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Mirror symmetry for weighted projective planes and their noncommutative deformations

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1. Introduction

The phenomenon of Mirror Symmetry, in its "classical" version, was first observed for Calabi-Yau manifolds, and mathematicians were introduced to it through a series of remarkable papers [20], [13], [38], [40], [15], [30]. Some very strong conjectures have been made about its topological interpretation – e.g. the Strominger-Yau-Zaslow conjecture. In a different direction, the framework of mirror symmetry was extended by Batyrev, Givental, Hori, Vafa, etc. to the case of Fano manifolds.

In this paper, we approach mirror symmetry for Fano manifolds from the point of view suggested by the work of Kontsevich and his remarkable Homological Mirror Symmetry (HMS) conjecture [27]. We extend the previous investigations in the following two directions:

- Building on recent works by Seidel [34], Hori and Vafa [23] (see also an earlier paper by Witten [41]), we prove HMS for some Fano manifolds, namely weighted projective lines and planes, and Hirzebruch surfaces. This extends, at a greater level of generality, a result of Seidel [35] concerning the case of the usual $\mathbb{CP}^2$.
- We obtain the first explicit description of the extension of HMS to non-commutative deformations of Fano algebraic varieties.

In the long run, the goal is to explore in greater depth the fascinating ties brought forth by HMS between complex algebraic geometry and symplectic geometry, hoping that the currently more developed algebro-geometric methods will open a fine opportunity for obtaining new interesting results in symplectic geometry. We first describe the results of this paper in more detail.

Most of the classical works on string theory deal with the case of $N = 2$ superconformal sigma models with a Calabi-Yau target space. In this situation the corresponding field theory has two topologically twisted versions, the A- and B-models, with D-branes of types A and B respectively. Mirror symmetry interchanges these two classes of D-branes. In mathematical terms, the category of B-branes on a Calabi-Yau manifold X is the derived category of coherent sheaves on X, $\mathbf{D}^b(\mathrm{coh}(X))$. The so-called (derived) Fukaya category $D\mathcal{F}(Y)$ has been proposed as a candidate for the category of A-branes on a Calabi-Yau manifold Y; in short this is a category whose objects are Lagrangian submanifolds equipped with flat vector bundles. The HMS conjecture claims that if two Calabi-Yau manifolds X and Y are mirrors to each other then $\mathbf{D}^b(\mathrm{coh}(X))$ is equivalent to $D\mathcal{F}(Y)$.

Physicists also consider more general $N = 2$ supersymmetric field theories and the corresponding D-branes; among these, two families of theories are of particular interest to us: on one hand, sigma models with a Fano variety as target space, and on the other hand, $N = 2$ Landau-Ginzburg models. Mirror symmetry relates the former with a certain subclass of the latter. In particular, B-branes on a Fano variety are described by the derived category of coherent sheaves, and under mirror symmetry they correspond to the A-branes of a mirror Landau-Ginzburg model. These A-branes are described by a suitable analogue of the Fukaya category, namely the derived category of Lagrangian vanishing cycles.

In order to demonstrate this feature of mirror symmetry, we use a procedure introduced by Batyrev [8], Givental [18], Hori and Vafa [23], which we will call the *toric mirror ansatz*. Starting from a complete intersection Y in a toric variety, this procedure yields a description of an affine subset of its mirror Landau-Ginzburg model (to obtain a full description of the mirror it is usually necessary to consider a partial (fiberwise) compactification) – an open symplectic manifold (X, ω) and a symplectic fibration $W : X \to \mathbb{C}$ (see e.g. [24]).

Following ideas of Kontsevich [28] and Hori-Iqbal-Vafa [22], Seidel rigorously defined (in the case of nondegenerate critical points) a derived category of Lagrangian vanishing cycles $D(\mathrm{Lag}_{\mathrm{vc}}(W))$ [34], whose objects represent A-branes on $W : X \to \mathbb{C}$.

