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Ramification theory for varieties over a perfect field

By Kazuya Kato and Takeshi Saito

Abstract

For an ℓ-adic sheaf on a variety of arbitrary dimension over a perfect field, we define the Swan class measuring the wild ramification as a 0-cycle class supported on the ramification locus. We prove a Lefschetz trace formula for open varieties and a generalization of the Grothendieck-Ogg-Shararevich formula using the Swan class.

Let F be a perfect field and U be a separated and smooth scheme of finite type purely of dimension d over F. In this paper, we study ramification of a finite étale scheme V over U along the boundary, by introducing a map (0.1) below.

We put $\mathrm{CH}_0(\overline{V} \setminus V) = \varprojlim \mathrm{CH}_0(Y \setminus V)$ where Y runs compactifications of V and the transition maps are proper push-forwards (Definition 3.1.1). The degree maps $\mathrm{CH}_0(Y \setminus V) \to \mathbb{Z}$ induce a map $\deg : \mathrm{CH}_0(\overline{V} \setminus V) \to \mathbb{Z}$. The fiber product $V \times_U V$ is smooth purely of dimension d and the diagonal $\Delta_V : V \to V \times_U V$ is an open and closed immersion. Thus the complement $V \times_U V \setminus \Delta_V$ is also smooth purely of dimension d and the Chow group $\mathrm{CH}_d(V \times_U V \setminus \Delta_V)$ is the free abelian group generated by the classes of connected components of $V \times_U V$ not contained in Δ_V. If U is connected and if $V \to U$ is a Galois covering, the scheme $V \times_U V$ is the disjoint union of the graphs Γ_σ for $\sigma \in G = \mathrm{Gal}(V/U)$ and the group $\mathrm{CH}_d(V \times_U V \setminus \Delta_V)$ is identified with the free abelian group generated by $G - \{1\}$.

The intersection of a connected component of $V \times_U V \setminus \Delta_V$ with Δ_V is empty. However, we define the intersection product with the logarithmic diagonal

$$(0.1) \qquad (\ , \Delta_{\overline{V}})^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \longrightarrow \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

using log product and alteration (Theorem 3.2.3). The aim of this paper is to show that the map (0.1) gives generalizations to an arbitrary dimension of the classical invariants of wild ramification of $f : V \to U$. The image of the map is in fact supported on the wild ramification locus (Proposition 3.3.5.2). If we have a strong form of resolution of singularities, we do not need $\otimes_{\mathbb{Z}}\mathbb{Q}$ to define the map (0.1). We prove a Lefschetz trace formula for open varieties

$$\sum_{q=0}^{2d}(-1)^q \mathrm{Tr}(\Gamma^* : H^q_c(V_{\bar F}, \mathbb{Q}_\ell)) = \deg\ (\Gamma, \Delta_{\overline{V}})^{\log} \tag{0.2}$$

in Proposition 3.2.4. If $V \to U$ is a Galois covering of smooth curves, the log Lefschetz class $(\Gamma_\sigma, \Delta_{\overline{V}})^{\log}$ for $\sigma \in \mathrm{Gal}(V/U) \setminus \{1\}$ is an equivalent of the classical Swan character (Lemma 3.4.7).

For a smooth ℓ-adic sheaf $\mathcal{F}$ on U where ℓ is a prime number different from the characteristic of F, we define the Swan class $\mathrm{Sw}(\mathcal{F}) \in \mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ (Definition 4.2.8) also using the map (0.1). From the trace formula (0.2), we deduce a formula

$$\chi_c(U_{\bar F}, \mathcal{F}) = \mathrm{rank}\ \mathcal{F} \cdot \chi_c(U_{\bar F}, \mathbb{Q}_\ell) - \deg\ \mathrm{Sw}(\mathcal{F}) \tag{0.3}$$

for the Euler characteristic $\chi_c(U_{\bar F}, \mathcal{F}) = \sum_{q=0}^{2d}(-1)^q \dim H^q_c(U_{\bar F}, \mathcal{F})$ in Theorem 4.2.9. If U is a smooth curve, we have $\mathrm{Sw}(\mathcal{F}) = \sum_{x \in \overline{U}\setminus U} \mathrm{Sw}_x(\mathcal{F})[x]$ by Lemma 4.3.6. Thus the formula (0.3) is nothing other than the Grothendieck-Ogg-Shafarevich formula [14], [26]. As a generalization of the Hasse-Arf theorem (Lemma 4.3.6), we state Conjecture 4.3.7 asserting that we do not need $\otimes_{\mathbb{Z}}\mathbb{Q}$ in the definition of the Swan class. We prove a part of Conjecture 4.3.7 in dimension 2 (Corollary 5.1.7.1).

The profound insight that the wild ramification gives rise to invariants as 0-cycle classes supported on the ramification locus is due to S. Bloch [4] and is developed by one of the authors in [17], [18]. Since a covering ramifies along a divisor in general, it is naturally expected that the invariants defined as 0-cycle classes should be computable in terms of the ramification at the generic points of irreducible components of the ramification divisor. For the log Lefschetz class $(\Gamma_\sigma, \Delta_{\overline{V}})^{\log}$, we give such a formula (3.31) in Lemma 3.4.11. For the Swan class of a sheaf of rank 1, we state Conjecture 5.1.1 in this direction and prove it assuming $\dim U \leq 2$ in Theorem 5.1.5. We expect that the log filtration by ramification groups defined in [3] should enable us to compute the Swan classes of sheaves of arbitrary rank.[1]

In a subsequent paper, we plan to study ramification of schemes over a discrete valuation ring and prove an analogue of Grothendieck-Ogg-Shafarevich formula for the Swan conductor of cohomology (cf. [1], [2]). In p-adic setting, the relation between the Swan conductor and the irregularities are studied in [6], [7], [23] and [33]. The relation between the Swan classes defined in this paper and the characteristic varieties of $\mathcal{D}$-modules defined in [5] should be investigated.[2]

[1] Added in Proof. See T. Saito, Wild ramification and the characteristic cycle of an ℓ-adic sheaf (preprint arXiv:0705.2799).

[2] Added in Proof. See T. Abe, Comparison between Swan conductors and characteristic cycles (preprint).

In Section 1, we recall a log product construction in [20]. In Section 2, we prove a Lefschetz trace formula Theorem 2.3.4 for algebraic correspondences on open varieties, under a certain assumption. In Section 3, we define and study the map (0.1) and prove the trace formula (0.2) in Proposition 3.2.4. In Section 4, we define the Swan class of an ℓ-adic sheaf and prove the formula (0.3) in Theorem 4.2.9. In Section 5, we compare the Swan class in rank 1 case with an invariant defined in [18]. We also compare the formula (0.3) with a formula of Laumon in dimension 2.

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Contents

Notation. In this paper, we fix a base field F. A scheme means a separated scheme of finite type over F unless otherwise stated explicitly. For schemes X and Y over F, the fiber product over F will be denoted by $X \times Y$.

The letter ℓ denotes a prime number invertible in F.

1. Log products

In Section 1.1, we introduce log products and establish elementary properties. In Section 1.2, we define and study admissible automorphisms.

1.1. *Log blow-up and log product.* We introduce log blow-ups and log products with respect to families of Cartier divisors.

Definition 1.1.1. Let F be a field and let X and Y be separated schemes of finite type over F. Let $\mathcal{D} = (D_i)_{i\in I}$ be a finite family of Cartier divisors of X and $\mathcal{E} = (E_i)_{i\in I}$ be a finite family of Cartier divisors of Y indexed by the same finite set I.

For $i \in I$, let $(X \times Y)'_i \to X \times Y$ be the blow-up at $D_i \times E_i \subset X \times Y$ and let $(X \times Y)^\sim_i \subset (X \times Y)'_i$ be the complement of the proper transforms of $D_i \times Y$ and $X \times E_i$.

1. We define the log blow-up

$$p : (X \times Y)' \longrightarrow X \times Y, \tag{1.1}$$

more precisely denoted by $((X, \mathcal{D}) \times (Y, \mathcal{E}))'$, to be the fiber product

$$\prod_{i\in I}{}_{X\times Y} (X \times Y)'_i \to X \times Y$$

of $(X \times Y)'_i$ $(i \in I)$ over $X \times Y$.

2. Similarly, we define the log product

$$(X \times Y)^\sim \subset (X \times Y)', \tag{1.2}$$

or more precisely denoted by $((X, \mathcal{D}) \times (Y, \mathcal{E}))^\sim$, to be the fiber product $\prod_{i\in I}{}_{X\times Y} (X \times Y)^\sim_i \to X \times Y$ of $(X \times Y)^\sim_i$ $(i \in I)$ over $X \times Y$.

3. If $X = Y$ and $\mathcal{D} = \mathcal{E}$, we call $(X \times X)^\sim$ the log self product of X with respect to $\mathcal{D}$. By the universality of blow-up, the diagonal map $\Delta : X \to X \times X$ induces an immersion

$$X \to (X \times X)^\sim$$

called the log diagonal map.

Locally on X and Y, the log blow-up, log self-product and the log diagonal maps are described as follows.

Lemma 1.1.2. *Let the notation be as in Definition* 1.1.1. *Assume that* $X = \operatorname{Spec} A$ *and* $Y = \operatorname{Spec} B$ *are affine and that the Cartier divisors* D_i *are defined by* $t_i \in A$ *and* E_i *are defined by* $s_i \in B$ *respectively.*

1. *The log product* $(X \times Y)'$ *is the union of*

$$\operatorname{Spec} \frac{A \otimes_F B[U_i\ (i \in I_1), V_j\ (j \in I_2)]}{(t_i \otimes 1 - U_i(1 \otimes s_i)\ (i \in I_1), 1 \otimes s_j - V_j(t_j \otimes 1)\ (j \in I_2))} \tag{1.3}$$

for decompositions $I = I_1 \amalg I_2$.

2. *The log product* $(X \times Y)^\sim$ *is given by*

$$\mathrm{Spec}\ A \otimes_F B[U_i^{\pm 1}\ (i \in I)]/(t_i \otimes 1 - U_i(1 \otimes s_i)\ (i \in I)) \tag{1.4}$$

3. *Assume further that* $A = B$, $D_i = E_i$ *and* $t_i = s_i$ *for each* $i \in I$. *Then in the notation* (1.4), *the log diagonal map* $\Delta : X \to (X \times X)^\sim$ *is defined by the map*

$$A \otimes_F A[U_i^{\pm 1}\ (i \in I)]/(t_i \otimes 1 - U_i(1 \otimes t_i)\ (i \in I)) \to A \tag{1.5}$$

sending $a \otimes 1$ *and* $1 \otimes a$ *to* $a \in A$ *and* U_i *to* 1 *for* $i \in I$.

Proof. For each $i \in I$, the Cartier divisors $D_i \times Y$ and $X \times E_i$ are locally defined by a regular sequence. Thus we obtain 1. The rest is clear from this and the definition. □

For the sake of readers familiar with log schemes, we recall an intrinsic definition using log structures given in [20]. The Cartier divisors $D_1, \ldots, D_m$ define a log structure M_X on X. In the notation in Lemma 1.1.2, the log structure M_X is defined by the chart $\mathbb{N}^m \to A$ sending the standard basis to $t_1, \ldots, t_m$. The local chart $\mathbb{N}^m \to A$ induces a map $\mathbb{N}^m \to \Gamma(X, M_X/O_X^\times)$ of monoids. Similarly, the Cartier divisors $E_1, \ldots, E_m$ defines a log structure on Y and a map $\mathbb{N}^m \to \Gamma(Y, M_Y/O_Y^\times)$. Then, the log product $(X \times Y)^\sim$ represents the functor attaching to an fs-log scheme T over F the set of pairs (f, g) of morphisms of log schemes $f : T \to X$ and $g : T \to Y$ over F such that the diagram

$$\begin{array}{ccc} \mathbb{N}^m & \longrightarrow & \Gamma(X, M_X/O_X^\times) \\ \downarrow & & \downarrow \\ \Gamma(Y, M_Y/O_Y^\times) & \longrightarrow & \Gamma(T, M_T/O_T^\times) \end{array}$$

is commutative. The log diagonal $\Delta : X \to (X \times X)^\sim$ corresponds to the pair $(\mathrm{id}, \mathrm{id})$.

The log product satisfies the following functoriality. Let X, X', Y and Y' be schemes over F and $\mathcal{D} = (D_i)_{i \in I}$, $\mathcal{D}' = (D'_i)_{i \in I}$, $\mathcal{E} = (E_j)_{j \in J}$, and $\mathcal{E}' = (E'_j)_{j \in J}$ be families of Cartier divisors of X, X', Y and of Y' respectively. Let $f : X \to Y$ and $g : X' \to Y'$ be morphisms over F and let $e_{ij} \geq 0, (i, j) \in I \times J$ be integers satisfying $f^*E_j = \sum_{i \in I} e_{ij} D_i$ and $f^*E'_j = \sum_{i \in I} e_{ij} D'_i$ for $j \in J$. Then, the maps f and g induces a map $(f \times g)^\sim : (X \times X')^\sim \to (Y \times Y')^\sim$. If $Y = Y'$ and $\mathcal{E} = \mathcal{E}'$, we define $(X \times_Y X')^\sim$, or more precisely $((X, \mathcal{D}) \times_{(Y, \mathcal{E})} (X', \mathcal{D}'))^\sim$, to be the fiber product $(X \times X')^\sim \times_{(Y \times Y)^\sim} Y$ with the log diagonal $Y \to (Y \times Y)^\sim$.

Lemma 1.1.3. *Let* F *be a field and* $n \geq 1$ *be an integer. Let* Y *be a separated scheme over* F. *Let* $\mathcal{L}$ *be an invertible* O_Y*-module and* $\mu : \mathcal{L}^{\otimes n} \to O_Y$ *be an injection of* O_Y*-modules. We define an* O_Y*-algebra* $\mathcal{A} = \bigoplus_{i=0}^{n-1} \mathcal{L}^{\otimes i}$ *with* *the multiplication defined by* $\mu : \mathcal{L}^{\otimes n} \to O_Y$ *and put* $X = \mathrm{Spec}\mathcal{A}$*. Let* E *be the Cartier divisor of* Y *defined by* $\mathcal{I}_E = \mathrm{Im}(\mathcal{L}^{\otimes n} \to O_Y)$ *and* D *be the Cartier divisor of* X *defined by* $\mathcal{L}O_X$*. Let* $(X \times_Y X)^\sim$ *be the log self product defined with respect to* D *and* E*.*

We define an action of the group scheme $\mu_n = \mathrm{Spec}F[t]/(t^n - 1)$ *on* X *over* Y *by the multiplication by* t *on* $\mathcal{L}$*. We consider the action of* μ_n *on* $(X \times_Y X)^\sim$ *by the action on the first factor* X*.*

Then, by the second projection $(X \times_Y X)^\sim \to X$*, the scheme* $(X \times_Y X)^\sim$ *is a* μ_n*-torsor on* X*. Further the log diagonal map* $X \to (X \times_Y X)^\sim$ *induces an isomorphism* $\mu_n \times X \to (X \times_Y X)^\sim$*.*

Proof. Since the question is local on Y, it is reduced to the case where $Y = \mathbf{A}^1 = \mathrm{Spec}\ F[T]$ and μ send a basis S^n of $\mathcal{L}^{\otimes n}$ to T. Then we have $X = \mathbf{A}^1 = \mathrm{Spec}\ F[S]$ and the map $X \to Y$ is given by $T \mapsto S^n$. Then, by Lemma 1.1.2.2, we have $(Y \times Y)^\sim = \mathrm{Spec}\ F[T, T', U^{\pm 1}]/(T' - UT) = \mathrm{Spec}\ F[T, U^{\pm 1}]$, $(X \times X)^\sim = \mathrm{Spec}\ F[S, S', V^{\pm 1}]/(S' - VS) = \mathrm{Spec}\ F[S, V^{\pm 1}]$, and the map $(X \times X)^\sim \to (Y \times Y)^\sim$ is given by $T \mapsto S^n$ and $U \mapsto V^n$. Since the log diagonal $Y \to (Y \times Y)^\sim$ is defined by $U = 1$, we have $(X \times_Y X)^\sim = \mathrm{Spec}\ F[S, V^{\pm 1}]/(V^n - 1)$. Thus the assertion is proved. $\square$

Let F be a field and X be a smooth scheme purely of dimension d over F. In this paper, we say a divisor D of X has simple normal crossings if the irreducible components D_i $(i \in I)$ are smooth over F and, for each subset $J \subset I$, the intersection $\bigcap_{i \in J} D_i$ is smooth purely of dimension $d - |J|$ over F. In other words, Zariski locally on X, there is an étale map to $\mathbf{A}^d_F = \mathrm{Spec}\ F[T_1, \ldots, T_d]$ such that D is the pull-back of the divisor defined by $T_1 \cdots T_r$ for some $0 \le r \le d$. If D_i is an irreducible component, D_i is smooth and $\bigcup_{j \ne i}(D_i \cap D_j)$ is a divisor of D_i with simple normal crossings.

Let X be a smooth scheme over a field F and D be a divisor of X with simple normal crossings. Let D_i $(i \in I)$ be the irreducible components of D. We consider the log blow-up $p : (X \times X)' \to X \times X$ with respect to the family D_i $(i \in I)$ of irreducible components of D, defined in Definition 1.1.1. Let $D^{(1)\prime} \subset (X \times X)'$ and $D^{(2)\prime} \subset (X \times X)'$ be the proper transforms of $D^{(1)} = D \times X$ and of $D^{(2)} = X \times D$ respectively. Let $E_i = (X \times X)' \times_{X \times X} (D_i \times D_i)$ be the exceptional divisors and $E = \bigcup_i E_i \subset (X \times X)'$ be the union.

The log blow-up $p : (X \times X)' \to X \times X$ is used in [10] and in [25] in the study of cohomology of open varieties. For an irreducible component D_i of D, the log blow-up $(D_i \times D_i)' \to D_i \times D_i$ is defined with respect to the family $D_i \cap D_j, j \ne i$ of Cartier divisors.

Lemma 1.1.4. *Let* X *be a smooth scheme over* F*,* D *be a divisor of* X *with simple normal crossings and* $U = X \setminus D$ *be the complement. Let* $p : (X \times X)' \to X \times X$ *be the log blow-up with respect to the family of irreducible components of* D*.*

1. *The scheme* $(X\times X)'$ *is smooth over* F. *The complement* $(X\times X)'\setminus(U\times U)=D^{(1)'}\cup D^{(2)'}\cup E$ *is a divisor with simple normal crossings. The log product* $(X\times X)^\sim$ *is equal to the complement*

$$(X\times X)'\setminus(D^{(1)\prime}\cup D^{(2)\prime}).$$

2. *Let* D_i *be an irreducible component of* D. *The projection* $E_i\to D_i\times D_i$ *induces a map* $E_i\to(D_i\times D_i)'$ *and further a map* $E_i^\circ=E_i\cap(X\times X)^\sim\to(D_i\times D_i)^\sim$. *We have a canonical isomorphism*

$$E_i \longrightarrow \mathbf{P}(N_{D_i\times D_i/X\times X})\times_{D_i\times D_i}(D_i\times D_i)' \tag{1.6}$$

to the pull-back of the $\mathbf{P}^1$*-bundle* $\mathbf{P}(N_{D_i\times D_i/X\times X})=\mathbf{Proj}(\mathrm{S}^\bullet N_{D_i\times D_i/X\times X})$ *associated to the conormal sheaf* $N_{D_i\times D_i/X\times X}$.

We identify E_i *with* $\mathbf{P}(N_{D_i\times D_i/X\times X})\times_{D_i\times D_i}(D_i\times D_i)'$ *by the isomorphism* (1.6). *Then the open subscheme* $E_i^\circ\subset E_i$ *is the complement of the two disjoint sections* $(D_i\times D_i)^\sim\to\mathbf{P}(N_{D_i\times D_i/X\times X})\times_{D_i\times D_i}(D_i\times D_i)^\sim$ *defined by the surjections* $N_{D_i\times D_i/X\times X}\to N_{D_i\times D_i/D_i\times X}$ *and* $N_{D_i\times D_i/X\times X}\to N_{D_i\times D_i/X\times D_i}$.

Proof. 1. It follows immediately from the definition and the description in Lemma 1.1.2.

2. Clear from the definition. □

Corollary 1.1.5. *Let the notation be as in Lemma* 1.1.4. *Let* D_i *be an irreducible component of* D *and let* $D_i\to(D_i\times D_i)^\sim$ *be the log diagonal map. Then the isomorphism* (1.6) *induces an isomorphism*

$$E^\circ_{i,D_i}=E_i^\circ\times_{(D_i\times D_i)^\sim}D_i \longrightarrow \mathbb{G}_{m,D_i}. \tag{1.7}$$

The section $D_i\to E^\circ_{i,D_i}$ *induced by the log diagonal* $X\to(X\times X)^\sim$ *is identified with the unit section* $D_i\to\mathbb{G}_{m,D_i}$.

Proof. The restrictions of the conormal sheaf $N_{D_i\times D_i/X\times X}$ to the diagonal $D_i\subset D_i\times D_i$ is the direct sum of the restrictions $N_{D_i\times D_i/D_i\times X}|_{D_i}$ and $N_{D_i\times D_i/X\times D_i}|_{D_i}$. Further the restrictions $N_{D_i\times D_i/D_i\times X}|_{D_i}$ and $N_{D_i\times D_i/X\times D_i}|_{D_i}$ are canonically isomorphic to $N_{D_i/X}$. Hence we have a canonical isomorphism $\mathbf{P}(N_{D_i\times D_i/X\times X})\times_{D_i\times D_i}D_i\to\mathbf{P}^1_{D_i}$ and the assertion follows from Lemma 1.1.4.2. □

Proposition 1.1.6. *Let* X *be a separated smooth scheme purely of dimension* d *over* F *and* $U=X\setminus D$ *be the complement of a divisor* $D=\bigcup_{i\in I}D_i$ *with simple normal crossings. Let* Y *be a separated scheme over* F *and* $V=Y\setminus B$ *be the complement of a Cartier divisor* B. *We consider a Cartesian* *diagram*

$$
\begin{array}{ccc}
U & \xrightarrow{\subset} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f} \\
V & \xrightarrow{\subset} & Y.
\end{array}
\tag{1.8}
$$

We put $\bar f^*B = \sum_{i\in I} e_iD_i$.

1. *Let* $(X\times X)^\sim$ *be the log product with respect to the family* $(D_i)_{i\in I}$ *of irreducible components and* $(Y\times Y)^\sim$ *be the log product with respect to* B. *Let* $(X\times_Y X)^\sim = (X\times X)^\sim \times_{(Y\times Y)^\sim} Y$ *be the inverse image of the diagonal. We keep the notation in Corollary* 1.1.5. *Let* D_i *be an irreducible component of* D. *We identify* $E^\circ_{i,D_i} = E^\circ_i \times_{(D_i\times D_i)^\sim} D_i$ *with* $\mathbb{G}_{m,D_i}$ *by the isomorphism* (1.7).

Then the intersection $E^\circ_{i,D_i}\cap (X\times_Y X)^\sim$ *is a closed subscheme of the subscheme* $\mu_{e_i,D_i}\subset \mathbb{G}_{m,D_i}$ *of* e_i*-th roots of* 1.

2. *The closure* $\overline{U\times_V U}$ *in the log product* $(X\times X)'$ *satisfies the equality*

$$\overline{U\times_V U}\cap D^{(1)\prime} = \overline{U\times_V U}\cap D^{(2)\prime} \tag{1.9}$$

of the underlying sets.

Proof. 1. The assertion is local on $D_i\subset (D_i\times D_i)^\sim$. Hence, we may assume that $X=\operatorname{Spec} A$ is affine and that the divisor D_k is defined by $t_k\in A$ for $k\in I$. We may also assume that the Cartier divisor B of Y is defined by a function s. Then, we have $f^*s = v\prod_{k\in I} t_k^{e_k}$ for a unit $v\in A^\times$. We identify $(X\times X)^\sim = \operatorname{Spec} A\otimes_F A[U_k^{\pm1}\ (k\in I)]/(t_k\otimes 1 - U_k(1\otimes t_k)\ \ (k\in I))$ as in (1.5). Then on the closed subscheme $(X\times_Y X)^\sim\subset (X\times X)^\sim$, we have an equation

$$\frac{v\otimes 1}{1\otimes v}\prod_{k\in I} U_k^{e_k} = 1.$$

On the log diagonal $D_i\subset (D_i\times D_i)^\sim$, we have $v\otimes 1 = 1\otimes v$ and $U_k = 1$ for $k\in I\setminus\{i\}$. Since the coordinate of the $\mathbb{G}_m$-bundle E_{i,D_i} is given by U_i, the assertion follows.

2. It suffices to show the equality $\overline{\Gamma}\cap D^{(1)\prime} = \overline{\Gamma}\cap D^{(2)\prime}$ for any integral closed subscheme $\Gamma\subset U\times_V U$. We regard $\overline{\Gamma}$ as a closed subscheme of $(X\times X)'$ with an integral scheme structure and let $p_1,p_2:\overline{\Gamma}\to X$ denote the compositions with the projections. We consider the Cartier divisors $p_1^*D_i$ and $p_2^*D_i$ of $\overline{\Gamma}$. We also consider the Cartier divisors $(D_i\times X)'\cap\overline{\Gamma}$ and $(X\times D_i)'\cap\overline{\Gamma}$.

By the Cartesian diagram (1.8), we have $e_i>0$ in $X\times_Y B = \sum_{i\in I} e_iD_i$ for all i. Since $\Gamma\subset U\times_V U$, the closure $\overline{\Gamma}$ is a closed subscheme of the pull-back $(X\times X)'\times_{Y\times Y} Y$ of the diagonal. Hence, we have an equality $\sum_i e_ip_1^*D_i = \sum_i e_ip_2^*D_i$ of Cartier divisors of $\overline{\Gamma}$. Thus, we have an equality

$\sum_i e_i(D_i \times X)' \cap \overline{\Gamma} = \sum_i e_i(X \times D_i)' \cap \overline{\Gamma}$. Since $e_i > 0$ for all i, we obtain

$$\overline{\Gamma} \cap D^{(1)\prime} = \bigcup_i (D_i \times X)' \cap \overline{\Gamma} = \bigcup_i (X \times D_i)' \cap \overline{\Gamma} = \overline{\Gamma} \cap D^{(2)\prime}. \qquad \square$$

We consider tamely ramified coverings.

Definition 1.1.7. 1. Let K be a complete discrete valuation field. We say a finite separable extension L of K is tamely ramified if the ramification index $e_{L/K}$ is invertible in the residue field and if the extension of the residue field is separable.

2. Let

$$\begin{array}{ccc} U & \xrightarrow{\subset} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \bar f} \\ V & \xrightarrow{\subset} & Y \end{array}$$

be a Cartesian diagram of locally noetherian normal schemes. We assume that Y is regular, V is the complement of a divisor with simple normal crossings and that U is a dense open subscheme of X. We also assume that the map $f : U \to V$ is finite étale and $\bar f : X \to Y$ is quasi-finite.

We say $\bar f : X \to Y$ is tamely ramified if, for each point $\xi \in X \setminus U$ such that $O_{X,\xi}$ is a discrete valuation ring, the extension of the complete discrete valuation fields $\mathrm{Frac}(\hat O_{X,\xi})$ over $\mathrm{Frac}(\hat O_{Y,\bar f(\xi)})$ is tamely ramified.

Lemma 1.1.8. *Let*

$$\begin{array}{ccc} U & \xrightarrow{\subset} & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \bar h} \\ V' & \xrightarrow{\subset} & Y' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \bar g} \\ V & \xrightarrow{\subset} & Y \end{array}$$

be a Cartesian diagram of separated normal schemes of finite type over F. We assume that X and Y are smooth over F, $U \subset X$ and $V \subset Y$ are the complements of divisors with simple normal crossings and V' is a dense open subscheme of Y'. We also assume that $g : V' \to V$ is finite étale and $\bar g : Y' \to Y$ is quasi-finite and tamely ramified.

Then, in $(X \times X)^\sim$, the intersection of the closure $\overline{U \times_V U \setminus U \times_{V'} U}$ with the log diagonal $X \subset (X \times X)^\sim$ is empty.

