Annals of Mathematics, **167** (2008), 945–991

Localization of modules for a semisimple Lie algebra in prime characteristic

By Roman Bezrukavnikov, Ivan Mirković, and Dmitriy Rumynin*

Abstract

We show that, on the level of derived categories, representations of the Lie algebra of a semisimple algebraic group over a field of finite characteristic with a given (generalized) regular central character are the same as coherent sheaves on the formal neighborhood of the corresponding (generalized) Springer fiber.

The first step is to observe that the derived functor of global sections provides an equivalence between the derived category of $\mathcal{D}$-modules (with no divided powers) on the flag variety and the appropriate derived category of modules over the corresponding Lie algebra. Thus the "derived" version of the Beilinson-Bernstein localization theorem holds in sufficiently large positive characteristic. Next, one finds that for any smooth variety this algebra of differential operators is an Azumaya algebra on the cotangent bundle. In the case of the flag variety it splits on Springer fibers, and this allows us to pass from $\mathcal{D}$-modules to coherent sheaves. The argument also generalizes to twisted $\mathcal{D}$-modules. As an application we prove Lusztig's conjecture on the number of irreducible modules with a fixed central character. We also give a formula for behavior of dimension of a module under translation functors and reprove the Kac-Weisfeiler conjecture.

The sequel to this paper [BMR2] treats singular infinitesimal characters.

To Boris Weisfeiler, missing since 1985

Contents

Introduction

1. Central reductions of the envelope $\mathcal{D}_X$ of the tangent sheaf
   1.1. Frobenius twist
   1.2. The ring of "crystalline" differential operators $\mathcal{D}_X$
   1.3. The difference ι of p^{th} power maps on vector fields
   1.4. Central reductions

*R.B. was partially supported by NSF grant DMS-0071967 and the Clay Institute, D.R. by EPSRC and I.M. by NSF grants.

2. The Azumaya property of $\mathcal{D}_X$
2.1. Commutative subalgebra $\mathcal{A}_X \subseteq \mathcal{D}_X$
2.2. Point modules δ^ζ
2.3. Torsors
3. Localization of $\mathfrak{g}$-modules to $\mathcal{D}$-modules on the flag variety
3.1. The setting
3.2. Theorem
3.3. Localization functors
3.4. Cohomology of $\widetilde{\mathcal{D}}$
3.5. Calabi-Yau categories
3.6. Proof of Theorem 3.2
4. Localization with a generalized Frobenius character
4.1. Localization on (generalized) Springer fibers
5. Splitting of the Azumaya algebra of crystalline differential operators on (generalized) Springer fibers
5.1. $\mathcal{D}$-modules and coherent sheaves
5.2. Unramified Harish-Chandra characters
5.3. $\mathfrak{g}$-modules and coherent sheaves
5.4. Equivalences on formal neighborhoods
5.5. Equivariance
6. Translation functors and dimension of U_χ-modules
6.1. Translation functors
6.2. Dimension
7. K-theory of Springer fibers
7.1. Bala-Carter classification of nilpotent orbits [Sp]
7.2. Base change from $\mathcal{K}$ to $\mathbb{C}$
7.3. The specialization map in 7.1.7(a) is injective
7.4. Upper bound on the K-group
References

Introduction

$\mathfrak{g}$-modules and $\mathcal{D}$-modules. We are interested in representations of a Lie algebra $\mathfrak{g}$ of a (simply connected) semisimple algebraic group G over a field $\Bbbk$ of positive characteristic. In order to relate $\mathfrak{g}$-modules and $\mathcal{D}$-modules on the flag variety $\mathcal{B}$ we use the sheaf $\mathcal{D}_\mathcal{B}$ of *crystalline* differential operators (i.e. differential operators without divided powers).

The basic observation is a version of the famous Localization Theorem [BB], [BrKa] in positive characteristic. The center of the enveloping algebra $U(\mathfrak{g})$ contains the "Harish-Chandra part" $\mathfrak{Z}_{\mathrm{HC}} \overset{\mathrm{def}}{=} U(\mathfrak{g})^G$ which is familiar from characteristic zero. $U(\mathfrak{g})$-modules where $\mathfrak{Z}_{\mathrm{HC}}$ acts by the same character as on the trivial $\mathfrak{g}$-module $\Bbbk$ are modules over the central reduction $U^0 \stackrel{\text{def}}{=} U(\mathfrak{g})\otimes_{\mathfrak{Z}_{\mathrm{HC}}}\Bbbk$. Abelian categories of U^0-modules and of $\mathcal{D}_{\mathcal{B}}$-modules are quite different. However, their bounded derived categories are canonically equivalent if the characteristic p of the base field $\Bbbk$ is sufficiently large, say, $p > h$ for the Coxeter number h. More generally, one can identify the bounded derived category of U-modules with a given regular (generalized) Harish-Chandra central character with the bounded derived category of the appropriately twisted $\mathcal{D}$-modules on $\mathcal{B}$ (Theorem 3.2).

$\mathcal{D}$-modules and coherent sheaves. The sheaf $\mathcal{D}_X$ of crystalline differential operators on a smooth variety X over $\Bbbk$ has a nontrivial center, canonically identified with the sheaf of functions on the Frobenius twist $T^*X^{(1)}$ of the cotangent bundle (Lemma 1.3.2). Moreover $\mathcal{D}_X$ is an Azumaya algebra over $T^*X^{(1)}$ (Theorem 2.2.3). More generally, the sheaves of twisted differential operators are Azumaya algebras on twisted cotangent bundles (see 2.3).

When one thinks of the algebra $U(\mathfrak{g})$ as the right translation invariant sections of $\mathcal{D}_G$, one recovers the well-known fact that the center of $U(\mathfrak{g})$ also has the "Frobenius part" $\mathfrak{Z}_{\mathrm{Fr}} \cong \mathcal{O}(\mathfrak{g}^{*(1)})$, the functions on the Frobenius twist of the dual of the Lie algebra.

For $\chi \in \mathfrak{g}^*$ let $\mathcal{B}_\chi \subset \mathcal{B}$ be a connected component of the variety of all Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$ such that $\chi|_{[\mathfrak{b},\mathfrak{b}]} = 0$; for nilpotent χ this is the corresponding Springer fiber. Thus $\mathcal{B}_\chi$ is naturally a subvariety of a twisted cotangent bundle of $\mathcal{B}$. Now, imposing the (infinitesimal) character $\chi \in \mathfrak{g}^{*(1)}$ on U-modules corresponds to considering $\mathcal{D}$-modules (set-theoretically) supported on ${\mathcal{B}_\chi}^{(1)}$.

Our second main observation is that the Azumaya algebra of twisted differential operators splits on the formal neighborhood of $\mathcal{B}_\chi$ in the twisted cotangent bundle. So, the category of twisted $\mathcal{D}$-modules supported on ${\mathcal{B}_\chi}^{(1)}$ is equivalent to the category of coherent sheaves supported on ${\mathcal{B}_\chi}^{(1)}$ (Theorem 5.1.1). Together with the localization, this provides an algebro-geometric description of representation theory – the derived categories are equivalent for U-modules with a generalized $\mathfrak{Z}$-character and for coherent sheaves on the formal neighborhood of ${\mathcal{B}_\chi}^{(1)}$ for the corresponding χ.

Representations. One representation theoretic consequence of the passage to algebraic geometry is the count of irreducible U_χ-modules with a given regular Harish-Chandra central character (Theorem 5.4.3). This was known previously when χ is regular nilpotent in a Levi factor ([FP]), and the general case was conjectured by Lusztig ([Lu1], [Lu]). In particular, we find a canonical isomorphism of Grothendieck groups of U^0_χ-modules and of coherent sheaves on the Springer fiber $\mathcal{B}_\chi$. Moreover, the rank of this K-group is the same as the dimension of cohomology of the corresponding Springer fiber in characteristic zero (Theorem 7.1.1); hence it is well understood. One of the purposes of this paper is to provide an approach to Lusztig's elaborate conjectural description of representation theory of $\mathfrak{g}$.

0.0.1. Sections 1 and 2 deal with algebras of differential operators $\mathcal{D}_X$. Equivalence $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U^0)) \xrightarrow{\cong} \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}_{\mathcal{B}}))$ and its generalizations are proved in Section 3. In Section 4 we specialize the equivalence to objects with the χ-action of the Frobenius center $\mathfrak{Z}_{\mathrm{Fr}}$. In Section 5 we relate $\mathcal{D}$-modules with the χ-action of $\mathfrak{Z}_{\mathrm{Fr}}$ to $\mathcal{O}$-modules on the Springer fiber $\mathcal{B}_\chi$. This leads to a dimension formula for $\mathfrak{g}$-modules in terms of the corresponding coherent sheaves in Section 6, here we also spell out compatibility of our functors with translation functors. Finally, in Section 7 we calculate the rank of the K-group of the Springer fiber, and thus of the corresponding category of $\mathfrak{g}$-modules.

0.0.2. The origin of this study was a suggestion of James Humphreys that the representation theory of U^0_χ should be related to geometry of the Springer fiber $\mathcal{B}_\chi$. This was later supported by the work of Lusztig [Lu] and Jantzen [Ja1], and by [MR].

0.0.3. We would like to thank Vladimir Drinfeld, Michael Finkelberg, James Humphreys, Jens Jantzen, Masaharu Kaneda, Dmitry Kaledin, Victor Ostrik, Cornelius Pillen, Simon Riche and Vadim Vologodsky for various information over the years; special thanks go to Andrea Maffei for pointing out a mistake in example 5.3.3(2) in the previous draft of the paper. A part of the work was accomplished while R.B. and I.M. visited the Institute for Advanced Study (Princeton), and the Mathematical Research Institute (Berkeley); in addition to excellent working conditions these opportunities for collaboration were essential. R.B. is also grateful to the Independent Moscow University where part of this work was done.

0.0.4. *Notation.* We consider schemes over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. For an affine S-scheme $X \xrightarrow{q} S$, we denote $q_*\mathcal{O}_X$ by $\mathcal{O}_{X/S}$, or simply by $\mathcal{O}_X$. For a subscheme $\mathfrak{Y}$ of $\mathfrak{X}$ the formal neighborhood $FN_{\mathfrak{X}}(\mathfrak{Y})$ is an ind-scheme (a formal scheme), the notation for the categories of modules on $\mathfrak{X}$ supported on $\mathfrak{Y}$ is introduced in 3.1.7, 3.1.8 and 4.1.1. The Frobenius neighborhood $\mathrm{Fr}\,N_{\mathfrak{X}}(\mathfrak{Y})$ is introduced in 1.1.2. The inverse image of sheaves is denoted f^{-1} and for $\mathcal{O}$-modules f^* (both direct images are denoted f_*). We denote by $\mathcal{T}_X$ and $\mathcal{T}^*_X$ the sheaves of sections of the (co)tangent bundles TX and T^*X.

1. Central reductions of the envelope $\mathcal{D}_X$ of the tangent sheaf

We will describe the center of differential operators (without divided powers) as functions on the Frobenius twist of the cotangent bundle. Most of the material in this section is standard.

1.1. *Frobenius twist.*

1.1.1. *Frobenius twist of a $\Bbbk$-scheme.* Let X be a scheme over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The Frobenius map of schemes $X \to X$ is defined as the identity on topological spaces, but the pull-back of functions is the p^{th} power: $\mathrm{Fr}_X^*(f) = f^p$ for $f \in \mathcal{O}_{X^{(1)}} = \mathcal{O}_X$. The Frobenius twist $X^{(1)}$ of X is the $\Bbbk$-scheme that coincides with X as a scheme (i.e. $X^{(1)} = X$ as a topological space and $\mathcal{O}_{X^{(1)}} = \mathcal{O}_X$ as a sheaf of rings), but with a different $\Bbbk$-structure: $a \underset{(1)}{\cdot} f \overset{\text{def}}{=} a^{1/p} \cdot f,\ a \in \Bbbk,\ f \in \mathcal{O}_{X^{(1)}}$. This makes the Frobenius map into a map of $\Bbbk$-schemes $X \xrightarrow{\mathrm{Fr}_X} X^{(1)}$. We will use the twists to keep track of using Frobenius maps. Since Fr_X is a bijection on $\Bbbk$-points, we will often identify $\Bbbk$-points of X and $X^{(1)}$. Also, since Fr_X is affine, we may identify sheaves on X with their $(\mathrm{Fr}_X)_*$-images. For instance, if X is reduced the p^{th} power map $\mathcal{O}_{X^{(1)}} \to (\mathrm{Fr}_X)_* \mathcal{O}_X$ is injective, and we think of $\mathcal{O}_{X^{(1)}}$ as a subsheaf $\mathcal{O}_X^p \overset{\text{def}}{=} \{f^p,\ f \in \mathcal{O}_X\}$ of $\mathcal{O}_X$.

1.1.2. *Frobenius neighborhoods.* The Frobenius neighborhood of a subscheme Y of X is the subscheme $(\mathrm{Fr}_X)^{-1} Y^{(1)} \subseteq X$; we denote it $\mathrm{Fr}\, N_X(Y)$ or simply $\underline{X}_Y$. It contains Y and $\mathcal{O}_{\underline{X}_Y} = \mathcal{O}_X \underset{\mathcal{O}_{X^{(1)}}}{\otimes} \mathcal{O}_{Y^{(1)}} = \mathcal{O}_X \underset{\mathcal{O}_X^p}{\otimes} \mathcal{O}_X^p / \mathcal{I}_Y^p = \mathcal{O}_X / \mathcal{I}_Y^p \cdot \mathcal{O}_X$ for the ideal of definition $\mathcal{I}_Y \subseteq \mathcal{O}_X$ of Y.

1.1.3. *Vector spaces.* For a $\Bbbk$-vector space V the $\Bbbk$-scheme $V^{(1)}$ has a natural structure of a vector space over $\Bbbk$; the $\Bbbk$-linear structure is again given by $a \underset{(1)}{\cdot} v \overset{\text{def}}{=} a^{1/p} v,\ a \in \Bbbk,\ v \in V$. We say that a map $\beta : V \to W$ between $\Bbbk$-vector spaces is p-linear if it is additive and $\beta(a \cdot v) = a^p \cdot \beta(v)$; this is the same as a linear map $V^{(1)} \to W$. The canonical isomorphism of vector spaces $(V^*)^{(1)} \xrightarrow{\cong} (V^{(1)})^*$ is given by $\alpha \to \alpha^p$ for $\alpha^p(v) \overset{\text{def}}{=} \alpha(v)^p$ (here, $V^{*(1)} = V^*$ as a set and $(V^{(1)})^*$ consists of all p-linear $\beta :\ V \to \Bbbk$). For a smooth X, canonical $\Bbbk$-isomorphisms $T^*(X^{(1)}) = (T^*X)^{(1)}$ and $(T(X))^{(1)} \xrightarrow{\cong} T(X^{(1)})$ are obtained from definitions.

1.2. *The ring of "crystalline" differential operators $\mathcal{D}_X$.* Assume that X is a smooth variety. Below we will occasionally compute in local coordinates: since X is smooth, any point a has a Zariski neighborhood U with étale coordinates $x_1, \ldots, x_n$; i.e., (x_i) define an étale map from U to $\mathbb{A}^n$ sending a to 0. Then the dx_i form a frame of T^*X at a; the dual frame $\partial_1, \ldots, \partial_n$ of $\mathcal{T}_X$ is characterized by $\partial_i(x_j) = \delta_{ij}$.

Let $\mathcal{D}_X = U_{\mathcal{O}_X}(\mathcal{T}_X)$ denote the enveloping algebra of the tangent Lie algebroid $\mathcal{T}_X$; we call $\mathcal{D}_X$ the sheaf of crystalline differential operators. Thus $\mathcal{D}_X$ is generated by the algebra of functions $\mathcal{O}_X$ and the $\mathcal{O}_X$-module of vector fields $\mathcal{T}_X$, subject to the module and commutator relations $f \cdot \partial = f\partial$,

$\partial\cdot f - f\cdot\partial = \partial(f)$, $\partial \in \mathcal{T}_X$, $f \in \mathcal{O}_X$, and the Lie algebroid relations $\partial'\cdot\partial'' - \partial''\cdot\partial' = [\partial', \partial'']$, $\partial', \partial'' \in \mathcal{T}_X$. In terms of a local frame ∂_i of vector fields we have $\mathcal{D}_X = \bigoplus_I \mathcal{O}_X\cdot\partial^I$. One readily checks that $\mathcal{D}_X$ coincides with the object defined (in a more general situation) in [BO, §4], and called there "PD differential operators".

By the definition of an enveloping algebra, a sheaf of $\mathcal{D}_X$ modules is just an $\mathcal{O}_X$ module equipped with a flat connection. In particular, the standard flat connection on the structure sheaf $\mathcal{O}_X$ extends to a $\mathcal{D}_X$-action. This action is not faithful: it provides a map from $\mathcal{D}_X$ to the "true" differential operators $\mathbb{D}_X \subseteq \mathcal{E}nd_{\Bbbk}(\mathcal{O}_X)$ which contain divided powers of vector fields; the image of this map is an $\mathcal{O}_X$-module of finite rank $p^{\dim X}$; see [BO] or 2.2.5 below.

For $f \in \mathcal{O}_X$ the p^{th} power f^p is killed by the action of $\mathcal{T}_X$, hence for any closed subscheme $Y \subseteq X$ we get an action of $\mathcal{D}_X$ on the structure sheaf $\mathcal{O}_{\underline{X}_Y}$ of the Frobenius neighborhood.

Being defined as an enveloping algebra of a Lie algebroid, the sheaf of rings $\mathcal{D}_X$ carries a natural "Poincaré-Birkhoff-Witt" filtration $\mathcal{D}_X = \cup \mathcal{D}_{X,\leq n}$, where $\mathcal{D}_{X,n+1} = \mathcal{D}_{X,\leq n} + \mathcal{T}_X \cdot \mathcal{D}_{X,\leq n}$, $\mathcal{D}_{X,\leq 0} = \mathcal{O}_X$. In the following Lemma parts (a,b) are proved similarly to the familiar statements in characteristic zero, while (c) can be proved by a straightforward use of local coordinates.

1.2.1. LEMMA. a) *There is a canonical isomorphism of the sheaves of algebras*: $\mathrm{gr}(\mathcal{D}_X) \cong \mathcal{O}_{T^*X}$.

b) $\mathcal{O}_{T^*X}$ *carries a Poisson algebra structure, given by* $\{f_1, f_2\} = [\tilde{f}_1, \tilde{f}_2] \mod \mathcal{D}_{X,\leq n_1+n_2-2}$, $\tilde{f}_i \in \mathcal{D}_{X,\leq n_i}$, $f_i = \tilde{f}_i \mod \mathcal{D}_{X,\leq n_i-1} \in \mathcal{O}_{T^*X}$, $i = 1, 2$.

This Poisson structure coincides with the one arising from the standard symplectic form on T^*X.

c) *The action of* $\mathcal{D}_X$ *on* $\mathcal{O}_X$ *induces an injective morphism* $\mathcal{D}_{X,\leq p-1} \hookrightarrow \mathcal{E}nd(\mathcal{O}_X)$.

We will use the familiar terminology, referring to the image of $d \in \mathcal{D}_{X,\leq i}$ in $\mathcal{D}_{X,\leq i}/\mathcal{D}_{X,\leq i-1} \subset \mathcal{O}_{T^*X}$ as its symbol.

1.3. *The difference ι of p^{th} power maps on vector fields.* For any vector field $\partial \in \mathcal{T}_X$, $\partial^p \in \mathcal{D}_X$ acts on functions as another vector field which one denotes $\partial^{[p]} \in \mathcal{T}_X$. For $\partial \in \mathcal{T}_X$ set $\iota(\partial) \stackrel{\text{def}}{=} \partial^p - \partial^{[p]} \in \mathcal{D}_X$. The map ι lands in the kernel of the action on $\mathcal{O}_X$; it is injective, since it is injective on symbols.

1.3.1. LEMMA. a) *The map* $\iota : \mathcal{T}_X{}^{(1)} \to \mathcal{D}_X$ *is* $\mathcal{O}_{X^{(1)}}$*-linear, i.e.,* $\iota(\partial) + \iota(\partial') = \iota(\partial + \partial')$ *and* $\iota(f\partial) = f^p\cdot\iota(\partial)$, $\partial, \partial' \in \mathcal{T}_{X^{(1)}}$, $f \in \mathcal{O}_{X^{(1)}}$.

b) *The image of ι is contained in the center of* $\mathcal{D}_X$.

Proof.[1] For each of the two identities in (a), both sides act by zero on $\mathcal{O}_X$. Also, they lie in $D_{X,\leq p}$, and clearly coincide modulo $D_{X,\leq p-1}$. Thus the identities follow from Lemma 1.2.1(c).

b) amounts to: $[f, \iota(\partial)] = 0$, $[\partial', \iota(\partial)] = 0$, for $f \in \mathcal{O}_X$, $\partial, \partial' \in \mathcal{T}_X$. In both cases the left-hand sides lie in $\mathcal{D}_{X,\leq p-1}$: this is obvious in the first case, and in the second one it follows from the fact that the p^{th} power of an element in a Poisson algebra in characteristic p lies in the Poisson center. The identities follow, since the left-hand sides kill $\mathcal{O}_X$. □

Since ι is p-linear, we consider it as a linear map $\iota : \mathcal{T}_X{}^{(1)} \to \mathcal{D}_X$.

1.3.2. Lemma. *The map $\iota : \mathcal{T}_X{}^{(1)} \to \mathcal{D}_X$ extends to an isomorphism of $\mathfrak{Z}_X \stackrel{\text{def}}{=} \mathcal{O}_{T^*X^{(1)}/X^{(1)}}$ and the center $Z(\mathcal{D}_X)$. In particular, $Z(\mathcal{D}_X)$ contains $\mathcal{O}_{X^{(1)}}$.*

Proof. For $f \in \mathcal{O}_X$ we have $f^p \in Z(\mathcal{D}_X)$, because the identity $ad(a)^p = ad(a^p)$ holds in an associative ring in characteristic p, which shows that $[f^p, \partial] = 0$ for $\partial \in \mathcal{T}_X$. This, together with Lemma 1.3.1, yields a homomorphism $\mathfrak{Z}_X \to Z(\mathcal{D}_X)$. This homomorphism is injective, because the induced map on symbols is the Frobenius map $\varphi \mapsto \varphi^p$, $\mathfrak{Z} = \mathcal{O}_{T^*X^{(1)}} \to \mathcal{O}_{T^*X}$. To prove that it is surjective it suffices to show that the Poisson center of the sheaf of Poisson algebras $\mathcal{O}_{T^*X}$ is spanned by the p^{th} powers. Since the Poisson structure arises from a nondegenerate two-form, a function $\varphi \in \mathcal{O}_{T^*X}$ lies in the Poisson center if and only if $d\varphi = 0$. It is a standard fact that a function φ on a smooth variety over a perfect field of characteristic p satisfies $d\varphi = 0$ if and only if $\varphi = \eta^p$ for some η. □

Example. If $X = \mathbb{A}^n$, so that $\mathcal{D}_X = \Bbbk\langle x_i, \partial_i\rangle$ is the Weyl algebra, then $Z(\mathcal{D}_X) = \Bbbk[x_i^p, \partial_i^p]$.

1.3.3. *The Frobenius center of enveloping algebras.* Let G be an algebraic group over $\Bbbk$, $\mathfrak{g}$ its Lie algebra. Then $\mathfrak{g}$ is the algebra of left invariant vector fields on G, and the p^{th} power map on vector fields induces the structure of a restricted Lie algebra on $\mathfrak{g}$. Considering left invariant sections of the sheaves in Lemma 1.3.2 we get an embedding $\mathcal{O}(\mathfrak{g}^{*(1)}) \xhookrightarrow{\iota_{\mathfrak{g}}} Z(U(\mathfrak{g}))$; we have $\iota_{\mathfrak{g}}(x) = x^p - x^{[p]}$ for $x \in \mathfrak{g}$. Its image is denoted $\mathfrak{Z}_{\text{Fr}}$ (the “Frobenius part” of the center).

From the construction of $\mathfrak{Z}_{\text{Fr}}$ we see that if G acts on a smooth variety X then $\mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$ extends to $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X)$ and the constant sheaf $(\mathfrak{Z}_{\text{Fr}})_X = \mathcal{O}(\mathfrak{g}^{*(1)})_X$ is mapped into the center $\mathfrak{Z}_X = \mathcal{O}_{T^*X^{(1)}}$. The last map comes from the moment map $T^*X \to \mathfrak{g}^*$.

[1] Another proof of the lemma follows directly from Hochschild's identity (see [Ho, Lemma 1]).

$U\mathfrak{g}$ is a vector bundle of rank $p^{\dim(\mathfrak{g})}$ over $\mathfrak{g}^{*(1)}$. Any $\chi \in \mathfrak{g}^*$ defines a point χ of $\mathfrak{g}^{*(1)}$ and a central reduction $U_\chi(\mathfrak{g}) \stackrel{\text{def}}{=} U(\mathfrak{g})\otimes_{\mathfrak{Z}_{\mathrm{Fr}}}\Bbbk_\chi$.

1.4. *Central reductions.* For any closed subscheme $\mathcal{Y} \subseteq T^*X$ one can restrict $\mathcal{D}_X$ to $\mathcal{Y}^{(1)} \subseteq T^*X^{(1)}$; we denote the restriction

$$\mathcal{D}_{X,\mathcal{Y}} \stackrel{\text{def}}{=} \mathcal{D}_X \underset{\mathcal{O}_{T^*X^{(1)}/X^{(1)}}}{\otimes} \mathcal{O}_{\mathcal{Y}^{(1)}/X^{(1)}}.$$

1.4.1. *Restriction to the Frobenius neighborhood of a subscheme of X.* A closed subscheme $Y\hookrightarrow X$ gives a subscheme $T^*X|Y\subseteq T^*X$, and the corresponding central reduction

$$\mathcal{D}_X \underset{\mathcal{O}_{T^*X^{(1)}}}{\otimes} \mathcal{O}_{(T^*X|Y)^{(1)}} = \mathcal{D}_X \underset{\mathcal{O}_{X^{(1)}}}{\otimes} \mathcal{O}_{Y^{(1)}} = \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\underline{X}_Y},$$

is just the restriction of $\mathcal{D}_X$ to the Frobenius neighborhood of Y. Alternatively, this is the enveloping algebra of the restriction $\mathcal{T}_X|\underline{X}_Y$ of the Lie algebroid $\mathcal{T}_X$. Locally, it is of the form $\underset{I}{\oplus}\ \mathcal{O}_{\underline{X}_Y}\partial^I$. As a quotient of $\mathcal{D}_X$ it is obtained by imposing $f^p = 0$ for $f \in \mathcal{I}_Y$. One can say that the reason we can restrict Lie algebroid $\mathcal{T}_X$ to the Frobenius neighborhood $\underline{X}_Y$ is that for vector fields (hence also for $\mathcal{D}_X$), the subscheme $\underline{X}_Y$ behaves as an open subvariety of X.