In the case of Fano manifolds the statement of the HMS conjecture is the following:

CONJECTURE 1.1. *The category of* A*-branes* $D(\mathrm{Lag}_{\mathrm{vc}}(W))$ *is equivalent to the derived category of coherent sheaves* (B*-branes*) *on* Y*.*

We will prove this conjecture for various examples.

There is also a parallel statement of HMS relating the derived category of B-branes on $W : X \to \mathbb{C}$, whose definition was suggested by Kontsevich and carried out algebraically in [33], and the derived Fukaya category of Y. Since very little is known about these Fukaya categories, we will not discuss the details of this statement in the present paper. Our hope in this direction is that algebro-geometric methods will allow us to look at Fukaya categories from a different perspective.

The case we will be mainly concerned with in this paper is that of the weighted projective plane $\mathbb{CP}^2(a, b, c)$ (where a, b, c are coprime positive integers). Its mirror is the affine hypersurface $X = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, equipped with an exact symplectic form ω and the superpotential $W = x+y+z$. Our main theorem is:

THEOREM 1.2. HMS *holds for* $\mathbb{CP}^2(a, b, c)$ *and its noncommutative deformations.*

Namely, we show that the derived category of coherent sheaves (B-branes) on the weighted projective plane $\mathbb{CP}^2(a, b, c)$ is equivalent to the derived category of vanishing cycles (A-branes) on the affine hypersurface $X \subset (\mathbb{C}^*)^3$. Moreover, we show that this mirror correspondence between derived categories can be extended to toric noncommutative deformations of $\mathbb{CP}^2(a,b,c)$ where B-branes are concerned, and their mirror counterparts, nonexact deformations of the symplectic structure of X where A-branes are concerned.

Observe that weighted projective planes are rigid in terms of commutative deformations, but have a one-dimensional moduli space of toric noncommutative deformations ($\mathbb{CP}^2$ also has some other noncommutative deformations; see §6.2). We expect a similar phenomenon to hold in many cases where the toric mirror ansatz applies. An interesting question will be to extend this correspondence to the case of general noncommutative toric vareties.

We will also consider some other examples besides weighted projective planes, in order to demonstrate the ubiquity of HMS:

- As a warm-up example, we give a proof of HMS for weighted projective lines (a result also announced by D. van Straten in [39]).
- We also discuss HMS for Hirzebruch surfaces $\mathbb{F}_n$. For $n \geq 3$, the canonical class is no longer negative ($\mathbb{F}_n$ is not Fano), and HMS does not hold directly, because some modifications of the toric mirror ansatz are needed, as already noticed in [22]. The direct application of the ansatz produces a Landau-Ginzburg model whose derived category of vanishing cycles is identical to that on the mirror of the weighted projective plane $\mathbb{CP}^2(1,1,n)$. In order to make the HMS conjecture work we need to restrict ourselves to an open subset in the target space X of this Landau-Ginzburg model.
- We will also outline an idea of the proof of HMS (missing only some Floer-theoretic arguments about certain moduli spaces of pseudo-holomorphic discs) for some higher-dimensional Fano manifolds, e.g. $\mathbb{CP}^3$.

A word of warning is in order here. We do not describe completely and do not make use of the full potential of the toric mirror ansatz in this paper. Indeed we do not compactify and desingularize the open manifold X. Compactification and desingularization procedures will be addressed in full detail in future papers [5], [6] dealing with the cases of more general Fano manifolds and manifolds of general type, where these extra steps are needed in order to exhibit the whole category of D-branes of the Landau-Ginzburg model. In this paper we work with specific examples for which compactification and desingularization are not needed (conjecturally this is the case for all toric varieties). However there are two principles which are readily apparent from these specific examples:

- Noncommutative deformations of Fano manifolds are related to variations of the cohomology class of the symplectic form on the mirror Landau-Ginzburg models.

- Even in the toric case, a fiberwise compactification of the Landau-Ginzburg model is required in order to obtain general noncommutative deformations. The noncompact case then arises as a limit where the symplectic form on the compactified fiber acquires poles along the compactification divisor.

