \oplus L^1(\Omega,\lambda_1^{\perp})$ and $L^1(\Omega,\lambda_1^{\perp}) = L^1(\Omega,\lambda_1') \oplus L^1(\Omega,\lambda_1'')$ (since $\lambda_1' \perp \lambda_1''$). The H_1-quasi-invariance of λ_1, λ_2 implies that $\lambda_1^{\perp}, \lambda_1', \lambda_1''$ are also H_1-quasi-invariant. This gives a decomposition $\mu_2 = \nu_2 + \nu_2' + \nu_2''$ with $\nu_2 \in L^1(\Omega,\lambda_1)$, $\nu_2' \in L^1(\Omega,\lambda_1')$, $\nu_2'' \in L^1(\Omega,\lambda_1'')$. Recall that $P_{H_1} = P_{H_1} \circ P_{H_2}$; hence $\Phi_1(x) = P_{H_1}(\Phi_2(x))$ for $x \in H_1$. Then $\Phi_2(x) = \mu_2 - x\circ\mu_2$ implies (since $\nu_2, \nu_2' \in M(\Omega)_{\mathrm{inf},H_1}$, $\lambda_1'' \in M(\Omega)_{\mathrm{fin},H_1}$) $\Phi_1(x) = \nu_2 - x\circ\nu_2 + \nu_2' - x\circ\nu_2'$. Because $\Phi_1(x) \ll \lambda_1$, we get $\Phi_1(x) = \nu_2 - x\circ\nu_2$ for $x \in H_1$ and from Remark 5.6, it follows that $\nu_2 = \mu$. Then $\|\mu\| = \|\mu_2\| = \dfrac{\|\Phi\|}{2}$ implies $\nu_2', \nu_2'' = 0$, thus $\mu_2 = \mu$, proving our claim. As explained above, this completes the proof of Proposition 6.2. □

Remark 6.3. It follows from the proof that there exists always a countable subgroup H_1 of G such that the restriction of Φ to H_1 determines μ uniquely.

7. The finite case

In this section, Theorem 1.1 is proved for bounded crossed homomorphisms with values in $M(\Omega)_{\mathrm{fin}}$ (Proposition 7.1). Here we employ the approach (that already appears in [J1, §3]) using the relation between crossed homomorphisms and affine actions of G, and then apply fixed point theorems. The proof of weak relative compactness of the range of Φ uses estimates with similar decomposition methods, as in the proof of Lemma 5.5.

PROPOSITION 7.1. *Let $\Phi : G \to M(\Omega)_{\mathrm{fin}}$ be a bounded crossed homomorphism. Then $\Phi(G)$ is weakly relatively compact and there exists μ in the closed convex hull of $\Phi(G)$ (in particular, it satisfies $\|\mu\| \le \|\Phi\|$) such that $\Phi(x) = \mu - x \circ \mu$ for all $x \in G$.*

Proof. (a) First, we want to show weak relative compactness of $\Phi(G)$. We assume that $\Phi(G)$ is not weakly relatively compact. As in the proof of Lemma 6.1, we may assume that G is countable. Since $\Phi(G) \subseteq M(\Omega)_{\mathrm{fin}}$, it follows that there exists a G-invariant probability measure $\lambda \in M(\Omega)$ such that $\Phi(G) \subseteq L^1(\Omega, \lambda)$. Put $\Psi(x) = |\Phi(x)|$, $K = \Psi(G)$ and let $\overline{K}$ be the w*-closure of K in $L^1(\Omega,\lambda)''$. By the Dunford-Pettis criterion (Result 3.1), K is not weakly relatively compact; hence $\overline{K} \not\subseteq L^1(\Omega, \lambda)$. Put

$$c_0 = \sup\{\, \|w - P_\lambda(w)\| : w \in \overline{K}\,\} \,. \tag{16}$$

Then $c_0 > 0$. Choose $w \in \overline{K}$ such that, putting

$$w_a = P_\lambda(w)\,, \quad w_s = w - w_a\,, \quad c = \|w_s\|\,, \tag{17}$$

we have

$$c > \frac{c_0}{2}\,. \tag{18}$$

Approximating w by a net from K, gives some $p \in \beta G$ such that

$$w \;=\; \text{w*-}\lim_{y \to p} \Psi(y) \tag{19}$$

(as in §5, y is restricted to elements of G in this limit). By absolute continuity, there exist δ_n $(n = 1, 2, \dots)$ satisfying