Any section ω of T^*X over $Y \subseteq X$ gives $\omega(Y)\subseteq T^*X|Y$, and a further reduction $\mathcal{D}_{X,\omega(Y)}$. The restriction to $\omega(Y)\subseteq T^*X|Y$ imposes $\iota(\partial) = \langle\omega,\partial\rangle^p$, i.e., $\partial^p = \partial^{[p]} + \langle\omega,\partial\rangle^p,\ \partial \in \mathcal{T}_X$. So, locally, $\mathcal{D}_{X,\omega(Y)} = \underset{I\in\{0,1,\dots,p-1\}^n}{\oplus} \mathcal{O}_{\underline{X}_Y}\partial^I$ and $\partial_i^p = \partial_i^{[p]} + \langle\omega,\partial_i\rangle^p = \langle\omega,\partial_i\rangle^p$.

1.4.2. *The "small" differential operators $\mathcal{D}_{X,0}$.* When $\mathcal{Y}$ is the zero section of T^*X (i.e., $X = Y$ and $\omega = 0$), we get the algebra $\mathcal{D}_{X,0}$ by imposing in $\mathcal{D}_X$ the relation $\iota\partial = 0$, i.e., $\partial^p = \partial^{[p]},\ \partial \in \mathcal{T}_X$ (in local coordinates ${\partial_i}^p = 0$). The action of $\mathcal{D}_X$ on $\mathcal{O}_X$ factors through $\mathcal{D}_{X,0}$ since ∂^p and $\partial^{[p]}$ act the same on $\mathcal{O}_X$. Actually, $\mathcal{D}_{X,0}$ is the image of the canonical map $\mathcal{D}_X\to\mathbb{D}_X$ from 1.2 (see 2.2.5).

2. The Azumaya property of $\mathcal{D}_X$

2.1. *Commutative subalgebra $\mathcal{A}_X\subseteq\mathcal{D}_X$.* We will denote the centralizer of $\mathcal{O}_X$ in $\mathcal{D}_X$ by $\mathcal{A}_X \stackrel{\text{def}}{=} Z_{\mathcal{D}_X}(\mathcal{O}_X)$, and the pull-back of $T^*X^{(1)}$ to X by $T^{*,1}X \stackrel{\text{def}}{=} X\times_{X^{(1)}}T^*X^{(1)}$.

2.1.1. Lemma. $\mathcal{A}_X = \mathcal{O}_X\cdot\mathfrak{Z}_X = \mathcal{O}_{T^{*,1}X/X}$.

Proof. The problem is local so assume that X has coordinates x_i. Then $\mathcal{D}_X = \oplus\ \mathcal{O}_X\partial^I$ and $\mathfrak{Z}_X = \oplus\ \mathcal{O}_{X^{(1)}}\partial^{pI}$ (recall that $\iota(\partial_i) = {\partial_i}^p$). So, $\mathcal{O}_X\cdot\mathfrak{Z}_X =$

$\oplus\ \mathcal{O}_X\partial^{pI} \overset{\cong}{\longleftarrow} \mathcal{O}_X\otimes_{\mathcal{O}_{X^{(1)}}}\mathfrak{Z}_X$, and this is the algebra $\mathcal{O}_X\otimes_{\mathcal{O}_{X^{(1)}}}\mathcal{O}_{T^*X^{(1)}}$ of functions on $T^{*,1}X$. Clearly, $Z_{\mathcal{D}_X}(\mathcal{O}_X)$ contains $\mathcal{O}_X{\cdot}\mathfrak{Z}_X$, and the converse $Z_{\mathcal{D}_X}(\mathcal{O}_X)\subseteq \oplus\ \mathcal{O}_X\partial^{pI}$ was already observed in the proof of Lemma 1.3.2. $\square$

2.1.2. *Remark.* In view of the lemma, any $\mathcal{D}_X$-module $\mathcal{E}$ carries an action of $\mathcal{O}_{T^{*,1}X}$; such an action is the same as a section ω of $\mathrm{Fr}^*(\Omega^1_X)\otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. As noted above $\mathcal{E}$ can be thought of as an $\mathcal{O}_X$ module with a flat connection; the section ω is known as *the p-curvature* of this connection. The section ω is parallel for the induced flat connection on $\mathrm{Fr}^*(\Omega^1_X)\otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.

2.2. *Point modules δ^ζ.* A cotangent vector $\zeta=(b,\omega)\in T^*X^{(1)}$ (i.e., $b\in X^{(1)}$ and $\omega\in T^*_aX^{(1)}$) defines a central reduction $\mathcal{D}_{X,\zeta}=\mathcal{D}_X\otimes_{\mathfrak{Z}_X}\mathcal{O}_{\zeta^{(1)}}$. Given a lifting $a\in T^*X$ of b under the Frobenius map (such a lifting exists since $\Bbbk$ is perfect and it is always unique), we get a $\mathcal{D}_X$-module $\delta^\xi \overset{\text{def}}{=} \mathcal{D}_X\otimes_{\mathcal{A}_X}\mathcal{O}_\xi$, where we have set $\xi=(a,\omega)\in T^{*,(1)}X$. It is a central reduction of the $\mathcal{D}_X$-module $\delta_a \overset{\text{def}}{=} \mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{O}_a$ of distributions at a, namely $\delta^\xi=\delta_a\otimes_{\mathfrak{Z}_X}\mathcal{O}_\zeta$. In local coordinates at a, 1.4.1 says that $\mathcal{D}_{X,\zeta}$ has a $\Bbbk$-basis $x^J\partial^I,\ I,J\in\{0,1,\dots,p-1\}^n$ with $x_i^p=0$ and $\partial_i^p=\langle\omega,\partial_i\rangle^p$.

2.2.1. Lemma. *Central reductions of $\mathcal{D}_X$ to points of $T^*X^{(1)}$ are matrix algebras. More precisely, in the above notations,*

$$\Gamma(X,\mathcal{D}_{X,\zeta}) \xrightarrow{\cong} \mathrm{End}_\Bbbk(\Gamma(X,\delta^\xi)).$$

Proof. Let $x_1,\dots,x_n$ be local coordinates at a. Near a,

$$\mathcal{D}_X=\oplus_{I\in\{0,\dots,p-1\}^n}\ \partial^I{\cdot}\mathcal{A}_X;$$

hence $\delta^\xi\cong\ \oplus_{I\in\{0,\dots,p-1\}^n}\ \Bbbk\partial^I$. Since $x_i(a)=0$,

$$x_k{\cdot}\partial^I\ =\ I_k{\cdot}\partial^{I-e_k}\qquad\text{and}\qquad \partial_k{\cdot}\partial^I=\begin{Bmatrix}\partial^{I+e_k} & \text{if } I_k+1<p,\\ \omega(\partial_i)^p{\cdot}\partial^{I-(p-1)e_k} & \text{if } I_k=p-1.\end{Bmatrix}.$$

Irreducibility of δ^ξ is now standard and x_i's act on polynomials in ∂_i's by derivations; so for $0\neq P=\sum_{I\in\{0,\dots,p-1\}^n}\ c_I\partial^I\ \in\ \delta^\xi$ and a maximal K with $c_K\neq 0$, $x^K{\cdot}P$ is a nonzero scalar. Now multiply with ∂^I's to get all of δ^ξ. Thus δ^ξ is an irreducible $\mathcal{D}_{X,\zeta}$-module. Since $\dim\mathcal{D}_{X,\zeta}=p^{2\dim(X)}=(\dim\delta^\xi)^2$ we are done. $\square$

Since the lifting $\xi\in T^{*,(1)}X$ of a point $\zeta\in T^*X^{(1)}$ exists and is unique, we will occasionally talk about point modules associated to a point in $T^*X^{(1)}$, and denote it by δ^ζ, $\zeta\in T^*X^{(1)}$.

2.2.2. Proposition (Splitting of $\mathcal{D}_X$ on $T^{*,1}X$). *Consider $\mathcal{D}_X$ as an $\mathcal{A}_X$-module $(\mathcal{D}_X)_{\mathcal{A}_X}$ via the right multiplication. Left multiplication by $\mathcal{D}_X$*

and right multiplication by $\mathcal{A}_X$ give an isomorphism

$$\mathcal{D}_X \underset{\mathfrak{Z}_X}{\otimes} \mathcal{A}_X \xrightarrow{\cong} \mathcal{E}nd_{\mathcal{A}_X}((\mathcal{D}_X)_{\mathcal{A}_X}).$$

Proof. Both sides are vector bundles over $T^{*,1}X = \mathrm{Spec}(\mathcal{A}_X)$; the $\mathcal{A}_X$-module $(\mathcal{D}_X)_{\mathcal{A}_X}$ has a local frame ∂^I, $I \in \{0,\dots,p-1\}^{\dim X}$; while $x^J\partial^I$, $J, I \in \{0,\dots,p-1\}^{\dim X}$ is a local frame for both the $\mathfrak{Z}_X$-module $\mathcal{D}_X$ and the $\mathcal{A}_X$-module $\mathcal{D}_X \otimes_{\mathfrak{Z}_X} \mathcal{A}_X$. So, it suffices to check that the map is an isomorphism on fibers. However, this is the claim of Lemma 2.2.1, since the restriction of the map to a $\Bbbk$-point ζ of $T^{*,1}X$ is the action of $(\mathcal{D}_X \otimes_{\mathfrak{Z}_X} \mathcal{A}_X) \otimes_{\mathcal{A}_X} \mathcal{O}_\zeta = \mathcal{D}_X \otimes_{\mathfrak{Z}_X} \mathcal{O}_\zeta = \mathcal{D}_{X,\zeta}$ on $(\mathcal{D}_X)_{\mathcal{A}_X} \otimes_{\mathcal{A}_X} \mathcal{O}_\zeta = \delta^\zeta$. □

2.2.3. Theorem. *$\mathcal{D}_X$ is an Azumaya algebra over $T^*X^{(1)}$ (nontrivial if* $\dim(X) > 0$).

Proof. One of the characterizations of Azumaya algebras is that they are coherent as $\mathcal{O}$-modules and become matrix algebras on a flat cover [MI]. The map $T^{*,1}X \to T^*X^{(1)}$ is faithfully flat; i.e., it is a flat cover, since the Frobenius map $X \to X^{(1)}$ is flat for smooth X (it is surjective and on the formal neighborhood of a point given by $\Bbbk[[x_i^p]] \hookrightarrow \Bbbk[[x_i]]$). If $\dim(X) > 0$, then $\mathcal{D}_X$ is nontrivial, i.e. it is not isomorphic to an algebra of the form $\mathrm{End}(V)$ for a vector bundle V, because locally in the Zariski topology of X, $\mathcal{D}_X$ has no zero-divisors, since $\mathrm{gr}(\mathcal{D}_X) = \mathcal{O}_{T^*X}$; while the algebra of endomorphisms of a vector bundle of rank higher than one on an affine algebraic variety has zero divisors. □

2.2.4. *Remarks.*(1) A related Azumaya algebra was considered in [Hur].

(2) One can give a different, somewhat shorter proof of Theorem 2.2.3 based on the fact that a function on a smooth $\Bbbk$-variety has zero differential if and only if it is a p^{th} power, which implies that any Poisson ideal in $\mathcal{O}_{T^*X}$ is induced from $\mathcal{O}_{T^*X^{(1)}}$. This proof applies to a more general situation of the so called Frobenius constant quantizations of symplectic varieties in positive characteristic, see [BeKa, Prop. 3.8].

(3) The statement of the theorem can be compared to the well-known fact that the algebra of differential operators in characteristic zero is simple: in characteristic p it becomes simple after a central reduction. Another analogy is with the classical Stone – von Neumann Theorem, which asserts that $L^2(\mathbb{R}^n)$ is the only irreducible unitary representation of the Weyl algebra: Theorem 2.2.3 implies, in particular, that the standard quantization of functions on the Frobenius neighborhood of zero in $\mathbb{A}^{2n}_\Bbbk$ has unique irreducible representation realized in the space of functions on the Frobenius neighborhood of zero in $\mathbb{A}^n_\Bbbk$.

(4) The class of the Azumaya algebra in the Brauer group can be described as follows. In [MI, II.4.14] one finds the following exact sequence of sheaves in étale topology available for any smooth variety M over a perfect field of characteristic p:

$$0 \to \mathcal{O}_M^* \xrightarrow{\mathrm{Fr}} \mathcal{O}_M^* \xrightarrow{d\log} \Omega^1_{M,cl} \xrightarrow{C-1} \Omega^1_M \to 0,$$

where $\mathrm{Fr} : f \mapsto f^p$, C is the Cartier operator and $\Omega^1_{M,cl}$ is the sheaf of closed 1-forms. This exact sequences produces a map $H^0(\Omega^1_M) \to H^2(\mathcal{O}_M^*)$. One can check that applying the map to the canonical 1-form on $M = T^*X$ one gets the class of the Azumaya algebra $\mathcal{D}_X$.

2.2.5. *Splitting on the zero section.* By a well known observation[2] the small differential operators, i.e., the restriction $\mathcal{D}_{X,0}$ of $\mathcal{D}_X$ to $X^{(1)} \subseteq T^*X^{(1)}$, form a sheaf of matrix algebras. In the notation above, this is the observation that the action map $(\mathrm{Fr}_X)_*\mathcal{D}_{X,0} \xrightarrow{\cong} \mathcal{E}nd_{\mathcal{O}_{X^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{O}_X)$ is an isomorphism by 2.2.1. Thus Azumaya algebra $\mathcal{D}_X$ splits on $X^{(1)}$, and $(\mathrm{Fr}_X)_*\mathcal{O}_X$ is a splitting bundle. The corresponding equivalence between $\mathrm{Coh}_{X^{(1)}}$ and $\mathcal{D}_{X,0}$ modules sends $\mathcal{F} \in \mathrm{Coh}_{X^{(1)}}$ to the sheaf $\mathrm{Fr}_X^*\mathcal{F}$ equipped with a standard flat connection (the one for which pull-back of a section of $\mathcal{F}$ is parallel).

2.2.6. *Remark.* Let $Z \subset T^*X^{(1)}$ be a closed subscheme, such that the Azumaya algebra $\mathcal{D}_X$ splits on Z (see Section 5 below for more examples of this situation); thus we have a splitting vector bundle $\mathcal{E}_Z$ on Z such that $\mathcal{D}_X|_Z \xrightarrow{\cong} \mathrm{End}(\mathcal{E}_Z)$. It is easy to see then that $\mathcal{E}_Z$ is a locally free, rank one module over $\mathcal{A}_X|_Z$, thus it can be thought of as a line bundle on the preimage Z' of Z in $T^{*(1)}X$ under the map $\mathrm{Fr} \times \mathrm{id} : X \times_{X^{(1)}} T^*X^{(1)} \to T^*X^{(1)}$. In the particular case when Z maps isomorphically to its image $\bar{Z}$ in X the scheme Z' is identified with the Frobenius neighborhood of $\bar{Z}$ in X. The action of $\mathcal{D}_X$ equips the resulting line bundle on $\mathrm{Fr}\, N(\bar{Z})$ with a flat connection. The above splitting on the zero-section corresponds to the trivial line bundle $\mathcal{O}_X$ with the standard flat connection.

2.3. *Torsors.* A torsor $\widetilde{X} \xrightarrow{\pi} X$ for a torus T defines a Lie algebroid $\widetilde{\mathcal{T}}_X \stackrel{\text{def}}{=} \pi_*(\mathcal{T}_{\widetilde{X}})^T$ with the enveloping algebra $\widetilde{\mathcal{D}}_X \stackrel{\text{def}}{=} \pi_*(\mathcal{D}_{\widetilde{X}})^T$. Let $\mathfrak{t}$ be the Lie algebra of T. Locally, any trivialization of the torsor splits the exact sequence $0 \to \mathfrak{t} \otimes \mathcal{O}_X \to \widetilde{\mathcal{T}}_X \to \mathcal{T}_X \to 0$ and gives $\widetilde{\mathcal{D}}_X \cong \mathcal{D} \otimes U\mathfrak{t}$. So the map of the constant sheaf $U(\mathfrak{t})_X$ into $\widetilde{\mathcal{D}}_X$, given by the T-action, is a central embedding and $\widetilde{\mathcal{D}}_X$ is a deformation of $\mathcal{D}_X \cong \widetilde{\mathcal{D}}_X \otimes_{S(\mathfrak{t})} \Bbbk_0$ over $\mathfrak{t}^*$. The center $\mathcal{O}_{T^*\widetilde{X}^{(1)}}$ of $\mathcal{D}_{\widetilde{X}}$ gives a central subalgebra $(\pi_*\mathcal{O}_{T^*\widetilde{X}^{(1)}})^T = \mathcal{O}_{\widetilde{T^*X}^{(1)}}$ of $\widetilde{\mathcal{D}}_X$. We combine the two into a map from functions on $\widetilde{T}^*X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$ to $Z(\widetilde{\mathcal{D}}_X)$ (the map $\mathfrak{t}^* \to \mathfrak{t}^{*(1)}$ is the Artin-Schreier map AS; the corresponding map on the rings of functions

[2] The second author thanks Paul Smith from whom he has learned this observation.

$S(\mathfrak{t}^{(1)}) \to S(\mathfrak{t})$ is given by $\iota(h) = h^p - h^{[p]},\ h \in \mathfrak{t}^{(1)}$). Local trivializations again show that this is an isomorphism and that $\widetilde{\mathcal{D}}_X$ is an Azumaya algebra on $\widetilde{T}^*X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$, which splits on $X \times_{X^{(1)}} (\widetilde{T}^*X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*)$.

In particular, for any $\lambda \in \mathfrak{t}^*$, specialization $\mathcal{D}_X^\lambda \stackrel{\text{def}}{=} \widetilde{\mathcal{D}}_X \otimes_{S(\mathfrak{t})} \Bbbk_\lambda$ is an Azumaya algebra on the twisted cotangent bundle

$$T^*_{\mathrm{AS}(\lambda)} X^{(1)} \stackrel{\text{def}}{=} \widetilde{T}^* X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathrm{AS}(\lambda),$$

which splits on $T^{*,(1)}_{\mathrm{AS}(\lambda)} X \stackrel{\text{def}}{=} X \times_{X^{(1)}} T^*_{\mathrm{AS}(\lambda)} X^{(1)}$. For instance, if $\lambda = d(\chi)$ is the differential of a character χ of T then $\mathrm{AS}(\lambda) = 0$; thus $T^*_{\mathrm{AS}(\lambda)} X = T^*X$. In this case $\mathcal{D}_X^\lambda$ is identified with the sheaf ${}^{\mathcal{O}_\chi}\mathcal{D}_X \cong \mathcal{O}_\chi \otimes \mathcal{D}_X \otimes \mathcal{O}_\chi{}^{-1}$ of differential operators on sections of the line bundle $\mathcal{O}_\chi$ on X, associated to $\widetilde{X}$ and χ.

By a straightforward generalization of 2.1, 2.2, $\widetilde{\mathcal{A}}_X \stackrel{\text{def}}{=} \mathcal{O}_{X \times_{X^{(1)}} \widetilde{T}^* X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}$ embeds into $\widetilde{\mathcal{D}}_X$. As in 2.2, for a point $\zeta = (a, \omega; \lambda)$ of $X \times_{X^{(1)}} \widetilde{T}^* X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$ we define the point module $\delta^\zeta = \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{A}}_X} \mathcal{O}_\zeta$. If $\zeta^{(1)} = (\omega, \lambda)$ is the corresponding point of $\widetilde{T}^* X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$ then we have $\widetilde{\mathcal{D}}_X \otimes_{Z(\widetilde{\mathcal{D}}_X)} \mathcal{O}_{\zeta^{(1)}} \xrightarrow{\cong} \mathcal{E}nd_{\Bbbk}(\delta^\zeta)$.

We finish the section with a technical lemma to be used in Section 5.

2.3.1. Lemma. *Let $\nu = d(\eta)$ be an integral character. Define a morphism τ_ν from $\widetilde{T}^* X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$ to itself by $\tau_\nu(x, \lambda) = (x, \lambda + \nu)$. Then the Azumaya algebras $\widetilde{\mathcal{D}}_X$ and $\tau_\nu^*(\widetilde{\mathcal{D}}_X)$ are canonically equivalent.*

Proof. Recall that to establish an equivalence between two Azumaya algebras $\mathcal{A}$, $\mathcal{A}'$ on a scheme Y (i.e. an equivalence between their categories of modules) one needs to provide a locally projective module M over $\mathcal{A} \otimes_{\mathcal{O}_Y} (\mathcal{A}')^{\mathrm{op}}$ such that $\mathcal{A} \xrightarrow{\cong} \mathrm{End}_{(\mathcal{A}')^{\mathrm{op}}}(M)$, $\mathcal{A}' \xrightarrow{\cong} \mathrm{End}_{\mathcal{A}}(M)$. The sheaf $\pi_*(\mathcal{D}_{\widetilde{X}})^{T,\eta}$ of sections of $\pi_*(\mathcal{D}_{\widetilde{X}})$ which transform by the character η under the action of T carries the structure of such a module. $\square$

3. Localization of $\mathfrak{g}$-modules to $\mathcal{D}$-modules on the flag variety

This crucial section extends the basic result of [BB], [BrKa] to positive characteristic.

3.1. *The setting.* We define relevant triangulated categories of $\mathfrak{g}$-modules and $\mathcal{D}$-modules and functors between them.

3.1.1. *Semisimple group G.* Let G be a semisimple simply-connected algebraic group over $\Bbbk$. Let $B = T \cdot N$ be a Borel subgroup with the unipotent radical N and a Cartan subgroup T. Let H be the (abstract) Cartan group of G so that B gives isomorphism $\iota_{\mathfrak{b}} = (T \xrightarrow{\cong} B/N \cong H)$. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{n}, \mathfrak{h}$ be the corresponding Lie algebras. The weight lattice $\Lambda = X^*(H)$ contains the set of roots Δ and of positive roots Δ^+. Roots in Δ^+ are identified with T-roots in $\mathfrak{g}/\mathfrak{b}$ via the above "$\mathfrak{b}$-identification" $\iota_{\mathfrak{b}}$. Also, Λ contains the root lattice Q generated by Δ, the dominant cone $\Lambda^+ \subseteq \Lambda$ and the semi-group Q^+ generated by Δ^+. Let $I \subseteq \Delta^+$ be the set of simple roots. For a root $\alpha \in \Delta$ let $\alpha \mapsto \check{\alpha} \in \check{\Delta}$ be the corresponding coroot.

Similarly, $\iota_{\mathfrak{b}}$ identifies $N_G(T)/T$ with the Weyl group $W \subseteq \mathrm{Aut}(H)$. Let $W_{\mathrm{aff}} \overset{\mathrm{def}}{=} W \ltimes Q \subseteq W'_{\mathrm{aff}} \overset{\mathrm{def}}{=} W \ltimes \Lambda$ be the affine Weyl group and the extended affine Weyl group. We have the standard action of W on Λ, $w : \lambda \mapsto w(\lambda) = w{\cdot}\lambda$, and the ρ-shift gives the dot-action $w : \lambda \mapsto w\bullet\lambda = w\bullet_\rho\lambda \overset{\mathrm{def}}{=} w(\lambda+\rho)-\rho$ which is centered at $-\rho$, where ρ is the half sum of positive roots. Both actions extend to W'_{aff} so that $\mu \in \Lambda$ acts by the $p\mu$-translation. We will indicate the dot-action by writing $(W, \bullet)$, this is really the action of the ρ-conjugate ${}^\rho W$ of the subgroup $W \subseteq W'_{\mathrm{aff}}$.

Any weight $\nu \in \Lambda$ defines a line bundle $\mathcal{O}_{\mathcal{B},\nu} = \mathcal{O}_\nu$ on the flag variety $\mathcal{B} \cong G/B$, and a standard G-module $V_\nu \overset{\mathrm{def}}{=} \mathrm{H}^0(\mathcal{B}, \mathcal{O}_{\nu^+})$ with extremal weight ν. Here ν^+ denotes the dominant W-conjugate of ν (notice that a dominant weight corresponds to a semi-ample line bundle in our normalization). We will also write $\mathcal{O}_\nu$ instead of $\pi^*(\mathcal{O}_\nu)$ for a scheme X equipped with a map $\pi : X \to \mathcal{B}$ (e.g. a subscheme of $\widetilde{\mathfrak{g}}^*$).

We let $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone, i.e. the zero set of invariant polynomials of positive degree.

3.1.2. *Restrictions on the characteristic p.* Let h be the maximum of Coxeter numbers of simple components of G. If G is simple then $h = \langle \rho, \check{\alpha}_0 \rangle + 1$ where $\check{\alpha}_0$ is the highest coroot. We mostly work under the assumption $p > h$, though some intermediate statements are proved under weaker assumptions; a straightforward extension of the main Theorem 3.2 with weaker assumptions on p is recorded in the sequel paper [BMR2]. The main result is obtained for a regular Harish-Chandra central character, and the most interesting case is that of an integral Harish-Chandra central character; integral regular characters exist only for $p \geq h$, hence our choice of restrictions[3] on p.

Recall that a prime is called good if it does not coincide with a coefficient of a simple root in the highest root [SS, §4], and p is very good if it is good and G does not contain a factor isomorphic to $\mathrm{SL}(mp)$ [Sl, 3.13]. We will need a crude observation that $p > h \Rightarrow$ very good $\Rightarrow$ good.

For p very good $\mathfrak{g}$ carries a nondegenerate invariant bilinear form; also $\mathfrak{g}$ is simple provided that G is simple [Ja, 6.4]. We will occasionally identify $\mathfrak{g}$ and $\mathfrak{g}^*$ as G-modules. This will identify the nilpotent cones $\mathcal{N}$ in $\mathfrak{g}$ and $\mathfrak{g}^*$.

[3]The case $p = h$ is excluded because for $G = \mathrm{SL}(p)$, $p = h$ is not very good and $\mathfrak{g} \not\cong \mathfrak{g}^*$ as G-modules.

3.1.3. *The sheaf* $\widetilde{\mathcal{D}}$. Our main object is the sheaf $\mathcal{D} = \mathcal{D}_{\mathcal{B}}$ on the flag variety. Along with $\mathcal{D}$ we will consider its deformation $\widetilde{\mathcal{D}}$ defined by the H-torsor $\widetilde{\mathcal{B}} \stackrel{\text{def}}{=} G/N \xrightarrow{\pi} \mathcal{B}$ as in subsection 2.3. Here $G \times H$ acts on $\widetilde{\mathcal{B}} = G/N$ by $(g,h)\cdot aN \stackrel{\text{def}}{=} gahN$, and this action differentiates to a map $\mathfrak{g}\oplus\mathfrak{h} \to \widetilde{\mathcal{T}}_{\mathcal{B}}$ which extends to $U(\mathfrak{g})\otimes U(\mathfrak{h}) \to \widetilde{\mathcal{D}}_{\mathcal{B}}$. Then $\widetilde{\mathcal{D}} = \pi_*(\mathcal{D}_{\widetilde{\mathcal{B}}})^H$ is a deformation over $\mathfrak{h}^*$ of $\mathcal{D} \cong \widetilde{\mathcal{D}} \otimes_{S(\mathfrak{h})} \mathbb{k}_0$.

