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An 8-dimensional nonformal, simply connected, symplectic manifold

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Abstract

We answer in the affirmative the question posed by Babenko and Taimanov [3] on the existence of nonformal, simply connected, compact symplectic manifolds of dimension 8.

1. Introduction

Simply connected compact manifolds of dimension less than or equal to 6 are formal [18], [11], and there are simply connected, compact manifolds of dimension greater than or equal to 7 which are nonformal [20], [10], [9], [6], [12]. If we are treating the symplectic case, the story is not so straightforward. Lupton and Oprea [15] conjectured that any simply connected, compact symplectic manifold is formal. Babenko and Taimanov [2], [3] disproved this conjecture giving examples of nonformal, simply connected, compact *symplectic* manifolds of any dimension bigger than or equal to 10, by using the symplectic blow-up [16]. They raise the question of the existence of nonformal, simply connected, compact, symplectic manifolds of dimension 8. The techniques of construction of symplectic manifolds used so far [1], [3], [6], [7], [11], [14], [21], [22] have not proved fruitful when addressing this problem. In this note, we answer the question in the affirmative by proving the following.

Theorem 1.1. *There is a simply connected, compact, symplectic manifold of dimension* 8 *which is nonformal.*

To construct such a manifold, we introduce a new technique to produce symplectic manifolds, which we hope can be useful for obtaining examples with interesting properties. We consider a nonformal, compact, symplectic 8-dimensional manifold with a symplectic nonfree action of a finite group such that the quotient space is a nonformal orbifold which is simply connected. Then we resolve symplectically the singularities to produce a *smooth* symplectic 8-manifold satisfying the required properties. The origin of the idea stems from our study of Guan's examples [13] of compact, holomorphic, symplectic manifolds which are not Kähler.

2. A simply-connected symplectic 8-manifold

Consider the complex Heisenberg group $H_{\mathbb{C}}$, that is, the complex, nilpotent Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $G = H_{\mathbb{C}} \times \mathbb{C}$, where $\mathbb{C}$ is the additive group of complex numbers. We denote by u_4 the coordinate function corresponding to this extra factor. In terms of the natural (complex) coordinate functions (u_1, u_2, u_3, u_4) on G, we have that the complex 1-forms $\mu = du_1$, $\nu = du_2$, $\theta = du_3 - u_2\, du_1$ and $\eta = du_4$ are left invariant, and

$$d\mu = d\nu = d\eta = 0, \quad d\theta = \mu \wedge \nu.$$

Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/3}$, and consider the discrete subgroup $\Gamma \subset G$ formed by the matrices in which $u_1, u_2, u_3, u_4 \in \Lambda$. We define the compact (parallelizable) nilmanifold

$$M = \Gamma \backslash G.$$

We can describe M as a principal torus bundle

$$T^2 = \mathbb{C}/\Lambda \hookrightarrow M \to T^6 = (\mathbb{C}/\Lambda)^3,$$

by the projection $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_4)$.

Now introduce the following action of the finite group $\mathbb{Z}_3$

$$\begin{aligned} \rho : G &\to G \\ (u_1, u_2, u_3, u_4) &\mapsto (\zeta\, u_1, \zeta\, u_2, \zeta^2\, u_3, \zeta\, u_4). \end{aligned}$$

This action satisfies $\rho(p \cdot q) = \rho(p) \cdot \rho(q)$, for $p, q \in G$, where the dot denotes the natural group structure of G. The map ρ is a particular case of a homothetic transformation (by ζ in this case) which is well defined for all nilpotent simply connected Lie groups with graded Lie algebra. Moreover $\rho(\Gamma) = \Gamma$; therefore ρ induces an action on the quotient $M = \Gamma \backslash G$. The action on the forms is given by

$$\rho^* \mu = \zeta\, \mu, \quad \rho^* \nu = \zeta\, \nu, \quad \rho^* \theta = \zeta^2\, \theta, \quad \rho^* \eta = \zeta\, \eta.$$

The complex 2-form

$$\omega = i\, \mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta} + i\, \eta \wedge \bar{\eta}$$

is actually a real form which is clearly closed and which satisfies $\omega^4 \neq 0$. Thus ω is a symplectic form on M. Moreover, ω is $\mathbb{Z}_3$-invariant. Hence the space

$$\widehat{M} = M/\mathbb{Z}_3$$

is a symplectic orbifold, with the symplectic form $\widehat{\omega}$ induced by ω. Our next step is to find a smooth symplectic manifold $\widetilde{M}$ that desingularises $\widehat{M}$.

