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On the classification problem for nuclear C^*-algebras

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Abstract

We exhibit a counterexample to Elliott's classification conjecture for simple, separable, and nuclear C*-algebras whose construction is elementary, and demonstrate the necessity of extremely fine invariants in distinguishing both approximate unitary equivalence classes of automorphisms of such algebras and isomorphism classes of the algebras themselves. The consequences for the program to classify nuclear C*-algebras are far-reaching: one has, among other things, that existing results on the classification of simple, unital AH algebras via the Elliott invariant of K-theoretic data are the best possible, and that these cannot be improved by the addition of continuous homotopy invariant functors to the Elliott invariant.

1. Introduction

Elliott's program to classify nuclear C*-algebras via K-theoretic invariants (see [E2] for an overview) has met with considerable success since his seminal classification of approximately finite-dimensional (AF) algebras via their scaled, ordered K_0-groups ([E1]). Classification results of this nature are *existence theorems* asserting that isomorphisms at the level of certain invariants for C*-algebras in a class $\mathcal{B}$ are liftable to $*$-isomorphisms at the level of the algebras themselves. Obtaining such theorems usually requires proving a *uniqueness theorem* for $\mathcal{B}$, i.e., a theorem which asserts that two $*$-isomorphisms between members A and B of $\mathcal{B}$ which agree at the level of the said invariants differ by a locally inner automorphism.

Elliott's program began in earnest with his classification of simple circle algebras of real rank zero in 1989 — he conjectured shortly thereafter that the topological K-groups, the Choquet simplex of tracial states, and the natural connections between these objects would form a complete invariant for the class of separable, nuclear C*-algebras. This invariant came to be known simply as the Elliott invariant, denoted by Ell($\bullet$). Elliott's conjecture held in the case of simple algebras throughout the 1990s, during which time several spectacular classification results were obtained: the Kirchberg-Phillips classification of simple, separable, nuclear, and purely infinite (Kirchberg) C*-algebras satisfying the Universal Coefficient Theorem, the Elliott-Gong-Li classification of simple unital AH algebras of very slow dimension growth, and Lin's classification of tracially AF algebras (see [K], [EGL], and [L], respectively).

In 2002, however, Rørdam constructed a simple, nuclear C*-algebra containing both a finite and an infinite projection ([R1]). Apart from answering negatively the question of whether simple, nuclear C*-algebras have a type decomposition similar to that of factors, his example provided the first counterexample to Elliott's conjecture in the simple nuclear case; it had the same Elliott invariant as a Kirchberg algebra — its tensor product with the Jiang-Su algebra $\mathcal{Z}$, to be precise — yet was not purely infinite. It could, however, be distinguished from its Kirchberg twin by its (nonzero) real rank ([R4]).

Later in the same year, the present author found independently a simple, nuclear, separable and stably finite counterexample to Elliott's conjecture ([T]). This algebra could again be distinguished from its tensor product with the Jiang-Su algebra $\mathcal{Z}$ by its real rank. These examples made it clear that the Elliott conjecture would not hold at its boldest, but the question of whether the addition of some small amount of new information to $\mathrm{Ell}(\bullet)$ could repair the defect in Elliott's conjecture remained unclear. The counterexamples above suggested the addition of the real rank, and such a modification would not have been without precedent: the discovery that the pairing between traces and the K_0-group was necessary for determining the isomorphism class of a nuclear C*-algebra was unexpected, yet the incorporation of this object into the Elliott invariant led to the classification of approximately interval (AI) algebras ([E3]).

The sequel clarifies the nature of the information not captured by the Elliott invariant. We exhibit a pair of simple, separable, nuclear, and nonisomorphic C*-algebras which agree not only on $\mathrm{Ell}(\bullet)$, but also on a host of other invariants including the real rank and continuous (with respect to inductive sequences) homotopy invariant functors. The Cuntz semigroup, employed to distinguish our algebras, is thus the minimum quantity by which the Elliott invariant must be enlarged in order to obtain a complete invariant, but we shall see that the question of range for this semigroup is out of reach. Any classification result for C*-algebras which includes this semigroup as part of the invariant will therefore lack the impact of the Elliott program's successes — the latter are always accompanied by range-of-invariant results. Our aim, however, is not to discourage work on the classification program. It is to demonstrate unequivocally the need for a new regularity assumption in Elliott's program, as opposed to an expansion of the invariant.

Let $\mathcal{F}$ denote the following collection of invariants for C*-algebras:

- all homotopy invariant functors from the category of C*-algebras which commute with countable inductive limits;

- the real rank (denoted by $\mathrm{rr}(\bullet)$);
- the stable rank (denoted by $\mathrm{sr}(\bullet)$);
- the Hausdorffized algebraic K_1-group;
- the Elliott invariant.

Let $\mathcal{F}_{\mathbf{R}}$ be the subcollection of $\mathcal{F}$ obtained by removing those continuous and homotopy invariant functors which do not have ring modules as their target category.

Our main results are:

THEOREM 1.1. *There exists a simple, separable, unital, and nuclear C^*-algebra A such that for any* UHF *algebra $\mathcal{U}$ and any $F \in \mathcal{F}$ one has*

$$F(A) \cong F(A \otimes \mathcal{U}),$$

yet A and $A \otimes \mathcal{U}$ are not isomorphic. A is moreover an approximately homogeneous (AH) *algebra, and $A \otimes \mathcal{U}$ is an approximately interval* (AI) *algebra.*

THEOREM 1.2. *There exist a simple, separable, unital, and nuclear C^*-algebra B and an automorphism α of B of period two such that α induces the identity map on $F(B)$ for every $F \in \mathcal{F}_{\mathbf{R}}$, yet α is not locally inner.*

Thus, both existence and uniqueness fail for simple, separable, and nuclear C*-algebras despite the scope of $\mathcal{F}$.