Proof. The assertion is étale local on X and on Y. We put $f = g \circ h$ and $\bar f = \bar g \circ \bar h$. Let $\bar x$ be a geometric point of X and $\bar y = \bar f(\bar x)$ be its image. We take étale maps $Y \to \mathbf{A}^d_F = \mathrm{Spec}\ F[T_1, \dots, T_d]$ and $X \to \mathbf{A}^n_F =$

Spec $F[S_1, \dots, S_n]$ such that $V = Y \times_{\mathbf{A}^d_F} \mathrm{Spec}\, F[T_1, \dots, T_d][(T_1 \cdots T_r)^{-1}]$ and $U = X \times_{\mathbf{A}^n_F} \mathrm{Spec}\, F[S_1, \dots, S_n][(S_1 \cdots S_q)^{-1}]$. Since the assertion is étale local on Y, we may assume that there exist an integer $e \geq 1$ invertible in F and a surjection $Y_e = Y \times_{\mathbf{A}^d_F} \mathrm{Spec}\, F[T_1, \dots, T_d][T_1^{1/e}, \dots, T_r^{1/e}] \to Y'$ over Y by Abhyankar's lemma. Further we may assume that there exists a surjection $X_e = X \times_{\mathbf{A}^n_F} \mathrm{Spec}\, F[S_1, \dots, S_n][S_1^{1/e}, \dots, S_r^{1/e}] \to X \times_{Y'} Y_e$ over X.

We put $V_e = V \times_Y Y_e$ and $U_e = U \times_X X_e$. Then, $(X_e \times X_e)^\sim \to (X \times X)^\sim$ is finite, $X_e \to X$ is surjective and the inverse image of $U \times_V U \setminus U \times_{V'} U$ is a subset of $U_e \times_V U_e \setminus U_e \times_{V_e} U_e$. Hence, it is reduced to the case where $X \to Y'$ is $X_e \to Y_e$ and further to the case $X_e = Y_e$. Since $(Y_e \times_Y Y_e)^\sim \to Y_e$ is finite étale as in Lemma 1.1.3, the assertion is proved. □

1.2. *Admissible automorphisms.* Let X be a smooth scheme over F, D be a divisor of X with simple normal crossings and $U = X \setminus D$ be the complement. We study an automorphism of X stabilizing U.

Definition 1.2.1. Let X be a smooth scheme over F, D be a divisor of X with simple normal crossings and $U = X \setminus D$ be the complement. Let $D_1, \dots, D_m$ be the irreducible components of D.

Let σ be an automorphism of X over F satisfying $\sigma(U) = U$. We say σ is admissible if, for each $i = 1, \dots, m$, we have either $\sigma(D_i) = D_i$ or $\sigma(D_i) \cap D_i = \emptyset$.

We define the blow-up $X_\Sigma \to X$ associated to the subdivision by baricenters and show that the induced action on X_Σ is admissible.

Definition 1.2.2. Let X be a smooth scheme purely of dimension d over F, D be a divisor of X with simple normal crossings and let $D_1, \dots, D_m$ be the irreducible components of D. For a subset $I \subset \{1, \dots, m\}$, we put $D_I = \bigcap_{i \in I} D_i$. We put $X = X_0$ and, for $0 \leq i < d$, we define $X_{i+1} \to X_i$ to be the blow-up at the proper transforms of D_I for $|I| = d - i$ inductively. We call $X_\Sigma = X_d \to X$ the blow-up associated to the subdivision by baricenters.

Lemma 1.2.3. *Let X be a smooth scheme over F, D be a divisor of X with simple normal crossings and let $D_1, \dots, D_m$ be the irreducible components of D. Let $U = X \setminus D$ be the complement and let $p : X_\Sigma \to X$ be the blow-up associated to the subdivision by baricenters.*

1. *The scheme X_Σ is smooth over F and the complement $D' = X_\Sigma \setminus U$ is a divisor with simple normal crossings. For an irreducible component D'_j of D', we put $I = \{i | D'_j \subset p^{-1}(D_i), 1 \leq i \leq m\}$ and $k = |I|$. Then there exists an irreducible component Z of D_I satisfying the following condition. Let $Z' \subset X_k$ be the proper transform of Z in X_k and $E_Z \subset X_{k+1}$ be the inverse image of Z'. Then D'_j is the proper transform of E_Z.*

2. *For an automorphism σ of X over F satisfying $\sigma(U) = U$, the induced action of σ on X_Σ is admissible.*

Proof. 1. It suffices to study étale locally on X. Hence, it suffices to consider the case where $X = \mathbf{A}^d = \mathrm{Spec}\, F[T_1, \dots, T_d]$ and D is defined by $T_1 \cdots T_m = 0$. Then X_Σ is obtained by patching $\mathrm{Spec}\, A_\varphi$ where

$$A_\varphi = F\left[T_{\varphi(1)}, \frac{T_{\varphi(2)}}{T_{\varphi(1)}}, \dots, \frac{T_{\varphi(m)}}{T_{\varphi(m-1)}}, T_{m+1}, \dots, T_d\right]$$

for bijections $\varphi : \{1, \dots, m\} \to \{1, \dots, m\}$. The assertion follows easily from this.

2. Let $D'_1, \dots, D'_{m'}$ be the irreducible components of D' and $\Sigma = \{I \subset \{1, \dots, m\}\}$ be the power set of $\{1, \dots, m\}$. We define a map $\psi : \{1, \dots, m'\} \to \Sigma$ by putting $\psi(j) = \{i | D'_j \subset p^{-1}(D_i), 1 \leq i \leq m\}$. Then by 1, for irreducible components $D'_j \neq D'_{j'}$ such that $D'_j \cap D'_{j'} \neq \emptyset$, we have either $\psi(j) \subsetneqq \psi(j')$ or $\psi(j) \supsetneqq \psi(j')$. The map $\psi : \{1, \dots, m'\} \to \Sigma$ is compatible with the natural actions of σ. Therefore, if $\sigma(D'_j) = D'_{\sigma(j)} \neq D'_j$, we have $|\psi(\sigma(j))| = |\sigma(\psi(j))| = |\psi(j)|$ and $\sigma(D'_j) \cap D'_j = \emptyset$. □

We define the log fixed part for an admissible automorphism.

LEMMA 1.2.4. *Let X be a separated and smooth scheme of finite type over F, D be a divisor of X with simple normal crossings and $U = X \setminus D$ be the complement. Let σ be an admissible automorphism of X over F satisfying $\sigma(U) = U$. Then, the closed immersion $(1, \sigma) : U \to U \times U$ is extended to a closed immersion*

$$\tilde{\Gamma}_\sigma : X \setminus \textstyle\bigcup_{i:\sigma(D_i) \neq D_i} D_i \longrightarrow (X \times X)^\sim. \tag{1.10}$$

Proof. By the assumption that σ is admissible, the closed immersion $(1, \sigma) : X \to X \times X$ induces a closed immersion $X \to (X \times X)'$. Let Γ'_σ denote X regarded as a closed subscheme of $(X \times X)'$ by this immersion. Then, it induces an isomorphism $X \setminus \bigcup_{i:\sigma(D_i) \neq D_i} D_i \to \Gamma'_\sigma \cap (X \times X)^\sim$. □

Definition 1.2.5. Let X be a separated and smooth scheme of finite type over F, D be a divisor of X with simple normal crossings and $U = X \setminus D$ be the complement. Let σ be an admissible automorphism of X over F satisfying $\sigma(U) = U$ and let $\tilde{\Gamma}_\sigma \subset (X \times X)^\sim$ denote the image of the closed immersion $\tilde{\Gamma}_\sigma : X \setminus \bigcup_{i:\sigma(D_i) \neq D_i} D_i \to (X \times X)^\sim$. We call the closed subscheme

$$X^\sigma_{\log} = \Delta_X \cap \tilde{\Gamma}_\sigma = X \times_{(X \times X)^\sim} \tilde{\Gamma}_\sigma \tag{1.11}$$

of X the log σ-fixed part.

LEMMA 1.2.6. *Let X be a separated and smooth scheme of finite type over F, D be a divisor of X with simple normal crossings and $U = X \setminus D$ be* *the complement. Let σ be an admissible automorphism of X over F satisfying $\sigma(U) = U$.*

1. *The closed subscheme $X^\sigma_{\log} \subset X$ is a closed subscheme of the σ-fixed part $X^\sigma = X \times_{X\times X \swarrow \Gamma_\sigma} X$.*

2. *Let $k \in \mathbb{Z}$ be an integer and assume σ^k is also admissible. Then, we have an inclusion*

$$X^\sigma_{\log} \subset X^{\sigma^k}_{\log}$$

of closed subschemes.

3. *Assume $U^\sigma = \emptyset$ and σ is of finite order invertible in F. Then, we have*

$$X^\sigma_{\log} = \emptyset.$$

Proof. 1. Clear from the commutative diagram

$$\begin{array}{ccc} X \setminus \bigcup_{i:\sigma(D_i)\neq D_i} D_i & \xrightarrow{\Gamma_\sigma} & (X \times X)^\sim \\ \cap\big\downarrow & & \big\downarrow \\ X & \xrightarrow{(1,\sigma)} & X \times X. \end{array}$$

2. Since $X^\sigma_{\log} = X^{\sigma^{-1}}_{\log}$ and $X^{\mathrm{id}}_{\log} = X$, we may assume $k \geq 1$. Let J_σ and J_{σ^k} be the ideals of O_X defining the closed subschemes $X^\sigma_{\log}$ and $X^{\sigma^k}_{\log}$ respectively. By 1, it is sufficient to show the inclusion $J_{\sigma^k,x} \subset J_{\sigma,x}$ of the ideals of $O_{X,x}$ for each $x \in X^\sigma$. Let x be a point of X^σ. The ideal $J_{\sigma,x}$ is generated by $\sigma(a)-a$ and $\sigma(b)/b-1$ for $a \in O_{X,x}$ and $b \in O_{X,x} \cap j_*O^\times_{U,x}$ where $j : U \to X$ is the open immersion. Similarly, $J_{\sigma^k,x}$ is generated by $\sigma^k(a)-a$ and $\sigma^k(b)/b-1$ for $a \in O_{X,x}$ and $b \in O_{X,x} \cap j_*O^\times_{U,x}$. Since σ is admissible, we have $\sigma(b)/b \in O^\times_{X,x}$ for $b \in O_{X,x} \cap j_*O^\times_{U,x}$. We have $\sigma^k(a)-a = \sum_{i=0}^{k-1}(\sigma(\sigma^i(a))-\sigma^i(a)) \in J_{\sigma,x}$ and $\sigma^k(b)/b-1 = \sum_{i=0}^{k-1}(\sigma(\sigma^i(b))/\sigma^i(b)-1)(\sigma^i(b)/b) \in J_{\sigma,x}$ for $a \in O_{X,x}$ and $b \in O_{X,x} \cap j_*O^\times_{U,x}$. Hence, we have $J_{\sigma^k,x} \subset J_{\sigma,x}$.

3. By 1, it is sufficient to show $J_{\sigma,x} = O_{X,x}$ for each closed point $x \in X^\sigma$. Let x be a closed point in X^σ and e be the order of σ. Since the question is étale local, we may assume F contains a primitive e-th root of unity. We take a regular system $t_1, \ldots, t_d$ of parameters of $O_{X,x}$ such that $t_1 \cdots t_r$ defines D at x. By replacing t_i's if necessary, we may assume there is a unique e-th root ζ_i of unity such that $\sigma(t_i) \equiv \zeta_i t_i \bmod m_x^2$ for each t_i. Replacing t_i by $\sum_{k=1}^{e} \zeta_i^{-k}\sigma^k(t_i)/e$, we may assume $\sigma(t_i) = \zeta_i t_i$. Then, the ideal $J_{\sigma,x}$ is generated by $\zeta_i - 1$ for $1 \leq i \leq r$ and $(\zeta_i-1)t_i$ for $r < i \leq d$. Since $\zeta_i - 1$ is invertible unless $\zeta_i = 1$, we have either $J_{\sigma,x} = O_{X,x}$ or $J_{\sigma,x} = ((\zeta_i-1)t_i, (r < i \leq d))$. By the assumption that $U^\sigma = \emptyset$, we have $J_{\sigma,x} = O_{X,x}$ and the assertion follows. □

Corollary 1.2.7. *Let the notation be as in Lemma* 1.2.6. *Assume σ is of finite order e and σ^j is admissible for each $j \in \mathbb{Z}$.*

1. *If j is prime to e, we have*

$$X^{\sigma}_{\log} = X^{\sigma^j}_{\log}.$$

2. *If $U^{\sigma} = \emptyset$ and if e is not a power of characteristic of F, then we have*

$$X^{\sigma}_{\log} = \emptyset.$$

Proof. Clear from Lemma 1.2.6.2 and 3. □

2. A Lefschetz trace formula for open varieties

In preliminary subsections 2.1 and 2.2, we recall some facts on the cycle class map and a lemma of Faltings on the cohomology of the log self product respectively. In Section 2.3, we prove a Lefschetz trace formula, Theorem 2.3.4, for open varieties.

In this section, we keep the notation that F denotes a field and ℓ denotes a prime number invertible in F.

2.1. *Complements on cycle maps.* We recall some facts on cycle maps. Let X be a smooth scheme over F and $i : Y \to X$ be a closed immersion of codimension d. Then, the cycle class $[Y] \in H^{2d}_Y(X, \mathbb{Z}_\ell(d))$ and the corresponding map $\mathbb{Z}_\ell \to Ri^!\mathbb{Z}_\ell(d)[2d]$ are defined in [13].

LEMMA 2.1.1. *Let X be a smooth scheme over F and $j : U \to X$ be an open immersion. Let $i : Y \to U$ be a closed immersion and assume that the composition $i' = j \circ i : Y \to X$ is also a closed immersion. Assume that Y is of codimension d in X. Then, for an integer $q \in \mathbb{Z}$, the composition*

$$H^q_c(X, \mathbb{Z}_\ell) \xrightarrow{i'^*} H^q_c(Y, \mathbb{Z}_\ell) \xrightarrow{i_*} H^{q+2d}_c(U, \mathbb{Z}_\ell(d))$$

is the cup-product with the image of the cycle class $[Y] \in H^{2d}_Y(X, \mathbb{Z}_\ell(d))$ by the map $H^{2d}_Y(X, \mathbb{Z}_\ell(d)) = H^{2d}_Y(X, j_!\mathbb{Z}_\ell(d)) \to H^{2d}(X, j_!\mathbb{Z}_\ell(d))$.

Proof. The cycle class $[Y] \in H^{2d}_Y(X, \mathbb{Z}_\ell(d))$ defines a map $\mathbb{Z}_\ell \to Ri'^!\mathbb{Z}_\ell(d)[2d]$. The push-forward map $i_* : H^q_c(Y, \mathbb{Z}_\ell) \to H^{q+2d}_c(U, \mathbb{Z}_\ell(d))$ is the composition of the map $H^q_c(Y, \mathbb{Z}_\ell) \to H^{q+2d}_{Y!}(X, j_!\mathbb{Z}_\ell(d))$ induced by $\mathbb{Z}_\ell \to Ri'^!\mathbb{Z}_\ell(d)[2d]$ with the canonical map $H^{q+2d}_{Y!}(X, j_!\mathbb{Z}_\ell(d)) \to H^{q+2d}_c(U, \mathbb{Z}_\ell(d))$ in the notation of [13, 1.2.5, 2.3.1]. Hence the assertion follows. □

LEMMA 2.1.2. *Let X and Y be smooth schemes purely of dimensions n and m over F and $f : X \to Y$ be a morphism over F. Let Z be a closed subscheme of Y of codimension d and put $W = Z \times_Y X$. Then, the image of the cycle class $[Z] \in H^{2d}_Z(Y, \mathbb{Z}_\ell(d))$ by the pull-back map $f^* : H^{2d}_Z(Y, \mathbb{Z}_\ell(d)) \to H^{2d}_W(X, \mathbb{Z}_\ell(d))$ is equal to the cycle class $[f^!(Z)]$ of the image $f^!(Z)$ of the Gysin map $f^! : \mathrm{CH}_{m-d}(Z) \to \mathrm{CH}_{n-d}(W)$.*

Proof. In the case $f : X \to Y$ is smooth, the assertion is in [13, Th. 2.3.8 (ii)]. By decomposing $f : X \to Y$ as the composition of the graph map $X \to X \times Y$ with the projection $X \times Y \to Y$, we may assume $f : X \to Y$ is a closed immersion. We prove this case using the deformation to normal cone.

Let $(Y \times \mathbf{A}^1)' \to Y \times \mathbf{A}^1$ be the blow-up at $X \times \{0\}$ and let Y' be the complement of the proper transform of $Y \times \{0\}$ in $(Y \times \mathbf{A}^1)'$. Let Z' be the proper transform of $Z \times \mathbf{A}^1$ in Y'. The fiber $Y' \times_{\mathbf{A}^1} \{0\}$ at 0 is naturally identified with the normal bundle $N = N_{X/Y}$ of X in Y and $Z' \times_{\mathbf{A}^1} \{0\}$ is also identified with the normal cone $C = C_W Z$ of $W = X \times_Y Z$ in Z [12, Ch. 5.1]. Let $f' : X \times \mathbf{A}^1 \to Y'$ denote the immersion and $g : X \to N$ be the 0-section. We consider the commutative diagram

$$\begin{array}{ccccc}
H^{2d}_Z(Y, \mathbb{Z}_\ell(d)) & \xleftarrow{1^*} & H^{2d}_{Z'}(Y', \mathbb{Z}_\ell(d)) & \xrightarrow{0^*} & H^{2d}_C(N, \mathbb{Z}_\ell(d)) \\
{\scriptstyle f^*}\downarrow & & {\scriptstyle f'^*}\downarrow & & \downarrow{\scriptstyle g^*} \\
H^{2d}_W(X, \mathbb{Z}_\ell(d)) & \xleftarrow{1^*} & H^{2d}_{W \times \mathbf{A}^1}(X \times \mathbf{A}^1, \mathbb{Z}_\ell(d)) & \xrightarrow{0^*} & H^{2d}_W(X, \mathbb{Z}_\ell(d)).
\end{array}$$

The lower horizontal arrows are the same and are isomorphisms. In the upper line, the images of the cycle class $[Z']$ in the middle are the cycle classes $[Z]$ and $[C]$ respectively by [13, Th. 2.3.8 (ii)], since Z' is flat over $\mathbb{A}^1$ ([12, B.6.7]). Since $f^!(Z)$ is defined as $g^!(C)$ [12, Ch. 6.1 (1)], it is reduced to showing the equality $g^*([C]) = [g^!(C)]$.

We put $N_W = N \times_X W$. Since $C \subset N_W$, the pull-back $g^* : H^{2d}_C(N, \mathbb{Z}_\ell(d)) \to H^{2d}_W(X, \mathbb{Z}_\ell(d))$ is the composition

$$H^{2d}_C(N, \mathbb{Z}_\ell(d)) \to H^{2d}_{N_W}(N, \mathbb{Z}_\ell(d)) \xrightarrow{g^*} H^{2d}_W(X, \mathbb{Z}_\ell(d)).$$

Thus it is reduced to showing that the diagram

$$\begin{array}{ccc}
\mathrm{CH}_{m-d}(N_W) & \xrightarrow{cl} & H^{2d}_{N_W}(N, \mathbb{Z}_\ell(d)) \\
{\scriptstyle g^!}\downarrow & & \downarrow{\scriptstyle g^*} \\
\mathrm{CH}_{n-d}(W) & \xrightarrow[cl]{} & H^{2d}_W(X, \mathbb{Z}_\ell(d))
\end{array}$$

is commutative. Let $p : N \to X$ be the projection. Then the maps $g^!$ and g^* are the inverse of the pull-back map p^*. Hence it is reduced to the case where $f = p$ is smooth. $\square$

2.2. *Cohomology of the log self products.* We recall a lemma of Faltings on the cohomology of the log self products. To state it, we introduce a notation. Let Y be a smooth scheme over F and D_1, D_2 be relatively prime divisors of Y such that the sum $D_1 \cup D_2$ has simple normal crossings. Let

$$\begin{tikzcd}
Y \setminus (D_1 \cup D_2) \arrow[r, "k_2"] \arrow[d, "k_1"'] & Y \setminus D_1 \arrow[d, "j_1"'] \\
Y \setminus D_2 \arrow[r, "j_2"] & Y
\end{tikzcd}$$

be open immersions. Let ℓ be a prime number invertible in F. Then, the base change map

$$j_{1!}Rk_{2*}\mathbb{Z}_\ell \longrightarrow Rj_{2*}k_{1!}\mathbb{Z}_\ell \tag{2.1}$$

is an isomorphism. We will identify $j_{1!}Rk_{2*}\mathbb{Z}_\ell = Rj_{2*}k_{1!}\mathbb{Z}_\ell$ by the isomorphism (2.1). We define

$$H^q(Y, D_{1!}, D_{2*}, \mathbb{Z}_\ell) = H^q(Y, D_{2*}, D_{1!}, \mathbb{Z}_\ell)$$

to be $H^q(Y, j_{1!}Rk_{2*}\mathbb{Z}_\ell) = H^q(Y, Rj_{2*}k_{1!}\mathbb{Z}_\ell)$. If D_1 or D_2 is empty, we write simply $H^q(Y, D_{1!}, \emptyset_*, \mathbb{Z}_\ell) = H^q(Y, D_{1!}, \mathbb{Z}_\ell)$ or $H^q(Y, D_{2*}, \emptyset_!, \mathbb{Z}_\ell) = H^q(Y, D_{2*}, \mathbb{Z}_\ell)$ respectively. With this convention, we have $H^q(Y, D_{1!}, \mathbb{Z}_\ell) = H^q_c(Y - D_1, \mathbb{Z}_\ell)$, if Y is proper, and $H^q(Y, D_{2*}, \mathbb{Z}_\ell) = H^q(Y - D_2, \mathbb{Z}_\ell)$.

Let X be a smooth scheme of finite type over a field F, D be a divisor of X with simple normal crossings and $U = X \setminus D$ be the complement. We consider a commutative diagram

(2.2)

$$\begin{tikzcd}
& (X \times X)' \setminus D^{(2)'} \arrow[dl, "j'_2"'] & & (X \times X)^\sim \arrow[ll, "k'_1"'] \arrow[dl, "k'_2"] \\
(X \times X)' \arrow[ddd, "p"'] & & (X \times X)' \setminus D^{(1)'} \arrow[ll, "j'_1"'] & \\
& X \times U \arrow[uu, "e_1"] \arrow[dl, "j_2"'] & & U \times U \arrow[ll, "k_1"'] \arrow[uu, "\tilde{j}"'] \arrow[dl, "k_2"] \\
X \times X & & U \times X. \arrow[ll, "j_1"'] \arrow[uu, "e_2"] &
\end{tikzcd}$$

All the arrows except the log blow-up $p : (X \times X)' \to X \times X$ are open immersions. The four faces consisting of open immersions are Cartesian. Let ℓ be a prime number invertible in F. The canonical maps $\tilde{j}_!\mathbb{Z}_\ell \to \mathbb{Z}_\ell \to R\tilde{j}_*\mathbb{Z}_\ell$ induce maps

$$\begin{aligned}
&(j'_1e_2)_!Rk_{2*}\mathbb{Z}_\ell \\
&= j'_{1!}Rk'_{2*}\tilde{j}_!\mathbb{Z}_\ell \longrightarrow j'_{1!}Rk'_{2*}\mathbb{Z}_\ell \\
&\qquad\qquad = Rj'_{2*}k'_{1!}\mathbb{Z}_\ell \longrightarrow Rj'_{2*}k'_{1!}R\tilde{j}_*\mathbb{Z}_\ell \\
&\qquad\qquad\qquad\qquad = R(j'_2e_1)_*k_{1!}\mathbb{Z}_\ell.
\end{aligned} \tag{2.3}$$

The equalities refer to the identification by (2.1).

Lemma 2.2.1 (Faltings). *Let X be a smooth scheme over F, D be a divisor of X with simple normal crossings and $p : (X \times X)' \to X \times X$ be the blow-up* (1.1)*. The maps* (2.3) *induce isomorphisms*

$$\begin{aligned} & j_{1!}Rk_{2*}\mathbb{Z}_\ell \\ (2.4) \quad & = Rp_*(j'_1e_2)_!Rk_{2*}\mathbb{Z}_\ell \longrightarrow Rp_*j'_{1!}Rk'_{2*}\mathbb{Z}_\ell \longrightarrow Rp_*R(j'_2e_1)_*k_{1!}\mathbb{Z}_\ell \\ & \qquad = Rj_{2*}k_{1!}\mathbb{Z}_\ell \end{aligned}$$

and the composition is the isomorphism (2.1)*.*

For the sake of completeness, we recall the proof in [10].

Proof. Since $j_1 = p \circ j'_1 \circ e_2$, $j_2 = p \circ j'_2 \circ e_1$ and p is proper, we have $j_{1!}Rk_{2*}\mathbb{Z}_\ell = Rp_*(j'_1e_2)_!Rk_{2*}\mathbb{Z}_\ell$ and $Rp_*R(j'_2e_1)_*k_{1!}\mathbb{Z}_\ell = Rj_{2*}k_{1!}\mathbb{Z}_\ell$. It is clear that the composition is the isomorphism (2.1). Thus, it is sufficient to show that the first arrow

$$Rp_*(j'_1e_2)_!Rk_{2*}\mathbb{Z}_\ell \longrightarrow Rp_*j'_{1!}Rk'_{2*}\mathbb{Z}_\ell \tag{2.5}$$

is an isomorphism. Since the question is étale local on $X \times X$, it is reduced to the case where $X = \operatorname{Spec} F[T_1, \dots, T_d]$ and D is defined by $T_1 \cdots T_r = 0$. Further by the Künneth formula, it is reduced to the case where $X = \mathbf{A}^1 = \operatorname{Spec} F[T]$ and D is defined by $T = 0$. In this case, by the proper base change theorem, the assertion follows from $H^q(\mathbf{A}^1_{\bar F}, j_!\mathbb{Z}_\ell) = 0$ for $q \in \mathbb{Z}$ where $j : \mathbf{A}^1 \setminus \{0\} \to \mathbf{A}^1$ is the open immersion. □

Corollary 2.2.2 (Faltings). *Let the notation be as in Lemma* 2.2.1*. If X is proper over F, the maps*

$$\begin{aligned} & H^q(X_{\bar F} \times X_{\bar F}, D^{(1)}_{\bar F !}, D^{(2)}_{\bar F *}, \mathbb{Z}_\ell(d)) = H^q((X \times X)'_{\bar F}, (D^{(1)\prime} \cup E)_{\bar F !}, D^{(2)\prime}_{\bar F *}, \mathbb{Z}_\ell(d)) \\ \longrightarrow & H^q((X \times X)'_{\bar F}, D^{(1)\prime}_{\bar F !}, D^{(2)\prime}_{\bar F *}, \mathbb{Z}_\ell(d)) \\ \longrightarrow & H^q((X \times X)'_{\bar F}, D^{(1)\prime}_{\bar F !}, (D^{(2)\prime} \cup E)_{\bar F *}, \mathbb{Z}_\ell(d)) \end{aligned}$$

are isomorphisms for $q \in \mathbb{Z}$.

Proof. Clear from Lemma 2.2.1. □

2.3. *A Lefschetz trace formula for open varieties.* Let F be a field, X be a proper scheme over F and U be a dense open subscheme of X. Let $\Gamma \subset U \times U$ be a closed subscheme. Let $p_1, p_2 : \Gamma \to U$ denote the compositions of the closed immersion $i : \Gamma \to U \times U$ with the projections $pr_1, pr_2 : U \times U \to U$.

Lemma 2.3.1. *Let X be a proper scheme over F. Let D be a closed subscheme and $U = X \setminus D \subset X$ be the complement. Let $\Gamma \subset U \times U$ be a closed subscheme and $\overline{\Gamma}$ be the closure of Γ in $X \times X$. We put $D^{(1)} = D \times X$ and*

$D^{(2)} = X \times D$. Then, the second projection $p_2 : \Gamma \to U$ is proper if and only if we have the inclusion

$$\overline{\Gamma} \cap D^{(1)} \subset \overline{\Gamma} \cap D^{(2)} \tag{2.6}$$

of the underlying sets.