The corresponding deformation of $T^*\mathcal{B}$ will be denoted $\tilde{\mathfrak{g}}^* = \widetilde{T}^*\mathcal{B} = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B},\ x|_{\mathrm{rad}(\mathfrak{b})} = 0\}$; we have projections $\mathbf{pr}_1 : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$, $\mathbf{pr}_1(\mathfrak{b}, x) = x$ and $\mathbf{pr}_2 : \tilde{\mathfrak{g}}^* \to \mathfrak{h}^*$ sending $(\mathfrak{b}, x)$ to $x|_{\mathfrak{b}} \in (\mathfrak{b}/\mathrm{rad}(\mathfrak{b}))^* = \mathfrak{h}^*$; they yield a map $\mathbf{pr} = \mathbf{pr}_1 \times \mathbf{pr}_2 : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^* \times_{\mathfrak{h}^*//W} \mathfrak{h}^*$. According to subsection 2.3 the sheaf $\widetilde{\mathcal{D}}$ is an Azumaya algebra on $\tilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$ where $\mathfrak{h}^*$ maps to $\mathfrak{h}^{*(1)}$ by the Artin-Schreier map.

We denote for any B-module Y by $Y^{\mathbf{0}}$ the sheaf of sections of the associated G-equivariant vector bundle on $\mathcal{B}$. For instance, vector bundle $\mathcal{T}_{\mathcal{B}} = [\mathfrak{g}/\mathfrak{b}]^{\mathbf{0}}$ is generated by the space $\mathfrak{g}$ of global sections, so that $\mathfrak{g}$ and $\mathcal{O}_{\mathcal{B}}$ generate $\mathcal{D}$ as an $\mathcal{O}_{\mathcal{B}}$-algebra; one finds that $\mathcal{D}$ is a quotient of the smash product $U^{\mathbf{0}} = \mathcal{O}_{\mathcal{B}} \# U(\mathfrak{g})$ (the semi-direct tensor product), by the two-sided ideal $\mathfrak{b}^{\mathbf{0}}\cdot U(\mathfrak{g})^{\mathbf{0}}$. So $\mathcal{D} = [U(\mathfrak{g})/\mathfrak{b}U(\mathfrak{g})]^{\mathbf{0}}$, and the fiber (with respect to the left $\mathcal{O}$-action) at $\mathfrak{b} \in \mathcal{B}$ is $\mathcal{O}_{\mathfrak{b}} \otimes_{\mathcal{O}} \mathcal{D} \cong U(\mathfrak{g})/\mathfrak{b}U(\mathfrak{g})$. Similarly, $\widetilde{\mathcal{D}} = [U(\mathfrak{g})/\mathfrak{n}U(\mathfrak{g})]^{\mathbf{0}}$.

3.1.4. *Baby Verma and point modules.* Here we show that $\widetilde{\mathcal{D}}$ can be thought of as the sheaf of endomorphisms of the "universal baby Verma module".

Recall the construction of the baby Verma module over $U(\mathfrak{g})$. To define it one fixes a Borel $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{t} \subset \mathfrak{g}$, and elements $\chi \in \mathfrak{g}^{*(1)}$, $\lambda \in \mathfrak{t}^*$, such that $\chi|_{\mathfrak{n}^{(1)}} = 0$, $\chi|_{\mathfrak{t}^{(1)}} = \mathrm{AS}(\lambda)$ (see 2.3 for notation). For such a triple $\zeta = (\mathfrak{b}, \chi; \lambda)$ one sets $M_\zeta = U_\chi(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{k}_\lambda$, where $U_\chi(\mathfrak{g})$ is as in 1.3.3, and $\mathbb{k}_\lambda$ is the one dimensional $\mathfrak{b}$-module given by the map $\mathfrak{b} \to \mathfrak{t} \xrightarrow{\lambda} \mathbb{k}$.

On the other hand, a triple $\zeta = (\mathfrak{b}, \chi; \lambda)$ as above defines a point of $\tilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$ (here we use the isomorphism $\mathfrak{t} \cong \mathfrak{h}$ defined by $\mathfrak{b}$); thus we have the corresponding point module δ^ζ over $\widetilde{\mathcal{D}}$ (see 2.3). Pulling back this module under the homomorphism $U(\mathfrak{g}) \to \Gamma(\widetilde{\mathcal{D}})$ we get a $U(\mathfrak{g})$-module (also denoted by δ^ζ).

PROPOSITION. $\delta^\zeta \cong M_{\mathfrak{b},\chi;\lambda+2\rho}$.

Proof. Let $\mathfrak{n}^- \subset \mathfrak{g}$ be a maximal unipotent subalgebra opposite to $\mathfrak{b}$, and set $U_\chi(\mathfrak{n}^-) = U_{\chi|_{\mathfrak{n}^-}}(\mathfrak{n}^-)$. It suffices to check that there exists a vector $v \in \delta^\zeta$ such that (1) the subspace $\mathbb{k}v$ is $\mathfrak{b}$-invariant, and $\mathbb{k}v \cong \mathbb{k}_{\lambda+2\rho}$; and (2) δ^ζ is a free $U_\chi(\mathfrak{n}^-)$-module with generator v. These two statements follow from the next lemma, which is checked by a straightforward computation in local coordinates.

Lemma. *Let $\mathfrak{a}$ be a Lie algebra acting*[4] *on a smooth variety X and let $\widetilde{X} \to X$ be an $\mathfrak{a}$-equivariant torsor for a torus T. Let $\zeta = (x, \chi; \lambda)$ be a point of $X \times_{X^{(1)}} \widetilde{T}^* X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$, and δ^ζ be the corresponding point module. Let $v \in \delta^\zeta$ be the canonical generator, $v = 1 \otimes 1$.*

a) *If x is fixed by $\mathfrak{a}$ then $\mathfrak{a}$ acts on v by $\lambda_x - \omega_x$, where*: (1) *the character $\lambda_x : \mathfrak{a} \to \Bbbk$ is the pairing of $\lambda \in \mathfrak{t}^*$ with the action of $\mathfrak{a}$ on the fiber $\widetilde{X}_x$, and* (2) *the character $\omega_x : \mathfrak{a} \to \Bbbk$ is the action of $\mathfrak{a}$ on the fiber at x of the canonical bundle ω_X.*[5]

b) *If, on the other hand, the action is simply transitive at x* (*i.e. it induces an isomorphism $\mathfrak{a} \xrightarrow{\cong} T_x X$*), *then the map $u \mapsto u(v)$ gives an isomorphism $U_{\chi_x}(\mathfrak{a}) \xrightarrow{\cong} \delta^\zeta$; here $\chi_x \in \mathfrak{a}^{*(1)}$ is the pull-back of $\chi \in \widetilde{T}^*_x X$ under the action map.*

3.1.5. *The "Harish-Chandra center" of $U(\mathfrak{g})$.* Now let $U = U\mathfrak{g}$ be the enveloping algebra of $\mathfrak{g}$. The subalgebra of G-invariants $\mathfrak{Z}_{\mathrm{HC}} \stackrel{\mathrm{def}}{=} (U\mathfrak{g})^G$ is clearly central in $U\mathfrak{g}$.

Lemma. *Let the characteristic p be arbitrary*; *the group G is simply-connected, as above.*

(a) *The map $U(\mathfrak{h}) \to \Gamma(\mathcal{B}, \widetilde{\mathcal{D}})$ defined by the H-action on $\widetilde{\mathcal{B}}$ gives an isomorphism $U(\mathfrak{h}) \xrightarrow{\cong} \Gamma(\mathcal{B}, \widetilde{\mathcal{D}})^G$.*

(b) *The map $U^G \to \Gamma(\mathcal{B}, \widetilde{\mathcal{D}})^G \cong S(\mathfrak{h})$ gives an isomorphism $U^G \xrightarrow{i_{\mathrm{HC}}} S(\mathfrak{h})^{(W,\bullet)}$* (*the "Harish-Chandra map"*). *For good p this isomorphism is strictly compatible with filtrations, where the filtration on Z_{HC} is induced by the canonical filtration on U, while the one on the target is induced by the filtration on $S(\mathfrak{h})$ by degree.*

(c) *The map $U(\mathfrak{g}) \otimes S(\mathfrak{h}) \to \Gamma(\mathcal{B}, \widetilde{\mathcal{D}})$ factors through $\widetilde{U} \stackrel{\mathrm{def}}{=} U \otimes_{\mathfrak{Z}_{\mathrm{HC}}} S(\mathfrak{h})$.*

Proof. We borrow the arguments from [Mi]. In (a),

$$\Gamma(\mathcal{B}, \widetilde{\mathcal{D}})^G = \Gamma(\mathcal{B}, [U/\mathfrak{n}U]^{\mathbf{0}})^G \cong [U/\mathfrak{n}U]^B \supseteq U(\mathfrak{b})/\mathfrak{n}U(\mathfrak{b}) \cong U(\mathfrak{h}),$$

and the inclusion is an equality, as one sees by calculating invariants for a Cartan subgroup $T \subseteq B$.

For (b), the map $U \to \Gamma(\mathcal{B}, \widetilde{\mathcal{D}})$ restricts to a map $U^G \xrightarrow{i_{\mathrm{HC}}} \Gamma(\mathcal{B}, \widetilde{\mathcal{D}})^G \cong U(\mathfrak{h})$, which fits into $U^G \subseteq U \twoheadrightarrow U/\mathfrak{n}U \supseteq U(\mathfrak{b})/\mathfrak{n}U(\mathfrak{b}) \cong U(\mathfrak{h})$. So, $U^G \subseteq \mathfrak{n}U + U(\mathfrak{b})$ and i_{HC} is the composition $U^G \subseteq \mathfrak{n}U + U(\mathfrak{b}) \twoheadrightarrow [\mathfrak{n}U + U(\mathfrak{b})]/\mathfrak{n}U \cong U(\mathfrak{h})$. On the other hand a choice of a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{b}$ defines an

[4] An action of a Lie algebra $\mathfrak{a}$ on a variety X is an action of $\mathfrak{a}$ on $\mathcal{O}_X$ by derivations. Equivalently, it is a Lie algebra homomorphism from $\mathfrak{a}$ to the algebra of vector fields on X.

[5] For a section Ω of ω_X near x and $\xi \in \mathfrak{a}$, $\mathrm{Lie}_\xi(\Omega)|_x = \omega_x(\xi) \cdot \Omega|_x$.

opposite Borel subalgebra $\overline{\mathfrak{b}}$ with $\overline{\mathfrak{b}}\cap\mathfrak{b}=\mathfrak{t}$ and $\overline{\mathfrak{b}}=\ \overline{\mathfrak{n}}\ltimes\mathfrak{t}$. Let us use the B-identification $\iota_{\mathfrak{b}}:\mathfrak{h}^*\cong\mathfrak{t}^*$ from 3.1.1 to carry over the dot-action of W to $\mathfrak{t}^*$ (now the shift is by $\iota_{\mathfrak{b}}(\rho)=\rho_{\overline{\mathfrak{n}}}$, the half sum of T-roots in $\overline{\mathfrak{n}}$). According to [Ja, 9.3], an argument of [KW] shows that for any simply-connected semisimple group, regardless of p, the projection $U=(\mathfrak{n}U+U\overline{\mathfrak{n}})\oplus U(\mathfrak{t})\to U(\mathfrak{t})$ restricts to the Harish-Chandra isomorphism $\mathfrak{Z}_{\mathrm{HC}}\xrightarrow{\iota_{\mathfrak{n},\overline{\mathfrak{n}}}}S(\mathfrak{t})^{W,\bullet}$. Therefore, $i_{\mathrm{HC}}=\iota_{\mathfrak{b}}\circ\iota_{\mathfrak{n},\overline{\mathfrak{n}}}$ is an isomorphism $\mathfrak{Z}_{\mathrm{HC}}\xrightarrow{\cong}S(\mathfrak{h})^{W,\bullet}$.

Strict compatibility with filtrations follows from the fact that the homomorphism $U\to\Gamma(\widetilde{\mathcal{D}})$ is strictly compatible with filtrations. The latter follows from injectivity of the induced map on the associated graded algebras: $S(\mathfrak{g})=\mathrm{gr}(U)\to\Gamma(\mathcal{O}_{\widetilde{\mathfrak{g}}^*})\cong\mathrm{gr}(\Gamma(\widetilde{\mathcal{D}}))$. Here the last isomorphism holds for good p, because of vanishing of higher cohomology $H^{>0}(\mathcal{B},\mathrm{gr}(\widetilde{\mathcal{D}}))=H^{>0}(\widetilde{\mathfrak{g}}^*,\mathcal{O})$. This cohomology vanishing for good p follows from [KLT], cf. the proof of Proposition 3.4.1 below. Injectivity of the map $\mathcal{O}(\mathfrak{g}^*)\to\Gamma(\mathcal{O}_{\widetilde{\mathfrak{g}}^*})$ follows from the fact that the morphism $\widetilde{\mathfrak{g}}^*\to\mathfrak{g}^*$ is dominant. This latter fact is a consequence of [Ja, 6.6], which claims that every element in $\mathfrak{g}^*$ annihilates the radical of some Borel subalgebra by a result of [KW].

Finally, (c) means that the two maps from $\mathfrak{Z}_{\mathrm{HC}}$ to $\Gamma(\mathcal{B},\widetilde{\mathcal{D}})$, via U and $S\mathfrak{h}$, are the same – but this is the definition of the second map. □

3.1.6. *The center of* $U(\mathfrak{g})$ [Ve], [KW], [MR1]. For a very good p the center $\mathfrak{Z}$ of U is a combination of the Harish-Chandra part (3.1.5) and the Frobenius part (1.3.3):

$$\mathfrak{Z}\xleftarrow{\cong}\mathfrak{Z}_{\mathrm{Fr}}\otimes_{\mathfrak{Z}_{\mathrm{Fr}}\cap\mathfrak{Z}_{\mathrm{HC}}}\mathfrak{Z}_{\mathrm{HC}}\cong\ \mathcal{O}(\mathfrak{g}^{*(1)}\times_{\mathfrak{h}^{*(1)}//W}\mathfrak{h}^*//(W,\bullet)).$$

Here, $//$ denotes the invariant theory quotient, the map $\mathfrak{g}^{*(1)}\to\mathfrak{h}^{*(1)}//W$ is the adjoint quotient, while the map $\mathfrak{h}^*//(W,\bullet)\to\mathfrak{h}^{*(1)}//W$ comes from the Artin-Schreier map $\mathfrak{h}^*\xrightarrow{\mathrm{AS}}\mathfrak{h}^{*(1)}$ defined in 2.3.

3.1.7. *Derived categories of sheaves supported on a subscheme.* Let $\mathcal{A}$ be a coherent sheaf on a Noetherian scheme $\mathfrak{X}$ equipped with an associative $\mathcal{O}_{\mathfrak{X}}$-algebra structure. We denote by $\mathrm{mod}^{\mathrm{c}}(\mathcal{A})$ the abelian category of coherent $\mathcal{A}$-modules. We also use notations $\mathcal{C}\mathrm{oh}(\mathfrak{X})$ if $\mathcal{A}=\mathcal{O}_{\mathfrak{X}}$ and $\mathrm{mod}^{\mathrm{fg}}(\mathcal{A})$ if $\mathfrak{X}$ is a point.

We denote by $\mathrm{mod}^{\mathrm{c}}_{\mathfrak{Y}}(\mathcal{A})$ the full subcategory of coherent $\mathcal{A}$-modules supported set-theoretically in $\mathfrak{Y}$, i.e., killed by some power of the ideal sheaf $\mathcal{I}_{\mathfrak{Y}}$. The following statement is standard.

Lemma. a) *The tautological functor identifies the bounded derived category* $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_{\mathfrak{Y}}(\mathcal{A}))$ *with a full subcategory in* $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{A}))$.

b) *For* $\mathcal{F}\in\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{A}))$ *the following conditions are equivalent*:

i) $\mathcal{F}\in\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_{\mathfrak{Y}}(\mathcal{A}))$;

ii) *$\mathcal{F}$ is killed by a power of the ideal sheaf $\mathcal{I}_{\mathfrak{Y}}$, i.e. the tautological arrow $\mathcal{I}_{\mathfrak{Y}}^n \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}$ is zero for some n;*

iii) *the cohomology sheaves of $\mathcal{F}$ lie in* $\mathrm{mod}^{\mathrm{c}}_{\mathfrak{Y}}(\mathcal{A})$.

Proof. In (a) we can replace $\mathrm{mod}^{\mathrm{c}}$ with $\mathrm{mod}^{\mathrm{qc}}$ (since $\mathcal{A}$ is coherent, $D(\mathrm{mod}^{\mathrm{c}}(\mathcal{A}))$ is a full subcategory of $D(\mathrm{mod}^{\mathrm{qc}}(\mathcal{A}))$, and the same proof works for $D(\mathrm{mod}^{\mathrm{c}}_{\mathfrak{Y}}(\mathcal{A}))$ and $D(\mathrm{mod}^{\mathrm{qc}}_{\mathfrak{Y}}(\mathcal{A})))$. Now it suffices to show that each sheaf in $\mathrm{mod}^{\mathrm{qc}}_{\mathfrak{Y}}(\mathcal{A})$ embeds into an object of $\mathrm{mod}^{\mathrm{qc}}_{\mathfrak{Y}}(\mathcal{A})$ which is injective in $\mathrm{mod}^{\mathrm{qc}}(\mathcal{A})$ ([Ha, Prop. I.4.8]). This follows from the corresponding statement for quasi-coherent sheaves of $\mathcal{O}$ modules (see e.g. [Ha, Th. I.7.18 and its proof]), since we can get a quasicoherent injective sheaf of $\mathcal{A}$-modules from an injective quasicoherent sheaf of $\mathcal{O}$-modules by coinduction.

b) Implications (i)⇒(ii)⇒(iii) are clear by definitions, and (iii)⇒(i) is clear from (a). □

3.1.8. *Categories of modules with a generalized Harish-Chandra character.* Let us apply 3.1.7 to $\widetilde{\mathcal{D}}$ and U (or $\widetilde{U}$), considered as coherent sheaves over the spectra $\widetilde{T}^*\mathcal{B}^{(1)}$ and $\mathfrak{g}^{*(1)}$ of central subalgebras. The interesting categories are $\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda) \subseteq \mathrm{mod}^{\mathrm{c}}_{\lambda}(\widetilde{\mathcal{D}}) \subseteq \mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}})$. Here, $\mathrm{mod}^{\mathrm{c}}_{\lambda}(\widetilde{\mathcal{D}}) \stackrel{\mathrm{def}}{=} \mathrm{mod}^{\mathrm{c}}_{\widetilde{T}^*_{\mathrm{AS}(\lambda)}\mathcal{B}^{(1)}}(\widetilde{\mathcal{D}})$ consists of those objects in $\mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}})$ which are killed by a power of the maximal ideal λ in $U\mathfrak{h}$.

For $\lambda \in \mathfrak{h}^*$, denote by U^λ the specialization of U at the image of λ in $\mathfrak{h}^*//W = \mathrm{Spec}(\mathfrak{Z}_{\mathrm{HC}})$, i.e., the specialization of $\widetilde{U}$ at $\lambda \in \mathfrak{h}^*$. There are analogous abelian categories $\mathrm{mod}^{\mathrm{fg}}(U^\lambda) \subseteq \mathrm{mod}^{\mathrm{fg}}_{\lambda}(U) \subseteq \mathrm{mod}^{\mathrm{fg}}(U)$, where the category $\mathrm{mod}^{\mathrm{fg}}_{\lambda}(U) \stackrel{\mathrm{def}}{=} \mathrm{mod}^{\mathrm{c}}_{\mathfrak{g}^{*(1)}_{\lambda}}(U)$ for $\mathfrak{g}^{*(1)}_{\lambda} \stackrel{\mathrm{def}}{=} \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^*//W^{(1)}} \mathrm{AS}(\lambda)$, consists of U-modules killed by a power of the maximal ideal in $\mathfrak{Z}_{\mathrm{HC}}$. The corresponding triangulated categories are $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U^\lambda)) \to \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_{\lambda}(U)) \subseteq \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U))$.

3.1.9. *The global section functors on D-modules.* Let $\Gamma = \Gamma_{\mathcal{O}}$ be the functor of global sections on the category $\mathrm{mod}^{\mathrm{qc}}(\mathcal{O})$ of quasicoherent sheaves on $\mathcal{B}$ and let $\mathrm{R}\Gamma = \mathrm{R}\Gamma_{\mathcal{O}}$ be the derived functor on $\mathrm{D}(\mathrm{mod}^{\mathrm{qc}}(\mathcal{O}))$. Recall from 3.1.5 that the action of $G \times H$ on $\widetilde{\mathcal{B}}$ gives a map $\widetilde{U} \to \Gamma(\widetilde{\mathcal{D}})$; this gives a functor $\mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}}) \xrightarrow{\Gamma_{\widetilde{\mathcal{D}}}} \mathrm{mod}(\widetilde{U})$, which can be derived to $\mathrm{D}^b(\mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}})) \xrightarrow{R\Gamma_{\widetilde{\mathcal{D}}}} D(\mathrm{mod}(\widetilde{U}))$ because the category of modules has direct limits. This derived functor commutes with the forgetful functors; i.e. $\mathrm{Forg}^{\widetilde{U}}_{\Bbbk} \circ \mathrm{R}\Gamma_{\widetilde{\mathcal{D}}} = \mathrm{R}\Gamma \circ \mathrm{Forg}^{\widetilde{\mathcal{D}}}_{\mathcal{O}}$ where $\mathrm{Forg}^{\widetilde{\mathcal{D}}}_{\mathcal{O}} : \mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}}) \to \mathrm{mod}^{\mathrm{qc}}(\mathcal{O})$, $\mathrm{Forg}^{\widetilde{U}}_{\Bbbk} : \mathrm{mod}(\widetilde{U}) \to \mathrm{Vect}_{\Bbbk}$ are the forgetful functors. This is true since the category $\mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}})$ has enough objects acyclic for the functor of global sections $\mathrm{R}\Gamma$ (derived in quasicoherent $\mathcal{O}$-modules). Namely, if $U_i \xrightarrow{j_i} \mathcal{B}$, $i \in I$, is an affine open cover then for any object $\mathcal{F}$ in $\mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}})$ one has $\mathcal{F} \hookrightarrow \oplus_{i \in I} (j_i)_*(j_i)^*(\mathcal{F})$. Since Γ has finite homological dimension, $\mathrm{R}\Gamma_{\widetilde{\mathcal{D}}}$ actually lands in the bounded derived category.

LEMMA. *The (derived) functor of global sections preserves coherence; i.e., it sends the full subcategory* $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}})) \subset \mathrm{D}^b(\mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}}))$ *into the full subcategory* $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(\widetilde{U})) \subset \mathrm{D}^b(\mathrm{mod}(\widetilde{U}))$.

Proof. First notice that since $\widetilde{U}$ is noetherian, $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(\widetilde{U}))$ is indeed identified with $\mathrm{D}^b_{fg}(\mathrm{mod}(\widetilde{U}))$, the full subcategory in $\mathrm{D}^b(\mathrm{mod}(\widetilde{U}))$ consisting of complexes with finitely generated cohomology.

The map $\widetilde{U} \to \Gamma\widetilde{\mathcal{D}}$ is compatible with natural filtrations and it produces a proper map μ from $\mathrm{Spec}(\mathrm{Gr}(\widetilde{\mathcal{D}})) = G\times_B \mathfrak{n}^{\perp}$ to the affine variety $\mathrm{Spec}(\mathrm{Gr}(\widetilde{U})) \cong \mathfrak{g}^*\times_{\mathfrak{h}^*//W}\mathfrak{h}^*$ (here, $\mathrm{gr}(\mathfrak{Z}_{\mathrm{HC}}) \cong \mathcal{O}(\mathfrak{h}^*)^W$ by Lemma 3.1.5(b)). Any coherent $\widetilde{\mathcal{D}}$-module M has a coherent filtration, i.e., a lift to a filtered $\widetilde{\mathcal{D}}$-module $M_\bullet$ such that $\mathrm{gr}(M_\bullet)$ is coherent for $\mathrm{Gr}(\widetilde{\mathcal{D}})$. Now, each $R^i\mu_*(\mathrm{gr}(M_\bullet))$ is a coherent sheaf on $\mathrm{Spec}(\mathrm{Gr}(\widetilde{U}))$, i.e, $\mathrm{H}^*(\mathcal{B}, \mathrm{gr}(M_\bullet))$ is a finitely generated module over $\mathrm{Gr}(\widetilde{U})$. The filtration on M leads to a spectral sequence $\mathrm{H}^*(\mathcal{B}, \mathrm{gr}(M)) \Rightarrow \mathrm{gr}(\mathrm{H}^*(\mathcal{B}, M))$, so $\mathrm{gr}(\mathrm{H}^*(\mathcal{B}, M))$ is a subquotient of $\mathrm{H}^*(\mathcal{B}, \mathrm{gr}(M))$, and therefore it is also finitely generated. Observe that the induced filtration on $\mathrm{H}^*(\mathcal{B}, M)$ makes it into a filtered module for $\mathrm{H}^*(\mathcal{B}, \mathcal{D})$ with its induced filtration. Since $\widetilde{U} \to \mathrm{H}^0(\mathcal{B}, \mathcal{D})$ is a map of filtered rings, $\mathrm{H}^*(\mathcal{B}, M)$ is also a filtered module for $\widetilde{U}$. Now, since $gr(\mathrm{H}^*(\mathcal{B}, M))$ is a finitely generated module for $\mathrm{gr}(\widetilde{U})$, we find that $\mathrm{H}^*(\mathcal{B}, M)$ is finitely generated for $\widetilde{U}$. This shows that $R\Gamma_{\widetilde{\mathcal{D}}}$ maps $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}}))$ to $\mathrm{D}^b_{fg}(\mathrm{mod}(\widetilde{U})) \cong \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(\widetilde{U}))$. □

From 3.1.5, the canonical map $\widetilde{U} \to \mathcal{D}^\lambda$ factors for any $\lambda \in \mathfrak{h}^*$ to $U^\lambda \to \mathcal{D}^\lambda$. So, as above, we get functors

$$\mathrm{mod}^{\mathrm{c}}_\lambda(\widetilde{\mathcal{D}}) \xrightarrow{\Gamma_{\widetilde{\mathcal{D}},\lambda}} \mathrm{mod}^{\mathrm{fg}}_\lambda(\widetilde{U}), \ \ \mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda) \xrightarrow{\Gamma_{\mathcal{D}^\lambda}} \mathrm{mod}^{\mathrm{fg}}(U^\lambda).$$

The derived functors

$$\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\lambda(\widetilde{\mathcal{D}})) \xrightarrow{\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda}} \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_\lambda(U)), \ \ \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda)) \xrightarrow{\mathrm{R}\Gamma_{\mathcal{D}^\lambda}} \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U^\lambda))$$

are defined and compatible with the forgetful functors.

3.2. THEOREM (The main result). *Suppose*[6] *that $p > h$. For any regular $\lambda \in \mathfrak{h}^*$ the global section functors provide equivalences of triangulated categories*:

$$\mathrm{R}\Gamma_{\mathcal{D}^\lambda} : \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda)) \xrightarrow{\cong} \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U^\lambda)); \tag{1}$$

$$\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda} : \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\lambda(\widetilde{\mathcal{D}})) \xrightarrow{\cong} \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_\lambda(U)). \tag{2}$$

[6]The restriction on p is discussed in 3.1.2 above.