Recall that a C*-algebra A is said to be $\mathcal{Z}$-stable if it absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially, i.e., $A \otimes \mathcal{Z} \cong A$. ($\mathcal{Z}$-stability is the regularity property alluded to above.) Theorem 1.1, or rather, its proof, has two immediate corollaries which are of independent interest.

Corollary 1.1. *There exists a simple, separable, and nuclear C^*-algebra with unperforated ordered* K_0*-group whose Cuntz semigroup fails to be almost unperforated.*

Corollary 1.2. *Say that a simple, separable, nuclear, and stably finite C^*-algebra has property* (M) *if it has stable rank one, weakly unperforated topological K-groups, weak divisibility, and property* (SP). *Then,* (M) *is strictly weaker than $\mathcal{Z}$-stability.*

Corollary 1.1 follows from the proof of Theorem 1.1, while Corollary 1.2 follows from Corollary 1.1 and Theorem 4.5 of [R3].

The counterexample to the Elliott conjecture constituted by Theorem 1.1 is more powerful and succinct than those of [R1] or [T]: A and $A \otimes \mathcal{U}$ agree on the distinguishing invariant for the counterexamples of [R1] and [T] and a host of others including K-theory with coefficients mod p, the homotopy groups of the unitary group, the stable rank, and all σ-additive homologies and cohomologies from the category of nuclear C*-algebras (cf. [B]); A and $A \otimes \mathcal{U}$ are simple, unital inductive limits of homogeneous algebras with contractible spectra, a class of algebras which forms the weakest and most natural extension of the very slow dimension growth AH algebras classified in [EGL]; both A and $A \otimes \mathcal{U}$ are stably finite, weakly divisible, and have property (SP), minimal stable rank, and next-to-minimal real rank; the proof of the theorem is elementary compared to the intricate constructions of [R1] and [T], and demonstrates the necessity of a distinguishing invariant for which no range results can be expected. Furthermore, one has in Theorem 1.2 a companion lack-of-uniqueness result. Together with Theorem 1.1, this yields what might be called a categorical counterexample — the structure of the category whose objects are isomorphism classes of simple, separable, nuclear, stably finite C*-algebras (let alone just nuclear algebras) and whose morphisms are locally inner equivalence classes of $*$-isomorphisms cannot be determined by $\mathcal{F}$.

The paper is organized as follows: Section 2 fixes notation and reviews the definition of the Cuntz semigroup $W(\bullet)$; in Section 3 we prove Theorem 1.1; in Section 4 we prove Theorem 1.2; Section 5 demonstrates the complexity of the Cuntz semigroup, and discusses the relevance of $\mathcal{Z}$-stability to the classification program.

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2. Preliminaries

For the remainder of the paper, let M_n denote the $n \times n$ matrices with complex entries, and let $\mathrm{C}(X)$ denote the continuous complex-valued functions on a topological space X.

Let A be a C*-algebra. We recall the definition of the Cuntz semigroup $W(A)$ from [C]. (Our synopsis is essentially that of [R3].) Let $\mathrm{M}_n(A)^+$ denote the positive elements of $\mathrm{M}_n(A)$, and let $\mathrm{M}_\infty(A)^+$ be the disjoint union $\cup_{i=n}^\infty \mathrm{M}_n(A)^+$. For $a \in \mathrm{M}_n(A)^+$ and $b \in \mathrm{M}_m(A)^+$ set $a \oplus b = \mathrm{diag}(a, b) \in \mathrm{M}_{n+m}(A)^+$, and write $a \preceq b$ if there is a sequence $\{x_k\}$ in $\mathrm{M}_{m,n}(A)$ such that $x_k^* b x_k \to a$. Write $a \sim b$ if $a \preceq b$ and $b \preceq a$. Put $W(A) = \mathrm{M}_\infty(A)^+/\sim$, and let $\langle a \rangle$ be the equivalence class containing a. Then, $W(A)$ is a positive ordered abelian semigroup when equipped with the relations:

$$\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle, \qquad \langle a\rangle \leq \langle b\rangle \Longleftrightarrow a \preceq b, \qquad a, b \in \mathrm{M}_\infty(A)^+.$$

The relation $\preceq$ reduces to Murray-von Neumann comparison when a and b are projections.

We will have occasion to use the following simple lemma in the sequel:

LEMMA 2.1. *Let p and q be projections in a C^*-algebra D such that*

$$||xpx^* - q|| < 1/2$$

for some $x \in D$. Then, q is equivalent to a subprojection of p.

Proof. We have that

$$\sigma(xpx^*) \subseteq (-1/2, 1/2) \cup (1/2, 3/2),$$

and that $\sigma(xpx^*)$ contains at least one point from $(1/2, 3/2)$. The C*-algebra generated by xpx^* contains a nonzero projection, say r, represented (via the functional calculus) by the function $r(t)$ on $\sigma(xpx^*)$ which is zero when $t \in (-1/2, 1/2)$ and one otherwise. This projection is dominated by

$$2xpx^* = \sqrt{2}xpx^*\sqrt{2}.$$

By the functional calculus one has $||xpx^* - r|| < 1/2$, so that $||r - q|| < 1$. Thus, r and q are Murray-von Neumann equivalent. By the definition of Cuntz equivalence we have $\sqrt{2}xpx^*\sqrt{2} \preceq p$, so that $q \sim r \preceq p$ by transitivity. Cuntz comparison agrees with Murray-von Neumann comparison on projections, and the lemma follows. □

