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The strong Macdonald conjecture and Hodge theory on the loop Grassmannian

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Abstract

We prove the strong Macdonald conjecture of Hanlon and Feigin for reductive groups G. In a geometric reformulation, we show that the Dolbeault cohomology $H^q(X;\Omega^p)$ of the loop Grassmannian X is freely generated by de Rham's forms on the disk coupled to the indecomposables of $H^\bullet(BG)$. Equating the two Euler characteristics gives an identity, independently known to Macdonald [M], which generalises Ramanujan's ${}_1\psi_1$ sum. For simply laced root systems at level 1, we also find a 'strong form' of Bailey's ${}_4\psi_4$ sum. Failure of Hodge decomposition implies the *singularity* of X, and of the algebraic loop groups. Some of our results were announced in [T2].

Introduction

This article address some basic questions concerning the cohomology of affine Lie algebras and their flag varieties. Its chapters are closely related, but have somewhat different flavours, and the methods used in each may well appeal to different readers. Chapter I proves the *strong Macdonald constant term conjectures* of Hanlon [H1] and Feigin [F1], describing the cohomologies of the Lie algebras $\mathfrak{g}[z]/z^n$ of truncated polynomials with values in a reductive Lie algebra $\mathfrak{g}$ and of the *graded* Lie algebra $\mathfrak{g}[z,s]$ of $\mathfrak{g}$-valued skew polynomials in an even variable z and an odd one s (Theorems A and B). The proof uses little more than linear algebra, and, while Nakano's identity (3.15) effects a substantial simplification, we have included a brutal computational by-pass in Appendix A, to avoid reliance on external sources.

Chapter II discusses the Dolbeault cohomology $H^q(\Omega^p)$ of flag varieties of loop groups. In addition to the "Macdonald cohomology", the methods and proofs draw heavily on [T3]. For the *loop Grassmannian* $X := G((z))/G[[z]]$, we obtain the free algebra generated by copies of the spaces $\mathbb{C}[[z]]$ and $\mathbb{C}[[z]]dz$, in bi-degrees $(p,q) = (m,m)$, respectively $(m+1,m)$, as m ranges over the exponents of $\mathfrak{g}$. Moreover, de Rham's operator $\partial : H^q(\Omega^p) \to H^q(\Omega^{p+1})$ is induced by the differential $d : \mathbb{C}[[z]] \to \mathbb{C}[[z]]dz$ on matching generators.

A noteworthy consequence of our computation is the *failure of Hodge decomposition*,

$$H^n(X;\mathbb{C}) \neq \bigoplus\nolimits_{p+q=n} H^q(X;\Omega^p).$$

Because X is a union of projective varieties, this implies that X is *not smooth*, in the sense that it is not locally expressible as an increasing union of smooth complex-analytic sub-varieties (Theorem 5.4). We are thus dealing with a *homogeneous variety* which is *singular everywhere*. We are unable to offer a geometric explanation of this striking fact.

Our results generalise to an arbitrary smooth affine curve Σ. The Macdonald cohomology involves now the Lie algebra $\mathfrak{g}[\Sigma, s]$ of $\mathfrak{g}[s]$-valued algebraic maps, while X is replaced by the *thick flag variety* $\mathbf{X}_\Sigma$ of Section 7. Answering the question in this generality requires more insight than is provided by the listing of co-cycles in Theorem B. Thus, after re-interpreting the Macdonald cohomology as the (algebraic) Dolbeault cohomology of the classifying stack $BG[[z]]$, and the flag varieties $\mathbf{X}_\Sigma$ as moduli of G-bundles on Σ trivialised near ∞, we give in Section 8 a uniform construction of all generating Dolbeault classes. Inspired by the Atiyah-Bott description of the cohomology generators for the moduli of G-bundles, our construction is a Dolbeault refinement thereof, based on the Atiyah class of the universal bundle, with the invariant polynomials on $\mathfrak{g}$ replacing the Chern classes.

The more geometric perspective leads us to study $H^q(X;\Omega^p\otimes\mathcal{V})$ for certain vector bundles $\mathcal{V}$; this ushers in Chapter III. In Section 12, we find a beautiful answer for simply laced groups and the level 1 line bundle $\mathcal{O}(1)$. In general, we can define, for each level $h \geq 0$ and G-representation V, the formal Euler series in t and z with coefficients in the character ring of G:

$$P_{h,V} = \sum\nolimits_{p,q} (-1)^q(-t)^p \mathrm{ch}\, H^q\left(X;\Omega^p(h)\otimes\mathcal{V}\right),$$

where the vector bundle $\mathcal{V}$ is associated to the G-module V as in Section 11.8 and z carries the weights of the $\mathbb{C}^\times$-scaling action on X. These series, expressible using the Kac character formula, are affine analogues of the Hall-Littlewood symmetric functions, and their complexity leaves little hope for an explicit description of the cohomologies. On the other hand, the finite Hall-Littlewood functions are related to certain filtrations on weight spaces of G-modules, studied by Kostant, Lusztig and Ranée Brylinski in general. We find in Section 12.2 that such a relationship persists in the affine case *at positive level*. Failure of the level zero theory is captured precisely by the Macdonald cohomology, or by its Dolbeault counterpart in Chapter II, whereas the good behaviour at positive level relies on a higher-cohomology vanishing (Theorem E).

We emphasise that finite-dimensional analogues of our results (Remarks 11.1 and 11.10), which are known to carry geometric information about the G-flag variety G/B and the nilpotent cone in $\mathfrak{g}$, can be deduced from standard Hodge theory or other cohomology vanishing results (the Grauert-Riemenschneider theorem, applied to the moment map $\mu : T^*(G/B) \to \mathfrak{g}^*$).

No such general theorems are available in the loop group case; our results provide a substitute for this. Developing the full theory would take us too far afield, and we postpone it to a future paper, but Section 11 illustrates it with a simple example.

Finally, just as the strong Macdonald conjecture refines a combinatorial identity, our new results also have combinatorial applications. Comparing our answer for $H^q(X; \Omega^p(h))$ with the Kac character formula for $P_{h,\mathbb{C}}$ leads to q-basic hyper-geometric summation identities. For SL_2, this is a specialisation of Ramanujan's ${}_1\psi_1$ sum. For general affine root systems, these identities were independently discovered by Macdonald [M]. The level one identity for SL_2 comes from a specialised Bailey ${}_4\psi_4$ sum; its extension to simply laced root systems seems new.

Most of the work for this paper dates back to 1998, and the authors have lectured on it at various times; the original announcement is in [T2], and a more leisurely survey is [Gr]. We apologise for the delay in preparing the final version.

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Contents

Definitions and notation

Our (Lie) algebras and vector spaces are defined over $\mathbb{C}$. Certain vector spaces, such as $\mathbb{C}[[z]]$, have natural inverse limit topologies, and $*$-superscripts will then indicate their *continuous* duals; this way, $\mathbb{C}[[z]]^{**} \cong \mathbb{C}[[z]]$. Completed tensor products or powers of such spaces will be indicated by $\widehat{\otimes}$, $\hat{\mathrm{S}}^p$, $\hat{\Lambda}^p$.

(0.1) *Lie algebra* (*co*)*homology.* The *Lie algebra homology Koszul complex*[1] [Ko] of a Lie algebra $\mathfrak{L}$ with coefficients in a module V is $\Lambda^\bullet\mathfrak{L} \otimes V$, homologically graded by $\bullet$, with differential

$$\begin{aligned}\delta(\lambda_1 \wedge \ldots \wedge \lambda_n \otimes v)& \\ = \sum\nolimits_p & (-1)^p \lambda_1 \wedge \ldots \wedge \hat{\lambda}_p \wedge \ldots \wedge \lambda_n \otimes \lambda_p(v) \\ + \sum\nolimits_{p<q} & (-1)^{p+q} [\lambda_p, \lambda_q] \wedge \lambda_1 \wedge \ldots \wedge \hat{\lambda}_p \wedge \ldots \wedge \hat{\lambda}_q \wedge \ldots \wedge \lambda_n \otimes v;\end{aligned}$$

hats indicate missing factors. Its homology $H_\bullet(\mathfrak{L}; V)$ is the *Lie algebra homology of $\mathfrak{L}$ with coefficients in V*. If $\mathfrak{g} \subseteq \mathfrak{L}$ is a sub-algebra, δ descends to the quotient $(\Lambda(\mathfrak{L}/\mathfrak{g}) \otimes V) / \mathfrak{g} (\Lambda(\mathfrak{L}/\mathfrak{g}) \otimes V)$ of co-invariants under $\mathfrak{g}$, which resolves the *relative homology* $H_\bullet(\mathfrak{L}, \mathfrak{g}; V)$. We denote by $H_\bullet(\mathfrak{L})$ the homology with coefficients in the trivial one-dimensional module.

Dual to these are the *cohomology complexes*, with underlying spaces $\mathrm{Hom}(\Lambda^\bullet\mathfrak{L}; W)$; the cohomology is denoted $H^\bullet(\mathfrak{L}; W)$, or $H^\bullet(\mathfrak{L}, \mathfrak{g}; W)$ in the relative case. They are the full duals of the homologies, when W is the full dual of V. If W is an algebra and $\mathfrak{L}$ acts by derivations, the Koszul complex is a differential graded algebra. Similarly, the homology complex is a differential graded co-algebra, when V is a co-algebra and $\mathfrak{L}$ acts by co-derivations.

0.2 *Remark.* More abstractly, $H_k(\mathfrak{L}; V) = \mathrm{Tor}_k^{\mathfrak{L}}(\mathbb{C}; V)$ and $H^k(\mathfrak{L}; V) = \mathrm{Ext}_{\mathfrak{L}}^k(\mathbb{C}; V)$ in the category of $\mathfrak{L}$-modules. If $\mathfrak{g} \subseteq \mathfrak{L}$ is reductive, and $\mathfrak{L}$ (via ad) and V are semi-simple $\mathfrak{g}$-modules, the relative homologies are the Tor groups in the category of $\mathfrak{g}$-*semi-simple* $\mathfrak{L}$-modules.

(0.3) *Exponents.* Either of the following statements defines the *exponents* $m_1, \ldots, m_\ell$ of a reductive Lie algebra $\mathfrak{g}$ of rank ℓ:

- the algebra $(\mathrm{S}\mathfrak{g}^*)^{\mathfrak{g}}$ of polynomials on $\mathfrak{g}$ which are invariant under the co-adjoint action is a free symmetric algebra generated in degrees $m_1 + 1, \ldots, m_\ell + 1$;

- the sub-algebra $(\Lambda\mathfrak{g})^{\mathfrak{g}}$ of ad-invariants in the full exterior algebra of $\mathfrak{g}$ is a free exterior algebra generated in degrees $2m_1 + 1, \ldots, 2m_\ell + 1$.

[1] Also called the *Chevalley complex.*

For instance, when $\mathfrak{g} = \mathfrak{gl}_n$, $\ell = n$ and $(m_1, \dots, m_n) = (0, \dots, n-1)$. The first algebra is also naturally isomorphic to the cohomology $H^\bullet(BG; \mathbb{C})$, if we set $\deg \mathfrak{g} = 2$.

(0.4) *Generators.* Most cohomologies in this paper will be free graded polynomial (or power series) algebras, which are canonically described by identifying their spaces of indecomposables[2] with those for $H^\bullet(BG)$, tensored with suitable graded vector spaces $V^\bullet$ (cf. Theorem B). However, we can choose once and for all a space $\mathrm{Gen}^\bullet(BG)$ spanned by homogeneous free generators for the cohomology, and identify our cohomologies as the free algebras on $\mathrm{Gen}^\bullet(BG) \otimes V^\bullet$. There are many choices of generators,[3] but our explicit constructions of cohomology classes from invariant polynomials serve to 'canonise' this second description.

(0.5) *Fourier basis.* When G is semi-simple, we will choose a compact form and a basis of self-adjoint elements ξ_a in $\mathfrak{g}$, orthonormal in the Killing form. Call, for $m \geq 0$, $\psi^a(-m)$ and $\sigma^a(-m)$ the elements of $\Lambda^1\mathfrak{g}[z]^*$ and $\mathrm{S}^1\mathfrak{g}[z]^*$ dual to the basis $z^m \cdot \xi_a$ of the Lie algebra $\mathfrak{g}[z]$. We abusively write $\xi_{[a,b]}$ for $[\xi_a, \xi_b]$, and similarly $\psi^{[a,b]}(m)$ for $\mathrm{ad}^*_{\xi_a}\psi^b(m)$, etc.

I. The strong Macdonald conjecture

1. Statements

(1.1) *Background.* The strong Macdonald conjectures describe the cohomologies of the truncated Lie algebras $\mathfrak{g}[z]/z^n$ and of the graded Lie algebra $\mathfrak{g}[z, s]$. The first conjecture is due to Hanlon [H1], who also proved it for $\mathfrak{gl}_n$ [H2]. The conjecture may have been independently known to Feigin [F1], who in [F2] related it to the cohomology of $\mathfrak{g}[z, s]$. Feigin also outlined a computation of the latter; but we are unsure whether it can be carried out as indicated.[4] While we could not fill the gap, we do confirm the conjectures by a different route: we compute the cohomology of $\mathfrak{g}[z, s]$ by finding the *harmonic co-cycles* in the Koszul complex, in a suitable metric. Feigin's argument then recovers the cohomology of the truncated Lie algebra.

The success of our Laplacian approach relies on the specific metric used on the Koszul complex and originates in the Kähler geometry of the loop

[2] Recall that the space of indecomposables of a nonnegatively graded algebra $A^\bullet$ is $A^{>0}/(A^{>0})^2$. If $A^\bullet$ is a free algebra over A^0, a graded A^0-lifting of the indecomposables in $A^\bullet$ gives a space of algebra generators.

[3] Natural examples for GL_n include the Chern classes and the traces $\mathrm{Tr}F^k$ of the universal curvature form F.

[4] One particular step, the lemma on p. 93 of [F2], seems incorrect: the analogous statement fails for absolute cohomology when $Q = \partial/\partial\xi$, and nothing in the suggested argument seems to account for that.

Grassmannian. The latter is responsible for an identity between two different Laplacians, far from obvious in Lie algebra form, which implies here that the harmonic co-cycles form a *sub-algebra* and allows their computation. We do not know of a computation in the more obvious Killing metric: its harmonic co-cycles are not closed under multiplication.

Truncated algebras. The following affirms Hanlon's original conjecture for reductive $\mathfrak{g}$. Note that the cohomology of $\mathfrak{g}[z]/z^n$ decomposes by z-weight, in addition to the ordinary grading.

THEOREM A. *$H^\bullet(\mathfrak{g}[z]/z^n)$ is a free exterior algebra on $n\cdot\ell$ generators, with n generators in cohomology degree $2m+1$ and z-weights equal to the negatives of $0, mn+1, mn+2, \dots, mn+n-1$, for each exponent $m = m_1, \dots, m_\ell$.*

1.2 *Remark.* (i) Ignoring z-weights leads to an *abstract* ring isomorphism $H^\bullet(\mathfrak{g}[z]/z^n) \cong H^\bullet(\mathfrak{g})^{\otimes n}$.

(ii) The degree-wise lower bound $\dim H^\bullet(\mathfrak{g}[z]/z^n) \geq \dim H^\bullet(\mathfrak{g})^{\otimes n}$ holds for *any* Lie algebra $\mathfrak{g}$. Namely, $\mathfrak{g}[z]/z^n$ is a degeneration of $\mathfrak{g}[z]/(z^n - \varepsilon)$, as $\varepsilon \to 0$. When $\varepsilon \neq 0$, the quotient is isomorphic to $\mathfrak{g}^{\oplus n}$, whose cohomology is $H^\bullet(\mathfrak{g})^{\otimes n}$, and the ranks are upper semi-continuous. However, this argument says nothing about the *ring* structure.

(iii) There is a *natural* factorisation $H^\bullet(\mathfrak{g}[z]/z^n) = H^\bullet(\mathfrak{g}) \otimes H^\bullet(\mathfrak{g}[z]/z^n, \mathfrak{g})$, and the first factor has z-weight 0. Indeed, reductivity of $\mathfrak{g}$ leads to a spectral sequence [Ko] with

$$E_2^{p,q} = H^q(\mathfrak{g}) \otimes H^p(\mathfrak{g}[z]/z^n, \mathfrak{g}) \Rightarrow H^{p+q}(\mathfrak{g}[z]/z^n),$$

whose collapse there is secured by the evaluation map $\mathfrak{g}[z]/z^n \to \mathfrak{g}$, which provides a lifting of the left edge $H^q(\mathfrak{g})$ in the abutment and denies the possibility of higher differentials.

(1.3) *Relation to cyclic homology.* A conceptual formulation of Theorem A was suggested independently by Feigin and Loday. Given a skew-commutative algebra A and any Lie algebra $\mathfrak{g}$, an invariant polynomial Φ of degree $(m+1)$ on $\mathfrak{g}$ determines a linear map from the dual of $HC_n^{(m)}(A)$, the mth *Adams component* of the nth *cyclic homology* group of A, to $H^{n+1}(\mathfrak{g} \otimes A)$ (see our Theorem B for the case of interest here, and [T2, (2.2)], or the comprehensive discussion in [L] in general). When $\mathfrak{g}$ is reductive, Loday suggested that these maps might be injective, and that $H^\bullet(\mathfrak{g}\otimes A)$ might be freely generated by their images, as Φ ranges over a set of generators of the ring of invariant polynomials. The Adams degree m will then range over the exponents $m_1, \dots, m_\ell$. Thus, for $A = \mathbb{C}$, $HC_n^{(m)} = 0$ for $n \neq 2m$, while $HC_{2m}^{(m)} = \mathbb{C}$; we recover the well-known description of $H^\bullet(\mathfrak{g})$. For $\mathfrak{g} = \mathfrak{gl}_\infty$ and any associative, unital, graded A, this is the theorem of Loday-Quillen [LQ] and Tsygan [Ts]. It emerges from its proof that Theorem A affirms Loday's conjecture for $\mathbb{C}[z]/z^n$, while (1.5) below does the same for the graded algebra $\mathbb{C}[z,s]$. (The conjecture fails in general [T2].)

(1.4) *The super-algebra.* The graded space $\mathfrak{g}[z,s]$ of $\mathfrak{g}$-valued skew polynomials in z and s, with $\deg z = 0$ and $\deg s = 1$, is an infinite-dimensional *graded* Lie algebra, isomorphic to the semi-direct product $\mathfrak{g}[z] \ltimes s\mathfrak{g}[z]$ (for the adjoint action), with zero bracket in the second factor. We shall give three increasingly concrete descriptions in Theorems 1.5, 1.10, B for its (co)homology. We start with *homology*, which has a natural *co-algebra* structure. As in Remark 1.2.iii, we factor $H_\bullet(\mathfrak{g}[z,s])$ as $H_\bullet(\mathfrak{g}) \otimes H_\bullet(\mathfrak{g}[z,s],\mathfrak{g})$; the first factor behaves rather differently from the rest and is best set aside.

1.5 Theorem. $H_\bullet(\mathfrak{g}[z,s],\mathfrak{g})$ *is isomorphic to the free, graded co-commutative co-algebra whose space of primitives is the direct sum of copies of* $\mathbb{C}[z] \cdot s^{\otimes(m+1)}$, *in total degree* $2m+2$, *and of* $\mathbb{C}[z]dz \cdot s^{\otimes m}$, *in total degree* $2m+1$, *as* m *ranges over the exponents* $m_1, \ldots, m_\ell$. *The isomorphism respects* (z,s)*-weights.*

1.6 *Remark.* (i) The *total degree* $\bullet$ includes that of s. As multi-linear tensors in $\mathfrak{g}[z,s]$, both types of cycles have degree $m+1$.
(ii) A free co-commutative co-algebra is isomorphic, as a vector space, to the graded symmetric algebra on its primitives; but there is no *a priori* algebra structure on homology.