Remark 1. In the characteristic zero case Beilinson-Bernstein ([BB]; see also [Mi]), proved that for a dominant λ the functor of global sections provides an equivalence between the abelian categories $\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^{\lambda}) \to \mathrm{mod}^{\mathrm{fg}}(U^{\lambda})$. The analogue for crystalline differential operators in characteristic p is evidently false: for any line bundle $\mathcal{L}$ on $\mathcal{B}$ the line bundle $\mathcal{L}^{\otimes p}$ carries a natural structure of a $\mathcal{D}$-module (2.2.5); however $\mathrm{R}^i\Gamma(\mathcal{L}^{\otimes p})$ may certainly be nonzero for $i > 0$. Heuristically, the analogue of characteristic zero results about dominant weights is not available in characteristic p, because a weight cannot be dominant (positive) modulo p.

However, for a generic $\lambda \in \mathfrak{h}^*$ it is very easy to see that global sections give an equivalence of abelian categories $\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^{\lambda}) \to \mathrm{mod}^{\mathrm{fg}}(U^{\lambda})$. If $\iota(\lambda)$ is regular, the twisted cotangent bundle $T^*_{\iota(\lambda)}\mathcal{B}$ is affine, so that $\mathcal{D}^{\lambda}$-modules are equivalent to modules for $\Gamma(\mathcal{B}, \mathcal{D}^{\lambda})$, and $\Gamma(\mathcal{B}, \mathcal{D}^{\lambda}) = U^{\lambda}$ is proved in 3.4.1.

Remark 2. Quasicoherent and "unbounded" versions of the equivalence, say $D^?(\mathrm{mod}^{\mathrm{qc}}(\mathcal{D}^{\lambda})) \xrightarrow{\mathrm{R}\Gamma_{\mathcal{D}^{\lambda}}} D^?(\mathrm{mod}(U^{\lambda}))$, $? = +, -$ or b, follow formally from the coherent versions since $R\Gamma_{\mathcal{D}^{\lambda}}$ and its adjoint (see 3.3) commute with homotopy direct limits. For completions to formal neighborhoods see 5.4.

3.2.1. *The strategy of the proof of Theorem* 3.2. We concentrate on the second statement, the first one follows (or can be proved in a similar way). First we observe that the functor of global sections

$$\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda} : \ \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_{\lambda}(\widetilde{\mathcal{D}})) \to \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_{\lambda}(U))$$

has left adjoint – the localization functor $\mathcal{L}^{\widehat{\lambda}}$. A straightforward modification of a known characteristic-zero argument shows that the composition of the two adjoint functors in one order is isomorphic to the identity. The theorem then follows from a certain abstract property of the category $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_{\lambda}(\widetilde{\mathcal{D}}))$ which we call the *(relative) Calabi-Yau* property (because the derived category of coherent sheaves on a Calabi-Yau manifold provides a typical example of such a category). This property of $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_{\lambda}(\widetilde{\mathcal{D}}))$ will be derived from the triviality of the canonical class of $\widetilde{\mathfrak{g}}^*$.

Remark 3. One can give another proof of Theorem 3.2 with a stronger restriction on characteristic p, which is closer to the original proof by Beilinson and Bernstein [BB] of the characteristic zero statement. (A similar proof appears in an earlier preprint version of this paper.) Namely, for fixed weights λ, μ and large p one can use the Casimir element in Z_{HC} to show that the sheaf $\mathcal{O}_{\mu} \otimes \mathcal{M}$ is a direct summand in the sheaf of $\mathfrak{g}$ modules $V_{\mu} \otimes \mathcal{M}$ for a $\mathcal{D}_{\lambda}$-module $\mathcal{M}$ (where λ is assumed to be integral and regular). Choosing p, such that this statement holds for a finite set of weights μ, such that $\mathcal{O}_{\mu}$ generates $\mathrm{D}^b(Coh(\mathcal{B}))$, we deduce from Proposition 3.4.1 that the functor $R\Gamma$ is fully faithful. Since the adjoint functor $\mathcal{L}$ is easily seen to be fully faithful as well (see Corollary 3.4.2), we get the result.

3.3. *Localization functors.*

3.3.1. *Localization for categories with generalized Harish-Chandra character.* We start with the localization functor Loc from (finitely generated) U-modules to $\widetilde{\mathcal{D}}$ modules, $\mathrm{Loc}(M) = \widetilde{\mathcal{D}} \otimes_U M$. Since U has finite homological dimension it has a left derived functor $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U)) \xrightarrow{\mathcal{L}} \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}}))$. Fix $\lambda \in \mathfrak{h}^*$, for any $M \in \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_\lambda(U))$ we have a canonical decomposition $\mathcal{L}(M) = \bigoplus_{\mu\in W\bullet\lambda} \mathcal{L}^{\lambda\to\mu}(M)$ with $\mathcal{L}^{\lambda\to\mu}(M) \in \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\mu(\widetilde{\mathcal{D}}))$. Localization with the generalized character λ is the functor $\mathcal{L}^{\widehat{\lambda}} \stackrel{\mathrm{def}}{=} \mathcal{L}^{\lambda\to\lambda} : \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_\lambda(U)) \to \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\lambda(\widetilde{\mathcal{D}}))$.

3.3.2. Lemma. *The functor $\mathcal{L}$ is left adjoint to* $\mathrm{R}\Gamma$, *and $\mathcal{L}^{\widehat{\lambda}}$ is left adjoint to* $\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda}$.

Proof. It is easy to check that the functors between abelian categories $\Gamma : \mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}}) \to \mathrm{mod}(U)$, $\mathrm{Loc} : \mathrm{mod}(U) \to \mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}})$ form an adjoint pair. Since $\mathrm{mod}^{\mathrm{qc}}(\widetilde{\mathcal{D}})$ (respectively, $\mathrm{mod}(U)$) has enough injective (respectively, projective) objects, and the functors Γ, Loc have bounded homological dimension it follows that their derived functors form an adjoint pair. Lemma 3.1.9 asserts that $R\Gamma$ sends $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}}))$ into $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U))$; and it is immediate to check that $\mathcal{L}$ sends $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U))$ to $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}}))$. This yields the first statement. The second one follows from the first one. □

3.3.3. *Localization for categories with a fixed Harish-Chandra character.* We now turn to the categories appearing in equivalence (1) of Theorem 3.2. The functor *Loc* from the previous subsection restricts to a functor $\mathrm{Loc}^\lambda : \mathrm{mod}^{\mathrm{fg}}(U^\lambda) \to \mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda)$, $\mathrm{Loc}^\lambda(M) = \mathcal{D}^\lambda \otimes_{U^\lambda} M$. It has a left derived functor $\mathcal{L}^\lambda : \mathrm{D}^-(\mathrm{mod}^{\mathrm{fg}}(U^\lambda) \to \mathrm{D}^-(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda))$, $\mathcal{L}^\lambda(M) = \mathcal{D}^\lambda \overset{L}{\otimes}_{U^\lambda} M$. Notice that the algebra U^λ may *a priori* have infinite homological dimension[7], so $\mathcal{L}^\lambda$ need not preserve the bounded derived categories. The next lemma shows that it *does* for regular λ.

3.3.4. Lemma. a) *$\mathcal{L}^\lambda$ is left adjoint to the functor*

$$\mathrm{D}^-(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda)) \xrightarrow{R\Gamma_{\mathcal{D}^\lambda}} \mathrm{D}^-(\mathrm{mod}^{\mathrm{fg}}(U^\lambda)).$$

b) *For regular λ the localizations at λ and the generalized character λ are compatible*, *i.e.*, *for the obvious functors* $\mathrm{D}^-(\mathrm{mod}^{\mathrm{fg}}(U^\lambda)) \xrightarrow{i} \mathrm{D}^-(\mathrm{mod}^{\mathrm{fg}}_\lambda(U))$ *and*

[7]For regular λ the finiteness of homological dimension will eventually follow from the equivalence 3.2.

$\mathrm{D}^-(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda)) \xrightarrow{\iota} \mathrm{D}^-(\mathrm{mod}^{\mathrm{c}}_\lambda(\widetilde{\mathcal{D}}))$, *there is a canonical isomorphism*

$$\iota \circ \mathcal{L}^\lambda \cong \mathcal{L}^{\widehat{\lambda}} \circ i,$$

and this isomorphism is compatible with the adjunction arrows in the obvious sense.

Proof. a) is standard. To check (b) observe that if λ is regular for the dot-action of W, then the projection $\mathfrak{h}^* \to \mathfrak{h}^*/(W, \bullet)$ is étale at λ; thus we have $\mathcal{O}(\mathfrak{h}^*)^{\widehat{\lambda}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}(\mathfrak{h}^*/(W,\bullet))} \Bbbk_\lambda = \Bbbk$, where $\mathcal{O}(\mathfrak{h}^*)^{\widehat{\lambda}}$ is the completion of $\mathcal{O}(\mathfrak{h}^*)$ at the maximal ideal of λ. It follows that $\widetilde{\mathcal{D}}^{\widehat{\lambda}} \overset{\mathrm{L}}{\otimes}_U U^\lambda = \mathcal{D}^\lambda$, where $\widetilde{\mathcal{D}}^{\widehat{\lambda}} = \widetilde{\mathcal{D}} \otimes_{\mathcal{O}(\mathfrak{h}^*)} \mathcal{O}(\mathfrak{h}^*)^{\widehat{\lambda}}$. It is easy to see from the definition that $\mathcal{L}^{\widehat{\lambda}}(M) \cong \mathcal{D}^{\widehat{\lambda}} \overset{L}{\otimes}_U M$ canonically, thus we obtain the desired isomorphism of functors. Compatibility of this isomorphism with adjunction follows from the definitions. □

3.3.5. Corollary. *The functor $\mathcal{L}^\lambda$ sends the bounded derived category* $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda))$ *to* $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U^\lambda))$ *provided λ is regular.*

3.4. *Cohomology of $\widetilde{\mathcal{D}}$.* The computation in this section will be used to check that $R\Gamma_{\widetilde{\mathcal{D}},\lambda} \circ \mathcal{L}^{\widehat{\lambda}} \cong \mathrm{id}$ for regular λ.

3.4.1. Proposition. *Assume that p is very good. Then we have* $\widetilde{U} \xrightarrow{\cong} \mathrm{R}\Gamma(\widetilde{\mathcal{D}})$ *and also* $U^\lambda \xrightarrow{\cong} \mathrm{R}\Gamma(\mathcal{D}^\lambda)$ *for $\lambda \in \mathfrak{h}^*$.*

Proof. The sheaves of algebras $\mathcal{D}^\lambda$, $\widetilde{\mathcal{D}}$ carry filtrations by the order of a differential operator; the associated graded sheaves are, respectively, $\mathcal{O}_{\widetilde{\mathcal{N}}}$ and $\mathcal{O}_{\widetilde{\mathfrak{g}}^*}$. Cohomology vanishing for $\mathcal{D}$, $\widetilde{\mathcal{D}}$ follows from cohomology vanishing of the associated graded sheaves. For $\mathcal{O}_{T^*\mathcal{B}}$ this is Theorem 2 of [KLT], which only requires p to be good for $\mathfrak{g}$. The case of $\widetilde{\mathfrak{g}}^*$ is a formal consequence. To see this consider a two-step B-invariant filtration on $(\mathfrak{g}/\mathfrak{n})^*$ with associated graded $\mathfrak{h}^* \oplus (\mathfrak{g}/\mathfrak{b})^*$. It induces a filtration on $\widetilde{\mathfrak{g}}^*$ considered as a vector bundle on $\mathcal{B}$. The associated graded of the corresponding filtration on $\mathcal{O}_{\widetilde{\mathfrak{g}}^*}$ (considered as a sheaf on $\mathcal{B}$) is $S(\mathfrak{h}) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}$. Cohomology vanishing of the last sheaf follows from the one for $\mathcal{O}_{\widetilde{\mathcal{N}}}$, and implies one for $\mathcal{O}_{\widetilde{\mathfrak{g}}^*}$.

Furthermore, higher cohomology vanishing for the associated graded sheaves $\mathcal{O}_{\widetilde{\mathcal{N}}} = \mathrm{gr}(\mathcal{D}^\lambda)$, $\mathcal{O}_{\widetilde{\mathfrak{g}}^*} = \mathrm{gr}(\widetilde{\mathcal{D}})$ implies that the natural maps $\mathrm{gr}(\Gamma(\mathcal{D}^\lambda)) \to \Gamma(\mathcal{O}_{\widetilde{\mathcal{N}}})$, $\mathrm{gr}(\Gamma(\widetilde{\mathcal{D}})) \to \Gamma(\widetilde{\mathfrak{g}}^*)$ are isomorphisms.

We will show that the maps $U^\lambda \to \Gamma(\mathcal{D}^\lambda)$, $\widetilde{U} \to \Gamma(\widetilde{\mathcal{D}})$ are isomorphisms by showing that the induced maps on the associated graded algebras are. Here the filtration on U^λ is induced by the canonical filtration on U, and the one on $\widetilde{\mathcal{D}}$ is induced by the canonical filtration on U and the degree filtration on $S(\mathfrak{h})$.

The associated graded rings of U^λ, $\widetilde{U}$ are quotients of, respectively, $S(\mathfrak{g})$ and $S(\mathfrak{g}) \otimes S(\mathfrak{h})$. Moreover, in view of Lemma 3.1.5(b), they are quotients of, respectively, $S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^G} \Bbbk$ and $S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^G} S(\mathfrak{h})$. It remains to show that the maps $S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^G} \Bbbk \to \Gamma(\mathcal{O}_{\widetilde{\mathcal{N}}})$, $S(\mathfrak{g}) \otimes_{S(\mathfrak{g})^G} S(\mathfrak{h}) \to \Gamma(\mathcal{O}_{\widetilde{\mathfrak{g}}^*})$ are isomorphisms. Here the maps are readily seen to be induced by the canonical morphisms $\widetilde{\mathcal{N}} \to \mathfrak{g}^*$ and $\widetilde{\mathfrak{g}}^* \to \mathfrak{g}^* \times_{\mathfrak{h}^*/W} \mathfrak{h}^*$.

Since p is very good, we have a G-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$; see 3.1.2. Thus it suffices to show that the global functions on the nilpotent variety $\mathcal{N} \subset \mathfrak{g}$ map isomorphically to the ring of global functions on $\widetilde{\mathcal{N}} \cong \mathfrak{n} \times^B G$. Moreover, the étale slice theorem of [BaRi] shows that for very good p there exists a G-equivariant isomorphism between $\mathcal{N}$ and the subscheme $\mathcal{U} \subset G$ defined by the G-invariant polynomials on G vanishing at the unit element; cf. [BaRi, 9.3]. Thus the task is reduced to showing that the ring of regular functions on $\mathcal{U}$ maps isomorphically to the ring of global functions on $N \times^B G$. This follows once we know that $\mathcal{U}$ is reduced and normal and the Springer map $N \times^B G \to \mathcal{U}$ is birational. These facts can be found in [St] for all p: $\mathcal{U}$ is reduced and normal by 3.8, Theorem 7, it is irreducible by 3.8, Theorem 1, while the Springer map is a resolution of singularities by 3.9, Theorem 1.

Finally, surjectivity of the map $S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^W} S(\mathfrak{h}) \to \Gamma(\mathcal{O}(\widetilde{\mathfrak{g}}^*))$ follows from surjectivity established in the previous paragraph by the graded Nakayama lemma; notice that higher cohomology vanishing for $\mathcal{O}_{\widetilde{\mathfrak{g}}^*}$ implies that $\Gamma(\mathcal{O}_{\widetilde{\mathcal{N}}}) = \Gamma(\mathcal{O}_{\widetilde{\mathfrak{g}}^*}) \otimes_{S(\mathfrak{h})} \Bbbk$. Injectivity of this map is clear from the fact that $S(\mathfrak{h})$ is free over $S(\mathfrak{h})^W$ for very good p [De]; cf. also [Ja, 9.6]. Hence $S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^W} S(\mathfrak{h})$ is free over $S(\mathfrak{g})$, while the map $\widetilde{\mathfrak{g}}^* \to \mathfrak{g}^* \times_{\mathfrak{h}^*/W} \mathfrak{h}^*$ is an isomorphism over the open set of regular semisimple elements in $\mathfrak{g}^*$ for any p. □

3.4.2. Corollary. a) *The composition* $\mathrm{R}\Gamma_{\widetilde{\mathcal{D}}} \circ \mathcal{L} : \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U)) \to \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(\widetilde{U}))$ *is isomorphic to the functor* $M \mapsto M \otimes_{\mathfrak{Z}_{\mathrm{HC}}} S(\mathfrak{h})$.

b) *For a regular weight* λ *the adjunction map* $\mathrm{id} \to \mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda} \circ \mathcal{L}^{\widehat{\lambda}}$ *is an isomorphism on* $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_{\lambda}(U))$.

c) *For any* λ, *the adjunction map is an isomorphism* $\mathrm{id} \to \mathrm{R}\Gamma_{\mathcal{D}^\lambda} \circ \mathcal{L}^\lambda$ *on* $\mathrm{D}^-(\mathrm{mod}^{\mathrm{fg}}(U^\lambda))$.

Proof. For any U-module M the action of U on $\Gamma_{\widetilde{\mathcal{D}}}(\mathcal{L}(M))$ extends to the action of $\Gamma(\widetilde{\mathcal{D}}) = \widetilde{U}$. So the adjunction map $M \to \Gamma_{\widetilde{\mathcal{D}}}(\mathcal{L}(M))$ extends to $S(\mathfrak{h}) \otimes_{\mathfrak{Z}_{\mathrm{HC}}} M = \widetilde{U} \otimes_U M \to \Gamma_{\widetilde{\mathcal{D}}} \circ \mathcal{L}(M)$. Proposition 3.4.1 implies that if M is a free module then this map is an isomorphism, while higher derived functors $R^i\Gamma_{\widetilde{\mathcal{D}}}(\mathcal{L}(M))$, $i > 0$, vanish. This yields statement (a) and (c) is proved in the same way by the second claim in Proposition 3.4.1.

To deduce (b) observe that for regular λ and $M \in \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_{\lambda}(U))$, we have canonically $M \otimes_{\mathfrak{Z}_{\mathrm{HC}}} S(\mathfrak{h}) \cong \oplus_W M$. The adjunction morphism viewed as $M \to \oplus_W M$, equals $\sum_W \mathrm{id}_M$ (when M is the restriction of U to the n^{th} infinitesimal neighborhood of λ this follows by restricting $\widetilde{U} \xrightarrow{\cong} R\Gamma(\widetilde{\mathcal{D}})$). Now the claim follows since $\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda}(\mathcal{L}^{\widehat{\lambda}}(M))$ is one of the summands. □

3.5. *Calabi-Yau categories.* We recall some generalities about Serre functors in triangulated categories; we refer to the original paper[8] [BK] for details.

Let $\mathcal{O}$ be a finite type commutative algebra over the field $\Bbbk$, D an $\mathcal{O}$-linear triangulated category. A structure of an $\mathcal{O}$-triangulated category on D is a functor $\mathrm{RHom}_{D/\mathcal{O}} : D^{\mathrm{op}} \times D \to \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(\mathcal{O}))$, together with a functorial isomorphism $\mathrm{Hom}_D(X,Y) \cong \mathrm{H}^0(\mathrm{RHom}_{D/\mathcal{O}}(X,Y))$.

For any quasi-projective variety Y, the triangulated category $\mathrm{D}^b(\mathcal{C}\mathrm{oh}(Y))$ is equipped with a canonical anti-auto-equivalence, namely the Grothendieck-Serre duality $\mathbb{D}_Y = \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{O}}(-, K_Y)$ for the dualizing complex $K_Y = (Y \to pt)^!\Bbbk$.

By an *$\mathcal{O}$-Serre functor* on D we will mean an auto-equivalence $S : D \to D$ together with a natural (functorial) isomorphism $\mathrm{RHom}_{D/\mathcal{O}}(X,Y) \cong \mathbb{D}_{\mathcal{O}}(\mathrm{RHom}_{D/\mathcal{O}}(Y, SX))$ for all $X, Y \in D$. If a Serre functor exists, it is unique up to a unique isomorphism. An $\mathcal{O}$-triangulated category will be called *Calabi-Yau* if for some $n \in \mathbb{Z}$ the shift functor $X \mapsto X[n]$ admits a structure of an $\mathcal{O}$-Serre functor.

For example, if X is a smooth variety over $\Bbbk$ equipped with a projective morphism $\pi : X \to \mathrm{Spec}(\mathcal{O})$ then $D = \mathrm{D}^b(\mathcal{C}\mathrm{oh}_X)$ is $\mathcal{O}$-triangulated by $\mathrm{RHom}_{D/\mathcal{O}}(\mathcal{F},\mathcal{G}) \overset{\mathrm{def}}{=} R\pi_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G})$. The functor $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X[\dim X]$ is naturally a Serre functor with respect to $\mathcal{O}$; this is true because Grothendieck-Serre duality commutes with proper direct images, and the dualizing complex for a smooth X is $K_X \xrightarrow{\cong} \omega_X[\dim(X)]$, so that

$$\begin{aligned}\mathbb{D}_{\mathcal{O}}(R\pi_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G})) &\cong R\pi_*(\mathbb{D}_X\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G})) \\ &\cong R\pi_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{F} \otimes \omega_X[\dim X]).\end{aligned}$$

We will need the following generalization of this fact. Its proof is straightforward and left to the reader.[9]

3.5.1. LEMMA. *Let $\mathcal{A}$ be an Azumaya algebra on a smooth variety X over k, equipped with a projective morphism $\pi : X \to \mathrm{Spec}(\mathcal{O})$. Then $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{A}))$ is naturally $\mathcal{O}$-triangulated and the functor $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X[\dim X]$ is naturally a Serre functor with respect to $\mathcal{O}$. In particular, if X is a Calabi-Yau manifold (i.e., $\omega_X \cong \mathcal{O}_X$) then the $\mathcal{O}$-triangulated category $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{A}))$ is Calabi-Yau.*

Application of the above notions to our situation is based on the following lemma. A similar argument was used e.g. in [BKR, Th. 2.3].

[8] We slightly generalize the definition of [BK]; cf. [BeKa].

[9] Details of the proof can also be found in the sequel paper [BMR2].

3.5.2. Lemma. *Let D be a Calabi-Yau $\mathcal{O}$-triangulated category for some commutative finitely generated algebra $\mathcal{O}$. Then a sufficient condition for a triangulated functor $L : C \to D$ to be an equivalence is given by*

i) *L has a right adjoint functor R and the adjunction morphism* $\mathrm{id} \to R \circ L$ *is an isomorphism, and*

ii) *D is indecomposable, i.e. D cannot be written as $D = D_1 \oplus D_2$ for nonzero triangulated categories D_1, D_2; and $C \neq 0$.*

Proof. Consider any full subcategory $\mathcal{C} \subseteq D$ invariant under the shift functor. The right orthogonal is the full subcategory $\mathcal{C}^\perp = \{y \in D;\ \mathrm{Hom}_D(c, y) = 0\ \forall c \in \mathcal{C}\}$. If S an $\mathcal{O}$-Serre functor for D then $S^{-1} : \mathcal{C}^\perp \to {}^\perp\mathcal{C}$ (the left orthogonal of $\mathcal{C}$), since for $y \in \mathcal{C}^\perp$ and $c \in \mathcal{C}$ one has $\mathrm{H}^n\mathrm{RHom}_{D/\mathcal{O}}(c, y) = \mathrm{Hom}_D(c, y[n]) = \mathrm{Hom}_D(c[-n], y) = 0,\ n \in \mathbb{Z}$, hence $\mathrm{RHom}_{D/\mathcal{O}}(c, y) = 0$, and then $D_{\mathcal{O}}\mathrm{RHom}_{D/\mathcal{O}}(S^{-1}y, c) = \mathrm{RHom}_{D/\mathcal{O}}(c, y) = 0$. In particular, if D is Calabi-Yau relative to $\mathcal{O}$, then ${}^\perp\mathcal{C} = \mathcal{C}^\perp$.

Now, condition (i) implies that L is a full embedding, so we will regard it as the inclusion of a full subcategory C into D. Moreover, for $d \in D$, any cone y of the map $LR(d) \to d$ is in $C^\perp$. Therefore, $y \in {}^\perp C$, and then $d \cong LR(d) \oplus y$. This yields a decomposition $D = C \oplus C^\perp$. Thus, condition (ii) implies that $C^\perp = 0$ and L is an equivalence. □

Another useful simple fact is:

3.5.3. Lemma (cf. [BKR, Lemma 4.2]). *Let X be a connected scheme quasiprojective over a field $\Bbbk$, and let $\mathcal{A}$ be an Azumaya algebra on X. Then the category $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{A}))$ is indecomposable. Moreover, if $Y \subset X$ is a connected closed subset then $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_Y(X, \mathcal{A}))$ is indecomposable.*

Proof. Assume that $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{A})) = D_1 \oplus D_2$ is a decomposition invariant under the shift functor. Let P be an indecomposable summand of the free $\mathcal{A}$-module. Let L be a very ample line bundle on X such that $0 \neq H^0(L \otimes \mathcal{H}om_{\mathcal{A}}(P, P)) = \mathrm{Hom}_{\mathcal{A}}(P, P \otimes L)$. For any $n \in \mathbb{Z}$ the $\mathcal{A}$-module $P \otimes L^{\otimes n}$ is indecomposable, hence belongs either to D_1 or to D_2. Moreover, all these modules belong to the same summand, because $\mathrm{Hom}_{\mathcal{A}}(P \otimes L^{\otimes n}, P \otimes L^{\otimes m}) \neq 0$ for $n \leq m$. If $\mathcal{F}$ is an object of the other summand, then we have $\mathrm{Ext}^\bullet_{\mathcal{A}}(P \otimes L^{\otimes n}, \mathcal{F}) = 0$ for all n. However, since $\mathcal{A}$ is Azumaya algebra, $P \neq 0$ is a locally projective $\mathcal{A}$-module and X is connected, $\mathcal{F} \neq 0$ would imply $\mathrm{R}\mathcal{H}om_{\mathcal{A}}(P, \mathcal{F}) \neq 0$ (this claim reduces to the case when $\mathcal{A}$ is a matrix algebra and then to $\mathcal{A} = \mathcal{O}_X$). So $\mathcal{F} = 0$ (otherwise $H^*(X, \mathrm{R}\mathcal{H}om_{\mathcal{A}}(P, \mathcal{F}) \otimes L^{\otimes -n})$ could not be zero for all n), and this proves the first statement. The second claim follows: for any closed subscheme $Y' \subset X$ whose topological space equals Y, the image of $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(Y', \mathcal{A}|_{Y'}))$ under the push-forward functor lies in one summand of any decomposition $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_Y(X, \mathcal{A})) = D_1 \oplus D_2$. □

3.6. *Proof of Theorem* 3.2. The canonical line bundle on $\widetilde{\mathfrak{g}}^*$ is trivial; hence the same is true for $\widetilde{\mathfrak{g}}^{*(1)}\times_{\mathfrak{h}^{*(1)}}\mathfrak{h}^*$, the spectrum of the center of $\widetilde{\mathcal{D}}$ (see 3.1.6). Thus Lemma 3.5.1 shows that $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}}))$ is Calabi-Yau with respect to $\mathcal{O}(\mathfrak{g}^*)$.