Proof. The projection $p_2 : \Gamma \to U$ is proper if and only if $\overline{\Gamma} \cap (X \times U) = \overline{\Gamma} \cap (U \times U) = \Gamma$. Taking the complement, it is equivalent to $\overline{\Gamma} \cap D^{(2)} = \overline{\Gamma} \cap (D^{(1)} \cup D^{(2)})$. It is further equivalent to $\bar{\Gamma} \cap D^{(1)} \subset \overline{\Gamma} \cap D^{(2)}$. $\square$

In the following, we assume that U is smooth purely of dimension d, that Γ is purely of dimension d and that $p_2 : \Gamma \to U$ is proper. For a prime number ℓ invertible in F, we define an endomorphism Γ^* of $H^q_c(U_{\bar F}, \mathbb{Z}_\ell)$ to be $p_{1*} \circ p_2^*$ and consider the alternating sum

$$\mathrm{Tr}(\Gamma^* : H^*_c(U_{\bar F}, \mathbb{Z}_\ell)) = \sum_{q=0}^{2d} (-1)^q \mathrm{Tr}(\Gamma^* : H^q_c(U_{\bar F}, \mathbb{Z}_\ell)).$$

Since p_2 is assumed proper, the pull-back $p_2^* : H^q_c(U_{\bar F}, \mathbb{Z}_\ell) \to H^q_c(\Gamma_{\bar F}, \mathbb{Z}_\ell)$ is defined. We briefly recall the definition of the push-forward map $p_{1*} : H^q_c(\Gamma_{\bar F}, \mathbb{Z}_\ell) \to H^q_c(U_{\bar F}, \mathbb{Z}_\ell)$. Let $f : U \to \mathrm{Spec}\ F$ and $g : \Gamma \to \mathrm{Spec}\ F$ denote the structural maps. Then the trace map $Rg_!\mathbb{Z}_\ell(d)[2d] \to \mathbb{Z}_\ell$ induces the cycle class map $\mathbb{Z}_\ell(d)[2d] \to Rg^!\mathbb{Z}_\ell$. Since U is smooth of dimension d, the cycle class map for U induces an isomorphism $Rp_1^!\mathbb{Z}_\ell(d)[2d] \to Rp_1^! Rf^!\mathbb{Z}_\ell \to Rg^!\mathbb{Z}_\ell$. Thus, we obtain a canonical map $\mathbb{Z}_\ell \to Rp_1^!\mathbb{Z}_\ell$ and hence $Rp_{1!}\mathbb{Z}_\ell \to \mathbb{Z}_\ell$ by adjunction. The map $Rp_{1!}\mathbb{Z}_\ell \to \mathbb{Z}_\ell$ induces the push-forward map $p_{1*} : H^q_c(\Gamma_{\bar F}, \mathbb{Z}_\ell) \to H^q_c(U_{\bar F}, \mathbb{Z}_\ell)$.

We give another description of the map $\Gamma^* = p_{1*} \circ p_2^*$ using the cycle class of Γ. We put $H^q_{!,*}(U_{\bar F} \times U_{\bar F}, \mathbb{Z}_\ell(d)) = H^q(X_{\bar F} \times U_{\bar F}, (j \times \mathrm{id})_!\mathbb{Z}_\ell(d))$. By the assumption that $p_2 : \Gamma \to U$ is proper, Γ is closed in $X \times U$ and hence the canonical maps

$$H^{2d}_\Gamma(X \times U, (j \times \mathrm{id})_!\mathbb{Z}_\ell(d)) \to H^{2d}_\Gamma(X \times U, \mathbb{Z}_\ell(d)) \to H^{2d}_\Gamma(U \times U, \mathbb{Z}_\ell(d))$$

are isomorphisms. Thus the cycle class $[\Gamma] \in H^{2d}_\Gamma(U \times U, \mathbb{Z}_\ell(d))$ defines a class $[\Gamma] \in H^{2d}(X_{\bar F} \times U_{\bar F}, (j \times \mathrm{id})_!\mathbb{Z}_\ell(d)) = H^{2d}_{!,*}(U_{\bar F} \times U_{\bar F}, \mathbb{Z}_\ell(d))$. By the Künneth formula and Poincaré duality, we have canonical isomorphisms

$$\begin{array}{ccc} \bigoplus_q H^q_c(U_{\bar F}, \mathbb{Q}_\ell) \otimes H^{2d-q}(U_{\bar F}, \mathbb{Q}_\ell(d)) & \longrightarrow & H^{2d}_{!,*}(U_{\bar F} \times U_{\bar F}, \mathbb{Q}_\ell(d)) \\ \downarrow & & \\ \bigoplus_q H^q_c(U_{\bar F}, \mathbb{Q}_\ell) \otimes Hom(H^q_c(U_{\bar F}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) & \longrightarrow & \prod_{q=0}^{2d} \mathrm{End}\ H^q_c(U_{\bar F}, \mathbb{Q}_\ell). \end{array}$$

Lemma 2.3.2. *Let $\Gamma \subset U \times U$ be a closed subscheme of dimension d. Assume that $p_2 : \Gamma \to U$ is proper. Then, by the canonical isomorphism*

$H^{2d}_{!,*}(U_{\bar F}\times U_{\bar F},\mathbb{Q}_\ell(d)) \to \prod_{q=0}^{2d}\mathrm{End}\ H^q_c(U_{\bar F},\mathbb{Q}_\ell)$, *the image of the cycle class* $[\Gamma]$ *is* Γ^*.

Proof. It is sufficient to show the equality

$$p_{1*}p_2^*\alpha = pr_{1*}([\Gamma]\cup pr_2^*\alpha)$$

in $H^q_c(U_{\bar F},\mathbb{Q}_\ell)$ for an arbitrary integer $q\in\mathbb{Z}$ and $\alpha\in H^q_c(U_{\bar F},\mathbb{Q}_\ell)$. Let $i:\Gamma\to U\times U$ be the immersion and $i':\Gamma\to U\times U\to X\times U$ be the composition. Since $p_{1*}p_2^*\alpha = pr_{1*}(i_*i'^*pr_2^*\alpha)$, it is reduced to showing the equality

$$i_*i'^*\beta = [\Gamma]\cup\beta$$

in $H^q_c(U_{\bar F}\times U_{\bar F},\mathbb{Q}_\ell)$ for $\beta\in H^q_c(X_{\bar F}\times U_{\bar F},\mathbb{Q}_\ell)$. By Lemma 2.1.1, the class $i_*i'^*\beta$ is the product with the class of Γ. Thus the assertion follows. □

Lemma 2.3.3. *Let U and V be connected separated smooth schemes of finite type purely of dimension d over F. Let $g:U\to V$ be a proper and generically finite morphism of constant degree $[U:V]$ over F. Then, for a cohomology class $\Gamma\in H^{2d}_{!,*}(V_{\bar F}\times V_{\bar F},\mathbb{Q}_\ell)=\prod_{q=0}^{2d}\mathrm{End}\ H^q_c(V_{\bar F},\mathbb{Q}_\ell)$, we have*

$$\mathrm{Tr}(\Gamma^*:H^*_c(V_{\bar F},\mathbb{Q}_\ell)) = \frac{1}{[U:V]}\mathrm{Tr}(((g\times g)^*\Gamma)^*:H^*_c(U_{\bar F},\mathbb{Q}_\ell)). \tag{2.7}$$

Proof. Since $g^*:H^*_c(V_{\bar F},\mathbb{Q}_\ell)\to H^*_c(U_{\bar F},\mathbb{Q}_\ell)$ is injective and $g_*\circ g^*$ is the multiplication by $[U:V]$, it is sufficient to show that $((g\times g)^*\Gamma)^*$ is the composition $g^*\circ\Gamma^*\circ g_*$. In other words, it suffices to show the equality

$$pr_{1*}(((g\times g)^*\Gamma\cup pr_2^*\alpha) = g^*(pr_{1*}(\Gamma\cup pr_2^*g_*\alpha))$$

for $q\in\mathbb{Z}$ and $\alpha\in H^q_c(U_{\bar F},\mathbb{Q}_\ell)$. In the commutative diagram

$$\begin{tikzcd}
U\times U \arrow[rr, "pr_2"] \arrow[d, "1\times g"'] & & U \arrow[d, "g"] \\
U\times V \arrow[r, "g\times 1"] \arrow[d, "pr_1"'] & V\times V \arrow[r, "pr_2"'] \arrow[d, "pr_1"] & V \\
U \arrow[r, "g"] & V, &
\end{tikzcd}$$

α lives on U in the northeast and Γ lives on $V\times V$. Thus, by the projection formula, we compute

$$\begin{aligned}
&pr_{1*}((g\times g)^*\Gamma\cup pr_2^*\alpha) = pr_{1*}((1\times g)^*(g\times 1)^*\Gamma\cup pr_2^*\alpha)\\
=&pr_{1*}((g\times 1)^*\Gamma\cup(1\times g)_*pr_2^*\alpha) = pr_{1*}((g\times 1)^*(\Gamma\cup pr_2^*g_*\alpha))\\
=&g^*pr_{1*}(\Gamma\cup pr_2^*g_*\alpha).
\end{aligned}$$

□

We prove a Lefschetz trace formula for open varieties.

THEOREM 2.3.4. *Let X be a proper and smooth scheme purely of dimension d over a field F and U be the complement of a divisor D with simple normal crossings. Let $\Gamma \subset U \times U$ be a closed subscheme purely of dimension d. Let $D^{(1)\prime}, D^{(2)\prime} \subset (X \times X)'$ denote the proper transforms of $D^{(1)}, D^{(2)}$ respectively and let $\overline{\Gamma}'$ be the closure of Γ in $(X \times X)'$. We assume that we have an inclusion*

$$\overline{\Gamma}' \cap D^{(1)\prime} \subset \overline{\Gamma}' \cap D^{(2)\prime} \tag{2.8}$$

of the underlying sets.

Then, the map $p_2 : \Gamma \to U$ is proper and we have an equality

$$\mathrm{Tr}(\Gamma^* : H^*_c(U_{\bar F}, \mathbb{Q}_\ell)) = \deg\ (\overline{\Gamma}', \Delta'_X)_{(X\times X)'}. \tag{2.9}$$

The right-hand side is the intersection product in $(X \times X)'$ of the closure $\overline{\Gamma}'$ with the image Δ'_X of the log diagonal closed immersion $\Delta' : X \to (X \times X)'$.

Proof. First, we show the map $p_2 : \Gamma \to U$ is proper. By the assumption (2.8) $\overline{\Gamma}' \cap D^{(1)\prime} \subset \overline{\Gamma}' \cap D^{(2)\prime}$, we have $\overline{\Gamma}' \cap (D^{(1)\prime} \cup E) \subset \overline{\Gamma}' \cap (D^{(2)\prime} \cup E)$. Hence we have (2.6) $\overline{\Gamma} \cap D^{(1)} \subset \overline{\Gamma} \cap D^{(2)}$ and the assertion follows by Lemma 2.3.1.

Since the restriction of $j_{1!}Rk_{2*}\mathbb{Z}_\ell(d)$ on the diagonal $X \subset X \times X$ is $j_!\mathbb{Z}_\ell(d)$, the pull-back map

$$\begin{aligned}\Delta^* : H^{2d}_{!,*}(U_{\bar F} \times U_{\bar F}, \mathbb{Z}_\ell(d)) &\longrightarrow H^{2d}_c(U_{\bar F}, \mathbb{Z}_\ell(d))\\ &= H^{2d}(X_{\bar F} \times X_{\bar F}, j_{1!}Rk_{2*}\mathbb{Z}_\ell(d)) = H^{2d}(X_{\bar F}, j_!\mathbb{Z}_\ell(d))\end{aligned}$$

by the diagonal is defined. Then, by Lemma 2.3.2 and by the standard argument (cf. [13, Prop. 3.3]) in the proof of Lefschetz trace formula, we have

$$\mathrm{Tr}(\Gamma^* : H^*_c(U_{\bar F}, \mathbb{Q}_\ell)) = \mathrm{Tr}(\Delta^*([\Gamma])). \tag{2.10}$$

In the notation introduced in the beginning of §2.2, we have

$$H^{2d}_{!,*}(U_{\bar F} \times U_{\bar F}, \mathbb{Z}_\ell(d)) = H^{2d}(X_{\bar F} \times X_{\bar F}, D^{(1)}_{\bar F!}, D^{(2)}_{\bar F*}, \mathbb{Z}_\ell(d))$$

and

$$H^{2d}_c(U_{\bar F}, \mathbb{Z}_\ell(d)) = H^{2d}(X_{\bar F}, D_{\bar F!}, \mathbb{Z}_\ell).$$

The canonical map $(X \times X)' \to X \times X$ induces an isomorphism $H^q(X_{\bar F} \times X_{\bar F}, D^{(1)}_{\bar F!}, D^{(2)}_{\bar F*}, \mathbb{Z}_\ell(d)) \to H^q((X \times X)'_{\bar F}, (D^{(1)\prime} \cup E)_{\bar F!}, D^{(2)\prime}_{\bar F*}, \mathbb{Z}_\ell(d))$. Thus the composition

$$\begin{aligned}H^{2d}_{!,*}(U_{\bar F} \times U_{\bar F}, \mathbb{Z}_\ell(d)) &= H^q(X_{\bar F} \times X_{\bar F}, D^{(1)}_{\bar F!}, D^{(2)}_{\bar F*}, \mathbb{Z}_\ell(d))\\ &\longrightarrow H^q((X \times X)'_{\bar F}, (D^{(1)\prime} \cup E)_{\bar F!}, D^{(2)\prime}_{\bar F*}, \mathbb{Z}_\ell(d))\\ &\longrightarrow H^q((X \times X)'_{\bar F}, D^{(1)\prime}_{\bar F!}, D^{(2)\prime}_{\bar F*}, \mathbb{Z}_\ell(d))\end{aligned} \tag{2.11}$$

is an isomorphism by Corollary 2.2.2.

We put $\Gamma' = \overline{\Gamma}' \backslash \overline{\Gamma}' \cap D^{(2)\prime}$. By the assumption (2.8), we have $\Gamma' \cap D^{(1)\prime} = \emptyset$. Thus the cycle class $[\Gamma'] \in H^{2d}((X \times X)'_{\bar F}, D^{(1)\prime}_{\bar F !}, D^{(2)\prime}_{\bar F *}, \mathbb{Z}_\ell(d))$ is defined. We show that the arrow (2.11) sends $[\Gamma]$ to $[\Gamma']$. By Corollary 2.2.2, the map

$$H^{2d}((X \times X)'_{\bar F}, D^{(1)\prime}_{\bar F !}, D^{(2)\prime}_{\bar F *}, \mathbb{Z}_\ell(d)) \to H^{2d}((X \times X)'_{\bar F}, D^{(1)\prime}_{\bar F !}, (E \cup D^{(2)\prime})_{\bar F *}, \mathbb{Z}_\ell(d))$$

is an isomorphism. By this isomorphism, both $[\Gamma']$ and the image of $[\Gamma]$ are sent to $[\Gamma]$. Hence the arrow (2.11) sends $[\Gamma]$ to $[\Gamma']$.

Since $\Delta'_X \cap D^{(2)\prime} = \emptyset$, the map $\Delta'^* : H^q((X \times X)'_{\bar F}, D^{(2)\prime}_{\bar F *}, \mathbb{Z}_\ell(d)) \to H^q(X_{\bar F}, \mathbb{Z}_\ell(d))$ is defined. We consider the commutative diagram

$$(2.12)\qquad \begin{array}{ccccc}
[\Gamma] \in & H^{2d}_{!,*}(U_{\bar F} \times U_{\bar F}, \mathbb{Z}_\ell(d)) & \xrightarrow{\Delta^*} & H^{2d}_c(U_{\bar F}, \mathbb{Z}_\ell(d)) \\
\downarrow & \Big\downarrow{\scriptstyle (2.11)} & & \Big\downarrow \\
[\Gamma'] \in & H^{2d}((X \times X)'_{\bar F}, D^{(1)\prime}_{\bar F !}, D^{(2)\prime}_{\bar F *}, \mathbb{Z}_\ell(d)) & \xrightarrow{\Delta'^*} & H^{2d}(X_{\bar F}, \mathbb{Z}_\ell(d)) \\
\downarrow & \Big\downarrow & & \Big\Vert \\
[\Gamma'] \in & H^{2d}((X \times X)'_{\bar F}, D^{(2)\prime}_{\bar F *}, \mathbb{Z}_\ell(d)) & \xrightarrow{\Delta'^*} & H^{2d}(X_{\bar F}, \mathbb{Z}_\ell(d)) \\
\uparrow & \Big\uparrow & & \Big\Vert \\
[\overline{\Gamma}'] \in & H^{2d}((X \times X)'_{\bar F}, \mathbb{Z}_\ell(d)) & \xrightarrow{\Delta'^*} & H^{2d}(X_{\bar F}, \mathbb{Z}_\ell(d)).
\end{array}$$

As we have shown above, the arrow (2.11) sends $[\Gamma]$ to $[\Gamma']$. Since the middle and the lower left vertical arrows send $[\Gamma']$ and $[\overline{\Gamma}']$ to $[\Gamma']$ respectively, we have

$$(2.13)\qquad \mathrm{Tr}(\Delta^*([\Gamma])) = \mathrm{Tr}(\Delta'^*([\overline{\Gamma}'])).$$

Since

$$\mathrm{Tr}(\Delta'^*([\overline{\Gamma}'])) = \deg\,(\overline{\Gamma}', \Delta'_X)_{(X \times X)'},$$

the assertion follows from the equalities (2.10) and (2.13). □

Remark 2.3.5. In Theorem 2.3.4, we can not replace the assumption (2.8) $\overline{\Gamma}' \cap D^{(1)\prime} \subset \overline{\Gamma}' \cap D^{(2)\prime}$ by a weaker assumption (2.6) $\overline{\Gamma} \cap D^{(1)} \subset \overline{\Gamma} \cap D^{(2)}$ as the following example shows. Let $X = \mathbf{P}^1$, $U = \mathbf{A}^1$, and $n \geq 1$ be an integer. Let $f : U \to U$ be the n-th power map and $\Gamma \subset U \times U$ be the transpose $\Gamma = \{(x, y) \in U \times U | x = y^n\}$ of the graph of f. Then, we have $\mathrm{Tr}(\Gamma^* : H^*_c(U_{\bar F}, \mathbb{Z}_\ell)) = \mathrm{Tr}(f_* : H^2_c(U_{\bar F}, \mathbb{Z}_\ell)) = 1$ while $(\Gamma, \Delta)_{(X \times X)'} = n$.

One can deduce a part of a conjecture of Deligne from Theorem 2.3.4 as follows. The conjecture of Deligne itself is proved assuming resolution of singularities by Pink in [25] and proved unconditionally by Fujiwara in [11] using rigid geometry. In the proof below, we will not use rigid geometry or assume resolution of singularities.[3]

We introduce some notation assuming F is a finite field. For a scheme over F, let Fr denote the Frobenius endomorphism over F. Let U be a separated smooth scheme of finite type of pure dimension d over F. Let $\Gamma \subset U \times U$ be a closed subscheme of dimension d and assume the composition $p_2 : \Gamma \to U$ with the projection is proper. For an integer $n \geq 0$ and a prime number ℓ different from the characteristic of F, we consider the alternating sum $\mathrm{Tr}(\mathrm{Fr}_F^{*n}\Gamma^* : H_c^*(U_{\bar{F}}, \mathbb{Q}_\ell))$. Let $i_n : \Gamma \to U \times U$ be the composition of the immersion $i : \Gamma \to U \times U$ with the endomorphism $1 \times Fr^n$ of $U \times U$. Let Γ_n denote the scheme Γ regarded as a scheme over $U \times U$ by i_n. If the fiber product $\Gamma_n \times_{U \times U} \Delta_U$ is proper over F, the degree of the intersection product $(\Gamma_n, \Delta_U)_{U \times U} \in \mathrm{CH}_0(\Gamma_n \times_{U \times U} \Delta_U)$ is defined.

Proposition 2.3.6 (cf. [11], [25]). *Let U be a separated smooth scheme of finite type of pure dimension d over a field F and ℓ be a prime number different from the characteristic of F. Let $\Gamma \subset U \times U$ be a closed subscheme of dimension d. Assume the composition $p_2 : \Gamma \to U$ with the projection is proper. Then, we have the following.*

1. *The alternating sum $\mathrm{Tr}(\Gamma^* : H_c^*(U_{\bar{F}}, \mathbb{Q}_\ell))$ is in $\mathbb{Z}[\frac{1}{p}]$ and is independent of ℓ invertible in F.*

2. *Assume F is a finite field. Then, there exists an integer $n_0 \geq 0$ satisfying the following property.*

For an integer $n \geq n_0$, the fiber product $\Gamma_n \times_{U \times U} \Delta_U$ is proper over F and we have

$$\mathrm{Tr}(\mathrm{Fr}_F^{*n}\Gamma^* : H_c^*(U_{\bar{F}}, \mathbb{Q}_\ell)) = \deg(\Gamma_n, \Delta_U)_{U \times U}. \tag{2.14}$$

Proof. 1. It is reduced to 2 by a standard argument using specialization.

2. By the main result of de Jong [9] and Lemma 2.3.3, we may assume that there exists a proper smooth scheme X containing U as the complement of a divisor with simple normal crossings. We will derive the proposition from Theorem 2.3.4 using the following lemma.

Lemma 2.3.7. *Let X be a proper smooth scheme over a finite field F of order q and $D \subset X$ be a divisor with simple normal crossings. Let $U = X \setminus D$ be the complement and let $\Gamma \subset U \times U$ be an integral closed subscheme. Assume $p_2 : \Gamma \to U$ is proper.*

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[3]Added in Proof. An unconditional proof without using rigid geometry is given in Y. Varshavsky, Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara, *Geom. Funct. Anal.* **17** (2007), 271–319.

Then, there exists an integer $n_0 \geq 0$ *such that, for all* $n \geq n_0$, *the closure* $\overline{i_n(\Gamma_n)} \subset (X \times X)'$ *of the image* $i_n(\Gamma_n) \subset U \times U$ *satisfies the inclusion*

$$\overline{i_n(\Gamma_n)} \setminus i_n(\Gamma_n) \subset D^{(2)\prime}. \tag{2.15}$$

Proof. Let $\overline{\Gamma} \subset (X \times X)'$ be the closure of Γ. By the main result of de Jong [9], there exist a proper smooth integral scheme Z of dimension d, a proper map $Z \to \overline{\Gamma}$ over F such that the inverse image $W = Z \times_{\overline{\Gamma}} \Gamma$ is the complement of a divisor B with simple normal crossings. Let $Z' \to Z$ be the blow-up associated to the subdivision by baricenters and $B' = Z' \setminus W$ be the complement.

Let $\bar{r}_1, \bar{r}_2 : Z' \to X$ be the compositions with the projections. Let D_i $(i \in I)$ be the irreducible components of D and B'_j $(j \in J)$ be the irreducible components of B'. We put $\bar{r}_1^* D_i = \sum_{j \in J} e_{ij}^{(1)} B'_j$ and $\bar{r}_2^* D_i = \sum_{j \in J} e_{ij}^{(2)} B'_j$ for $i \in I$. By the assumption $p_2 : \Gamma \to U$ is proper, the composition $r_2 : W \to U$ is proper and hence the support of $\bar{r}_2^* D = \sum_{j \in J} (\sum_{i \in I} e_{ij}^{(2)}) B'_j$ equals B'. In other words, for every $j \in J$, there exists an index $i \in I$ such that $e_{ij}^{(2)} > 0$.

Let $J_0 \subset J$ be the subset $\{j \in J | B'_j$ is the proper transform of an irreducible component of $B\}$. Then, if $B'_j \cap B'_{j'} \neq \emptyset$ and if $j \in J_0$, we have $e_{ij}^{(2)} \leq e_{ij'}^{(2)}$. Hence, if $e_{ij}^{(2)} = 0$ and $e_{ij'}^{(2)} > 0$ for every $B'_{j'}$ such that $B'_j \cap B'_{j'} \neq \emptyset$, then we have $j \in J_0$.

We show that, for every $z \in B'$, there exists an index $i \in I$ such that $e_{ij}^{(2)} > 0$ for all $B'_j \ni z$. We prove this by contradiction. Assume there exists $z \in B'$ such that, for every $i \in I$, there exists a component $B'_j \ni z$ such that $e_{ij}^{(2)} = 0$. First, we show that there exists an element $j_0 \in J_0$ such that $z \in B'_{j_0}$. Let B'_j be a component containing z. Then, as we have seen above, there exists an index $i \in I$ such that $e_{ij}^{(2)} > 0$. By the hypothesis, we also have an index $j_0 \in J$ such that $z \in B'_{j_0}$ and $e_{ij_0}^{(2)} = 0$. Since $z \in B'_{j_0} \cap B'_j$, we have $j_0 \in J_0$. We show $e_{ij_0}^{(2)} = 0$ for every $i \in I$, to get a contradiction. For $i \in I$, by the hypothesis, there exists $B'_j \ni z$ such that $e_{ij}^{(2)} = 0$. Since $z \in B'_{j_0} \cap B'_j \neq \emptyset$, we have $0 = e_{ij}^{(2)} \geq e_{ij_0}^{(2)} \geq 0$. Thus we get a contradiction.

We take $n_0 \geq 0$ such that $q^{n_0} > \max_{i \in I, j \in J} e_{ij}^{(1)}$. Then, for every $z \in B'$, there exists an index $i \in I$ such that $q^{n_0} e_{ij}^{(2)} > e_{ij}^{(1)}$ for all $B'_j \ni z$. Namely, we have a strict inequality

$$q^{n_0} \bar{r}_2^* D_i > \bar{r}_1^* D_i \tag{2.16}$$

of germs of Cartier divisors at z.

We show the inclusion (2.15) for $n \geq n_0$. We consider the product $\bar{i}_n : W \to (X \times X)' \times Z'$ of the composition $W \to \Gamma$ with $i_n : \Gamma_n \to U \times U \subset (X \times X)'$ and the inclusion $W \to Z'$. Let Z_n be the closure of the image of the immersion $\bar{i}_n : W \to (X \times X)' \times Z'$ with the reduced scheme structure. Let $\bar{r}_n : Z_n \to (X \times X)'$ and $f_n : Z_n \to Z'$ be the projections. Further, let $\bar{r}_{1,n}, \bar{r}_{2,n} : Z_n \to X$ be the compositions of $\bar{r}_n$ with the projections. Then, since $W \subset Z_n$ is dense, the diagram

$$\begin{tikzcd} Z_n \arrow[r, "f_n"] \arrow[d, "\bar{r}_{1,n}\times\bar{r}_{2,n}"'] & Z' \arrow[d, "\bar{r}_1\times\bar{r}_2"] \\ X \times X & X \times X \arrow[l, "1\times Fr^n"'] \end{tikzcd}$$

is commutative. Thus, we have equalities $\bar{r}_{1,n}^* D_i = f_n^* \bar{r}_1^* D_i$ and $\bar{r}_{2,n}^* D_i = q^n f_n^* \bar{r}_2^* D_i$ of Cartier divisors on Z_n for each $i \in I$.