The description (1.5) is not quite canonical. If $P_{(k)}$ is the space of kth degree primitives in the quotient co-algebra $\mathrm{S}\mathfrak{g}/[\mathfrak{g},\mathrm{S}\mathfrak{g}]$, canonical descriptions of our primitives are

$$\begin{aligned}&\textstyle\bigoplus_m P_{(m+1)} \otimes \mathbb{C}[z] \cdot s(ds)^m,\\ &\textstyle\bigoplus_m P_{(m+1)} \otimes \dfrac{\mathbb{C}[z]\cdot(ds)^m + \mathbb{C}[z]dz \cdot s(ds)^{m-1}}{d\left(\mathbb{C}[z]\cdot s(ds)^{m-1}\right)}.\end{aligned} \tag{1.7}$$

The right factors are the cyclic homology components $HC_{2m+1}^{(m)}$ and $HC_{2m}^{(m)}$ of the *nonunital* algebra $\mathbb{C}[z,s] \ominus \mathbb{C}$. The last factor, $HC_{2m}^{(m)}$, is identifiable with $\mathbb{C}[z]dz \cdot s(ds)^{m-1}$, for $m \neq 0$, and with $\mathbb{C}[z]/\mathbb{C}$ if $m=0$. This description is compatible with the action of *super-vector fields* in z and s (see Remark 2.5 below), whereas (1.5) only captures the action of vector fields in z.

(1.8) *Restatement without super-algebras.* There is a natural isomorphism between $H_\bullet(\mathfrak{L};\Lambda^\bullet V)$ and the homology of the semi-direct product Lie algebra $\mathfrak{L} \ltimes V$, with zero bracket on V [Ko]. Its graded version, applied to $\mathfrak{L} = \mathfrak{g}[z]$ and the *odd* vector space $V = s\mathfrak{g}[z]$, is the equality

$$H_n(\mathfrak{g}[z,s],\mathfrak{g}) = \bigoplus_{p+q=n} H_{q-p}\left(\mathfrak{g}[z],\mathfrak{g};\mathrm{S}^p(s\mathfrak{g}[z])\right); \tag{1.9}$$

note that elements of $s\mathfrak{g}[z]$ carry homology degree 2 (Remark 1.6.i). We can restate Theorem 1.5 as follows:

1.10 THEOREM. *$H_\bullet(\mathfrak{g}[z], \mathfrak{g}; \mathrm{S}(s\mathfrak{g}[z]))$ is isomorphic to the free graded co-commutative co-algebra with primitive space $\mathbb{C}[z] \cdot s^{\otimes(m+1)}$, in degree 0, and primitive space $\mathbb{C}[z]dz \cdot s^{\otimes m}$ in degree 1, as m ranges over the exponents $m_1, \dots, m_\ell$. The isomorphism preserves z-and s-weights.*

(1.11) *Cohomology.* While $H^\bullet(\mathfrak{g}[z,s], \mathfrak{g})$ is obtained from (1.9) by duality, infinite-dimensionality makes it a bit awkward, and we opt for a *restricted duality*, defined using the direct sum of the (s,z)-weight spaces in the dual of the Koszul complex (0.1). These weight spaces are finite-dimensional and are preserved by the Koszul differential. The resulting restricted Lie algebra cohomology $H^\bullet_{\mathrm{res}}(\mathfrak{g}[z,s], \mathfrak{g})$ is the direct sum of weight spaces in the full dual of (1.9).

THEOREM B. *$H^\bullet_{\mathrm{res}}(\mathfrak{g}[z], \mathfrak{g}; \mathrm{S}\mathfrak{g}[z]^*)$ is isomorphic to the free graded commutative algebra generated by the restricted duals of $\bigoplus_m P_{(m+1)} \otimes \mathbb{C}[z]$ and $\bigoplus_m P_{(m+1)} \otimes \mathbb{C}[z]dz$, in cohomology degrees 0 and 1 and symmetric degrees $m+1$ and m, respectively.*

Specifically, an invariant linear map $\Phi : \mathrm{S}^{m+1}\mathfrak{g} \to \mathbb{C}$ determines linear maps

$$\begin{aligned} S_\Phi &: \mathrm{S}^{m+1}\mathfrak{g}[z] \to \mathbb{C}[z], \\ &\qquad\qquad \sigma_0 \cdot \sigma_1 \cdot \ldots \cdot \sigma_m \mapsto \Phi\left(\sigma_0(z), \sigma_1(z), \dots, \sigma_m(z)\right) \\ E_\Phi &: \Lambda^1\left(\mathfrak{g}[z]/\mathfrak{g}\right) \otimes \mathrm{S}^m\mathfrak{g}[z] \to \mathbb{C}[z]dz, \\ &\qquad\qquad \psi \otimes \sigma_1 \cdot \ldots \cdot \sigma_m \mapsto \Phi\left(d\psi(z), \sigma_1(z), \dots, \sigma_m(z)\right). \end{aligned}$$

The coefficients $S_\Phi(-n)$, $E_\Phi(-n)$ of z^n, resp. $z^{n-1}dz$ are restricted 0- and 1-cocycles and $H^\bullet_{\mathrm{res}}$ is freely generated by these, as Φ ranges over a generating set of invariant polynomials on $\mathfrak{g}$.

To illustrate, here are the cocycles associated to the Killing form on $\mathfrak{g}$ (notation as in §0.5):

$$S(-n) = \sum_{\substack{1 \le a \le \dim G \\ 0 \le p \le n}} \sigma^a(-p)\sigma^a(p-n), \qquad E(-n) = \sum_{\substack{1 \le a \le \dim G \\ 0 < p \le n}} p\psi^a(-p)\sigma^a(p-n).$$

We close this section with two generalisations of Theorem B. The first will be proved in Section 4; the second relies on more difficult techniques, and will only be proved in Section 10.

(1.12) *The Iwahori sub-algebra.* Let us replace $\mathfrak{g}[z]$ with an *Iwahori sub-algebra* $\mathfrak{B} \subset \mathfrak{g}[z]$, the inverse image of a Borel sub-algebra $\mathfrak{b} \subset \mathfrak{g}$ under the evaluation at $z = 0$. Note that the cocycles $S_\Phi(0)$ generate a copy of $(\mathrm{S}^\bullet\mathfrak{g}^*)^{\mathfrak{g}}$

within $H^{2\bullet}_{\mathrm{res}}(\mathfrak{g}[z,s],\mathfrak{g})$. With $\mathfrak{h} := \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, isomorphic to a Cartan sub-algebra, a similar inclusion $\mathrm{S}^\bullet\mathfrak{h}^* \to H^{2\bullet}_{\mathrm{res}}(\mathfrak{B}[s],\mathfrak{h})$ results from identifying $\mathfrak{h}^*$ with the $\mathfrak{B}$-invariants in $\mathfrak{B}^*$. Recall that $(\mathrm{S}\mathfrak{g}^*)^{\mathfrak{g}}$ embeds in $\mathrm{S}\mathfrak{h}^*$ (as the Weyl-invariant sub-algebra). It turns out that, when passing from $\mathfrak{g}[z]$ to $\mathfrak{B}$, the factor $(\mathrm{S}\mathfrak{g}^*)^{\mathfrak{g}}$ is replaced with $\mathrm{S}\mathfrak{h}^*$.

1.13 THEOREM. $H^\bullet_{\mathrm{res}}(\mathfrak{B}[s],\mathfrak{h}) \cong H^\bullet_{\mathrm{res}}(\mathfrak{g}[z,s],\mathfrak{g}) \otimes_{(\mathrm{S}(s\mathfrak{g})^*)^G} \mathrm{S}(s\mathfrak{h})^*$.

(1.14) *Affine curves.* Our second generalisation replaces $\mathfrak{g}[z]$ by the $\mathfrak{g}$-valued algebraic functions on a smooth affine curve Σ. The space $\mathfrak{g}[\Sigma]$ has no restricted dual as in Section 1.11, so we use full duals in the Koszul complex; consequently, the cohomology will be a power series algebra. Moreover, there is now a contribution from the cohomology with constant coefficients, whereas before we had $H^\bullet(\mathfrak{g}[z],\mathfrak{g};\mathbb{C}) = \mathbb{C}$, by [GL]. The last cohomology is described in (10.6).

1.15 THEOREM. *For a smooth affine curve* Σ, *the cohomology*

$$H^\bullet(\mathfrak{g}[\Sigma];(\mathrm{S}\mathfrak{g}[\Sigma])^*)$$

is densely generated over $H^\bullet(\mathfrak{g}[\Sigma];\mathbb{C})$ *by the full duals of* $P_{(m+1)} \otimes \Omega^0[\Sigma]$ *and* $P_{(m+1)} \otimes \Omega^1[\Sigma]$, *in cohomology degrees* 0 *and* 1 *and symmetric degrees* $m+1$ *and* m, *respectively. Generating co-cycles are constructed as in Theorem* B, *and the algebra is completed in the inverse limit topology defined by the order-of-pole filtration on* $\Omega^i[\Sigma]$.

2. Proof for truncated algebras

Assuming Theorem B, we now explain how Feigin's construction in [F2] proves Theorem A, the conjecture for truncated Lie algebras. Its shadow is the specialisation $t = q^n$ in the combinatorial literature ($s = z^n$ in our notation). We can resolve $\mathfrak{g}[z]/z^n$ by the *differential graded Lie algebra* $(\mathfrak{g}[z,s],\partial)$ with differential $\partial s = z^n$,

$$\left\{ s\mathfrak{g}[z] \xrightarrow{\partial: s\mapsto z^n} \mathfrak{g}[z] \right\} \xrightarrow{\sim} \mathfrak{g}[z]/z^n. \tag{2.1}$$

This identifies $H^*(\mathfrak{g}[z]/z^n)$ with the *hyper-cohomology* of $(\mathfrak{g}[z,s],\partial)$, and $H^\bullet(\mathfrak{g}[z]/z^n,\mathfrak{g})$ with the relative one of the pair $((\mathfrak{g}[z,s],\partial),\mathfrak{g})$. Recall that hyper-cohomology is computed by a *double complex*, where Koszul's differential is supplemented by the one induced by ∂. This leads to a convergent spectral sequence, with

$$E_1^{p,q} = H^{q-p}_{\mathrm{res}}(\mathfrak{g}[z],\mathfrak{g};\mathrm{S}^p(\mathfrak{g}[z])^*_{\mathrm{res}}) \Rightarrow H^{p+q}(\mathfrak{g}[z]/z^n,\mathfrak{g}). \tag{2.2}$$

The $E_1^{p,q}$ term arises by ignoring ∂, and is the portion of $H^{p+q}_{\mathrm{res}}(\mathfrak{g}[z,s],\mathfrak{g})$ with s-weight $(-p)$, cf. (1.9). If we assign weight 1 to z and weight n to s, then $(\mathfrak{g}[z,s],\partial)$ carries this additional z-*grading*, preserved by ∂ and hence by the spectral sequence.

2.3 Lemma. *Let $n > 0$. $E_2^{p,q}$ is the free skew-commutative algebra generated by the dual of the sum of vector spaces $s^{\otimes m}\mathbb{C}[z]dz/d\left(z^n\mathbb{C}[z]\right)$, placed in bi-degrees $(p,q) = (m, m+1)$, as m ranges over $m_1, \ldots, m_\ell$. The z-weight of s is n.*

Proof of Theorem A. The E_2 term of Lemma 2.3 already meets the dimensional lower bound for our cohomology (Remark 1.2.iii). Therefore, $E_2 = E_\infty$ is the associated graded ring for a filtration on $H^\bullet\left(\mathfrak{g}[z]/z^n, \mathfrak{g}\right)$, compatible with the z-grading. However, freedom of E_∞ as an algebra forces $H^\bullet$ to be isomorphic to the same, and we get the desired description of $H^\bullet\left(\mathfrak{g}[z]/z^n\right)$ from the factorisation (1.2.i). $\square$

Proof of Lemma 2.3. The description in Theorem B of the generating cocycles E_Φ and S_Φ of E_1 allows us to compute δ_1. The S_Φ have nowhere to go, but for $E_\Phi : \Lambda^1 \otimes \mathrm{S}^m \to \mathbb{C}[z]dz$, we get

$$
\begin{aligned}
\left(\delta_1 E_\Phi\right)\left(\sigma_0 \cdot \ldots \cdot \sigma_m\right) &= E_\Phi\left(\partial(\sigma_0 \cdot \ldots \cdot \sigma_m)\right) \\
&= \sum\nolimits_k E_\Phi\left(z^n\sigma_k \otimes \sigma_0 \cdot \ldots \cdot \hat{\sigma}_k \cdot \ldots \cdot \sigma_m\right) \\
&= \sum\nolimits_k \Phi\left(\sigma_0 \cdot \ldots \cdot d(z^n\sigma_k) \cdot \ldots \cdot \sigma_m\right) \\
&= (m+1)n \cdot z^{n-1}dz \cdot \Phi(\sigma_0 \cdot \ldots \cdot \sigma_m) \\
&\quad + z^n \cdot d\Phi\left(\sigma_0 \cdot \ldots \cdot \sigma_m\right) \\
&= \left((m+1)n \cdot z^{n-1}dz \cdot S_\Phi + z^n \cdot dS_\Phi\right)\left(\sigma_0 \cdot \ldots \cdot \sigma_m\right),
\end{aligned}
\tag{2.4}
$$

and so δ_1 is the transpose of the linear operator $(m+1)n \cdot z^{n-1}dz \wedge + z^n \cdot d$, from $\mathbb{C}[z]$ to $\mathbb{C}[z]dz$. This has no kernel for $n > 0$, and its co-kernel is $\mathbb{C}[z]dz/d\left(z^n\mathbb{C}[z]\right)$. $\square$

2.5 *Remark.* (i) On $\mathfrak{g}[z,s]$, ∂ is given by the super-vector field $z^n\partial/\partial s$. This acts on the presentation (1.7) of the homology primitives,

$$
z^n\partial/\partial s : \mathbb{C}[z] \cdot s(ds)^m \to \frac{\mathbb{C}[z] \cdot (ds)^m + \mathbb{C}[z]dz \cdot s(ds)^{m-1}}{d\left(\mathbb{C}[z] \cdot s(ds)^{m-1}\right)}.
\tag{2.6}
$$

Identifying the target space with $\mathbb{C}[z]dz \cdot s(ds)^{m-1}$ by projection, we can check that $z^k \cdot s(ds)^m$ maps to $(mn + n + k) \cdot z^{n+k-1}dz \cdot s(ds)^{m-1}$. This map agrees with (the dual of) the differential δ_1 in the preceding lemma, confirming our claim that the description (1.7) was natural.

(ii) If $n = 0$, the map in (2.6) is surjective, with 1-dimensional kernel; so $E_\infty^{p,q}$ now lives on the diagonal, and equals $(\mathrm{S}^p\mathfrak{g}^*)^{\mathfrak{g}}$. This is, in fact, a correct interpretation of $H^*(0, \mathfrak{g}; \mathbb{C})$.

3. The Laplacian on the Koszul complex

In preparation for the proof of Theorem B, we now study the Koszul complex for the pair $(\mathfrak{g}[z,s],\mathfrak{g})$ and establish the key formula (3.11) for its Laplacian.

(3.1) For explicit work with $\mathfrak{g}[z]$-co-chains, we introduce the following derivations on $\Lambda\otimes\mathrm{S} := \Lambda(\mathfrak{g}[z]/\mathfrak{g})^*_{\mathrm{res}}\otimes\mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}}$, describing the brutally truncated adjoint action of $\mathfrak{g}[z,z^{-1}]$:

$$\mathrm{ad}_a(m):\psi^b(n)\mapsto\begin{cases}\psi^{[a,b]}(m+n), & \text{if } m+n<0,\\ 0, & \text{if } m+n\geq 0;\end{cases}\tag{3.2}$$

$$R_a(m):\sigma^b(n)\mapsto\begin{cases}\sigma^{[a,b]}(m+n) & \text{if } m+n\leq 0,\\ 0, & \text{if } m+n>0.\end{cases}\tag{3.3}$$

Notation is as in Section 0.5, $m\in\mathbb{Z}$ and a,b range over $A:=\{1,\dots,\dim\mathfrak{g}\}$. Let

$$\bar\partial=\sum_{a\in A;m>0}\left\{\psi^a(-m)\otimes R_a(m)+\psi^a(-m)\cdot\mathrm{ad}_a(m)\otimes 1/2\right\},\tag{3.4}$$

where $\psi^a(-m)$ doubles notationally for the appropriate multiplication operator. The notation $\bar\partial$ stems from its geometric origin as a Dolbeault operator on the loop Grassmannian of G.

3.5 *Definition.* The *restricted Koszul complex* $\left(C^\bullet,\bar\partial\right)$ for the pair $(\mathfrak{g}[z],\mathfrak{g})$ with coefficients in $\mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}}$ is the $\mathfrak{g}$-invariant part of $\Lambda^\bullet\otimes\mathrm{S}$, with differential (3.4).

(3.6) *The metric and the Laplacian.* Define a hermitian metric on $\Lambda\otimes\mathrm{S}$ by setting

$$\langle\sigma^a(m)|\sigma^b(n)\rangle=1,\quad\langle\psi^a(m)|\psi^b(n)\rangle=-1/n,\qquad\text{if } m=n \text{ and } a=b,$$

and both products to zero otherwise; we then take the multi-linear extension. For example, $\|\sigma^a(m)^n\|^2=n!$. The hermitian adjoints to (3.2) are the derivations defined by

$$\mathrm{ad}_a(m)^*\psi^b(n)=\frac{n-m}{n}\psi^{[a,b]}(n-m),\quad\text{or zero, if } n\geq m.\tag{3.2*}$$

The R's of (3.3) satisfy the simpler relation $R_a(m)^*=R_a(-m)$. The adjoint of (3.4) is

$$\bar\partial^*=\sum_{a\in A;m>0}\left\{\psi^a(-m)^*\otimes R_a(-m)+\mathrm{ad}_a(m)^*\circ\psi^a(-m)^*\otimes 1/2\right\}.\tag{3.4*}$$

A (restricted) Koszul cocycle in the kernel of the *Laplacian* $\overline{\Box}:=\left(\bar\partial+\bar\partial^*\right)^2=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$ is called *harmonic*. Since $\bar\partial$, $\bar\partial^*$ and $\overline{\Box}$ preserve the orthogonal decomposition into the finite-dimensional (z,s)-weight spaces, elementary linear algebra gives the following "Hodge decomposition":

3.7 PROPOSITION. *The map from harmonic cocycles* $\mathcal{H}^k \subset C^k$ *to their cohomology classes, via the decompositions* $\ker\bar\partial = \operatorname{Im}\bar\partial \oplus \mathcal{H}^k$, $C^k = \operatorname{Im}\bar\partial \oplus \mathcal{H}^k \oplus \operatorname{Im}\bar\partial^*$, *is a linear isomorphism.*

To investigate $\overline{\square}$, we introduce the following adjoint pairs of operators:

$$d_a(m) : \sigma^b(n) \mapsto \psi^{[a,b]}(m+n), \text{ or zero, if } m+n \geq 0, \qquad d_a(m)\psi^b(n) = 0; \tag{3.8}$$

$$d_a(m)^* : \psi^b(n) \mapsto -\sigma^{[a,b]}(n-m)/n, \text{ or zero, if } n > m, \qquad d_a(m)^*\sigma^b(n) = 0, \tag{3.8*}$$

extended to odd-degree derivations of $\Lambda \otimes \mathrm{S}$. Finally, let

$$D := \sum_{m>0; a\in A} d_a(-m) d_a(-m)^*, \tag{3.9}$$

$$\square := \sum_{a\in A; m>0} \frac{1}{m} \left[R_a(-m) + \mathrm{ad}_a(-m)\right]\left[R_a(m) + \mathrm{ad}_a(-m)^*\right]. \tag{3.10}$$

3.11 THEOREM. *On* $C^\bullet$, $\overline{\square} = \square + D$. *In particular, the harmonic forms are the joint kernel in* $\Lambda\otimes\mathrm{S}$ *of the derivations* $d_a(-m)^*$, *as* $a \in A$, $m > 0$, *and* $R_a(m) + \mathrm{ad}_a(-m)^*$, *as* $a \in A$, $m \geq 0$.