3. The proof of Theorem 1.1

Proof. We construct A as an inductive limit $\lim_{i\to\infty}(A_i, \phi_i)$ where, for each $i \in \mathbb{N}$, A_i is of the form

$$\mathrm{M}_{k_i} \otimes \mathrm{C}\left([0,1]^{6(\Pi_{j\leq i} n_j)}\right), \quad n_i, k_i \in \mathbb{N},$$

and ϕ_i is a unital $*$-homomorphism. Our construction is essentially that of [V1]. Put $k_1 = 4$, $n_1 = 1$, and $N_i = \Pi_{j\leq i} n_j$. Let

$$\pi_l^i : [0,1]^{6N_i} \to [0,1]^{6N_{i-1}}, \quad l \in \{1, \ldots, n_i\},$$

be the co-ordinate projections, and let $f \in A_{i-1}$. Define ϕ_{i-1} by

$$\phi_{i-1}(f)(x) = \mathrm{diag}\left(f(\pi_1^i(x)), \ldots, f(\pi_{n_i}^i(x)), f(x_1^{i-1}), \ldots, f(x_{m_i}^{i-1})\right),$$

where $x_1^{i-1}, \ldots, x_{m_i}^{i-1}$ are points in $X_{i-1} \stackrel{\mathrm{def}}{=} [0,1]^{6N_{i-1}}$. With $m_i = i$, the $x_1^{i-1}, \ldots, x_{m_i}^{i-1}$, $i \in \mathbb{N}$, can be chosen so as to make $\lim_{i\to\infty}(A_i, \phi_i)$ simple (cf. [V2]). The multiplicity of ϕ_{i-1} is $n_i + m_i$ by construction. We impose two conditions on the n_i and m_i: first, $n_i \gg m_i$ as $i \to \infty$, and second, given any natural number r, there is an $i_0 \in \mathbb{N}$ such that r divides $n_{i_0} + m_{i_0}$.

Note that $(\mathrm{K}_0 A_i, \mathrm{K}_0^+ A_i, [1_{A_i}]) = (\mathbb{Z}, \mathbb{Z}^+, k_i)$ since X_i is contractible for all $i \in \mathbb{N}$. The second condition on the n_i above implies that

$$(\mathrm{K}_0 A, \mathrm{K}_0 A^+, [1_A]) = \lim_{i \to \infty} (\mathrm{K}_0 A_i, \mathrm{K}_0 A_i^+, [1_{A_i}]) \cong (\mathbb{Q}, \mathbb{Q}^+, 1).$$

Since $\mathrm{K}_1 A_i = 0$, $i \in \mathbb{N}$, we have $\mathrm{K}_1 A = 0$. Thus, A has the same Elliott invariant as some AI algebra, say B. Tensoring A with a UHF algebra $\mathfrak{U}$ does not disturb the K_0-group or the tracial simplex ($\mathfrak{U}$ has a unique normalized tracial state). The tensor product $A \otimes \mathfrak{U}$ is a simple, unital AH algebra with very slow dimension growth in the sense of [EGL], and is thus isomorphic to B by the classification theorem of [EGL].

Let us now prove that A and B are shape equivalent. By the range-of-invariant theorem of [Th] we may write B as an inductive limit of full matrix algebras over the closed unit interval (as opposed to direct sums of such), say

$$B \cong \lim_{i \to \infty} (B_i, \psi_i).$$

From K-theory considerations we may assume that $B_i = \mathrm{M}_{k_i} \otimes \mathrm{C}([0,1])$, i.e., that the dimension of the unit of B_i is the same as the dimension of the unit of A_i. Let $s_i = \mathrm{mult}\phi_i = \mathrm{mult}\psi_i$. Define maps

$$\eta_i : A_i \to B_{i+1}, \quad \eta_i(f) = \bigoplus_{j=1}^{s_i} f((0, \ldots, 0))$$

and

$$\gamma_i : B_i \to A_i, \quad \gamma_i(g) = g(0).$$

Both $\gamma_{i+1} \circ \eta_i$ and $\eta_i \circ \gamma_{i-1}$ are diagonal maps, and so are homotopic to ϕ_i and ψ_i, respectively, since $[0,1]$ and X_i are contractible.

Finally, A has stable rank one and real rank one by [V2], and therefore so also does B.

To complete the proof of the theorem, we must show that A and B are nonisomorphic. Since B is approximately divisible, we have that $W(B)$ is almost unperforated, i.e., if $mx \precsim ny$ for natural numbers $m > n$ and elements $x, y \in W(B)$, then $x \precsim y$ ([R2]). We claim that the Cuntz semigroup of A fails to be almost unperforated. We proceed by extending Villadsen's Euler class obstruction argument (cf. [V1], [V2]) to positive elements of a particular form.

To show that $W(A)$ fails to be almost unperforated, it will suffice to exhibit positive elements $x, y \in A_1$ such that, for all $i \in \mathbb{N}$, for some $\delta > 0$

$$m\langle \phi_{1i}(x) \rangle \precsim n\langle \phi_{1i}(y) \rangle, \quad m > n, \quad m, n \in \mathbb{N}$$

and

$$||r\phi_{1i}(y)r^* - \phi_{1i}(x)|| > \delta, \quad \forall r \in A_i, \quad \forall i \in \mathbb{N}.$$

The second statement is stronger than the requirement that $\langle \phi_{1i}(x) \rangle$ is not less than $\langle \phi_{1i}(y) \rangle$ in $W(A_i)$, since $W(\bullet)$ does not commute with inductive limits. Clearly, we need only establish this second statement over some closed subset Y of the spectrum of A_i.