PROPOSITION 2.1. *There exists a smooth, compact, symplectic manifold* $(\widetilde{M}, \widetilde{\omega})$ *which is isomorphic to* $(\widehat{M}, \widehat{\omega})$ *outside the singular points.*

Proof. Let $p \in M$ be a fixed point of the $\mathbb{Z}_3$-action. Translating by a group element $g \in G$ taking p to the origin, we may suppose that $p = (0,0,0,0)$ in our coordinates. At p, the symplectic form is

$$\omega_0 = i\, du_1 \wedge d\bar{u}_1 + du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3 + i\, du_4 \wedge d\bar{u}_4.$$

Take now $\mathbb{Z}_3$-equivariant Darboux coordinates around p,

$$\Phi\colon (B, \omega) \longrightarrow (B_{\mathbb{C}^4}(0, \varepsilon), \omega_0),$$

for some $\varepsilon > 0$. This means that $\Phi^*\omega_0 = \omega$ and $\Phi \circ \rho = d\rho_p \circ \Phi$, where we interpret $(B_{\mathbb{C}^4}(0, \varepsilon), \omega_0) \subset (T_pM, \omega_0) \cong (\mathbb{C}^4, \omega_0)$ in the natural way. (The proof of the existence of usual Darboux coordinates in [17, pp. 91–93] carrries over to this case, only we must be careful that all the objects constructed are $\mathbb{Z}_3$-equivariant.) We denote the new coordinates given by Φ as (u_1, u_2, u_3, u_4) again (although they are not the same coordinates as before).

Now introduce the new set of coordinates:

$$(w_1, w_2, w_3, w_4) = (u_1, \frac{1}{\sqrt{2}}(u_2 + \bar{u}_3), \frac{i}{\sqrt{2}}(u_3 - \bar{u}_2), u_4).$$

Then the symplectic form ω can be expressed as

$$\omega = i\,(dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2 + dw_3 \wedge d\bar{w}_3 + dw_4 \wedge d\bar{w}_4).$$

Moreover, with respect to these coordinates, the $\mathbb{Z}_3$-action ρ is given as

$$\rho(w_1, w_2, w_3, w_4) = (\zeta\, w_1, \zeta\, w_2, \zeta^2\, w_3, \zeta\, w_4).$$

With this Kähler model for a neighbourhood B of p, we may resolve the singularity of $B/\mathbb{Z}_3$ with a nonsingular Kähler model. Basically, blow up B at p to get $\widetilde{B}$. This replaces the point with a complex projective space $\mathbb{P}^3$ in which $\mathbb{Z}_3$ acts as

$$[w_1, w_2, w_3, w_4] \mapsto [\zeta\, w_1, \zeta\, w_2, \zeta^2\, w_3, \zeta\, w_4] = [w_1, w_2, \zeta\, w_3, w_4].$$

Therefore there are two components of the fix-point locus of the $\mathbb{Z}_3$-action on $\widetilde{B}$, namely the point $q = [0,0,1,0]$ and the complex projective plane $H = \{[w_1, w_2, 0, w_4]\} \subset F = \mathbb{P}^3$. Next blow up $\widetilde{B}$ at q and at H to get $\widetilde{\widetilde{B}}$. The point q is substituted by a projective space $H_1 = \mathbb{P}^3$. The normal bundle of $H \subset \widetilde{B}$ is the sum of the normal bundle of $H \subset F$, which is $\mathcal{O}_{\mathbb{P}^2}(1)$, and the restriction of the normal bundle of $F \subset \widetilde{B}$ to H, which is $\mathcal{O}_{\mathbb{P}^3}(-1)|_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-1)$. Therefore the second blow-up replaces the plane H by the $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ defined as $H_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. The strict transform of $F \subset \widetilde{B}$ under the second blow-up is the blow up $\widetilde{F}$ of $F = \mathbb{P}^3$ at q, which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, actually $\widetilde{F} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. See Figure 1 below.

