Annals of Mathematics, **167** (2008), 1099–1108

A generic property of families of Lagrangian systems

By Patrick Bernard* and Gonzalo Contreras**

Abstract

We prove that a generic Lagrangian has finitely many minimizing measures for every cohomology class.

1. Introduction

Let M be a compact boundaryless smooth manifold.
Let $\mathbb{T}$ be either the group $(\mathbb{R}/\mathbb{Z}, +)$ or the trivial group $(\{0\}, +)$.
A *Tonelli Lagrangian* is a C^2 function $L : \mathbb{T} \times TM \to \mathbb{R}$ such that

- The restriction to each fiber of $\mathbb{T} \times TM \to \mathbb{T} \times M$ is a *convex* function.
- It is fiberwise *superlinear*:
$$\lim_{|\theta|\to+\infty} L(t,\theta)/|\theta| = +\infty, \qquad (t,\theta) \in \mathbb{T} \times TM.$$
- The Euler-Lagrange equation
$$\tfrac{d}{dt} L_v = L_x$$
defines a *complete* flow $\varphi : \mathbb{R} \times (\mathbb{T} \times TM) \longrightarrow \mathbb{T} \times TM$.

We say that a Tonelli Lagrangian L is *strong Tonelli* if $L + u$ is a Tonelli Lagrangian for each $u \in C^\infty(\mathbb{T} \times M, \mathbb{R})$. When $\mathbb{T} = \{0\}$ we say that the lagrangian is *autonomous*.

Let $\mathcal{P}(L)$ be the set of Borel probability measures on $\mathbb{T} \times TM$ which are invariant under the Euler-Lagrange flow φ. The action functional $A_L : \mathcal{P}(L) \to \mathbb{R} \cup \{+\infty\}$ is defined as
$$A_L(\mu) := \langle L, \mu \rangle := \int_{\mathbb{T}\times TM} L \, d\mu.$$

*Membre de l'IUF.
**Partially supported by CONACYT-México grant # 46467-F.

The functional A_L is lower semi-continuous and the minimizers of A_L on $\mathcal{P}(L)$ are called *minimizing measures*. The ergodic components of a minimizing measure are also minimizing, and they are mutually singular, so that the set $\mathfrak{M}(L)$ of minimizing measures is a simplex whose extremal points are the ergodic minimizing measures.

In general, the simplex $\mathfrak{M}(L)$ may be of infinite dimension. The goal of the present paper is to prove that this is a very exceptional phenomenon. The first results in that direction were obtained by Mañé in [4]. His paper has been very influential to our work.

We say that a property is *generic* in the sense of Mañé if, for each strong Tonelli Lagrangian L, there exists a residual subset $\mathcal{O} \subset C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that the property holds for all the Lagrangians $L - u, u \in \mathcal{O}$. A set is called residual if it is a countable intersection of open and dense sets. We recall which topology is used on $C^\infty(\mathbb{T} \times M, \mathbb{R})$. Denoting by $\|u\|_k$ the C^k-norm of a function $u : \mathbb{T} \times M \longrightarrow \mathbb{R}$, define

$$\|u\|_\infty := \sum_{k \in \mathbb{N}} \frac{\arctan(\|u\|_k)}{2^k}.$$

Note that $\|.\|_\infty$ is not a norm. Endow the space $C^\infty(\mathbb{T} \times M, \mathbb{R})$ with the translation-invariant metric $\|u - v\|_\infty$. This metric is complete, hence the Baire property holds: any residual subset of $C^\infty(\mathbb{T} \times M, \mathbb{R})$ is dense.

THEOREM 1. *Let A be a finite-dimensional convex family of strong Tonelli Lagrangians. Then there exists a residual subset $\mathcal{O}$ of $C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that,*

$$u \in \mathcal{O}, \quad L \in A \quad \Longrightarrow \quad \dim \mathfrak{M}(L - u) \leqslant \dim A.$$

In other words, there exist at most $1 + \dim A$ ergodic minimizing measures of $L - u$.