(20)

$$0 < \delta_n \le \frac{1}{2^n} \qquad \text{and} \qquad \langle\, w_a\,, c_A\,\rangle < \frac{1}{n} \;\text{ for all } A \in \mathcal{B} \text{ with } \lambda(A) < \delta_n\,.$$

By (17) and Result 3.4, there exist $A_n \in \mathcal{B}$ such that

$$\langle\, w_s\,, c_{A_n}\rangle = c \qquad \text{and} \qquad \lambda(A_n) < \delta_n\,. \tag{21}$$

Since $w \ge 0$, it follows that $w_a, w_s \ge 0$; hence $\langle\, w\,, c_{A_n}\rangle \ge c$. By approximation (19), there exist $y_n \in G$ such that

$$\langle\, \Psi(y_n)\,, c_{A_n}\rangle \;>\; c - \frac{1}{n}\,. \tag{22}$$

Again by absolute continuity, there exist δ_n' satisfying

(23)

$$0 < \delta_n' \le \frac{1}{2^n} \qquad \text{and} \qquad \langle\, \Psi(y_n)\,, c_A\,\rangle < \frac{1}{n} \;\text{ for all } A \in \mathcal{B} \text{ with } \lambda(A) < \delta_n'\,.$$

Again by (17) and Result 3.4, there exist $A_n' \in \mathcal{B}$ such that

$$\langle\, w_s\,, c_{A_n'}\rangle = c \qquad \text{and} \qquad \lambda(A_n') < \delta_n'\,. \tag{24}$$

G-invariance of λ and (21) imply $\lambda(y_n^{-1}A_n) = \lambda(A_n) < \delta_n$. This gives

$$\langle\, w\,, c_{y_n^{-1}A_n\setminus A_n'}\rangle \underset{(24)}{=} \langle\, w_a\,, c_{y_n^{-1}A_n\setminus A_n'}\rangle \underset{(20)}{<} \frac{1}{n}\ .$$

Again by approximation (19), there exist $y_n' \in G$ such that

$$\text{(25)} \qquad \langle\, \Psi(y_n')\,, c_{A_n'}\rangle > c - \frac{1}{n} \qquad \text{and} \qquad \langle\, \Psi(y_n')\,, c_{y_n^{-1}A_n\setminus A_n'}\rangle < \frac{1}{n}\ .$$

Put $z_n = y_n\, y_n'$ and $B_n = A_n \cup y_n A_n'$. Then by (20), (21), (23), (24), we have

$$\text{(26)} \qquad B_n \in \mathcal{B} \qquad \text{and} \qquad \lambda(B_n) < \frac{1}{2^{n-1}}\ .$$

Since $\Phi(z_n) = \Phi(y_n) + y_n \circ \Phi(y_n')$ and the right action of G on $L^\infty(\Omega, \lambda)$ satisfies $c_A \circ y = c_{y^{-1}A}$ for $A \in \mathcal{B}$, we get

$$\text{(27)} \qquad \begin{aligned} \langle\, |\Phi(z_n)|\,, c_{y_nA_n'}\rangle \quad &\geq \quad \langle\, |\Phi(y_n')|\,, c_{A_n'}\rangle - \langle\, |\Phi(y_n)|\,, c_{y_nA_n'}\rangle \\ &\underset{(25),(23),(24)}{>} \quad c - \frac{1}{n} \qquad\qquad - \frac{1}{n}\ . \end{aligned}$$