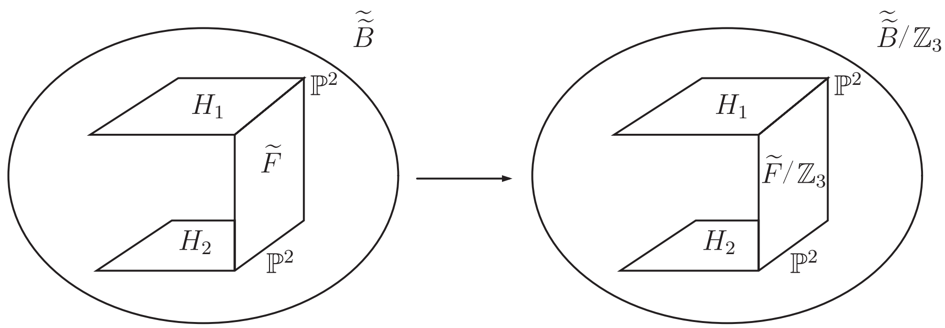

Figure 1: Desingularisation process

The fix-point locus of the $\mathbb{Z}_3$-action on $\widetilde{\widetilde{B}}$ are exactly the two disjoint divisors H_1 and H_2. Therefore the quotient $\widetilde{\widetilde{B}}/\mathbb{Z}^3$ is a smooth Kähler manifold [4, p. 82]. This provides a symplectic resolution of the singularity $B/\mathbb{Z}_3$. To glue this Kähler model to the symplectic form in the complement of the singular point we use Lemma 2.2 below. We do this at every fixed point to get a smooth symplectic resolution of $\widehat{M}$. $\boxed{QED}$

LEMMA 2.2. *Let* (B, ω_0) *be the standard Kähler ball in* $\mathbb{C}^n$, $n > 1$, *and let* Π *be a finite group acting linearly* (*by complex isometries*) *on* B *whose only fixed point is the origin. Let* $\phi : (\widetilde{B}, \omega_1) \to (B/\Pi, \omega_0)$ *be a Kähler resolution of the singularity of the quotient. Then there is a symplectic form* Ω *on* $\widetilde{B}$, *which coincides with* ω_0 *near the boundary, with a positive multiple of* ω_1 *near the exceptional divisor* $E = \phi^{-1}(0)$. *Moreover* Ω *is tamed by the complex structure.*

Proof. Since $\phi : (\widetilde{B}, \omega_1) \to (B/\Pi, \omega_0)$ is holomorphic, ω_0 and ω_1 are Kähler forms in $\widetilde{B} - E = B - \{0\}$ with respect to the same complex structure J. Therefore $(1-t)\omega_0 + t\omega_1$ is a Kähler form on $\widetilde{B}$, for any number $0 < t < 1$. (Note that $\omega_0|_E = 0$, where we denote again by ω_0 the pull-back to $\widetilde{B}$.)

Fix $\delta > 0$ small and let $A = \{z \in B \,|\, \delta < |z| < 2\delta\} \subset B$. Since A is simply connected, we may write $\omega_1 - \omega_0 = d\alpha$, with $\alpha \in \Omega^1(A)$, which we can furthermore suppose Π-invariant.

Let $\rho : [0, \infty) \to [0, 1]$ be a smooth function whose value is 1 for $r \leq 1.1\delta$ and 0 for $r \geq 1.9\delta$. Define

$$\Omega = \omega_0 + \varepsilon \, d(\rho(|z|)\alpha).$$

This equals ω_0 for $|z| \geq 1.9\delta$, and $\omega_0 + \varepsilon(\omega_1 - \omega_0) = (1-\varepsilon)\omega_0 + \varepsilon\, \omega_1$ for $|z| \leq 1.1\delta$. For $1.1\delta \leq |z| \leq 1.9\delta$, let $C > 0$ be a bound of $d(\rho\alpha)(u, Ju)$, for any u unitary tangent vector (with respect to the Kähler form ω_0). Choose $0 < \varepsilon < \min\{1, C^{-1}\}$. Then $\Omega(u, Ju) > 0$ for any nonzero u. $\boxed{QED}$