The main result of Mañé in [4] is that having a unique minimizing measure is a generic property. This corresponds to the case where A is a point in our statement. Our generalization of Mañé's result is motivated by the following construction due to John Mather:

We can view a 1-form on M as a function on TM which is linear on the fibers. If λ is closed, the Euler-Lagrange equation of the Lagrangian $L - \lambda$ is the same as that of L. However, the minimizing measures of $L - \lambda$, are not the same as the minimizing measures of L. Mather proves in [5] that the set $\mathfrak{M}(L - \lambda)$ of minimizing measures of the lagrangian $L - \lambda$ depends only on the cohomology class c of λ. If $c \in H^1(M, \mathbb{R})$ we write $\mathfrak{M}(L - c) := \mathfrak{M}(L - \lambda)$, where λ is a closed form of cohomology c.

It turns out that important applications of Mather theory, such as the existence of orbits wandering in phase space, require understanding not only of the set $\mathfrak{M}(L)$ of minimizing measures for a fixed or generic cohomology classes but of the set of all Mather minimizing measures for every $c \in H^1(M, L)$. The following corollaries are crucial for these applications.

COROLLARY 2. *The following property is generic in the sense of Mañé: For all* $c \in H^1(M, \mathbb{R})$, *there are at most* $1 + \dim H^1(M, \mathbb{R})$ *ergodic minimizing measures of* $L - c$.

We say that a property is of infinite codimension if, for each finite-dimensional convex family A of strong Tonelli Lagrangians, there exists a residual subset $\mathcal{O}$ in $C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that none of the Lagrangians $L - u$, $L \in A$, $u \in \mathcal{O}$ satisfy the property.

COROLLARY 3. *The following property is of infinite codimension: There exists* $c \in H^1(M, \mathbb{R})$, *such that* $L - c$ *has infinitely many ergodic minimizing measures.*

Another important issue concerning variational methods for Arnold diffusion questions is the total disconnectedness of the quotient Aubry set. John Mather proves in [7, § 3] that the quotient Aubry set $\overline{\mathcal{A}}$ of any Tonelli Lagrangian on $\mathbb{T} \times TM$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\dim M \leqslant 2$ (or with $\mathbb{T} = \{0\}$ and $\dim M \leqslant 3$) is totally disconnected. See [7] for its definition.

The elements of the quotient Aubry set are called *static classes*. They are disjoint subsets of $\mathbb{T} \times TM$ and each static class supports at least one ergodic minimizing measure. We then get

COROLLARY 4. *The following property is generic in the sense of Mañé: For all* $c \in H^1(M, \mathbb{R})$ *the quotient Aubry set* $\overline{\mathcal{A}}_c$ *of* $L - c$ *has at most* $1 + \dim H^1(M, \mathbb{R})$ *elements.*

2. Abstract results

Assume that we are given

- Three topological vector spaces E, F, G.
- A continuous linear map $\pi : F \to G$.
- A bilinear pairing $\langle u, \nu \rangle : E \times G \to \mathbb{R}$.
- Two metrizable convex compact subsets $H \subset F$ and $K \subset G$ such that $\pi(H) \subset K$.

Suppose that

(i) The map

$$E \times K \ni (u, \nu) \longmapsto \langle u, \nu \rangle$$

is continuous.

We will also denote $\langle u, \pi(\mu)\rangle$ by $\langle u, \mu\rangle$ when $\mu \in H$. Observe that each element $u \in E$ gives rise to a linear functional on F

$$F \ni \mu \longmapsto \langle u, \mu\rangle$$

which is continuous on H. We shall denote by H^* the set of affine and continuous functions on H and use the same symbol u for an element of E and for the element $\mu \longmapsto \langle u, \mu\rangle$ of H^* which is associated to it.