Since $W \to \Gamma$ is proper and surjective, we have $\overline{i_n(\Gamma_n)} \setminus i_n(\Gamma_n) = \bar{r}_n(Z_n \setminus W)$. For every point $z \in Z_n \setminus W$, there exists an index $i \in I$ satisfying a strict inequality

$$\bar{r}_{2,n}^* D_i = q^n f_n^* \bar{r}_2^* D_i > f_n^* \bar{r}_1^* D_i = \bar{r}_{1,n}^* D_i$$

of germs of Cartier divisors at z by (2.16). Namely, we have $z \in \bar{r}_n^{-1}(X \times D_i)'$. Thus, we have $\bar{r}_n(Z_n \setminus W) \subset D^{(2)\prime} = \bigcup_{i \in I}(X \times D_i)'$ and the assertion follows. $\square$

We complete the proof of Proposition 2.3.6. Take a proper scheme $\overline{\Gamma_n}$ over F containing Γ as a dense open subscheme and a map $\bar{i}_n : \overline{\Gamma_n} \to (X \times X)'$ extending the map $i_n : \Gamma_n \to U \times U$. The intersection of the log diagonal $\Delta'_X \subset (X \times X)'$ with $D^{(2)\prime}$ is empty. Hence by the inclusion (2.15) in Lemma 2.3.7, the intersection $\overline{i_n(\Gamma_n)} \cap \Delta'_X$ with the log diagonal equals $i_n(\Gamma_n) \cap \Delta_U$. Hence the fiber product $\Gamma_n \times_{U \times U} \Delta_U = \overline{\Gamma_n} \times_{(X \times X)'} \Delta'_X$ is proper over F and we have $(\overline{\Gamma_n}, \Delta'_X)_{(X \times X)'} = (\Gamma_n, \Delta_U)_{U \times U}$.

Also by the inclusion (2.15) in Lemma 2.3.7, the assumption (2.8) of Theorem 2.3.4 is satisfied for the support of the cycle $\bar{i}_{n*}(\overline{\Gamma}_n)$. Thus, by Theorem 2.3.4, we have $\mathrm{Tr}(\mathrm{Fr}_F^{*n}\Gamma^* : H_c^*(U_{\bar{F}}, \mathbb{Q}_\ell)) = \deg(\overline{\Gamma_n}, \Delta'_X)_{(X \times X)'} = \deg(\Gamma_n, \Delta_U)_{U \times U}$. $\square$

3. Intersection product with the log diagonal and a trace formula

We introduce the target group $\mathrm{CH}_0(\overline{V} \setminus V)$ of the map (0.1) in Section 3.1. We define the map (0.1) and prove the trace formula (0.2) in Section 3.2. We establish elementary properties of the map (0.1) in Section 3.3. We define and compute the wild different of a covering and the log Lefschetz class of an automorphism using the map (0.1) in Section 3.4.

In this section, F denotes a perfect field and $f : V \to U$ is a finite étale morphism of separated and smooth schemes of finite type purely of dimension d over F.

3.1. *Chow group of* 0*-cycles on the boundary.* In this subsection, we introduce the target group $\mathrm{CH}_0(\overline{V} \setminus V)$ of the map (0.1).

Definition 3.1.1. Let V be a separated smooth scheme of finite type over a field F.

1. Let $\mathcal{C}_V$ be the following category. An object of $\mathcal{C}_V$ is a proper scheme Y over F containing V as a dense open subscheme. A morphism $Y' \to Y$ in $\mathcal{C}_V$ is a morphism $Y' \to Y$ over F inducing the identity on V.

Let $\mathcal{C}_V^{\mathrm{sm}}$ be the full subcategory of $\mathcal{C}_V$ consisting of smooth objects. Let $\mathcal{C}_V^{\mathrm{sm},0}$ be the full subcategory of $\mathcal{C}_V$ consisting of smooth objects Y such that V is the complement of a divisor with simple normal crossings.

2. We put

$$\mathrm{CH}_0(\overline{V} \setminus V) = \varprojlim_{\mathcal{C}_V} \mathrm{CH}_0(Y \setminus V). \tag{3.1}$$

The transitions maps are proper push-forwards. Let

$$\deg : \mathrm{CH}_0(\overline{V} \setminus V) \longrightarrow \mathbb{Z} \tag{3.2}$$

be the limit of the degree maps $\mathrm{CH}_0(Y \setminus V) \to \mathbb{Z}$.

Recall that we assume F is perfect. The resolution of singularities means that the full subcategory $\mathcal{C}_V^{\mathrm{sm}}$ is cofinal in $\mathcal{C}_V$. A strong form of the resolution of singularities means that $\mathcal{C}_V^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_V$. Thus, it is known that $\mathcal{C}_V^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_V$ if dimension V is at most 2. More precisely, if dimension is at most 2, we have a strong form of equivariant resolution of singularities as follows.

Lemma 3.1.2. *Let V be a separated smooth scheme of finite type of dimension ≤ 2 over a perfect field F and G be a finite group of automorphisms of V over F.*

Then the full subcategory of $\mathcal{C}_V^{\mathrm{sm},0}$ consisting of Y with an admissible action of G extending that on V is cofinal in $\mathcal{C}_V$.

Proof. Let Y_0 be an object of $\mathcal{C}_V$. Let Y_1 be the closure of the image of the map $V \to \prod_{\sigma \in G} V \subset \prod_{\sigma \in G} Y_0$ sending v to $(\sigma(v))_{\sigma \in G}$. Let Y_2 be the minimal resolution of the normalization of Y_1. By blowing-up Y_2 successively at the closed points where the complement $Y_2 \setminus V$ does not have simple normal crossing, we obtain Y_3 in $\mathcal{C}_V^{\mathrm{sm},0}$ with an action of G. The action of G on the blow-up Y of Y_3 associated to the subdivision by baricenters is admissible by Lemma 1.2.3.2. $\square$

Let Y be a separated scheme of finite type over F containing V as a dense open subscheme. Then there exists a unique map $\mathrm{CH}_0(\overline{V} \setminus V) \to \mathrm{CH}_0(Y \setminus V)$ satisfying the following property. Let Y' be an object of $\mathcal{C}_V$ containing Y as a dense open subscheme. Then it is the same as the composition of the projection $\mathrm{CH}_0(\overline{V} \setminus V) \to \mathrm{CH}_0(Y' \setminus V)$ and the restriction $\mathrm{CH}_0(Y' \setminus V) \to \mathrm{CH}_0(Y \setminus V)$.

Let $f : V \to U$ be a finite flat morphism of smooth schemes over F. The push-forward maps induces a map $f_* : \mathrm{CH}_0(\overline{V} \setminus V) \to \mathrm{CH}_0(\overline{U} \setminus U)$. The flat pull-back map $f^* : \mathrm{CH}_0(\overline{U} \setminus U) \to \mathrm{CH}_0(\overline{V} \setminus V)$ is defined as follows.

LEMMA 3.1.3. *Let $f : V \to U$ be a finite flat morphism of smooth schemes over F. Then, the following holds.*

1. *Let X be an object of $\mathcal{C}_U$ and Y be an object of $\mathcal{C}_V$. Then there exist a morphism $X' \to X$ in $\mathcal{C}_U$, a morphism $Y' \to Y$ in $\mathcal{C}_V$ and a finite flat morphism $\bar{f}' : Y' \to X'$ over F extending $f : V \to U$.*

If $f : V \to U$ is a Galois covering of group G, there exists $\bar{f}' : Y' \to X'$ as above such that the action of G is extended to an action on Y'.

2. *Let $g : X' \to X$ be a morphism in $\mathcal{C}_U$ and $h : Y' \to Y$ be a morphism in $\mathcal{C}_V$. Let*

$$\begin{array}{ccc} Y' & \xrightarrow{h} & Y \\ {\scriptstyle \bar{f}'}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}} \\ X' & \xrightarrow{g} & X \end{array} \tag{3.3}$$

be a commutative diagram of morphisms over F where the vertical arrows are finite flat morphisms extending $f : V \to U$. Then, the diagram

$$\begin{array}{ccc} \mathrm{CH}_0(Y' \setminus V) & \xrightarrow{h_*} & \mathrm{CH}_0(Y \setminus V) \\ {\scriptstyle \bar{f}'^*}\big\uparrow & & \big\uparrow{\scriptstyle \bar{f}^*} \\ \mathrm{CH}_0(X' \setminus U) & \xrightarrow{g_*} & \mathrm{CH}_0(X \setminus U) \end{array}$$

is commutative.

Proof. 1. By replacing Y by the closure of the graph $\Gamma_f \subset V \times U \subset Y \times X$, we may assume that there exists a proper map $\bar{f} : Y \to X$ extending $f : V \to U$. Then, we obtain a finite flat morphism $\bar{f}' : Y' \to X'$ by applying Théorème (5.2.2) of [27].

Assume $V \to U$ is a Galois covering. Then, by replacing Y by the closure of the image $V \to \prod_{\sigma \in G} V \subset \prod_{\sigma \in G} Y$ sending v to $(\sigma(v))_{\sigma \in G}$, we may assume the action of G on V is extended to an action on Y. It suffices to apply the construction in the paragraph above.

2. Since the assertion is clear if the diagram (3.3) is Cartesian, we may assume $X' = X$. Let $x \in X \setminus U$ be a closed point and let A be the completion $\hat{O}_{X,x}$. For $y \in \bar{f}^{-1}(x)$, we put $B_y = \hat{O}_{Y,y}$. For $y' \in \bar{f}'^{-1}(x)$, we put $B'_{y'} = \hat{O}_{Y',y'}$. Then, we have $\bar{f}^*([x]) = \sum_{y \in \bar{f}^{-1}(x)} \mathrm{rank}_A B_y / [\kappa(y) : \kappa(x)] \cdot [y]$ and $\bar{f}'^*([x]) = \sum_{y' \in \bar{f}'^{-1}(x)} \mathrm{rank}_A B'_{y'} / [\kappa(y') : \kappa(x)] \cdot [y']$. For each $y \in \bar{f}^{-1}(x)$, we have $\mathrm{rank}_A B_y = \sum_{y' \in h^{-1}(y)} \mathrm{rank}_A B'_{y'}$. Thus the assertion follows. □

By Lemma 3.1.3, the flat pull-back maps $\bar{f}^* : \mathrm{CH}_0(X \setminus U) \to \mathrm{CH}_0(Y \setminus V)$ induce $f^* : \mathrm{CH}_0(\overline{U} \setminus U) \to \mathrm{CH}_0(\overline{V} \setminus V)$.

Corollary 3.1.4. *Let $f : V \to U$ be a finite flat morphism of smooth schemes of constant degree N.*

1. *Then, the composition $f_* \circ f^* : \mathrm{CH}_0(\overline{U} \setminus U) \to \mathrm{CH}_0(\overline{U} \setminus U)$ is the multiplication by N.*

2. *Assume further that $V \to U$ is a Galois covering of Galois group G. Then, the composition $f^* \circ f_* : \mathrm{CH}_0(\overline{V} \setminus V) \to \mathrm{CH}_0(\overline{V} \setminus V)$ is equal to $\sum_{\sigma \in G} \sigma^*$.*

The pull-back map f^* induces an isomorphism $f^* : \mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q} \to (\mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q})^G$ to the G-fixed part. The inverse is given by $\frac{1}{|G|} f_*$.

Proof. Clear from Lemma 3.1.3. □

If we admit resolution of singularities, the projective limit $\mathrm{CH}_0(\overline{V} \setminus V)$ is computed by a smooth object in $\mathcal{C}_V$ as we see in Corollary 3.1.6 below.

Lemma 3.1.5. *Let V be a separated smooth scheme of finite type over F. Let Y and Y' be separated smooth schemes over F containing V as dense open subschemes and $g : Y' \to Y$ be a morphism over F inducing the identity on V.*

Then, the Gysin map $g^! : \mathrm{CH}_0(Y \setminus V) \to \mathrm{CH}_0(Y' \setminus V)$ is a surjection. Further if $g : Y' \to Y$ is proper, the map $g^! : \mathrm{CH}_0(Y \setminus V) \to \mathrm{CH}_0(Y' \setminus V)$ is an isomorphism and is the inverse of $g_* : \mathrm{CH}_0(Y' \setminus V) \to \mathrm{CH}_0(Y \setminus V)$.

Proof. Let $\mathcal{K}_d$ denote the Zariski sheaf of Quillen's K-theory. Then, by the Gersten resolution, the Chow group $\mathrm{CH}_0(Y \setminus V)$ is identified with the cohomology $H^d_{Y \setminus V}(Y, \mathcal{K}_d)$ with support and the Gysin map $g^! : \mathrm{CH}_0(Y \setminus V) \to \mathrm{CH}_0(Y' \setminus V)$ is identified with the pull-back map $g^* : H^d_{Y \setminus V}(Y, \mathcal{K}_d) \to H^d_{Y' \setminus V}(Y', \mathcal{K}_d)$. Thus, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
H^{d-1}(V, \mathcal{K}_d) & \longrightarrow & \mathrm{CH}_0(Y \setminus V) & \longrightarrow & \mathrm{CH}_0(Y) & \longrightarrow & \mathrm{CH}_0(V) \\
\| & & \downarrow{\scriptstyle g^!} & & \downarrow{\scriptstyle g^!} & & \| \\
H^{d-1}(V, \mathcal{K}_d) & \longrightarrow & \mathrm{CH}_0(Y' \setminus V) & \longrightarrow & \mathrm{CH}_0(Y') & \longrightarrow & \mathrm{CH}_0(V).
\end{array}$$

Since $\mathrm{CH}_0(Y')$ is generated by the 0-cycles on the dense open $V \subset Y'$, the map $g^! : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(Y')$ is surjective. Thus a diagram chasing shows the surjectivity of $g^! : \mathrm{CH}_0(Y \setminus V) \to \mathrm{CH}_0(Y' \setminus V)$.

If g is proper, we have $g_* \circ g^! = \mathrm{id}$ by the projection formula. Hence $g^!$ is an isomorphism and is the inverse of g_*. □

Corollary 3.1.6. *Let V be a separated smooth scheme of finite type over F. Assume the full subcategory $\mathcal{C}_V^{\mathrm{sm}}$ consisting of smooth objects is cofinal in $\mathcal{C}_V$.*

1. *Then, the projection* $\mathrm{CH}_0(\overline{V}\backslash V)\to\mathrm{CH}_0(Y\backslash V)$ *is an isomorphism for an object* Y *of* $\mathcal{C}_V^{\mathrm{sm}}$*. Their inverses induce an isomorphism* $\varinjlim_{\mathcal{C}_V^{\mathrm{sm,opp}}}\mathrm{CH}_0(Y\backslash V)\to\mathrm{CH}_0(\overline{V}\setminus V)$ *where the transition maps are Gysin maps.*

2. *Let* $f : V \to U$ *be a finite flat morphism of smooth schemes. Assume the full subcategory* $\mathcal{C}_U^{\mathrm{sm}}$ *is also cofinal in* $\mathcal{C}_U$*.*

Then, the pull-back map $f^* : \mathrm{CH}_0(\overline{U}\setminus U)\to\mathrm{CH}_0(\overline{V}\setminus V)$ *is the same as the map* $\varinjlim_{\mathcal{C}_U^{\mathrm{sm,opp}}}\mathrm{CH}_0(X\setminus U)\to\varinjlim_{\mathcal{C}_V^{\mathrm{sm,opp}}}\mathrm{CH}_0(Y\setminus V)$ *induced by the Gysin maps.*

Proof. 1. Clear from Lemma 3.1.5.

2. Let X and Y be objects of $\mathcal{C}_U^{\mathrm{sm}}$ and of $\mathcal{C}_V^{\mathrm{sm}}$ respectively and let $\bar{f} : Y\to X$ be a morphism over F extending $f : V \to U$. It is sufficient to show that $\bar{f}^!([x]) = f^*([x])$ for an arbitrary closed point $x \in X\setminus U$. Let $X'\to X$ be the blow-up at x and Y' be an object of $\mathcal{C}_V^{\mathrm{sm}}$ dominating $Y\times_X X'$. Replacing $Y\to X$ by $Y'\to X'$, we may assume that the map $\bar{f} : Y\to X$ is finite flat on a neighborhood of x. Then, we have $\bar{f}^!([x]) = [\bar{f}^{-1}(x)]$. By applying Théorème (5.2.2) of [27], we also get $f^*([x]) = [\bar{f}^{-1}(x)]$. □

3.2. *Definition of the intersection product with the log diagonal.* First, we recall the existence of alteration.

Lemma 3.2.1. *Let* $f : V \to U$ *be a finite étale morphism of separated and smooth schemes of finite type purely of dimension* d *over a perfect field* F*. Let* Y *be a separated scheme of finite type over* F *containing* V *as a dense open subscheme.*

Then, there exists a commutative diagram

(3.4)
$$\begin{tikzcd} W \arrow[r, "\subset"] \arrow[d, "g"] & Z \arrow[d, "\bar{g}"] \arrow[ddr, "\bar{h}"] & \\ V \arrow[r, "\subset"] \arrow[d, "f"] & Y & \\ U \arrow[rr, "\subset"] & & X \end{tikzcd}$$

satisfying the following conditions:

(3.4.1) *U is the complement of a Cartier divisor B of X.*

(3.4.2) *Z is smooth purely of dimension d over F and W is the complement of a divisor D of Z with simple normal crossings.*

(3.4.3) *The two quadrangles are Cartesian.*

(3.4.4) *$\bar{g} : Z\to Y$ is proper. The map $g : W\to V$ is a generically finite surjection of constant degree $[W : V]$.*

Proof. By [24], there exists a proper scheme X over F containing U as a dense open subscheme. By replacing X by its blow-up at a closed subscheme whose support is the complement of U, the condition (3.4.1) is satisfied. By replacing Y by the closure of the graph of $f : V \to U$ in $Y \times X$, we may assume there is a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\subset} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ U & \xrightarrow{\subset} & X. \end{array} \tag{3.5}$$

Since V is proper over U and is dense in $U \times_X Y$, the diagram (3.5) is Cartesian. Now, it is sufficient to apply the main result of de Jong [9] to $V \subset Y$ to find $W \subset Z$. □

Next, we study the intersection product with the log diagonal on the level of alteration. We consider a Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{\subset} & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \bar{h}} \\ U & \xrightarrow{\subset} & X \end{array} \tag{3.6}$$

of separated schemes of finite type over F satisfying the conditions:

(3.4.1) U is the complement of a Cartier divisor B of X.

(3.4.2) Z is smooth purely of dimension d over F and W is the complement of a divisor D of Z with simple normal crossings.

Let $D_1, \dots, D_m$ be the irreducible components of D and let $(Z \times Z)^\sim$ be the log product with respect to the divisors $D_1, \dots, D_m$. The scheme $(Z \times Z)^\sim$ is smooth over F and contains $W \times W$ as the complement of a divisor with simple normal crossings by Lemma 1.1.4. The log diagonal map $\Delta_Z : Z \to (Z \times Z)^\sim$ is a regular closed immersion of codimension d. Let $(Z \times_X Z)^\sim = (Z \times Z)^\sim \times_{(X \times X)^\sim} X$ be the relative log product defined with respect to the Cartier divisor B and the family $D_1, \dots, D_m \subset Z$ of Cartier divisors.

Let T be an open neighbourhood of Δ_W in $W \times_U W$. Then the closure $\overline{W \times_U W \setminus T}$ in $(Z \times Z)^\sim$ satisfies $\overline{W \times_U W \setminus T} \cap \Delta_Z \subset Z \setminus W$ since $\overline{W \times_U W \setminus T} \cap W \times W = W \times_U W \setminus T$. Thus the intersection product in $(Z \times Z)^\sim$ defines a map

$$(\ , \Delta_Z)_{(Z \times Z)^\sim} : \mathrm{CH}_d(\overline{W \times_U W \setminus T}) \longrightarrow \mathrm{CH}_0(Z \setminus W). \tag{3.7}$$

Proposition 3.2.2. *Let Z be a smooth scheme purely of dimension d over F and $W \subset Z$ be the complement of a divisor D with simple normal crossings. Let $W \to U$ be a morphisms of schemes of finite type over F and*

$T \subset W \times_U W$ be an open neighborhood of the diagonal Δ_W. Assume there exists a Cartesian diagram (3.6) *satisfying the conditions* (3.4.1) *and* (3.4.2).

1. *Let $\overline{W \times_U W \setminus T}$ be the closure in $(Z \times Z)^\sim$. Then, there exists a unique map*

$$(\ , \Delta_Z)^{\log} : \mathrm{CH}_d(W \times_U W \setminus T) \longrightarrow \mathrm{CH}_0(Z \setminus W) \tag{3.8}$$

making the diagram

$$\begin{tikzcd} \mathrm{CH}_d(\overline{W \times_U W \setminus T}) \arrow[d, "\text{restriction}"'] \arrow[dr, "(\ ,\Delta_Z)_{(Z\times Z)^\sim}"] & \\ \mathrm{CH}_d(W \times_U W \setminus T) \arrow[r, "(\ ,\Delta_Z)^{\log}"'] & \mathrm{CH}_0(Z \setminus W) \end{tikzcd}$$

commutative.

2. *Further, let*

$$\begin{tikzcd} W' \arrow[r, "\subset"] \arrow[d, "k"'] & Z' \arrow[d, "\bar{k}"] \\ W \arrow[r, "\subset"] & Z \end{tikzcd}$$

be a Cartesian diagram of schemes over F. We assume that Z' is smooth over F and that W' is the complement of a divisor of Z' with simple normal crossings. Then, we have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}_d(W \times_U W \setminus T) \arrow[r, "(\ ,\Delta_Z)^{\log}"] \arrow[d, "(k\times k)^!"'] & \mathrm{CH}_0(Z \setminus W) \arrow[d, "\bar{k}^!"] \\ \mathrm{CH}_d(W' \times_U W' \setminus (k\times k)^{-1}(T)) \arrow[r, "(\ ,\Delta_{Z'})^{\log}"] & \mathrm{CH}_0(Z' \setminus W') \end{tikzcd} \tag{3.9}$$

where the left vertical arrow is the Gysin map for $k \times k : W' \times W' \to W \times W$.

Proof. 1. Take a Cartesian diagram (3.6) satisfying the conditions (3.4.1) and (3.4.2). Then $(Z \times_X Z)^\sim$ is a closed subscheme of $(Z \times Z)^\sim$ containing $W \times_U W$ as an open subscheme. Hence, $\overline{W \times_U W \setminus T}$ is closed in $(Z \times_X Z)^\sim \setminus T$ and $W \times_U W \setminus T$ is open in $(Z \times_X Z)^\sim \setminus T$. Thus, it suffices to show that the map $(\ , \Delta_Z)_{(Z\times Z)^\sim} : \mathrm{CH}_d((Z \times_X Z)^\sim \setminus T) \to \mathrm{CH}_0(Z \setminus W)$ factors through the restriction map $\mathrm{CH}_d((Z \times_X Z)^\sim \setminus T) \to \mathrm{CH}_d(W \times_U W \setminus T)$. The kernel of the surjection $\mathrm{CH}_d((Z \times_X Z)^\sim \setminus T) \to \mathrm{CH}_d(W \times_U W \setminus T)$ is generated by the image of $\mathrm{CH}_d((Z \times_X Z)^\sim \setminus W \times_U W)$.

We use the notation in Lemma 1.1.4 replacing $X \to Y$ by $Z \to X$. Then, the complement $(Z \times Z)^\sim \setminus (W \times W)$ is the union of divisors E_i°. Hence the complement

$$(Z \times_X Z)^\sim \setminus (W \times_U W) = (Z \times_X Z)^\sim \cap ((Z \times Z)^\sim \setminus (W \times W))$$

is the union of $(Z \times_X Z)^\sim \cap E_i^\circ$. Thus the kernel of the restriction map $\mathrm{CH}_d((Z \times_X Z)^\sim \setminus T) \to \mathrm{CH}_d(W \times_U W \setminus T)$ is generated by the images of $\mathrm{CH}_d((Z \times_X Z)^\sim \cap E_i^\circ)$.

The pull-back of the Cartier divisor $E_i^\circ \subset (Z \times Z)^\sim$ by the log diagonal map $\Delta_Z \to (Z \times Z)^\sim$ is the Cartier divisor $\Delta_{D_i} \subset \Delta_Z$. Hence we have $(C, \Delta_Z)_{(Z\times Z)^\sim} = (C, \Delta_{D_i})_{E_i^\circ}$ for a cycle C in E_i°. Thus, it is sufficient to show that the map

$$(\ , \Delta_{D_i})_{E_i^\circ} : \mathrm{CH}_d((Z \times_X Z)^\sim \cap E_i^\circ) \longrightarrow \mathrm{CH}_0(D_i) \tag{3.10}$$

is the 0-map.

The log diagonal map $D_i \to (D_i \times D_i)^\sim$ is a regular immersion of codimension $d-1$. The restriction E_{i,D_i}° of the $\mathbb{G}_m$-bundle $E_i^\circ \to (D_i \times D_i)^\sim$ to the log diagonal $D_i \subset (D_i \times D_i)^\sim$ has a canonical isomorphism $E_{i,D_i}^\circ \to \mathbb{G}_{m,D_i}$ (1.7). The immersion $\Delta_{D_i} = \Delta_Z \cap E_i^\circ \to E_i^\circ$ gives the unit section $D_i \to E_{i,D_i}^\circ \to \mathbb{G}_{m,D_i}$. Hence the map (3.10) is the composition of the maps

$$\begin{aligned} \mathrm{CH}_d((Z \times_X Z)^\sim \cap E_i^\circ) &\xrightarrow{(\ , D_i)_{(D_i\times D_i)^\sim}} \mathrm{CH}_1((Z \times_X Z)^\sim \cap E_{i,D_i}^\circ) \\ &\xrightarrow{(\ , D_i)_{E_{i,D_i}^\circ}} \mathrm{CH}_0(D_i). \end{aligned} \tag{3.11}$$

By Proposition 1.1.6.1, the intersection $(Z \times_X Z)^\sim \cap E_{i,D_i}^\circ$ is a closed subscheme of $\mu_{e_i,D_i} \subset E_{i,D_i}^\circ = \mathbb{G}_{m,D_i}$. Hence the second map in (3.11) is the composition

$$\mathrm{CH}_1((Z \times_X Z)^\sim \cap E_{i,D_i}^\circ) \to \mathrm{CH}_1(\mu_{e_i,D_i}) \to \mathrm{CH}_1(\mathbb{G}_{m,D_i}) \xrightarrow{(\ , D_i)_{\mathbb{G}_{m,D_i}}} \mathrm{CH}_0(D_i).$$

Since the composition of the last two maps is the 0-map, the map $(\ , \Delta_Z)_{(Z\times Z)^\sim} : \mathrm{CH}_d((Z \times_X Z)^\sim \setminus T) \to \mathrm{CH}_0(Z \setminus W)$ induces a map $\mathrm{CH}_d(W \times_U W \setminus T) \to \mathrm{CH}_0(Z \setminus W)$. Thus the assertion follows.

2. We consider the commutative diagram

$$\begin{array}{ccccc} (Z' \times Z')^\sim & \longleftarrow & (Z' \times_{X'} Z')^\sim & \longleftarrow & Z' \\ {\scriptstyle (\bar k\times\bar k)^\sim}\downarrow & & {\scriptstyle (\bar k\times\bar k)^\sim}\downarrow & & \downarrow{\scriptstyle \bar k} \\ (Z \times Z)^\sim & \longleftarrow & (Z \times_{X'} Z)^\sim & \longleftarrow & Z \end{array}$$

where the right horizontal arrows are the log diagonal maps. Then, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_d((Z \times_{X'} Z)^\sim \setminus T) & \xrightarrow{(\ , \Delta_Z)_{(Z\times Z)^\sim}} & \mathrm{CH}_0(Z \setminus W) \\ {\scriptstyle (\bar k\times\bar k)^{\sim !}}\downarrow & & \downarrow{\scriptstyle \bar k^!} \\ \mathrm{CH}_d((Z' \times_{X'} Z')^\sim \setminus (k \times k)^{-1}(T)) & \xrightarrow{(\ , \Delta_{Z'})_{(Z'\times Z')^\sim}} & \mathrm{CH}_0(Z' \setminus W') \end{array}$$

and the assertion follows. $\square$

THEOREM 3.2.3. *Let $f : V \to U$ be a finite étale morphism of separated and smooth schemes of finite type purely of dimension d over a perfect field F.*

1. *There exists a unique map*

$$(0.1)\qquad (\ ,\Delta_{\overline{V}})^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \longrightarrow \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

that makes the diagram

(3.12)

$$\begin{tikzcd} \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \arrow[r, "(\ ,\Delta_{\overline{V}})^{\log}"] \arrow[d, "(g\times g)^!"'] & \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] & \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \mathrm{CH}_d(W \times_U W \setminus W \times_V W) \arrow[r, "(\ ,\Delta_Z)^{\log}"] & \mathrm{CH}_0(Z \setminus W) \arrow[ur, "\frac{1}{[W:V]}\bar{g}_*"'] & \end{tikzcd}$$

commutative for an arbitrary commutative diagram (3.4) *satisfying the condition*

(3.4.0) *Y contains V as a dense open subscheme.*

and the conditions (3.4.1)–(3.4.4).