PROPOSITION 2.3. *The manifold $\widetilde{M}$ is simply connected.*

Proof. Fix the base points: let $p_0 \in M = \Gamma\backslash G$ be the image of $(0,0,0,0) \in G$ and let $\hat{p}_0 \in \widehat{M}$ be the image of p_0 under the projection $M \to \widehat{M}$. There is an epimorphism of fundamental groups

$$\Gamma = \pi_1(M) \twoheadrightarrow \pi_1(\widehat{M}),$$

since the $\mathbb{Z}_3$-action has a fixed point [5, Cor. 6.3]. Now the nilmanifold M is a principal 2-torus bundle over the 6-torus T^6, and so we have an exact sequence

$$\mathbb{Z}^2 \hookrightarrow \Gamma \to \mathbb{Z}^6.$$

Let $\bar{p}_0 = \pi(p_0)$, where $\pi\colon M \to T^6$ denotes the projection of the torus bundle. Clearly, $\mathbb{Z}_3$ acts on $\pi^{-1}(\bar{p}_0) \cong T^2 = \mathbb{C}/\Lambda$ with three fixed points, and the quotient space $T^2/\mathbb{Z}_3$ is a 2-sphere S^2. So the restriction to $\mathbb{Z}^2 = \pi_1(T^2) \subset \pi_1(M) = \Gamma$ of the map $\Gamma \twoheadrightarrow \pi_1(\widehat{M})$ factors through $\pi_1(T^2/\mathbb{Z}_3) = \{1\}$; hence it is trivial. Thus the map $\Gamma \twoheadrightarrow \pi_1(\widehat{M})$ factors through the quotient $\mathbb{Z}^6 \twoheadrightarrow \pi_1(\widehat{M})$. But M contains three $\mathbb{Z}_3$-invariant 2-tori, T_1, T_2 and T_3 (which are the images of $\{(u_1,0,0,0)\}$, $\{(0,u_2,0,0)\}$ and $\{(0,0,0,u_4)\}$, respectively) such that $\pi_1(\widehat{M})$ is generated by the images of $\pi_1(T_1)$, $\pi_1(T_2)$ and $\pi_1(T_3)$. Again, each quotient $T_i/\mathbb{Z}_3$ is a 2-sphere; hence $\pi_1(\widehat{M})$ is generated by $\pi_1(T_i/\mathbb{Z}_3) = \{1\}$, which proves that $\pi_1(\widehat{M}) = \{1\}$.

Finally, the resolution $\widetilde{M} \to \widehat{M}$ consists of substituting, for each singular point p, a neighbourhood $B/\mathbb{Z}_3$ of it by a nonsingular model $\widetilde{\widetilde{B}}/\mathbb{Z}_3$. The fiber over the origin of $\widetilde{\widetilde{B}}/\mathbb{Z}_3 \to B/\mathbb{Z}_3$ is simply connected: it consists of the union of the three divisors $H_1 = \mathbb{P}^3$, $H_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ and $\widetilde{F}/\mathbb{Z}_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3))$; all of them are simply connected spaces, and their intersection pattern forms no cycles (see Figure 1). Therefore, a simple Seifert-Van Kampen argument proves that $\widetilde{M}$ is simply connected

$\boxed{QED}$

LEMMA 2.4. *The odd degree Betti numbers of $\widetilde{M}$ are $b_1(\widetilde{M}) = b_3(\widetilde{M}) = b_5(\widetilde{M}) = b_7(\widetilde{M}) = 0$.*