If $a \in \mathrm{M}_n \otimes \mathrm{C}(X)$ is a constant positive element and X is compact, then $\langle a \rangle$ is the class of a projection in $W(\mathrm{M}_n \otimes \mathrm{C}(X))$. Indeed, a is unitarily equivalent (hence Cuntz equivalent) to a diagonal positive element:

$$uau* = \mathrm{diag}(a_1, \ldots, a_m, 0, \ldots, 0), \quad \text{some } u \in \mathcal{U}(\mathrm{M}_n),$$

where $a_l \neq 0$, $l \in \{1, \ldots, m\}$. Let $r = \mathrm{diag}(a_1^{-1}, \ldots, a_m^{-1}, 0, \ldots, 0)$. Then,

$$r^{1/2} uau * r^{1/2} = (r^{1/2}u)a(r^{1/2}u)^* = \mathrm{diag}(\underbrace{1, \ldots, 1}_{m \text{ times}}, 0, \ldots, 0).$$

Set

$$S \overset{\mathrm{def}}{=} \left\{ \overline{x} \in [0,1]^3 \ : \ \frac{1}{8} < \mathrm{dist}\left(\overline{x}, \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right) < \frac{3}{8} \right\}.$$

Note that $\mathrm{M}_4(\mathrm{C}_0(S \times S))$ is a hereditary subalgebra of A_1. Let ξ be a line bundle over S^2 with nonzero Euler class (the Hopf line bundle, for instance). Let θ_1 denote the trivial line bundle. By Lemma 1 of [V2], we have that θ_1 is not a sub-bundle of $\xi \times \xi$ over $\mathrm{S}^2 \times \mathrm{S}^2$. Both $\xi \times \xi$ and θ_1 can be considered as projections in $\mathrm{M}_4(\mathrm{S}^2 \times \mathrm{S}^2)$. By Lemma 2.1 we have

$$||x(\xi \times \xi)x^* - \theta_1|| \geq 1/2, \ \forall x \in \mathrm{M}_4(\mathrm{S}^2 \times \mathrm{S}^2).$$

On the other hand, the stability properties of vector bundles imply that

$$11\langle \theta_1 \rangle \leq 10 \langle \xi \times \xi \rangle.$$

Consider the closure S^- of $S \subseteq [0,1]^3$, and let τ be the projection of S^- onto

$$S_{1/4} \overset{\mathrm{def}}{=} \left\{ \overline{x} \in S \ : \ \mathrm{dist}\left(\overline{x}, \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right) = \frac{1}{4} \right\} \subseteq [0,1]^3$$

along rays emanating from $(1/2, 1/2, 1/2) \in [0,1]^3$. Let $\tau^*(\xi)$ be the pullback of ξ via τ. Fix a positive scalar function $f \in A_1$ of norm one which is equal to $1 \in \mathrm{M}_4$ on $S_{1/4} \times S_{1/4}$ and has support $S \times S$. It follows that $f(\tau^*(\xi) \times \tau^*(\xi))$ $\in A_1$. By Lemma 2.1 we have

$$||xf(\tau^*(\xi) \times \tau^*(\xi))x^* - f\theta_1|| \geq 1/2$$

for any $x \in A_1$ — one simply restricts to $S_{1/4} \times S_{1/4} \subseteq S \times S$. We may pull the inequality

$$11\langle \theta_1 \rangle \leq 10 \langle \xi \times \xi \rangle.$$

back via τ to conclude that

$$11\langle\theta_1\rangle \leq 10\langle\tau^*(\xi) \times \tau^*(\xi)\rangle.$$

This last inequality is equivalent to the existence of a sequence (r_j) in the appropriately sized matrix algebra over $\mathrm{C}(S^- \times S^-)$ with the property that

$$r_j \left(\oplus_{i=1}^{10} \tau^*(\xi) \times \tau^*(\xi)\right) r_j^* \stackrel{j\to\infty}{\longrightarrow} \theta_{11}.$$

Since f is central in $\mathrm{C}_0(S \times S)$, we have that

$$r_j \left(\oplus_{i=1}^{10} f(\tau^*(\xi) \times \tau^*(\xi))\right) r_j^* \stackrel{j\to\infty}{\longrightarrow} f\theta_{11}.$$

In other words,

$$11\langle f\theta_1\rangle \leq 10\langle f(\tau^*(\xi) \times \tau^*(\xi))\rangle$$

and $W(A_1)$ fails to be weakly unperforated.

Since

$$11\langle\phi_{1i}(f\theta_1)\rangle \leq 10\langle\phi_{1i}(f(\tau^*(\xi) \times \tau^*(\xi)))\rangle$$

via $\phi_{1i}(r_j)$, we need only show that

$$||x\phi_{1i}(f(\tau^*(\xi) \times \tau^*(\xi)))x^* - \phi_{1i}(f\theta_1)||\geq 1/2$$

for each natural number i and any $x \in A_i$. Fix i. One can easily verify that the restriction of $\phi_{1i}(f \cdot \tau^*(\xi) \times \tau^*(\xi))$ to $(S^-)^{2N_i} \subseteq [0,1]^{6N_i}$ is

$$(\tau^*(\xi) \times \tau^*(\xi))^{\times N_i} \oplus f_{\theta_l},$$

where f_{θ_l} is a constant positive element of rank l (hence Cuntz equivalent to θ_l), and the direct sum decomposition separates the summands of ϕ_{i-1} which are point evaluations from those which are not. The similar restricted decomposition of $\phi_{1i}(f \cdot \theta_1)$ is

$$\theta_{k-l/2} \oplus g_{\theta_{l/2}},$$

where $g_{\theta_{l/2}}$ is a constant positive element Cuntz equivalent to a trivial projection of dimension $l/2$, and k is greater than $3l/2$ (this last inequality follows from the fact that $n_i \gg m_i$). Suppose that there exists $x \in A_i|_{(S^-)^{2N_i}}$ such that

$$||x((\tau^*(\xi) \times \tau^*(\xi))^{\times N_i} \oplus f_{\theta_l})x^* - \theta_{k-l/2} \oplus g_{\theta_{l/2}}|| < 1/2.$$