Moreover there are two features of HMS for toric varieties, which become apparent in this paper and which we would like to emphasize:

- It is important to think of singular toric varieties as smooth quotient stacks. As a consequence of the work of Cox [14] this characterization is possible in many cases.
- As suggested by our specific examples, we would like to conjecture that the derived category of coherent sheaves over a smooth toric quotient stack is always generated by an exceptional collection of line bundles.

The paper is organized as follows. In Chapter 2 we give a detailed description of derived categories of coherent sheaves over weighted projective spaces and some of their noncommutative deformations. After recalling the definition of the weighted projective space $\mathbb{P}(\overline{\mathbf{a}})$ as a quotient stack, we describe the category of coherent sheaves over $\mathbb{P}(\overline{\mathbf{a}})$ and its noncommutative deformations $\mathbb{P}_\theta(\overline{\mathbf{a}})$, and describe explicitly generating exceptional collections for $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta(\overline{\mathbf{a}})))$ (Theorem 2.12 and Corollary 2.27). This is a novel result, and we believe that it suggests a procedure that applies to many other examples of noncommutative toric varieties. We also discuss derived categories of coherent sheaves over Hirzebruch surfaces.

In Chapter 3 we introduce the category of Lagrangian vanishing cycles associated to a Lefschetz fibration, and outline the main steps involved in its determination; to illustrate the definitions, we treat the case of the mirror of a weighted projective line. After this warm-up, in Chapter 4 we turn to our main examples, namely the Landau-Ginzburg models mirror to weighted projective planes and their nonexact symplectic deformations. More precisely we start by studying the vanishing cycles and their intersection properties, which allows us to determine all the morphisms in $\mathrm{Lag}_{\mathrm{vc}}$ (Lemma 4.3). Next we study moduli spaces of pseudo-holomorphic discs in the fiber in order to determine Floer products (Lemmas 4.4 and 4.5); this gives formulas for compositions of morphisms and higher products in $\mathrm{Lag}_{\mathrm{vc}}$ (the latter turn out to be identically zero). Finally, after a discussion of Maslov index and grading, we establish an explicit correspondence between deformation parameters on both sides (noncommutative deformation of the weighted projective plane, and complexified Kähler class on the mirror) and complete the proof of Theorem 1.2.

Chapter 5 deals with the case of mirrors to Hirzebruch surfaces, showing how their categories of Lagrangian vanishing cycles relate to those of mirrors to weighted projective planes $\mathbb{CP}^2(n,1,1)$. In particular we prove HMS for $\mathbb{F}_n$ when $n \in \{0,1,2\}$, and show how for $n \geq 3$ a certain degenerate limit of the Landau-Ginzburg model singles out a full subcategory of $\mathrm{Lag}_{\mathrm{vc}}$ whose derived category is equivalent to that of coherent sheaves on the Hirzebruch surface.

Finally, in Chapter 6 we make various observations and concluding remarks, related to the following directions for future research:

- HMS for Del Pezzo surfaces, and for higher-dimensional weighted projective spaces (cf. §6.1 for a discussion of the case of $\mathbb{CP}^3$);
- HMS for general (non toric) noncommutative deformations (cf. §6.2 for a discussion of the case of $\mathbb{CP}^2$);
- the "other side" of HMS – relating derived Fukaya categories to derived categories of B-branes on the mirror Landau-Ginzburg model.

Another topic that will be investigated in a forthcoming paper [6] is HMS for products: our considerations for $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ suggest a certain product formula on both sides of HMS: if we consider two manifolds Y_1, Y_2 with mirror Landau-Ginzburg models (X_1, W_1) and (X_2, W_2), then the mirror of $Y_1 \times Y_2$ is simply $(X_1 \times X_2, W_1 + W_2)$, and we have the following general conjecture:

CONJECTURE 1.3. $D(\mathrm{Lag}_{\mathrm{vc}}(W_1 + W_2))$ *is equivalent to the product* $D(\mathrm{Lag}_{\mathrm{vc}}(W_1) \otimes \mathrm{Lag}_{\mathrm{vc}}(W_2))$.