It follows from the definitions that a full triangulated subcategory in a Calabi-Yau category with respect to some algebra $\mathcal{O}$ is again a Calabi-Yau category with respect to $\mathcal{O}$. Therefore, (2) follows from Corollary 3.4.2(b) and Lemmas 3.5.2, 3.5.3.

To deduce (1) from (2) we use Lemma 3.3.4(b). It says that the functors i, ι send the adjunction arrows into adjunction arrows; since i, ι kill no objects, and the adjunction arrows in $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_{\lambda}(\widetilde{\mathcal{D}}))$, $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_{\lambda}(U))$ are isomorphisms, we conclude that the adjunction arrows in $\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^{\lambda}))$, $\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}(U^{\lambda}))$ are isomorphisms, which implies (1). □

4. Localization with a generalized Frobenius character

4.1. *Localization on* (*generalized*) *Springer fibers.* The map $U \to \widetilde{\mathcal{D}}$ restricts to a map of central algebras $\mathcal{O}(\mathfrak{g}^{*(1)}) \to \mathcal{O}_{\widetilde{\mathfrak{g}}^{*(1)}}$. So, the commutative part of the localization mechanism is the resolution $\widetilde{\mathfrak{g}}^{*(1)} \to \mathfrak{g}^{*(1)}$. Therefore, the specialization of the algebra U to $\chi \in \mathfrak{g}^{*(1)}$ will correspond to the restriction of $\widetilde{\mathcal{D}}$ to the corresponding Springer fiber.

From here on we keep in mind that the Weyl group always acts by the dot action and we write $\mathfrak{X}//W$ instead of $\mathfrak{X}//(W,\bullet)$ for the invariant theory quotients.

4.1.1. *Categories with a generalized character χ of the Frobenius center.* Recall that the center $\mathfrak{Z} = \mathcal{O}(\mathfrak{g}^{*(1)}\times_{\mathfrak{h}^{*(1)}//W}\mathfrak{h}^*//W)$ of U is generated by subalgebras $\mathfrak{Z}_{\mathrm{Fr}} = \mathcal{O}(\mathfrak{g}^{*(1)})$ and $\mathfrak{Z}_{\mathrm{HC}} = \mathcal{O}(\mathfrak{h}^*//W)$ which the map $U(\mathfrak{g})\to \Gamma\widetilde{\mathcal{D}}$ sends to the central subalgebras $\mathcal{O}(\widetilde{T}^*\mathcal{B}^{(1)})$ and $S\mathfrak{h}$ of $\widetilde{\mathcal{D}}$ (3.1.6).

For $\lambda \in \mathfrak{h}^*$, $\chi \in \mathfrak{g}^*$, the notation $U^{\lambda}, U_{\chi}, U^{\lambda}_{\chi}$ denotes the specializations of U to the characters $\lambda, \chi, (\lambda,\chi)$ of $\mathfrak{Z}_{\mathrm{HC}}, \mathfrak{Z}_{\mathrm{Fr}}, \mathfrak{Z}$. Similarly, the sheaf of algebras $\widetilde{\mathcal{D}}$ has specializations $\mathcal{D}^{\lambda} \overset{\mathrm{def}}{=} \widetilde{\mathcal{D}}^{\lambda}, \widetilde{\mathcal{D}}_{\chi}, \mathcal{D}^{\lambda}_{\chi}$. As in 3.1.7, we denote the full subcategories with a generalized character $\zeta \in \{\lambda, \chi, (\lambda,\chi)\}$ of $\mathfrak{Z}_{\mathrm{HC}}, \mathfrak{Z}_{\mathrm{Fr}}$ or $\mathfrak{Z}$, by $\mathrm{mod}^c_{\zeta}(-) \subseteq \mathrm{mod}^c(-)$, and one has $\mathrm{D}^b(\mathrm{mod}^c_{\zeta}(-)) \subseteq \mathrm{D}^b(\mathrm{mod}^c(-))$. For later use we notice that $\mathrm{mod}^{\mathrm{fg}}_{\chi}(U)$ can be viewed as the category $\mathrm{mod}^{\mathrm{fl}}(U^{\lambda}_{\widehat{\chi}})$ of finite length modules for the completion $U^{\lambda}_{\widehat{\chi}}$ of U_{λ} at χ.

According to 3.1.6 the specialization $\mathfrak{Z}^{\lambda}$ of the center $\mathfrak{Z}$ of U is the space of functions on ${\mathfrak{g}^*}^{(1)}_{\lambda} \overset{\mathrm{def}}{=} (\mathfrak{g}^{*(1)}\times_{\mathfrak{h}^{*(1)}//W}\mathfrak{h}^*//W)\times_{\mathfrak{h}^*//W}\lambda = \mathfrak{g}^{*(1)}\times_{\mathfrak{h}^{*(1)}//W}\mathrm{AS}(\lambda)$. For instance, any integral λ is killed by the Artin-Schreier map, so ${\mathfrak{g}^*}^{(1)}_{\lambda} = \mathcal{N}^{(1)}$ and U^{λ} is an $\mathcal{O}(\mathcal{N}^{(1)})$-algebra.

4.1.2. (*Generalized*) *Springer fibers.* Fix $(\chi,\nu) \in \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}//W} \mathfrak{h}^*$, and define $\mathcal{B}_{\chi}, \mathcal{B}_{\chi,\nu} \subset \widetilde{\mathfrak{g}}^*$ by $\mathcal{B}_{\chi} = \mathbf{pr}_1^{-1}(\chi)$, $\mathcal{B}_{\chi,\nu} = \mathbf{pr}^{-1}(\chi,\nu)$ (notation of 3.1.3);

we equip $\mathcal{B}_\chi$, $\mathcal{B}_{\chi,\nu}$ with the *reduced*[10] subscheme structure. When χ is nilpotent (so that $\nu = 0$ and $\mathcal{B}_{\chi,\nu} = \mathcal{B}_\chi$) it is called a Springer fiber; otherwise we call it a generalized Springer fiber.

One can show that $\mathcal{B}_{\chi,\nu}$ is connected; in fact it is a Springer fiber for the centralizer of χ_{ss} where $\chi = \chi_{ss} + \chi_{\rm nil}$ is the Jordan decomposition. Thus $\mathcal{B}_{\chi,\nu}$ is a connected component of $\mathcal{B}_\chi$. Via the projection $\widetilde{\mathfrak{g}}^* \xrightarrow{\pi} \mathcal{B}$ the (generalized) Springer fiber can be identified with a subscheme $\pi(\mathcal{B}_{\chi,\nu})$ of $\mathcal{B}$, and $\mathcal{B}_{\chi,\nu}$ is a section of $\widetilde{\mathfrak{g}}^*$ over $\pi(\mathcal{B}_{\chi,\nu})$.

4.1.3. Lemma. *If $\lambda \in \mathfrak{h}^*$ is regular and $(\chi, \mathrm{AS}(\lambda)) \in \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}//W} \mathfrak{h}^{*(1)}$, the equivalences in Theorem* 3.2 *restrict to*

$$\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\chi(\mathcal{D}^\lambda)) \cong \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_\chi(U^\lambda)), \quad \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_{\lambda,\chi}(\widetilde{\mathcal{D}})) \cong \mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_{\lambda,\chi}(U)).$$

Proof. $\mathcal{O}(\mathfrak{g}^{*(1)})$ acts on the categories $\mathrm{mod}^{\mathrm{c}}(\widetilde{\mathcal{D}})$, $\mathrm{mod}^{\mathrm{fg}}(U)$, etc., and on their derived categories. The equivalences in Theorem 3.2 are equivariant under $\mathcal{O}(\mathfrak{g}^{*(1)})$ and therefore they restrict to the full subcategories of objects on which the p-center acts by the generalized character χ (cf. Lemma 3.1.7).

4.1.4. Corollary. *If λ is regular and $(\chi, \mathrm{AS}(\lambda)) \in \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}//W} \mathfrak{h}^{*(1)}$, the localization gives a canonical isomorphism $K(U^\lambda_\chi) \cong K(\mathcal{D}^\lambda_\chi)$.*

Proof. By Lemma 4.1.3, the localization gives an isomorphism

$$K(\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_\chi(U^\lambda))) \xrightarrow{\cong} K(\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\chi(\mathcal{D}^\lambda))).$$

This simplifies to the desired isomorphism since

$$K(U^\lambda_\chi) \stackrel{\mathrm{def}}{=} K(\mathrm{mod}^{\mathrm{fg}}(U^\lambda_\chi)) \xrightarrow{\cong} K(\mathrm{mod}^{\mathrm{fg}}_\chi(U^\lambda)) \cong K(\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_\chi(U^\lambda))),$$

the first isomorphism is the fact that the subcategory $\mathrm{mod}^{\mathrm{fg}}(U^\lambda_\chi)$ generates $\mathrm{mod}^{\mathrm{fg}}_\chi(U^\lambda)$ under extensions, and the second is the equality of K-groups of a triangulated category (with a bounded t-structure), and of its heart. Similarly,

$$K(\mathcal{D}^\lambda_\chi) \stackrel{\mathrm{def}}{=} K(\mathrm{mod}^{\mathrm{c}}(\mathcal{D}^\lambda_\chi)) \xrightarrow{\cong} K(\mathrm{mod}^{\mathrm{c}}_\chi(\mathcal{D}^\lambda)) = K(\mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\chi(\mathcal{D}^\lambda))). \quad \square$$

5. Splitting of the Azumaya algebra of crystalline differential operators on (generalized) Springer fibers

5.1. *$\mathcal{D}$-modules and coherent sheaves.* Since $\widetilde{\mathcal{D}}$ is an Azumaya algebra over $\widetilde{T}^*\mathcal{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$, for $\lambda \in \mathfrak{h}^*$, we will view $\mathcal{D}^\lambda$ as an Azumaya algebra over $T^*_\nu\mathcal{B}^{(1)}$ where $\underline{\nu = \mathrm{AS}(\lambda)}$ (see 2.3). The aim of this section is the following:

[10] "Reduced" will only be used in lemma 7.1.5c. It is irrelevant in §4 and §5 since we only use formal neighborhoods of the fiber.

5.1.1. THEOREM. a) *For any $\lambda \in \mathfrak{h}^*$, Azumaya algebra $\widetilde{\mathcal{D}}$ splits on the formal neighborhood in $\widetilde{T}^*\mathcal{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$ of ${\mathcal{B}_\chi}^{(1)} \times_{\mathfrak{h}^{*(1)}} \lambda \cong {\mathcal{B}_{\chi,\nu}}^{(1)}$, i.e., there is a vector bundle $\mathcal{M}_\chi^\lambda$ on this formal neighborhood, such that the restriction of $\widetilde{\mathcal{D}}$ to the neighborhood is isomorphic to $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}_\chi^\lambda)$.*

b) *The functor $\mathcal{F} \mapsto \mathcal{M}_\chi^\lambda \otimes_{\mathcal{O}} \mathcal{F}$ provides equivalences*

$$\mathcal{C}\mathrm{oh}_{{\mathcal{B}_{\chi,\nu}}^{(1)}}(\widetilde{T}^*\mathcal{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*) \xrightarrow{\cong} \mathrm{mod}^{\mathrm{c}}_{\chi,\lambda}(\widetilde{\mathcal{D}}),$$
$$\mathcal{C}\mathrm{oh}_{{\mathcal{B}_{\chi,\nu}}^{(1)}}(T^*_\nu\mathcal{B}^{(1)}) \xrightarrow{\cong} \mathrm{mod}^{\mathrm{c}}_{\chi}(\mathcal{D}^\lambda).$$

Proof. (b) follows from (a). Lemma 2.3.1 shows that to check statement (a) for particular (χ, λ) it suffices to check it for $(\chi, \lambda + d\eta)$ for some character $\eta : H \to \mathbb{G}_m$.

Let us say that $\lambda \in \mathfrak{h}^*$ is *unramified* if for any coroot α we have either $\langle \alpha, \lambda + \rho \rangle = 0$, or $\langle \alpha, \lambda \rangle \notin \mathbb{F}_p$. We claim that for any $\lambda \in \mathfrak{h}^*$ one can find a character $\eta : H \to \mathbb{G}_m$ such that $\lambda + d\eta$ is unramified. For this it suffices to show the existence of $\mu \in \mathfrak{h}^*(\mathbb{F}_p)$, such that $\langle \alpha, \lambda + \rho \rangle = \langle \alpha, \mu \rangle$ for any coroot α, such that $\langle \alpha, \lambda \rangle \in \mathbb{F}_p$. These conditions constitute a system of linear equations over $\mathbb{F}_p$, which have a solution over the bigger field $\Bbbk$. By standard linear algebra they also have a solution over $\mathbb{F}_p$.

Thus it suffices to check (a) when λ is unramified. The next proposition shows that for unramified λ the restriction of $\widetilde{\mathcal{D}}$ to the formal neighborhood of ${\mathcal{B}_\chi}^{(1)} \times_{\mathfrak{h}^{*(1)}} \lambda$ is isomorphic to the pull-back of an Azumaya algebra on the formal neighborhood $\widehat{\chi}^{(1)} = FN_{\mathfrak{g}^*}(\chi)^{(1)}$ of χ in $\mathfrak{g}^{*(1)}$. The latter splits by [MI, IV.1.7] (vanishing of the Brauer group of a complete local ring with a separably closed residue field). ☐

5.2. *Unramified Harish-Chandra characters.* Let $\mathfrak{h}^*_{\mathrm{unr}} \subset \mathfrak{h}^*$ be the open set of all unramified weights. Let $\mathfrak{Z}_{\mathrm{unr}}$ be the algebra of functions on $\mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}//W} \mathfrak{h}^*_{\mathrm{unr}} \subseteq \mathrm{Spec}(\mathfrak{Z})$ (see 3.1.6).

5.2.1. PROPOSITION. a) *$U \otimes_{\mathfrak{Z}} \mathfrak{Z}_{\mathrm{unr}}$ is an Azumaya algebra over $\mathfrak{Z}_{\mathrm{unr}}$.*

b) *The action map $U \otimes_{\mathfrak{Z}} \mathcal{O}(\widetilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*) \to \widetilde{\mathcal{D}}$ induces an isomorphism*

$$U \otimes_{\mathfrak{Z}} \mathcal{O}(\widetilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*_{\mathrm{unr}}) \xrightarrow{\cong} \widetilde{\mathcal{D}}|_{\widetilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*_{\mathrm{unr}}}.$$

Proof. (a) is proved in [BG, Cor. 3.11]; moreover, it is shown in *loc. cit.* that for $\mathfrak{z} \in \mathfrak{Z}_{\mathrm{unr}}$ and a baby Verma module M with central character $\mathfrak{z}$ we have an isomorphism $U(\mathfrak{g}) \otimes_{\mathfrak{Z}} \Bbbk_{\mathfrak{z}} \xrightarrow{\cong} \mathrm{End}_{\Bbbk}(M)$. This implies (b) in view of Proposition 3.1.4. ☐

5.2.2. *Remarks.* 1) Consider the restriction of $\mathcal{M}^0_\chi$ to the reduced subscheme ${\mathcal{B}_\chi}^{(1)}$. In view of Remark 2.2.6 it defines (and is defined by) a line bundle with a flat connection on the Frobenius neighborhood of $\mathcal{B}_\chi$ in $\mathcal{B}$. The requirement that the sheaf on $T^*X^{(1)}$ arising from the bundle with connection lives on ${\mathcal{B}_\chi}^{(1)}$ is equivalent to the equality between the p-curvature of the connection and the section of $\Omega^1_{\mathcal{B}}|_{\mathcal{B}_\chi}$ defined by χ (cf. Remark 2.1.2).[11]

For some particular cases, such a line bundle with a flat connection was constructed in [MR]. Notice that already in the case $G = \mathrm{SL}(3)$, and χ subregular this line bundle is nontrivial for any choice of the splitting bundle $\mathcal{M}^\lambda_\chi$ (see, however, equality (5) in the proof of Lemma 6.2.5 below).

2) The choice of a character $\eta \in \Lambda$ such that $\lambda + d\eta$ is unramified, provides a particular splitting line bundle $\mathcal{M}^\lambda_\chi = \mathcal{M}^\lambda_\chi(\eta)$ in Theorem 5.1.1(a): apply the equivalence of Lemma 2.3.1 to the trivial (equivalently, lifted from $\widehat{\nu}^{(1)}$) splitting vector bundle on the formal neighborhood of ${\mathcal{B}_\chi}^{(1)} \times_{\mathfrak{h}^{*(1)}} (\lambda + d\eta)$. It is easy to see then that $\mathcal{M}^\lambda_\chi(\eta + p\zeta) = \mathcal{M}^\lambda_\chi(\eta) \otimes \mathcal{O}_{-\zeta}$.

3) One can show that the Azumaya algebra $U \otimes_{\mathfrak{Z}} \mathfrak{Z}_{\mathrm{unr}}$ splits on some closed subvarieties of $\mathrm{Spec}(\mathfrak{Z}_{\mathrm{unr}})$; e.g. the Verma module $M^{\mathfrak{b}}(-\rho) \overset{\mathrm{def}}{=} \mathrm{ind}^{U\mathfrak{g}}_{U\mathfrak{b}}\, \Bbbk_{-\rho}$ is easily seen to be a splitting module on $\mathfrak{n} \times \{-\rho\}$.

5.3. $\mathfrak{g}$-*modules and coherent sheaves.* By putting together known equivalences (Theorem 4.1.3 and Theorem 5.1.1(b)), we get

5.3.1. THEOREM. *If* $\lambda \in \mathfrak{h}^*$ *is regular and* $(\chi, \lambda) \in \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}//W} \mathfrak{h}^*$ *with* $(\chi, W \bullet \lambda) \in \mathrm{Spec}(\mathfrak{Z})$, *then there are equivalences* (*set* $\nu = \mathrm{AS}(\lambda)$)

$$\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_\chi(U^\lambda)) \cong \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\chi(\mathcal{D}^\lambda)) \cong \mathrm{D}^b(\mathcal{C}\mathrm{oh}_{{\mathcal{B}_{\chi,\nu}}^{(1)}}(T^*_\nu \mathcal{B}^{(1)}))\ ;$$
$$\mathrm{D}^b(\mathrm{mod}^{\mathrm{fg}}_{(\lambda,\chi)}(U)) \cong \mathrm{D}^b(\mathrm{mod}^{c}_{(\lambda,\chi)}(\widetilde{\mathcal{D}})) \cong \mathrm{D}^b(\mathcal{C}\mathrm{oh}_{{\mathcal{B}_{\chi,\nu}}^{(1)}}(\widetilde{T}^*\mathcal{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*))\ .$$

5.3.2. *Remark.* The equivalences depend on the choice of the splitting bundle $\mathcal{M}^\lambda_\chi$ in Theorem 5.1.1(a), thus on the choice of $\eta \in \Lambda$ such that $\lambda + d\eta$ is unramified (see Remark 5.2.2(2)). Replacing η by $\eta + p\zeta$ we get another equivalence, which is the composition of the first one with twist by $\mathcal{O}_\zeta$.

5.3.3. *Examples.* Let us list some objects in $\mathrm{mod}^{\mathrm{fg}}_\chi(U^\lambda)$ whose image in the derived category of coherent sheaves can be computed explicitly. We leave the proofs as an exercise to the reader.

0) A baby Verma module $M_{\mathfrak{b},\chi;\lambda+2\rho}$ corresponds to a skyscraper sheaf, see section 3.1.4.

Notice that our conventions about weights are chosen to make ample line bundles correspond to positive weights, which leads to a non-standard enumer-

[11] As is pointed out in Remark 2.1.2 the p-curvature of a $\mathcal{D}_X$-module $\mathcal{E}$ is a parallel section of $\mathrm{Fr}^*(\Omega^1) \otimes \mathcal{E}nd(\mathcal{E})$. If $\mathcal{E}$ is a line bundle we get a parallel section of $\mathrm{Fr}^*(\Omega^1)$, i.e. a section of Ω^1; for a line bundle with a flat connection on $\mathrm{Fr}\, N_X(Y)$ its p-curvature is a section of $\Omega^1_X|_Y$.

ation of baby Verma modules. In parallel notations in characteristic zero an irreducible Verma module has a dominant highest weight.

1) Let G be simple and simply-laced, and χ a subregular nilpotent.

Recall that the irreducible components of the (reduced) Springer fiber are indexed by the simple roots of G, each component is a projective line.

Consider the equivalence of the previous theorem corresponding to the choice $\lambda = -2\rho$, $\eta = \rho$ in the notations of the last remark. The images of irreducible objects of $\mathrm{mod}_\chi(U^{-2\rho}) = \mathrm{mod}_\chi(U^0)$ are as follows: $\mathcal{O}_{\mathbb{P}^1_\alpha}(-1)[1]$; and $\mathcal{O}_{\pi^{-1}(\chi)}$. Here $\mathbb{P}^1_\alpha$ runs over the set of irreducible components of $\mathcal{B}_\chi{}^{(1)}$, $\pi : T^*\mathcal{B}^{(1)} \to \mathcal{N}^{(1)}$ is the projection, and π^{-1} stands for the scheme-theoretic preimage. Notice that the same objects appear in the geometric theory of McKay correspondence, [KV].

2) $G = \mathrm{SL}(3)$, $\chi = 0$. See the appendix for a description of this example.

5.4. *Equivalences on formal neighborhoods.* We will extend Theorem 5.3.1 to the formal neighborhood of χ.[12] For λ, χ, ν as in 5.3.1, denote by $\widehat{\chi}$ and $\widehat{\mathcal{B}_{\chi,\nu}}$ the formal neighborhoods of χ in $\mathbf{pr}_1(T^*_\nu\mathcal{B})$ and $\mathcal{B}_{\chi,\nu}$ in $T^*_\nu\mathcal{B}$.

5.4.1. THEOREM. *There are canonical equivalences* $\mathrm{D}^b_{fg}(U^\lambda_{\widehat{\chi}}) \cong \mathrm{D}^b_c(\mathcal{D}^\lambda_{\widehat{\chi}}) \cong \mathrm{D}^b_c(\mathcal{O}_{\widehat{\mathcal{B}_{\chi,\nu}{}^{(1)}}})$.

Proof. Our main reference for sheaves on a formal scheme $\mathfrak{X}$ is [TL]. We consider the full subcategory $D^b_c(\mathcal{O}_\mathfrak{X})$ of the derived category $D(\mathcal{O}_\mathfrak{X})$ of the abelian category of *all* $\mathcal{O}_\mathfrak{X}$-modules by requiring that cohomology sheaves are coherent (and almost all vanish). Denote by $U^\lambda_{\widehat{\chi}}, \mathcal{D}^\lambda_{\widehat{\chi}}$ the restrictions of the coherent $\mathcal{O}$-algebras $U^\lambda, \mathcal{D}^\lambda$ to $\widehat{\chi}, \widehat{\mathcal{B}_{\chi,\nu}}$. Now, (coherent) $\mathcal{D}^\lambda_{\widehat{\chi}}$-modules are (coherent) $\mathcal{O}_{\widehat{\mathcal{B}_{\chi,\nu}}}$-modules with extra structure, and this allows us to lift the direct image functor $R\mu_* : \mathrm{D}^b_c(\mathcal{O}_{\widehat{\mathcal{B}_{\chi,\nu}{}^{(1)}}}) \to \mathrm{D}^b_c(\mathcal{O}_{\widehat{\chi}})$ to $R\mu_* : \mathrm{D}^b_c(\mathcal{D}^\lambda_{\widehat{\chi}}) \to \mathrm{D}^b_c(U^\lambda_{\widehat{\chi}})$ (as in 3.1.9). The proof that this is an equivalence follows the proof of Theorem 3.2. First, $R\mu_*(\mathcal{D}^\lambda_{\widehat{\chi}}) \cong U^\lambda_{\widehat{\chi}}$ follows from 3.4.1 by the formal base change for proper maps ([EGA, Th. 4.1.5]). Then, for the Calabi-Yau trick (3.5) one uses the Grothendieck duality for formal schemes ([TL, Th. 8.4, Prop. 2.5.11.c and 2.4.2.2]). The second equivalence follows from Theorem 5.1.1 above. $\square$

5.4.2. In the remainder of the section, for simplicity, λ *is integral regular and* $\chi \in \mathcal{N}$.

5.4.3. COROLLARY. *For* $p > h$ *there is a natural isomorphism of Grothendieck groups* $K(U^\lambda_\chi) \cong K(\mathcal{B}_\chi{}^{(1)})$. *In particular, the number of irre-*

[12]The same argument gives extension to the formal neighborhood of λ.

ducible U_χ^λ-modules is the rank of $K(\mathcal{B}_\chi)$. (This rank is calculated below in Theorem 7.1.1.)

Proof. It is well known that for a closed embedding $\iota : \mathfrak{X} \hookrightarrow \mathfrak{Y}$ of Noetherian schemes we have an isomorphism $K(\mathfrak{X}) \xrightarrow{\cong} K(\mathcal{C}\mathrm{oh}_{\mathfrak{X}}(\mathfrak{Y}))$ induced by the functor ι_*. In particular,

$$K({\mathcal{B}_\chi}^{(1)}) \cong K(\mathcal{C}\mathrm{oh}_{{\mathcal{B}_\chi}^{(1)}}(T^*\mathcal{B}^{(1)})) \cong K(\mathcal{C}\mathrm{oh}_{{\mathcal{B}_\chi}^{(1)}}(\widetilde{T}^*\mathcal{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*)). \qquad \square$$

5.4.4. *Remarks.* (a) In the case when χ is regular nilpotent in a Levi factor the corollary is a fundamental observation of Friedlander and Parshall ([FP]). The general case was conjectured by Lusztig ([Lu1], [Lu]).

(b) Theorem 5.1.1 provides a natural isomorphism of K-groups. However, if one is only interested in the number of irreducible modules (i.e., the size of the K-group), one does not need the splitting. Indeed, one can show that for any Noetherian scheme X, and an Azumaya algebra $\mathcal{A}$ over X of rank d^2, the forgetful functor from the category of $\mathcal{A}$-modules to the category of coherent sheaves induces an isomorphism $K(\mathcal{A}-\mathrm{mod}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{d}] \xrightarrow{\cong} K(\mathcal{C}\mathrm{oh}(X)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{d}]$.

5.5. *Equivariance.* Let H be a group. An H-category[13] is a category $\mathcal{C}$ with functors $[g] : \mathcal{C} \to \mathcal{C}$, $g \in H$, such that $[e_H]$ is isomorphic to the identity functor, and $[gh]$ to $[g] \circ [h]$ for all $g, h \in H$. If $\mathcal{C}$ is abelian or triangulated H-category we ask that the functors $[g]$ preserve the additional structure, and then $K(\mathcal{C})$ is an H-module. An H-functor is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ between H-categories such that $[g] \circ \mathcal{F} \cong \mathcal{F} \circ [g]$ for $g \in H$. If it induces a map of K-groups $K(\mathcal{F}) : K(\mathcal{C}) \to K(\mathcal{C}')$, then this is a homomorphism of H-modules.