Proof. As $\widetilde{M}$ is simply connected, $b_1(\widetilde{M}) = 0$. Next, using Nomizu's theorem [19] to compute the cohomology of the nilmanifold M, we easily find that $H^3(M) = W \oplus \overline{W}$, where

$$\begin{aligned} W = \langle & [\mu\wedge\bar{\mu}\wedge\eta], [\nu\wedge\bar{\nu}\wedge\eta], [\mu\wedge\bar{\nu}\wedge\eta], [\bar{\mu}\wedge\nu\wedge\eta], [\mu\wedge\eta\wedge\bar{\eta}], [\nu\wedge\eta\wedge\bar{\eta}], \\ & [\mu\wedge\nu\wedge\theta], [\mu\wedge\bar{\nu}\wedge\bar{\theta}], [\bar{\mu}\wedge\nu\wedge\bar{\theta}], [\mu\wedge\bar{\mu}\wedge\bar{\theta}], [\nu\wedge\bar{\nu}\wedge\bar{\theta}], [\mu\wedge\eta\wedge\theta], \\ & [\nu\wedge\eta\wedge\theta], [\bar{\mu}\wedge\eta\wedge\bar{\theta}], [\bar{\nu}\wedge\eta\wedge\bar{\theta}]\rangle \end{aligned}$$

and $\overline{W}$ is its complex conjugate. (Here $H^*(X)$ denotes cohomology with complex coefficients.) Clearly ρ acts as multiplication by ζ on W and as multiplication by $\zeta^2 = \bar{\zeta}$ on $\overline{W}$. Therefore $H^3(\widehat{M}) = H^3(M)^{\mathbb{Z}_3} = 0$.

The desingularisation process of Proposition 2.1 consists of removing contractible neighborhoods of the form $B_i/\mathbb{Z}_3$, $B_i \cong B_{\mathbb{C}^4}(0,\varepsilon)$, around each fixed point p_i, and inserting a nonsingular Kähler model $\widetilde{\widetilde{B}}_i/\mathbb{Z}_3$ which retracts to the "exceptional divisor" $E_i = \phi^{-1}(0)$, $\phi : \widetilde{\widetilde{B}}_i/\mathbb{Z}_3 \to B_i/\mathbb{Z}_3$. We glue along the region $A/\mathbb{Z}_3$ which retracts into $S^7/\mathbb{Z}_3$, a rational homology 7-sphere. An easy Mayer-Vietoris argument then shows that $H^j(\widetilde{M}) = H^j(\widehat{M}) \oplus (\bigoplus_i H^j(E_i))$ for $0 < j < 7$. All the E_i are diffeomorphic to the 6-dimensional complex manifold depicted in Figure 1, which consists of the union of $H_1 = \mathbb{P}^3$, $H_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ (a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$) and $\widetilde{F}/\mathbb{Z}_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3))$ (another $\mathbb{P}^1$-bundle over $\mathbb{P}^2$), intersecting in copies of the complex projective plane. So $H^3(E_i) = 0$ and hence $H^3(\widetilde{M}) = 0$.

The statement $b_5(\widetilde{M}) = b_7(\widetilde{M}) = 0$ follows from Poincaré duality. $\boxed{QED}$

3. Nonformality of the constructed manifold

Formality for a simply connected manifold M means that its rational homotopy type is determined by its cohomology algebra. Let us recall its definition (see [8], [22] for more details). Let X be a simply connected smooth manifold and consider its algebra of differential forms $(\Omega^*(X), d)$. Let $\psi : (\bigwedge V, d) \to (\Omega^*(X), d)$ be a minimal model for this algebra [8]. Then X is formal if there is a quasi-isomorphism $\psi' : (\bigwedge V, d) \to (H^*(X), d = 0)$, i.e. a morphism of differential algebras, inducing the identity on cohomology.

Lemma 3.1. *Let X be a simply connected smooth manifold with $H^3(X) = 0$, and let $a, x_1, x_2, x_3 \in H^2(X)$ be cohomology classes satisfying that $a \cup x_i = 0$, $i = 1, 2, 3$. Choose forms $\alpha, \beta_i \in \Omega^2(X)$ and $\xi_i \in \Omega^3(X)$, with $a = [\alpha]$, $x_i = [\beta_i]$ and $\alpha \wedge \beta_i = d\xi_i$, $i = 1, 2, 3$. If the cohomology class*

$$[\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] \in H^8(X) \tag{1}$$

is nonzero, then X is nonformal.