More precisely, the vanishing cycles of $W_1 + W_2$ are in one-to-one correspondence with pairs of vanishing cycles of W_1 and W_2, and it can be checked (cf. §6.3) that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Lag}_{\mathrm{vc}}(W_1+W_2)}&((A_1,A_2),(B_1,B_2))\\ &\simeq \mathrm{Hom}_{\mathrm{Lag}_{\mathrm{vc}}(W_1)}(A_1,B_1) \otimes \mathrm{Hom}_{\mathrm{Lag}_{\mathrm{vc}}(W_2)}(A_2,B_2).\end{aligned}$$

The conjecture asserts that Floer products behave in the expected manner with respect to these isomorphisms.

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2. Weighted projective spaces

2.1. *Weighted projective spaces as stacks.* We start by reviewing definitions from the theory of weighted projective spaces.

Let $\mathbf{k}$ be a base field. Let $a_0, \dots, a_n$ be positive integers. Define the graded algebra $S = S(a_0, \dots, a_n)$ to be the polynomial algebra $\mathbf{k}[x_0, \dots, x_n]$ graded by $\deg x_i = a_i$. Classically the projective variety $\mathbf{Proj}\, S$ is called the weighted projective space with weights $a_0, \dots, a_n$ and is denoted by $\mathbf{P}(a_0, \dots, a_n)$. Consider the action of the algebraic group $\mathbf{G}_m = \mathbf{k}^*$ on the affine space $\mathbf{A}^{n+1}$ given in some affine coordinates $x_0, \dots, x_n$ by the formula

$$\lambda(x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n). \tag{2.1}$$

In geometric terms, the weighted projective space $\mathbf{P}(a_0, \dots, a_n)$ is the quotient variety $(\mathbf{A}^{n+1} \backslash \mathbf{0}) / \mathbf{G}_m$ under the induced action of the group $\mathbf{G}_m$.

The variety $\mathbf{P}(a_0, \dots, a_n)$ is a rational n-dimensional projective variety, singular in general, whose affine charts $x_i \neq 0$ are isomorphic to $\mathbf{A}^n / \mathbb{Z}_{a_i}$. For example, the variety $\mathbf{P}(1, 1, n)$ is the projective cone over a twisted rational curve of degree n in $\mathbf{P}^n$.

Denote by $\overline{\mathbf{a}}$ the vector $(a_0, \dots, a_n)$ and write $\mathbf{P}(\overline{\mathbf{a}})$ instead $\mathbf{P}(a_0, \dots, a_n)$ for brevity.

There is also another way to define the quotient of the action above: in the category of stacks. The quotient stack

$$\left[(\mathbf{A}^{n+1} \backslash \mathbf{0}) / \mathbf{G}_m\right]$$

will be denoted by $\mathbb{P}(\overline{\mathbf{a}})$ and will also be called the weighted projective space. The stack $\mathbb{P}(\overline{\mathbf{a}})$ is smooth, and from many points of view it is a more natural object than $\mathbf{P}(\overline{\mathbf{a}})$.

We now review the notion of an algebraic stack as needed to understand our main example – weighted projective spaces. Detailed treatment of algebraic stacks can be found in [29] and [17].

There are two ways of thinking about an algebraic stack:

a) as a category $\mathcal{X}$, with additional properties;

b) as a presentation $R \rightrightarrows U$, with R and U schemes, R determining an equivalence relation on U.

From the categorical point of view a *stack* is a category $\mathcal{X}$ fibered in groupoids $p : \mathcal{X} \to \mathrm{Sch}$ over the category Sch of $\mathbf{k}$-schemes, satisfying two descent (sheafy) properties in the étale topology. An *algebraic stack* has to satisfy some additional representability conditions. For the precise definition see [29], [17].

Any scheme $X \in \mathrm{Sch}$ defines a category Sch/X: its objects are pairs (S, ϕ) with $\{S \xrightarrow{\phi} X\}$ a map in Sch, and a morphism from (S, ϕ) to (T, ψ) is a morphism $f : T \to S$ such that $\phi f = \psi$. The category Sch/X comes with a natural functor to Sch. Thus, any scheme is an algebraic stack.