It follows that the harmonic co-cycles form a sub-algebra, since they are cut out by derivations. We shall identify them in Section 4; the rest of this section is devoted to proving (3.11).

First proof of (3.11). Introduce yet another operator

$$K := \sum_{a,b\in A;\, m>0} \big(R_{[a,b]}(0) + \mathrm{ad}_{[a,b]}(0)\big) \cdot \psi^a(-m) \wedge \psi^b(-m)^*. \tag{3.12}$$

Note that the $\psi \wedge \psi^*$ factor could equally well be written in first position, because

$$\begin{aligned}
&\sum_{a,b} \Big[\mathrm{ad}_{[a,b]}(0), \psi^a(-m) \wedge \psi^b(-m)^*\Big] \\
&\quad = \sum_{a,b} \Big(\psi^{[[a,b],a]}(-m) \wedge \psi^b(-m)^* + \psi^a(-m) \wedge \psi^{[[a,b],b]}(-m)^*\Big) \\
&\quad = \sum_{a,b} \Big(\psi^{[a,b]}(-m) \wedge \psi^{[a,b]}(-m)^* - \psi^{[a,b]}(-m) \wedge \psi^{[a,b]}(-m)^*\Big) = 0.
\end{aligned}$$

As the first factor represents the total co-adjoint action of $\mathfrak{g}$ on $\Lambda\otimes\mathrm{S}$, $K = 0$ on the sub-complex $C^\bullet$ of $\mathfrak{g}$-invariants, and Theorem 3.11 is a special case of the following lemma. □

3.13 LEMMA. $\overline{\square} = \square + D + K$ *on* $\Lambda\otimes\mathrm{S}$.

Proof. All the terms are second-order differential operators on $\Lambda \otimes \mathrm{S}$. It suffices, then, to verify the identity on quadratic germs. The brutal calculations are performed in the appendix. $\square$

Second proof of (3.11). Let V be a negatively graded $\mathfrak{g}[z]$-module, such that $z^m\mathfrak{g}$ maps $V(n)$ to $V(n+m)$. Assume that V carries a hermitian inner product, compatible with the hermitian involution on the zero-modes $\mathfrak{g} \subseteq \mathfrak{g}[z]$, for which the graded pieces are mutually orthogonal. For us, V will be $\mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}}$. Write $R_a(m)$ for the action of $z^m\xi_a$ on V and define, for $m \geq 0$, $R_a(-m) := R_a(m)^*$. Define $\square$ and $\overline{\square}$ as before; our conditions on V ensure the finiteness of the sums. Define an endomorphism of $V \otimes \Lambda(\mathfrak{g}[z]/\mathfrak{g})^*_{\mathrm{res}}$ by the formula

$$(3.14)\quad T_V^\Lambda := \sum_{\substack{a,b\in A\\ m,n>0}} \left\{[R_a(m), R_b(-n)] - R_{[a,b]}(m-n)\right\} \otimes \psi^a(-m) \wedge \psi^b(-n)^*.$$

Our theorem now splits up into the two propositions that follow; the first is known as *Nakano's Identity*, the second describes T_V^Λ when $V = \mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}}$. $\square$

3.15 Proposition ([T1, Prop. 2.4.7]). *On C^k, $\overline{\square} = \square + T_S^\Lambda + k$.*

3.16 *Remark.* (i) Our $R_a(m)$ is the $\theta_a(m)$ of [T1, §2.4], whereas the operators $R_a(m)$ there are zero here, as is the level h. The constant $2c$ from [T1] is replaced here by 1, because of our use of the Killing form, instead of the basic inner product. A sign discrepancy in the definition of T_V^Λ arises, because our ξ_a here are self-adjoint, and not skew-adjoint as in [T1].

(ii) [T1] assumed finite dimensionality of V, but our grading condition is an adequate substitute.

3.17 Proposition. *On $\Lambda^k \otimes \mathrm{S}$, $D = T_S^\Lambda + k$.*

Proof. Both sides are second-order differential operators on $\Lambda \otimes \mathrm{S}$ and kill $1 \otimes \mathrm{S}$, so it suffices to check the equality on the following three terms of degree ≤ 2. Note that $T_S^\Lambda = 0$ on $\Lambda \otimes 1$, and that $\sum_a \psi^{[a,[a,b]]}(-n) = \psi^b(-n)$, because $\sum_a \mathrm{ad}(\xi_a)^2 = \mathbf{1}$ on $\mathfrak{g}$.

$$\begin{aligned}
D\psi^b(-n) &= \sum_{\substack{a\in A\\ 0<m\leq n}} d_a(-m)\sigma^{[a,b]}(m-n)/n \\
&= \sum_{\substack{a\in A\\ 0<m\leq n}} \psi^{[a,[a,b]]}(-n)/n = \psi^b(-n); \\
D\left(\psi^b(-n) \wedge \psi^c(-p)\right) &= D\psi^b(-n) \wedge \psi^c(-p) + \psi^b(-n) \wedge D\psi^c(-p) \\
&= 2 \cdot \psi^b(-n) \wedge \psi^c(-p);
\end{aligned}$$

$$\begin{aligned}D\left(\sigma^c(-p)\cdot\psi^d(-q)\right) &= \sigma^c(-p)\cdot D\psi^d(-q)\\ &\quad+\frac{1}{q}\sum_{\substack{a\in A\\0<m\leq q}}\sigma^{[a,d]}(m-q)\cdot\psi^{[a,c]}(-m-p)\\ &= \sigma^c(-p)\cdot\psi^d(-q)+T_S^\Lambda\left(\sigma^c(-p)\cdot\psi^d(-q)\right),\end{aligned}$$

with the last equality following from

$$\begin{aligned}&T_S^\Lambda\left(\sigma^c(-p)\cdot\psi^d(-q)\right)\\ &\quad=\frac{1}{q}\sum_{\substack{a\in A\\0<m}}\left\{[R_a(m),R_d(-q)]-R_{[a,d]}(m-q)\right\}\sigma^c(-p)\cdot\psi^a(-m)\\ &\quad=\frac{1}{q}\sum_{\substack{a\in A\\0<m\leq p+q}}\sigma^{[a,[d,c]]}(m-q-p)\cdot\psi^a(-m)\\ &\qquad-\frac{1}{q}\sum_{\substack{a\in A\\0<m\leq p}}\sigma^{[d,[a,c]]}(m-q-p)\cdot\psi^a(-m)\\ &\qquad-\frac{1}{q}\sum_{\substack{a\in A\\0<m\leq p+q}}\sigma^{[[a,d],c]}(m-q-p)\cdot\psi^a(-m)\\ &\quad=\frac{1}{q}\sum_{\substack{b\in A\\0<m\leq p+q}}\sigma^{[d,[a,c]]}(m-q-p)\cdot\psi^a(-m)\\ &\qquad+\frac{1}{q}\sum_{\substack{a\in A\\0<m\leq p}}\sigma^{[d,[a,c]]}(m-q-p)\cdot\psi^a(-m)\\ &\quad=\frac{1}{q}\sum_{\substack{a\in A\\p<m\leq q}}\sigma^{[d,a]}(m-q-p)\cdot\psi^{[c,a]}(-m).\end{aligned}$$

□

4. The harmonic forms and proof of Theorem B

We now use Theorem 3.11 to identify the harmonic forms in $C^\bullet$; version (B) of the strong Macdonald conjecture follows by assembling Propositions 4.5, 4.8 and 4.10.

(4.1) *Relabelling* ψ. It will help to identify $\Lambda\left(\mathfrak{g}[z]/\mathfrak{g}\right)^*_{\text{res}}$ with $\Lambda\mathfrak{g}[z]^*_{\text{res}}$ by the isomorphism $d/dz:\mathfrak{g}[z]/\mathfrak{g}\cong\mathfrak{g}[z]$. This amounts to relabelling the exterior generators, with $\psi^a(-m)$ now denoting what used to be $(m+1)\cdot\psi^a(-m-1)$ ($m\geq 0$). Relations (3.2^*) and (3.8^*) now become

$$\left.\begin{aligned}\mathrm{ad}_a(-m)^*\psi^b(-n) &= \psi^{[a,b]}(m-n),\\ d_a(-m-1)^*\psi^b(-n) &= \sigma^{[a,b]}(m-n),\end{aligned}\right\}\qquad\text{or zero, if } m>n. \tag{4.2}$$

According to (3.11), the harmonic forms in the relative Koszul complex (3.4) are the forms in $\Lambda\mathfrak{g}[z]^*_{\mathrm{res}}\otimes \mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}}$ killed by $d_a(-m-1)^*$ and $R_a(m)+\mathrm{ad}_a(-m)^*$, as $m\geq 0$ and $a\in A$.

(4.3) *The harmonic forms.* The graded vector space $\mathfrak{g}[[z,s]] := \mathfrak{g}[[z]] \oplus s\mathfrak{g}[[z]]$ carries the structure of a *super-scheme*, if we declare functions to be the skew polynomials in finitely many of the components $z^m\mathfrak{g}$, $sz^m\mathfrak{g}$. It carries the adjoint action of the super-group scheme $G[[z,s]]$, which is a semi-direct product $G[[z,s]]\cong G[[z]]\ltimes s\mathfrak{g}[[z]]$.

4.4 Lemma. *Identifying* $\Lambda\mathfrak{g}[z]^*_{\mathrm{res}}\otimes \mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}}$ *with the* (*skew*) *polynomials on* $\mathfrak{g}[[z,s]]$, *the operators* $d_a(-m-1)^*$ *and* $R_a(m)+\mathrm{ad}_a(-m)^*$, *as* $m\geq 0$, *generate the co-adjoint action of* $\mathfrak{g}[z,s]$.

Proof. This is clear from (4.2): $d_a(-m-1)^*$ is the co-adjoint action of $s\cdot z^m\xi_a$. □

4.5 Proposition. *The harmonic forms in* $C^\bullet$ *correspond to those skew polynomials on* $\mathfrak{g}[[z,s]]$ *which are invariant under the adjoint action of* $G[[z,s]]$.

Proof. Lie algebra and group invariance of functions are equivalent, because the action is locally finite and factors, locally, through the finite-dimensional quotients $\mathfrak{g}[z,s]/z^N$. □

4.6 *Remark.* The super-language can be avoided by identifying $\mathfrak{g}[[z,s]]$ with the tangent bundle to its even part $\mathfrak{g}[[z]]$, after we have declared the tangent spaces to be odd: the skew polynomials become the *polynomial differential forms* on $\mathfrak{g}[[z]]$, and the invariant skew functions under $G[[z,s]]$ correspond to the *basic* forms under the Ad-action of $G[[z]]$.

(4.7) *The invariant skew polynomials.* The (GIT) quotient $\mathfrak{g}/\!/G := \mathrm{Spec}(\mathrm{S}\mathfrak{g}^*)^G$ is the space P of primitives in the co-algebra $\mathrm{S}\mathfrak{g}/[\mathfrak{g},\mathrm{S}\mathfrak{g}]$. The quotient map $q:\mathfrak{g}\to P$ induces a morphism $Q:\mathfrak{g}[[z,s]]\to P[[z,s]]$, which is invariant under the adjoint action of $G[[z,s]]$.

4.8 Proposition. *The* ad-*invariant skew polynomials on* $\mathfrak{g}[[z,s]]$ *are precisely the pull-backs by* Q *of the skew polynomials on* $P[[z,s]]$.

Proof. Elements $\Lambda\mathfrak{g}[z]^*_{\mathrm{res}}\otimes \mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}}$ are algebraic sections of the vector bundle $\Lambda\mathfrak{g}[z]^*_{\mathrm{res}}$ over $\mathfrak{g}[[z]]$. As such, they are uniquely determined by their restriction to Zariski open subsets. The analogue holds for P. Now, the open subset $\mathfrak{g}^{rs}\subset\mathfrak{g}$ of *regular semi-simple* elements is an algebraic fibre bundle, via q, over the open subset $P^r\subset P$ of regular conjugacy classes. Let $\mathfrak{g}^{rs}[[z,s]]$ be the pull-back of $\mathfrak{g}^{rs}$ under the evaluation morphism $s=z=0$. Because of the local product structure, it is clear that ad-invariant polynomials over $\mathfrak{g}^{rs}[[z,s]]$

are precisely the pull-backs by Q of functions on $P^r[[z,s]]$. In particular, the pull-back of polynomials from $P[[z,s]]$ to $\mathfrak{g}[[z,s]]$ is injective.

Now, let f be an invariant polynomial on $\mathfrak{g}[[z,s]]$. Its restriction to $\mathfrak{g}^{rs}[[z,s]]$ has the form $g \circ Q$, for some regular function g on $P^r[[z,s]]$. Let $\mathfrak{g}^r \subset \mathfrak{g}$ be the open subset of *regular* elements. A theorem of Kostant's ensures that $q : \mathfrak{g}^r \to P$ is a submersion. In particular, it has local sections everywhere, so the morphism $Q : \mathfrak{g}^r[[z,s]] \to P[[z,s]]$ has local sections also. We can use local sections to extend our g from $P^r[[z,s]]$ to $P[[z,s]]$, because f was everywhere defined upstairs. The extension of g is unique, and its Q-lifting must agree with f everywhere, as it does so on an open set. So we have written g as a pull-back. $\square$

(4.9) *Relation to S_Φ and E_Φ.* A polynomial Φ on P defines a map $P[[z]] \to \mathbb{C}[[z]]$ by point-wise evaluation, and the mth coefficient $\Phi(-m)$ of the image series is a polynomial on $P[[z]]$. The analogue holds for differential forms, or skew polynomials on our super-schemes (4.6).

4.10 Proposition. *Let $\Phi_1, \dots, \Phi_\ell$ be a basis of linear functions on P and let $\Phi_k(m)$ $(m \le 0)$ be the associated Fourier mode basis of linear functions on $P[[z]]$. After ψ-relabelling as in* Section 4.1, *the cocycles $S_k(m)$ and $E_k(m)$ associated to Φ_k in* (B) *are the Q-lifts of $\Phi_k(m)$ and $d\Phi_k(m)$.*

Proof. For $S_k(m)$, this is the obvious equality $\Phi_k(m) \circ Q = (\Phi_k \circ q)(m)$, the $(-m)$th Fourier mode of $\Phi \circ q$ on $\mathfrak{g}[[z]]$. For $E_k(m)$, observe that when replacing skew polynomials on $X[[z,s]]$ by forms on $X[[z]]$ as in Remark 4.6 $(X = \mathfrak{g}, P)$, Q is the differential of its restriction $\mathfrak{g}[[z]] \to P[[z]]$, while $E_k(m-1) = dS_k(m)$, after our relabelling. $\square$

(4.11) *The super-Iwahori algebra.* We now deduce Theorem 1.13 from B. Let $\exp(\mathfrak{B})$ be the closed Iwahori subgroup of $G[[z]]$, whose Lie algebra is the z-adic completion $\mathfrak{B}_z$ of $\mathfrak{B}$. We write $H^\bullet_{\exp(\mathfrak{B})}(V)$, $H^\bullet_{G[[z]]}(V)$ for the *algebraic group cohomologies* of $\exp(\mathfrak{B})$, resp. $G[[z]]$ with coefficients in a representation V. Applying van Est's spectral sequence gives

$$H^\bullet\left(\mathfrak{B}, \mathfrak{h}; \mathrm{S}\mathfrak{B}^*_{\mathrm{res}}\right) = H^\bullet_{\exp(\mathfrak{B})}\left(\mathrm{S}\mathfrak{B}^*_{\mathrm{res}}\right),$$
$$H^\bullet_{\mathrm{res}}\left(\mathfrak{g}[z], \mathfrak{g}; \mathrm{S}\mathfrak{g}[[z]]^*_{\mathrm{res}}\right) = H^\bullet_{G[[z]]}\left(\mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}}\right).$$

We now relate the right-hand terms using *Shapiro's spectral sequence*

$$E_2^{p,q} = H^p_{G[[z]]}\left(R^q \operatorname{Ind}^{G[[z]]}_{\exp(\mathfrak{B})} \mathrm{S}\mathfrak{B}^*_{\mathrm{res}}\right) \Rightarrow H^{p+q}_{\exp(\mathfrak{B})}\left(\mathrm{S}\mathfrak{B}^*_{\mathrm{res}}\right),$$

whose collapse is a consequence of the following lemma, which, combined with the freedom of $\mathrm{S}\mathfrak{h}^*$ as a $(\mathrm{S}\mathfrak{g}^*)^{\mathfrak{g}}$-module, also completes the proof of Theorem 1.13. Write $R^q \operatorname{Ind}$ for $R^q \operatorname{Ind}^{G[[z]]}_{\exp(\mathfrak{B})}$.

4.12 Lemma. Ind $\mathrm{S}\mathfrak{B}^*_{\mathrm{res}} = \mathrm{S}\mathfrak{g}[z]^*_{\mathrm{res}} \otimes_{(\mathrm{S}\mathfrak{g}^*)^{\mathfrak{g}}} \mathrm{S}\mathfrak{h}^*$, *with the adjoint action of* $G[[z]]$ *on the first factor on the right*; *whereas* $R^q\,\mathrm{Ind}\,\mathrm{S}\mathfrak{B}^*_{\mathrm{res}} = 0$ *for* $q > 0$.

Proof. $R^q\,\mathrm{Ind}\,\mathrm{S}\mathfrak{B}^*_{\mathrm{res}}$ is the qth sheaf cohomology of the algebraic vector bundle $\mathrm{S}\mathfrak{B}^*_{\mathrm{res}}$ over the quotient variety $G[[z]]/\exp(\mathfrak{B}) \cong G/B$, and hence also the qth cohomology of the structure sheaf $\mathcal{O}$ over the variety $G[[z]] \times_{\exp(\mathfrak{B})} \mathfrak{B}_z$, with the adjoint action of $\exp(\mathfrak{B})$ on $\mathfrak{B}_z$. Splitting $\mathfrak{B}_z$ as $\mathfrak{b} \times z\mathfrak{g}[[z]]$ and shearing off the second factor identifies this variety with $(G \times_B \mathfrak{b}) \times z\mathfrak{g}[[z]]$. The factor $G \times_B \mathfrak{b}$ maps properly and generically finitely to $\mathfrak{g}$ via $\mu : (g, \beta) \mapsto g\beta g^{-1}$. The canonical bundle upstairs is trivial, and a theorem of Grauert and Riemenschneider ensures the vanishing of higher cohomology of $\mathcal{O}$, and thus of the higher $R^q\,\mathrm{Ind}$'s.

The functions on $G \times_B \mathfrak{b}$ are identified with $\mathrm{S}\mathfrak{h}^* \otimes_{(\mathrm{S}\mathfrak{g}^*)^G} \mathrm{S}\mathfrak{g}^*$ by the Stein factorisation of μ,

$$G \times_B \mathfrak{b} \xrightarrow{(\pi,\mu)} \mathfrak{h} \times_{\mathfrak{g}//G} \mathfrak{g} \to \mathfrak{g},$$

where $\pi : \mathfrak{b} \to \mathfrak{h}$ is the natural projection and the second arrow the second projection. (The middle space is regular in co-dimension three, therefore normal.) Using this and evaluation at $z = 0$, we can factor the conjugation morphism $G[[z]] \times_{\exp(\mathfrak{B})} \mathfrak{B}_z \to \mathfrak{g}[[z]]$ into the $G[[z]]$-equivariant maps below, of which the first has proper and connected fibres,

$$G[[z]] \times_{\exp(\mathfrak{B})} \mathfrak{B}_z \to \mathfrak{h} \times_{\mathfrak{g}//G} \mathfrak{g}[[z]] \to \mathfrak{g}[[z]].$$

This exhibits the space of functions Ind $\mathrm{S}\mathfrak{B}^*_{\mathrm{res}}$ on $G[[z]] \times_{\exp(\mathfrak{B})} \mathfrak{B}_z$ to be as claimed. □

II. Hodge theory

We now turn to a remarkable application of the strong Macdonald theorem: the determination of Dolbeault cohomologies $H^q(\Omega^p)$ and the Hodge-de Rham sequence for flag varieties of loop groups. For the loop Grassmannian X, these are described formally from $H^\bullet(BG)$ and de Rham's operator $d : \mathbb{C}[[z]] \to \mathbb{C}[[z]]dz$ on the formal disk (Theorem C). In particular, we find that the sequence collapses at E_2, and not at E_1, as in the case of smooth projective varieties. This failure of Hodge decomposition is unexpected, given the (ind-)projective nature of X; surprisingly for a homogeneous space, the explanation lies in the lack of smoothness.