Proof. First, notice that

$$\begin{aligned} d(\xi_1 \wedge \xi_2 \wedge \beta_3 &+ \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2) \\ &= \alpha \wedge \beta_1 \wedge \xi_2 \wedge \beta_3 - \xi_1 \wedge \alpha \wedge \beta_2 \wedge \beta_3 \\ &\quad + \alpha \wedge \beta_2 \wedge \xi_3 \wedge \beta_1 - \xi_2 \wedge \alpha \wedge \beta_3 \wedge \beta_1 \\ &\quad + \alpha \wedge \beta_3 \wedge \xi_1 \wedge \beta_2 - \xi_3 \wedge \alpha \wedge \beta_1 \wedge \beta_2 = 0\,, \end{aligned}$$

so that (1) is a well-defined cohomology class.

Second, let us see that the cohomology class (1) does not depend on the particular forms $\alpha, \beta_i \in \Omega^2(X)$ and $\xi_i \in \Omega^3(X)$ chosen. If we write $a = [\alpha + df]$, with $f \in \Omega^1(X)$, then $(\alpha + df) \wedge \beta_i = d(\xi_i + f \wedge \beta_i)$ and

$$\begin{aligned}
&(\xi_1 + f \wedge \beta_1) \wedge (\xi_2 + f \wedge \beta_2) \wedge \beta_3 \\
&\quad + (\xi_2 + f \wedge \beta_2) \wedge (\xi_3 + f \wedge \beta_3) \wedge \beta_1 + (\xi_3 + f \wedge \beta_3) \wedge (\xi_1 + f \wedge \beta_1) \wedge \beta_2 \\
&= \xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2,
\end{aligned}$$

so the cohomology class (1) does not change by changing the representative of a. If we change the representatives of x_i, say for instance $x_1 = [\beta_1 + df]$, $f \in \Omega^1(X)$, then $\alpha \wedge (\beta_1 + df) = d(\xi_1 + \alpha \wedge f)$ and

$$\begin{aligned}
&(\xi_1 + \alpha \wedge f) \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge (\beta_1 + df) + \xi_3 \wedge (\xi_1 + \alpha \wedge f) \wedge \beta_2 \\
&\qquad = \xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2 + d(f \wedge \xi_2 \wedge \xi_3),
\end{aligned}$$

so the cohomology class (1) does not change again. Finally, if we change the form ξ_1 to $\xi_1 + g$, $g \in \Omega^3(X)$ closed, then

$$\begin{aligned}
&(\xi_1 + g) \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge (\xi_1 + g) \wedge \beta_2 \\
&\qquad = \xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2 + g \wedge (\xi_2 \wedge \beta_3 - \xi_3 \wedge \beta_2),
\end{aligned}$$

and $\xi_2 \wedge \beta_3 - \xi_3 \wedge \beta_2 \in \Omega^3(X)$ is closed, hence exact since $H^3(X) = 0$. Also in this case the cohomology class (1) does not change.

To see that X is nonformal, consider the minimal model $\psi : (\bigwedge V, d) \to (\Omega^*(X), d)$ for X. Then there are closed elements $\hat{a}, \hat{x}_i \in (\bigwedge V)^2$ whose images are 2-forms α, β_i representing a, x_i. Since $[\hat{a} \cdot \hat{x}_i] = 0$, there are elements $\hat{\xi}_i \in (\bigwedge V)^3$ such that $d\hat{\xi}_i = \hat{a} \cdot \hat{x}_i$. Let $\xi_i = \psi(\hat{\xi}_i) \in \Omega^3(X)$. Now, $d\xi_i = \alpha \wedge \beta_i$, $i = 1, 2, 3$.