2. *Assume the full subcategory $\mathcal{C}_V^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_V$. Then, there exists a unique map*

$$(3.13)\qquad (\ ,\Delta_{\overline{V}})^{\log}_{\mathbb{Z}} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \longrightarrow \mathrm{CH}_0(\overline{V} \setminus V)$$

satisfying the following property.

Let Y be an arbitrary smooth separated scheme of finite type containing V as the complement of a divisor with simple normal crossings and let $(\ ,\Delta_Y)^{\log}$: $\mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(Y \setminus V)$ be the map (3.8) *for $Z = Y$. Then the diagram*

$$(3.14)\qquad \begin{tikzcd} \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \arrow[r, "(\ ,\Delta_{\overline{V}})^{\log}_{\mathbb{Z}}"] \arrow[dr, "(\ ,\Delta_Y)^{\log}"'] & \mathrm{CH}_0(\overline{V} \setminus V) \arrow[d] \\ & \mathrm{CH}_0(Y \setminus V) \end{tikzcd}$$

is commutative if there exists a Cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ U & \longrightarrow & X \end{array}$$

of separated scheme of finite type where X contains U as the complement of a Cartier divisor.

Proof. 1. We consider an arbitrary commutative diagram (3.4) satisfying the conditions (3.4.0)–(3.4.4). By the assumption that $V \to U$ is étale, the fiber product $T = W \times_V W$ is an open neighborhood of Δ_W in $W \times_U W$ and the map $(\ ,\Delta_Z)^{\log} : \mathrm{CH}_d(W \times_U W \setminus W \times_V W) \to \mathrm{CH}_0(Z \setminus W)$ is defined by Proposition 3.2.2.1.

For an object Y of $\mathcal{C}_V$, there exists a commutative diagram (3.4) satisfying the conditions (3.4.1)-(3.4.4) by Lemma 3.2.1. The composition $\mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ via the lower line in (3.12) is independent of the choice of diagram (3.4) by Proposition 3.2.2.2. We define the map $(\ ,\Delta_{\overline{V}})^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ as the limit. Then it is clear that the map $(\ ,\Delta_{\overline{V}})^{\log}$ satisfies the condition.

2. By the assumption and Corollary 3.1.6.1, the group $\mathrm{CH}_0(\overline{V} \setminus V)$ is identified with the inductive limit $\varinjlim_{\mathcal{C}_V^{\mathrm{sm},0,\mathrm{opp}}} \mathrm{CH}_0(Y \setminus V)$ with respect to the Gysin maps. Hence it follows from Proposition 3.2.2. □

If $\mathcal{C}_V^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_V$, the map $(\ ,\Delta_{\overline{V}})^{\log}$ is induced by $(\ ,\Delta_{\overline{V}})^{\log}_{\mathbb{Z}}$.

We prove the trace formula (0.2) in Proposition 3.2.4. Let $V \to U$ be a finite étale morphism of separated smooth schemes of dimension d over F. Let ℓ be a prime number invertible in F and $\bar{F}$ be an algebraic closure of F. For an open and closed subscheme Γ of $V \times_U V \setminus \Delta_V$, we define an endomorphism Γ^* of $H^q_c(V_{\bar{F}}, \mathbb{Q}_\ell)$ to be $p_{1*} \circ p_2^*$. We put

$$\mathrm{Tr}(\Gamma^* : H^*_c(V_{\bar{F}}, \mathbb{Q}_\ell)) = \sum_{q=0}^{2d} (-1)^q \mathrm{Tr}(\Gamma^* : H^q_c(V_{\bar{F}}, \mathbb{Q}_\ell)).$$

Proposition 3.2.4. *Let $f : V \to U$ be a finite étale morphism of separated and smooth schemes of finite type purely of dimension d over a perfect field F. Let ℓ be a prime number invertible in F.*

Then, for an open and closed subscheme Γ of $V \times_U V \setminus \Delta_V$, we have

$$\mathrm{Tr}(\Gamma^* : H^*_c(V_{\bar{F}}, \mathbb{Q}_\ell)) = \deg(\Gamma, \Delta_{\overline{V}})^{\log}. \tag{3.15}$$

Proof. Take a diagram (3.4) with X, Y and Z proper over F satisfying the conditions (3.4.0)-(3.4.4). By Lemma 2.3.3, we have

$$\mathrm{Tr}(\Gamma^* : H^*_c(V_{\bar{F}}, \mathbb{Q}_\ell)) = \frac{1}{[W:V]} \mathrm{Tr}(((g \times g)^*\Gamma)^* : H^*_c(W_{\bar{F}}, \mathbb{Q}_\ell)).$$

By Lemma 2.1.2, we have $(g \times g)^*[\Gamma] = [(g \times g)^!(\Gamma)]$. Take an element $\tilde{\Gamma} = \sum_i n_i [C_i] \in Z_d(W \times_U W \setminus W \times_V W)$ representing $[(g \times g)^!(\Gamma)] \in \mathrm{CH}_d(W \times_U W \setminus W \times_V W)$. By Proposition 1.1.6.2, the closures $\overline{C}_i \subset (Z \times Z)'$ satisfy the condition (2.8). Hence by Theorem 2.3.4, we have

$$\begin{aligned}\mathrm{Tr}(((g \times g)^*\Gamma)^* : H^*_c(W_{\bar{F}}, \mathbb{Q}_\ell)) &= \deg\ (\textstyle\sum_i n_i [\overline{C}_i], \Delta_Z)_{(Z \times Z)'} \\ &= \deg\ ((g \times g)^! \Gamma, \Delta_Z)^{\log}.\end{aligned}$$

By the definition of $(\Gamma, \Delta_{\overline{V}})^{\log}$, we have

$$\deg\,(\Gamma, \Delta_{\overline{V}})^{\log} = \frac{1}{[W:V]} \cdot \deg\,((g \times g)^! \Gamma, \Delta_Z)^{\log}.$$

Thus the equality (3.15) is proved. □

3.3. *Properties of the intersection product with the log diagonal.* We keep the notation that $f : U \to V$ denotes a finite étale morphism of separated smooth schemes of finite type purely of dimension d over a perfect field F.

The maps $(\ ,\Delta_{\overline{V}})^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfies the following functoriality.

Lemma 3.3.1. *Let U be a separated smooth scheme of finite type purely of dimension d over a perfect field F.*

1. *Let $V \to U'$ be a morphism of finite and étale schemes over U. Then the map $(\ ,\Delta_{\overline{V}})^{\log} : \mathrm{CH}_d(V \times_{U'} V \setminus \Delta_V) \to \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to the restriction of $(\ ,\Delta_{\overline{V}})^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

2. *Let $g : V \to V'$ be a morphism of finite and étale schemes over U. Then, the diagram*

$$\begin{array}{ccc} \mathrm{CH}_d(V' \times_U V' \setminus \Delta_{V'}) & \xrightarrow{(\ ,\Delta_{\overline{V'}})^{\log}} & \mathrm{CH}_0(\overline{V'} \setminus V') \otimes_{\mathbb{Z}} \mathbb{Q} \\ {\scriptstyle (g\times g)^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathrm{CH}_d(V \times_U V \setminus \Delta_V) & \xrightarrow{(\ ,\Delta_{\overline{V}})^{\log}} & \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \end{array} \tag{3.16}$$

is commutative.

Proof. 1. Clear from the definition and Proposition 3.2.2.2.

2. We may assume U, V and V' are connected. Then by Corollary 3.1.4.1, the right vertical arrow g^* in (3.16) is injective. Hence, we may replace V by its Galois closure over U and may assume $V \to U$ is a Galois covering. Let G be the Galois group and $H \subset G$ be the subgroup corresponding to V'. Then, the images of the both compositions are in the H-fixed part of $\mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence, by Corollary 3.1.4.2, it suffices to show the diagram

$$\begin{array}{ccc} \mathrm{CH}_d(V' \times_U V' \setminus \Delta_{V'}) & \xrightarrow{(\ ,\Delta_{\overline{V'}})^{\log}} & \mathrm{CH}_0(\overline{V'} \setminus V') \otimes_{\mathbb{Z}} \mathbb{Q} \\ {\scriptstyle (g\times g)^*}\downarrow & & \uparrow{\scriptstyle \frac{1}{|H|}g_*} \\ \mathrm{CH}_d(V \times_U V \setminus \Delta_V) & \xrightarrow{(\ ,\Delta_{\overline{V}})^{\log}} & \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \end{array}$$

is commutative. This is clear from the definition and Proposition 3.2.2.2. □

If the subcategory $\mathcal{C}_V^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_V$, we can remove $\otimes_{\mathbb{Z}}\mathbb{Q}$ in 1. Further if $\mathcal{C}_{V'}^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_{V'}$, we can remove $\otimes_{\mathbb{Z}}\mathbb{Q}$ in 2. We will omit to state remarks on integrality of this type in the sequel.

Lemma 3.3.2. *Let $V \to U' \to U$ be finite étale morphisms of separated and smooth schemes of finite type purely of dimension d over a perfect field F. Let $n \geq 1$ be an integer invertible in F and assume $g : U' \to U$ is a $\mathbb{Z}/n\mathbb{Z}$-torsor over U.*

Then, the restriction

$$\mathrm{CH}_d(V \times_U V \setminus V \times_{U'} V) \subset \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \xrightarrow{(\ ,\Delta_{\overline{V}})^{\log}} \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is the 0*-map.*

Proof. By enlarging F, we may assume F contains a primitive n-th root of 1. Let $\chi : \mathbb{Z}/n\mathbb{Z} \to F^\times$ be a character of order n. Then the χ-part $\mathcal{L}_U$ of $g_* O_{U'}$ is an invertible O_U-module. The multiplication defines an isomorphism $\mu_U : \mathcal{L}_U^{\otimes n} \to O_U$ of O_U-modules. The O_U-algebra $g_* O_{U'}$ is isomorphic to $\bigoplus_{i=0}^{n-1} \mathcal{L}_U^{\otimes i}$ with the multiplication defined by μ_U. We take a proper scheme X over F containing U as a dense open subscheme. Replacing X by a blow-up, we may assume $\mathcal{L}_U$ is extended to an invertible O_X-module $\mathcal{L}$ and the map $\mu_U : \mathcal{L}_U^{\otimes n} \to O_U$ is extended to an injection $\mu : \mathcal{L}^{\otimes n} \to O_X$. We define a finite flat scheme $\bar{g} : X' \to X$ over X by the O_X-algebra $\bigoplus_{i=0}^{n-1} \mathcal{L}^{\otimes i}$ with the multiplication defined by μ. By Lemma 1.1.3, the diagonal $X' \to (X' \times_X X')^\sim$ is an open immersion.

We take a proper scheme Y containing V as a dense open subscheme such that the map $V \to U'$ is extended to $Y \to X'$ and an alteration $Z \to Y$ as in Lemma 3.2.1. Then, the inverse image of $V \times_U V \setminus V \times_{U'} V$ in $(Z \times Z)^\sim$ is contained in the inverse image of $(X' \times_X X')^\sim \setminus X'$. Thus the assertion follows from the definition of the map (0.1). □

For a separated scheme Y of finite type over F containing V as a dense open subscheme, let

$$(\ ,\Delta_Y)^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \longrightarrow \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{3.17}$$

denote the composition of the maps in the upper line of the diagram (3.12). The map $(\ ,\Delta_Y)^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is characterized by the commutativity of the diagram

$$\begin{tikzcd} \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \arrow[r, "(\ ,\Delta_Y)^{\log}"] \arrow[d, "(g\times g)^!"'] & \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \mathrm{CH}_d(W \times_U W \setminus W \times_V W) \arrow[r, "(\ ,\Delta_Z)^{\log}"] & \mathrm{CH}_0(Z \setminus W) \arrow[u, "\frac{1}{[W:V]}\bar{g}_*"'] \end{tikzcd} \tag{3.18}$$

for an arbitrary commutative diagram (3.4) satisfying the conditions (3.4.1)-(3.4.4). If Y is smooth and V is the complement of a divisor with simple normal crossings, the map $(\ ,\Delta_Y)^{\log}$ is given by the map (3.8) for $Z = Y$.

We give sufficient conditions for the vanishing of the map $(\ ,\Delta_Y)^{\log}$: $\mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$.

LEMMA 3.3.3. *Let*

$$\begin{array}{ccc} V & \xrightarrow{\subset} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f} \\ U & \xrightarrow{\subset} & X \end{array} \tag{3.5}$$

be a Cartesian diagram of separated schemes of finite type over a perfect field F. We assume $U \subset X$ and $V \subset Y$ are dense open subschemes, U is smooth purely of dimension d over F and $f : V \to U$ is finite and étale.

Let $\Gamma \subset V \times_U V$ be an open and closed subscheme. If the intersection $\overline{\Gamma} \cap \Delta_Y$ of the closure $\overline{\Gamma} \subset Y \times_X Y$ of Γ and the diagonal $Y \subset Y \times_X Y$ is empty, we have $(\Gamma, \Delta_Y)^{\log} = 0$ in $\mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. By replacing X by a blow-up, we may assume $U \subset X$ is the complement of a Cartier divisor. We take a Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{\subset} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar g} \\ V & \xrightarrow{\subset} & Y \end{array}$$

satisfying the conditions (3.4.2) and (3.4.4). We consider the natural map $\bar g \times \bar g : (Z \times_X Z)^{\sim} \to Y \times_X Y$ induced by $\bar g : Z \to Y$. The closure of $(g \times g)^{-1}(\Gamma)$ is in $(\bar g \times \bar g)^{-1}(\overline{\Gamma})$ and does not meet the log diagonal $\Delta_Z \subset (\bar g \times \bar g)^{-1}(\Delta_Y)$ by the assumption. Hence we have $((g \times g)^!(\Gamma), \Delta_Z)_{(Z\times Z)^{\sim}} = 0$ and the assertion follows. $\square$

COROLLARY 3.3.4. *Let the notation be as in Lemma* 3.3.3.

1. *If $\bar f : Y \to X$ is étale, the map $(\ ,\Delta_Y)^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the 0-map.*

2. *Let $\bar\sigma$ be an automorphism of Y over X and σ be the restriction on V. Let $\Gamma_\sigma \subset V \times_U V$ and $\Gamma_\sigma \subset Y \times_X Y$ be the graphs. If $Y^\sigma = \Gamma_\sigma \cap \Delta_Y$ is empty, we have $(\Gamma_\sigma, \Delta_Y)^{\log} = 0$ in $\mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. Clear from Lemma 3.3.3. $\square$

We show that the image of the map $(\ ,\Delta_Y)^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is supported on the wild ramification locus.

PROPOSITION 3.3.5. *Let*

$$\begin{array}{ccc} V & \xrightarrow{\subset} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f} \\ U & \xrightarrow{\subset} & X \end{array} \tag{3.5}$$

be a Cartesian diagram of separated schemes of finite type over a perfect field F. We assume X is smooth purely of dimension d over F, U is the complement of a divisor B with simple normal crossings, $V \subset Y$ is a dense open subscheme, and $f : V \to U$ is finite and étale.

1. *Let $V \subset V' \subset Y$ be an open normal subscheme. If V' is tamely ramified over X, then the map $(\ ,\Delta_Y)^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is decomposed as the composition*

$$\mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(Y \setminus V') \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

2. *Suppose there exists a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{\subset} & Y \\ \downarrow & & \downarrow \\ U' & \xrightarrow{\subset} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{g}} \\ U & \xrightarrow{\subset} & X \end{array}$$

of separated normal schemes of finite type over F, $g : U' \to U$ is finite étale and $\bar{g} : X' \to X$ is tamely ramified. Then, the restriction

$$\mathrm{CH}_d(V \times_U V \setminus V \times_{U'} V) \subset \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \xrightarrow{(\ ,\Delta_Y)^{\log}} \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is the 0-map.

Proof. It follows from the characterization of the map (3.17) and Lemma 1.1.8. □

3.4. *Wild differents and log Lefschetz classes.*

Definition 3.4.1. Let $f : V \to U$ be a finite étale morphism of separated and smooth schemes of finite type purely of dimension d over a perfect field F.

1. We call the 0-cycle class

$$D^{\log}_{V/U} = (V \times_U V \setminus \Delta_V, \Delta_{\overline{V}})^{\log} \in \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{3.19}$$

the wild different of V over U.

2. Let σ be an automorphism of V over U that is not the identity on any component of V. Let $\Gamma_\sigma \subset V \times_U V$ be the graph of σ. Then, we call the 0-cycle class

$$(\Gamma_\sigma, \Delta_{\overline{V}})^{\log} \in \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{3.20}$$

the log Lefschetz class of σ.

If the subcategory $\mathcal{C}^{\mathrm{sm},0}_V$ is cofinal in $\mathcal{C}_V$, the wild different $D^{\log}_{V/U}$ and the log Lefschetz class $(\Gamma_\sigma, \Delta_{\overline{V}})^{\log}$ are defined in $\mathrm{CH}_0(\overline{V} \setminus V)$.

LEMMA 3.4.2. *For a morphism $g : V \to V'$ of finite and étale schemes over U, we have*

$$D^{\log}_{V/U} = D^{\log}_{V/V'} + g^* D^{\log}_{V'/U}. \tag{3.21}$$

Proof. We have $V \times_U V \setminus \Delta_V = (V \times_{V'} V \setminus \Delta_V) \amalg (g \times g)^{-1}(V' \times_U V' \setminus \Delta_{V'})$. Hence, the equalities follow from Lemma 3.3.1.2. □

PROPOSITION 3.4.3. *Let $f : V \to U$ be a finite and étale morphism of connected separated and smooth schemes of finite type purely of dimension d over a perfect field F and let σ be an automorphism of V over U.*

If the order of σ is not a power of the characteristic p of F, we have

$$(\Gamma_\sigma, \Delta_{\overline{V}})^{\log} = 0.$$

Proof. Let n be the prime-to-p part of the order e of σ. By Lemma 3.3.1.1, we may replace U by the quotient $V/\langle\sigma\rangle$. Then it suffices to apply Lemma 3.3.2 to $V \to U' = V/\langle\sigma^n\rangle \to U = V/\langle\sigma\rangle$. □

We expect the following holds.

CONJECTURE 3.4.4. *Let $f : V \to U$ be a finite and étale morphism of connected separated and smooth schemes of finite type purely of dimension d over a perfect field F and let σ be a nontrivial automorphism of V over U.*

If j is an integer prime to the order of σ, we have

$$(\Gamma_\sigma, \Delta_{\overline{V}})^{\log} = (\Gamma_{\sigma^j}, \Delta_{\overline{V}})^{\log}.$$

We will prove Conjecture 3.4.4 assuming $\dim \leq 2$ in Lemma 3.4.13.

LEMMA 3.4.5. *Let the notation be as in Definition* 3.4.1 *and let ℓ be a prime number invertible in a perfect field F.*

1. *If $f : V \to U$ is of constant degree $[V : U]$, we have*

$$\deg\ D^{\log}_{V/U} = [V : U]\chi_c(U_{\bar F}, \mathbb{Q}_\ell) - \chi_c(V_{\bar F}, \mathbb{Q}_\ell) \tag{3.22}$$

2. *Let σ be an automorphism of V over U that is not the identity on any component of V. Then, we have*

$$\deg\ (\Gamma_\sigma, \Delta_{\overline{V}})^{\log} = \mathrm{Tr}(\sigma^* : H^*_c(V_{\bar F}, \mathbb{Q}_\ell)). \tag{3.23}$$

Proof. 1. By Proposition 3.2.4, we have

$$\deg\ D^{\log}_{V/U} = \mathrm{Tr}((V \times_U V)^* : H^*_c(V_{\bar F}, \mathbb{Q}_\ell)) - \mathrm{Tr}(\Delta_V^* : H^*_c(V_{\bar F}, \mathbb{Q}_\ell)).$$

By Lemma 2.3.3, we have $\mathrm{Tr}((V \times_U V)^* : H^*_c(V_{\bar F}, \mathbb{Q}_\ell)) = [V : U] \cdot \chi_c(U_{\bar F}, \mathbb{Q}_\ell)$. Hence the assertion follows.

2. It suffices to apply Proposition 3.2.4 to the graph Γ_σ. □

Corollary 3.4.6 ([15, Lemma 2.5]). *Let U be a separated scheme of finite type over a field F and $V \to U$ be an étale $\mathbb{Z}/n\mathbb{Z}$-torsor. Let σ be the automorphism defined by the generator $1 \in \mathbb{Z}/n\mathbb{Z}$ and assume n is not a power of p. Then, we have*

$$\mathrm{Tr}(\sigma^* : H_c^*(V_{\bar{F}}, \mathbb{Q}_\ell)) = 0.$$

Proof. We may assume F is perfect. If the assertion holds for the base changes to a closed subscheme $Z \subset U$ and to the complement $U \setminus Z$, it holds for U. Hence, by induction on dimension, it is reduced to the case where U is smooth. Then it follows from Lemma 3.4.5.2 and Proposition 3.4.3. □

In the rest of this subsection, we give some computations of wild differents and log Lefschetz classes.

In the classical case where U is a smooth curve over F, Definition 3.4.1 gives the classical invariants of wild ramifications as follows. Let A be a complete discrete valuation ring and B be the integral closure of A in a finite separable extension L of the fraction field K. Let $e_{L/K}$ be the ramification index of L over K. Then the wild different $D^{\log}_{B/A} \in \mathbb{N}$ is defined by

$$D^{\log}_{B/A} = \mathrm{length}_B \Omega^1_{B/A} - (e_{L/K} - 1).$$

For a nontrivial automorphism σ of L over K, we put

$$j_B(\sigma) = \mathrm{length}_B B / \left(\frac{\sigma(b)}{b} - 1; b \in B \setminus \{0\} \right).$$

Lemma 3.4.7. *Let U be a smooth connected curve over a perfect field F and $f : V \to U$ be a finite étale morphism over F. Let X be the proper smooth curve containing U as a dense open subscheme and $\bar{f} : Y \to X$ be the normalization in V. We put $B = X \setminus U$ and $D = Y \setminus V$ and identify $\mathrm{CH}_0(\overline{V} \setminus V) = \bigoplus_{y \in D} \mathbb{Z}$.*

1. *We have*

$$D^{\log}_{V/U} = [\mathrm{Coker}(\bar{f}^* \Omega^1_{X/F}(\log B) \to \Omega^1_{Y/F}(\log D))] = \sum_{y \in D} D^{\log}_{\hat{O}_{Y,y}/\hat{O}_{X,\bar{f}(y)}} \cdot [y].$$

2. *Let σ be a nontrivial automorphism of V over U. Then, we have*

$$(\Gamma_\sigma, \Delta_{\overline{V}})^{\log}_{\mathbb{Z}} = \sum_{y \in D, \sigma(y) = y} j_{\hat{O}_{Y,y}}(\sigma) \cdot [y].$$

Proof. Follows from Proposition 3.4.10 and Lemma 3.4.11 below. □

We compute the wild different assuming a strong form of resolution. Before doing it, we recall some general facts on intersection theory and localized Chern classes.

Let X be a scheme of finite type over F and $Z \subset X$ be a closed subscheme. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free O_X-modules of rank d and $f : \mathcal{E} \to \mathcal{F}$ be an O_X-linear map. We assume that $f : \mathcal{E} \to \mathcal{F}$ is an isomorphism on $X \setminus Z$. We consider the complex $\mathcal{K} = [\mathcal{E} \to \mathcal{F}]$ of O_X-modules by putting $\mathcal{F}$ on degree 0. Then, the localized Chern class $c_Z^X(\mathcal{K}) - 1$ is defined as an element of $\mathrm{CH}^*(Z \to X)$ in [12, Ch. 18.1]. We define an element $c(\mathcal{F} - \mathcal{E})_Z^X = (c_i(\mathcal{F} - \mathcal{E})_Z^X)_{i>0}$ of $\mathrm{CH}^*(Z \to X)$ by

$$c(\mathcal{F} - \mathcal{E})_Z^X = c(\mathcal{E}) \cap (c_Z^X(\mathcal{K}) - 1). \tag{3.24}$$

In other words, we put $c_i(\mathcal{F} - \mathcal{E})_Z^X = \sum_{j=0}^{\min(d,i-1)} c_j(\mathcal{E}) \cap c_{i-j}{}_Z^X(\mathcal{K})$ for $i > 0$. The image of $c(\mathcal{F} - \mathcal{E})_Z^X$ in $\mathrm{CH}^*(X)$ is the difference $c(\mathcal{F}) - c(\mathcal{E})$ of Chern classes.

LEMMA 3.4.8. *Let X be a scheme of finite type over F and $Z \subset X$ be a closed subscheme. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free O_X-modules of rank d and $f : \mathcal{E} \to \mathcal{F}$ be an O_X-linear map such that $f : \mathcal{E} \to \mathcal{F}$ is an isomorphism on $X \setminus Z$. Then,*

1. *We have $c_i(\mathcal{F} - \mathcal{E})_Z^X = 0$ for $i > d$.*
2. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of exact sequences of locally free O_X-modules. We assume that the maps f' and f'' are isomorphism on $X \setminus Z$. We assume $\mathcal{E}'$ and $\mathcal{F}'$ are of rank d' and $\mathcal{E}''$ and $\mathcal{F}''$ are of rank d'' for some integers $d' + d'' = d$. Then, we have

$$c(\mathcal{F} - \mathcal{E})_Z^X = c(\mathcal{F}' - \mathcal{E}')_Z^X \cap c(\mathcal{F}'') + c(\mathcal{F}'' - \mathcal{E}'')_Z^X \cap c(\mathcal{E}'). \tag{3.25}$$

Proof. 1. The localized Chern classes $c_i{}_Z^X(\mathcal{F})$ and $c_i{}_Z^X(\mathcal{E})$ are defined for $i > d$ in [4, §1]. Further they are equal to 0 since $\mathcal{F}$ and $\mathcal{E}$ are locally free of rank d. Hence, by the distinguished triangle $\to \mathcal{E} \to \mathcal{F} \to \mathcal{K} \to$, we have an equality $0 = c_i{}_Z^X(\mathcal{F}) = \sum_{j=0}^{d} c_j(\mathcal{E}) \cap c_{i-j}{}_Z^X(\mathcal{K})$ as in Proposition 1.1 (iii) *loc. cit*. Since the right-hand side is $c_i(\mathcal{F} - \mathcal{E})_Z^X$, the assertion follows.

2. We put $\mathcal{K}' = [\mathcal{E}' \to \mathcal{F}']$ and $\mathcal{K}'' = [\mathcal{E}'' \to \mathcal{F}'']$ as above. Then, by the assumption, we have $c_i{}_Z^X(\mathcal{K}) = \sum_{j=0}^{i-1} c_j(\mathcal{K}'') \cap c_{i-j}{}_Z^X(\mathcal{K}') + c_i{}_Z^X(\mathcal{K}'')$ for $i > 0$ (cf. [12, Example 18.1.3, Proposition 18.1 (b)] and [4, Proposition 1.1 (iii)]). In other words, we have $c_Z^X(\mathcal{K}) - 1 = (c_Z^X(\mathcal{K}') - 1) \cap c(\mathcal{K}'') + (c_Z^X(\mathcal{K}'') - 1)$. Multiplying $c(\mathcal{E}) = c(\mathcal{E}') \cap c(\mathcal{E}'')$ and substituting $c(\mathcal{E}'') \cap c(\mathcal{K}'') = c(\mathcal{F}'')$, we obtain $c(\mathcal{E}) \cap (c_Z^X(\mathcal{K}) - 1) = c(\mathcal{E}') \cap (c_Z^X(\mathcal{K}') - 1) \cap c(\mathcal{F}'') + c(\mathcal{E}') \cap c(\mathcal{E}'') \cap (c_Z^X(\mathcal{K}'') - 1)$ and the assertion follows. $\square$

LEMMA 3.4.9. *Let*

$$\begin{array}{ccc} W & \longrightarrow & Y \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ V & \longrightarrow & X \end{array}$$

be a commutative diagram of separated schemes of finite type over F. We assume that Y is purely of dimension n and the horizontal arrows $V \to X$ and $W \to Y$ are regular closed immersions of codimension d. Let $N_{V/X}$ and $N_{W/Y}$ be the conormal sheaves.