(ii) The compact K is separated by E. This means that, if η and ν are two different points of K, then there exists a point u in E such that $\langle u, \eta\rangle \neq \langle u, \nu\rangle \neq 0$.

Note that the topology on K is then the weak topology associated to E. A sequence η_n of elements of K converges to η if and only if we have $\langle u, \eta_n\rangle \longrightarrow \langle u, \eta\rangle$ for each $u \in E$. We shall, for notational conveniences, fix once and for all a metric d on K.

(iii) E is a Frechet space. It means that E is a topological vector space whose topology is defined by a translation-invariant metric, and that E is complete for this metric.

Note then that E has the Baire property. We say that a subset is residual if it is a countable intersection of open and dense sets. The Baire property says that any residual subset of E is dense.

Given $L \in H^*$ denote by

$$M_H(L) := \arg\min_H L$$

the set of points $\mu \in H$ which minimize $L|_H$, and by $M_K(L)$ the image $\pi(M_H(L))$. These are compact convex subsets of H and K.

Our main abstract result is:

THEOREM 5. *For every finite-dimensional affine subspace A of H^*, there exists a residual subset $\mathcal{O}(A) \subset E$ such that, for all $u \in \mathcal{O}(A)$ and all $L \in A$, we have*

$$\dim M_K(L - u) \leq \dim A. \tag{1}$$

Proof. We define the ε-neighborhood V_ε of a subset V of K as the union of all the open balls in K which have radius ε and are centered in V. Given a subset $D \subset A$, a positive number ε, and a positive integer k, denote by $\mathcal{O}(D, \varepsilon, k) \subset E$ the set of points $u \in E$ such that, for each $L \in D$, the convex set $M_K(L-u)$ is contained in the ε-neighborhood of some k-dimensional convex subset of K.

We shall prove that the theorem holds with

$$\mathcal{O}(A) = \bigcap_{\varepsilon>0} \mathcal{O}(A, \varepsilon, \dim A).$$

If u belongs to $\mathcal{O}(A)$, then (1) holds for every $L \in A$. Otherwise, for some $L \in A$, the convex set $M_K(L-u)$ would contain a ball of dimension $\dim A + 1$, and, if ε is small enough, such a ball is not contained in the ε-neighborhood of any convex set of dimension $\dim A$.

So we have to prove that $\mathcal{O}(A)$ is residual. In view of the Baire property, it is enough to check that, for any compact subset $D \subset A$ and any positive ε, the set $\mathcal{O}(D, \varepsilon, \dim A)$ is open and dense. We shall prove in 2.1 that it is open, and in 2.2 that it is dense. □

2.1. *Open.* We prove that, for any $k \in \mathbb{Z}^+$, $\varepsilon > 0$ and any compact $D \subset A$, the set $\mathcal{O}(D, \varepsilon, k) \subset E$ is open. We need a Lemma.

LEMMA 6. *The set-valued map* $(L, u) \longmapsto M_H(L-u)$ *is upper semi-continuous on* $A \times E$. *This means that for any open subset* U *of* H, *the set*

$$\{(L, u) \in A \times E \ : \ M_H(L-u) \subset U\} \subset A \times E$$

is open in $A \times E$. *Consequently, the set-valued map* $(L, u) \longmapsto M_K(L-u)$ *is also upper semi-continuous.*

Proof. This is a standard consequence of the continuity of the map

$$A \times E \times H \ni (L, u, \mu) \longmapsto (L-u)(\mu) = L(\mu) - \langle u, \mu \rangle. \qquad \square$$