Another example, the most important one for us, comes from an action of an algebraic group G on a scheme X. The *quotient stack* $[X/G]$ is defined to be the category whose objects are those G-torsors (principal homogeneous right G-schemes) $\mathcal{G} \to S$ which are locally trivial in the étale topology, together with a G-equivariant map from $\mathcal{G}$ to X.

In order to work with coherent sheaves on a stack it is convenient to use an atlas for the stack. We describe, very briefly, groupoid presentations (or atlases) of algebraic stacks. A pair of schemes R and U with morphisms s, t, e, m, i, satisfying certain group-like properties, is called a groupoid in Sch or an *algebraic groupoid*. For any scheme S the morphisms $s, t : R \to U$ ("source" and "target") determine two maps from the set $\mathrm{Hom}(S, R)$ to the set $\mathrm{Hom}(S, U)$. A quick way to state all relations between s, t, e, m, i is to say that the induced morphisms make the "objects" $\mathrm{Hom}(S, U)$ and "morphisms" $\mathrm{Hom}(S, R)$ into a category in which all arrows are invertible. We will denote an algebraic groupoid by $R \rightrightarrows U$ (the two arrows being the source and target maps), omitting the notations for e, m, and i.

Any scheme X determines a groupoid $X \rightrightarrows X$, whose morphisms are identity maps. The main example for us is the transformation groupoid associated to an algebraic group action $X \times G \to X$, which provides an atlas for the quotient stack $[X/G]$. The *transformation groupoid* $X \times G \rightrightarrows X$ is defined by

$$s(x, g) = x, \quad t(x, g) = x \cdot g, \quad m((x, g), (x \cdot g, h)) = (x, g \cdot h),$$
$$e(x) = (x, e_G), \quad i(x, g) = (x \cdot g, g^{-1}).$$

If $R \rightrightarrows U$ is a presentation for a stack $\mathcal{X}$, giving a coherent sheaf on $\mathcal{X}$ is equivalent to giving a coherent sheaf $\mathcal{F}$ on U, together with an isomorphism $s^*\mathcal{F} \xrightarrow{\sim} t^*\mathcal{F}$ on R satisfying a cocycle condition on $R \underset{t,U,s}{\times} R$. In particular, for a quotient stack $[X/G]$ the category of coherent sheaves is equivalent to the category of G-equivariant sheaves on X due to effective descent for strictly flat morphisms of algebraic stacks (see, e.g., [29, Th. 13.5.5]). Applying this fact to weighted projective spaces, we obtain that

$$\mathrm{coh}(\mathbb{P}(\overline{\mathbf{a}})) \cong \mathrm{coh}^{\mathbf{G}_m}_{\overline{\mathbf{a}}}(\mathbf{A}^{n+1} \backslash \mathbf{0}), \tag{2.2}$$

where $\operatorname{coh}_{\mathbf{a}}^{\mathbf{G}_m}(\mathbf{A}^{n+1}\backslash\mathbf{0})$ is the category of $\mathbf{G}_m$-equivariant coherent sheaves on $(\mathbf{A}^{n+1}\backslash\mathbf{0})$ with respect to the action given by rule (2.1).

2.2. *Coherent sheaves on weighted projective spaces.* Let $A = \bigoplus_{i\geq 0} A_i$ be a finitely generated graded algebra. Denote by $\operatorname{mod}(A)$ the category of finitely generated right A-modules and by $\operatorname{gr}(A)$ the category of finitely generated graded right A-modules in which morphisms are the homomorphisms of degree zero. Both are abelian categories.

Denote by $\operatorname{tors}(A)$ the full subcategory of $\operatorname{gr}(A)$ which consists of those graded A-modules which have finite dimension over $\mathbf{k}$.