Similar results hold for other flag varieties, associated as in Section 7 below to a smooth affine curve Σ; the Dolbeault groups and first differentials in the Hodge sequence arise from $d : \Omega^0[\Sigma] \to \Omega^1[\Sigma]$ (Theorem D). This is in concordance with the Hodge decomposition established in [T4] for the *closed curve* analogue of our flag varieties, the moduli stack of G-bundles over a smooth projective curve. Evidently, the failure of Hodge decomposition for flag varieties is rooted in the same phenomenon for open curves, but we do not feel that we have a satisfactory explanation.

The description of Dolbeault groups is unified conceptually in Section 8, where we construct generating co-cycles. We also interpret the Macdonald cohomology of Chapter I as the Dolbeault cohomology of the *classifying stack* $BG[[z]]$. That is also the moduli stack of principal G-bundles on the formal disk; its relevance arises by viewing the flag varieties as moduli spaces of G-bundles over the completion of Σ, trivialised in a formal neighbourhood of the divisor at infinity. The construction leads to the proofs in Section 9, and our arguments feed back in Section 10 into some new Lie algebra results, including the proof of Theorem 1.15 on the cohomology of $\mathfrak{g}[\Sigma, s]$.

To keep the statements straightforward, G will be *simple* and *simply connected.*

5. Dolbeault cohomology of the loop Grassmannian

(5.1) *The loop Grassmannian.* By the *loop group* LG of G we mean the group $G(\!(z)\!)$ of formal Laurent loops; it is an *ind-group-scheme*, filtered by the order of the pole. (The order, but not the ind-structure, depends on a choice of closed embedding G into affine space.) The *loop Grassmannian* of G is the quotient (ind-)variety $X := LG/G[[z]]$ of LG. This is *ind-projective*—an increasing union of closed projective varieties—and in fact Kodaira-embeds in a direct limit projective space [Ku]. The largest ind-projective quotient of LG is the *full flag variety* $LG/\exp(\mathfrak{B})$, which is a bundle over X with fibre the full G-flag manifold G/B; the other ind-projective quotients correspond to the subgroups of LG containing $\exp(\mathfrak{B})$.

As a homogeneous space, X is formally smooth, so there is an obvious meaning for the algebraic differentials Ω^p. The Dolbeault cohomologies[5] $H^q(X;\Omega^p)$ carry a translation action of the loop group, and a grading from the $\mathbb{C}^\times$-action scaling z (the loop rotation).

5.2 Proposition. *$H^\bullet(X;\Omega^\bullet)$ is the direct product of its z-weight spaces, and the action of LG is trivial.*

Proof. The sheaves Ω^p are sections of the pro-vector bundles associated to the co-adjoint action of $G[[z]]$ on the full duals of the exterior powers of $\mathfrak{g}(\!(z)\!)/\mathfrak{g}[[z]]$. These bundles carry a decreasing filtration $Z^n\Omega^p$ $(n > 0)$ by z-weight, and are complete thereunder. The associated sheaves $\mathrm{Gr}^n\Omega^p$ are

[5]We retain the analytic term Dolbeault cohomology to indicate the presence of differential forms, even when using *algebraic* sheaf cohomology; the distinction is immaterial for X, by GAGA.

sections of finite-dimensional bundles, stemming from the co-adjoint action of $G[[z]]$ on $\mathrm{Gr}^n\Lambda^p\left\{\mathfrak{g}((z))/\mathfrak{g}[[z]]\right\}^*$. This action factors through G by the evaluation $z=0$. The cohomologies of the $\mathrm{Gr}^n\Omega^p$ are then finite-dimensional, trivial LG-representations [Ku]; so, then, are the cohomologies $H^*(X;\Omega^p/Z^n\Omega^p)$ of the z-truncations, which are finite extensions of such representations.

The $\Omega^p/Z^n\Omega^p$ give a surjective system of sections over any ind-affine open subset of X. The Mittag-Leffler condition for their cohomologies is clear by finite-dimensionality; we conclude the equality

$$H^*(X;\Omega^p) = \lim_n H^*(X;\Omega^p/Z^n\Omega^p)$$

and the proposition. □

Our main theorem describes the Dolbeault groups of X and the action thereon of de Rham's operator $\partial : \Omega^p \to \Omega^{p+1}$. The z-adic completeness, ensured by the previous proposition, stems from the close relation of X with the formal disk (cf. the discussion of thick flag varieties in §7).

THEOREM C. (i) *$H^\bullet(X;\Omega^\bullet)$ is the z-adically completed skew power series ring generated by copies of $\mathbb{C}[[z]]$ and $\mathbb{C}[[z]]dz$, lying in $H^m(\Omega^m)$ and $H^m(\Omega^{m+1})$, respectively.*

(ii) *De Rham's differential $\partial : H^q(X;\Omega^p) \to H^q(X;\Omega^{p+1})$ is the derivation induced by $d : \mathbb{C}[[z]] \to \mathbb{C}[[z]]dz$ on generators. Its cohomology is the free algebra on ℓ generators in bi-degrees (m,m).*

In both cases, m ranges over the exponents $m_1,\dots,m_\ell$ of $\mathfrak{g}$.

The generators are constructed in Theorem 8.5, and the theorem will be proved in Section 9.

(5.3) *Failure of Hodge decomposition.* In the analytic topology, de Rham's complex $(\Omega^\bullet,\partial)$ resolves the constant sheaf $\mathbb{C}$. GAGA implies that the hypercohomology $\mathbb{H}^\bullet(X;\Omega^\bullet,\partial)$ agrees with the complex cohomology $H^\bullet(X;\mathbb{C})$. Recall [GR] that X is homotopy equivalent to the group ΩG of based continuous loops, or again, to the double loop space $\Omega^2 BG$ of the classifying space. Its complex cohomology is freely generated by the S^2-transgressions of the generators of $H^{2\bullet}(BG;\mathbb{C}) \cong (\mathrm{S}^\bullet\mathfrak{g}^*)^G$. Theorem C implies that the differential ∂_1 on $H^q(\Omega^p)$ resolves the complex cohomology of X. In other words, the *Hodge-de Rham spectral sequence* induced by ∂ on $\Omega^\bullet$ collapses at E_2.

As X is ind-projective, formally smooth and reduced [LS], we might have expected a Hodge decomposition of its complex cohomology into the $H^q(X;\Omega^p)$. Failure of this has the following consequence, as announced in [T2]. The proof is lifted from [ST, §7]. We emphasise that the result asserts more than the absence of a *global* expression for X as a union of smooth projective sub-varieties (indeed, there is a cleaner argument for this last fact, [Gr]).

5.4 THEOREM. *X is not a smooth complex manifold: that is, it cannot be expressed, locally in the analytic topology, as an increasing union of smooth complex sub-manifolds.*

Because X is homogeneous, it is singular everywhere. The same is true for the full flag variety $LG/\exp(\mathfrak{B})$, and for the loop group LG itself.

Proof of (5.4). Expressing X as a union of projective sub-varieties Y_n (for instance, the closed Bruhat varieties) gives an equivalence of X with the (0-stack) represented, over the category of complex schemes of finite type, by the groupoid $\coprod Y_n \rightrightarrows \coprod Y_n$. The two structural maps are the identity and the family of inclusions $Y_n \hookrightarrow Y_{n+1}$. In more traditional terms, this gives a *simplicial resolution* $Y_\bullet \xrightarrow{\varepsilon} X$ of X by a simplicial variety whose space of n-simplices is a union of projective varieties, for each n. Resolution of singularities and the method of hyper-coverings in [D] allows us to replace $Y_\bullet$ by a smooth simplicial resolution $X_\bullet \xrightarrow{\varepsilon} X$ (in the topology generated by proper surjective maps). The total direct image $R\varepsilon_*$ of de Rham's complex $(\Omega^\bullet, \partial; F)$ with its Hodge filtration

$$F^p\Omega^\bullet := \left[\Omega^p \xrightarrow{\partial} \Omega^{p+1} \xrightarrow{\partial} \dots\right]$$

is the *DuBois complex* [DuB] on X. The associated graded complex $\underline{\Omega}^p := \mathrm{Gr}^p R\varepsilon_*(\Omega^\bullet, \partial; F)$ is the 'correct' singular-variety analogue of the pth Hodge-graded sheaf of the constant sheaf $\mathbb{C}$. Because $X_\bullet$ is simplicially projective, the cohomology of $\underline{\Omega}^p$ satisfies the Hodge decomposition

$$H^n(X;\mathbb{C}) \cong \bigoplus_{p+q=n} H^q(X;\underline{\Omega}^p). \tag{5.5}$$

The key properties of the DuBois complex are *locality in the analytic topology* and *independence of simplicial resolution*. The restriction in [DuB] to finite-dimensional varieties need not trouble us: the arguments there show that $\underline{\Omega}^p$ is well-defined, up to canonical isomorphism, in the bounded-below derived category of coherent sheaves over the site of analytic spaces, in the topology generated by both projective morphisms and open covers. Here, we are studying the hyper-cohomology of these $\underline{\Omega}^p$ in the restricted site of analytic spaces over X. These properties would lead to a quasi-isomorphism $\underline{\Omega}^p \sim \Omega^p$, If X was a complex manifold in the sense of Theorem 5.4. But then, (5.5) conflicts with Theorem C. □

6. Application: A ${}_1\psi_1$ summation

The $H^q(X;\Omega^p)$ are graded by z-weight, with finite-dimensional weight spaces. The z-weighted holomorphic Euler characteristics for all p can be collected in the *E-series*

$$E(z,t) := \sum\nolimits_{p,q} (-1)^q (-t)^p \dim_z H^q\left(X,\Omega^p\right) \in \mathbb{Z}[[z,t]]. \tag{6.1}$$

(6.2) *The Kac formula.* The Mittag-Leffler conditions in the proof of Proposition 5.2 imply the convergence of the spectral sequence for the Z-filtration,

$$E_1^{r,s} = H^{r+s}\left(X; \mathrm{Gr}^r\Omega^p\right) \Rightarrow H^{r+s}\left(X;\Omega^p\right),$$

whence it follows that our Euler characteristic is already computed by E_1. Because $\mathrm{Gr}\,\Omega^p$ is a product of bundles associated to irreducible representations of $G[[z]]$, the $E(z,t)$ can be described explicitly using the *Kac character formula* [K]. Choose a maximal torus $T \subset G$ and recall that the *affine Weyl group* W_{aff} is the semi-direct product of the finite Weyl group by the co-root lattice. This W_{aff} acts on Fourier polynomials on T and in z, whereby a co-root γ sends the Fourier mode e^λ of T to $z^{\langle\lambda|\gamma\rangle}\mathrm{e}^\lambda$. (The Weyl group acts in the obvious way, and z is unaffected.) The desired formula is the infinite sum of infinite products, where α ranges over the roots of $\mathfrak{g}$,

$$\sum_{w\in W_{\mathrm{aff}}} \prod_{\substack{n>0\\ \alpha}} w\left(\frac{1-tz^n\mathrm{e}^\alpha}{1-z^n\mathrm{e}^\alpha}\right)\cdot \prod_{\alpha>0} w(1-\mathrm{e}^\alpha)^{-1}\cdot \prod_{n>0}\left[\frac{1-tz^n}{1-z^n}\right]^\ell. \tag{6.3}$$

The summands are the w-transforms of the quotient of the (T,z,t)-character of the fibre $\sum_p(-t)^p\mathrm{Gr}\,\Omega^p$ at the base-point of X by the Kac denominator. The sum expands into a formal power series in z and t, with characters of T as coefficients.

(6.4) *Relation to Ramanujan's ${}_1\psi_1$ sum.* Factoring affine Weyl elements as $\gamma\cdot w$ (co-root times finite Weyl element) and leaving out, for now, we see that the third factor converts (6.3) into

$$\sum_{\gamma}\prod_{\substack{n>0\\ \alpha}} \frac{1-tz^{n+\langle\alpha|\gamma\rangle}\mathrm{e}^\alpha}{1-z^{n+\langle\alpha|\gamma\rangle}\mathrm{e}^\alpha}\cdot \sum_{w\in W}\prod_{\alpha>0}(1-z^{\langle w\alpha|\gamma\rangle}\mathrm{e}^{w\alpha})^{-1},$$

where we have substituted $\alpha\mapsto w\alpha$ in the first product, in order to make it w-independent. The second factor, the sum over W, is identically 1, by the Weyl denominator formula. Equating now (6.3) with our answer in Theorem C gives the following identity:

$$\sum_{\gamma} \prod_{\substack{n>0 \\ \alpha}} \frac{1 - tz^{n+\langle\alpha|\gamma\rangle} \mathrm{e}^{\alpha}}{1 - z^{n+\langle\alpha|\gamma\rangle} \mathrm{e}^{\alpha}} = \prod_{\substack{1 \le k \le \ell \\ n \ge 0}} \frac{(1 - z^{n+1})(1 - t^{m_k+1} z^{n+1})}{(1 - tz^{n+1})(1 - t^{m_k} z^{n})}.$$

The third factor in 6.3 has been moved to the right side. It is part of the statement that the left-hand side is constant, as a function on T.

For $G = \mathrm{SL}_2$, we obtain, after setting $\mathrm{e}^{\alpha} = u$, the identity

$$\sum_{m} \prod_{n>0} \frac{(1 - tz^{n+2m}u^2)(1 - tz^{n-2m}u^{-2})}{(1 - z^{n+2m}u^2)(1 - z^{n-2m}u^{-2})} = \frac{1}{1-t} \prod_{n>0} \frac{(1 - z^n)(1 - t^2 z^n)}{(1 - tz^n)^2}$$

which also follows from a 3-variable specialisation of Ramanujan's ${}_1\psi_1$ sum [T2, §5]. (Note that our sum contains the even terms only; the "other half" of the specialised ${}_1\psi_1$ sum is carried by the twisted SL_2 loop Grassmannian, the odd component of $LG/G[[z]]$ for $G = \mathrm{PSL}_2$.) Thus, Theorem C is a *strong form* of (specialised) ${}_1\psi_1$ summation, generalised to (untwisted) affine root systems. We later learned that (the "weak" forms of) such generalised summation formulae, for all affine root systems, were independently discovered and proved by Macdonald [M].

7. Thick flag varieties

Related and, in a sense, opposite to X is the quotient variety $\mathbf{X} := LG/G[z^{-1}]$. This is a scheme covered by translates of the open cell $\mathbf{U} \cong G[[z]]/G$, the $G[[z]]$-orbit of 1. Generalisations of $\mathbf{X}$ are associated to smooth affine curves Σ, with divisor at infinity D in their smooth completion $\overline{\Sigma}$. These generalised flag varieties are the quotients $\mathbf{X}_\Sigma := L^D G/G[\Sigma]$ of a product $L^D G$ of loop groups, defined by local coordinates centred at the points of D, by the ind-subgroup $G[\Sigma]$ of G-valued regular maps. Variations decorated by bundles of G-flag varieties, attached to points of Σ, also exist, and our results can be easily extended to those, but we shall not spell that out. When a distinction is needed, we call the $\mathbf{X}_\Sigma$ and their variations *thick flag varieties* of LG.

(7.1) *Relation to moduli spaces.* One formulation of the *uniformisation theorem* of [LS] equates X_Σ with the moduli space pairs $(\mathcal{P}, \sigma)$ of algebraic principal G-bundles $\mathcal{P}$ over $\overline{\Sigma}$, equipped with a section σ over the formal neighbourhood $\widehat{D}$ of the divisor at infinity. In other words, $\mathbf{X}_\Sigma$ is the moduli space of relative G-bundles over the pair $(\overline{\Sigma}, \widehat{D})$, and we also denote it by $\mathfrak{M}(\overline{\Sigma}, \widehat{D})$. Here, $\mathfrak{M}$ stands for the *stack of morphisms to* BG, the classifying stack of G [T3, App. B]; thus, $\mathfrak{M}(\overline{\Sigma})$ is the moduli stack of G-bundles over the closed curve. The corresponding description of X is the moduli space of pairs, consisting of a G-bundle over $\mathbb{P}^1$ and a section over $\mathbb{P}^1 \setminus \{0\}$; this is the moduli space $\mathfrak{M}(\mathbb{P}^1, \mathbb{P}^1 \setminus \{0\})$ of bundles over the respective pair. In this sense, X is the $\mathbf{X}$ associated to the formal disk around 0. Slightly more generally, $\mathfrak{M}(\overline{\Sigma}, \Sigma)$ is the product of loop Grassmannians associated to the points of D.

The thick flag varieties are smooth in an obvious geometric sense: the open cell in $\mathbf{X}$ is isomorphic to the vector space $\mathfrak{g}[[z]]/\mathfrak{g}$, while $\mathbf{X}$ is a principal $G[[z]]$-bundle over $\mathfrak{M}(\overline{\Sigma})$. In their case, failure of Hodge decomposition in Theorem D below should be attributed to their "noncompactness".

(7.2) *Technical note on spaces.* We shall use the terms *space* or, abusively, *variety*, for the homogeneous spaces of LG. They live in a suitable world of contravariant functors on complex schemes: thus, the functor $\mathbf{X}_\Sigma$ sends a scheme S to the set $\operatorname{Hom}(S, \mathbf{X}_\Sigma)$ of isomorphism classes of bundles over $(S \times \overline{\Sigma}, S \times \widehat{D})$, and the ambient world is the category of sheaves over the topos of complex schemes, in the smooth (or étale) topology. To include stacks, we must enrich the structure to include the simplicial sheaves and their homotopy category; [T3] gives a brief introduction to this jargon. For the stack $\mathfrak{M}$ and the thin flag variety X, we can confine ourselves to the sub-category of schemes of finite type, because the two are covered by sub-stacks, respectively varieties of finite type. This restriction to finite type is necessary when discussing the Hodge structure.

(7.3) *Cohomology and Hodge structure.* Recall now the analogue of the homotopy equivalence $X \sim \Omega G$ for thick varieties $\mathbf{X}_\Sigma$. The natural morphism from $\mathbf{X}_\Sigma = \mathfrak{M}(\overline{\Sigma}, \widehat{D})$ to the stack $\mathfrak{M}(\overline{\Sigma}, D)$ of G-bundles on $(\overline{\Sigma}, D)$ (trivialised over D) is a fibre bundle in affine spaces; in particular, it is a homotopy equivalence. Similarly to [T3, Th. 1′], in which $D = \emptyset$, this last stack has the homotopy type of the space of the continuous maps from $\overline{\Sigma}$ to BG, based at D; the equivalence is the forgetful functor from the stack of (D-based) analytic bundles to that of continuous bundles.[6]

Generators of the algebra $H^\bullet(\mathfrak{M}(\overline{\Sigma}, D), \mathbb{Q})$ arise by transgressing those of $H^\bullet(BG)$ along a basis of cycles in $H_\bullet(\overline{\Sigma}, D)$; the latter is also the *Borel-Moore homology* $H_\bullet^{BM}(\Sigma)$. As the classifying morphism $(\overline{\Sigma}, D) \times \mathfrak{M}(\overline{\Sigma}, D) \to BG$ for the universal bundle is algebraic, the construction of generating classes is compatible with Hodge structures and we obtain as in [T3, Ch. IV]

7.4 Proposition. *$H^\bullet(\mathfrak{M}(\overline{\Sigma}, D))$, with its Hodge structure, is the free algebra generated by* $\operatorname{Gen} H^\bullet(BG) \otimes H_\bullet^{BM}(\Sigma)$, *with the natural Hodge structures on the factors.*

Recall [D] that the Hodge structure on BG is pure of type (p,p). We can use the isomorphism $H^\bullet(\mathbf{X}_\Sigma) \cong H^\bullet(\mathfrak{M}(\overline{\Sigma}, D))$ to define the Hodge structure on $\mathbf{X}_\Sigma$, which is a scheme of infinite type. (By the argument in §7.2, it agrees with the structure of the functor represented by $\mathbf{X}_\Sigma$ over the schemes of finite type.)