Now let u_0 be a point of $\mathcal{O}(D, \varepsilon, k)$. For each $L \in D$, there exists a k-dimensional convex set $V \subset K$ such that $M_K(L-u_0) \subset V_\varepsilon$. In other words, the open sets of the form

$$\{(L, u) \in D \times E \ : \ M_H(L-u) \subset V_\varepsilon\} \subset D \times E,$$

where V is some k-dimensional convex subset of K, cover the compact set $D \times \{u_0\}$. So there exists a finite subcovering of $D \times \{u_0\}$ by open sets of the form $\Omega_i \times U_i$, where Ω_i is an open set in A and $U_i \subset \mathcal{O}(\Omega_i, \varepsilon, k)$ is an open set in E containing u_0. We conclude that the open set $\cap U_i$ is contained in $\mathcal{O}(D, \varepsilon, k)$, and contains u_0. This ends the proof. □

2.2. *Dense.* We prove the density of $\mathcal{O}(A, \varepsilon, \dim A)$ in E for $\varepsilon > 0$. Let w be a point in E. We want to prove that w is in the closure of $\mathcal{O}(A, \varepsilon, \dim A)$.

LEMMA 7. *There exists an integer* m *and a continuous map*

$$T_m = (w_1, \dots, w_m) : K \longrightarrow \mathbb{R}^m,$$

with $w_i \in E$ such that

$$\forall\, x \in \mathbb{R}^m \qquad \operatorname{diam} T_m^{-1}(x) < \varepsilon, \tag{2}$$

where the diameter is taken for the distance d on K.

Proof. In $K \times K$, to each element $w \in E$ we associate the open set

$$U_w = \{(\eta, \mu) \in K \times K \ : \ \langle w, \eta - \mu \rangle \neq 0\}.$$

Since E separates K, the open sets $U_w, w \in E$ cover the complement of the diagonal in $K \times K$. Since this complement is open in the separable metrizable set $K \times K$, we can extract a countable subcovering from this covering. So we have a sequence U_{w_k}, with $w_k \in E$, which covers the complement of the diagonal in $K \times K$. This amounts to say that the sequence w_k separates K. Defining $T_m = (w_1, \dots, w_m)$, we have to prove that (2) holds for m large enough. Otherwise, we would have two sequences η_m and μ_m in K such that

$$T_m(\mu_m) = T_m(\eta_m) \quad \text{ and } \quad d(\mu_m, \eta_m) \geqslant \varepsilon.$$

By extracting a subsequence, we can assume that the sequences μ_m and η_m have different limits μ and η, which satisfy $d(\eta, \mu) \geqslant \varepsilon$. Take m large enough, so that $T_m(\eta) \neq T_m(\mu)$. Such a value of m exists because the linear forms w_k separate K. We have that

$$T_m(\mu_k) = T_m(\eta_k) \quad \text{ for } \quad k \geq m.$$

Hence at the limit $T_m(\eta) = T_m(\mu)$. This is a contradiction. □

Define the function $F_m : A \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ as

$$F_m(L, x) := \min_{\substack{\mu \in H \\ T_m \circ \pi(\mu) = x}} (L - w)(\mu),$$

when $x \in T_m(\pi(H))$ and $F_m(L, x) = +\infty$ if $x \in \mathbb{R}^m \setminus T_m(\pi(H))$. For $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, let

$$M_m(L, y) := \operatorname*{arg\,min}_{x \in \mathbb{R}^m} \big[F_m(L, x) - y \cdot x \big] \subset \mathbb{R}^m$$

be the set of points which minimize the function $x \longmapsto F_m(L, x) - y \cdot x$. We have that

$$M_K\big(L - w - y_1 w_1 - \cdots - y_m w_m\big) \subset T_m^{-1}(M_m(L, y)).$$

Let

$$\mathcal{O}_m(A, \dim A) := \{ y \in \mathbb{R}^m \mid \forall\, L \in A \ : \ \dim M_m(L, y) \leqslant \dim A \}.$$

From Lemma 7 it follows that

$$y \in \mathcal{O}_m(A, \dim A) \quad \Longrightarrow \quad w + y_1 w_1 + \cdots + y_m w_m \in \mathcal{O}(A, \varepsilon, \dim A).$$

Therefore, in order to prove that w is in the closure of $\mathcal{O}(A, \varepsilon, \dim A)$, it is enough to prove that 0 is in the closure of $\mathcal{O}_m(A, \dim A)$, which follows from the next proposition.