If X is formal, then there exists a quasi-isomorphism $\psi' : (\bigwedge V, d) \to (H^*(X), 0)$. Note that $\psi'(\hat{\xi}_i) = 0$ since $H^3(X) = 0$. Then

$$[\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] = \psi'(\hat{\xi}_1 \wedge \hat{\xi}_2 \wedge \hat{x}_3 + \hat{\xi}_2 \wedge \hat{\xi}_3 \wedge \hat{x}_1 + \hat{\xi}_3 \wedge \hat{\xi}_1 \wedge \hat{x}_2) = 0,$$

contradicting our assumption. This proves that X is nonformal. $\boxed{QED}$

THEOREM 3.2. *The manifold $\widetilde{M}$ is nonformal.*

Proof. We start by considering the nilmanifold M. Consider the closed forms:

$$\alpha = \mu \wedge \bar{\mu}, \quad \beta_1 = \nu \wedge \bar{\nu}, \quad \beta_2 = \nu \wedge \bar{\eta}, \quad \beta_3 = \bar{\nu} \wedge \eta.$$

Then

$$\alpha \wedge \beta_1 = d(-\theta \wedge \bar{\mu} \wedge \bar{\nu}), \quad \alpha \wedge \beta_2 = d(-\theta \wedge \bar{\mu} \wedge \bar{\eta}), \quad \alpha \wedge \beta_3 = d(\bar{\theta} \wedge \mu \wedge \eta).$$

All the forms α, β_1, β_2, β_3, $\xi_1 = -\theta \wedge \bar{\mu} \wedge \bar{\nu}$, $\xi_2 = -\theta \wedge \bar{\mu} \wedge \bar{\eta}$ and $\xi_3 = \bar{\theta} \wedge \mu \wedge \eta$ are $\mathbb{Z}_3$-invariant. Hence they descend to the quotient $\widehat{M} = M/\mathbb{Z}_3$. Let $q : M \to \widehat{M}$

denote the projection, and define $\hat{\alpha} = q_*\alpha$, $\hat{\beta}_i = q_*\beta_i$, $\hat{\xi}_i = q_*\xi_i$, $i = 1, 2, 3$. Now take a $\mathbb{Z}_3$-equivariant map $\varphi : M \to M$ which is the identity outside some small balls around the fixed points, and contracts some smaller balls into the fixed points. This induces a map $\hat{\varphi} : \widehat{M} \to \widehat{M}$ such that $\hat{\varphi} \circ q = q \circ \varphi$. The forms $\tilde{\alpha} = \hat{\varphi}^*\hat{\alpha}$, $\tilde{\beta}_i = \hat{\varphi}^*\hat{\beta}_i$, $\tilde{\xi}_i = \hat{\varphi}^*\hat{\xi}_i$, $i = 1, 2, 3$, are zero in a neighbourhood of the fixed points; therefore they define forms on $\widetilde{M}$, by extending them by zero along the exceptional divisors. Note that $\tilde{\alpha}, \tilde{\beta}_i \in \Omega^2(\widetilde{M})$ are closed forms and $\tilde{\xi}_i \in \Omega^3(\widetilde{M})$ satisfies $d\tilde{\xi}_i = \tilde{\alpha} \wedge \tilde{\beta}_i$, $i = 1, 2, 3$.

By Lemma 2.4, $H^3(\widetilde{M}) = 0$, so we may apply Lemma 3.1 to the cohomology classes $a = [\alpha], b_i = [\beta_i] \in H^2(\widetilde{M})$, $i = 1, 2, 3$. The cohomology class

$$\begin{aligned}
&[\tilde{\xi}_1 \wedge \tilde{\xi}_2 \wedge \tilde{\beta}_3 + \tilde{\xi}_2 \wedge \tilde{\xi}_3 \wedge \tilde{\beta}_1 + \tilde{\xi}_3 \wedge \tilde{\xi}_1 \wedge \tilde{\beta}_2] \\
&\quad = [\hat{\varphi}^* q_*(\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2)] \\
&\quad = \hat{\varphi}^* q_*(2[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}]) \neq 0,
\end{aligned}$$

since its integral is

$$\begin{aligned}
&\int_{\widetilde{M}} \hat{\varphi}^* q_*(2[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}]) \\
&\qquad = \int_{\widehat{M}} \hat{\varphi}^* q_*(2[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}]) \\
&\qquad = 3 \int_M \varphi^*(2[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}]) \\
&\qquad = 6 \int_M [\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}] \neq 0\,.
\end{aligned}$$