Next,we use

$$\text{(28)} \qquad \begin{aligned} \langle\, |\Phi(z_n)|\,, c_{B_n\setminus y_nA_n'}\rangle &= \langle\, |\Phi(z_n)|\,, c_{A_n\setminus y_nA_n'}\rangle \\ &\geq \langle\, |\Phi(y_n)|\,, c_{A_n\setminus y_nA_n'}\rangle - \langle\, y_n \circ |\Phi(y_n')|\,, c_{A_n\setminus y_nA_n'}\rangle\ . \end{aligned}$$

Now

$$\langle y_n \circ |\Phi(y_n')|, c_{A_n\setminus y_nA_n'}\rangle \quad = \quad \langle\, |\Phi(y_n')|, c_{y_n^{-1}A_n\setminus A_n'}\rangle \underset{(25)}{<} \frac{1}{n}$$

and

$$\begin{aligned} \langle\, |\Phi(y_n)|, c_{A_n\setminus y_nA_n'}\rangle \quad &\geq \quad \langle\, |\Phi(y_n)|, c_{A_n}\rangle - \langle\, |\Phi(y_n)|, c_{y_nA_n'}\rangle \\ &\underset{(22),(23),(24)}{>} c - \frac{1}{n} - \frac{1}{n}\ . \end{aligned}$$

Combining these statements, we get from (28),

$$\text{(29)} \qquad \langle\, |\Phi(z_n)|\,, c_{B_n\setminus y_nA_n'}\rangle \ >\ c - \frac{3}{n}\ .$$

Together with (27), this gives

$$\text{(30)} \qquad \langle\, \Psi(z_n)\,, c_{B_n}\rangle \ >\ 2\,c - \frac{5}{n}\ .$$

By Lemma 3.5, it follows from (26), (30) that any w*-cluster point w' of $\big(\Psi(z_n)\big)$ should satisfy $\|w' - P_\lambda w'\| \geq 2\,c$. But this contradicts the choice of c in (18).

(b) For $x \in G$, put $A(x)\,\mu = x \circ \mu + \Phi(x)$. In this way ([J1, Prop. 3.1]), $A(x) : M(\Omega) \to M(\Omega)$ is a continuous affine transformation and we get an action of G on $M(\Omega)$. It is easy to see that $A(x)\,\Phi(y) = \Phi(xy)$; thus $\Phi(G)$ is invariant under the action. Let K_1 be the closed convex hull of $\Phi(G)$. Then K_1 is also invariant under the action of G and by (a) it is weakly compact. Therefore we can apply the fixed point theorem (Result 3.8). Let $\mu \in K_1$ be a fixed point. Obviously, $A(x)\,\mu = \mu$ is equivalent to $\Phi(x) = \mu - x \circ \mu$ which finishes the proof of the Proposition 7.1 (and also that of Theorem 1.1). □

Remark 7.2. (a) Comparing the Propositions 6.2 and 7.1, one can see a difference in the norm estimates for μ. In the case of C^*- algebras, some work has been done on the norm of inner derivations (see e.g., [AS]). Clearly, $\Phi_\nu = 0$ for $\nu \in M(\Omega)_{\mathrm{inv}}$, thus Φ_μ depends only on the coset of μ in $M(\Omega)/M(\Omega)_{\mathrm{inv}}$ (which will be denoted again by μ), i.e., $\|\,\Phi_\mu\| \leq 2\|\mu\|_{M(\Omega)/M(\Omega)_{\mathrm{inv}}}$ holds in general and for $\mu \in M(\Omega)_{\mathrm{inf}}$, we get equality by Proposition 6.2. We want to give an example showing that there are compact groups G for which $\inf\{\|\,\Phi_\mu\| : \mu \in M(G)_{\mathrm{fin}}\,, \|\mu\|_{M(G)/Z(M(G))} = 1\} = 1$ (for the action $x \circ \omega = x\,\omega\,x^{-1}$ used in Corollary 1.2; in this case $M(G)_{\mathrm{inv}}$ coincides with the centre $Z\big(M(G)\big)$ of the algebra $M(G)$); i.e., the norm estimate in Proposition 7.1 cannot be improved in general. Since $\|\,\Phi_\mu\| = \|\,\mathrm{ad}_\mu\|$, this applies also to the corresponding derivations of $L^1(G)$.