PROPOSITION 8. *The set $\mathcal{O}_m(A, \dim A)$ is dense in $\mathbb{R}^m$.*

Proof. Consider the Legendre transform of F_m with respect to the second variable,

$$\begin{aligned} G_m(L, y) &= \max_{x \in \mathbb{R}^m} y \cdot x - F_m(L, x) \\ &= \max_{\mu \in H} \langle w + y_1 w_1 + \cdots + y_m w_m, \mu \rangle - L(\mu). \end{aligned}$$

It follows from this second expression that the function G_m is convex and finite-valued, hence continuous on $A \times \mathbb{R}^m$.

Consider the set $\tilde{\Sigma} \subset A \times \mathbb{R}^m$ of points (L, y) such that $\dim \partial G_m(L, y) \geq \dim A + 1$, where ∂G_m is the subdifferential of G_m. It is known, see the appendix, that this set has Hausdorff dimension at most

$$(m + \dim A) - (\dim A + 1) = m - 1.$$

Consequently, the projection Σ of the set $\tilde{\Sigma}$ on the second factor $\mathbb{R}^m$ also has Hausdorff dimension at most $m - 1$. Therefore, the complement of Σ is dense in $\mathbb{R}^m$. So it is enough to prove that

$$y \notin \Sigma \quad \Longrightarrow \quad \forall\, L \in A \; : \quad \dim M_m(L, y) \leq \dim A.$$

Since we know by definition of Σ that $\dim \partial G_m(L, y) \leq \dim A$, it is enough to observe that

$$\dim M_m(L, y) \leq \dim \partial G_m(L, y).$$

The last inequality follows from the fact that the set $M_m(L, y)$ is the subdifferential of the convex function

$$\mathbb{R}^m \ni z \longmapsto G_m(L, z)$$

at the point y. □

3. Application to Lagrangian dynamics

Let C be the set of continuous functions $f : \mathbb{T} \times TM \to \mathbb{R}$ with linear growth, i.e.

$$\|f\|_\ell := \sup_{(t,\theta) \in \mathbb{T} \times TM} \frac{|f(t, \theta)|}{1 + |\theta|} < +\infty,$$

endowed with the norm $\|\cdot\|_\ell$.

We apply Theorem 5 to the following setting:

- $F = C^*$ is the vector space of continuous linear functionals $\mu : C \to \mathbb{R}$ provided with the weak-$\star$ topology. Recall that

$$\lim_n \mu_n = \mu \quad \Longleftrightarrow \quad \lim_n \mu_n(f) = \mu(f), \quad \forall\, f \in C.$$

- $E = C^\infty(\mathbb{T} \times M, \mathbb{R})$ provided with the C^∞ topology.

- G is the vector space of finite Borel signed measures on $\mathbb{T} \times M$, or equivalently the set of continuous linear forms on $C^0(\mathbb{T} \times M, \mathbb{R})$, provided with the weak-$\star$ topology.

- The pairing $E \times G \to \mathbb{R}$ is given by integration:

$$\langle u, \nu \rangle = \int_{\mathbb{T}\times M} u \, d\nu.$$

- The continuous linear map $\pi : F \longrightarrow G$ is induced by the projection $\mathbb{T} \times TM \longrightarrow \mathbb{T} \times M$.

- The compact $K \subset G$ is the set of Borel probability measures on $\mathbb{T} \times M$, provided with the weak-$\star$ topology. Observe that K is separated by E.