The actions of the group $G(\mathbb{k})$ on U and $\mathcal{B}$ make all categories in Theorem 3.2 into $G(\mathbb{k})$-categories, while the categories appearing in Theorem 5.1.1(b) (for $\nu = 0$) are $G_\chi(\mathbb{k})$ categories. The action of $G_\chi(\mathbb{k})$ on these K-groups factors through $A_\chi = \pi_0(G_\chi)$.

5.5.1. Proposition. *The isomorphism $K(U_\chi^\lambda) \cong K({\mathcal{B}_\chi}^{(1)})$ in Corollary* 5.4.3 *is an isomorphism of A_χ-modules.*

Proof. The functors $\mathrm{R}\Gamma_{\mathcal{D}^\lambda}$ and $\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda}$ are clearly $G(\mathbb{k})$-functors. Thus it suffices to check that the Morita equivalences in Theorem 5.1.1 are $G_\chi(\mathbb{k})$-functors.

We will use a general observation that if a group H acts on a split Azumaya algebra A with a center Z and a splitting module E is H-invariant (in the sense that ${}^gE \cong E$ for any $g \in H$), then the Morita equivalence defined by E is an

[13] The term "a weak H-category" would be more appropriate here, since we do not fix isomorphisms between $[gh]$ and $[g] \circ [h]$; we use the shorter expression, since the more rigid structure does not appear in this paper.

H-functor. Indeed, for $g \in H$ a choice of an A-isomorphism $\psi_g : {}^gE \xrightarrow{\cong} E$ gives for each A-module M a Z-isomorphism

$$^g(E \otimes_A M) \xrightarrow{\mathrm{Id}} {}^gE \otimes_A ({}^gM) \xrightarrow{\psi_g \otimes \mathrm{Id}} E \otimes_A ({}^gM).$$

Thus we have to check that the splitting bundle $\mathcal{M}^\lambda_\chi$ of Theorem 5.1.1 is $G_\chi(\Bbbk)$ invariant. The equivalence between the Azumaya algebras $\mathcal{D}^\lambda$ and $\mathcal{D}^{\lambda+d\eta}$ from Lemma 2.3.1 is clearly $G(\Bbbk)$, and hence $G_\chi(\Bbbk)$ equivariant. Then our Azumaya algebra is $G_\chi(\Bbbk)$ equivariantly identified with the pull-back of an Azumaya algebra on $\widehat{\chi}^{(1)}$ (see the proof of Theorem 5.1.1), and $\mathcal{M}^\lambda_\chi$ is the pull-back of a splitting bundle from $\widehat{\chi}^{(1)}$; thus it is enough to see that the latter is $G_\chi(\Bbbk)$ invariant. This is obvious, since any two vector bundles (and also any two modules over a given Azumaya algebra) on $\widehat{\chi}^{(1)}$ of a given rank are isomorphic. $\square$

5.5.2. *Remarks.* (1) Proposition 5.5.1 can be used to sort out how many simple modules in a regular block are twists of each other, a question raised by Jantzen ([Ja3]). For instance, if G is of type G_2 and $p > 6$, we find that three out of five simple modules in a regular block are twists of each other.

(2) We expect that Proposition 5.5.1 can be strengthened: the splitting bundle $\mathcal{M}^\lambda_\chi$ can be shown to carry a natural $G_\chi(\Bbbk)$ equivariant structure; thus the equivalences of Theorem 5.1.1(b) can be enhanced to equivalences of strong $G_\chi(\Bbbk)$ categories (the isomorphisms $[gh] \cong [g] \circ [h]$ are fixed and satisfy natural compatibilities). We neither prove nor use this fact here.

6. Translation functors and dimension of U_χ-modules

In this section we spell out compatibility between the localization functor and translation functors, and use our results to derive some rough information about the dimension of U_χ-modules for $\chi \in \mathcal{N}$. We consider only integral elements of $\mathfrak{h}^*$ and we view them as differentials of elements of Λ. Similar methods can be applied to computation of the characters of the maximal torus in the centralizer of χ acting on an irreducible U_χ-module. We keep the assumption $p > h$.

6.1. *Translation functors.* For $\lambda \in \Lambda$, $\mathcal{D}^\lambda \overset{\mathrm{def}}{=}{}^{\mathcal{O}_\lambda}\mathcal{D}$ is canonically isomorphic to $\mathcal{D}^{d\lambda}$ for the differential $d\lambda$ and we also denote $U^\lambda \overset{\mathrm{def}}{=} U^{d\lambda}$ etc. We denote by $M \to [M]_\lambda$ the projection of the category of finitely generated $\mathfrak{g}$-modules with a locally finite action of $\mathfrak{Z}_{\mathrm{HC}}$ to its direct summand $\mathrm{mod}^{\mathrm{fg}}_\lambda(U) \overset{\mathrm{def}}{=} \mathrm{mod}^{\mathrm{fg}}_{d\lambda}(U)$. For $\lambda, \mu \in \Lambda$ the translation functor $T^\mu_\lambda : \mathrm{mod}^{\mathrm{fg}}_\lambda(U) \to \mathrm{mod}^{\mathrm{fg}}_\mu(U)$ is defined by $T^\mu_\lambda(M) \overset{\mathrm{def}}{=} [V_{\mu-\lambda} \otimes M]_\mu$ where $V_{\mu-\lambda}$ is the standard G-module with an extremal weight $\mu - \lambda$ as defined in 3.1.1.

Notice that the translation functor is well-defined. First, $V_{\mu-\lambda}\otimes M$ is finitely generated by [Ko, Prop. 3.3]. To show that the action of $\mathfrak{Z}_{\mathrm{HC}}$ on $V_{\mu-\lambda}\otimes M$ is locally finite we can assume that M is annihilated by a maximal ideal I_η of $\mathfrak{Z}_{\mathrm{HC}}$. By [MR1, Th. 1], for a very good p there is a ring homomorphism $\Upsilon : \mathfrak{Z}_{\mathbb{Z}} \to \mathfrak{Z}_{\mathrm{HC}} = \mathfrak{Z}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ where $\mathfrak{Z}_{\mathbb{Z}}$ is the center of $U(\mathfrak{g}_{\mathbb{Z}})$. By [Ko, Th. 5.1], for each $x \in \mathrm{im}(\Upsilon)$, on $V_{\mu-\lambda}\otimes M$

$$\prod_{\nu}(x-\eta(x)-\nu(x)) = 0, \tag{3}$$

where ν runs over the weights of $V_{\mu-\lambda}$. Thus $\mathfrak{Z}_{\mathrm{HC}}$ is spanned by elements satisfying equation (3). It follows that the action of $\mathfrak{Z}_{\mathrm{HC}}$ on $V_{\mu-\lambda}\otimes M$ is locally finite.

We review some standard ideas. For $\lambda, \mu, \eta \in \Lambda$ we denote by $\mathcal{W}_\eta$ the weights of V_η and $\mathcal{W}^\mu_\lambda \overset{\mathrm{def}}{=} (\lambda + \mathcal{W}_{\mu-\lambda}) \cap W'_{\mathrm{aff}}\bullet\mu$. Since we assume $p > h$, $\mathcal{W}^\mu_\lambda = (\lambda + \mathcal{W}_{\mu-\lambda}) \cap W_{\mathrm{aff}}\bullet\mu$.

6.1.1. For $\mathcal{M} \in \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\lambda(\widetilde{\mathcal{D}}))$, the sheaf of $\mathfrak{g}$-modules $V_\eta\otimes\mathcal{M} = (V_\eta\otimes\mathcal{O})\otimes_{\mathcal{O}}\mathcal{M}$ is an extension of terms $V_\eta(\nu)\otimes(\mathcal{O}_\nu\otimes_{\mathcal{O}}\mathcal{M})$ where ν runs over the set of weights $\mathcal{W}_\eta$ and $V_\eta(\nu)$ is the corresponding weight space. Since $\mathcal{O}_\nu\otimes_{\mathcal{O}}\mathcal{M} \in \mathrm{D}^b(\mathrm{mod}^{\mathrm{c}}_\lambda(\widetilde{\mathcal{D}}))$ we get the local finiteness of the $\mathfrak{Z}_{\mathrm{HC}}$-action on the sheaf $V_\eta\otimes\mathcal{M}$. Therefore, translation functors commute with taking the cohomology of $\mathcal{D}$-modules:

$$\begin{aligned} T^\mu_\lambda(\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda}\mathcal{M}) &= [V_{\mu-\lambda}\otimes\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda}\mathcal{M}]_\mu \\ &= [\mathrm{R}\Gamma_{\widetilde{\mathcal{D}}}(V_{\mu-\lambda}\otimes\mathcal{M})]_\mu \cong \mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\mu}([V_{\mu-\lambda}\otimes\mathcal{M})]_\mu). \end{aligned}$$

Moreover, $[V_{\mu-\lambda}\otimes_{\mathcal{O}}\mathcal{M}]_\mu$ is a successive extension of terms $V_{\mu-\lambda}(\nu)\otimes(\mathcal{O}_\nu\otimes_{\mathcal{O}}\mathcal{M})$ for weights $\nu \in \mathcal{W}^\mu_\lambda - \lambda \subseteq \mathcal{W}_{\lambda-\mu}$. There are two simple special cases:

6.1.2. Lemma. *Let λ, μ lie in the same closed alcove $\mathcal{A}$.*

(a) ("Down".) *If μ is in the closure of the facet of λ then*

$$T^\mu_\lambda(\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda}\mathcal{M}) \cong \mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\mu}(\mathcal{O}_{\mu-\lambda}\otimes_{\mathcal{O}}\mathcal{M}).$$

(b) ("Up".) *Let λ lie on the single wall H of $\mathcal{A}$ and μ be regular. If $s_H(\mu) < \mu$ for the reflection s_H in the H-wall, then*

$$\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},s_H(\mu)}(\mathcal{O}_{\lambda-\mu}\otimes_{\mathcal{O}}\mathcal{M}) \to T^\mu_\lambda(\mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\lambda}\mathcal{M}) \to \mathrm{R}\Gamma_{\widetilde{\mathcal{D}},\mu}(\mathcal{O}_{\mu-\lambda}\otimes_{\mathcal{O}}\mathcal{M}).$$

Proof. This follows from 6.1.1 and the following combinatorial observation from [Ja0, Lemmas 7.7 and 7.8]: if $\lambda, \mu \in \Lambda$ lie in the same alcove then

$$\mathcal{W}^\mu_\lambda = (\lambda + \mathcal{W}_{\mu-\lambda}) \cap W_{\mathrm{aff}}\bullet\mu = (W_{\mathrm{aff}})_\lambda\bullet\mu \subseteq \lambda + W\cdot(\mu-\lambda).$$

Indeed, the assumption in (a) implies that $(W_{\mathrm{aff}})_\mu \subseteq (W_{\mathrm{aff}})_\lambda$, hence $\mathcal{W}^\mu_\lambda = \{\mu\}$, while in (b) we assume $(W_{\mathrm{aff}})_\lambda = \{1, s_H\}$; hence $\mathcal{W}^\mu_\lambda = \{\mu, s_H(\mu)\}$, and $s_H(\mu)$ appears earlier in the filtration since $s_H(\mu) < \mu$. ∎

6.2. *Dimension.* We set $R = \prod_{\alpha} \langle \rho, \check{\alpha} \rangle$ where α runs over the set of positive roots of G.

6.2.1. THEOREM. *Fix $\chi \in \mathcal{N}$ and a regular weight $\lambda \in \Lambda$. For any module $M \in \mathrm{mod}^{\mathrm{fg}}_{(\lambda,\chi)}(U)$ there exists a polynomial $\mathbf{d}_M \in \frac{1}{R}\mathbb{Z}[\Lambda^*]$ of degree less or equal to $\dim(\mathcal{B}_\chi)$, such that for any $\mu \in \Lambda$ in the closure of the alcove of λ,*

$$\dim(T^\mu_\lambda(M)) = \mathbf{d}_M(\mu).$$

Moreover, $\mathbf{d}_M(\mu) = p^{\dim \mathcal{B}} \mathbf{d}^0_M(\frac{\mu+\rho}{p})$ for another polynomial $\mathbf{d}^0_M \in \frac{1}{R}\mathbb{Z}[\Lambda^*]$, such that $\mathbf{d}^0_M(\mu) \in \mathbb{Z}$ for $\mu \in \Lambda$.

6.2.2. *Remarks.* (0) The theorem is suggested by the experimental data kindly provided by J. Humphreys and V. Ostrik.

(1) The proof of the theorem gives an explicit description of $\mathbf{d}_M$ in terms of the corresponding coherent sheaf $\mathcal{F}_M$ on ${\mathcal{B}_\chi}^{(1)}$.

(2) For μ and λ as above, any module $N \in \mathrm{mod}^{\mathrm{fg}}_{(\mu,\chi)}(U)$ is of the form $T^\mu_\lambda M$ for some $M \in \mathrm{mod}^{\mathrm{fg}}_{(\lambda,\chi)}(U)$.[14] Indeed, according to Lemma 6.1.2.a and Proposition 3.4.2.c, $T^\mu_\lambda \mathrm{R}\Gamma(\mathcal{O}_{\lambda-\mu} \otimes \mathcal{L}^\mu N) = N$. Since T^μ_λ is exact we can choose M as the zero cohomology of $\mathrm{R}\Gamma(\mathcal{O}_{\lambda-\mu} \otimes \mathcal{L}^\mu N)$.

6.2.3. COROLLARY. *The dimension of any $N \in \mathrm{mod}^{\mathrm{fg}}_\chi(U)$ is divisible by $p^{\mathrm{codim}_{\mathcal{B}} \mathcal{B}_\chi}$.*

Proof. To apply the theorem observe that $\dim(N) < \infty$, so we may assume that $\mathfrak{Z}_{\mathrm{HC}}$ acts by a generalized eigencharacter. Since $\chi \in \mathcal{N}$ eigencharacter is necessarily integral, it lifts to some $\mu \in \Lambda$. We choose a regular λ so that μ is in the closure of the λ-facet, and $M \in \mathrm{mod}^{\mathrm{fg}}_{(\lambda,\chi)}(U)$ as in the remark 6.2.2(2). Then Theorem 6.2.1 says that $\dim(N) = p^{\dim \mathcal{B}} \cdot \mathbf{d}^0_M(\frac{\mu+\rho}{p})$. For $\delta = \deg(d^0_M) = \deg(d_M) \leq \dim(\mathcal{B}_\chi)$, the rational number $\dim(N)/p^{\dim(\mathcal{B})-\delta} = p^\delta \cdot \mathbf{d}^0_M(\frac{\mu+\rho}{p})$ is an integer since the denominator divides both R and a power of p, but R is prime to p for $p > h$ (for any root α, $\langle \rho, \check{\alpha} \rangle < h$). ∎

6.2.4. *Remark.* The statement of the corollary was conjectured by Kac and Weisfeiler [KW], and proved by Premet [Pr] under less restrictive assumptions on p. We still found it worthwhile to point out how this famous fact is related to our methods.

Our basic observation is

6.2.5. LEMMA. *Let $\mathcal{M}^\lambda_\chi$ be the splitting vector bundle for the restriction of the Azumaya algebra $\mathcal{D}^\lambda$ to ${\mathcal{B}_\chi}^{(1)}$, that was constructed in the proof of*

[14] Also, exactness of T^μ_λ implies that if N is irreducible we can choose M to be irreducible.

Theorem 5.1.1. *We have an equality in* $K^0({\mathcal{B}_\chi}^{(1)})$:

$$[\mathcal{M}_\chi^\lambda] = [(\mathrm{Fr}_{\mathcal{B}})_* \mathcal{O}_{p\rho+\lambda}|_{{\mathcal{B}_\chi}^{(1)}}]. \tag{4}$$

Proof. Since $\mathcal{D}^\lambda$ contains the algebra of functions on $\mathcal{B}\times_{\mathcal{B}^{(1)}} T^*\mathcal{B}^{(1)}$, any $\mathcal{D}^\lambda$-module $\mathcal{F}$ can be viewed as a quasicoherent sheaf $\mathcal{F}'$ on $\mathcal{B}\times_{\mathcal{B}^{(1)}} T^*\mathcal{B}^{(1)}$. If $\mathcal{F}$ is a splitting bundle of the restriction $\mathcal{D}^\lambda\big|_{Z^{(1)}}$ for a closed subscheme $Z \subset T^*\mathcal{B}$, then $\mathcal{F}'$ is a line bundle on $\mathcal{B}\times_{\mathcal{B}^{(1)}} Z^{(1)}$. It remains to show that the equality

$$[(\mathcal{M}_\chi^\lambda)'] = [\mathcal{O}_{p\rho+\lambda}|_{\mathrm{Fr}\, N(\mathcal{B}_\chi)}] \tag{5}$$

holds in $K(\mathrm{Fr}\, N(\mathcal{B}_\chi))$. The construction in the proof of Theorem 5.1.1 shows that $(\mathcal{M}_\chi^\lambda)' = \mathcal{O}_\lambda \otimes (\mathcal{M}_\chi^0)'$, thus it suffices to check (5) for one λ. We will do it for $\lambda = -\rho$ by constructing a line bundle $\mathcal{L}$ on $\mathrm{Fr}\, N(\mathcal{B}_\chi) \times \mathbb{A}^1$ such that the restriction of $\mathcal{L}$ at $1 \in \mathbb{A}^1$ is $(\mathcal{M}_\chi^{-\rho})'$, and at 0 it is $\mathcal{O}_{(p-1)\rho}|_{\mathrm{Fr}\, N(\mathcal{B}_\chi)}$; existence of such a line bundle implies the desired statement by rational invariance of K^0.

Let $\widetilde{\mathfrak{n}} \subset T^*\mathcal{B}$ be the preimage of $\mathfrak{n} \subset \mathcal{N}$ under the Springer map. Remark 5.2.2(3) together with Proposition 5.2.1(b) show that there exists a splitting bundle $\widetilde{\mathcal{M}}$ for $\mathcal{D}^{-\rho}\big|_{\widetilde{\mathfrak{n}}^{(1)}}$ whose restriction to ${\mathcal{B}_\chi}^{(1)}$ is $\mathcal{M}$; we thus get a line bundle $\widetilde{\mathcal{M}}'$ on $\mathcal{B}\times_{\mathcal{B}^{(1)}} \widetilde{\mathfrak{n}}^{(1)}$. Its restriction to the zero section $\mathcal{B} \subset \mathcal{B}\times_{\mathcal{B}^{(1)}} T^*\mathcal{B}^{(1)}$ is a line bundle on $\mathcal{B}$ whose direct image under Frobenius is isomorphic to $\mathcal{O}_{\mathcal{B}}^{p^{\dim \mathcal{B}}}$. It is easy to see that the only such line bundle is $\mathcal{O}_{(p-1)\rho}$. Thus we can let $\mathcal{L}$ be the pull-back of $\widetilde{\mathcal{M}}'$ under the map $\mathrm{Fr}\, N(\mathcal{B}_\chi) \times \mathbb{A}^1 \to \mathcal{B}\times_{\mathcal{B}^{(1)}} \widetilde{\mathfrak{n}}^{(1)}$, $(x,t) \mapsto (x, (Fr(x), t\chi))$. □

We also recall the standard numerics of line bundles on the flag variety.

6.2.6. LEMMA. *For any* $\mathcal{F} \in \mathrm{D}^b(\mathcal{C}oh(\mathcal{B}))$ *there exists a polynomial* $\mathbf{d}_{\mathcal{F}} \in \frac{1}{R}\mathbb{Z}[\Lambda^*]$ *such that for* $\lambda \in \Lambda$ *the Euler characteristic of* $R\Gamma(\mathcal{F}\otimes\mathcal{O}_\lambda)$ *equals* $\mathbf{d}(\lambda)$. *Moreover, we have*

$$\deg(\mathbf{d}_{\mathcal{F}}) \le \dim \mathrm{supp}(\mathcal{F}); \tag{6}$$

$$\mathbf{d}_{\mathrm{Fr}^*(\mathcal{F})}(\mu) = p^{\dim \mathcal{B}} \mathbf{d}_{\mathcal{F}}\Big(\frac{\mu + (1-p)\rho}{p}\Big). \tag{7}$$

Proof. The existence of $\mathbf{d}_{\mathcal{F}}$ is well-known, for line bundles it is given by the Weyl dimension formula, and the general case follows since the classes of line bundles generate $K(\mathcal{B})$. The degree estimate follows from Grothendieck-Riemann-Roch once we recall that $ch_i(\mathcal{F}) = 0$ for $i < \mathrm{codim}\,\mathrm{supp}(\mathcal{F})$ because the restriction map $\mathrm{H}^{2i}(\mathcal{B}) \to \mathrm{H}^{2i}(\mathcal{B} - \mathrm{supp}(\mathcal{F}))$ is an isomorphism for such i. To prove the polynomial identity (7) it suffices to check it for $\mu = p\nu - \rho$, $\nu \in \Lambda$. Then it follows from the well-known isomorphism $\mathrm{Fr}_*(\mathcal{O}_{-\rho}) \cong \mathcal{O}_{-\rho}^{\oplus\, p^{\dim(\mathcal{B})}}$ which implies that

$$\mathrm{Fr}_*(\mathrm{Fr}^*(\mathcal{F}) \otimes \mathcal{O}_{p\nu-\rho}) \cong \mathrm{Fr}_*(\mathrm{Fr}^*(\mathcal{F}\otimes\mathcal{O}_\nu) \otimes \mathcal{O}_{-\rho}) \cong \mathcal{F}\otimes\mathcal{O}_\nu \otimes \mathrm{Fr}_*(\mathcal{O}_{-\rho})$$

is isomorphic to the sum of $p^{\dim \mathcal{B}}$ copies of $\mathcal{F}\otimes\mathcal{O}_{\nu-\rho}$. □

6.2.7. *Proof of Theorem* 6.2.1. Let $\mathcal{F}_M \in \mathrm{D}^b(\mathcal{C}\mathrm{oh}_{\mathcal{B}_{\chi,\nu}{}^{(1)}}(\widetilde{T}^*\mathcal{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*))$ be the image of M under the equivalence of Theorem 5.3.1, i.e., $\mathcal{L}^\lambda M \cong \mathcal{M}_\lambda \otimes \mathcal{F}_M$; and let $[\mathcal{F}_M] \in K(\mathcal{C}\mathrm{oh}_{\mathcal{B}_{\chi,\nu}{}^{(1)}}(\widetilde{T}^*\mathcal{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*)) = K(\mathcal{B}_\chi{}^{(1)})$ be its class. According to Lemma 6.1.2(a)

$$T_\lambda^\mu(M) = \mathrm{R}\Gamma(\mathcal{O}_{\mu-\lambda} \otimes \mathcal{L}^\lambda M) = \mathrm{R}\Gamma(\mathcal{O}_{\mu-\lambda} \otimes \mathcal{M}_\lambda \otimes \mathcal{F}_M) = \mathrm{R}\Gamma(\mathcal{M}_\mu \otimes \mathcal{F}_M).$$

Let $\int$ stand for Euler characteristic of $\mathrm{R}\Gamma$, so that

$$\dim(T_\lambda^\mu(M)) = \int_{\mathcal{B}_\chi{}^{(1)}} [\mathcal{M}_\mu] \cdot [\mathcal{F}_M],$$

where the multiplication sign stands for the action of K^0 on K. Now, by Lemma 6.2.5 we may rewrite this as (denoting by f^*, f_* the standard functoriality of Grothendieck groups and $\mathcal{B}_\chi{}^{(1)} \overset{i}{\hookrightarrow} \mathcal{B}^{(1)}$),

$$\begin{aligned} \int_{\mathcal{B}_\chi{}^{(1)}} i^*[(\mathrm{Fr}_\mathcal{B})_* \mathcal{O}_{p\rho+\mu}] \cdot [\mathcal{F}_M] &= \int_{\mathcal{B}^{(1)}} [(\mathrm{Fr}_\mathcal{B})_* \mathcal{O}_{p\rho+\mu}] \cdot i_*[\mathcal{F}_M] \\ &= \int_{\mathcal{B}} \mathcal{O}_{p\rho+\mu} \cdot \mathrm{Fr}_\mathcal{B}^*(i_*[\mathcal{F}_M]). \end{aligned}$$

So, Lemma 6.2.6 shows that

$$\dim(T_\lambda^\mu M) = \mathbf{d}_{\mathrm{Fr}_\mathcal{B}^*(i_*\mathcal{F}_M)}(p\rho + \mu) = p^{\dim \mathcal{B}} \cdot \mathbf{d}_{\mathcal{F}_M}(\frac{\mu+\rho}{p}).$$

Taking into account (6), (7) we see that the polynomial $\mathbf{d}_M^0 = \mathbf{d}_{i_*\mathcal{F}_M}$ satisfies the conditions of the theorem. □

7. K-theory of Springer fibers

In this section we prove Theorem 7.1.1.

7.1. *Bala-Carter classification of nilpotent orbits* [Sp]. Let $G_\mathbb{Z}$ (with the Lie algebra $\mathfrak{g}_\mathbb{Z}$) be the split reductive group scheme over $\mathbb{Z}$ that gives G by extension of scalars: $(G_\mathbb{Z})_\Bbbk = G$. Fix a split Cartan subgroup $T_\mathbb{Z} \subseteq G_\mathbb{Z}$ and a Bala-Carter datum, i.e., a pair (L, λ) where L is Levi factor of $G_\mathbb{Z}$ that contains $T_\mathbb{Z}$, and λ is a cocharacter of $T_\mathbb{Z} \cap L'$ (for the derived subgroup L' of L), such that the λ-weight spaces $(\mathfrak{l}')^0$ and $(\mathfrak{l}')^2$ (in the Lie algebra $\mathfrak{l}'$ of L'), have the same rank. To such data one associates for any closed field k of good characteristic a nilpotent orbit in $\mathfrak{g}_k$ which we will denote α_k. It is characterized by the fact that α_k is dense in $(\mathfrak{l}'_k)^2$. This gives a bijection between W-orbits of Bala-Carter data and nilpotent orbits in $\mathfrak{g}_k$. In particular the classification of nilpotent orbits over a closed field is uniform for all good characteristics (including zero). This is used in the formulation of:

7.1.1. THEOREM. *For $p > h$ the Grothendieck group of $\mathcal{C}\mathrm{oh}(\mathcal{B}_\chi)$ has no torsion and its rank coincides with the dimension of the cohomology of the corresponding Springer fiber over a field of characteristic zero.*

7.1.2. The absence of torsion is clear from Corollary 5.4.3. The rank will be found from known favorable properties of K-theory and cohomology of Springer fibers using the Riemann-Roch Theorem. We start with recalling some standard basic facts about the K-groups.