Recall that

$$(\tau^*(\xi) \times \tau^*(\xi))^{\times N_i} \oplus f_{\theta_l} = a((\tau^*(\xi) \times \tau^*(\xi))^{\times N_i} \oplus \theta_l)a$$

for some positive $a \in A_i$. Cutting down by $\theta_{k-l/2}$, we have

$$||\theta_{k-l/2}xa((\tau^*(\xi) \times \tau^*(\xi))^{\times N_i} \oplus \theta_l)ax^*\theta_{k-l/2} - \theta_{k-l/2}|| < 1/2.$$

By Lemma 2.1, we must conclude that

$$\theta_{k-l/2} \preceq (\tau^*(\xi) \times \tau^*(\xi))^{\times N_i} \oplus \theta_l$$

over $(S^-)^{2N_i}$. But this is impossible by Lemma 1 of [V2]. Hence

$$||x(\phi_{1i}(f \cdot \tau^*(\xi) \times \tau^*(\xi)))x^* - \phi_{1i}(f \cdot \theta_1)|| \geq 1/2 \ \forall x \in A_i,$$

as desired. □

4. The proof of Theorem 1.2

Proof. We perturb the construction of a simple, unital AH algebra by Villadsen ([V1]) to obtain the algebra B of Theorem 1.2, and construct α as an inductive limit automorphism. Let X and Y be compact connected Hausdorff spaces, and let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable Hilbert space. Projections in the C*-algebra $\mathrm{C}(Y) \otimes \mathcal{K}$ can be identified with finite-dimensional complex vector bundles over Y, and two such bundles are stably isomorphic if and only if the corresponding projections in $\mathrm{C}(Y) \otimes \mathcal{K}$ have the same K_0-class.

Given a set of mutually orthogonal projections

$$P = \{p_1, \ldots, p_n\} \subseteq \mathrm{C}(Y) \otimes \mathcal{K}$$

and continuous maps $\lambda_i : Y \to X$, $1 \leq i \leq n$, one may define a $*$-homomorphism

$$\lambda : \mathrm{C}(X) \to \mathrm{C}(Y) \otimes \mathcal{K}, \quad f \to \bigoplus_{i=1}^{n} (f \circ \lambda_i) p_i.$$

A $*$-homomorphism of this form is called *diagonal*. We say that λ comes from the set $\{(\lambda_i, p_i)\}_{i=1}^n$.

Let I denote the closed unit interval in $\mathbb{R}$, and put

$$X_i = \mathrm{I} \times \mathrm{CP}^{\sigma(1)} \times \mathrm{CP}^{\sigma(2)} \times \cdots \times \mathrm{CP}^{\sigma(i)},$$

where the $\sigma(i)$ are natural numbers to be specified. Let

$$\pi^1_{i+1} : X_{i+1} \to X_i; \quad \pi^2_{i+1} : X_{i+1} \to \mathrm{CP}^{\sigma(i+1)}$$

be the co-ordinate projections. Let $B_i = p_i(\mathrm{C}(X_i) \otimes \mathcal{K})p_i$, where p_i is a projection in $\mathrm{C}(X_i) \otimes \mathcal{K}$ to be specified. The algebra B of Theorem 1.2 will be realized as the inductive limit of the B_i with diagonal connecting $*$-homomorphisms $\gamma_i : B_i \to B_{i+1}$.

Let p_1 be a projection corresponding to the vector bundle

$$\theta_1 \times \xi_{\sigma(1)},$$

over X_1, where θ_1 denotes the trivial complex line bundle, ξ_k denotes the universal line bundle over CP^k for a given natural number k, and $\sigma(1) = 1$. Put $\eta_i = \pi_i^{2*}(\xi_{\sigma(i)})$.

We now specify, inductively, the maps $\gamma_i : B_i \to B_{i+1}$. Let $\tilde{\psi}$ be the homeomorphism of I given by

$$\tilde{\psi}(x) = 1 - x.$$

Abusing notation, we will also take $\tilde{\psi}$ to be the homeomorphism of $X_i \stackrel{\mathrm{def}}{=} \mathrm{I} \times Y_i$ given by $(x, y) \mapsto (\tilde{\psi}(x), y)$. Choose a dense sequence $(z_i^l)_{l=1}^{\infty}$ in X_i and choose for each $j = 1, 2, \ldots, i+1$ a point $y_i^j \in X_i$ such that $y_i^{i+1} = z_i^1$, $y_i^i = z_i^2$ and $\pi_{j+1}^1 \circ \pi_j^1 \circ \cdots \circ \pi_i^1(y_i^j) = z_j^{i-j+2}$ for $1 \le j \le i-1$. Let

$$\tilde{\gamma_i} : \mathrm{C}(X_i \otimes \mathcal{K}) \longrightarrow \mathrm{C}(X_{i+1} \otimes \mathcal{K})$$

be a diagonal $*$-homomorphism coming from

$$(\pi_{i+1}^1, \theta_1) \cup \{(y_i^j, \eta_{i+1})\}_{j=1}^{i+1} \cup \{(\tilde{\psi}(y_i^j), \eta_{i+1})\}_{j=1}^{i+1}.$$

Let $\tilde{\gamma_{1i}}$ be the composition $\tilde{\gamma_i} \circ \cdots \circ \tilde{\gamma_1}$, and put $p_{i+1} = \tilde{\gamma_{1i}}(p_1)$ for all natural numbers i. Let $\gamma_i : B_i \to B_{i+1}$ be the restriction of $\tilde{\gamma_i}$. Let $B = \lim_{\to}(B_i, \gamma_i)$. It follows from [V1] that B is a simple, unital AH-algebra. (Apart from the choice of point evaluations in the $\tilde{\gamma_i}$, the construction above is precisely that of [V1]. The reason for the specific choice of point evaluations will be made clear shortly.)