Let U be a dense open subscheme of Y. We assume that $W \cap U$ is dense in W and that the closed immersion $W \cap U \to V \times_X U$ is an open immersion. We put $Z = W \setminus (W \cap U)$ and $Z' = (V \times_X Y) \setminus (W \cap U)$. Then, we have the following.

1. *The canonical map $g^*N_{V/X} \to N_{W/Y}$ is an isomorphism on $W \cap U = W \setminus Z$ and $c(g^*N_{V/X} - N_{W/Y})^W_Z \in \mathrm{CH}^*(Z \to W)$ is defined.*

2. *The canonical map $Z_{n-d}(W) \oplus \mathrm{CH}_{n-d}(Z') \to \mathrm{CH}_{n-d}(V \times_X Y)$ is an isomorphism. The projection $\mathrm{CH}_{n-d}(V \times_X Y) \to Z_{n-d}(W)$ is given by the restriction map $\mathrm{CH}_{n-d}(V \times_X Y) \to \mathrm{CH}_{n-d}(W \cap U) = Z_{n-d}(W \cap U) \simeq Z_{n-d}(W)$.*

3. *There exists a unique element $[f^!V - W] \in \mathrm{CH}_{n-d}(Z')$ satisfying $[W] + [f^!V - W] = [f^!V]$ in $\mathrm{CH}_{n-d}(V \times_X Y)$. Further, we have an equality*

$$(W, [f^!V - W])_Y = (-1)^{d-1} c_d(N_{W/Y} - g^*N_{V/X})^W_Z \cap [W] \tag{3.26}$$

in $\mathrm{CH}_{n-2d}(Z)$.

Proof. 1. By the assumption that the closed immersion $W \cap U \to V \times_X U$ is an open immersion, the canonical map $g^*N_{V/X} \to N_{W/Y}$ is an isomorphism on $W \cap U = W \setminus Z$. Hence $c(g^*N_{V/X} - N_{W/Y})^W_Z \in \mathrm{CH}^*(Z \to W)$ is defined.

2. By the assumption, the canonical maps $Z_{n-d}(W) \oplus Z_{n-d}(Z') \to Z_{n-d}(V \times_X Y)$ and $Z_{n-d}(W) \to \mathrm{CH}_{n-d}(W)$ are isomorphisms. Thus the assertion follows.

3. By the assumption, the restriction of $[f^!V]$ to the open subscheme $W \cap U \subset V \times_X Y$ is $[W \cap U]$. Hence, by 2, there exists a unique element $[f^!V - W] \in \mathrm{CH}_{n-d}(Z')$ satisfying $[W] + [f^!V - W] = [f^!V]$.

Let $p : Y' \to Y$ be the blow-up at $V \times_X Y \subset Y$ and at $W \subset Y$. Let $D = V \times_X Y'$ and $D' = W \times_Y Y'$ be the exceptional divisors. We compute $(W, [f^!V - W])_Y$ using $p : Y' \to Y$. Let $h : D \to V$ and $h' : D' \to W$ be the canonical maps and let $N_1 = \mathrm{Ker}(h^*N_{V/X} \to N_{D/Y'})$ and $N_1' = \mathrm{Ker}(h'^*N_{W/Y} \to N_{D'/Y'})$ be the excess conormal sheaves. The O_D-module N_1 and the $O_{D'}$-module N_1' are locally free of rank $d-1$. By the excess intersection formula, we have $f^!V = p_*(V, Y')_X = (-1)^{d-1} p_* c_{d-1}(N_1) \cap [D]$ and $W = p_*(W, Y')_Y = (-1)^{d-1} p_* c_{d-1}(N_1') \cap [D']$.

Let $i : D' \to D$ be the immersion. Since $[f^!V - W] \in \mathrm{CH}_{n-d}(Z')$ is characterized by the property that $[f^!V - W] + [W] = [f^!V]$ in $\mathrm{CH}_{n-d}(V \times_X Y)$, we obtain

$$[f^!V - W] = (-1)^{d-1} p_* \Big(c_{d-1}(N_1) \cap ([D] - [D']) + c_{d-1}(N_1' - i^*N_1)^{D'}_{Z_{D'}} \cap [D'] \Big)$$

in $\mathrm{CH}_{n-d}(Z)$. Further by the excess intersection formula, we have

$$(W, [f^!V - W])_Y = p_* \Big(c_{d-1}(N_1') \cap [D'] \cap \Big(c_{d-1}(N_1) \cap ([D] - [D']) + c_{d-1}(N_1' - i^*N_1)^{D'}_{Z_{D'}} \cap [D'] \Big) \Big) \tag{3.27}$$

in $\mathrm{CH}_{n-2d}(Z)$.

Since

$$[D'] \cdot ([D] - [D']) = ([D] - [D']) \cdot [D'] = c_1(N_{D'/Y'} - i^*N_{D/Y'})^{D'}_{Z_{D'}} \cap [D'],$$

the right-hand side of (3.27) is equal to

$$p_* \Big(\big(c_{d-1}(N_1) \cap c_1(N_{D'/Y'} - i^*N_{D/Y'})^{D'}_{Z_{D'}} + c_{d-1}(N_1' - i^*N_1)^{D'}_{Z_{D'}} \cap c_1(N_{D'/Y'}) \big) \cap c_{d-1}(N_1') \cap [D'] \Big).$$

By the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i^*N_1 & \longrightarrow & i^*h^*N_{V/X} & \longrightarrow & i^*N_{D/Y'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_1' & \longrightarrow & h'^*N_{W/Y} & \longrightarrow & N_{D'/Y'} & \longrightarrow & 0, \end{array}$$

and by Lemma 3.4.8.2, it is further equal to

$$p_*(c_d(h'^*N_{W/Y} - i^*h^*N_{V/X})^{D'}_{Z_{D'}} \cap c_{d-1}(N_1') \cap [D']) = c_d(N_{W/Y} - g^*N_{V/X})^W_Z \cap p_*(c_{d-1}(N_1') \cap [D']).$$

Since $(-1)^{d-1} p_*(c_{d-1}(N_1') \cap [D']) = [W]$, the assertion follows. □

Let $f : V \to U$ be a finite étale morphism of smooth separated schemes of finite type over F and Y be a separated smooth scheme of finite type containing V as the complement of a divisor with simple normal crossings. We put $D^{\log}_{V/U,Y} = (V \times_U V \setminus \Delta_V, \Delta_Y)^{\log}_{\mathbb{Z}} \in \mathrm{CH}_0(Y \setminus V)$. Its image in $\mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the same as the image of $D^{\log}_{V/U}$.

Proposition 3.4.10. *Let*

$$\begin{array}{ccc} V & \xrightarrow{\subset} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f} \\ U & \xrightarrow{\subset} & X \end{array} \tag{3.5}$$

be a Cartesian diagram of separated schemes of finite type over F. We assume X and Y smooth purely of dimension d over F, $U \subset X$ and $V \subset Y$ are the complements of divisors B and D with simple normal crossings respectively and $f : V \to U$ is finite and étale.

Then, the canonical map $f^*\Omega^1_{X/F}(\log B) \to \Omega^1_{Y/F}(\log D)$ is an isomorphism on $V = Y \setminus D$ and we have

$$D^{\log}_{V/U,Y} = (-1)^{d-1} c_d \left(\Omega^1_{Y/F}(\log D) - f^*\Omega^1_{X/F}(\log B)\right)^Y_D \cap [Y]. \tag{3.28}$$

Proof. We consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & (Y \times Y)^\sim \arrow[d, "(f\times f)^\sim"] \\ X \arrow[r] & (X \times X)^\sim. \end{tikzcd}$$

As in [20, Cor. 4.2.8], the conormal sheaves $N_{X/(X\times X)^\sim}$ and $N_{Y/(Y\times Y)^\sim}$ are naturally identified with $\Omega^1_{X/F}(\log B)$ and $\Omega^1_{Y/F}(\log D)$ respectively. Hence, it is sufficient to apply Lemma 3.4.9 to the diagram by taking $V \times V \subset (Y \times Y)^\sim$ as the open subscheme $U \subset Y$ in Lemma 3.4.9. □

We compute the log Lefschetz class assuming an equivariant resolution. For a closed immersion $Z \to Y$, let $s(Z/Y) \in \bigoplus_i \mathrm{CH}_i(Y)$ be the Segre class. For a locally free O_Y-module $\mathcal{E}$, let $c(\mathcal{E})^* = c(\mathcal{E}^*) = \sum_i (-1)^i c_i(\mathcal{E}) \in \bigoplus_i \mathrm{CH}^i(Y \to Y)$ be the bivariant Chern class [12] Chapter 17.3 of the dual $\mathcal{E}^* = \mathcal{H}om(\mathcal{E}, O_Y)$, loc. cit. Remark 3.2.3 (a).

Lemma 3.4.11. *Let Y be a separated and smooth scheme of finite type purely of dimension d over a perfect field F and $V \subset Y$ be the complement of a divisor D with simple normal crossings. Let σ be an automorphism of Y over F. We assume that σ induces an automorphism of V, σ is admissible and that $V^\sigma = \emptyset$. Then, we have*

$$(\Gamma_\sigma, \Delta_Y)^{\log}_{\mathbb{Z}} = \{c(\Omega^1_{Y/F}(\log D))^* \cap s(Y^\sigma_{\log}/Y)\}_{\dim 0} \tag{3.29}$$

in $\mathrm{CH}_0(Y^\sigma_{\log})$. In particular, if $Y^\sigma_{\log}$ is a Cartier divisor D_σ of Y, we have

$$\begin{aligned}(\Gamma_\sigma, \Delta_Y)^{\log}_{\mathbb{Z}} &= \{c(\Omega^1_{Y/F}(\log D))^* \cap (1 + D_\sigma)^{-1} \cap D_\sigma\}_{\dim 0} \\ &= (-1)^{d-1}\{c(\Omega^1_{Y/F}(\log D)) \cap (1 - D_\sigma)^{-1} \cap D_\sigma\}_{\dim 0}.\end{aligned} \tag{3.30}$$

Proof. Clear from the definition of the intersection product [12, Prop. 6.1 (a)] and $N_{Y/(Y\times Y)^\sim} = \Omega^1_{Y/F}(\log D)$. □

Corollary 3.4.12. *Let $f : V \to U$ be a finite and étale morphism of connected separated and smooth scheme of finite type purely of dimension d over a perfect field F and let σ be an automorphism of V over U of order e.*

Let Y be a smooth separated scheme of finite type over F containing V as the complement of a divisor D with simple normal crossings. If σ is extended to an automorphism of Y over F, the following holds.

1. *If j is an integer prime to e, we have $(\Gamma_\sigma, \Delta_Y)^{\log}_{\mathbb{Z}} = (\Gamma_{\sigma^j}, \Delta_Y)^{\log}_{\mathbb{Z}}$ in $\mathrm{CH}_0(Y \setminus V)$.*
2. *If e is not a power of p, we have $(\Gamma_\sigma, \Delta_Y)^{\log}_{\mathbb{Z}} = 0$ in $\mathrm{CH}_0(Y \setminus V)$.*

Proof. Let $g : Y' \to Y$ be the blow-up associated to the subdivision by baricenters. Since $g_* : \mathrm{CH}_0(Y' \setminus V) \to \mathrm{CH}_0(Y \setminus V)$ is an isomorphism, by replacing Y by Y', we may assume that the action of σ^j on Y is admissible for each $j \in \mathbb{Z}$ by Lemma 1.2.3.2. Then it follows from Lemma 3.4.11 and Corollary 1.2.7. □

Lemma 3.4.13. *Conjecture 3.4.4 is true if $\dim U \leq 2$.*

Proof. It follows from Lemma 3.1.2 and Corollary 3.4.12. □

We consider the case of isolated fixed point.

Lemma 3.4.14. *Let Y be a separated and smooth scheme of finite type purely of dimension d over F, y be a closed point of Y and σ be an automorphism of Y over a perfect field F. Assume that the underlying set of the fixed part Y^σ is $\{y\}$.*

Let $f : Y' \to Y$ be the blow-up at y and D be the exceptional divisor. Let $g : (Y' \times Y')' \to (Y' \times Y')$ be the blow-up at $D \times D$. Then the automorphism σ' of Y' induced by σ is admissible. Let $\Gamma'_{\sigma'} \subset (Y' \times Y')'$ denote the proper transform of the graph $\Gamma_{\sigma'} \subset Y' \times Y'$ of σ and $\Delta_{Y'} \subset (Y' \times Y')'$ be the log diagonal. Then, we have

$$f_*(\Gamma'_{\sigma'}, \Delta_{Y'})_{(Y' \times Y')'} = [O_{Y^\sigma}] - [y] \tag{3.31}$$

in $\mathrm{CH}_0(y) = \mathbb{Z}$ where $[O_{Y^\sigma}] = \mathrm{length}\ O_{Y^\sigma} \cdot [y]$.

Proof. We have $[O_{Y^\sigma}] = (\Gamma_\sigma, \Delta_Y)_{Y \times Y}$. By the projection formula, we have

$$(\Gamma_\sigma, \Delta_Y)_{Y \times Y} = f_*(g^!(f \times f)^! \Gamma_\sigma, \Delta_{Y'})_{(Y' \times Y')'}.$$

Thus it is sufficient to show the equality

$$(g^!(f \times f)^! \Gamma_\sigma - \Gamma'_{\sigma'}, \Delta_{Y'})_{(Y' \times Y')'} = [y']$$

in $\mathrm{CH}_0(D)$ for a $\kappa(y)$-rational point $y' \in D$.

We compute $g^!(f \times f)^! \Gamma_\sigma$. Since the irreducible components of

$$(f \times f)^{-1}(\Gamma_\sigma) = \Gamma_\sigma \times_{Y \times Y} (Y' \times Y')$$

are $\Gamma_{\sigma'}$ and $D \times D$, we have

$$\begin{aligned}(f \times f)^![\Gamma_\sigma] &= [\Gamma_{\sigma'}] + \{c(\Omega^1_{Y/F})^* s(D \times D/Y \times Y)\}_{\dim d} \\ &= [\Gamma_{\sigma'}] + \{(1 + D^{(1)})^{-1}(1 + D^{(2)})^{-1} D^{(1)} \cdot D^{(2)}\}_{\dim d}.\end{aligned}$$

Here $D^{(1)} = D \times Y$ and $D^{(2)} = Y \times D$. The irreducible components of $g^{-1}(\Gamma_{\sigma'}) = \Gamma_{\sigma'} \times_{Y' \times Y'} (Y' \times Y')'$ are $\Gamma'_{\sigma'}$ and the inverse image E_D of the diagonal $D \subset D \times D$. Hence we have

$$g^!(f \times f)^![\Gamma_\sigma] = [\Gamma'_{\sigma'}] + [E_D] + \{(1+g^*D^{(1)})^{-1}(1+g^*D^{(2)})^{-1} g^*D^{(1)} \cdot g^*D^{(2)}\}_{\dim d}.$$

Thus we obtain

$$(g^!(f \times f)^!\Gamma_\sigma - \Gamma'_{\sigma'}, \Delta_{Y'})_{(Y' \times Y')'} = (E_D, \Delta_{Y'})_{(Y' \times Y')'} + \{(1 + D)^{-2} D^2\}_{\dim 0}.$$

By

$$\begin{aligned}(E_D, \Delta_{Y'})_{(Y' \times Y')'} &= (E_D, \Delta_D)_E = (\Delta_D, \Delta_D)_{D \times D} = d[y'], \\ \{(1 + D)^{-2} D^2\}_{\dim 0} &= (-1)^d (d - 1) D^d = -(d - 1)[y'],\end{aligned}$$

the assertion follows. □

4. Swan class and Euler characteristic of a sheaf

We keep the following notation in this section. Let U be a connected, separated and smooth scheme of finite type purely of dimension d over a perfect field F. Let ℓ be a prime number different from the characteristic p of F.

We consider a smooth $\bar{\mathbb{F}}_\ell$-sheaf $\mathcal{F}$ on U and a finite étale Galois covering $f : V \to U$ trivializing $\mathcal{F}$. We define and study the Swan character class in Section 4.1. Using it, we define the Swan classes $\mathrm{Sw}_{V/U}(\mathcal{F}) \in \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{Sw}(\mathcal{F}) \in \mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ in Section 4.2. We also prove the formula (0.3) in Section 4.2. In Section 4.3, we state an integrality conjecture (Conjecture 4.3.7) that is a generalization of the Hasse-Arf theorem (Lemma 4.3.6).

4.1. *Swan character class.* We define the Swan character class for a ramified Galois covering using the map (0.1) $(\ , \Delta_{\overline{V}})^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 4.1.1. Let $f : V \to U$ be a finite and étale Galois covering of Galois group G of connected separated and smooth schemes of finite type purely of dimension d over a perfect field F. For $\sigma \in G$, we define the Swan character class $s_{V/U}(\sigma) \in \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$s_{V/U}(\sigma) = \begin{cases} D^{\log}_{V/U} & \text{if } \sigma = 1 \\ -(\Gamma_\sigma, \Delta_{\overline{V}})^{\log} & \text{if } \sigma \neq 1. \end{cases} \tag{4.1}$$

If $\mathcal{C}_V^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_V$, the Swan character class $s_{V/U}(\sigma)$ is defined in $\mathrm{CH}_0(\overline{V} \setminus V)$.

We show basic properties of Swan character classes.

LEMMA 4.1.2. *Let the notation be as in Definition* 4.1.1.

1. *We have*

$$\sum_{\sigma \in G} s_{V/U}(\sigma) = 0. \tag{4.2}$$

2. *If the order of σ is not a power of the characteristic of F, we have $s_{V/U}(\sigma) = 0$.*

3. *Let $H \subset G$ be a subgroup and $g : V \to U'$ be the corresponding intermediate covering. Then, for $\sigma \in H$, we have*

$$s_{V/U}(\sigma) = \begin{cases} s_{V/U'}(\sigma) & \textit{if } \sigma \neq 1 \\ s_{V/U'}(1) + \bar{g}^* D^{\log}_{U'/U} & \textit{if } \sigma = 1. \end{cases} \tag{4.3}$$

4. *Let $N \subset G$ be a normal subgroup, and $g : V \to V'$ be the corresponding intermediate covering. Then, we have*

$$\bar{g}^* s_{V'/U}(\sigma) = \sum_{\tilde{\sigma} \in G, \mapsto \sigma} s_{V/U}(\tilde{\sigma}) \tag{4.4}$$

for $\sigma \in G/N$.

Proof. 1. Clear from the definition and $V \times_U V \setminus \Delta_V = \coprod_{\sigma \neq 1} \Gamma_\sigma$.

2. Clear from Proposition 3.4.3.

3. For $\sigma \neq 1$, it is clear from Lemma 3.3.1.1. For $\sigma = 1$, it is nothing but (3.21).

4. For $\sigma \neq 1$, the equality (4.4) is clear from Lemma 3.3.1.2. For $\sigma = 1$, it follows from the case $\sigma \neq 1$ and the equality (4.2). □

Remark 4.1.3. If Conjecture 3.4.4 is true, we have $s_{V/U}(\sigma) = s_{V/U}(\sigma^j)$ for an integer j prime to the order e of $\sigma \in G$.

We have the following trace formula.

LEMMA 4.1.4. *Let the notation be as in Definition* 4.1.1. *Then, we have*

$$\deg\, s_{V/U}(\sigma) = \begin{cases} [V:U]\chi_c(U_{\bar{F}}, \mathbb{Q}_\ell) - \chi_c(V_{\bar{F}}, \mathbb{Q}_\ell) & \textit{if } \sigma = 1 \\ -\mathrm{Tr}(\sigma^* : H^*_c(V_{\bar{F}}, \mathbb{Q}_\ell)) & \textit{if } \sigma \neq 1. \end{cases} \tag{4.5}$$

Proof. Clear from the definition and Lemma 3.4.5. □

COROLLARY 4.1.5. *If j is prime to the order of $\sigma \in G$,*

$$\deg\, s_{V/U}(\sigma) = \deg\, s_{V/U}(\sigma^j).$$

Proof. It suffices to consider the case $\sigma \neq 1$. Since j is prime to the order of σ, $\mathrm{Tr}(\sigma^{j*} : H^*_c(V_{\bar F}, \mathbb{Q}_\ell))$ is a conjugate of $\mathrm{Tr}(\sigma^* : H^*_c(V_{\bar F}, \mathbb{Q}_\ell))$ over $\mathbb{Q}$. Hence, by the equality $\deg s_{V/U}(\sigma) = -\mathrm{Tr}(\sigma^* : H^*_c(V_{\bar F}, \mathbb{Q}_\ell))$, the degree $\deg s_{V/U}(\sigma^j)$ is a conjugate of $\deg s_{V/U}(\sigma)$ over $\mathbb{Q}$. Since $\deg s_{V/U}(\sigma) \in \mathbb{Q}$, the assertion follows. □

If Y is a separated scheme of finite type containing U as a dense open subscheme, let $s_{V/U,Y}(\sigma) \in \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the image of $s_{V/U}(\sigma)$. Let $f : V \to U$ be a finite étale Galois covering of separated smooth schemes of finite type over F. Let G be the Galois group. Let X be a normal scheme containing U as a dense open subscheme and Y be the normalization of X in V. For a geometric point $\bar y$ of $Y \setminus V$, let $I_{\bar y} \subset G$ be the inertia group at y. For a geometric point $\bar x$ of $X \setminus U$, let $I_{\bar x} \subset G$ be the inertia group $I_{\bar y}$ at a geometric point $\bar y$ of $Y \setminus V$ lifting $\bar x$, that is defined modulo conjugate.

Lemma 4.1.6. *Let $f : V \to U$ be a finite étale Galois covering of Galois group G of connected, separated smooth schemes of finite type purely of dimension d over a perfect field F. Let X be a separated normal scheme of finite type containing U as a dense open subscheme and let Y be the normalization of X in V.*

Let $\sigma \in G$ be a nontrivial element and p be the characteristic of F. Assume that σ is not in any conjugate of any p-Sylow group of the inertia subgroup $I_{\bar x} \subset G$ for any geometric point $\bar x$ of $X \setminus U$.

Then, we have $s_{V/U,Y}(\sigma) = 0$.

Proof. If the order of σ is not a power of p, it follows from Lemma 4.1.2.2. Thus, it suffices to show $s_{V/U,Y}(\sigma) = 0$ assuming σ is not in any conjugate of the $I_{\bar x} \subset G$ for any geometric point $\bar x$ of $X \setminus U$. The assumption means that the σ-fixed part Y^σ is empty. Hence, the assertion follows from Corollary 3.3.4.2. □

For an isolated fixed point, the following is a special case of a conjecture of Serre.

Conjecture 4.1.7 (Serre [28, (1) p. 418]). *Let Y be a separated smooth scheme over a perfect field F purely of dimension d and y be a closed point of Y. Let G be a finite group of automorphisms of Y over F such that, for $\sigma \neq 1$, the underlying set of the fixed part Y^σ is $\{y\}$. Then, the function $a_G : G \to \mathbb{Z}$ defined by*

$$a_G(\sigma) = \begin{cases} -\text{length } O_{Y^\sigma,y} & \text{if } \sigma \neq 1 \\ -\sum_{\tau \in G\setminus\{1\}} a_G(\tau) & \text{if } \sigma = 1 \end{cases} \tag{4.6}$$

is a character of the group G.

Serre conjectures more precisely that the character a_G is rational over $\mathbb{Q}_\ell$ for all $\ell \neq p$ in loc. cit. (2). Conjecture 4.1.7 is proved in [19] assuming $\dim Y = 2$. In Corollary 5.1.7.3, we give a new proof by deducing it from a generalization, Conjecture 4.3.7.1, assuming $\dim Y = 2$.

We compare $a_G(\sigma)$ with the Swan character class $s_{V/U,Y}(\sigma)$.

LEMMA 4.1.8. *Let Y and G be as in Conjecture 4.1.7. We assume the quotient $\bar{f} : Y \to X = Y/G$ exists. Let $x = \bar{f}(y)$ be the image of y. Then, the map $f : V = Y \setminus \{y\} \to U = X \setminus \{x\}$ is finite étale and V is a Galois covering of Galois group G. Further, for $\sigma \in G$, we have*

$$a_G(\sigma) = \begin{cases} s_{V/U,Y}(\sigma) - 1 & \text{if } \sigma \neq 1 \\ s_{V/U,Y}(\sigma) + |G| - 1 & \text{if } \sigma = 1 \end{cases}$$

in $\mathrm{CH}_0(y) = \mathbb{Z}$.

Proof. We keep the notation in the proof of Lemma 3.4.14. Then the natural map $X' = Y'/G \to X$ is an isomorphism on the complement $U = X \setminus \{x\}$ and U is the complement of a Cartier divisor of X'. Hence the map $(\ , \Delta_Y)^{\log} : \mathrm{CH}_d(V \times_U V \setminus \Delta_V) \to \mathrm{CH}_0(y) = \mathbb{Z}$ is induced by the intersection product $(\ , \Delta_{Y'})_{(Y' \times Y')'}$ and the assertion is clear from Lemma 3.4.14. □

4.2. *Swan class and Euler characteristic of a sheaf.* We define the Swan class of an $\bar{\mathbb{F}}_\ell$-sheaf $\mathcal{F}$ as a 0-cycle class on the boundary of a covering trivializing $\mathcal{F}$. For a finite group G and a prime number p, let $G_{(p)} \subset G$ be the set of elements of order a power of p. If $p = 0$, we put $G_{(p)} = \emptyset$. For a representation M of G and $\sigma \in G$, let M^σ denote the fixed part $\{m \in M | \sigma(m) = m\}$.

Definition 4.2.1. Let U be a smooth connected scheme of dimension d over a perfect field F of characteristic p and $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U. Let $f : V \to U$ be a finite étale Galois covering of Galois group G trivializing $\mathcal{F}$. Let M be the $\bar{\mathbb{F}}_\ell$-representation of G corresponding to $\mathcal{F}$.

Then, we define the Swan class $\mathrm{Sw}_{V/U}(\mathcal{F}) \in \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$\mathrm{Sw}_{V/U}(\mathcal{F}) = \sum_{\sigma \in G_{(p)}} \left(\dim_{\bar{\mathbb{F}}_\ell} M^\sigma - \frac{\dim_{\bar{\mathbb{F}}_\ell} M^{\sigma^p}/M^\sigma}{p-1} \right) \cdot s_{V/U}(\sigma). \tag{4.7}$$

Recall that we have $s_{V/U}(\sigma) = 0$ if the order of σ is not a power of p by Lemma 4.1.2.2. Thus we take the sum over $\sigma \in G_{(p)}$. If $p = 0$, we have $\mathrm{Sw}_{V/U}(\mathcal{F}) = 0$.

We define a variant of the Swan class expected to be the same as that defined above. For an $\bar{\mathbb{F}}_\ell$-automorphism σ of an $\bar{\mathbb{F}}_\ell$-vector space M of dimension m, the Brauer trace $\mathrm{Tr}^{Br}(\sigma : M) \in \mathbb{Z}[\zeta_\infty] \subset \bar{\mathbb{Q}}_\ell$ is defined as follows. Let $\alpha_1, \ldots, \alpha_m$ be the eigenvalues of σ counted with multiplicities and let $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m \in \mathbb{Z}[\zeta_\infty] \subset \bar{\mathbb{Q}}_\ell$ be the roots of unity of order prime to ℓ lifting

$\alpha_1, \dots, \alpha_m$. Then, we define $\mathrm{Tr}^{Br}(\sigma : M) = \sum_{i=1}^m \tilde{\alpha}_i$. If σ is an automorphism of order p^e of M, one can easily verify the equality

$$(4.8)\quad |(\mathbb{Z}/p^e\mathbb{Z})^\times| \cdot \left(\dim_{\mathbb{F}_\ell} M^\sigma - \frac{\dim_{\mathbb{F}_\ell} M^{\sigma^p}/M^\sigma}{p-1}\right) = \sum_{i \in (\mathbb{Z}/p^e\mathbb{Z})^\times} \mathrm{Tr}^{Br}(\sigma^i : M).$$

Definition 4.2.2. Let the notation be as in Definition 4.2.1. Then, we define the naive Swan class $\mathrm{Sw}'_{V/U}(\mathcal{F}) \in \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}(\zeta_{p^\infty})$ by

$$(4.9)\qquad \mathrm{Sw}'_{V/U}(\mathcal{F}) = \sum_{\sigma \in G_{(p)}} s_{V/U}(\sigma) \otimes \mathrm{Tr}^{Br}(\sigma : M).$$

LEMMA 4.2.3. *Let the notation be as in Definition* 4.2.1.