7.1.3. *Specialization in* K*-theory.* Let X be a Noetherian scheme, flat over a discrete valuation ring $\mathcal{O}$. Let $\eta = \mathrm{Spec}(k_\eta), s = \mathrm{Spec}(k_s)$ be respectively the generic and the special point of $\mathrm{Spec}(\mathcal{O})$ and denote $X_s \overset{i_s}{\to} X \overset{i_\eta}{\leftarrow} X_\eta$. The specialization map $\mathrm{sp} : K(X_\eta) \to K(X_s)$ is defined by $\mathrm{sp}(a) \overset{\mathrm{def}}{=} (i_s)^*(\widetilde{a})$ for $a \in K(X_s)$ and any extension $\widetilde{a} \in K(X)$ of a (i.e. $(i_\eta)^*\widetilde{a} = a$). To see that this makes sense we use the excision sequence

$$K(X_s) \xrightarrow{(i_s)_*} K(X) \xrightarrow{(i_\eta)^*} K(X_\eta) \to 0$$

and observe that $(i_s)^*(i_s)_* = 0$ on $K(X_s)$ since the flatness of X gives exact triangle $\mathcal{F}[1] \to (i_s)^*(i_s)_*(\mathcal{F}) \to \mathcal{F}$ for $\mathcal{F} \in \mathrm{D}^b(\mathcal{C}\mathrm{oh}_{X_s})$.

7.1.4. *A lift to the formal neighborhood of p.* Assume now that $\mathcal{O}$ is the ring of integers in a finite extension $\mathcal{K} = k_\eta$ of $\mathbb{Q}_p$, with an embedding of the residue field k_s into $\Bbbk$.

Let $G_\mathcal{O}$ be the group scheme $(G_\mathbb{Z})_\mathcal{O}$ over $\mathcal{O}$ (extension of scalars), so that $(G_\mathcal{O})_\Bbbk = G$, and similarly for the Lie algebras. By a result of Spaltenstein [Sp], one can choose $x_\mathcal{O} \in \mathfrak{g}_\mathcal{O}$ so that (1) its images in $\mathfrak{g}_\mathcal{K}$ and in $\mathfrak{g}_{k_s}$ lie in nilpotent orbits $\alpha_\mathcal{K}$ and α_{k_s}, (2) the $\mathcal{O}$-submodule $[x_\mathcal{O}, \mathfrak{g}_\mathcal{O}] \subseteq \mathfrak{g}_\mathcal{O}$ has a complementary submodule $Z_\mathcal{O}$, (3) for the Bala-Carter cocharacter $G_{m,\mathbb{Z}} \overset{\lambda}{\to} G_\mathbb{Z}$ (see 7.1), $x_\mathcal{O}$ has weight 2 and the sum of all positive weight spaces $\mathfrak{g}_\mathcal{O}^{>0}$ lies in $[x_\mathcal{O}, \mathfrak{g}_\mathcal{O}]$. We denote by $\mathcal{B}_\chi^\mathcal{O}$ the Springer fiber at $x_\mathcal{O}$ (i.e., the $\mathcal{O}$-version of $\mathcal{B}_\chi$ from 4.1.2), and so it is defined as the reduced part of the inverse of $x_\mathcal{O}$ under the moment map.

7.1.5. LEMMA. (a) *$Z_\mathcal{O}$ can be chosen G_m-invariant and with weights ≤ 0.*

(b) *Now $S_\mathcal{O} = x_\mathcal{O} + Z_\mathcal{O}$ is a slice to the orbit α in the sense that*:

(i) *the conjugation $G_\mathcal{O} \times_\mathcal{O} S_\mathcal{O} \to \mathfrak{g}_\mathcal{O}$ is smooth,*

(ii) *the G_m-action on $\mathfrak{g}$ by $c \bullet y \overset{\mathrm{def}}{=} c^{-2} \cdot {}^{\lambda(c)}y$, contracts $S_\mathcal{O}$ to $x_\mathcal{O}$.*

(c) *The Springer fiber* $X=\mathcal{B}^{\mathcal{O}}_{\chi}$ *of* $x_{\mathcal{O}}$ *is flat*[15] *over* $\mathcal{O}$ *and the Slodowy scheme* $\widetilde{S}_{\mathcal{O}}$ (*defined as the preimage of* $S_{\mathcal{O}}$ *under the Springer map*), *is smooth over* $\mathcal{O}$.

Proof. (a) is elementary: if $M\subseteq A\subseteq B$ and M has a complement C in B then it has a complement $A\cap C$ in A. Now $[x_{\mathcal{O}},\mathfrak{g}_{\mathcal{O}}]$ is G_m-invariant and each weight space $[x_{\mathcal{O}},\mathfrak{g}_{\mathcal{O}}]^n$ has a complement in $[x_{\mathcal{O}},\mathfrak{g}_{\mathcal{O}}]$, hence in $\mathfrak{g}_{\mathcal{O}}$, and then also a complement $Z^n_{\mathcal{O}}$ in $\mathfrak{g}^n_{\mathcal{O}}$. So, $Z_{\mathcal{O}}=\oplus_n Z^n_{\mathcal{O}}$ is a G_m-invariant complement. Claim (b_{ii}) is clear. The smoothness in (b_{i}) is valid on a neighborhood of $G_{\mathcal{O}}\times_{\mathcal{O}}x_{\mathcal{O}}$ by (2) (the image of the differential at a point in $G_{\mathcal{O}}\times_{\mathcal{O}}S_{\mathcal{O}}$ is $[x_{\mathcal{O}},\mathfrak{g}_{\mathcal{O}}]+Z_{\mathcal{O}}$). Then the general case follows from the contraction in (b_{ii}).

In (c), the smoothness of $\widetilde{S}_{\mathcal{O}}$ follows from (b_{i}) by a formal base change argument ([Sl, §5.3]). Finally, to see that $\mathcal{B}^{\mathcal{O}}_{\chi}$ is flat we use the cocharacter λ to define a parabolic subgroup $P_{\mathcal{O}}\subseteq G_{\mathcal{O}}$ such that its Lie algebra is $\mathfrak{g}^{\geq 0}_{\mathcal{O}}$. Let $\mathcal{B}_{x_{\mathcal{O}}}$ be the scheme theoretic Springer fiber at $x_{\mathcal{O}}$, i.e., the scheme theoretic inverse of $x_{\mathcal{O}}$ under the moment map. Following Proposition 3.2 in [DLP] we will see that the intersection of $\mathcal{B}_{x_{\mathcal{O}}}$ with each $P_{\mathcal{O}}$ orbit in the flag variety $\mathcal{B}_{\mathcal{O}}$ is smooth over $\mathcal{O}$.

Each $w\in W$ defines a Borel subalgebra ${}^{w}\mathfrak{b}_{\mathcal{O}}$ of $\mathfrak{g}_{\mathcal{O}}$. We view it also as an $\mathcal{O}$-point $p^w_{\mathcal{O}}$ of the flag variety $\mathcal{B}_{\mathcal{O}}$ over $\mathcal{O}$, and use it to generate a $P_{\mathcal{O}}$-orbit $\mathfrak{O}_w\subseteq\mathcal{B}_{\mathcal{O}}$. Consider the maps

$$\mathfrak{O}_w \xleftarrow{\psi_w} P_{\mathcal{O}} \xrightarrow{\phi} \mathfrak{g}^{\geq 2}_{\mathcal{O}},$$

where ϕ is given by $P_{\mathcal{O}}\cong P_{\mathcal{O}}\times_{\mathcal{O}}x_{\mathcal{O}}\to\mathfrak{g}^{\geq 2}_{\mathcal{O}}$, $(g,y)\mapsto {}^{g^{-1}}y$, and ψ_w by $P_{\mathcal{O}}\cong P_{\mathcal{O}}\times_{\mathcal{O}}p^w_{\mathcal{O}}\to\mathfrak{g}^{\geq 2}_{\mathcal{O}}$, $(g,p)\mapsto gp$. Here, ψ_w is smooth as the quotient map of a group scheme by a smooth group subscheme, and ϕ is smooth since property (3) implies that $\mathfrak{g}^{\geq 2}_{\mathcal{O}}\subseteq[x_{\mathcal{O}},\mathfrak{g}_{\mathcal{O}}]^{\geq 2}=[x_{\mathcal{O}},\mathfrak{g}^{\geq 0}_{\mathcal{O}}]=[x_{\mathcal{O}},\mathfrak{p}_{\mathcal{O}}]$. Now, $\mathcal{B}_{x_{\mathcal{O}}}\cap\mathfrak{O}_w$ is smooth over $\mathcal{O}$ since the scheme theoretic inverses ${\psi_w}^{-1}(\mathcal{B}_{x_{\mathcal{O}}}\cap\mathfrak{O}_w)$ and $\phi^{-1}(\mathfrak{g}^{\geq 2}_{\mathcal{O}}\cap{}^{w}\mathfrak{b}_{\mathcal{O}})$ coincide.

Now we see that any p-torsion function f on an open affine piece U of $\mathcal{B}_{x_{\mathcal{O}}}$ has to be nilpotent (so the functions on the reduced scheme $\mathcal{B}^{\mathcal{O}}_{\chi}$ have no p-torsion and $\mathcal{B}^{\mathcal{O}}_{\chi}$ is flat over $\mathcal{O}$). The restriction of f to each stratum is zero (strata are smooth, in particular flat). However any closed point of U lies in the restriction U_s to the special point, hence in one of the strata. Since f vanishes at closed points of U it is nilpotent. □

7.1.6. We will use the rational K-groups $K(\mathfrak{X})_{\mathbb{Q}}\overset{\mathrm{def}}{=}K(\mathfrak{X})\otimes_{\mathbb{Z}}\mathbb{Q}$ where $\mathfrak{X}$ is a Springer fiber $\mathcal{B}^{A}_{\chi}$ over A which could be $\mathbb{C},\mathcal{O},\eta,s,\Bbbk$ etc. The main claim in this section is

[15] Though one expects that the scheme theoretic fiber is also flat, this version is good enough for the specialization machinery.

7.1.7. PROPOSITION. *Assume that* $\oplus_i\ \mathrm{H}^{2i}_{\mathrm{et}}(\mathcal{B}^{\overline{\mathcal{K}}}_\chi, \mathbb{Q}_l(-i))$ *is a trivial* $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ *module.*[16]

(a) *The specialization* $\mathrm{sp} : K(\mathcal{B}^\eta_\chi)_\mathbb{Q} \xrightarrow{\cong} K(\mathcal{B}^s_\chi)_\mathbb{Q}$ *identifies the* K*-groups over generic and special points.*

(b) *The base change map identifies the* K*-groups over the special point and over* $\Bbbk$*. Also, for any embedding* $\mathcal{K} \hookrightarrow \mathbb{C}$ *the corresponding base change map identifies* K*-groups over the generic point and over* $\mathbb{C}$:

$$K(\mathcal{B}^\eta_\chi)_\mathbb{Q} \xrightarrow{\cong} K(\mathcal{B}^\mathbb{C}_\chi)_\mathbb{Q}, \tag{8}$$

$$K(\mathcal{B}^s_\chi)_\mathbb{Q} \xrightarrow{\cong} K(\mathcal{B}^\Bbbk_\chi)_\mathbb{Q}. \tag{9}$$

7.1.8. *Proposition* 7.1.7 *implies Theorem* 7.1.1. In the chain of isomorphisms

$$K(\mathcal{B}^\Bbbk_\chi)_\mathbb{Q} \xleftarrow{\cong} K(\mathcal{B}^s_\chi)_\mathbb{Q} \xleftarrow[\mathrm{sp}]{\cong} K(\mathcal{B}^\eta_\chi)_\mathbb{Q} \xrightarrow{\cong} K(\mathcal{B}^\mathbb{C}_\chi)_\mathbb{Q} \underset{\tau}{\cong} A_\bullet(\mathcal{B}^\mathbb{C}_\chi)_\mathbb{Q} \cong \mathrm{H}^*(\mathcal{B}^\mathbb{C}_\chi, \mathbb{Q}),$$

the first three are provided by the proposition. It is shown in [DLP] that the Chow group $A_\bullet(\mathcal{B}^\mathbb{C}_\chi)$ is a free abelian group of finite rank equal to $\dim \mathrm{H}^*(\mathcal{B}^\mathbb{C}_\chi, \mathbb{Q})$. Finally, by [Fu], Corollary 18.3.2, the "modified Chern character" $\tau_{\mathcal{B}^\mathbb{C}_\chi}$ provides the fourth isomorphism. □

7.2. *Base change from* $\mathcal{K}$ *to* $\mathbb{C}$. The remainder is devoted to the proof of Proposition 7.1.7. We need two standard auxiliary lemmas on Galois action.

7.2.1. LEMMA. *Let* L/K *be a field extension. Let* X *be a scheme of finite type over* K*. Then the base change map* $\mathrm{bc} = \mathrm{bc}^L_K : K(X)_\mathbb{Q} \to K(X_L)_\mathbb{Q}$ *is injective. If* L/K *is a composition of a purely transcendental and a normal algebraic extension* (*e.g. if* L *is algebraically closed*) *then the image of* bc *is the space of invariants* $K(X_L)^{\mathrm{Gal}(L/K)}_\mathbb{Q}$.

Proof. If L/K is a finite normal extension, then the direct image (restriction of scalars) functor induces a map $\mathrm{res} : K(X_L) \to K(X)$, such that $\mathrm{res} \circ \mathrm{bc} = \deg(L/K) \cdot \mathrm{id}$, and $\mathrm{bc} \circ \mathrm{res}(x) = n \cdot \sum_{\gamma \in \mathrm{Gal}(L/K)} \gamma(x)$, where n is the inseparability degree of the extension L/K. This implies our claim in this case; injectivity of bc for any finite extension follows.

If $L = K(\alpha)$ where α is transcendental over K, then $K(X) \xrightarrow{\cong} K(X_L)$; this follows from the excision sequence

$$\oplus_{t \in \mathbb{A}^1} K(X \times t) \to K(X \times \mathbb{A}^1) \to K(X_{K(\alpha)}) \to 0$$

(where t runs over the closed points in $\mathbb{A}^1_K$), since the first map is zero and $K(X \times \mathbb{A}^1) \cong K(X)$.

[16] A finite extension $\mathcal{K}/\mathbb{Q}_p$ satisfying this assumption exists by Lemma 7.2.2.

If L is finitely generated over K, so that there exists a purely transcendental subextension $K \subset K' \subset L$ with $|L/K| < \infty$, then injectivity follows by comparing the previous two special cases; if L/K' is normal we also get the description of the image of bc.

Finally, the general case follows from the case of a finitely generated extension by passing to the limit. □

7.2.2. LEMMA. *For all i the Galois group* $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ *acts on the l-adic cohomology* $\mathrm{H}^{2i}_{\mathrm{et}}(\mathcal{B}^{\overline{\mathcal{K}}}_{\chi}, \mathbb{Q}_l(-i))$ *through a finite quotient.*

Proof. The cycle map $c_{\mathbb{Q}_l} : A_i(\mathcal{B}^{\overline{\mathcal{K}}}_{\chi})_{\mathbb{Q}_l} \to \mathrm{H}^{2i}_{\mathrm{et}}(\mathcal{B}^{\overline{\mathcal{K}}}_{\chi}, \mathbb{Q}_l(-i))^*$, defined by $\langle c_{\mathbb{Q}_l}([Z]), h\rangle = \int h|_Z$ for an i-dimensional cycle Z (here $\int : \mathrm{H}^{2i}_{\mathrm{et}}(Z, \mathbb{Q}_l(-i)) \to \mathbb{Q}_l$ is the canonical map), is compatible with the $\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$ action. It is an isomorphism since $\bar{\mathcal{K}} \cong \mathbb{C}$ and the results of [DLP] show that the cycle map $c : A_i(\mathcal{B}^{\mathbb{C}}_{\chi}) \to \mathrm{H}_{2i}(\mathcal{B}^{\mathbb{C}}_{\chi}, \mathbb{Z})$ is an isomorphism.

In order to factor the action of $\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$ on $A_*(\mathcal{B}^{\overline{\mathcal{K}}}_{\chi})$ through $\mathrm{Gal}(\mathcal{K}'/\mathcal{K})$ we choose a finite set of cycles Z_i whose classes form a basis in $A_*(\mathcal{B}^{\mathbb{C}}_{\chi})_{\mathbb{Q}}$, and then a finite subextension $\mathcal{K}' \subset \bar{\mathcal{K}}$ such that all Z_i are defined over $\mathcal{K}'$. □

7.2.3. *Proof of* (8). Lemma 7.2.1 says that $K(\mathcal{B}^{\mathcal{K}}_{\chi})_{\mathbb{Q}} = K(\mathcal{B}^{\overline{\mathcal{K}}}_{\chi})_{\mathbb{Q}}^{\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})}$ so it suffices to see that the Galois action on $K(\mathcal{B}^{\overline{\mathcal{K}}}_{\chi})_{\mathbb{Q}}$ is trivial. However, 7.1.8 and the proof of 7.2.2 provide $\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$-equivariant isomorphisms $K(\mathcal{B}^{\overline{\mathcal{K}}}_{\chi})_{\mathbb{Q}} \xrightarrow[\tau]{\cong} A_\bullet(\mathcal{B}^{\overline{\mathcal{K}}}_{\chi})_{\mathbb{Q}} \xrightarrow[c_{\mathbb{Q}_l}]{\cong} \mathrm{H}^\bullet_{\mathrm{et}}(\mathcal{B}^{\bar{\mathcal{K}}}_{\chi}, \mathbb{Q}_l(-i))^*$. □

7.3. *The specialization map in* 7.1.7(a) *is injective.* For this we will use the pairing of K-groups of the Springer fiber and of the Slodowy variety. Let $\mathfrak{X}$ be a proper variety over a field k, and $i : \mathfrak{X} \hookrightarrow \mathfrak{Y}$ be a closed embedding, where $\mathfrak{Y}$ is smooth over k. We have a bilinear pairing $\mathcal{E}\mathrm{ul} = \mathcal{E}\mathrm{ul}_k : K(\mathfrak{Y}) \times K(\mathfrak{X}) \to \mathbb{Z}$, where $\mathcal{E}\mathrm{ul}([\mathcal{F}], [\mathcal{G}])$ is the Euler characteristic of $\mathrm{Ext}^\bullet(\mathcal{F}, i_*\mathcal{G})$.

Let us now return to the situation of 7.1.3, and assume that X is proper over $\mathcal{O}$, and that $i : X \hookrightarrow Y$ is a closed embedding, where Y is smooth over $\mathcal{O}$. For $a \in K(Y^\eta)$, $b \in K(X^\eta)$ we have

$$\mathcal{E}\mathrm{ul}_s(\mathrm{sp}(a), \mathrm{sp}(b)) = \mathcal{E}\mathrm{ul}_\eta(a, b)$$

since $(\mathrm{L}i_s^*)\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \cong \mathrm{R}\mathcal{H}\mathrm{om}(\mathrm{L}i_s^*\mathcal{F}, \mathrm{L}i_s^*\mathcal{G})$ for

$$\mathcal{F} \in \mathrm{D}^b(\mathcal{C}\mathrm{oh}(Y)), \quad \mathcal{G} \in \mathrm{D}^b(\mathcal{C}\mathrm{oh}(Y)).$$

In particular, if the pairing $\mathcal{E}\mathrm{ul}_\eta$ is nondegenerate in the second variable, specialization $\mathrm{sp} : K(X^\eta) \to K(X^s)$ is injective.

Since the Slodowy scheme $\widetilde{S}_{\mathcal{O}}$ is smooth (in particular flat) over $\mathcal{O}$ (Lemma 7.1.5), we can apply these considerations to $X = \mathcal{B}^{\mathcal{O}}_{\chi}$, and $Y = \widetilde{S}_{\mathcal{O}}$. It is proved in [Lu, II, Th. 2.5], that the pairing $(\mathcal{E}\mathrm{ul}_{\mathbb{C}})_{\mathbb{Q}} : K(Y^{\mathbb{C}})_{\mathbb{Q}} \times K(X^{\mathbb{C}})_{\mathbb{Q}}$

$\to \mathbb{Q}$ is nondegenerate. Since $K(X^\eta)_\mathbb{Q} \xrightarrow{\cong} K(X^\mathbb{C})_\mathbb{Q}$ is proved in 7.3 and the same argument shows that $K(Y^\eta)_\mathbb{Q} \xrightarrow{\cong} K(Y^\mathbb{C})_\mathbb{Q}$, the pairing $\mathcal{E}\mathrm{ul}_\eta$ is also nondegenerate and then sp is injective. □

Remark 2. The proof of Lemma 7.4.1 below can be adapted to give a proof that $\mathcal{E}\mathrm{ul}_\Bbbk$ is nondegenerate if $\Bbbk$ has large positive characteristic. One can then deduce that the same holds for $\Bbbk = \mathbb{C}$. This would give an alternative proof of the result from [Lu, II] mentioned above.

7.4. *Upper bound on the K-group.* Here we use another Euler pairing to prove that

$$\dim_\mathbb{Q} K(\mathcal{B}^\Bbbk_\chi)_\mathbb{Q} \;\le\; \dim_\mathbb{Q} \mathrm{H}^\bullet(\mathcal{B}^\mathbb{C}_\chi, \mathbb{Q}). \tag{10}$$

Besides $K(\mathfrak{X}) = K(\mathcal{C}\mathrm{oh}(\mathfrak{X}))$ one can consider $K^0(\mathfrak{X})$, the Grothendieck group of vector bundles (equivalently, of complexes of finite homological dimension) on $\mathfrak{X}$. When $\mathfrak{X}$ is proper over a field we have another Euler pairing $\mathrm{Eul}_\mathfrak{X}$: $K^0(\mathfrak{X}) \times K(\mathfrak{X}) \to \mathbb{Z}$ by $\mathrm{Eul}_\mathfrak{X}([\mathcal{F}], [\mathcal{G}]) = [\mathrm{RHom}(\mathcal{F}, \mathcal{G})]$.

7.4.1. Lemma. *The Euler pairing* $\mathrm{Eul}_\mathfrak{X}$ *for* $\mathfrak{X} = \mathcal{B}^\Bbbk_\chi$ *is nondegenerate in the second factor*; *i.e.*, *it yields an injective map* $K(\mathfrak{X}) \hookrightarrow \mathrm{Hom}(K^0(\mathfrak{X}), \mathbb{Z})$.

Proof. Let $\mathcal{B}_\chi \overset{\iota}{\hookrightarrow} \widehat{\mathcal{B}}_\chi$ be the formal neighborhood of $\mathcal{B}_\chi$ in $T^*\mathcal{B}$. For any vector bundle V on $\widehat{\mathcal{B}}_\chi$ and $\mathcal{G} \in \mathrm{D}^b(\mathcal{B}_\chi)$, one has $\mathrm{RHom}^\bullet(V, \iota_*\mathcal{G}) \cong \mathrm{RHom}^\bullet(\iota^*V, \mathcal{G})$. So it suffices to show that the Euler pairing $\mathcal{E}\mathrm{ul} : K(\widehat{\mathcal{B}}_\chi) \times K(\mathcal{B}_\chi) \to \mathbb{Z}$, $\mathcal{E}\mathrm{ul}([V], [\mathcal{G}]) = [\mathrm{RHom}^\bullet(V, \iota_*\mathcal{G})]$, is a perfect pairing.

Let us interpret this pairing using localization. The first of the isomorphisms (see 4.1.1 for notation)

$$K(\mathcal{B}_\chi) \cong K(\mathrm{mod}^{\mathrm{fl}}(U^0_{\widehat{\chi}})) \quad \text{and} \quad K(\mathcal{C}\mathrm{oh}(\widehat{\mathcal{B}}_\chi)) \cong K(\mathrm{mod}^{\mathrm{fg}}(U^0_{\widehat{\chi}})),$$

comes from Theorem 5.3.1 (notice that $\mathrm{mod}^{\mathrm{fl}}(U^0_{\widehat{\chi}}) = \mathrm{mod}_\chi(U^0)$; see 4.1.1), and the second one from Theorem 5.4.1 (notice that $K^0(\widehat{\mathcal{B}}_\chi) \xrightarrow{\cong} K(\widehat{\mathcal{B}}_\chi)$ because $T^*\mathcal{B}$ is smooth). The above Euler pairing now becomes the Euler pairing

$$K(\mathrm{mod}^{\mathrm{fg}}(U^0_{\widehat{\chi}})) \times K(\mathrm{mod}^{\mathrm{fl}}(U^0_{\widehat{\chi}})) \to \mathbb{Z}.$$

However, the completion $U^0_{\widehat{\chi}}$ of U^0 at χ is a complete multi-local algebra of finite homological dimension: this follows from finiteness of homological dimension of U^0, which is clear from Theorem 3.2. Thus the latter pairing is perfect, because the classes of irreducible and of indecomposable projective modules provide dual bases in $K(\mathrm{mod}^{\mathrm{fl}}(U^0_{\widehat{\chi}}))$ and $K(\mathrm{mod}^{\mathrm{fg}}(U^0_{\widehat{\chi}}))$ respectively. □

7.4.2. Lemma. *If* $\mathfrak{X}$ *is a projective variety over a field*, *such that the pairing* $\mathrm{Eul}_\mathfrak{X}$ *is nondegenerate in the second factor* $K(\mathfrak{X})$, *then the following composition of the modified Chern character* τ *and the* l*-adic cycle map* $c_{\mathbb{Q}_l}$, *is* *injective*:

$$K(\mathfrak{X})_{\mathbb{Q}_l} \xrightarrow{\tau} A_\bullet(\mathfrak{X})_{\mathbb{Q}_l} \xrightarrow{c_{\mathbb{Q}_l}} \bigoplus_i (\mathrm{H}^{2i}_{\mathrm{et}}(\mathfrak{X}, \mathbb{Q}_l(-i)))^*.$$

Proof. The pairing $\mathrm{Eul}_{\mathfrak{X}}$ factors through the modified Chern character by the Riemann-Roch-Grothendieck Theorem [Fu, 18.3], and then through the cycle map by [Fu, Prop. 19.1.2, and the text after Lemma 19.1.2] (this reference uses the cycle map for complex varieties and ordinary Borel-Moore homology; however the proofs adjust to the l-adic cycle map). ☐

7.4.3. LEMMA. $\dim_{\bar{\mathbb{Q}}_l}\ \mathrm{H}^*_{\mathrm{et}}(\mathcal{B}^{\Bbbk}_\chi, \bar{\mathbb{Q}}_l) = \dim_{\mathbb{Q}}\ \mathrm{H}^*(\mathcal{B}^{\mathbb{C}}_\chi, \mathbb{Q})$.