- The compact $H_n \subset F$ is the set of holonomic probability measures which are supported on

$$B_n := \{(t, \theta) \in \mathbb{T} \times TM \mid |\theta| \leqslant n\}.$$

Holonomic probabilities are defined as follows: Given a C^1 curve $\gamma : \mathbb{R} \to M$ of period $T \in \mathbb{N}$ define the element μ_γ of F by

$$\langle f, \mu_\gamma \rangle = \frac{1}{T} \int_0^T f(s, \gamma(s), \dot\gamma(s)) \, ds$$

for each $f \in C$. Let

$$\Gamma := \{\mu_\gamma \mid \gamma \in C^1(\mathbb{R}, M) \text{ is periodic of integral period}\} \subset F.$$

The set $\mathcal{H}$ of holonomic probabilities is the closure of Γ in F. One can see that $\mathcal{H}$ is convex (cf. Mañé [4, Prop. 1.1(a)]). The elements μ of $\mathcal{H}$ satisfy $\langle 1, \mu \rangle = 1$ therefore we have $\pi(\mathcal{H}) \subset K$.

Note that each Tonelli Lagrangian L gives rise to an element of H_n^*.

Let $\mathfrak{M}(L)$ be the set of minimizing measures for L and let $\operatorname{supp}\mathfrak{M}(L)$ be the union of their supports. Recalling that we have defined $M_{H_n}(L)$ as the set of measures $\mu \in H_n$ which minimize the action $\int L \, d\mu$ on H_n, we have:

Lemma 9. *If L is a Tonelli lagrangian then there exists $n \in \mathbb{N}$ such that*

$$\dim \pi(M_{H_n}(L)) = \dim \mathfrak{M}(L).$$

Proof. Birkhoff theorem implies that $\mathfrak{M}(L) \subset \mathcal{H}$ (cf. Mañé [4, Prop. 1.1.(b)]). In [5, Prop. 4, p. 185] Mather proves that $\operatorname{supp}\mathfrak{M}(L)$ is compact, therefore $\mathfrak{M}(L) \subset H_n$ for some $n \in \mathbb{N}$.

In [4, §1] Mañé proves that minimizing measures are also all the minimizers of the action functional $A_L(\mu) = \int L \, d\mu$ on the set of holonomic measures, therefore $\mathfrak{M}(L) = M_{H_n}(L)$ for some $n \in \mathbb{N}$.

In [5, Th. 2, p. 186] Mather proves that the restriction $\operatorname{supp}\mathfrak{M}(L) \to M$ of the projection $TM \to M$ is injective. Therefore the linear map $\pi : \mathfrak{M}(L) \to G$ is injective, so that $\dim \pi(M_{H_n}(L)) = \dim \pi(\mathfrak{M}(L)) = \dim \mathfrak{M}(L)$. □

Proof of Theorem 1. Given $n \in \mathbb{N}$ apply Theorem 5 and obtain a residual subset $\mathcal{O}_n(A) \subset E$ such that

$$L \in A, \quad u \in \mathcal{O}_n(A) \implies \quad \dim \pi(M_{H_n}(L-u)) \leqslant \dim A.$$

Let $\mathcal{O}(A) = \cap_n \mathcal{O}_n(A)$. By the Baire property $\mathcal{O}(A)$ is residual. We have that

$$L \in A, \quad u \in \mathcal{O}(A), \quad n \in \mathbb{N} \implies \quad \dim \pi(M_{H_n}(L-u)) \leqslant \dim A.$$

Then by Lemma 9, $\dim \mathfrak{M}(L-u) \leqslant \dim A$ for all $L \in A$ and all $u \in \mathcal{O}(A)$. □

Appendix A. Convex functions

Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$, define its subdifferential as

$$\partial f(x) := \{\ell : \mathbb{R}^n \to \mathbb{R} \ \text{ linear} \mid f(y) \geqslant f(x) + \ell(y-x)\,,\ \forall y \in \mathbb{R}^n\}.$$

Then the sets $\partial f(x) \subset \mathbb{R}^n$ are convex. If $k \in \mathbb{N}$, let

$$\Sigma_k(f) := \{x \in \mathbb{R}^n \mid \dim \partial f(x) \geqslant k\}.$$

The following result is standard.