1. *We have*

$$(4.10)\qquad \deg\, \mathrm{Sw}_{V/U}(\mathcal{F}) = \deg\, \mathrm{Sw}'_{V/U}(\mathcal{F}).$$

2. *If Conjecture* 3.4.4 *holds, we have*

$$(4.11)\qquad \mathrm{Sw}_{V/U}(\mathcal{F}) = \mathrm{Sw}'_{V/U}(\mathcal{F}).$$

Proof. 1. It follows from the equality (4.8) for an element $\sigma \in G$ of order p^e and Corollary 4.1.5.

2. It follows from the equality (4.8) for an element $\sigma \in G$ of order p^e. □

LEMMA 4.2.4. *Let $f : V \to U$ be a finite and étale Galois covering of connected separated and smooth schemes of finite type purely of dimension d over a perfect field F of Galois group G. Let ℓ be a prime number different form $p =$ char F.*

1. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of smooth $\bar{\mathbb{F}}_\ell$-sheaves on U trivialized on V. Then, we have*

$$(4.12)\qquad \mathrm{Sw}_{V/U}(\mathcal{F}) = \mathrm{Sw}_{V/U}(\mathcal{F}') + \mathrm{Sw}_{V/U}(\mathcal{F}'').$$

2. *Let $N \subset G$ be a normal subgroup and $g : V \to V'$ be the corresponding intermediate covering. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U trivialized on V'. Then, we have*

$$(4.13)\qquad \mathrm{Sw}_{V/U}(\mathcal{F}) = g^*\mathrm{Sw}_{V'/U}(\mathcal{F}).$$

Proof. 1. Clear from the definition.

2. It is clear from Lemma 4.1.2.4. □

COROLLARY 4.2.5. *Let U be a separated smooth scheme of finite type over F. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U. Let $f : V \to U$ be a finite étale Galois covering of Galois group G trivializing $\mathcal{F}$.*

1. *Then,*

$$\frac{1}{|G|} f_* \mathrm{Sw}_{V/U}(\mathcal{F}) \in \mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is independent of the choice of V.

2. *We have*

$$\mathrm{Sw}_{V/U}(\mathcal{F}) = \frac{1}{|G|} f^* f_* \mathrm{Sw}_{V/U}(\mathcal{F}).$$

Proof. 1. Clear from Lemma 4.2.4.2 and Corollary 3.1.4.2.

2. The Swan class $\mathrm{Sw}_{V/U}(\mathcal{F})$ is invariant by the Galois group G. Hence it follows from Corollary 3.1.4.2. □

Thus, we define the Swan class $\mathrm{Sw}(\mathcal{F})$ in $\mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ as follows.

Definition 4.2.6. Let U be a separated smooth scheme of finite type over F. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U.

We define the Swan class $\mathrm{Sw}(\mathcal{F}) \in \mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$\mathrm{Sw}(\mathcal{F}) = \frac{1}{|G|} f_* \mathrm{Sw}_{V/U}(\mathcal{F}) \tag{4.14}$$

that is independent of a finite étale Galois covering $V \to U$ trivializing $\mathcal{F}$ by Corollary 4.2.5.

Similarly, we define the naive Swan class by

$$\mathrm{Sw}'(\mathcal{F}) = \frac{1}{|G|} f_* \mathrm{Sw}'_{V/U}(\mathcal{F}) \tag{4.15}$$

We also define the Swan class for a smooth $\bar{\mathbb{Q}}_\ell$-sheaf.

LEMMA 4.2.7. *Let ℓ be a prime number invertible in F. Assume U is connected. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on U. Then the class* $\mathrm{Sw}(\mathcal{F}_0 \otimes_{\bar{\mathbb{Z}}_\ell} \bar{\mathbb{F}}_\ell) \in \mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is indepenent of the choice of a smooth $\bar{\mathbb{Z}}_\ell$-sheaf $\mathcal{F}_0$ on U satisfying $\mathcal{F} = \mathcal{F}_0 \otimes_{\bar{\mathbb{Z}}_\ell} \bar{\mathbb{Q}}_\ell$.*

Proof. Clear from Lemma 4.2.4.1. □

Definition 4.2.8. Let ℓ be a prime number invertible in F. Assume U is connected. For a smooth $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U, we define the Swan class $\mathrm{Sw}(\mathcal{F}) \in \mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ to be the class $\mathrm{Sw}_{V/U}(\mathcal{F}_0 \otimes_{\bar{\mathbb{Z}}_\ell} \bar{\mathbb{F}}_\ell)$ in Lemma 4.2.7 that is independent of $\mathcal{F}_0$.

We prove the formula (0.3) for the Euler characteristic. For a smooth $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on U, we put

$$\chi_c(U_{\bar{F}}, \mathcal{F}) = \sum_{q=0}^{2d} (-1)^q \dim_{\bar{\mathbb{Q}}_\ell} H^q_c(V_{\bar{F}}, \mathcal{F}).$$

We define $\chi_c(U_{\bar{F}}, \mathcal{F})$ similarly for a smooth $\bar{\mathbb{F}}_\ell$-sheaf $\mathcal{F}$ on U.

Theorem 4.2.9. *Let U be a connected separated smooth scheme of dimension d of finite type over F. Let ℓ be a prime number invertible in F. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf or a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on U. Then, we have*

$$\chi_c(U_{\bar F}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar F}, \mathbb{Q}_\ell) - \deg \text{Sw}(\mathcal{F}). \tag{0.3}$$

Proof. It is sufficient to show the case where $\mathcal{F}$ is a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U. Let the notation be as in Definition 4.2.1. Let $G_{\ell\text{-reg}}$ be the subset of G consisting of elements of order prime to ℓ. By Lemma 2.3 [15], we have

$$\chi_c(U_{\bar F}, \mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in G_{\ell\text{-reg}}} \text{Tr}(\sigma^* : H_c^*(V_{\bar F}, \mathbb{Q}_\ell)) \cdot \text{Tr}^{Br}(\sigma : M).$$

By Corollary 3.4.6, we may replace $G_{\ell\text{-reg}}$ in the summation by $G_{(p)}$. Thus by Lemma 4.1.4, we have

$$\chi_c(U_{\bar F}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar F}, \mathbb{Q}_\ell) - \deg \text{Sw}'(\mathcal{F})$$

where $\text{Sw}'(\mathcal{F})$ is the naive Swan class. By Lemma 4.2.3.1, we have $\deg \text{Sw}(\mathcal{F}) = \deg \text{Sw}'(\mathcal{F})$ and the assertion follows. $\square$

4.3. *Properties of Swan classes.* We keep the notation that U denotes a connected smooth scheme purely of dimension d over a perfect field F and ℓ is a prime number different from the characteristic of F.

We define the wild discriminant and show the induction formula for Swan classes.

Definition 4.3.1. Let $f : V \to U$ be a finite étale morphism of connected, separated and smooth scheme of finite type purely of dimension d over F. Then we define the wild discriminant $d^{\log}_{V/U} \in \text{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ of V over U by

$$d^{\log}_{V/U} = f_* D^{\log}_{V/U}. \tag{4.16}$$

Lemma 4.3.2. *Let $V \to U' \to U$ be finite étale morphism of separated and smooth schemes of finite type purely of dimension d over F. Assume $V \to U'$ is of constant degree $[V : U']$ and let $h : U' \to U$ denote the map. Then, we have*

$$d^{\log}_{V/U} = [V : U'] \cdot d^{\log}_{U'/U} + h_* d^{\log}_{V/U'}. \tag{4.17}$$

Proof. Clear from Lemma 3.4.2. $\square$

Proposition 4.3.3. *Let $f : V \to U$ be a finite and étale Galois covering of connected separated schemes of of dimension d of finite type over F. Let G be the Galois group and let $h : U' \to U$ be the intermediate covering corresponding to a subgroup $H \subset G$.*

Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U'. Assume that the pull-back $g^*\mathcal{F}$ by the map $g : V \to U'$ is constant. Then, if $T \subset G$ is a complete set of representatives of G/H, we have

$$\mathrm{Sw}_{V/U}(h_*\mathcal{F}) = \sum_{\tau\in T} \tau^*(\mathrm{Sw}_{V/U'}(\mathcal{F}) + \mathrm{rank}\ \mathcal{F}\cdot g^* D^{\log}_{U'/U}). \tag{4.18}$$

In particular, we have

$$\mathrm{Sw}_{V/U}(h_*\mathbb{F}_\ell) = \sum_{\tau\in T} \tau^* g^* D^{\log}_{U'/U}.$$

Proof. As in Definition 4.2.1, let p be the characteristic of F and $G_{(p)} \subset G$ be the subset consisting of elements of order a power of p. Let M be the $\bar{\mathbb{F}}_\ell$-representation of H corresponding to $\mathcal{F}$. For $\sigma \in G$, we have

$$\dim(\mathrm{Ind}_H^G M)^\sigma = \sum_{\tau\in T} \frac{\dim M^{\langle\tau\sigma\tau^{-1}\rangle\cap H}}{[\langle\tau\sigma\tau^{-1}\rangle : \langle\tau\sigma\tau^{-1}\rangle\cap H]}.$$

Thus, we have

$$\begin{aligned}
&\dim_{\mathbb{F}_\ell}(\mathrm{Ind}_H^G M)^\sigma - \frac{\dim_{\mathbb{F}_\ell}(\mathrm{Ind}_H^G M)^{\sigma^p}/(\mathrm{Ind}_H^G M)^\sigma}{p-1}\\
&\qquad= \sum_{\tau\in T, \tau\sigma\tau^{-1}\in H_{(p)}} \Big(\dim_{\mathbb{F}_\ell} M^{\tau\sigma\tau^{-1}} - \frac{\dim_{\mathbb{F}_\ell} M^{\tau\sigma^p\tau^{-1}}/M^{\tau\sigma\tau^{-1}}}{p-1}\Big).
\end{aligned}$$

Hence, the Swan class

$$\begin{aligned}
&\mathrm{Sw}_{V/U}(h_*\mathcal{F})\\
&\quad= \sum_{\sigma\in G_{(p)}} \Big(\dim_{\mathbb{F}_\ell}(\mathrm{Ind}_H^G M)^\sigma - \frac{\dim_{\mathbb{F}_\ell}(\mathrm{Ind}_H^G M)^{\sigma^p}/(\mathrm{Ind}_H^G M)^\sigma}{p-1}\Big)\cdot s_{V/U}(\sigma)
\end{aligned}$$

is equal to

$$\begin{aligned}
&\sum_{\sigma\in G_{(p)}} \sum_{\tau\in T, \tau\sigma\tau^{-1}\in H_{(p)}} \Big(\dim_{\mathbb{F}_\ell} M^{\tau\sigma\tau^{-1}} - \frac{\dim_{\mathbb{F}_\ell} M^{\tau\sigma^p\tau^{-1}}/M^{\tau\sigma\tau^{-1}}}{p-1}\Big)\cdot s_{V/U}(\sigma)\\
&= \sum_{\sigma'\in H_{(p)}} \sum_{\tau\in T} \Big(\dim_{\mathbb{F}_\ell} M^{\sigma'} - \frac{\dim_{\mathbb{F}_\ell} M^{\sigma'^p}/M^{\sigma'}}{p-1}\Big)\cdot s_{V/U}(\tau^{-1}\sigma'\tau)\\
&= \sum_{\tau\in T} \tau^*\Big(\sum_{\sigma'\in H_{(p)}} \Big(\dim_{\mathbb{F}_\ell} M^{\sigma'} - \frac{\dim_{\mathbb{F}_\ell} M^{\sigma'^p}/M^{\sigma'}}{p-1}\Big)\cdot s_{V/U}(\sigma')\Big).
\end{aligned}$$

By Lemma 4.1.2.3, the content of the big paranthese is equal to

$$\begin{aligned}
&\sum_{\sigma'\in H_{(p)}} \Big(\dim_{\mathbb{F}_\ell} M^{\sigma'} - \frac{\dim_{\mathbb{F}_\ell} M^{\sigma'^p}/M^{\sigma'}}{p-1}\Big)\cdot s_{V/U'}(\sigma') + \dim M\cdot g^* D^{\log}_{U'/U}\\
&= \mathrm{Sw}_{V/U'}(\mathcal{F}) + \mathrm{rank}\ \mathcal{F}\cdot g^* D^{\log}_{U'/U}.
\end{aligned}$$

Thus the assertion follows. ∎

Corollary 4.3.4. *Let $h : U' \to U$ be a finite and étale morphism of connected separated schemes of dimension d finite type over F. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U'.*

Then, we have

$$\mathrm{Sw}(h_*\mathcal{F}) = h_*\mathrm{Sw}(\mathcal{F}) + \mathrm{rank}\ \mathcal{F} \cdot d^{\log}_{U'/U}. \tag{4.19}$$

In particular, we have

$$\mathrm{Sw}(h_*\mathbb{F}_\ell) = d^{\log}_{U'/U}.$$

Proof. Clear from Proposition 4.3.3. □

We study the integrality of Swan classes. For a finite group G, let $C_p(G)$ denote the set of cyclic subgroups $C \subset G$ of order a power of p. For a cyclic subgroup $C \in C_p(G)$, we put $C^p = \langle \sigma^p \rangle \in C_p(G)$ for a generator σ of C and $C^\times = \{\text{generator of } C\}$. Further, for an $\bar{\mathbb{F}}_\ell$-representation M of G, we put $M^C = \{m \in M | \sigma(m) = m \text{ for all } \sigma \in C\}$. It is clear that the product $(\dim_{\mathbb{F}_\ell} M^C - \frac{\dim_{\mathbb{F}_\ell} M^{C^p}/M^C}{p-1}) \cdot |C^\times|$ is an integer.

Definition 4.3.5. Let U be a smooth connected scheme of dimension d over a perfect field F and $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U. Let $f : V \to U$ be a finite étale Galois covering of Galois group G trivializing $\mathcal{F}$. Let M be the $\bar{\mathbb{F}}_\ell$-representation of G corresponding to $\mathcal{F}$. We assume that $\mathcal{C}^{\mathrm{sm},0}_V$ is cofinal in $\mathcal{C}_V$ and that Conjectures 3.4.4 holds for $\sigma \in G$.

Then, we define the integral Swan class $\mathrm{Sw}_{V/U}(\mathcal{F})_{\mathbb{Z}} \in \mathrm{CH}_0(\overline{V} \setminus V)$ by

(4.20)

$$\mathrm{Sw}_{V/U}(\mathcal{F})_{\mathbb{Z}} = \sum_{C \in C_p(G)} \left(\dim_{\mathbb{F}_\ell} M^C - \frac{\dim_{\mathbb{F}_\ell} M^{C^p}/M^C}{p-1}\right) \cdot |C^\times| \cdot s_{V/U}(\sigma_C),$$

where σ_C denotes an arbitrary generator of $C \in C_p(G)$.

The assumptions that $\mathcal{C}^{\mathrm{sm},0}_V$ is cofinal in $\mathcal{C}_V$ and that Conjectures 3.4.4 holds for $\sigma \in G$ are satisfied if $\dim U \leq 2$.

We recall the classical theorem of Hasse-Arf for curves.

Lemma 4.3.6. *Let U be a smooth connected curve over a perfect field F and $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U trivialized by a finite étale Galois covering $f : V \to U$ of Galois group G. Let X be the proper smooth curve containing U as a dense open subscheme and $\bar{f} : Y \to X$ be the normalization in V. We identify* $\mathrm{CH}_0(X \setminus U) = \mathrm{CH}_0(\overline{U} \setminus U)$ *and* $\mathrm{CH}_0(Y \setminus V) = \mathrm{CH}_0(\overline{V} \setminus V)$.

Then, the integral Swan class $\mathrm{Sw}_{V/U}(\mathcal{F})_{\mathbb{Z}} \in \mathrm{CH}_0(Y \setminus V) = \bigoplus_{y \in Y \setminus V} \mathbb{Z} \cdot [y]$ *is in the image of the injection* $\bar{f}^* : \mathrm{CH}_0(X \setminus U) \to \mathrm{CH}_0(Y \setminus V)$.

Proof. Since Conjecture 3.4.4 holds in dimension 1, the Swan class $\mathrm{Sw}_{V/U}(\mathcal{F})$ is equal to the naive Swan class $\mathrm{Sw}'_{V/U}(\mathcal{F})$ by Lemma 4.2.3.2. For

$y \in Y \setminus V$, let $I_y \subset G$ be the inertia group at y. Let M be the corresponding $\bar{\mathbb{F}}_\ell$-representation of G. Then, by Lemma 3.4.7 and [29], the Swan conductor

$$\mathrm{Sw}_y(\mathcal{F}) = \frac{1}{|I_y|}\sum_{\sigma \in I_y} s_{V/U,y}(\sigma)\mathrm{Tr}^{Br}(\sigma : M)$$

is in $\mathbb{N}$. For $x \in X \setminus U$, $\mathrm{Sw}_y(\mathcal{F})$ is independent of the inverse image y of x. We put $\mathrm{Sw}_x(\mathcal{F}) = \mathrm{Sw}_y(\mathcal{F})$ for $x \in X \setminus U$ and $\mathrm{Sw}(\mathcal{F}) = \sum_{x \in X\setminus U} \mathrm{Sw}_x(\mathcal{F}) \cdot [x] \in \mathrm{CH}_0(X \setminus U)$. Then, we have

$$\mathrm{Sw}'_{V/U}(\mathcal{F}) = \sum_{y \in Y\setminus V} |I_y|\mathrm{Sw}_y(\mathcal{F}) \cdot [y] = \bar{f}^* \sum_{x \in X\setminus U} \mathrm{Sw}_x(\mathcal{F}) \cdot [x] = \bar{f}^*\mathrm{Sw}(\mathcal{F})$$

and the assertion is proved. □

We expect that Lemma 4.3.6 holds in higher dimension.

Conjecture 4.3.7. *Let U be a smooth connected scheme of dimension d over a perfect field F and $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U.*

1. *The Swan class $\mathrm{Sw}(\mathcal{F}) \in \mathrm{CH}_0(\overline{U}\setminus U)\otimes_{\mathbb{Z}}\mathbb{Q}$ is in the image of $\mathrm{CH}_0(\overline{U}\setminus U)$.*

2. *Let $f : V \to U$ be a finite étale Galois covering trivializing $\mathcal{F}$. Assume that $\mathcal{C}_V^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_V$ and that Conjecture 3.4.4 holds as in Definition 4.3.5.*

Then, the integral Swan class $\mathrm{Sw}_{V/U}(\mathcal{F})_{\mathbb{Z}} \in \mathrm{CH}_0(\overline{V} \setminus V)$ is in the image of $f^* : \mathrm{CH}_0(\overline{U} \setminus U) \to \mathrm{CH}_0(\overline{V} \setminus V)$.

Conjecture 4.3.7.1 is equivalent to the assertion that the Swan class $\mathrm{Sw}_{V/U}(\mathcal{F}) \in \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ is in the image of

$$f^* : \mathrm{CH}_0(\overline{U} \setminus U) \to \mathrm{CH}_0(\overline{V} \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for a finite étale Galois covering $f : V \to U$ trivializing $\mathcal{F}$, by Corollary 4.2.5.2.

By Lemma 4.3.6, Conjecture 4.3.7 is true if $\dim U = 1$. We prove Conjecture 4.3.7.1 assuming $\dim U \leq 2$ in Corollary 5.1.7.1. Conjecture 4.3.7.1 is reduced to the rank 1 case by the induction formula as follows.

Lemma 4.3.8. *Let $f : V \to U$ be a finite étale Galois covering of Galois group G. We assume that $\mathcal{C}_{U'}^{\mathrm{sm},0}$ is cofinal in $\mathcal{C}_{U'}$ for every intermediate covering $V \to U' \to U$. We also assume that $\mathrm{Sw}\ \mathcal{G} \in \mathrm{CH}_0(\overline{U'} \setminus U')\otimes_{\mathbb{Z}}\mathbb{Q}$ is in the image of $\mathrm{CH}_0(\overline{U'}\setminus U')$ for every smooth $\bar{\mathbb{F}}_\ell$-sheaf of rank 1 on an intermediate covering U' trivialized on V.*

Then, for every smooth $\bar{\mathbb{F}}_\ell$-sheaf $\mathcal{F}$ on U trivialized on V, the Swan class $\mathrm{Sw}\ \mathcal{F} \in \mathrm{CH}_0(\overline{U} \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ is in the image of $\mathrm{CH}_0(\overline{U} \setminus U)$.

Proof. By Brauer's theorem [29], we may assume $\mathcal{F} = h_*\mathcal{G}$ where $h : U' \to U$ is an intermediate covering and $\mathcal{G}$ is a smooth $\bar{\mathbb{F}}_\ell$-sheaf of rank 1 on U'. Since $\mathcal{C}_{U'}^{\mathrm{sm},0}$ is assumed cofinal in $\mathcal{C}_{U'}$, the wild different $D^{\log}_{U'/U}$ is defined in $\mathrm{CH}_0(\overline{U'} \setminus U')$ by Proposition 3.4.10. Hence, the wild discriminant $d^{\log}_{U'/U}$ is in the image of $\mathrm{CH}_0(\overline{U} \setminus U)$. Thus it follows from the assumption that Sw $\mathcal{G}$ is in the image of $\mathrm{CH}_0(\overline{U'} \setminus U')$ and the induction formula Corollary 4.3.4. □

If X is a separated scheme of finite type containing U as a dense open subscheme, let $\mathrm{Sw}_X(\mathcal{F}) \in \mathrm{CH}_0(X \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the image of $\mathrm{Sw}(\mathcal{F})$. Similarly, if Y is a separated scheme of finite type containing V as a dense open subscheme, let $\mathrm{Sw}_{V/U,Y}(\mathcal{F}) \in \mathrm{CH}_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the image of $\mathrm{Sw}_{V/U}(\mathcal{F})$.

Lemma 4.3.9. *Conjecture* 4.3.7.1 *implies Conjecture* 4.1.7.

Proof. Let the notation be as in Conjecture 4.1.7. Since $|G|a_G$ is a character of G by [28, Prop. 7], it is sufficient to show that the Artin conductor

$$a_G(M) = \frac{1}{|G|} \sum_{\sigma \in G} a_G(\sigma)\mathrm{Tr}(\sigma : M) \tag{4.21}$$

defined in $\mathbb{Q}$ is in $\mathbb{Z}$ for every $\bar{\mathbb{Q}}_\ell$-representation M of G. We may assume Y is affine and the quotient $X = Y/G$ exists. Let $x \in X$ be the image of y and $\mathcal{F}$ be the smooth sheaf on $U = X \setminus \{x\}$ corresponding to the representation M. Then, by Corollary 4.1.8 and Corollary 3.4.12.2, we have $a_G(M) = \mathrm{Sw}_X(\mathcal{F}) + \dim M - \dim M^G$ in $\mathrm{CH}_0(x) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$. Thus the assertion is proved. □

We give a refinement of Théorème 2.1 of [15].

Lemma 4.3.10. *Let the notation be as in Lemma* 4.1.6. *Let p be the characteristic of a perfect field F. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be smooth $\bar{\mathbb{F}}_\ell$-sheaves on U corresponding to $\bar{\mathbb{F}}_\ell$-representations M_1 and M_2 of G. Assume that X is normal and that, for each geometric point $\bar{x}$ of $X \setminus U$, the restrictions of M_1 and M_2 to a p-Sylow subgroup of the inertia subgroup $I_{\bar{x}}$ are isomorphic to each other. Then, we have*

$$\mathrm{Sw}_{V/U,Y}(\mathcal{F}_1) = \mathrm{Sw}_{V/U,Y}(\mathcal{F}_2)$$

Proof. Clear from Lemma 4.1.6 and Definition 4.2.1. □

If the base field is finite, we expect to have the following refinement of Theorem 4.2.9.

Conjecture 4.3.11. *Let U be a connected separated smooth scheme of dimension d of finite type over a finite field F. Let X be a proper normal scheme over F containing U as a dense open subscheme. Let $\mathrm{Fr}_F \in \mathrm{Gal}(\bar{F}/F)$ be the geometric Frobenius and let $\rho_X : \mathrm{CH}_0(X) \to \pi_1(X)^{\mathrm{ab}}$ be the reciprocity map sending $[x]$ to the geometric Frobenius Fr_x for closed points $x \in X$.*

Let ℓ be a prime number invertible in F. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf or a smooth $\bar{\mathbb{Q}}_\ell$-sheaf on U. We assume Conjecture 4.3.7.1 *holds and* $\mathrm{Sw}_X(\mathcal{F}) \in \mathrm{CH}_0(X \setminus U)$ *is defined.*

Let $\mathcal{G}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf or $\bar{\mathbb{Q}}_\ell$-sheaf on X and let $\det \mathcal{G} : \pi_1(X)^{\mathrm{ab}} \to \bar{\mathbb{F}}_\ell^\times$ *or* $\pi_1(X)^{\mathrm{ab}} \to \bar{\mathbb{Q}}_\ell^\times$ *be the character corresponding to the smooth sheaf* $\det \mathcal{G}$ *of rank* 1. *We put* $\det(-\mathrm{Fr}_F : H_c^*(U_{\bar{F}}, \mathcal{F})) = \prod_{q=0}^{2d} \det(-\mathrm{Fr}_F : H_c^q(U_{\bar{F}}, \mathcal{F}))^{(-1)^q}$.

Then, we have

$$\begin{aligned} &\det(-\mathrm{Fr}_F : H_c^*(U_{\bar{F}}, \mathcal{F} \otimes \mathcal{G})) \\ &\qquad\qquad = \det(-\mathrm{Fr}_F : H_c^*(U_{\bar{F}}, \mathcal{F}))^{\mathrm{rank}\ \mathcal{G}} \cdot \det \mathcal{G}(\rho_X(\mathrm{Sw}_X(\mathcal{F}))). \end{aligned}$$

If $\dim U = 1$, Conjecture 4.3.11 is a consequence of the product formula for the constant term of the functional equation of L-functions [8], [22].

5. Computations of Swan classes

We compare the Swan classes $\mathrm{Sw}(\mathcal{F})$ of sheaves of rank 1 with an invariant defined in [18] in Section 5.1. Using the computation, we prove the integrality conjecture Conjecture 4.3.7.1 assuming $\dim U \leq 2$. We also compare the formula (0.3) with Laumon's formula in [21].

We keep the notation that U denotes a connected smooth scheme purely of dimension d over a perfect field F and ℓ is a prime number different from the characteristic p of F.

5.1. *Rank* 1 *case.* Let X be a smooth separated scheme of finite type purely of dimension d over F and $U \subset X$ be the complement of a divisor D with simple normal crossings. Let ℓ be a prime number invertible in F. We identify $\mu_p(\bar{\mathbb{F}}_\ell) = \mathbb{Z}/p\mathbb{Z}$.

Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf of rank 1 on U. We briefly recall the definition of the 0-cycle class $c_{\mathcal{F}}$ in [18]. Let $D_1, \ldots, D_m$ be the irreducible components of D. Let $\chi \in H^1(U, \bar{\mathbb{F}}_\ell^\times)$ be the element corresponding to $\mathcal{F}$. In loc. cit., the Swan divisor $D_\chi = \sum_{i=1}^m \mathrm{sw}_i(\chi) D_i \geq 0$ is defined. Also the refined Swan character map

$$\mathrm{rsw}_i(\chi) : O(-D_\chi)|_{D_i} \to \Omega^1_{X/F}(\log D)|_{D_i}$$

is defined for each irreducible component D_i such that $\mathrm{sw}_i(\chi) > 0$.