[6] This can be seen from the Atiyah-Bott construction of $\mathfrak{M}(\overline{\Sigma})$.

(7.5) *Differentials.* Denote by Ω^p the sheaf of algebraic differential p-forms on any of our flag varieties. On X, this is the sheaf of sections of a pro-vector bundle, dual to $\Lambda^p TX$, but on thick flag varieties, it corresponds to an honest vector bundle, albeit of infinite rank. There is a de Rham differential $\partial : \Omega^p \to \Omega^{p+1}$.

7.6 PROPOSITION (Algebraic de Rham). $\mathbb{H}^\bullet(\mathbf{X}_\Sigma; (\Omega^\bullet, \partial)) = H^\bullet(\mathbf{X}_\Sigma; \mathbb{C})$, *the former being the algebraic sheaf (hyper)cohomology, the latter defined in the analytic topology.*

Proof. For $\mathbf{X}$, we use the standard Čech argument for the covering by the affine Weyl translates of the open cell; each finite intersection of the covering sets is a complement of finitely many coordinate hyperplanes in $\mathfrak{g}[[z]]/\mathfrak{g}$, where de Rham's theorem is obvious. The more general $\mathbf{X}_\Sigma$ are bundles in affine spaces over the (smooth, locally Artin) stacks $\mathfrak{M}(\overline{\Sigma}, D)$; de Rham's theorem for the total space follows from its knowledge on the fibres and on the base. $\square$

There results a convergent Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^q(\mathbf{X}; \Omega^p), \qquad E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}(\mathbf{X}; \mathbb{C}), \tag{7.7}$$

with the graded parts Gr^p of H^* associated to the *naïve Hodge filtration*, the images of the truncated hyper-cohomologies $\mathbb{H}^*\big(\mathbf{X}; (\Omega^{\geq p}, \partial)\big)$. We note in passing that, just as in the case of X, the LG-action on $H^q(\Omega^p)$ is trivial [T3, Rem. 8.10].

THEOREM D. (i) $H^\bullet(\mathbf{X}_\Sigma; \Omega^\bullet)$ *is the free skew-commutative algebra generated by copies of* $\Omega^0[\Sigma]$ *and of* $\Omega^1[\Sigma]$, *in* $H^m(\Omega^m)$, *respectively* $H^m(\Omega^{m+1})$, *as* m *ranges over the exponents of* $\mathfrak{g}$.

(ii) *The first Hodge-de Rham differential* ∂_1 *is induced by de Rham's operator* $d : \Omega^0[\Sigma] \to \Omega^1[\Sigma]$ *on generators, and the spectral sequence collapses at* E_2.

The theorem will be proved in Section 9. Assuming it, we see that Proposition 7.4 implies that E_2 already has the size of $H^\bullet(\mathbf{X}_\Sigma; \mathbb{C})$; this forces the vanishing of ∂_2 and higher differentials.

8. Uniform description of the cohomologies

We now relate the Dolbeault and Macdonald cohomologies. In the process, we give a unified construction for the generating Dolbeault classes in Theorems B, C and D.

(8.1) *Moduli spaces and stacks.* In Section 7, we identified the thick flag variety $\mathbf{X}_\Sigma$ and the loop Grassmannian X with the moduli spaces $\mathfrak{M}(\overline{\Sigma}, \widehat{D})$

and $\mathfrak{M}(\mathbb{P}^1, \mathbb{P}^1 \setminus \{0\})$ of G-bundles over the respective pairs. Their Dolbeault groups are described in Theorems C and D. For $\mathfrak{M}(\overline{\Sigma})$, Hodge decomposition [T4] implies that $H^\bullet(\Omega^\bullet)$ is the free algebra on the bi-graded vector space $H^{\bullet,\bullet}(\overline{\Sigma})^* \otimes \mathrm{Gen}H^{\bullet,\bullet}(BG)$; this is Proposition 7.4 with $D = \emptyset$.

We now give a Dolbeault interpretation of Theorem B. Consider the stack $\mathfrak{M}(\widehat{D})$ of G-bundles on $\widehat{D}$. Such bundles are trivial (locally in any family), but their automorphisms are locally represented by the group $G[\widehat{D}]$ of regular formal loops. So $\mathfrak{M}(\widehat{D})$ is the classifying stack $BG[\widehat{D}]$. Cathelineau [C] identified the Hodge-de Rham sequence for the classifying stack of a complex Lie group $\mathcal{G}$ (defined, say, from the simplicial realisation) with the holomorphic Bott-Shulman-Stasheff spectral sequence [BSS]

$$E_1^{p,q} = H^{q-p}(B\mathcal{G}; \mathcal{O}^{an} \otimes \mathrm{S}^p\mathrm{Lie}(\mathcal{G})^*) \Rightarrow H^{p+q}(B\mathcal{G}; \mathbb{C}), \tag{8.2}$$

in which E_1 is the group cohomology with $\mathrm{SLie}(\mathcal{G})^*$-valued analytic co-chains and the abutment is the cohomology with constant coefficients. The result applies to any group sheaf $\mathcal{G}$ over the site of algebraic or analytic spaces: indeed, (8.2) is the *descent spectral sequence* for the following fibration of classifying stacks, where $\widehat{\mathcal{G}}$ denotes the formal group of $\mathcal{G}$ at the identity:

$$B\widehat{\mathcal{G}} \hookrightarrow B\mathcal{G} \twoheadrightarrow B(\mathcal{G}/\widehat{\mathcal{G}}).$$

The base of this fibration has the property that $H^\bullet(B(\mathcal{G}/\hat{\mathcal{G}}); \mathcal{O}) = H^\bullet(B\mathcal{G}; \mathbb{C})$, by de Rham's theorem in the category of spaces. The first differential $H^n(\mathrm{S}^p) \to H^{n-1}(\mathrm{S}^{p+1})$ sends a group cocycle $\chi : \mathcal{G}^{n+1} \to \mathrm{S}^p$ to the sum of transposes of its derivatives $d_i\chi : \mathcal{G}^n \times \mathrm{Lie}(\mathcal{G}) \to \mathrm{S}^p$ at 1 along the components $i = 0, \dots, n$, symmetrised to land in S^{p+1}.

For simplicity, let D be a single point, so that $G[\widehat{D}] \cong G[[z]]$. Contractibility of $G[[z]]/G$ leads to a van Est isomorphism [T3] between the cohomology $H^\bullet(BG[[z]]; \mathrm{S}^p\mathfrak{g}[[z]]^*)$ over the algebraic site of $BG[[z]]$ and the Lie algebra cohomology $H^\bullet(\mathfrak{g}[[z]], \mathfrak{g}; \mathrm{S}^p\mathfrak{g}[[z]]^*)$, computed using continuous duals in the Koszul complex (this is the restricted cohomology of §2). Theorem B then says that $E_1^{p,q}$ in the Hodge-de Rham sequence for $BG[[z]]$ is the algebra generated by the continuous duals of $\mathbb{C}[[z]]dz$ and $\mathbb{C}[[z]]dz$, in bi-degrees $(p,q) = (m,m)$ and $(m+1, m)$, respectively. The first Hodge-to-de Rham differential converts an odd generator in $\Lambda \otimes \mathrm{S}$ to its even partner: this is induced by $s \mapsto 1$, or equivalently, $n = 0$ in (2.1). We showed in Section 2 that this leads to the dual of de Rham's operator $d^* : (\mathbb{C}[[z]]dz)^* \to \mathbb{C}[z]^*$ on generators.

(8.3) *Sheaf cohomology for a pair.* For a coherent sheaf $\mathcal{S}$ on $\overline{\Sigma}$, define the *cohomology* $H^\bullet(\overline{\Sigma}, \widehat{D}; \mathcal{S})$ *relative to* $\widehat{D}$ as the hyper-cohomology of the 2-term complex $\mathcal{S} \to \mathcal{S}_{\widehat{D}}$, starting in degree 0, mapping $\mathcal{S}$ to its completion at D.[7] If

[7] This is also the coherent sheaf cohomology *with proper supports* on the open curve $\Sigma = \overline{\Sigma} \setminus D$; it only depends on the restriction of $\mathcal{S}$ to Σ.

$D = \emptyset$, this is the ordinary sheaf cohomology on $\overline{\Sigma}$; else, H^0 is the torsion of $\mathcal{S}$ over Σ, and H^1 is identified with $\mathrm{Hom}_{\overline{\Sigma}}(\mathcal{S}, \Omega^1)^*$ by Serre duality. The groups relevant for us are

$$H^1(\overline{\Sigma}, \widehat{D}; \mathcal{O}) \cong \Omega^1[\Sigma]^*, \qquad H^1(\overline{\Sigma}, \widehat{D}; \Omega^1) \cong \Omega^0[\Sigma]^*,$$

Serre dual to the opposite-degree differentials on Σ. Similarly, $H^\bullet(\overline{\Sigma}, \Sigma; \mathcal{S})$ is the hyper-cohomology of $\mathcal{S} \to i_* i^* \mathcal{S}$, where $i : \Sigma \hookrightarrow \overline{\Sigma}$ is the inclusion. Again, we want $\mathcal{S} = \Omega^{0,1}$, when H^0 vanishes and Serre duality describes the H^1's as the continuous duals

$$H^1(\overline{\Sigma}, \Sigma; \mathcal{O}) \cong \Omega^1[\widehat{D}]^*, \qquad H^1(\overline{\Sigma}, \Sigma; \Omega^1) \cong \Omega^0[\widehat{D}]^*,$$

also known as the $\mathcal{O}$- and Ω^1-valued residues on $\overline{\Sigma}$ at D. When $\Sigma = \mathbb{P}^1 \setminus \{0\}$, these are the restricted duals of $\mathbb{C}[z]dz$ and $\mathbb{C}[z]$.

(8.4) *Dolbeault generators from the Atiyah class.* For a principal G-bundle $\mathcal{P}$ over a smooth base B, the tangent bundle to the total space of $\mathcal{P}$ is G-equivariant; it thus descends to B, where it gives an extension $\mathrm{ad}_{\mathcal{P}} \to T\mathcal{P}/G \to TB$. This extension defines the *Atiyah class* in $H^1(B; \mathrm{ad}_{\mathcal{P}} \otimes \Omega^1)$. With S standing for $\widehat{D}$, $\overline{\Sigma}$ or one of the pairs $(\overline{\Sigma}, \Sigma)$ or $(\overline{\Sigma}, \widehat{D})$ and the universal G-bundle $\mathcal{P}$ over $S \times \mathfrak{M}(S)$, we obtain the universal Atiyah class

$$\alpha_S \in \mathbb{H}^1\left(S \times \mathfrak{M}(S); \mathrm{ad}_{\mathcal{P}} \otimes \Omega^1\right).$$

(Keep in mind that differentials form a complex when $\mathfrak{M}$ is a stack.) An invariant polynomial Φ of degree d on $\mathfrak{g}$ defines a class $\Phi(\alpha) \in \mathbb{H}^d\left(S \times \mathfrak{M}(S); \Omega^d\right)$. We are now in a position to describe the Dolbeault cohomology generators in Theorems B, C and D.

8.5 THEOREM. *Let S stand for $\widehat{D}$, $\overline{\Sigma}$ or one of the pairs $(\overline{\Sigma}, \Sigma)$ or $(\overline{\Sigma}, \widehat{D})$. Then, $\mathbb{H}^q(\mathfrak{M}(S); \Omega^p)$ is the free skew-commutative algebra on*

$$H^\bullet(S; \Omega^\bullet)^* \otimes \mathrm{Gen}^{\bullet,\bullet}(BG).$$

More precisely, as Φ ranges over $\mathrm{Gen}^{d,d}(BG)$, Serre duality contraction of $\Phi(\alpha)$ with $H^a(S; \Omega^b)^*$ gives the Dolbeault generators in $\mathbb{H}^{d-a}\left(\mathfrak{M}(S); \Omega^{d-b}\right)$. The first Hodge-de Rham differential ∂_1 is induced by de Rham's differential on generators, and all higher differentials vanish.

This construction is clearly a Dolbeault refinement of the topological one in Proposition 7.4. For $S = \overline{\Sigma}$, Hodge decomposition on $\overline{\Sigma}$ and $\mathfrak{M}$ equates Dolbeault and de Rham cohomologies, so the result is clear. Of course, in that case $\partial_1 = 0$. For $S = (\overline{\Sigma}, \Sigma)$ or $(\overline{\Sigma}, \widehat{D})$ (the flag varieties), the new statement refines Theorems C and D, and will be proved in the next section. For the remaining case $S = \widehat{D}$, we must relate the newly constructed generators to those of Theorem B. To do so, we must say more about α.

(8.6) *The Atiyah class spelled out.* Splitting $\Omega^1 = \Omega^1_S \oplus \Omega^1_{\mathfrak{M}(S)}$, the two components of α_S can be interpreted as the Kodaira-Spencer deformation maps for the principal bundle $\mathcal{P}$, first regarded as a family of bundles over $\mathfrak{M}(S)$ parametrised by S, and then as a family of bundles on S parametrised by $\mathfrak{M}(S)$. Now, $\Omega^1_{\mathfrak{M}(S)}$ is the dual of $R\pi_*\mathrm{ad}_{\mathcal{P}}[1]$, for the projection π along S to $\mathfrak{M}(S)$, and from the very definition of $\mathfrak{M}(S)$ as the stack of all G-bundles, α_S has a tautological component

$$(8.7) \qquad \mathrm{Id} \in R\mathrm{Hom}_{\mathfrak{M}(S)}\left(R\pi_*(\mathrm{ad}_{\mathcal{P}}); R\pi_*(\mathrm{ad}_{\mathcal{P}})\right) \cong \mathbb{H}^1\left(S \times \mathfrak{M}(S); \Omega^1_{\mathfrak{M}(S)}\right).$$

The more geometric component lives in $H^1(S\times\mathfrak{M}(S); \Omega^1_S \otimes \mathrm{ad}_{\mathcal{P}})$. Locally on S, we see its "leading term" in $\Gamma\left(S; \Omega^1_S \otimes R^1\pi'_*\mathrm{ad}_{\mathcal{P}}\right)$, with the projection π' to S: this represents the local Kodaira-Spencer deformation for $\mathcal{P}$, as a bundle on $\mathfrak{M}(S)$.

8.8 *Remark.* The remaining information, with respect to the Leray sequence for π', would live in $H^1\left(S; \Omega^1 \otimes \pi'_*\mathrm{ad}_{\mathcal{P}}\right)$; however, the sheaf $\pi'_*\mathrm{ad}_{\mathcal{P}}$ is null in all our examples. On the other hand, our results show that the sheaf $R^1\pi'_*\mathrm{ad}_{\mathcal{P}}$ is the tangent sheaf T over $S = \widehat{D}$ or $\overline{\Sigma}$, and one can show that it is i_*T, when $S = (\overline{\Sigma}, \Sigma)$, and j_*T for the inclusion $j : \widehat{D} \to \overline{\Sigma}$, in the remaining case $S = \widehat{D}$. In this picture, one can show the first component of α is always the constant section 1 of $\Omega^1_S \otimes R^1\pi'_*\mathrm{ad}_{\mathcal{P}}$.

Let us spell out $\alpha_{\widehat{D}}$ when $\widehat{D} = \mathrm{Spf}\,\mathbb{C}[[z]]$. The cotangent complex of $\widehat{D} \times BG[\widehat{D}]$ splits as $\mathbb{C}dz \oplus \mathfrak{g}[[z]]^*[-1]$, the second summand carrying the co-adjoint action of $G[[z]]$. This gives

$$(8.9) \quad H^1\left(\widehat{D} \times BG[\widehat{D}]; \mathrm{ad} \otimes \Omega^1\right) = H^1\left(BG[[z]]; \mathfrak{g}[[z]]dz\right) \oplus \mathrm{Hom}_{G[[z]]}\left(\mathfrak{g}[[z]]; \mathfrak{g}[[z]]\right).$$

The second, tautological component of α is the identity in (8.7). We claim that the first component is the group co-cycle $\gamma \mapsto -d\gamma \cdot \gamma^{-1}$. (In fact, both groups are free $\mathbb{C}[[z]]$-modules of rank one generated by the named classes, but we do not need this.)

Now, $T\mathcal{P}/G$, over $\widehat{D} \times BG[[z]]$, is a $G[[z]]$-equivariant complex over $\widehat{D}$. Ignoring for a moment the group action, this complex is $\mathfrak{g}[\widehat{D}] \xrightarrow{e\oplus 0} (\mathfrak{g} \oplus T\widehat{D})$: the evaluation map $e : \mathfrak{g}[[z]] \times \widehat{D} \to \mathfrak{g}$ is the differential of the structural $G[[z]]$-action on $\mathcal{P}$ (and represents the transpose of the tautological component of α). Now, $G[[z]]]$ acts by Ad on the first term of the complex, while the second term is a $G[[z]]$-equivariant extension

$$\left[\mathfrak{g} \to \left(T\mathcal{P}|_{\widehat{D}}\right)/G \to T\widehat{D}\right] \in \mathrm{Ext}^1_{\widehat{D}\times BG[[z]]}(T\widehat{D}; \mathfrak{g}),$$

and this Ext-group is isomorphic to our first summand in (8.9). To understand the extension class, observe that a $\gamma \in G[[z]]$ changes a splitting $\mathfrak{g} \oplus T\widehat{D}$ of the tangent bundle $T\mathcal{P}/G$ by sending a section (ξ, v) to $(\gamma\xi\gamma^{-1} - v(\gamma)\cdot\gamma^{-1}, v)$. The derivative term represents the class of the group co-cycle $\gamma \mapsto -d\gamma\cdot\gamma^{-1}$ in (8.9).

Proof of (8.5) *when* $S = \widehat{D}$. Applying Φ to the tautological component of α in (8.9) gives

$$\Phi(\mathrm{Id}) \in \mathrm{Hom}_{G[[z]]}\left(\hat{\mathrm{S}}^d\mathfrak{g}[[z]]; \mathbb{C}[[z]]\right),$$

and contracting with the Fourier mode $z^n \in \mathbb{C}[[z]]$ gives the co-cycle $S(-n)$ of Theorem B, viewed as an element of

$$H^d(BG[[z]]; \Omega^d) = \left(\mathrm{S}^d\mathfrak{g}[[z]]^*\right)^{G[[z]]}.$$

The first factor of α in (8.9) squares to 0, and so Φ takes no more than one entry from there. Absorbing one entry from $T^*\widehat{D}\otimes\mathrm{ad} = \mathfrak{g}[[z]]dz$ and contracting against $z^{n-1}dz$ lead to the group 1-cocycle $G[[z]] \to \hat{\mathrm{S}}^{d-1}\mathfrak{g}[[z]]^*$ which is the contraction of Φ with $-d\gamma\cdot\gamma^{-1}$. Via the van Est isomorphism with $H^1(\mathfrak{g}[[z]], \mathfrak{g};\ .\ )$, this becomes the odd generator $E(-n)$. ☐

9. Proof of Theorems C and D

We now compute the Dolbeault cohomology for thick flag varieties. For convenience, in this section we write $\mathbf{X}$ for $\mathbf{X}_\Sigma$ and $\mathfrak{M}$ for $\mathfrak{M}(\overline{\Sigma})$, and continue to assume that D is a single point; the changes needed for the general case are obvious. A small modification then gives us Theorem C.