Proposition 10. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function then for all $0 \leqslant k \leqslant n$ the Hausdorff dimension $HD(\Sigma_k(f)) \leqslant n-k$.*

We recall here an elegant proof due to Ambrosio and Alberti; see [1]. Note that much more can be said on the structure of Σ_k, see [2], [9] for example.

By adding $|x|^2$ if necessary (which does not change Σ_k) we can assume that f is superlinear and that

$$f(y) \geqslant f(x) + \ell(y-x) + \tfrac{1}{2}\,|y-x|^2 \quad \forall x, y \in \mathbb{R}^n, \quad \forall \ell \in \partial f(x). \tag{3}$$

Lemma 11. $\qquad \ell \in \partial f(x), \quad \ell' \in \partial f(x') \implies |x - x'| \leqslant \|\ell - \ell'\|.$

Proof. From inequality (3) we have that

$$f(x') \geqslant f(x) + \ell(x'-x) \ + \ \tfrac{1}{2}\,|x'-x|^2\,,$$
$$f(x) \geqslant f(x') + \ell'(x-x') + \tfrac{1}{2}\,|x-x'|^2\,.$$

Then

$$0 \geqslant (\ell' - \ell)(x - x') + |x - x'|^2 \tag{4}$$

$$\|\ell - \ell'\| \, |x - x'| \geqslant (\ell - \ell')(x - x') \geqslant |x - x'|^2 . \tag{5}$$

Therefore $\|\ell - \ell'\| \geqslant |x - x'|$. □

Since f is superlinear, the subdifferential ∂f is surjective and we have:

Corollary 12. *There exists a Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\ell \in \partial f(x) \quad \Longrightarrow \quad x = F(\ell).$$

Proof of Proposition 10. Let A_k be a set with $HD(A_k) = n - k$ such that A_k intersects any convex subset of dimension k. For example,

$$A_k = \{x \in \mathbb{R}^n \,|\, x \text{ has at least } k \text{ rational coordinates}\}.$$

Observe that

$$x \in \Sigma_k \quad \Longrightarrow \quad \partial f(x) \text{ intersects } A_k \quad \Longrightarrow \quad x \in F(A_k).$$

Therefore $\Sigma_k \subset F(A_k)$. Since F is Lipschitz, we have that $HD(\Sigma_k) \leqslant HD(A_k) = n - k$. □

Universit Paris-Dauphine and CNRS, Paris, France
E-mail address: patrick.bernard@ceremade.dauphine.fr

CIMAT, Guanajuato, GTO, México
E-mail address: gonzalo@cimat.mx

References

[1] G. Alberti and L. Ambrosio, A geometrical approach to monotone functions in R^n, *Math. Z.* **230** (1999), 259–316.

[2] G. Alberti, On the structure of singular sets of convex functions, *Calc. Var. & Partial Differential Equations* **2** (1994), 17–27.

[3] R. Mañé, On the minimizing measures of Lagrangian dynamical systems, *Nonlinearity* **5** (1992), 623–638.

[4] ———, Generic properties and problems of minimizing measures of Lagrangian systems, *Nonlinearity* **9** (1996), 273–310.

[5] J. N. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, *Math. Z.* **207** (1991), 169–207.

[6] ———, Variational construction of connecting orbits, *Ann. Inst. Fourier (Grenoble)* **43** (1993), 1349–1386.

[7] ———, A property of compact, connected, laminated subsets of manifolds, *Ergodic Theory Dynam. Systems* **22** (2002), 1507–1520.

[8] ———, Examples of Aubry sets, *Ergodic Theory Dynamic. Systems* **24** (2004), 1667–1723.

[9] L. Zajček, On the points of multiplicity of monotone operators, *Comment. Math. Univ. Carolinae* **19** (1978), 179–189.

(Received February 22, 2006)