We put $E = \sum_{i; \mathrm{sw}_i(\chi) > 0} D_i \subset D$. If $\bar{f} : Y \to X$ is the normalization in the cyclic étale covering $f : V \to U$ corresponding to χ, the closed subscheme $E \subset X$ is the wild ramification locus of the covering $Y \to X$. The sheaf $\mathcal{F}$ is said to be clean with respect to X if the map $\mathrm{rsw}_i(\chi) : O(-D_\chi)|_{D_i} \to \Omega^1_{X/F}(\log D)|_{D_i}$ is a locally splitting injection for each component D_i of E. If $\mathcal{F}$ is clean with respect to X, the 0-cycle class $c_{\mathcal{F}} = c_\chi \in \mathrm{CH}_0(E)$ is defined by

$$c_{\mathcal{F}} = c_{\chi} = \{c(\Omega^1_{X/F}(\log D))^* \cap (1 + D_{\chi})^{-1} \cap D_{\chi}\}_{\dim 0} = (-1)^{d-1} \sum_{i=1}^{m} \mathrm{sw}_i(\chi) c_{d-1}(\mathrm{Coker}(\mathrm{rsw}_i(\chi))) \cap [D_i]. \tag{5.1}$$

If one wants to specify X, we write $c_{\mathcal{F},X}$ for $c_{\mathcal{F}}$.

CONJECTURE 5.1.1. *Let X be a separated scheme of finite type over a perfect field F and $U \subset X$ be a dense open subscheme of X. Assume U is connected and smooth purely of dimension d over F. Let ℓ be a prime number invertible in F and $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf of rank 1 on U.*

1. *Let*

$$\begin{array}{ccc} V & \xrightarrow{\subset} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f} \\ U & \xrightarrow{\subset} & X \end{array} \tag{3.5}$$

be a Cartesian diagram of smooth separated schemes of finite type over F. We assume $U \subset X$ and $V \subset Y$ are the complement of divisors with simple normal crossings, $f : V \to U$ is a connected finite étale Galois covering of Galois group G and $\mathcal{F}$ is constant on V. If $\mathcal{F}$ is clean with respect to X, we have

$$\mathrm{Sw}_{V/U,Y}(\mathcal{F}) = \bar f^* c_{\mathcal{F},X} \tag{5.2}$$

in $\mathrm{CH}_0(E \times_X Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

2. *There exists a Cartesian diagram*

$$\begin{array}{ccc} U & \xrightarrow{\subset} & X' \\ \| & & \downarrow{\scriptstyle \bar f} \\ U & \xrightarrow{\subset} & X \end{array}$$

satisfying the following conditions: the map $\bar f : X' \to X$ is proper, X' is smooth over F, U is the complement of a divisor with simple normal crossings in X' and $\mathcal{F}$ is clean with respect to X'.

Conjecture 5.1.1.2 is proved if $\dim U \le 2$ in [18, Th. 4.1]. We prove Conjecture 5.1.1.1 assuming $\dim U \le 2$ later in Theorem 5.1.5.

LEMMA 5.1.2. *Conjecture* 5.1.1 *implies Conjecture* 4.3.7.1.

Proof. Note that Conjecture 5.1.1.2 is stronger than the strong resolution of singularities. Hence the assertion follows from Lemma 4.3.8. □

We prove Conjecture 5.1.1.1 in some cases. We say $\mathcal{F}$ is s-clean with respect to X if it is clean and further if the composition

$$O(-D_{\chi})|_{D_i} \xrightarrow{\mathrm{rsw}_i(\chi)} \Omega^1_{X/F}(\log D)|_{D_i} \xrightarrow{\mathrm{res}_{D_i}} O_{D_i}$$

is either an isomorphism or the 0-map for each component D_i of E, depending on D_i. We recall results in [32].

LEMMA 5.1.3 ([32, Lemmas 1 and 2]). *Let $p>0$ be the characteristic of F and $e\geq 1$ be an integer. Let X be a separated and smooth scheme of finite type over F, U be the complement of a divisor with simple normal crossings. Let $f: V\to U$ be a finite étale connected cyclic covering of degree p^e and let $f_1: U_1\to U$ be the intermediate covering of degree p. Let $\mathcal{F}$ and $\mathcal{G}$ be the smooth $\bar{\mathbb{F}}_\ell$-sheaves of rank 1 corresponding to characters $\chi,\theta:\mathrm{Gal}(V/U)\to\bar{\mathbb{F}}_\ell^\times$ of degree p^e and p respectively. We assume that the sheaf $\mathcal{G}$ is s-clean with respect to X. Let $E\subset X$ be the union of irreducible components of $X\setminus U$ where $\mathcal{F}$ has wild ramification.*

Then, there exists a Cartesian diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{\subset} & Y_1 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \bar{f}_1} \\ U & \xrightarrow{\subset} & X \end{array}$$

of smooth separated scheme of finite type satisfying the following condition:

[5.1.3] *The map $\bar{f}_1: Y_1\to X$ is proper and $U_1\subset Y_1$ is the complement of a divisor with simple normal crossings. If σ is a generator of $\mathrm{Gal}(U_1/U)$, the action of σ on U_1 is extended to an admissible action on Y_1 over X and we have*

$$p\cdot s_{U_1/U}(\sigma)=-\bar{f}_1^*c_{\mathcal{G}} \tag{5.3}$$

in $\mathrm{CH}_0(Y_1\times_X E)$. Further if $\mathcal{F}$ is clean with respect to X and if $\mathcal{F}_1=f_1^*\mathcal{F}$ is clean with respect to Y_1, we have

$$\bar{f}_1^*c_{\mathcal{F}}=c_{\mathcal{F}_1}+D^{\log}_{U_1/U}. \tag{5.4}$$

PROPOSITION 5.1.4. *Let the notation be as in Conjecture 5.1.1.1. Let $\chi: G\to\bar{\mathbb{F}}_\ell^\times$ be the character corresponding to $\mathcal{F}$. Let n be the order of χ and $e=\mathrm{ord}_p n$ be the p-adic valuation. For $0\leq i\leq e$, let U_i be the intermediate étale covering corresponding to the subgroup $G_i\subset G$ of index p^i.*

We assume that the diagram (3.5) is inserted in a Cartesian diagram

$$\begin{array}{ccccccccccccc} V & \xrightarrow{h} & U_e & \longrightarrow\cdots\longrightarrow & U_{i+1} & \longrightarrow & U_i & = & U_i & \longrightarrow\cdots = & U_0=U \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \downarrow \\ Y & \xrightarrow{\bar{h}} & X_e & \xrightarrow{\bar{g}_e}\cdots\xrightarrow{\bar{g}_{i+1}} & Y_{i+1} & \xrightarrow{\bar{f}_{i+1}} & X_i & \xrightarrow{\bar{g}_i} & Y_i & \xrightarrow{\bar{f}_i}\cdots\xrightarrow{\bar{g}_0} & Y_0=X \end{array} \tag{5.5}$$

satisfying the following conditions (5.5.1)–(5.5.3):

(5.5.1) *For* $0 \leq i \leq e$, X_i *and* Y_i *are separated and smooth over* F *and contain* U_i *as the complement of divisors with simple normal crossings. The pull-back* $\mathcal{F}_i = \mathcal{F}|_{U_i}$ *is clean with respect to* X_i *and to* Y_i *and we have* $\bar{g}_i^*(c_{\mathcal{F}_i,Y_i}) = c_{\mathcal{F}_i,X_i}$.

(5.5.2) *For* $0 \leq i < e$, *the smooth* $\bar{\mathbb{F}}_\ell$*-sheaf* $\mathcal{G}_i$ *on* U_i *corresponding to a nontrivial character* $\mathrm{Gal}(U_{i+1}/U_i) \to \bar{\mathbb{F}}_\ell^\times$ *is s-clean with respect to* X_i *and* $\bar{f}_{i+1} : Y_{i+1} \to X_i$ *satisfies the condition* [5.1.3] *in Lemma* 5.1.3.

(5.5.3) *The actions of* G *on* $U_1, \dots, U_e$ *and on* V *are extended to admissible actions on* $X_1, \dots, X_e$ *and on* Y.

Then, we have

$$\mathrm{Sw}_{V/U,Y}(\mathcal{F})_{\mathbb{Z}} = \bar{f}^* c_{\mathcal{F},X} \tag{5.2}$$

in $\mathrm{CH}_0(E \times_X Y)$.

Proof. First, we reduce it to the case where $n = p^e$. We decompose $G = \mathrm{Gal}(V/U) = \mathrm{Gal}(U_e/U) \times \mathrm{Gal}(U'/U)$ to the p-part $\mathrm{Gal}(U_e/U)$ and the non-p-part $\mathrm{Gal}(U'/U)$. Let χ' be the restriction to the p-part $\mathrm{Gal}(U_e/U)$ and let $\mathcal{F}'$ be the corresponding sheaf on U. By the definition in [18], we have $c_{\mathcal{F}} = c_{\mathcal{F}'}$. By Lemma 4.1.2.2, we have $\mathrm{Sw}_{V/U,Y}(\mathcal{F})_{\mathbb{Z}} = \mathrm{Sw}_{V/U,Y}(\mathcal{F}')_{\mathbb{Z}}$. By Lemma 4.2.4.2, we have $\mathrm{Sw}_{V/U,Y}(\mathcal{F}')_{\mathbb{Z}} = \bar{h}^* \mathrm{Sw}_{V_e/U,X_e}(\mathcal{F}')_{\mathbb{Z}}$. Thus the assertion is reduced to the case where n is a power of p.

We assume $n = p^e$ and prove the assertion by induction on e. We prove the case $n = p$. By the condition (5.5.3) and Corollary 3.4.12.1, we have $\mathrm{Sw}_{V/U,Y}(\mathcal{F})_{\mathbb{Z}} = -p \cdot s_{V/U}(\sigma)$ for a generator σ of $\mathrm{Gal}(V/U)$. Hence the assertion follows from the equality (5.3) in Lemma 5.1.3 and the assumption $\bar{g}_0^* c_{\mathcal{F},Y_0} = c_{\mathcal{F},X_0}$ (5.5.1) in the case $n = p$.

We assume $e \geq 2$. By the induction hypothesis, we may assume $\mathrm{Sw}_{V/U_1,Y}(\mathcal{F}_1)_{\mathbb{Z}} = \bar{g}^* c_{\mathcal{F}_1}$ where $\bar{g} : Y \to Y_1$ denotes the composition. By the equality (5.4) in Lemma 5.1.3 and the assumption $\bar{g}_0^* c_{\mathcal{F},Y_0} = c_{\mathcal{F},X_0}$ (5.5.1), we have $\bar{f}^* c_{\mathcal{F}} = \bar{g}^* c_{\mathcal{F}_1} + \bar{g}^* D^{\log}_{U_1/U}$ in $\mathrm{CH}_0(Y \setminus V)$. By the condition (5.5.3), Corollary 3.4.12.1 and Lemma 4.1.2.3, we have $\mathrm{Sw}_{V/U,Y}(\mathcal{F})_{\mathbb{Z}} = \mathrm{Sw}_{V/U_1,Y}(\mathcal{F}_1)_{\mathbb{Z}} + \bar{g}^* D^{\log}_{U_1/U}$. Thus the assertion is proved. □

THEOREM 5.1.5. *Conjecture* 5.1.1.1 *is true if* $\dim U \leq 2$. *More precisely, we have*

$$\mathrm{Sw}_{V/U,Y}(\mathcal{F})_{\mathbb{Z}} = \bar{f}^* c_{\mathcal{F},X} \tag{5.2}$$

in $\mathrm{CH}_0(E \times_X Y)$.

Proof. Without loss of generality, we may assume X and Y are proper over F, since the strong resolution is known in dimension ≤ 2. If $\dim U = 1$, we obtain a diagram (5.5) satisfying the conditions (5.5.1)–(5.5.3) in Proposition 5.1.4 by taking the normalizations $X_i = Y_i$ of X in U_i and the assertion follows.

To prove the case $\dim U = 2$, first we recall some results from [18].

LEMMA 5.1.6. *Let X and X' be smooth surfaces of finite type over F containing U as the complement of divisors with simple normal crossings and $g : X' \to X$ be a morphism over F inducing the identity on U. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf of rank 1 on U clean with respect to X.*

1. *The sheaf $\mathcal{F}$ is also clean with respect to X' and we have $g^* c_{\mathcal{F},X} = c_{\mathcal{F},X'}$.*

2. *Assume $\mathcal{F}$ corresponds to a character of order p. Then $\mathcal{F}$ is s-clean with respect to the complement of at most finitely many closed points of $X \setminus U$. If $g : X' \to X$ is the blow-up at the points where $\mathcal{F}$ is not s-clean, then $\mathcal{F}$ is s-clean with respect to X'.*

Proof. 1. It is sufficient to consider the case where $g : X' \to X$ is the blow-up at a closed point of the complement of U. Then, $\mathcal{F}$ is clean with respect to X' by [18] Remark 4.13. Further, we have $c_{\mathcal{F},X} = g_* c_{\mathcal{F},X'}$ by [18, Th. 5.2]. Hence by Lemma 3.1.5, we have $g^* c_{\mathcal{F},X} = c_{\mathcal{F},X'}$.

2. The first assertion is clear from the definition of s-cleanness. We show the second assertion. We may assume $\mathcal{F}$ is s-clean with respect to $X \setminus \{x\}$ where $x \in X \setminus U$ is a closed point. Then, the characterization given in [18] (3.6) shows that $\mathcal{F}$ is defined by an Artin-Schreier equation $T^p - T = s/t^n$ where (s,t) is a local coordinate at x and n is prime to $p = \operatorname{char} F$, on an étale neighborhood of x. (In [18, p. 773], $h = gf$ in line 7 should read $h = g^{-1}f$ and $\pi_{1 \le i \le r} \pi_i$ in line 12 should read $\prod_{1 \le i \le r} \pi_i$.) Then the assertion is easily checked. □

We go back to the proof of Theorem 5.1.5 in the case $\dim U = 2$. By Lemma 3.1.5, we may replace Y by a successive blow-up $Y' \to Y$ at closed points in the complement of V. By Lemma 5.1.6, we may also replace X by a successive blow-up $X' \to X$ at closed points in the complement of U. By Proposition 5.1.4, it is sufficient to construct a diagram (5.5) satisfying the conditions (5.5.1)–(5.5.3) after possibly replacing X and Y by successive blow-ups at closed points in the complements.

For $0 \le i < e$, there exist a proper smooth surface X'_i containing U_i as the complement of a divisor with simple normal crossing such that $\mathcal{F}_i$ and $\mathcal{G}_i$ are clean with respect to X'_i and that the map $U_i \to U$ is extended to a map $X'_i \to X$ for $0 \le i < e$, by [18, Th. 4.1]. By Lemma 3.1.2, there exist a proper smooth surface Y' containing V as the complement of a divisor with simple normal crossing such that the maps $V \to U_i$ are extended to maps $Y' \to X'_i$ and that the action of G on V is extended to an admissible action on Y' over X.

We define a diagram (5.5) satisfying the conditions (5.5.1)-(5.5.3) in Proposition 5.1.4 inductively after possibly replacing X and Y by a successive blow-up. Applying Lemma 3.1.2 to the quotient Y'/G, we obtain a proper smooth surface Y_0 containing U_0 as the complement of a divisor with simple normal crossing with a map $Y_0 \to Y'/G$ extending the identity of U. Since the identity of $U = U_0$ is extended to a map $Y'/G \to X'_0$, the identity of U_0 is extended to a map $Y_0 \to X'_0$. By replacing X by Y_0, we put $X = Y_0$.

We define Y_{i+1} and X_i inductively by assuming that Y_i is already defined and that the identity of U_i is extended to a map $Y_i \to X'_i$. Applying Lemma 3.1.2 to Y_i, we obtain a proper smooth scheme Y'_i that contains U_i as the complement of a divisor with simple normal crossings and that the action of G on U_i is extended to an admissible action on Y'_i over X. Since Y'_i dominates X'_i, the sheaves $\mathcal{F}_i$ and $\mathcal{G}_i$ are clean with respect to Y'_i, by Lemma 5.1.6.1. Let $X_i \to Y'_i$ be the blowing-up at the closed points where $\mathcal{G}_i$ are not s-clean and $\bar{g}_i : X_i \to Y_i$ be the composition. Then the sheaf $\mathcal{G}_i$ is s-clean with respect to X_i by Lemma 5.1.6.2. Further $\mathcal{F}_i$ is clean with respect to X_i and the condition $\bar{g}_i^* c_{\mathcal{F}_i,Y_i} = c_{\mathcal{F}_i,X_i}$ is satisfied by Lemma 5.1.6.1. Applying Lemma 5.1.3, we obtain $Y_{i+1} \to X_i$.

By the construction, we see that Y_{i+1} dominates X'_{i+1}. Repeating this construction inductively, we obtain a diagram (5.5) except the map $\bar{h} : Y \to X_e$. We define Y'' by applying the construction in Lemma 3.1.2 to the normalization of X_e in V. Then the action of G on V is extended to an admissible action on Y'' over X. Replacing Y by Y'', we obtain a diagram (5.5) satisfying the conditions (5.5.1)-(5.5.3) in Proposition 5.1.4. Thus the assertion is proved. □

Corollary 5.1.7. 1. *Conjecture* 4.3.7.1 *is true if* $\dim U \le 2$.

2. ([19]) *Conjecture* 4.1.7 *is true if* $\dim Y \le 2$.

Proof. Clear from Lemmas 5.1.2 and 4.3.9 respectively. □

5.2. *Comparison with Laumon's formula.* In [21], Laumon proves a generalization of the Grothendieck-Ogg-Shafarevich formula for surfaces under the assumption (NF) below on ramification. We compare the formula (0.3) with Laumon's formula in [21].

For simplicity, we assume F is an algebraically closed field. Let X be a proper normal connected surface over F and U be a smooth dense open subscheme. Let $\mathcal{F}$ be a smooth $\bar{\mathbb{F}}_\ell$-sheaf on U. Let $B_1, \dots, B_m$ be the irreducible components of dimension 1 of the complement $B = X \setminus U$, let ξ_i be the generic point of B_i, and let K_i be the field of fractions of the completion of O_{X,ξ_i}. We assume the following condition.

(NF) For each i, the finite Galois extension of K_i that trivializes $\mathcal{F}$ has *separable* residue extension.

By this assumption, the Swan conductor $\mathrm{Sw}_i(\mathcal{F}) \in \mathbb{N}$ of $\mathcal{F}$ for the local field K_i is defined by the classical ramification theory, as in the proof of Lemma 4.3.6. In [21], a smooth dense open subscheme $B_i^\circ \subset B_i$ for each i and an integer

$\mathrm{Sw}_x(\mathcal{F}) \in \mathbb{Z}$ for each closed point $x \in \Sigma = B \setminus \bigcup_{i=1}^m B_i^\circ$ are defined and the formula

$$\chi_c(U,\mathcal{F}) = \mathrm{rank}\ \mathcal{F} \cdot \chi_c(U,\mathbb{Q}_\ell) - \sum_{i=1}^m \mathrm{Sw}_i(\mathcal{F}) \cdot \chi_c(B_i^\circ,\mathbb{Q}_\ell) + \sum_{x\in\Sigma} \mathrm{Sw}_x(\mathcal{F}) \tag{5.6}$$

is proved.

To compare the formula (5.6) with (0.3), we give a slight reformulation. Let $\pi_i : \overline{B}_i \to B_i$ be the normalization for each irreducible component of dimension 1. For each closed point $x \in \Sigma$, we put

$$S_x(\mathcal{F}) = -\mathrm{Sw}_x(\mathcal{F}) + \sum_{i=1}^m \mathrm{Sw}_i(\mathcal{F}) \cdot |\pi_i^{-1}(x)|.$$

Then, the formula (5.6) is equivalent to

$$\chi_c(U,\mathcal{F}) = \mathrm{rank}\,\mathcal{F} \cdot \chi_c(U,\mathbb{Q}_\ell) - \left(\sum_{i=1}^m \mathrm{Sw}_i(\mathcal{F})\chi(\overline{B}_i,\mathbb{Q}_\ell) + \sum_{x\in\Sigma} S_x(\mathcal{F}) \right). \tag{5.7}$$

We compute the Swan class $\mathrm{Sw}(\mathcal{F})$ assuming the condition (NF) and give a relation with $S_x(\mathcal{F})$. By Lemma 3.1.2, there exist a finite étale Galois covering $V \to U$ that trivializes $\mathcal{F}$ and a Cartesian diagram

$$\begin{tikzcd} V \arrow[r, "\subset"] \arrow[d, "f"'] & Y \arrow[d, "\bar f"] \\ U \arrow[r, "\subset"] & X \end{tikzcd} \tag{3.5}$$

such that Y is smooth, $Y \to X$ is proper, $V \subset Y$ is the complement of a divisor with simple normal crossings and that the action of $G = \mathrm{Gal}(V/U)$ is extended to an admissible action on Y. We may further assume that there exist a proper scheme X' containing U as the complement of a Cartier divisor and that $f : V \to U$ is extended to a morphism $Y \to X'$. Furthermore, by the assumption (NF), we may assume the following condition (NF$'$) is satisfied where $\{\eta_{i1},\ldots,\eta_{ik_i}\}$ denotes the inverse image of ξ_i in Y for $i = 1,\ldots,m$.

(NF$'$) For each i, j, the extension $\kappa(\eta_{ij})$ is *separable* over $\kappa(\xi_i)$.

Let $\sigma \neq 1$ be an element of the Galois group $G = \mathrm{Gal}(V/U)$. For a generic point η_{ij} as above, we put $m_{ij}(\sigma) = \mathrm{length}\ O_{Y^\sigma_{\log},\eta_{ij}}$. We define a divisor D_σ of Y by $D_\sigma = \sum_{i,j} m_{ij}(\sigma) D_{ij}$ where D_{ij} is the closure of $\{\eta_{ij}\}$. The Cartier divisor D_σ is a closed subscheme of $Y^\sigma_{\log}$. We define the residual subscheme $R_\sigma \subset Y^\sigma_{\log}$ to be the closed subscheme of Y satisfying $I_{Y^\sigma_{\log}} = I_{D_\sigma} I_{R_\sigma}$ where I_Z denotes the ideal sheaf of O_Y defining a closed subscheme $Z \subset Y$. Then, by the residual intersection formula [12, Th. 9.2], we have

$$\begin{aligned}(\Gamma_\sigma, \Delta_Y)^{\log}_{\mathbb{Z}} &= -\, s_{V/U}(\sigma) \\ &= \{c(\Omega^1_{Y/F}(\log D))^* \cap (1+D_\sigma)^{-1} \cap D_\sigma\}_{\dim 0} + \mathbb{R}_\sigma \\ &= -\sum_{i,j} m_{ij}(\sigma)(c_1(\Omega^1_{Y/F}(\log D)) + D_\sigma) \cap D_{ij} + \mathbb{R}_\sigma\end{aligned}$$

where $\mathbb{R}_\sigma = \{c(\Omega^1_{Y/F}(\log D) \otimes O_Y(-D_\sigma))^* \cap s(R_\sigma/Y)\}_{\dim 0}$ is a 0-cycle class supported on the inverse image of finitely many closed points of B.

To compute the first term in the right-hand side, we define a complex $\mathcal{K}_{ij}$ of $O_{D_{ij}}$-modules by

$$\mathcal{K}_{ij} = [\varphi^*_{ij}(\Omega^1_{\overline{B}_i/F}) \longrightarrow \Omega^1_{Y/F}(\log D)|_{D_{ij}} \xrightarrow{\ \alpha\ } O_Y(-D_\sigma)|_{D_{ij}}]. \tag{5.8}$$

The sheaf $\Omega^1_{Y/F}(\log D)|_{D_{ij}}$ is put on degree 0, the map $\varphi_{ij} : D_{ij} \to \overline{B}_i$ is the natural one and the map α is defined by $da \mapsto \sigma(a) - a$ and $d\log b \mapsto \sigma(b)/b - 1$. By the assumption (NF′), the cohomology sheaves $\mathcal{H}^q(\mathcal{K}_{ij})$ are 0 except for $q = 0, 1$ and are supported on finitely many closed points for $q = 0, 1$. Thus we have

$$(c_1(\Omega^1_{Y/F}(\log D)) + D_\sigma) \cap D_{ij} = \varphi^*_{ij} c_1(\Omega^1_{\overline{B}_i/F}) + [\mathcal{H}^*(\mathcal{K}_{ij})]$$

where $[\mathcal{H}^*(\mathcal{K}_{ij})] = [\mathcal{H}^0(\mathcal{K}_{ij})] - [\mathcal{H}^1(\mathcal{K}_{ij})]$. Let $Z_0(B)$ denote the free abelian group generated by the classes of the closed points in B. We define a 0-cycle $S_\sigma \in Z_0(B)$ by

$$S_\sigma = \bar{f}_*(\sum_{i,j} m_{ij}(\sigma)[\mathcal{H}^*(\mathcal{K}_{ij})] - \mathbb{R}_\sigma)$$

and put $m_i(\sigma) = \sum_j m_{ij}(\sigma)[\kappa(\eta_{ij}) : \kappa(\xi_i)]$. Then, we obtain

$$\bar{f}_* s_{V/U}(\sigma) = \sum_{i=1}^{m} m_i(\sigma) g_{i*}(c_1(\Omega^1_{\overline{B}_i/F}) \cap [\overline{B}_i]) + S_\sigma$$

where $g_i : \overline{B}_i \to X$ is the natural map. We define a 0-cycle $S_{\mathcal{F}} \in Z_0(B) \otimes \mathbb{Q}$ by

$$\frac{1}{|G|} \sum_{\sigma \in G_{(p)} \setminus \{1\}} \left(\dim_{\mathbb{F}_\ell} M^\sigma - \frac{\dim_{\mathbb{F}_\ell} M^{\sigma^p}/M^\sigma}{p-1} - \dim M \right) \cdot S_\sigma.$$

By $\mathrm{Sw}_i(\mathcal{F}) = -\frac{1}{|G|} \sum_{\sigma \in G_{(p)} \setminus \{1\}} m_i(\sigma)(\dim_{\mathbb{F}_\ell} M^\sigma - \dim_{\mathbb{F}_\ell}(M^{\sigma^p}/M^\sigma)/(p-1) - \dim M)$ and Lemma 4.1.2.1, we have

$$\begin{aligned}\mathrm{Sw}(\mathcal{F}) &= \frac{1}{|G|} \sum_{\sigma \in G_{(p)} \setminus \{1\}} (\dim_{\mathbb{F}_\ell} M^\sigma - \frac{\dim_{\mathbb{F}_\ell} M^{\sigma^p}/M^\sigma}{p-1} - \dim_{\mathbb{F}_\ell} M) \cdot \bar{f}_* s_{V/U}(\sigma) \\ &= -\sum_{i=1}^{m} \mathrm{Sw}_i(\mathcal{F}) g_{i*}(c_1(\Omega^1_{\overline{B}_i/F}) \cap [\overline{B}_i]) + S_{\mathcal{F}}.\end{aligned}$$

Since $\chi(\overline{B}_i, \mathbb{Q}_\ell) = -\deg(c_1(\Omega^1_{\overline{B}_i/F}) \cap [\overline{B}_i])$, the formula (0.3) together with the following proposition will imply the formula (5.7).

Proposition 5.2.1. *Under the notation above, we have an equality*

$$S_{\mathcal{F}} = \sum_{x \in \Sigma} S_x(\mathcal{F})[x] \tag{5.9}$$

in $Z_0(B)$.

In [19, Th. (6.7)], the invariant $\mathrm{Sw}_x(\mathcal{F})$ is shown to be equal to another invariant that is defined in [31] using intersection classes without introducing log products. A similar computation gives a proof of Proposition 5.2.1 but we leave the detail to the reader.

Kyoto University, Kyoto, Japan
E-mail address: kazuya@math.kyoto-u.ac.jp

University of Tokyo, Tokyo, Japan
E-mail address: t-saito@ms.u-tokyo.ac.jp

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