By Lemma 3.1, $\widetilde{M}$ is nonformal. $\boxed{QED}$

Remark 3.3. The symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ is not hard-Lefschetz. The $\mathbb{Z}_3$-invariant form $\nu \wedge \bar{\nu}$ on M is not exact, but $\omega^2 \wedge \nu \wedge \bar{\nu} = 2d(\theta \wedge \bar{\mu} \wedge \bar{\eta} \wedge \eta \wedge \bar{\nu})$. This form descends to the quotient $\widehat{M}$ and can be extended to $\widetilde{M}$ via the process done at the end of the proof of the previous theorem. Therefore the map $[\omega]^2 \colon H^2(\widetilde{M}) \to H^6(\widetilde{M})$ is not injective.

Cavalcanti [7] gave the first examples of simply connected compact symplectic manifolds of dimension ≥ 10 which are hard Lefschetz and nonformal. Yet examples of nonformal simply connected compact symplectic 8-manifolds satisfying the hard Lefschetz property have not been constructed.

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References

[1] I. K. Babenko and I. A. Taimanov, On the formality problem for symplectic manifolds, *Contemp. Math.* **288** (2001), 1–9.

[2] ———, On existence of nonformal simply connected symplectic manifolds, *Russ. Math. Surv.* **53** (1998), 1082–1083.

[3] ———, On nonformal simply connected symplectic manifolds, *Siberian Math. J.* **41** (2000), 204–217.

[4] W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, New York, 1984.

[5] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[6] G. R. Cavalcanti, Formality of k-connected spaces in $4k+3$ and $4k+4$ dimensions, *Math. Proc. Cambridge Philos. Soc.* **141** (2006), 101–112.

[7] ———, The Lefschetz property, formality and blowing up in symplectic geometry, *Trans. Amer. Math. Soc.* **359** (2007), 333–348.

[8] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, Real homotopy theory of Kähler manifolds, *Invent. Math.* **29** (1975), 245–274.

[9] A. Dranishnikov and Y. Rudyak, Examples of nonformal closed $(k-1)$-connected manifolds of dimensions $4k-1$ and more, *Proc. Amer. Math. Soc.* **133** (2005), 1557–1561.

[10] M. Fernández and V. Muñoz, On nonformal simply connected manifolds, *Topol. Appl.* **135** (2004), 111–117.

[11] ———, Formality of Donaldson submanifolds, *Math. Zeit.* **250** (2005), 149–175.

[12] ———, Non-formal compact manifolds with small Betti numbers, *Proc. Conf. Contemporary Geometry and Related Topics* (Belgrade 2005), Public. Faculty of Math., University of Belgrade (2006), 231–246.

[13] D. Guan, Examples of compact holomorphic symplectic manifolds which are not Kählerian II, *Invent. Math.* **121** (1995), 135–145.

[14] R. Ibáñez, Y. Rudyak, A. Tralle, and L. Ugarte, On certain geometric and homotopy properties of closed symplectic manifolds, *Topol. Appl.* **127** (2002), 33–45.

[15] G. Lupton and J. Oprea, Symplectic manifolds and formality, *J. Pure Appl. Algebra* **91** (1994), 193–207.

[16] D. McDuff, Examples of symplectic simply connected manifolds with no Kähler structure, *J. Differential Geom.* **20** (1984), 267–277.

[17] D. McDuff and D. Salamon, *Introduction to Symplectic Geometry*, second edition, *Oxford Math. Monographs*, Clarendon Press, Oxford, 1998.

[18] J. Neisendorfer and T. J. Miller, Formal and coformal spaces. *Illinois J. Math.* **22** (1978), 565–580.

[19] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* **59** (1954), 531–538.

[20] J. Oprea, The Samelson space of a fibration, *Mich. Math. J.* **34** (1987), 127–141.

[21] Y. Rudyak and A. Tralle, On Thom spaces, Massey products and nonformal symplectic manifolds, *Internat. Math. Res. Notices* **10** (2000), 495–513.

[22] A. Tralle and J. Oprea, *Symplectic manifolds with no Kähler structure*, *Lecture Notes in Math*. **1661**, Springer-Verlag, New York, 1997.

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