Definition 2.1. Define the category $\operatorname{qgr}(A)$ to be the *quotient category* $\operatorname{gr}(A)/\operatorname{tors}(A)$. The objects of $\operatorname{qgr}(A)$ are the objects of the category $\operatorname{gr}(A)$ (we denote by $\widetilde{M}$ the object in $\operatorname{qgr}(A)$ which corresponds to a module M). The morphisms in $\operatorname{qgr}(A)$ are defined to be

$$\operatorname{Hom}_{\operatorname{qgr}}(\widetilde{M}, \widetilde{N}) = \varinjlim_{M'} \operatorname{Hom}_{\operatorname{gr}}(M', N),$$

where M' runs over all submodules of M such that M/M' is finite dimensional over $\mathbf{k}$.

The category $\operatorname{qgr}(A)$ is an abelian category and there is a shift functor on it: for a given graded module $M = \bigoplus_{i\geq 0} M_i$ the shifted module $M(p)$ is defined by $M(p)_i = M_{p+i}$, and the induced shift functor on the quotient category $\operatorname{qgr}(A)$ sends $\widetilde{M}$ to $\widetilde{M}(p) = \widetilde{M(p)}$.

Similarly, we can consider the category $\operatorname{Gr}(A)$ of all graded right A-modules. It contains the subcategory $\operatorname{Tors}(A)$ of torsion modules. Recall that a module M is called torsion if for any element $x \in M$ one has $xA_{\geq s} = 0$ for some s, where $A_{\geq s} = \bigoplus_{i\geq s} A_i$. We denote by $\operatorname{QGr}(A)$ the quotient category $\operatorname{Gr}(A)/\operatorname{Tors}(A)$. It is clear that the intersection of the categories $\operatorname{gr}(A)$ and $\operatorname{Tors}(A)$ in the category $\operatorname{Gr}(A)$ coincides with $\operatorname{tors}(A)$. In particular, the category $\operatorname{QGr}(A)$ contains $\operatorname{qgr}(A)$ as a full subcategory. Sometimes it is convenient to work with $\operatorname{QGr}(A)$ instead of $\operatorname{qgr}(A)$.

In the case when the algebra $A = \bigoplus_{i\geq 0} A_i$ is a commutative graded algebra generated over $\mathbf{k}$ by its degree-one component (which is assumed to be finite dimensional) J-P. Serre [37] proved that the category of coherent sheaves $\operatorname{coh}(X)$ on the projective variety $X = \mathbf{Proj}\, A$ is equivalent to the category $\operatorname{qgr}(A)$. Such an equivalence also holds for the category of quasicoherent sheaves on X and the category $\operatorname{QGr}(A) = \operatorname{Gr}(A)/\operatorname{Tors}(A)$.

This theorem can be extended to general finitely generated commutative algebras if we work at the level of quotient stacks.

Let $S = \bigoplus_{p=0}^{\infty} S_p$ be a commutative graded $\mathbf{k}$-algebra which is connected, i.e. $S_0 = \mathbf{k}$. The grading on S induces an action of the group $\mathbf{G}_m$ on the affine scheme $\mathbf{Spec}\, S$. Let $\mathbf{0}$ be the closed point of $\mathbf{Spec}\, S$ that corresponds to the ideal $S_+ = S_{\geq 1} \subset S$. This point is invariant under the action.

Definition 2.2. Denote by $\mathbb{P}\mathrm{roj}\, S$ the *quotient stack* $\left[(\mathbf{Spec}\, S \backslash \mathbf{0})/\mathbf{G}_m\right]$.

There is a natural map $\mathbb{P}\mathrm{roj}\, S \to \mathbf{Proj}\, S$, which is an isomorphism when the algebra S is generated by its degree one component S_1.

Proposition 2.3. *Let $S = \underset{i\geq 0}{\oplus} S_i$ be a graded finitely generated algebra. Then the category of* (*quasi*)*coherent sheaves on the quotient stack* $\mathbb{P}\mathrm{roj}\,(S)$ *is equivalent to the quotient category* $\mathrm{qgr}(S)$ (*resp.* $\mathrm{QGr}(S)$).