(9.1) *Setting up the spectral sequence.* Uniformisation (§7.1) realises $\mathfrak{M}$ as the quotient stack $G[[z]]\setminus\mathbf{X}$. Equivariance under the translation $G[[z]]$-action on $\mathbf{X}$ makes the bundle Ω^p of differential p-forms descend to a bundle on $\mathfrak{M}$; we denote the descended bundle by $\Omega^p_{\mathbf{X}}$. The complex of differentials $\Omega^r = \Omega^r_{\mathfrak{M}}$ on $\mathfrak{M}$ is represented by a Koszul-style complex of bundles

$$\Omega^r \sim \left(\Omega^r_{\mathbf{X}} \xrightarrow{\kappa} \mathrm{S}^1\mathfrak{g}[[z]]^*\otimes\Omega^{r-1}_{\mathbf{X}} \xrightarrow{\kappa} \mathrm{S}^2\mathfrak{g}[[z]]^*\otimes\Omega^{r-2}_{\mathbf{X}} \xrightarrow{\kappa} \cdots\right), \tag{9.2}$$

cohomologically graded by symmetric degree. To describe the differential, observe that a choice of a loop $\gamma \in LG$ identifies the tangent space to $\mathbf{X}$ at $\gamma G[\Sigma]$ with $L\mathfrak{g}/\mathfrak{g}[\Sigma]$; thereunder, κ at $[\gamma] = G[[z]]\gamma G[\Sigma] \in \mathfrak{M}$ is induced by the γ-twisted dual to the natural projection $\mathfrak{g}[[z]] \to L\mathfrak{g}/\mathfrak{g}[\Sigma]$.

The complex (9.2) has finite length, and so it leads to a convergent spectral sequence with

$$E_1^{k,l} = H^l\left(\mathfrak{M}; \mathrm{S}^k\mathfrak{g}[[z]]^*\otimes\Omega^{r-k}_{\mathbf{X}}\right) \Rightarrow H^{k+l}(\mathfrak{M};\Omega^r). \tag{9.3}$$

There is one such spectral sequence for each $r \geq 0$, but the product, which is compatible with the differentials, mixes them. We have an identification of cohomologies

$$H^l(\mathfrak{M}; \mathrm{S}^k \otimes \Omega^{r-k}_{\mathbf{X}}) = H^l_{G[[z]]}\left(\mathbf{X}; \mathrm{S}^k \otimes \Omega^{r-k}\right),$$

where $H^k_{G[[z]]}$ is the (algebraic) equivariant cohomology.

(9.4) *The Key Factorisation.* Our E_1 term (9.3) factors as

$$E_1^{k,l} = \bigoplus\nolimits_s H^s_{G[[z]]}\left(\mathrm{S}^k\mathfrak{g}[[z]]^*\right) \otimes H^{l-s}\left(\mathbf{X}; \Omega^{r-k}\right). \tag{9.5}$$

A priori, the right-hand side is the ${}^LE_2^{s,l-s}$ term in the Leray sequence for the sheaf $\mathrm{S}^k \otimes \Omega^{r-k}$ and the morphism $\mathfrak{M} \to BG[[z]]$. However, no differentials are present, because ${}^LE_2$ is generated from the bottom edge ${}^LE_2^{s,0}$ by cup-product with classes which live on the total space: indeed, because $G[[z]]$ acts trivially on the cohomology and $H^{>0}_{BG[[z]]}(\mathcal{O}) = 0$, we have an isomorphism

$$H^l\left(\mathbf{X}; \Omega^{r-k}\right) \cong H^l_{G[[z]]}\left(\mathbf{X}; \Omega^{r-k}\right).$$

This also shows that (9.5) is a natural isomorphism, and not just the Gr of one.

(9.6) *Determining the spectral sequence.* The factor $H^s(BG[[z]]; \mathrm{S}^k\mathfrak{g}[[z]]^*)$ is isomorphic to the Macdonald cohomology of Theorem B. The abutment $H^s(\mathfrak{M}; \Omega^r) = H^{r,s}(\mathfrak{M}; \mathbb{C})$ is also known, by Proposition 7.4. We now construct an obvious candidate for the spectral sequence, with a map to (9.3), and prove by induction on r that the obvious candidate is correct. This last part of the argument is a variation on Zeeman's comparison theorem [Z].

9.7 Proposition. *The sum over all r of the spectral sequences* (9.3) *is the commutative differential bi-graded algebra freely generated by copies of the differential bi-graded vector spaces $\Omega^0[\Sigma] \to \Omega^1[\widehat{D}]^*$, in bi-degrees $(k,l) = (0,m)$ and $(m,1)$, and $\Omega^1[\Sigma] \to \Omega^0[\widehat{D}]^*$ in bi-degrees $(k,l) = (0,m)$ and $(m+1,0)$, respectively, as m ranges over the exponents of $\mathfrak{g}$.*

The arrows above are dual to the connecting maps

$$H^0(\widehat{D}; \Omega^i) \to H^1(\overline{\Sigma}, \widehat{D}; \Omega^i), \qquad i = 1, 0.$$

Concretely,

$$\Omega^1[\widehat{D}]^* = \mathbb{C}(\!(z)\!)/\mathbb{C}[[z]], \qquad \Omega^0[\widehat{D}]^* = \{\mathbb{C}(\!(z)\!)/\mathbb{C}[[z]]\} \otimes dz,$$

and the maps are the principal parts at $z = 0$ on $\overline{\Sigma}$ (cf. §8.3). Here is the location of the generators, with respect to the decomposition (9.5):

space	k	l	r	s
$\Omega^0[\Sigma]$	0	m	m	0
$\Omega^1[\Sigma]$	0	m	$m+1$	0
$\Omega^1[\widehat{D}]^*$	m	1	m	1
$\Omega^0[\widehat{D}]^*$	$m+1$	0	$m+1$	0

The spectral sequence differential which originates at $\Omega^i[\Sigma]$ has length $m+i$.

Proof. The candidate generators are mapped to E_1 as explained in Theorem 8.5. We will show at the end of the section that the terms $\Omega^i[\Sigma]$ survive to E_{m+i-1}, and that the differential δ_{m+i} maps them into the $\Omega^{1-i}[\widehat{D}]^*$ in the way indicated. Assuming this, observe that the kernels and co-kernels of these differentials are the Dolbeault groups of $\overline{\Sigma}$, so the fact that they define the generating classes for $H^s(\mathfrak{M};\Omega^r)$, and therefore survive to E_∞, is already known from the Hodge decomposition of $\mathfrak{M}$ [T4] and the Atiyah-Bott construction of its cohomology generators.

Let now $'E_n^{k,l}$, $n \geq 1$, be the spectral sequence with multiplicative generators and differentials as in (9.7). We will show by induction on r that the map to $E_n^{k,l}$ we constructed is an isomorphism. For $r = 0$, this merely says that $H^0(\mathbf{X};\mathcal{O}) = \mathbb{C}$ and $H^{>0}(\mathbf{X};\mathcal{O}) = 0$, which was shown in [T3]. If the assumption holds up to r, then the multiplicative decomposition (9.5) shows that, for $r+1$, $'E_1^{k,l} \cong E_1^{k,l}$, except perhaps on the left edge $k = 0$.

The assumption also implies that the spectral sub-sequence of $'E_n^{k,l}$, $k > 0$, obtained by deleting the left edge, converges to the hyper-cohomology of the sub-complex of Ω^{r+1}

$$\Omega_+^{r+1} := \mathrm{S}^1\mathfrak{g}[[z]]^* \otimes \Omega_{\mathbf{X}}^r \xrightarrow{\kappa} \mathrm{S}^2\mathfrak{g}[[z]]^* \otimes \Omega_{\mathbf{X}}^{r-1} \xrightarrow{\kappa} \cdots .$$

Our construction gives a map between the long exact sequences of cohomologies over $\mathfrak{M}$,

$$\ldots \to \mathbb{H}^l\left(\mathfrak{M};\Omega_+^{r+1}\right) \to \mathbb{H}^l\left(\mathfrak{M};\Omega^{r+1}\right) \to {}'E_1^{0,l} \to \mathbb{H}^{l+1} \to \ldots ,$$

obtained from the spectral sub-sequence, and

$$\ldots \to \mathbb{H}^l\left(\mathfrak{M};\Omega_+^{r+1}\right) \to \mathbb{H}^l\left(\mathfrak{M};\Omega^{r+1}\right) \to H^l_{G[[z]]}\left(\mathbf{X};\Omega^{r+1}\right) \to \mathbb{H}^{l+1} \to \ldots$$

arising from the sub-complex. As explained in (9.5), we can omit the $G[[z]]$-subscript in the third cohomology, and the Five Lemma gives the desired isomorphism $'E_1^{0,l} \cong H^l\left(\mathbf{X};\Omega^{r+1}\right)$. □

(9.8) *The Hodge differentials.* Since the construction of generators is compatible with de Rham's operator, the first Hodge-de Rham differentials are those described in Theorem D.

(9.9) *The loop Grassmannian.* To prove Theorem C, we repeat the argument above, but use the presentation $\mathfrak{M}(\mathbb{P}^1) = G[z^{-1}] \setminus X$ of the stack of G-bundles. The complex (9.2) representing the differentials is now a pro-vector bundle, completed for the z^{-1}-adic filtration on $\mathrm{S}^k\mathfrak{g}[z^{-1}] \otimes \Omega_X^{r-k}$. The key factorisation result (9.5) continues to apply (completed in the filtration topology), this time by Proposition 5.2.

Proof of (9.7), *concluded.* We now check the good behaviour of the leading differentials in $E_\bullet$ on the Dolbeault generators. The argument is a convoluted tautology, but we include it nonetheless for completeness. An invariant $\Phi \in \mathrm{S}^{m+1}\mathfrak{g}^*$, applied to the Atiyah class

$$\alpha_{\mathfrak{M}} := \alpha_{\overline{\Sigma}} \in \mathbb{H}^1\left(\overline{\Sigma} \times \mathfrak{M}; \Omega^1_{\overline{\Sigma}} \otimes \mathrm{ad}_{\mathcal{P}} \oplus (\Omega^1_{\mathbf{X}} \to \mathrm{S}^1\mathfrak{g}[[z]]^*) \otimes \mathrm{ad}_{\mathcal{P}}\right),$$

accepts at most one nontautological (first) entry, because $\dim \Sigma = 1$. For the same reason, this entry will be detected by contraction with the first set of generators $\Omega^0[\Sigma]$ in Proposition 9.7, but killed by the $\Omega^1[\Sigma]$. So the 0-form generators contain the tautological component (8.7) of α to degree m, and the 1-forms, to degree $m+1$.

Project the tautological component of $\alpha_{\mathfrak{M}}$ to $\mathbb{H}^1\left(\overline{\Sigma} \times \mathfrak{M}; \Omega^1_{\mathbf{X}} \otimes \mathrm{ad}_{\mathcal{P}}\right)$. Lifted to $\mathbf{X}$, this is the tautological component of $\alpha_{\mathbf{X}}$, and these components of $\alpha_{\mathfrak{M}}$ and $\alpha_{\mathbf{X}}$ have the common refinement

$$\mathrm{Id} \in R\mathrm{Hom}_{\mathfrak{M}}\left(R\pi_*\mathrm{ad}_{\mathcal{P}}; R\pi_*\mathrm{ad}_{\mathcal{P}}\right) \cong \mathbb{H}^1\left((\overline{\Sigma}, \widehat{D}) \times \mathfrak{M}; \Omega^1_{\mathbf{X}} \otimes \mathrm{ad}_{\mathcal{P}}\right) \tag{9.10}$$

(notation as in §8.6). Note that cup-product of (9.10) with classes in $H^\bullet(\overline{\Sigma} \times \dots)$ lands in $H^\bullet((\overline{\Sigma}, \widehat{D}) \times \dots)$, and such classes can be contracted with (= integrated against) all functions and forms on Σ. Let $\mathrm{S}^{[m+i]}$ denote $\mathrm{S}^{m+i}\mathfrak{g}[[z]]^*$, placed in degree $m+i$; this is a sub-complex of Ω^{m+i} in (9.2). *We have shown that contraction of $\Phi(\alpha)$ with $\Omega^i[\Sigma]$ gives well-defined classes in the truncated complex $\mathbb{H}^m(\mathfrak{M}; \Omega^{m+i}/\mathrm{S}^{[m+i]})$.* In particular, these Dolbeault generators survive to E_{m+i-1}.

To conclude, we must identify the differentials δ_{m+i}. These arise from the failure of

$$\Phi(\alpha) \in \mathbb{H}^{m+1}\left((\overline{\Sigma}, \widehat{D}) \times \mathfrak{M}; \Omega^{1-i}_{\overline{\Sigma}} \otimes \Omega^{m+i}/\mathrm{S}^{[m+i]}\right)$$

to lift to $\mathbb{H}^{m+1}(\,.\,; \Omega^{1-i}_{\overline{\Sigma}} \otimes \Omega^{m+i})$, in the distinguished triangle

$$\mathrm{S}^{[m+i]} \to \Omega^{m+i} \to \Omega^{m+i}/\mathrm{S}^{m+i} \to \mathrm{S}^{[m+i]}[1].$$

The obstruction is detected by a connecting homomorphism to $H^{2-i}(\,.\,; \Omega^{1-i}_{\overline{\Sigma}} \otimes \mathrm{S}^{m+i})$. Contraction with functions and forms on Σ leads to our differentials, which land in

$$H^{1-i}\left(\mathfrak{M}; \mathrm{S}^{m+i}\mathfrak{g}[[z]]^*\right) \cong H^{1-i}\left(BG[[z]]; \mathrm{S}^{m+i}\mathfrak{g}[[z]]^*\right); \tag{9.11}$$

we have used the key factorisation (9.4) for the isomorphism.

We can identify the connecting map in a different way. The class $\Phi(\alpha)$ *does* lift to the full differentials Ω^{m+i}, but only over $\overline{\Sigma} \times \mathfrak{M}$. A diagram chase then shows that our obstruction is the image of the restricted $\Phi(\alpha)$ under the connecting map

$$H^{1-i}\left(\widehat{D} \times \mathfrak{M}; \Omega^{1-i}_{\overline{\Sigma}} \otimes \mathrm{S}^{[m+i]}\mathfrak{g}[[z]]^*\right) \xrightarrow{\partial} H^{2-i}\left((\overline{\Sigma}, \widehat{D}) \times \mathfrak{M}; \Omega^{1-i}_{\overline{\Sigma}} \otimes \mathrm{S}^{m+i}\mathfrak{g}[[z]]^*\right).$$

However, the universal bundle $\mathcal{P}$ over $\overline{\Sigma} \times \mathfrak{M}$, when restricted to $\widehat{D} \times \mathfrak{M}$, is pulled back from the universal bundle on $\widehat{D} \times BG[\widehat{D}]$; hence, so is its Atiyah class and $\Phi(\alpha)$, and we can replace $\mathfrak{M}$ with $BG[\widehat{D}]$ above. Moreover, the ∂'s are the residue maps appearing in Proposition 9.7. This identifies the δ_{m+i} as asserted. □

10. Related Lie algebra results

(10.1) *Dolbeault cohomology as Lie algebra cohomology.* We now give a Lie algebra interpretation of the Dolbeault cohomology of the loop Grassmannian $X = LG/G[[z]]$. The dual of $\mathfrak{g}((z))/\mathfrak{g}[[z]]$ is identified with $\mathfrak{g}[[z]]dz$ by the residue pairing. The p-forms on X are then sections of the pro-vector bundle associated to the adjoint action of $G[[z]]$ on $\hat{\Lambda}^p\mathfrak{g}[[z]]dz$. Recall that the latter is z-adic completion of the exterior power. For modules thus completed, it is sensible to form the *continuous* $\mathfrak{g}[[z]]$-cohomology, resolved by the Koszul complex of *continuous* linear maps

$$\operatorname{Hom}\left(\hat{\Lambda}^\bullet\mathfrak{g}[[z]]; \hat{\Lambda}^p(\mathfrak{g}[[z]]dz)\right); \tag{10.2}$$

in this case, we get the *inverse limit* of cohomologies[8] of the z-adic truncations of the coefficients. We emphasise, however, that the complex (10.2) has infinite-dimensional $(z, \mathfrak{g})$-eigenspaces, which is a serious obstacle to a direct computation of its cohomology as in Chapter I.

10.3 Proposition. *The continuous Lie algebra cohomology*

$$H^q_{cts}\left(\mathfrak{g}[[z]], \mathfrak{g}; \hat{\Lambda}^p\mathfrak{g}[[z]]dz\right)$$

resolved by the complex (10.2) *is naturally isomorphic to* $H^q(X; \Omega^p)$.

Proof. Contractibility of $G[[z]]/G$ gives a natural "van Est" isomorphism [T3]

$$H^q_{cts}\left(\mathfrak{g}[[z]], \mathfrak{g}; \hat{\Lambda}^p\mathfrak{g}[[z]]dz\right) = H^q_{G[[z]]}\left(\hat{\Lambda}^p\mathfrak{g}[[z]]dz\right);$$

by the fact that the $H^q(X; \Omega^p)$ are the qth derived functors of induction from $G[[z]]$ to LG, the group and Dolbeault cohomologies are related by Shapiro's spectral sequence

$$E_2^{r,s} = H^r_{LG}\left(H^s(X; \Omega^p)\right) \Rightarrow H^{r+s}_{G[[z]]}\left(\hat{\Lambda}^p(\mathfrak{g}[[z]]dz)\right).$$

Alternatively, this is the Leray sequence for the morphism $BG[[z]] \to BLG$, with fibre X. Either way, $H^s(X; \Omega^p)$ is a trivial LG-module, so its higher LG-cohomology vanishes; the spectral sequence collapses and we obtain the asserted equality. □

[8]Finite-dimensionality of cohomology shows that there are no $\lim^1$ terms.

(10.4) *Thick flag varieties.* An obvious variation replaces the formal disk $\mathrm{Spf}\mathbb{C}[[z]]$ by a smooth affine curve Σ. We consider the Lie algebra cohomology $H^q\left(\mathfrak{g}[\Sigma];\Lambda^p\,\Omega^1(\Sigma;\mathfrak{g})\right)$. The answer carries now a contribution from the nontrivial topology of the group $G[\Sigma]$. As in [T3], the van Est sequence collapses at E_2, leading, by the same argument as above, to

10.5 PROPOSITION. $H^\bullet\left(\mathfrak{g}[\Sigma];\Lambda^p\,\Omega^1(\Sigma;\mathfrak{g})\right) \cong H^\bullet(\mathbf{X}_\Sigma;\Omega^p)\otimes H^\bullet(G[\Sigma];\mathbb{C})$, *naturally.*

The homotopy equivalence of $G[\Sigma]$ to the corresponding group of continuous maps shows that the topological factor $H^*(G[\Sigma];\mathbb{C})$ is isomorphic to

$$H^\bullet(G;\mathbb{C})\otimes H^\bullet(\Omega G;\mathbb{C})^{1-\chi(\Sigma)}, \tag{10.6}$$

with ΩG denoting the space of based continuous loops and χ the Euler characteristic. According to [T3], this is also isomorphic to the Lie algebra cohomology $H^\bullet(\mathfrak{g}[\Sigma];\mathbb{C})$.

(10.7) *Strong Macdonald for smooth curves.* The method of Section 9 allows us to carry out the long-postponed proof of the higher-genus version of the strong Macdonald theorem.

Proof of Theorem 1.15. We use the construction of Section 9, but realise the moduli stack $\mathfrak{M}$ of G-bundles on $\overline{\Sigma}$ as the quotient $X/G[\Sigma]$, and present the differentials $\Omega^r_{\mathfrak{M}}$ by a complex of pro-vector bundles

$$\Omega_X^{r-k}\,\widehat{\otimes}\,\hat{\mathrm{S}}^k\mathfrak{g}[\Sigma]^*.$$

The factorisation replacing (9.5) now reads

$$E_1^{k,l} = \bigoplus\nolimits_s H^s_{G[\Sigma]}\left(\hat{\mathrm{S}}^k\mathfrak{g}[\Sigma]^*\right)\widehat{\otimes}H^{l-s}\left(X;\Omega^{r-k}\right);$$

the Dolbeault cohomologies of X being known, the desired group cohomologies are again determined by induction on r, with the difference that it is the right $(r+1)$st edge of the sequence that is new, in the inductive step. Collapse of the van Est sequence leads to the factor $H^\bullet(\mathfrak{g}[\Sigma];\mathbb{C})$ when switching from group to Lie algebra cohomology. □

III. Positive level

The loop group LG admits central extensions by the circle; when G is semi-simple, these are parametrised up to isomorphism by a *level* in $H^3_G(G;\mathbb{Z})$.[9] When G is simple and simply connected, $H^3_G(G;\mathbb{Z})\cong\mathbb{Z}$, and positive levels

[9] There are additional choices for the torus factors, but only one of them is interesting [PS].

lead to the interesting class of *highest-weight representations* of LG, also called *integrable highest-weight modules* of $\mathfrak{g}((z))$. These have a Borel-Weil realisation as spaces of sections of vector bundles over the genus-zero thick flag variety $\mathbf{X}$ (see §11.8 below), and carry a semi-simple $\mathbb{C}^\times$-action intertwining with the z-scaling. The eigenvalues of its infinitesimal generator, called *energies* or *z-weights*, are bounded above.