First, we claim that it is sufficient to construct finite groups H_n and $\mu_n \in M(H_n)$ for which $\|\mu_n\|_{M(H_n)/Z(M(H_n))} = 1$ and $\|\,\Phi_{\mu_n}\| \to 1$ for $n \to \infty$. Then one can take $G = \prod_{n=1}^{\infty} H_n$. We put $H'_m = \prod_{n \neq m} H_n$ and, as usual, H_m is identified with a subgroup of G ($\cong\ H_m \times H'_m$). In this way, $\mu_m \in M(G)$ and we consider $\bar\mu_m = \mu_m * \lambda_{H'_m}$ (where $\lambda_{H'}$ denotes the normalized Haar measure of a compact group H'). Then $\bar\mu_m \in L^1(G) \subseteq M(G)_{\mathrm{fin}}$ and $\|\bar\mu_m\|_1 = \|\mu_m\|_{M(H_m)}$. Recall that $\lambda_{H'_m}$ is a central idempotent in $M(G)$. Thus for $\nu \in Z\big(M(G)\big)$, we have $\nu * \lambda_{H'_m} \in Z\big(M(G)\big)$ and $\|\bar\mu_m + \nu * \lambda_{H'_m}\| = \|(\bar\mu_m + \nu) * \lambda_{H'_m}\| \leq \|\bar\mu_m + \nu\|$. To compute the quotient norm, it is therefore enough to consider ν with $\nu = \nu * \lambda_{H'_m}$, and then $\nu = \nu_0 * \lambda_{H'_m}$ with $\nu_0 \in Z\big(M(H_m)\big)$. It follows that $\|\bar\mu_m\|_{M(G)/Z(M(G))} = \|\mu_m\|_{M(H_m)/Z(M(H_m))} = 1$. Similarly, $\Phi_{\bar\mu_m}(x) = \Phi_{\mu_m}(x_m) * \lambda_{H'_m}$ for all $x = (x_n) \in G$ and this implies $\|\,\Phi_{\bar\mu_m}\| = \|\,\Phi_{\mu_m}\| \to 1$, proving our claim.

We will now specialize to semidirect products of finite groups $H = K \ltimes L$, i.e., K acts on L by automorphisms. Any subset A of L determines a measure ρ on $L < H$, by putting $\rho(\{v\}) = 1$ for $v \in A$, $\rho(\{v\}) = 0$ otherwise. Then it is easy to see that $\|\rho\| = |A|$ (= cardinality of A) with the assumptions that L is abelian, $\|\,\Phi_\rho\| = \max_{x\in K} |(x \circ A) \bigtriangleup A\,|$ ($\bigtriangleup$ denoting the symmetric difference). Since $|(x \circ A) \bigtriangleup A\,| = 2\,(|A| - |(x \circ A) \cap A\,|)$, we get $\|\,\Phi_\rho\| = 2\,\big(|A| - \min_{x\in K} |(x \circ A) \cap A\,|\big)$. Furthermore, if $A \subseteq K \circ v$ for some $v \in L$, then $\|\rho\|_{M(H)/Z(M(H))} = \min\big(|A|, |(K \circ v) \setminus A|\big)$.

We choose $K = \mathbb{Z}_s^t$ ($\mathbb{Z}_s$ denotes the additive group of integers $\bmod s$, represented as the interval of integers $[0, s-1]$; $s \geq 2$) with $t = [s \ln 2] + 1$ ($[\cdot]$ denoting the integral part), $L = K \times K$ with the action $x \circ (y, z) = (y, z + x\,y)$ (the product being taken coordinatewise), $A = \{(1, \dots, 1)\} \times [0, s-2]^t$ ($\subseteq L$). Then $|A| = (s-1)^t < \frac{1}{2}\, s^t$, hence $\|\rho\|_{M(H)/Z(M(H))} = |A|$ (note that $\{(1, \dots, 1)\} \times K$ is the K-orbit containing A). Since $\big|([0, s-2] + u) \cap [0, s-2]\big| \geq s - 2$ for any $u \in \mathbb{Z}_s$, we get $\min_{x \in K} |(x \circ A) \cap A| = (s-2)^t$; thus $\|\, \Phi_\rho\| = 2\,\big((s-1)^t - (s-2)^t\big)$. Finally, putting $\mu = \dfrac{1}{|A|}\, \rho$, we arrive at $\|\mu\|_{M(H)/Z(M(H))} = 1$, $\big\|\, \Phi_\mu\big\| = 2\left(1 - \big(1 - \dfrac{1}{s-1}\big)^t\right)$ and since for $s \to \infty$ this approaches 1, we find the desired example.