Straightforward calculation shows that the projection $p_i \in B_i$ corresponds to a complex vector bundle over X_i of the form $\theta_1 \oplus \omega_i$. In fact, with $X_i = \mathrm{I} \times Y_i$ and with τ_1^i, τ_2^i the co-ordinate projections, we have that $\omega_i = \tau_2^{i*}(\tilde{\omega_i})$ for a vector bundle $\tilde{\omega_i}$ over Y_i. Thus, the homeomorphism $\tilde{\psi}$ of X_i fixes p_i, and so induces an automorphism ψ_i of B_i.

Let π_{im}^1 be the composition $\pi_m^1 \circ \cdots \circ \pi_{i+1}^1$. Let $f \in B_i$. Then, with (x, y) an element of $X_{i+1} = X_i \times \mathrm{CP}^{\sigma(i+1)}$, we have

$$\gamma_i(f)(x, y) = f(\pi_{i+1}^1(x)) \oplus \left(\bigoplus_{j=1}^{i+1} f(\tilde{\psi}(y_i^j)) \otimes \eta_{i+1} \oplus f(y_i^j) \otimes \eta_{i+1} \right),$$

so that

$$\psi_{i+1}\left(\gamma_i(f)(x, y)\right) = f\left(\tilde{\psi}(\pi_{i+1}^1(x))\right) \oplus \left(\bigoplus_{j=1}^{i+1} f(\tilde{\psi}(y_i^j)) \otimes \eta_{i+1} \oplus f(y_i^j) \otimes \eta_{i+1} \right).$$

On the other hand,

$$\gamma_i \circ \psi_i(f)(x, y) = f\left(\tilde{\psi}(\pi_{i+1}^1(x))\right) \oplus \left(\bigoplus_{j=1}^{i+1} f(\tilde{\psi}(y_i^j)) \otimes \eta_{i+1} \oplus f(y_i^j) \otimes \eta_{i+1} \right).$$

Thus, $\gamma_i \circ \psi_i$ and $\psi_{i+1} \circ \gamma_i$ differ only in the order of their direct summands, and so are unitarily equivalent. The unitary element implementing this equivalence squares to the identity. Conjugating ψ_{i+1} by said unitary element, we may assume that $\gamma_i \circ \psi_i = \psi_{i+1} \circ \gamma_i$. This process may be repeated inductively for ψ_m, $m > i$, yielding an inductive limit automorphism α of B via the ψ_i.

We now show that α is not locally inner, yet induces the identity map on Inv_F for any $F \in \mathcal{F}$. Recall that the Euler class $e(\omega)$ of a complex vector bundle ω over a connected finite CW-complex X is an element of $\mathrm{H}^{2\dim\omega}(X)$. For a trivial complex vector bundle θ_l of dimension $l \in \mathbb{N}$ we have $e(\theta_l) = 0$. We also have $e(\omega_1 \oplus \omega_2) = e(\omega_1) \cdot e(\omega_2)$ for two complex vector bundles ω_1 and ω_2 over X, where the product is the cup product in the integral cohomology ring $\mathrm{H}^*(X)$. Thus, if $e(\omega) \neq 0$, then ω has no trivial sub-bundles. Alternatively, ω does not admit an everywhere-nonzero cross section.

It follows from the construction of the $p_i = \theta_1 \oplus \tau_2^{i*}(\tilde{\omega_i})$ that $\tilde{\omega_i}$ is a vector bundle over Y_i with nonzero Euler class ([V2]).

It will suffice to find an element f of B_i such that $||\alpha(f) - f|| \geq 1$ and

$$||\mathrm{Ad}(u) \circ \alpha \circ \gamma_{im}(f) - \gamma_{im}(f)|| \geq 1$$

for all unitaries $u \in B_m$ and natural numbers $m \in \mathbb{N}$.

Let $\tilde{f}$ be a continuous function on I taking values in $[0, 1]$ such that $\tilde{f}(0) = 0$ and $\tilde{f}(1) = 1$. Pull this function back to a function on $X_i = \mathrm{I} \times Y_i$ via the co-ordinate projection onto I, keeping the same notation. Put $f = \tilde{f}\theta_1 \in B_i$. Thus chosen, the element $f \in B_i$ has the desired property:

$$||\alpha(f) - f|| \geq 1.$$

Notice that $\theta_1\gamma_{im}(f)\theta_1 = (\tilde{f} \circ \pi_{im})\theta_1$ inside B_m for all natural numbers $m \geq i$, and that $\alpha|_{B_i}(\theta_1) = \theta_1$ for every $i \in \mathbb{N}$.