In Section 11 we give a positive-level analogue of Theorem B, which includes in the coefficients of Macdonald cohomology a highest-weight LG-representation $\mathbf{H}$. This entails the *vanishing of higher cohomology.* As a result, the analogue of Macdonald's constant term—the (z,s)-weighted Lie algebra Euler characteristic in (1.5)—refines the z-dimension of the G-invariant part of $\mathbf{H}$, revealing an affine analogue of *R. Brylinski's filtration* [B], originally defined on weight spaces of G-representations (Remark 11.1).

Central extensions of LG lead to algebraic line bundles over the loop Grassmannians X and $\mathbf{X}$. The sections of the level h line bundle $\mathcal{O}(h)$ over $\mathbf{X}$ span the highest-weight *vacuum representation* $\mathbf{H}_0$. In Section 12, we determine the level 1 Dolbeault cohomologies $H^q(\mathbf{X};\Omega^p(1))$, for simply laced G. A combinatorial application is given in Section 12.7.

11. Brylinski filtration on loop group representations

Let $\mathbf{H}$ be a highest-weight representation of LG; it is the *direct sum* of its z-weight spaces $\mathbf{H}(n)$. We assume that the level is positive on each simple or central factor of $\mathfrak{g}$; the only level-zero representation is trivial, and has been discussed already.

THEOREM E. *$H^k(\mathfrak{g}[[z]],\mathfrak{g};\mathbf{H}\otimes \mathrm{S}^p\mathfrak{g}[[z]]^*)$ vanishes for positive k.*

With respect to Chapter I, this is the restricted cohomology $H^k_{\mathrm{res}}(\mathfrak{g}[z],\mathfrak{g};\mathbf{H}\otimes \mathrm{S}^p\mathfrak{g}[z]^*_{\mathrm{res}})$.

Proof. For abelian $\mathfrak{g}$, $\mathbf{H}$ is a sum of Fock representations, and so is injective for $(\mathfrak{g}[z],\mathfrak{g})$. Assume now that $\mathfrak{g}$ is simple and $\mathbf{H}$ has level $h>0$. In the notation of Section 3, with the operator $\bar\partial$ on $\mathbf{H}\otimes\Lambda\otimes\mathrm{S}$ modified to include the $\mathfrak{g}[z]$-action on $\mathbf{H}$, Theorem 2.4.7 from [T1] becomes

$$\overline{\square} = \square + T_S^\Lambda + k\cdot(1+\frac{h}{2c}) = \square + D + k\cdot h/2c,$$

where the second identity follows as in Proposition 3.17. This is strictly positive if $k>0$. $\square$

11.1 *Remark.* Theorem E has finite-dimensional analogues for G-modules V and the Borel and Cartan sub-algebras $\mathfrak{b},\mathfrak{h}\subset\mathfrak{g}$: the higher cohomologies $H^{>0}(\mathfrak{b},\mathfrak{h};V\otimes \mathrm{S}\mathfrak{b}^*)$ and $H^{>0}(\mathfrak{b},\mathfrak{h};V\otimes\mathrm{S}\mathfrak{n}^*)$ vanish ($\mathfrak{n}=[\mathfrak{b},\mathfrak{b}]$). By the Peter-

Weyl decomposition of the polynomial functions on G, this is equivalent to the vanishing of higher cohomology of $\mathcal{O}$ over $G \times_B \mathfrak{b}$ and $T^*G/B = G \times_B \mathfrak{n}$, and follows from the Grauert-Riemenschneider theorem (cf. the proof of Lemma 4.12).

(11.2) *Shift in the grading.* For reasons that will be clear below, we now replace $\mathfrak{g}[[z]]$ in the symmetric algebra by the differentials $\mathfrak{g}[[z]]dz$. This does not alter the $\mathfrak{g}[[z]]$-module structure, but shifts z-weights by the symmetric degree. To match the usual conventions, we set $q = z^{-1}$ and consider the q-Euler characteristic in the restricted Koszul complex (3.4), capturing the symmetric degree by means of a dummy variable t. After our shift, the isomorphism in Theorem B leads to the following identity, where CT denotes the G-constant term, after we expand the product into a formal (q,t)-series with coefficients in the representation ring of G:

$$\mathrm{CT}\left[\bigotimes_{n>0} \frac{1-q^n\cdot\mathfrak{g}}{1-tq^n\cdot\mathfrak{g}}\right] = \prod_{k=1}^{\ell}\prod_{n>m_k} \frac{1-t^{m_k}q^n}{1-t^{m_k+1}q^n}. \tag{11.3}$$

(11.4) *The constant term at positive level.* The *q-dimension* $\dim_q \mathbf{H} := \sum_n q^{-n}\dim \mathbf{H}(n)$, convergent for $|q|<1$, captures the z^{-1}-weights. Using the Koszul resolution of cohomology, Theorem E equates the q-dimensions of the invariants with a G-constant term,

$$\mathrm{CT}\left[\bigotimes_{n>0} \frac{1-q^n\mathfrak{g}}{1-tq^n\mathfrak{g}} \otimes \mathbf{H}\right] = \sum_{p\geq 0} t^p \dim_q \left\{\mathbf{H}\otimes \mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\right\}^{\mathfrak{g}[z]}. \tag{11.5}$$

When G is simple and $\mathbf{H}$ is irreducible, with highest energy zero and highest weight λ, the Kac character formula [K] converts the q-representation $\mathbf{H}\otimes \prod_{n>0}(1-q^n\mathfrak{g})$ of G into the sum

$$\sum_{\mu\in\lambda+(h+c)P} \pm q^{\frac{c(\mu)-c(\lambda)}{h+c}} V_\mu, \tag{11.6}$$

where $c(\mu) = (\mu+\rho)^2/2$, c is the dual Coxeter number of $\mathfrak{g}$, $P\subset\mathfrak{h}^*$ the integer lattice and $\pm V_\mu$ is the signed G-module induced from the weight μ, the sign depending, as usual, on the Weyl chamber of $\mu+\rho$. So the left side in (11.5) is also

$$\sum_{\mu\in\lambda+(h+c)P} \pm q^{\frac{c(\mu)-c(\lambda)}{h+c}}\, \mathrm{CT}\left[\frac{V_\mu}{\bigotimes_{n>0}(1-tq^n\mathfrak{g})}\right]. \tag{11.7}$$

The analogy with Macdonald's constant term becomes compelling, if we use the Kac denominator identity to convert the left side in (11.3) into the sum (11.7) for $\lambda = h = 0$.

(11.8) *Brylinski filtration.* Recall the Borel-Weil construction of $\mathbf{H}$. The thick loop Grassmannian $\mathbf{X} = G(\!(z)\!)/G[z^{-1}]$ carries the level h line bundle $\mathcal{O}(h)$ and the vector bundle $\mathcal{V}_\lambda$, the latter defined from the action of $G[z^{-1}]$ on V_λ by evaluation at $z = \infty$. Then, $\mathbf{H}$ is the space of algebraic sections of $\mathcal{V}_\lambda(h) := \mathcal{V}_\lambda \otimes \mathcal{O}(h)$ over $\mathbf{X}$.

Restricted to the *big cell* $\mathbf{U} \subset \mathbf{X}$, the orbit of the base-point under $G[[z]]$, $\mathcal{V}_\lambda(h)$ is trivialised by the action of the subgroup $\exp(z\mathfrak{g}[[z]])$. Now, sending $\gamma \in G[[z]]/G$ to $d\gamma\cdot\gamma^{-1}$ identifies $\mathbf{U}$ with $\mathfrak{g}[[z]]dz$, and the resulting affine space structure on $\mathbf{U}$ is preserved by the left translation action of $G[[z]]$. Sections of $\mathcal{V}_\lambda(h)$, having been identified with V_λ-valued polynomials, are increasingly filtered by degree, and this gives an increasing, $G[[z]]$-stable filtration of $\mathbf{H}$.

11.9 Theorem. *We have a natural isomorphism*

$$\mathrm{Gr}_p\, \mathbf{H}^G \simeq \{\mathbf{H} \otimes \mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\}^{\mathfrak{g}[z]}\,.$$

Proof. With the co-adjoint action of $G[[z]]$, $\mathrm{S}(\mathfrak{g}[[z]]dz)^*$ is the associated graded space of $\mathbb{C}[\mathbf{U}]$, the space of polynomials on the open cell $\mathbf{U} \simeq \mathfrak{g}[[z]]dz$, filtered by degree. In the Borel-Weil realisation, $\mathbf{H} \otimes \mathbb{C}[\mathbf{U}]$ is a subspace of the V-valued functions on $\mathbf{U} \times \mathbf{U}$, filtered by the degree on the second factor. It follows that the subspace of invariants under the diagonal translation action of $\mathfrak{g}[z]$ gets identified, by restriction to the first $\mathbf{U}$, with the G-invariants in $\mathbf{H}$, endowed with the Brylinski filtration described above. Cohomology vanishing gives an isomorphism

$$\mathrm{Gr}_p\, \{\mathbf{H} \otimes \mathbb{C}[\mathbf{U}]\}^{\mathfrak{g}[z]} = \{\mathbf{H} \otimes \mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\}^{\mathfrak{g}[z]}\,,$$

with the first space isomorphic to $\mathrm{Gr}_p\mathbf{H}^G$, leading to our theorem. ∎

11.10 *Remark.* Applied to a G-representation V and the cohomology vanishing in Remark 11.1, the same argument defines Brylinski's filtration on the zero-weight space $V^{\mathfrak{h}} \cong (V \otimes \mathrm{S}\mathfrak{n}^*)^{\mathfrak{b}}$.

(11.11) *The basic representation.* When G is simply laced, we can give a product expansion for the generating function of the Brylinski filtration on the G-invariants in the *basic representation* $\mathbf{H}_0$, the highest-weight module of level 1 and highest weight 0.

11.12 Theorem. *For simply laced G, the vacuum vector $\omega \in \mathbf{H}_0$ gives an isomorphism*

$$(*) \qquad \omega\otimes : \{\mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\}^{\mathfrak{g}[z]} \xrightarrow{\sim} \{\mathbf{H}_0 \otimes \mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\}^{\mathfrak{g}[z]}\,.$$

Consequently, with $q = z^{-1}$,

$$\sum_{p\geq 0} t^p \dim_q \mathrm{Gr}_p\, \mathbf{H}_0^G = \prod_{k=1}^{\ell} \prod_{n>m_k} (1 - t^{m_k+1}q^n)^{-1}.$$

Proof. After summing over p, we see that the q-dimension of the left side in (*) is $\prod_{k=1}^{\ell}\prod_{n>m_k}(1-q^n)^{-1}$ (Theorem B). According to [S, Prop. 6.8] this is also the q-dimension of $\mathbf{H}_0^G$. However, the map (*) is an inclusion; hence, by Theorem 11.9, it is an isomorphism, and then it is so in each p-degree separately. □

12. Line bundle twists

Let G be simple and simply connected and call $\mathcal{O}(h)$ the level h line bundle on X or $\mathbf{X}_\Sigma$. The loop group acts projectively on $\mathcal{O}(h)$, and hence on its Dolbeault cohomologies $H^q(X;\Omega^p(h))$. These turn out to be duals of integrable highest-weight modules at level h, and direct products of their z-weight spaces. (This follows as in Proposition 5.2, except that the cohomologies of $\mathrm{Gr}^n\Omega^p(h)$ are now finite sums of duals of irreducible highest-weight modules; this suffices for the Mittag-Leffler conditions, as their z-graded components are finite-dimensional.) For thick flag varieties, we obtain instead sums of highest-weight modules [T3, Remark 8.10].

The Dolbeault groups of $\mathcal{O}(h)$ also assemble to a bi-graded module over the Dolbeault algebra $H^\bullet(\Omega^\bullet)$. For simply laced G at level 1, our knowledge of the basic invariants (Theorem 11.12) allows us to describe the entire structure: $H^\bullet(\Omega^\bullet(1))$ is the free module generated from $H^0(X;\mathcal{O}(1))$ under the action of the *odd* Dolbeault generators. We prove the theorem for $\mathbf{X} = LG/G[z^{-1}]$; the thin X can be handled as in Section 9.9. For convenience, we use the coordinate $q = z^{-1}$ on $\mathbb{P}^1 \setminus \{0\}$; note that $\mathbf{X} = \mathbf{X}_{\mathbb{P}^1\setminus\{0\}}$.

THEOREM F. *For simply laced G, $H^\bullet(\mathbf{X},\Omega^\bullet(1))$ is freely generated from $\mathbf{H}_0 = H^0(\mathbf{X};\mathcal{O}(1))$ by the cup-product action of the odd generators $\mathbb{C}[q]dq \subset H^m(\mathbf{X},\Omega^{m+1})$, $m = m_1,\dots,m_\ell$. The multiplication action of the even generators of $H^\bullet(\Omega^\bullet)$ is nil.*

Proof. The centre of G acts trivially on $H^q(\Omega^p)$. For simply laced groups at level 1, this only allows the basic representation $\mathbf{H}_0$ to appear. The argument now parallels the level zero case. From the cohomology vanishing Theorem E, the $E_1^{k,l}$ term replacing (9.5) in the sequence converging to $H^{k+l}\left(\mathfrak{M}(\mathbb{P}^1);\Omega^r(1)\right)$ is now

$$H^0_{G[[z]]}\left(H^l\left(\mathbf{X};\Omega^{r-k}(1)\right)\otimes \mathrm{S}^k\mathfrak{g}[[z]]^*\right) \cong H^l\left(\mathbf{X};\Omega^{r-k}(1)\right)^{G[[z]]}\otimes H^0_{G[[z]]}\left(\mathrm{S}^k\mathfrak{g}[[z]]^*\right),$$

where we have used the isomorphism of Theorem 11.12. According to [T4, Th. 7.1], the Dolbeault cohomology $H^\bullet\left(\mathfrak{M}(\mathbb{P}^1);\Omega^\bullet(1)\right)$ is isomorphic to $H^{\bullet,\bullet}(BG;\mathbb{C})$, under restriction to the semi-stable sub-stack BG of $\mathfrak{M}(\mathbb{P}^1)$. Using the module structure over $H^\bullet(\Omega^\bullet)$, the argument of Section 9 shows that our new sequence is freely generated by $\mathbf{H}_0$ over the second family of level 0 generators in Proposition 9.7. $\square$

12.1 *Remark.* This result has an obvious analogue, with parallel proof, for the thick flag varieties $\mathbf{X}_\Sigma$, when Σ has genus 0. Extension to a higher genus would require us to equate $H^\bullet\left(\mathfrak{M}(\overline{\Sigma});\Omega^\bullet(1)\right)$ with the free module spanned by $H^0\left(\mathfrak{M}(\overline{\Sigma});\mathcal{O}(1)\right)$ on half the generators of $H^{p,q}\left(\mathfrak{M}(\overline{\Sigma})\right)$. While we believe that to be true, additional arguments seem to be needed.

(12.2) *Affine Hall-Littlewood functions.* For a G-representation V with associated vector bundle $\mathcal{V}$ on $\mathbf{X}$, the series of characters for the G-translation and the z-scaling actions

$$P_{h,V}(q,t) := \sum\nolimits_{r,s}(-1)^s(-t)^r \mathrm{ch} H^s(\mathbf{X},\Omega^r(h)\otimes\mathcal{V}) \in R_G[[q,t]] \tag{12.3}$$

are affine analogues of the Hall-Littlewood symmetric functions.[10] Decomposing the $H^\bullet(\mathbf{X};\Omega^\bullet(h))$ into the irreducible characters $\mathrm{ch}(\mathbf{H})$ at level h allows us to write

$$P_{h,V}(q,t) = \sum\nolimits_{\mathbf{H}}\langle P_{h,V}|\mathbf{H}\rangle(q,t)\cdot\mathrm{ch}(\mathbf{H}),$$

with co-factors $\langle P_{h,V}|\mathbf{H}\rangle(q,t)\in\mathbb{Z}[[q,t]]$. Thus, for simply laced G at level 1, Theorem F gives for the trivial representation $V=\mathbb{C}$

$$\langle P_{h,\mathbb{C}}|\mathbf{H}_0\rangle(q,t) = \prod_{k=1}^{\ell}\prod_{n>0}(1-t^{m_k+1}q^n). \tag{12.4}$$

Little seems to be known about the cohomology of $\Omega^p(h)\otimes\mathcal{V}$ in general, but the $\langle P_{h,V}|\mathbf{H}\rangle(q,t)$ are closely related to the Brylinski filtration of Section 11, as we now illustrate in a simple example.

(12.5) *Hall-Littlewood co-factors and Brylinski filtration.* For any simply connected G, the spectral sequence of Section 9.1 becomes, at level $h>0$

$$E_1^{k,l} = \sum\nolimits_{\mathbf{H}}\left\langle H^l\left(\mathbf{X};\Omega^{r-k}(h)\right)|\mathbf{H}\right\rangle\cdot\left\{\mathbf{H}\otimes \mathrm{S}^k\mathfrak{g}[[z]]^*\right\}^{G[[z]]} \Rightarrow H^{k+l}(BG;\Omega^r),$$

because $H^\bullet\left(\mathfrak{M}_{\mathbb{P}^1};\Omega^r(h)\right) = H^\bullet(BG;\Omega^r)$. We now form the (q,t)-characteristic by multiplying the left side by $(-1)^{k+l}(-t)^r$ and summing over k,l,r. Theorem 11.9 (with the substitution $t\mapsto tq^{-1}$, to undo the shift introduced in §11.2) gives the near-orthogonality relation

$$\sum\nolimits_{\mathbf{H}}\langle P_{h,\mathbb{C}}|\mathbf{H}\rangle(q,t)\cdot\sum\nolimits_p (tq^{-1})^p\dim_q\mathrm{Gr}_p\,\mathbf{H}^G = \prod\nolimits_{k=1}^{\ell}(1-t^{m_k+1})^{-1}; \tag{12.6}$$

[10]The true affine Hall-Littlewood functions involve the full flag variety $LG/\exp(\mathfrak{B})$ in lieu of the loop Grassmannian, but there is a close relation between the two.

the right-hand side is $\sum_{r,s}(-1)^s(-t)^r h^s(BG;\Omega^r) = \sum_r t^r h^{2r}(BG)$. Implications of (12.6) will be explored in future work; instead, we conclude with a combinatorial application.