(b) Let Φ be given as in Proposition 7.1 and construct $\mu \in M(\Omega)$ according to the method of the proof of Result 3.8. Take a probability measure $\lambda \in M(\Omega)_{\text{inv}}$ with $\mu \ll \lambda$. Then $\Phi(G) \subseteq L^1(\Omega, \lambda)$ and for every $h \in L^\infty(\Omega, \lambda)$ the function $x \mapsto \langle\, h\,, \Phi(x)\rangle$ is weakly almost periodic (consider $T_{v'}(v)$ as in the proof of Result 3.8 with $v = 0 = \Phi(e)$, $v' = h$) and it has the mean $\langle h, \mu\rangle$ (this is the immediate analogue of the proof of [J1, Th. 2.5] for amenable groups; see also [GRW, L. 2.1]). It follows easily from the invariance of the mean that for any μ' in the closed convex hull of $\Phi(G)$ (by classical results, the weak closure coincides with the norm closure) the function $x \mapsto \langle\, h\,, x \circ \mu'\rangle$ has mean zero. This implies that the measure μ is the unique element in the closed convex hull of $\Phi(G)$ which satisfies $\Phi = \Phi_\mu$. But observe that in general this will not give the measure $\mu' \in L^1(\Omega, \lambda)$ of minimal norm for which $\Phi = \Phi_{\mu'}$ (e.g., in (a), $\rho = c_A$ has minimal norm when $|A| \leq \frac{1}{2}|K \circ v|$, but the corresponding measure in the convex hull of $\Phi_\rho(H)$ is $c_A - \dfrac{|A|}{|K \circ v|}\, c_{K \circ v}$; its norm is $2\,|A| \left(1 - \dfrac{|A|}{|K \circ v|}\right)$ and for $|A| < \frac{1}{2}|K \circ v|$ this is greater than $|A|$).

Since μ is obtained by applying the invariant mean for weakly almost periodic functions to Φ , the mapping $\Phi \mapsto \mu$ is linear and gives a right inverse to the mapping $\mu \mapsto \Phi_\mu$. Hence $M(\Omega)_{\text{fin}}$ decomposes into a direct sum of $M(\Omega)_{\text{inv}}$ and the space of all μ for which $x \mapsto x \circ \mu$ has mean zero. The corresponding projection of $M(\Omega)_{\text{fin}}$ to $M(\Omega)_{\text{inv}}$ has norm 1 (this is related to [GRW, Cor. 2.6]). We have given an isomorphism between this complementary subspace to $M(\Omega)_{\text{inv}}$ and the space of bounded crossed homomorphisms with values in $M(\Omega)_{\text{fin}}$ (similarly for $M(\Omega)$). One can show that this isomorphism is *nonisometrical*, unless $M(\Omega)_{\text{fin}} = M(\Omega)_{\text{inv}}$ (i.e., the G-action is trivial on the points in the supports of the invariant measures). On the infinite part $M(\Omega)_{\text{inf}}$ the corresponding isomorphism has norm 2 whenever $M(\Omega)_{\text{inf}} \neq (0)$.