Let u be a unitary element in B_m. We claim that there is a $y_0 \in Y_m$ such that conjugation by u fixes the corner

$$\theta_1(\mathrm{C}(X_m) \otimes \mathcal{K})\theta_1$$

of B_m at $(0, y_0) \in X_m = \mathrm{I} \times Y_m$, i.e.,

$$(u^*\theta_1 g\theta_1 u)(0, y_0) = (\theta_1 g\theta_1)(0, y_0)$$

for all $g \in \mathrm{C}(X_m \otimes \mathcal{K})$. Let $\Gamma = (x, y) \mapsto v_{(x,y)}$ be an everywhere nonzero cross section of θ_1 over $\{0\} \times Y_m \subseteq X_m$. Suppose that there is no point $(0, y_0)$ as above. Let $R_{(x,y)}$ denote the fibre of the vector bundle corresponding to $p_m|_{\{0\}\times Y_m}$ at $(0, y)$, and let $W_{(x,y)}$ denote the subspace of $R_{(x,y)}$ corresponding to $\tilde{\omega}_m$. By assumption, the angle between $v_{(x,y)}$ and $u^*v_{(x,y)}$ is nonzero for every $(0, y) \in \{0\} \times Y_m$. But this implies that the projection of $u^*v_{(x,y)}$ onto $W_{(x,y)}$ is an everywhere nonzero cross section of $\tilde{\omega}_{i+1}$, contradicting $e(\tilde{\omega}_{i+1}) \neq 0$ and proving the claim.

Let $(0, y_0)$ be a point in $\{0\} \times Y_m$ at which u fixes the corner

$$\theta_1(\mathrm{C}(X_m) \otimes \mathcal{K})\theta_1.$$

Then,

$$(\mathrm{Ad}(u) \circ \alpha \circ \gamma_{im}(f))(0, y_0) = \theta_1 \alpha \circ \gamma_{im}(f)(0, y_0)\theta_1 \oplus g(0, y_0),$$

where $g \in \omega_m B_m \omega_m$. We conclude that

$$||\gamma_{im}(f) - \mathrm{Ad}u \circ \alpha \circ \gamma_{im}(f)||$$

is bounded below by

$$\begin{aligned} ||\tilde{f}(\pi_{im}(0, y_0))\theta_1 - \alpha(\tilde{f}(\pi_{im}(0, y_0)\theta_1)|| &= ||\tilde{f}(0, y^{'}) - \tilde{f}(\tilde{\psi}(0, y^{'}))|| \\ &= 1, \end{aligned}$$

as desired.

Note that ψ_i is homotopic to the identity map on B_i via unital endomorphisms of B_i for all $i \in \mathbb{N}$ — it is the composition two maps: the first is an automorphism of B_m induced by a map on X_m, which is itself homotopic to the identity map on X_m; the second is an inner automorphism implemented by a unitary in the connected component of $1 \in B_m$. Thus, α induces the identity map on any $F \in \mathcal{F}_{\mathbf{R}}$ — the restriction to functors whose target category consists of R-modules is sufficient to ensure that an inductive limit morphism in the target category uniquely determines an automorphism of a fixed limit object. Since B has a unique trace, α also induces the identity map on $\mathrm{Ell}(B)$.

Following [NT], one sees that the absence of topological K_1 and the fact that α induces the identity map on the Elliott invariant force α to induce the identity map at the level of the Hausdorffized algebraic K_1-group. The KK-class of α is the same as that of the identity map on B by virtue of its inducing the identity map on topological K-theory — since B is in the bootstrap class, $\mathrm{K}_0 B$ is free, and $\mathrm{K}_1 B = 0$ we have that

$$\mathrm{KK}^*(B, B) \simeq \mathrm{Hom}(\mathrm{K}_* B, \mathrm{K}_* B)$$

by the Universal Coefficient Theorem ([RS]).

The stable and the real rank of a C*-algebra are not relevant to the problem of distinguishing automorphsims of the algebra. The automorphism α squares to the identity map on B, whence the various notions of entropy for automorphisms of C*-algebras cannot distinguish it from the identity map. □

It is not clear to the author whether the Cuntz semigroup can distinguish α from the identity map on B, although it seems plausible. One can, with some industry, modify the construction of B so that there exists an embedding $\iota : S_\infty \to \mathrm{Aut}(B)$ with the following properties: the induced map

$$\bar{\iota} : S_\infty \to \mathrm{Out}(B) := \mathrm{Aut}(B)/\overline{\mathrm{Inn}(B)}$$

is a monomorphism, and, for each $g \in S_\infty$, $\iota(g)$ acts trivially on each $F \in \mathcal{F}_{\mathbf{R}}$. The information which goes undetected by $\mathcal{F}_{\mathbf{R}}$ is thus complicated indeed.

5. Some remarks on the classification problem

A classification theorem for a category $\mathcal{C}$ amounts to proving that $\mathcal{C}$ is equivalent to a second concrete category $\mathcal{D}$ whose objects and morphisms are well understood. Take, for instance, the case of AF algebras: the category $\mathcal{C}$ has AF algebras as its objects and approximate unitary equivalence classes of isomorphisms as its morphisms, while the equivalent (classifying) category $\mathcal{D}$ has dimension groups as its objects and order isomorphisms of such as its morphisms. If one does not understand $\mathcal{D}$ any better than $\mathcal{C}$, then one has achieved little; the range of a classifying invariant is an essential part of any classification result.

Theorems 1.1 and 1.2 show that any classifying invariant for simple nuclear separable C*-algebras will either be discontinuous with respect to inductive limits, or not homotopy invariant even modulo traces. A discontinuous classifying invariant would all but exclude the possibility of obtaining its range; existing range results for $\mathrm{Ell}(\bullet)$ require its continuity. The only current candidates for nonhomotopy invariant functors from the category of C*-algebras which are not captured by $\mathcal{F}$ are the Cuntz semigroup $W(\bullet)$ or its Grothendieck enveloping group. Neither of these invariants is continuous with respect to inductive limits, but this defect can perhaps be repaired by considering these invariants as objects in the correct category. An invariant obtained in this manner would, while exceedingly fine, have at least the advantage of continuity with respect to countable inductive limits. On the other hand, the question of range for such an invariant remains daunting, as the following lemma shows.