(12.7) *Lattice hyper-geometric sums.* There is a Kac formula for $P_{1,\mathbb{C}}$, established as in Section 6.2 (but now with the *increasing* filtration on Ω^p, as we work on the thick Grassmannian $\mathbf{X}$):

$$\sum_{w\in W_{\mathrm{aff}}} w\left[\prod_{n>0;\,\alpha}\frac{1-tq^n\mathrm{e}^\alpha}{1-q^n\mathrm{e}^\alpha}\cdot\prod_{\alpha>0}(1-\mathrm{e}^\alpha)^{-1}\right]\cdot\prod_{n>0}\left(\frac{1-tq^n}{1-q^n}\right)^\ell. \tag{12.8}$$

At level 1, a lattice element $\gamma\in W_{\mathrm{aff}}$ sends $q^n\mathrm{e}^\lambda$ to $q^{n+\gamma^2/2+\langle\lambda|\gamma\rangle}\mathrm{e}^{\lambda+\gamma}$, in which the basic inner product is used to convert γ to a weight. The manipulation in Section 6.4 converts (12.8) into

$$\sum_\gamma q^{\gamma^2/2}\mathrm{e}^\gamma\cdot\prod_{n>0;\,\alpha}\frac{1-tq^{n+\langle\alpha|\gamma\rangle}\mathrm{e}^\alpha}{1-q^{n+\langle\alpha|\gamma\rangle}\mathrm{e}^\alpha}\cdot\prod_{n>0}\left(\frac{1-tq^n}{1-q^n}\right)^\ell. \tag{12.9}$$

For simply laced G, another expression is provided by (12.4) and any of the character formulae for $\mathbf{H}_0$; thus, the basic bosonic realisation gives

$$P_{1,\mathbb{C}} = \prod_{k=1}^{\ell}\prod_{n>0}\frac{1-t^{m_k+1}q^n}{1-q^n}\cdot\sum_\gamma q^{\gamma^2/2}\mathrm{e}^\gamma \tag{12.10}$$

where we sum over the co-root lattice (which is also the root lattice). Equating the last two expressions gives the identity

(12.11)

$$\sum_\gamma q^{\gamma^2/2}\mathrm{e}^\gamma\cdot\prod_{n>0;\,\alpha}\frac{1-tq^{n+\langle\alpha|\gamma\rangle}\mathrm{e}^\alpha}{1-q^{n+\langle\alpha|\gamma\rangle}\mathrm{e}^\alpha} = \prod_{k=1}^{\ell}\prod_{n>0}\frac{1-t^{m_k+1}q^n}{1-tq^n}\cdot\sum_\gamma q^{\gamma^2/2}\mathrm{e}^\gamma.$$

With $G=\mathrm{SL}_2$, replacing q by $\sqrt{q}$ leads to

$$\sum_{m\in\mathbb{Z}} q^{m^2/2}u^{2m}\cdot\prod_{n>0}\frac{(1-tq^{n/2+m}u^2)(1-tq^{n/2-m}u^{-2})}{(1-q^{n/2+m}u^2)(1-q^{n/2-m}u^{-2})} = \prod_{n>0}\frac{1-t^2q^{n/2}}{1-tq^{n/2}}\cdot\sum_{m\in\mathbb{Z}}q^{m^2/2}u^{2m},$$

which, using the notation $(a)_\infty = \prod_{n\geq 0}(1-aq^n)$, $(a)_n = (a)_\infty/(aq^n)_\infty$, $(a_1,\dots,a_k)_n = \prod_i(a_i)_n$, becomes the hyper-geometric summation formula

$$\sum_{m\in\mathbb{Z}} q^{m^2/2}u^{2m}\cdot\frac{(\sqrt{q}u^2)_m(\sqrt{q}u^{-2})_{-m}(qu^2)_m(qu^{-2})_{-m}}{(\sqrt{q}tu^2)_m(\sqrt{q}tu^{-2})_{-m}(qtu^2)_m(qtu^{-2})_{-m}} = \frac{(\sqrt{q}u^2,\sqrt{q}u^{-2},qu^2,qu^{-2},\sqrt{q}t^2,qt^2)_\infty}{(\sqrt{q}tu^2,\sqrt{q}tu^{-2},qtu^2,qtu^{-2},\sqrt{q}t,qt)_\infty}\cdot\sum_{m\in\mathbb{Z}}q^{m^2/2}u^{2m};$$

most factors in the numerator on the left cancel out, and the series can then be summed by specialising Bailey's ${}_4\psi_4$ summation formula [GaR, V].

Appendix

A. Proof of Lemma 3.13

It is clear that both sides in (3.13) annihilate the constant line in $\Lambda \otimes \mathrm{S}$, and it is also easy to see that they agree on the symmetric part $1 \otimes \mathrm{S}$, where D, K, ad and ad* all vanish. So we must only check equality on the linear ψ terms, and on the quadratic $\psi \wedge \psi$ and $\sigma\psi$ terms.

(A.1) *The linear ψ terms.* Fix $b \in A$, $n > 0$. We compute:

$$\bar{\partial}\psi^b(-n) = \frac{1}{2} \sum_{\substack{a \in A \\ 0<m<n}} \psi^a(-m) \wedge \psi^{[a,b]}(m-n),$$

$$\begin{aligned}
\bar{\partial}^*\bar{\partial}\psi^b(-n) &= \frac{1}{4} \sum_{\substack{a \in A \\ 0<m<n}} \frac{n}{m(n-m)} \psi^{[a,[a,b]]}(-n) \\
&\quad - \frac{1}{4} \sum_{\substack{a \in A \\ 0<m<n}} \frac{n}{m(n-m)} \psi^{[[a,b],a]}(-n) \\
&= \frac{1}{2} \sum_{\substack{a \in A \\ 0<m<n}} \frac{n}{m(n-m)} \psi^{[a,[a,b]]}(-n) = \frac{1}{2} \sum_{0<m<n} \frac{n}{m(n-m)} \psi^b(-n) \\
&= \frac{1}{2} \sum_{0<m<n} \left(\frac{1}{m} + \frac{1}{n-m}\right) \cdot \psi^b(-n) = \sum_{0<m<n} 1/m \cdot \psi^b(-n).
\end{aligned}$$

Further, $\bar{\partial}^*\psi^b(-n) = 0$, so that $\overline{\Box}\psi^b(-n)$ is as just computed. Next,

$$\begin{aligned}
\Box\psi^b(-n) &= \sum_{\substack{a \in A \\ 0<m<n}} \frac{1}{m} \mathrm{ad}_a(-m)\mathrm{ad}_a(-m)^*\psi^b(-n) \\
&= \sum_{\substack{a \in A \\ 0<m<n}} \frac{1}{m} \cdot \frac{n-m}{n} \cdot \psi^{[a,[a,b]]}(-n) \\
&= \sum_{0<m<n} \left(\frac{1}{m} - \frac{1}{n}\right) \cdot \psi^b(-n) \\
&= \sum_{0<m<n} \frac{1}{m} \cdot \psi^b(-n) - \psi^b(-n) + \psi^b(-n)/n \\
D\psi^b(-n) &= \sum_{\substack{a \in A \\ 0<m<n}} \psi^{[a,[a,b]]}(-n)/n = \psi^b(-n),
\end{aligned}$$

$$K\psi^b(-n) = \frac{1}{n}\sum_{a\in A}\mathrm{ad}_{[a,b]}(0)\psi^a(-n) = -\frac{1}{n}\sum_{a\in A}\psi^{[a,[a,b]]}(-n) = -\psi^b(-n)/n,$$

and the last three terms sum up to $\overline{\Box}\psi^b(-n)$, as claimed. $\square$

(A.2) *The quadratic $\psi\wedge\psi$ terms.* Fix $b,c\in A$ and $n,p>0$. For each second-order differential operator P involved, we focus on the *cross-term* $P\left(\psi^b(-n)\wedge\psi^c(-p)\right) - P\psi^b(-n)\wedge\psi^c(-p) - \psi^b(-n)\wedge P\psi^c(-p)$; equality of cross-terms and the identity (3.13) on the linear factors imply the identity for quadratic terms.

A.3 Lemma. *The cross-term in $\overline{\Box}\left(\psi^b(-n)\wedge\psi^c(-p)\right)$ is the following three-term sum*:

$$\begin{aligned}
&\sum_{\substack{a\in A\\0<m<n}}\left(\frac{1}{m}-\frac{1}{n}\right)\cdot\psi^{[a,b]}(m-n)\wedge\psi^{[a,c]}(-m-p)\\
&\qquad+\sum_{\substack{a\in A\\0<m<p}}\left(\frac{1}{m}-\frac{1}{p}\right)\cdot\psi^{[a,b]}(-m-n)\wedge\psi^{[a,c]}(m-p)\\
&\qquad-\sum_{a\in A}\left(\frac{1}{n}+\frac{1}{p}\right)\cdot\psi^{[a,b]}(-n)\wedge\psi^{[a,c]}(-p).
\end{aligned}$$

Proof. We rewrite the sum by adding and subtracting terms:

$$\begin{aligned}
&\sum_{\substack{a\in A\\0<m<n}}\left(\frac{1}{m}+\frac{1}{p}\right)\cdot\psi^{[a,b]}(m-n)\wedge\psi^{[a,c]}(-m-p)\\
&\qquad+\sum_{\substack{a\in A\\0<m<p}}\left(\frac{1}{m}+\frac{1}{n}\right)\cdot\psi^{[a,b]}(-m-n)\wedge\psi^{[a,c]}(m-p)\\
&\qquad-\sum_{\substack{a\in A\\0<m<n+p}}\left(\frac{1}{n}+\frac{1}{p}\right)\cdot\psi^{[a,b]}(-m)\wedge\psi^{[a,c]}(m-n-p).
\end{aligned}\tag{A.4}$$

We now track, in turn, the source of each of the three terms in (A.4). We have

$$\begin{aligned}
\bar\partial\left(\psi^b(-n)\wedge\psi^c(-p)\right) &= \frac{1}{2}\sum_{\substack{a\in A\\0<m<n}}\psi^a(-m)\wedge\psi^{[a,b]}(m-n)\wedge\psi^c(-p)\\
&\quad+\frac{1}{2}\sum_{\substack{a\in A\\0<m<p}}\psi^a(-m)\wedge\psi^b(-n)\wedge\psi^{[a,c]}(m-p),
\end{aligned}\tag{A.5}$$

and applying $\bar{\partial}^*$ to the first sum gives the following, after collecting the terms where $\psi^c(-p)$ survives intact into the first summand:

$$\begin{aligned}\bar{\partial}^*\bar{\partial}\psi^b(-n)\wedge\psi^c(-p) &+ \frac{1}{4}\sum_{\substack{a\in A\\0<m<n}}\frac{1}{m}\psi^{[a,b]}(m-n)\wedge\mathrm{ad}_a(m)^*\psi^c(-p)\\ &-\frac{1}{4}\sum_{\substack{a\in A\\0<m<n}}\frac{1}{n-m}\psi^a(-m)\wedge\mathrm{ad}_{[a,b]}(n-m)^*\psi^c(-p)\\ &+\frac{1}{4}\sum_{\substack{a\in A\\0<m<n}}\frac{1}{p}\mathrm{ad}_c(p)^*\left(\psi^a(-m)\wedge\psi^{[a,b]}(m-n)\right).\end{aligned}\tag{A.6}$$

The first term is $\overline{\Box}\psi^b(-n)\wedge\psi^c(-p)$. The two middle-line terms agree, after substituting $m\leftrightarrow(n-m)$, and sum to

$$\frac{1}{2}\sum_{\substack{a\in A\\0<m<n}}\frac{p+m}{mp}\psi^{[a,b]}(m-n)\wedge\psi^{[a,c]}(-m-p).\tag{A.7}$$

Amusingly, the third line takes the same value (A.7); so the sum in (A.6) equals

$$\overline{\Box}\psi^b(-n)\wedge\psi^c(-p)+\sum_{\substack{a\in A\\0<m<n}}\frac{p+m}{mp}\psi^{[a,b]}(-m-p)\wedge\psi^{[a,c]}(m-n),\tag{A.8}$$

and so the cross-term in (A.8) accounts for the first term in (A.4). Substituting $b\leftrightarrow c$, $n\leftrightarrow p$ shows that the $\bar{\partial}^*$-image of the second term in (A.5) is

$$\psi^b(-n)\wedge\overline{\Box}\psi^c(-p)+\sum_{\substack{a\in A\\0<m<n}}\frac{n+m}{mn}\psi^{[a,b]}(-m-n)\wedge\psi^{[a,c]}(m-p),\tag{A.9}$$

whose cross-term is the second term in (A.4).

Finally, $\bar{\partial}^*\left(\psi^b(-n)\wedge\psi^c(-p)\right)=\frac{n+p}{np}\cdot\psi^{[b,c]}(-p-n)$, whence by applying $\bar{\partial}$ we get

$$\begin{aligned}\bar{\partial}\bar{\partial}^*\left(\psi^b(-n)\wedge\psi^c(-p)\right)&=\frac{1}{2}\frac{n+p}{np}\sum_{\substack{a\in A\\0<m<n+p}}\psi^a(-m)\wedge\psi^{[a,[b,c]]}(m-p-n)\\ &=\frac{n+p}{2np}\sum_{\substack{a\in A\\0<m<n+p}}\Big(\psi^a(-m)\wedge\psi^{[[a,b],c]}(m-p-n)\\ &\qquad\qquad+\psi^a(-m)\wedge\psi^{[b,[a,c]]}(m-p-n)\Big)\\ &=-\frac{n+p}{np}\sum_{\substack{a\in A\\0<m<n+p}}\psi^{[a,b]}(-m)\wedge\psi^{[a,c]}(m-p-n),\end{aligned}$$

which is the third term in (A.4). The proposition is proved. $\square$

Now $D\left(\psi^b(-n)\wedge\psi^c(-p)\right) = D\psi^b(-n)\wedge\psi^c(-p)+\psi^b(-n)\wedge D\psi^c(-p)$, with no cross-term, while

(A.10)

$$K\Big(\psi^b(-n)\wedge\psi^c(-p)\Big) = K\psi^b(-n)\wedge\psi^c(-p)+\psi^b(-n)\wedge K\psi^c(-p) \\ -\frac{1}{n}\psi^{[a,b]}(-n)\wedge\psi^{[a,c]}(-p)-\frac{1}{p}\psi^{[a,b]}(-n)\wedge\psi^{[a,c]}(-p),$$

(A.11)

$$\square\Big(\psi^b(-n)\wedge\psi^c(-p)\Big) = \psi^b(-n)\wedge\psi^c(-p)+\psi^b(-n)\wedge\psi^c(-p) \\ +\sum_{\substack{a\in A\\0<m<n}}\frac{1}{m}\frac{n-m}{n}\psi^{[a,b]}(m-n)\wedge\psi^{[a,c]}(-m-p) \\ +\sum_{\substack{a\in A\\0<m<p}}\frac{1}{m}\frac{p-m}{p}\psi^{[a,b]}(-m-n)\wedge\psi^{[a,c]}(m-p),$$

and the cross-terms in (A.10) and (A.11) add up to the expression in (A.3).

(A.12) *The $\sigma\psi$ terms.* As before, fix $b,c\in A$ and $n,p>0$. Then,

$$\bar\partial\Big(\sigma^b(-n)\psi^c(-p)\Big) = \sum_{\substack{a\in A\\0<m\le n}}\sigma^{[a,b]}(m-n)\cdot\psi^a(-m)\wedge\psi^c(-p) \\ +\frac{1}{2}\sum_{\substack{a\in A\\0<m<p}}\sigma^b(-n)\psi^a(-m)\wedge\psi^{[a,c]}(m-p),$$

and applying $\bar\partial^*$ yields the following sum:

$$\overline{\square}\sigma^b(-n)\cdot\psi^c(-p)-\frac{1}{p}\sum_{\substack{a\in A\\0<m\le n}}\sigma^{[a,c]}(m-p-n)\psi^{[a,b]}(-m) \\ +\frac{1}{2}\sum_{\substack{a\in A\\0<m<p}}\frac{1}{m}\sigma^{[a,b]}(-m-n)\psi^{[a,c]}(m-p) \\ -\frac{1}{2}\sum_{\substack{a\in A\\0<m<p}}\frac{1}{m-p}\sigma^{[[a,c],b]}(m-p-n)\psi^a(-m) \\ +\frac{1}{2}\sum_{\substack{a\in A\\0<m\le n}}\frac{m+p}{mp}\sigma^{[a,b]}(m-n)\psi^{[a,c]}(-m-p) \\ +\frac{1}{2}\sum_{\substack{a\in A\\0<m\le n}}\frac{m+p}{mp}\sigma^{[a,b]}(m-n)\psi^{[a,c]}(-m-p) \\ +\sigma^b(-n)\overline{\square}\psi^c(-n).$$

The first two lines come from the R^*-terms in $\bar\partial^*$, the last two lines from the ad*-terms. The two terms in each of the middle rows are equal, so the cross-term can be rewritten as follows:

$$-\frac{1}{p}\sum_{\substack{a\in A\\ 0<m\le n}} \sigma^{[a,c]}(m-p-n)\cdot\psi^{[a,b]}(-m)+\sum_{\substack{a\in A\\ 0<m<p}}\frac{1}{m}\sigma^{[a,b]}(-m-n)\cdot\psi^{[a,c]}(m-p)$$
$$+\sum_{\substack{a\in A\\ 0<m\le n}}\left(\frac{1}{m}+\frac{1}{p}\right)\sigma^{[a,b]}(m-n)\cdot\psi^{[a,c]}(-m-p). \tag{A.13}$$

Now, $\bar\partial^*\left(\sigma^b(-n)\cdot\psi^c(-p)\right)=\sigma^{[c,b]}(-n-p)/p$, whence

$$\bar\partial\bar\partial^*\left(\sigma^b(-n)\cdot\psi^c(-p)\right)=\frac{1}{p}\sum_{\substack{a\in A\\ 0<m\le n+p}}\sigma^{[a,[c,b]]}(m-n-p)\cdot\psi^a(-m) \tag{A.14}$$
$$=\frac{1}{p}\sum_{\substack{a\in A\\ 0<m<n+p}}\sigma^{[a,c]}(m-n-p)\cdot\psi^{[a,b]}(-m)-\frac{1}{p}\sum_{\substack{a\in A\\ 0<m<n+p}}\sigma^{[a,b]}(m-n-p)\cdot\psi^{[a,c]}(-m).$$

Summing (A.13) and (A.14) gives

$$\begin{aligned}\frac{1}{p}&\sum_{\substack{a\in A\\ 0<m\le p}}\sigma^{[a,c]}(m-p)\cdot\psi^{[a,b]}(-m-n)\\ &+\sum_{\substack{a\in A\\ 0<m<p}}\left(\frac{1}{m}-\frac{1}{p}\right)\sigma^{[a,b]}(-m-n)\cdot\psi^{[a,c]}(m-p)\\ &+\sum_{\substack{a\in A\\ 0<m<n}}\frac{1}{m}\sigma^{[a,b]}(m-n)\cdot\psi^{[a,c]}(-m-p)\\ &-\frac{1}{p}\sum_{a\in A}\sigma^{[a,b]}(-n)\cdot\psi^{[a,c]}(-p);\end{aligned} \tag{A.15}$$

here, the first term is the sum of the first terms in (A.13) and (A.14), the second and third incorporate the second and third terms in (A.13) and the $0<m<p$, resp. the $p<m<p+n$ portions of the second term in (A.14), and the final term is the $m=p$ contribution of the same.

Moving on to the right-hand side of (3.13), the cross-term in

$$\Box\left(\sigma^b(-n)\psi^c(-p)\right)$$

is

$$\sum_{\substack{a\in A\\ 0<m<n}}\frac{1}{m}\sigma^{[a,b]}(m-n)\cdot\psi^{[a,c]}(-m-p)$$
$$+\sum_{\substack{a\in A\\ 0<m<p}}\frac{p-m}{mp}\sigma^{[a,b]}(-m-n)\cdot\psi^{[a,c]}(m-p), \tag{A.16}$$

the two terms coming from the $\mathrm{ad}\cdot R^*$ and $R\cdot\mathrm{ad}^*$ cross-terms, respectively. Further,

$$D\left(\sigma^b(-n)\cdot\psi^c(-p)\right)=\sigma^b(-n)\cdot D\psi^c(-p)+\frac{1}{p}\sum_{\substack{a\in A\\ 0<m\le p}}\sigma^{[a,c]}(m-p)\cdot\psi^{[a,b]}(-m-n),\tag{A.17}$$

$$K\left(\sigma^b(-n)\cdot\psi^c(-p)\right)=\sigma^b(-n)\cdot K\psi^c(-p)-\frac{1}{p}\sum_{a\in A}\sigma^{[a,b]}(-n)\cdot\psi^{[a,c]}(-p).\tag{A.18}$$

It is now clear that the cross-terms in (A.16)–(A.18) sum to (A.15). □

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