Proof.[17] Since the decomposition of the Springer sheaf into irreducible perverse sheaves is independent of p, the calculation of the cohomology of Springer fibers (i.e., the stalks of the Springer sheaf), reduces to the calculation of stalks of intersection cohomology sheaves of irreducible local systems on nilpotent orbits. However, Lusztig proved that the latter one is independent of p for good p ([Lu2, §24, in particular Th. 24.8 and Subsection 24.10]). ☐

7.4.4. *Proof of the upper bound* (10) . Lemmas 7.4.1 and 7.4.2 give the embedding $K(\mathcal{B}^{\Bbbk}_\chi)_{\mathbb{Q}_l} \xrightarrow{c_{\mathbb{Q}_l}\circ\tau} \bigoplus_i \mathrm{H}^{2i}_{\mathrm{et}}(\mathcal{B}^{\Bbbk}_\chi, \mathbb{Q}_l(-i))^*$. Together with Lemma 7.4.3 this gives $\dim_{\mathbb{Q}} K(\mathcal{B}^{\Bbbk}_\chi)_{\mathbb{Q}} \leq\ \dim_{\mathbb{Q}_l} \mathrm{H}^*_{\mathrm{et}}(\mathcal{B}^{\Bbbk}_\chi, \mathbb{Q}_l(-i)) = \dim_{\mathbb{Q}} \mathrm{H}^*(\mathcal{B}^{\mathbb{C}}_\chi, \mathbb{Q})$. ☐

7.4.5. *End of the proof of Proposition* 7.1.7. We compare the K-groups via

$$K(\mathcal{B}^{\mathbb{C}}_\chi)_{\mathbb{Q}} \cong K(\mathcal{B}^{\overline{\mathcal{K}}}_\chi)_{\mathbb{Q}} \xleftarrow[\cong]{\mathrm{bc}^{\overline{\mathcal{K}}}_{\mathcal{K}}} K(\mathcal{B}^{\eta}_\chi)_{\mathbb{Q}} \overset{\mathrm{sp}}{\hookrightarrow} K(\mathcal{B}^{s}_\chi)_{\mathbb{Q}} \overset{\mathrm{bc}^{\Bbbk}_{k_s}}{\hookrightarrow} K(\mathcal{B}^{\Bbbk}_\chi)_{\mathbb{Q}}.$$

The first two isomorphisms are a particular case of (8) proved in 7.2.3; specialization is injective by 7.3, and the base change $\mathrm{bc}^{\Bbbk}_{k_s}$ is injective by Lemma 7.2.1. Actually, all maps have to be isomorphisms since (10) says that $\dim_{\mathbb{Q}} K(\mathcal{B}^{\Bbbk}_\chi)_{\mathbb{Q}}$ is bounded above by $\dim_{\mathbb{Q}} \mathrm{H}^\bullet(\mathcal{B}^{\mathbb{C}}_\chi, \mathbb{Q}) = \dim_{\mathbb{Q}} K(\mathcal{B}^{\mathbb{C}}_\chi)_{\mathbb{Q}}$. ☐

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA
E-mail address: bezrukav@math.mit.edu

UNIVERSITY OF MASSACHUSETTS, AMHERST, MA
E-mail address: mirkovic@math.umass.edu

UNIVERSITY OF WARWICK, COVENTRY, ENGLAND
E-mail address: rumynin@maths.warwick.ac.uk

REFERENCES

[BaRi] P. BARDSLEY and R. W. RICHARDSON, Étale slices for algebraic transformation groups in characteristic p, *Proc. London Math. Soc.* **51** (1985), 295–317.

[17] This argument was explained to us by Michael Finkelberg.

[BB] A. Beilinson and A. Bernstein, Localisation de g-modules, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 15–18.

[BO] P. Berthelot and A. Ogus, *Notes on Crystalline Cohomology*, *Mathematical Notes* **21**, Princeton University Press, Princeton, 1978.

[BeKa] R. Bezrukavnikov and D. Kaledin, McKay equivalence for symplectic resolutions of quotient singularities, Tr. Mat. Inst. Steklova **246** (2004), *Algebr. Geom. Metody, Svyazi i Prilozh.*, 20–42; translation in *Proc. Steklov Inst. Math.* **246** (2004), 13–33.

[BMR2] R. Bezrukavnikov, I. Mirković, and D. Rumynin, Singular localization and intertwinig functors for semisimple Lie algebras in prime characteristic, *Nagoya J. Math.* **184** (2006), 1–55.

[BK] A. Bondal and M. Kapranov, Representable functors, Serre functors, and mutations, *Izv. Ak. Nauk* **35** (1990), 519–541.

[BKR] T. Bridgeland, A. King, and M. Reid, The McKay correspondence as an equivalence of derived categories, *J. Amer. Math. Soc.* **14** (2001), 535–554 (preprint version: Mukai implies McKay, available as math.AG/9908027 at xxx.lanl.gov).

[BG] K. Brown and I. Gordon, The ramification of centres: Lie algebras in positive characteristic and quantised enveloping algebras, *Math. Z.* **238** (2001), 733–779.

[BrKa] J.-L. Brylinski and M. Kashiwara, Kazhdan-Lusztig conjecture and holonomic systems, *Invent. Math.* **64** (1981), 387–410.

[DLP] C. De Concini, G. Lusztig, and C. Procesi, Homology of the zero-set of a nilpotent vector field on a flag manifold, *J. Amer. Math. Soc.* **1** (1988), 15–34.

[De] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, *Invent. Math.* **21** (1973), 287–301.

[FP] E. Friedlander and B. Parshall, Deformations of Lie algebra representations, *Amer. J. Math.* **3** (1990), 375–395.

[Fu] W. Fulton, *Intersection Theory,* Springer-Verlag, New York, 1998.

[EGA] A. Grothendieck, *Éléments de Géométrie Algébrique.* III. *Étude Cohomologique des Faisceaux Cohérents.* I. *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 167 pp.

[Ha] R. Hartshorne, *Residues and Duality, Lecture Notes in Mathematics* **20**, Springer-Verlag, New York, 1966.

[Ho] G. Hochschild, Simple algebras with purely inseparable splitting fields of exponent 1, *Trans. Amer. Math. Soc.* **79** (1955), 477–489.

[Hur] W. Hürlimann, Sur le groupe de Brauer d'un anneau de polynômes en caractéristique p et la théorie des invariants, in *The Brauer Group* (Sem., Les Plans-sur-Bex, 1980), *Lecture Notes in Math.* **844**, 229–274, Springer-Verlag, New York, 1981.

[Ja] J. Jantzen, Representations of Lie algebras in prime characteristic, in *Proc. NATO ASI Representation Theory and Algebraic Geometry*, Montreal, 1997, 185–235 (A. Broer, ed.) Kluwer, Dordrecht (1998).

[Ja0] ———, Representations of algebraic groups, *Pure and Applied Math.* **131** (1987), Academic Press, Boston, MA.

[Ja1] ———, Subregular nilpotent representations of Lie algebras in prime characteristic, *Represent. Theory* **3** (1999), 153–222.

[Ja2] ———, Nilpotent orbits in representation theory, in *Lie Theory*, *Progr. Math.* **228** (2004), 1–211, Birkhäuser, Boston.

[Ja3] J. Jantzen, Representations of Lie algebras in positive characteristic, in *Representation Theory of Algebraic Groups and Quantum Groups*, *Adv. Stud. Pure Math.* **40** (2004), 175–218, Math. Soc. Japan, Tokyo.

[KW] V. Kac and B. Weisfeiler, Coadjoint action of a semisimple algebraic group and the center of the enveloping algebra in characteristic p, *Indag. Math.* **38** (1976), 136–151.

[KV] M. Kapranov and E. Vasserot, Kleinian singularities, derived categories and Hall algebras, *Math. Ann.* **316** (2000), 565–576.

[Ko] B. Kostant, On the tensor product of a finite and an infinite dimensional representation, *J. Funct. Anal.* **20** (1975), 257-285.

[KLT] S. Kumar, N. Lauritzen, and J. F. Thomsen, Frobenius splitting of cotangent bundles of flag varieties, *Invent. Math.* **136** (1999), 603–621.

[Lu] G. Lusztig, Bases in equivariant K-theory, *Represent. Theory* **2** (1998), 298–369; Bases in equivariant K-theory II, *Represent. Theory* **3** (1999), 281–353.

[Lu1] ———, Representation theory in characteristic p, in *Taniguchi Conference on Mathematics Nara* 1998, *Adv. Stud. Pure Math.* **31** (2001) 167–178.

[Lu2] ———, Character sheaves V, *Adv. in Math.* **61** (1986), 103–155.

[Mi] D. Miličić, Localization and representation theory of reductive Lie groups, available at http://www.math.utah.edu/~milicic.

[MI] J. S. Milne, *Étale Cohomology*, *Princeton Math. Series* **33** (1980) Princeton Univ. Press, Princeton, NJ.

[MR] I. Mirković and D. Rumynin, Geometric representation theory of restricted Lie algebras of classical type, *Transformation Groups* **2** (2001), 175–191.

[MR1] ———, Centers of reduced enveloping algebras, *Math. Z*. **231** (1999), 123–132.

[Pr] A. Premet, Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture, *Invent. Math.* **121** (1995), 79–117.

[Sl] P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, *Lecture Notes in Mathematics* **815**. Springer-Verlag, New York, 1980.

[Sp] N. Spaltenstein, Existence of good transversal slices to nilpotent orbits in good characteristic, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **31** (1984), 283–286.

[SS] T. A. Springer and R. Steinberg, Conjugacy classes, in *Seminar on Algebraic Groups and Related Finite Groups*, *Lecture Notes in Math.* **131**, 167–266, Springer-Verlag, New York.

[St] R. Steinberg, Conjugacy classes in algebraic groups, notes by Vinay V. Deodhar, *Lecture Notes in Math.* **366** (1974), Springer-Verlag, New York.

[TL] L. Alonso Tarrio, L. Jeremias López, and J. Lipman, Duality and flat base change on formal schemes, in *Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes*, *Contemporary Math.* **244** (1999), Amer. Math. Soc.; see also errata in: *Proc. Amer. Math. Soc.* **131** (2003), 351–357 (electronic).

[Ve] F. Veldkamp, The center of the universal enveloping algebra of a Lie algebra in characteristic p, *Ann. Sci. École Norm. Sup.* **5** (1972), 217–240.

(Received April 28, 2004)

Appendix: Computations for $\mathfrak{sl}(3)$

By Roman Bezrukavnikov and Simon Riche

For $\mathfrak{g} = \mathfrak{sl}(3)$ we compute coherent sheaves corresponding to irreducible representations in $\mathrm{mod}_0^{\mathrm{fg}}(U^0)$ and their projective covers under the equivalence $\mathcal{D}^b\mathcal{C}\mathrm{oh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \xrightarrow{\Upsilon} \mathcal{D}^b(\mathrm{mod}_0^{\mathrm{fg}}(U^0))$. We normalize the equivalences by setting $\eta = (p-1)\rho$ (notations of Remark 5.3.2); notice that for $\chi = 0$ this choice gives the splitting on the zero section $\mathcal{B}_0$ from 2.2.5, so that for every $\mathcal{F} \in \mathrm{Coh}(\mathcal{B}^{(1)})$ we have $\Upsilon(i_*\mathcal{F}) = R\Gamma(\mathcal{B}, \mathrm{Fr}_{\mathcal{B}}^*\mathcal{F})$.

1. Notations

We keep the notations of the article, with $G = \mathrm{SL}(3, \Bbbk)$, and denote α_1, α_2 the simple roots of G and ω_1, ω_2 the fundamental weights. Let s_j be the reflection $s_{\alpha_j} \in W$. We denote by $\mathcal{B} \xrightarrow{i} \widetilde{\mathcal{N}} \xrightarrow{p} \mathcal{B}$ the inclusion of the zero section and the natural projection. There are two natural maps $\pi_j : \mathcal{B} \to \mathbb{P}^2$ mapping a flag $0 \subset V_1 \subset V_2 \subset \Bbbk^3$ to V_j, $j = 1, 2$. For $n \in \mathbb{Z}$ we have isomorphisms: $\pi_i^*\mathcal{O}_{\mathbb{P}^2}(n) \cong \mathcal{O}_{\mathcal{B}}(n\omega_i)$, $i = 1, 2$, and $\mathrm{Fr}_{\mathcal{B}}^*\mathcal{O}_{\mathcal{B}^{(1)}}(\lambda) \cong \mathcal{O}_{\mathcal{B}}(p\lambda)$ for $\lambda \in \Lambda$. We will study irreducible G-modules $L(\lambda)$ of highest weight λ for reduced dominant weights λ in $W'_{\mathrm{aff}} \bullet 0$. Recall the exact sequence

$$(*) \qquad 0 \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

For simplicity, in what follows we will omit the Frobenius twist $^{(1)}$ (except in the proof of theorem 2.1, where we have to be more careful); it should appear on (almost) every variety we consider.

2. Irreducible modules

THEOREM 2.1. *The irreducible $U^0_{\hat{0}}$-modules and the corresponding coherent sheaves are*:

$L(0) = \Bbbk$	$i_*\mathcal{O}_{\mathcal{B}}$	$L((p-2)\omega_1 + \omega_2)$	$i_*\pi_1^*(\Omega^1_{\mathbb{P}^2}(1))[1]$
$L((p-3)\omega_2)$	$i_*\mathcal{O}_{\mathcal{B}}(-\omega_1)[2]$	$L(\omega_1 + (p-2)\omega_2)$	$i_*\pi_2^*(\Omega^1_{\mathbb{P}^2}(1))[1]$
$L((p-3)\omega_1)$	$i_*\mathcal{O}_{\mathcal{B}}(-\omega_2)[2]$	$L((p-2)\rho)$	$\mathcal{L}$

where $\mathcal{L}$ is the cone of the only (up to a constant) nonzero morphism $i_*\mathcal{O}_{\mathcal{B}} \to i_*\mathcal{O}_{\mathcal{B}}(-\rho)[3]$.

Proof. We have $\Upsilon(i_*\mathcal{O}_{\mathcal{B}^{(1)}}) = R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) = \Bbbk$. Also, $\Upsilon(i_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\omega_j)) = R\Gamma(\mathcal{O}_{\mathbb{P}^2}(-p))$, which gives the claim for $L((p-3)\omega_j)$, $j = 1, 2$.

Similarly $\Upsilon(i_*\pi_1^*(\Omega^1_{(\mathbb{P}^2)^{(1)}}(1))[1]) = R\Gamma(\mathcal{B}, \mathrm{Fr}^*_{\mathcal{B}}\pi_1^*(\Omega^1_{(\mathbb{P}^2)^{(1)}}(1)))[1]$. Using the exact sequence $(*)$ we obtain a distinguished triangle

$$R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}})^{\oplus 3} \to R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(p\omega_1)) \to \Upsilon(i_*\pi_1^*(\Omega^1_{(\mathbb{P}^2)^{(1)}}(1))[1]).$$

Here the first arrow is the inclusion of G-modules $L(\omega_1)^{(1)} \hookrightarrow H^0(p\omega_1)$. Hence $\Upsilon(i_*\pi_1^*(\Omega^1_{(\mathbb{P}^2)^{(1)}}(1))[1]) \cong L((p-2)\omega_1 + \omega_2)$. The claim for $L(\omega_1 + (p-2)\omega_2)$ follows by applying the outer automorphism of $\mathfrak{sl}(3)$.

Finally, the last irreducible module $L((p-2)\rho)$ is a quotient of the Weyl module $[H^0((p-2)\rho)]^*$, moreover, we have a short exact sequence $0 \to \mathbb{k} \to [H^0((p-2)\rho)]^* \to L((p-2)\rho) \to 0$. Applying Υ^{-1}, we get distinguished triangle $i_*\mathcal{O}_{\mathcal{B}^{(1)}} \to i_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)[3] \to \mathcal{L}$, where we used that

$$\Upsilon(i_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)) = R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(-p\rho)) = [H^0((p-2)\rho)]^*[-3]$$

by Serre duality. Since $\mathrm{Hom}(\mathbb{k}, [H^0((p-2)\rho)]^*)$ is one dimensional, we see that the first arrow in this triangle is the unique (up to a constant) map between the two objects. $\square$

Remark. We have just shown, using equivalence Υ, that

$$\mathrm{Ext}^3_{\widetilde{\mathcal{N}}}(i_*\mathcal{O}_{\mathcal{B}}, i_*\mathcal{O}_{\mathcal{B}}(-\rho))$$

is one dimensional. One can compute this Ext group more directly: using the Koszul resolution of $\mathcal{O}_{\mathcal{B}}$ over $\mathrm{S}(\mathcal{T}_{\mathcal{B}})$ one can identify it with

$$H^3(-\rho) \oplus H^2(\Omega^1_{\mathcal{B}}(-\rho)) \oplus H^1(\Omega^2_{\mathcal{B}}(-\rho)) \oplus H^0(\Omega^3_{\mathcal{B}}(-\rho)).$$

One can show that $H^3(-\rho)$, $H^0(\Omega^3_{\mathcal{B}}(-\rho))$ and $H^1(\Omega^2_{\mathcal{B}}(-\rho))$ vanish, while $H^2(\Omega^1_{\mathcal{B}}(-\rho)) \cong \mathbb{k}$: by Serre duality the last claim is equivalent to $H^1(\mathcal{T}_{\mathcal{B}}(-\rho)) \cong \mathbb{k}$, which is checked below.

3. Projective covers

THEOREM 3.1. *The coherent sheaves corresponding to the projective covers of the irreducible modules are*:

$i_*\mathcal{O}_{\mathcal{B}}$	$\mathcal{P}$	$i_*\pi_1^*(\Omega^1_{\mathbb{P}^2}(1))[1]$	$\mathcal{O}_{\widetilde{\mathcal{N}}}(\omega_1)$
$i_*\mathcal{O}_{\mathcal{B}}(-\omega_1)[2]$	$p^*((\pi_2^*\Omega^1_{\mathbb{P}^2})(\omega_1 + 2\omega_2))$	$i_*\pi_2^*(\Omega^1_{\mathbb{P}^2}(1))[1]$	$\mathcal{O}_{\widetilde{\mathcal{N}}}(\omega_2)$
$i_*\mathcal{O}_{\mathcal{B}}(-\omega_2)[2]$	$p^*((\pi_1^*\Omega^1_{\mathbb{P}^2})(2\omega_1 + \omega_2))$	$\mathcal{L}$	$\mathcal{O}_{\widetilde{\mathcal{N}}}(\rho)$

where $\mathcal{P}$ is the nontrivial extension of $\mathcal{O}_{\widetilde{\mathcal{N}}}(\rho)$ by $\mathcal{O}_{\widetilde{\mathcal{N}}}$ given by a non-zero element in the one dimensional space $H^1(\mathcal{T}_{\mathcal{B}}(-\rho)) \subset H^1(\mathcal{O}_{\widetilde{\mathcal{N}}}(-\rho))$.

Remark. In fact, the sheaves corresponding to the projective covers are vector bundles on the formal completion of $\widetilde{\mathcal{N}}$ at $\mathcal{B}$. The objects displayed in the above table are vector bundles on $\widetilde{\mathcal{N}}$. The former are obtained from the latter by pull-back to the formal completion.

Proof. We only have to check that for each $\mathcal{P}_i$ in the list and each irreducible $\mathcal{L}_j$, we have $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{P}_i, \mathcal{L}_j) = \Bbbk^{\delta_{ij}}$. Let us begin with $\mathcal{O}_{\widetilde{\mathcal{N}}}(\rho)$. We have $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\rho), i_*\mathcal{O}_{\mathcal{B}}) \cong \mathrm{Ext}^*_{\mathcal{B}}(\mathcal{O}_{\mathcal{B}}(\rho), \mathcal{O}_{\mathcal{B}}) \cong H^*(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(-\rho)) = 0$ by adjunction. Similarly for $i_*\mathcal{O}_{\mathcal{B}}(-\omega_j)[2]$ $(j=1,2)$. The sequence $(*)$ gives $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\rho), i_*\pi_j^*(\Omega^1_{\mathbb{P}^2}(1))[1]) = \mathrm{Ext}^*_{\mathcal{B}}(\mathcal{O}_{\mathcal{B}}(\rho), \pi_j^*(\Omega^1_{\mathbb{P}^2}(1))[1]) = 0$ $(j=1,2)$. Using the distinguished triangle from the definition of $\mathcal{L}$ we get $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\rho), \mathcal{L}) = \Bbbk$. The cases of $\mathcal{O}_{\widetilde{\mathcal{N}}}(\omega_j)$ $(j=1,2)$ are similar.

Now let us consider $p^*((\pi_1^*\Omega^1_{\mathbb{P}^2})(2\omega_1+\omega_2))$. The exact sequence $(*)$ and Borel-Weil-Bott Theorem [Ja] give the result for the first 5 irreducible modules. For $\mathcal{L}$, we have $\mathrm{Ext}^*_{\mathcal{B}}((\pi_1^*\Omega^1_{\mathbb{P}^2})(2\omega_1+\omega_2), \mathcal{O}_{\mathcal{B}}) = 0$, and in computing $\mathrm{Ext}^*_{\mathcal{B}}((\pi_1^*\Omega^1_{\mathbb{P}^2})(2\omega_1+\omega_2), \mathcal{O}_{\mathcal{B}}(-\rho)[3])$, two non-zero modules appear in degree 0: $[H^3(-2\rho)]^{\oplus 3}$ and $H^0(\omega_1)$. The map between these two modules is an isomorphism as in the proof of Theorem 2.1, hence $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(p^*((\pi_1^*\Omega^1_{\mathbb{P}^2})(2\omega_1+\omega_2)), \mathcal{L}) = 0$.

We claim that $H^1(\mathcal{T}_{\mathcal{B}}(-\rho)) \cong \Bbbk$, this follows by the Borel-Weil-Bott Theorem from the exact sequence $0 \to \mathcal{O}_{\mathcal{B}}(\alpha_1) \to \mathcal{T}_{\mathcal{B}} \to \pi_2^*(\mathcal{T}_{\mathbb{P}^2}) \to 0$, and vanishing of $R\Gamma(\pi_2^*(\mathcal{T}_{\mathbb{P}^2})(-\rho))$ (see, e.g., [D]). Thus we have the line $H^1(\mathcal{T}_{\mathcal{B}}(-\rho)) \subset H^1(\mathrm{S}(\mathcal{T}_{\mathcal{B}})(-\rho)) = \mathrm{Ext}^1_{\widetilde{\mathcal{N}}}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\rho), \mathcal{O}_{\widetilde{\mathcal{N}}})$, which defines a triangle $\mathcal{O}_{\widetilde{\mathcal{N}}} \to \mathcal{P} \to \mathcal{O}_{\widetilde{\mathcal{N}}}(\rho)$. Standard calculations give the result for $\mathcal{P}$ and the first three irreducible modules. The triangle defining $\mathcal{P}$ gives $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{P}, i_*\pi_1^*(\Omega^1_{\mathbb{P}^2}(1))[1]) = H^*(\pi_1^*(\Omega^1_{\mathbb{P}^2}(1)))[1]$. Using $(*)$, we have an exact sequence $0 \to H^0(\pi_1^*(\Omega^1_{\mathbb{P}^2}(1))) \to \Bbbk^3 \to H^0(\omega_1) \to H^1(\pi_1^*(\Omega^1_{\mathbb{P}^2}(1))) \to 0$ with invertible middle arrow (the other cohomology modules vanish).

Finally, let us show that $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{P}, \mathcal{L}) = 0$. We have $R\mathrm{Hom}_{\widetilde{\mathcal{N}}}(\mathcal{P}, i_*\mathcal{O}_{\mathcal{B}}) \cong R\Gamma(\mathcal{O}_{\mathcal{B}}) \cong \Bbbk$, $R\mathrm{Hom}_{\widetilde{\mathcal{N}}}(\mathcal{P}, i_*\mathcal{O}_{\mathcal{B}}(-\rho)[3]) \cong R\Gamma(\mathcal{O}_{\mathcal{B}}(-2\rho)[3]) \cong \Bbbk$, thus we only need to check that for nonzero morphisms $b : i_*\mathcal{O}_{\mathcal{B}} \to i_*\mathcal{O}_{\mathcal{B}}(-\rho)[3]$, $\phi : \mathcal{P} \to i_*\mathcal{O}_{\mathcal{B}}$ we have $b \circ \phi \neq 0$. It is clear from Remark after Theorem 2.1 that $b = i_*(\beta) \circ \delta$, where $\delta : i_*\mathcal{O}_{\mathcal{B}} \to i_*\mathcal{T}_{\mathcal{B}}[1]$ is the class of the extension $0 \to i_*\mathcal{T}_{\mathcal{B}} \to \mathcal{O}_{\widetilde{\mathcal{N}}}/\mathcal{J}^2_{\mathcal{B}} \to i_*\mathcal{O}_{\mathcal{B}} \to 0$, and $\beta : \mathcal{T}_{\mathcal{B}}[1] \to \mathcal{O}_{\mathcal{B}}(-\rho)[3]$ is a non-zero morphism; here $\mathcal{J}_{\mathcal{B}}$ is the ideal sheaf on the zero section in $\widetilde{\mathcal{N}}$.

We claim that $\delta \circ \phi = i_*(\gamma) \circ \psi$, where $\psi : \mathcal{P} \twoheadrightarrow i_*\mathcal{O}_{\mathcal{B}}(\rho)$ and $\gamma : \mathcal{O}_{\mathcal{B}}(\rho) \to \mathcal{T}_{\mathcal{B}}[1]$ are nonzero morphisms. This follows from the definition of $\mathcal{P}$, which implies that $\mathcal{P}$ has a quotient, which is an extension of $i_*\mathcal{O}_{\mathcal{B}} \oplus i_*\mathcal{O}_{\mathcal{B}}(\rho)$ by $i_*\mathcal{T}_{\mathcal{B}}$, such that the corresponding class in $\mathrm{Ext}^1(i_*\mathcal{O}_{\mathcal{B}}, i_*(\mathcal{T}_{\mathcal{B}}))$ equals δ, while the corresponding class in $\mathrm{Ext}^1(i_*\mathcal{O}_{\mathcal{B}}(\rho), i_*(\mathcal{T}_{\mathcal{B}}))$ is non-trivial and is an image under i_* of an extension of coherent sheaves on $\mathcal{B}$.

It remains to show that the composition $i_*\beta \circ i_*\gamma \circ \psi$ is nonzero. The composition $\beta \circ \gamma \in \mathrm{Ext}^3(\mathcal{O}_{\mathcal{B}}(\rho), \mathcal{O}_{\mathcal{B}}(-\rho)) = H^3(\mathcal{B}, \mathcal{O}(-2\rho)) = \Bbbk$ is nonzero, because it coincides with the Serre duality pairing of nonzero elements β, γ in the two dual one-dimensional spaces $H^1(\mathcal{T}_{\mathcal{B}}(-\rho))$, $H^2(\Omega^1_{\mathcal{B}}(-\rho))$. Consequently, the composition $i_*(\beta \circ \gamma) \circ \psi$ is also nonzero, since under the isomorphism $\mathrm{Hom}(\mathcal{P}, i_*\mathcal{O}_{\mathcal{B}}(-\rho)[3]) \cong \mathrm{Hom}(i^*\mathcal{P}, \mathcal{O}_{\mathcal{B}}(-\rho)[3]) \cong \mathrm{Hom}(\mathcal{O}_{\mathcal{B}} \oplus \mathcal{O}_{\mathcal{B}}(\rho), \mathcal{O}_{\mathcal{B}}(-\rho)[3])$ it corresponds to the composition of $\beta \circ \gamma$ and projection to the second summand. □

Acknowledgement. We thank Patrick Polo for helpful discussions.

Massachusetts Institute of Technology, Cambridge, MA
E-mail address: bezrukav@math.mit.edu

Université Pierre et Marie Curie, Institut de Mathématiques de Jussieu (UMR 7586 du CNRS), Paris, France
E-mail address: riche@math.jussieu.fr

References

[Ja] J. C. Jantzen, *Representations of Algebraic Groups, second edition*, *Mathematical Surveys and Monographs* **107**, Amer. Math. Soc., Providence, RI (2003).

[D] M. Demazure, A very simple proof of Bott's theorem, *Invent. Math.* **33** (1976), 271–272.

(Received October 16, 2006)