Lemma 5.1. *Let $S^{n_1},\ldots,S^{n_k}$ be a finite collection of spheres. Put*

$$Y = S^{n_1}\times\cdots\times S^{n_k},\ \ N = k+\sum_{i=1}^{k} n_i,$$

and let $D(Y)$ be the semigroup of Murray-von Neumann equivalence classes of projections in $\mathrm{M}_\infty(\mathrm{C}(Y))$. Then, there is an order embedding

$$\iota : D(Y)\to W\left(\mathrm{C}\left([0,1]^N\right)\right).$$

Proof. S^{n_i} can be embedded more or less canonically into $[0,1]^{n_i+1}$ as the n_i-sphere with centre $\left(\frac{1}{2},\ \frac{1}{2},\ldots,\frac{1}{2}\right)$ and radius $\frac{1}{4}$. Let $S_0^{n_i}\subseteq[0,1]^{n_i+1}$ be the hollow ball

$$S_0^{n_i}\stackrel{\mathrm{def}}{=}\left\{\overline{x}\in[0,1]^{n_i+1}\ :\ \frac{1}{8}<\mathrm{dist}\left(\overline{x},\left(\frac{1}{2},\ \frac{1}{2},\ldots,\frac{1}{2}\right)\right)<\frac{3}{8}\right\},$$

and let

$$\pi_i : S_0^{n_i}\to S^{n_i}$$

be the projection along rays emanating from $\left(\frac{1}{2},\ \frac{1}{2},\dots,\frac{1}{2}\right) \in [0,1]^{n_i+1}$. Put

$$Y_0 = S_0^{n_1} \times \cdots \times S_0^{n_k} \subseteq [0,1]^N; \quad \pi = \pi_1 \times \cdots \times \pi_k.$$

Notice that for every natural number n, $\mathrm{M}_n \otimes \mathrm{C}_0(Y_0)$ is a hereditary subalgebra of $\mathrm{M}_n \otimes \mathrm{C}\left([0,1]^N\right)$. Let $p, q \in \mathrm{M}_n \otimes \mathrm{C}(Y)$ be projections, and let $\pi^*(p), \pi^*(q)$ be their pullbacks to Y_0. Let $f \in \mathrm{M}_n \otimes \mathrm{C}\left([0,1]^N\right)$ be a scalar function taking values in $[0,1]$ which vanishes off Y_0 and is equal to one on Y. Then, $f\pi^*(p), f\pi^*(q)$ are positive elements of $\mathrm{C}\left([0,1]^N\right)$. If $f\pi^*(p)$ and $f\pi^*(q)$ are Cuntz equivalent, then upon restriction to Y we have that p and q are Cuntz equivalent. This in turn implies that p and q are Murray-von Neumann equivalent. Now suppose that p and q are Murray-von Neumann equivalent. Since this implies Cuntz equivalence, there exist sequences (x_i) and (y_i) in $\mathrm{M}_n \otimes \mathrm{C}(Y)$ such that

$$x_i p x_i^* \stackrel{i\to\infty}{\longrightarrow} q; \quad y_i q y_i^* \stackrel{i\to\infty}{\longrightarrow} p.$$

Let (g_i) be an approximate unit of scalar functions for $\mathrm{M}_n \otimes \mathrm{C}_0(Y_0)$. It follows that

$$g_i \pi^*(x_i) f\pi^*(p) \pi^*(x_i^*) g_i \stackrel{i\to\infty}{\longrightarrow} f\pi^*(q)$$

and

$$g_i \pi^*(y_i) f\pi^*(q) \pi^*(y_i^*) g_i \stackrel{i\to\infty}{\longrightarrow} f\pi^*(p),$$

whence $\pi^*(p)$ and $\pi^*(q)$ are Cuntz equivalent. The desired embedding is

$$\iota([p]) \stackrel{\mathrm{def}}{=} \langle f\pi^*(p) \rangle. \qquad \square$$

Lemma 5.1 shows that the problem of determining $W\left(\mathrm{C}\left([0,1]^N\right)\right)$ for general $N \in \mathbb{N}$ is at least as difficult as determining the isomorphism classes of all complex vector bundles over an arbitrary Cartesian product of spheres; this, in turn, is a difficult unsolved problem in its own right. Any attempt to use $W(\bullet)$ to prove a classification theorem for, say, all simple, unital AH algebras — even, as Theorem 1.1 shows, if one restricts to limits of full matrix algebras over contractible spaces, a class for which the ranges of $\mathrm{Ell}(\bullet)$, $\mathrm{sr}(\bullet)$, $\mathrm{rr}(\bullet)$, K-theory with coefficients, and the Hausdorffized algebraic K_1-group are known — will not enjoy a salient advantage over the slow dimension growth case: the luxury of building blocks whose invariants can be easily and concretely described. (Other technical obstacles are also sure to be much more complicated than those faced in the work of Elliott, Gong, and Li, and their proof already runs to several hundred pages.) The Cuntz semigroup is at once necessary for classification, and unlikely to admit a range result.

But rather than end on a pessimistic note, we enjoin the reader to view our results as further evidence that the Elliott invariant *will* turn out to be complete for a sufficiently well behaved class of C^*-algebras. We have proved that the moment one relaxes the slow dimension growth condition for AH algebras (and therefore, *a fortiori* for ASH algebras), one obtains counterexamples to the Elliott conjecture of a particularly forceful nature, so that slow dimension growth is connected essentially to the classification problem. There is evidence that slow dimension growth and $\mathcal{Z}$-stability are equivalent for ASH algebras — in the case of simple and unital AH algebras with unique trace this has recently been proved ([TW1], [TW2]). Optimistically, $\mathcal{Z}$-stability is an abstraction of slow dimension growth, and the Elliott conjecture will be confirmed for all simple, separable, and nuclear C*-algebras having this property.

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