On the other hand, there exist examples of systems G, Ω and $\mu \neq 0$ of finite type such that $\|\mu\| = \|\Phi_\mu\|$, $x \mapsto x \circ \mu$ has mean zero and μ is the measure of minimal norm representing Φ_μ: When $G = \mathbb{Z}$, $\Omega = \{0, 1\}^{\mathbb{Z}}$, G acts on Ω by shifting coordinates. Let λ be the product measure on Ω giving weight $\frac{1}{2}$ to

0 and 1 ("Bernoulli shift") and put $u(\omega) = (-1)^{\omega_0}$ for $\omega = (\omega_n) \in \Omega$. Then an easy computation gives that in $L^1(\Omega, \lambda)$ we have $\|u - n \circ u\|_1 = 1$ for all $n \in \mathbb{Z} \setminus \{0\}$. Thus $\|\Phi_u\| = \|u\|_1 = 1$. Ergodicity of the shift implies that the mean of $x \mapsto x \circ u$ is given by $\int u \, d\lambda = 0$ (constant function). Furthermore, if $\mu_0 \in M_{\rm inv}(\Omega)$, its λ-continuous component must be a multiple of λ. Since $\|u - c\|_1 > 1$ for all $c \in \mathbb{C} \setminus \{0\}$, it follows that $\|\mu'\| > 1$ for all $\mu' \in M(\Omega)$ such that $\Phi_{\mu'} = \Phi_u$ and $\mu' \neq u$.

It appears very likely that such examples do not exist in the original group case (i.e., $\Omega = G$ with the action $x \circ \omega = x \, \omega \, x^{-1}$) used in Corollary 1.2.

Universität Wien, Wien, Austria
E-mail address: viktor.losert@univie.ac.at

References

[A] C. A. Akemann, The dual space of an operator algebra, *Trans. Amer. Math. Soc.* **126** (1967), 286–302.

[AS] R. J. Archbold and D. W. B. Somerset, Inner derivations and primal ideals of C^*-algebras. II, *Proc. London Math. Soc.* **88** (2004), 225–250.

[BB] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges. A Study of Finitely Additive Meausres*, Academic Press, New York, 1983.

[D] H. G. Dales, *Banach Algebras and Automatic Continuity*, *London Math. Soc. Monographs* **24**, Oxford Univ. Press, New York, 2000.

[DS] N. Dunford and J. T. Schwartz, *Linear Operators* I, Interscience, New York, 1958.

[GRW] F. Ghahramani, V. Runde, and G. Willis, Derivations on group algebras, *Proc. London Math. Soc.* **80** (2000), 360–390.

[Gr] F. P. Greenleaf, *Invariant Means on Topological Groups and their Applications*, van Nostrand Reinhold, New York, 1969.

[HS] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification*, Walter de Gruyter, Berlin, 1998.

[J1] B. E. Johnson, *Cohomology in Banach Algebras*, *Memoirs Amer. Math. Soc.* **127**, A.M.S., Providence, RI, 1972.

[J2] ———, The derivation problem for group algebras of connected locally compact groups, *J. London Math. Soc.* **63** (2001), 441–452.

[JR] B. E. Johnson and J. R. Ringrose, Derivations of operator algebras and discrete group algebras, *Bull. London Math. Soc.* **1** (1969), 70–74.

[JS] B. E. Johnson and A. M. Sinclair, Continuity of derivations and a problem of Kaplansky, *Amer. J. Math.* **90** (1968), 1067–1073.

[KS] I. Kovács and J. Szücs, Ergodic type theorems in von Neumann algebras, *Acta Sci. Math.* (*Szeged*) **27** (1966), 233–246.

[La] A. T. Lau, Some fixed point theorems and their applications to W^*-algebras, in *Fixed Point Theory and its Applications* (Halifax 1975), 121–129, Academic Press, New York, 1976.

[Lo] V. Losert, Weakly compact multipliers on group algebras, *J. Funct. Anal.* **213** (2004), 466–472.

[Sa] S. Sakai, Derivations of W^*-algebras, *Ann. of Math.* **83** (1966), 273–279.

[Sch] H. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York, 1971.

[Se] Z. Semadeni, *Banach Spaces of Continuous Functions*, Monografie Mathematyczne, PWN-Polish Scientific Publishers, Warsaw, 1971.

[T] M. Takesaki, *Theory of Operator Algebras* I, Springer-Verlag, New York, 1979.

(Received January 3, 2005)