Proof. Let $\mathbf{0}$ be the closed point on the affine scheme $\mathbf{Spec}\, S$ which corresponds to the maximal ideal $S_+ \subset S$. Denote by U the scheme $(\mathbf{Spec}\, S \backslash \mathbf{0})$. We know that the category of (quasi)coherent sheaves on the stack $\mathbb{P}\mathrm{roj}\, S$ is equivalent to the category of $\mathbf{G}_m$-equivariant (quasi)coherent sheaves on U. The category of (quasi)coherent sheaves on U is equivalent to the quotient of the category of (quasi)coherent sheaves on $\mathbf{Spec}\, S$ by the subcategory of (quasi)coherent sheaves with support on $\mathbf{0}$. This is also true for the categories of $\mathbf{G}_m$-equivariant sheaves. But the category of (quasi)coherent $\mathbf{G}_m$-equivariant sheaves on $\mathbf{Spec}\, S$ is just the category $\mathrm{gr}(S)$ (resp. $\mathrm{Gr}(S)$) of graded modules over S, and the subcategory of (quasi)coherent sheaves with support on $\mathbf{0}$ coincides with the subcategory $\mathrm{tors}(S)$ (resp. $\mathrm{Tors}(S)$). Thus, we obtain that $\mathrm{coh}(\mathbb{P}\mathrm{roj}\, S)$ is equivalent to the quotient category $\mathrm{qgr}(S) = \mathrm{gr}(S)/\operatorname{tors}(S)$ (and $\mathrm{Qcoh}(\mathbb{P}\mathrm{roj}\, S)$ is equivalent to $\mathrm{QGr}(S) = \mathrm{Gr}(S)/\operatorname{Tors}(S)$). □

Corollary 2.4. *The category of* (*quasi*)*coherent sheaves on the weighted projective space* $\mathbb{P}(\overline{\mathbf{a}})$ *is equivalent to the category* $\mathrm{qgr}(S(a_0, \dots, a_n))$ (*resp.* $\mathrm{QGr}(S(a_0, \dots, a_n))$).

We conclude this section by giving the definition of noncommutative weighted projective spaces and the categories of coherent sheaves on them. Consider a matrix $\theta = (\theta_{ij})$ of dimension $(n+1) \times (n+1)$ with entries $\theta_{ij} \in \mathbf{k}^*$ for all i, j. The set of all such matrices will be denoted by $\mathrm{M}(n+1, \mathbf{k}^*)$. Consider the graded algebra $S_\theta = S_\theta(a_0, \dots, a_n)$ generated by elements $x_i, i = 0, \dots, n$ of degree a_i and with relations

$$\theta_{ij} x_i x_j = \theta_{ji} x_j x_i$$

for all i and j. This algebra is a noncommutative deformation of the algebra $S(a_0, \dots, a_n)$. It can be easily checked that the algebra S_θ depends only on the matrix θ^{an}, with entries

$$\theta^{\mathrm{an}}_{ij} := \theta_{ij}\theta_{ji}^{-1} \qquad \text{for all} \quad 0 \le i, j \le n. \tag{2.3}$$

Thus, if $(\theta')^{\mathrm{an}} = \theta^{\mathrm{an}}$ for two matrices θ' and θ, then $S_{\theta'} \cong S_\theta$.

As before, denote by $\mathrm{qgr}(S_\theta)$ the quotient category $\mathrm{gr}(S_\theta)/\operatorname{tors}(S_\theta)$, where $\mathrm{gr}(S_\theta)$ is the category of finitely generated graded right S_θ-modules and $\operatorname{tors}(A)$ is the full subcategory of $\mathrm{gr}(S_\theta)$ consisting of graded modules of finite dimension over $\mathbf{k}$.

Corollary 2.4 suggests that the category $\mathrm{qgr}(S_\theta)$ should be considered as the category of coherent sheaves on a noncommutative weighted projective space. We will denote this space by $\mathbb{P}_\theta(\overline{\mathbf{a}})$ and will write $\mathrm{coh}(\mathbb{P}_\theta)$ instead of $\mathrm{qgr}(S_\theta)$. Similarly, the category of quasi-coherent sheaves $\mathrm{Qcoh}(\mathbb{P}_\theta)$ is defined as the quotient $\mathrm{QGr}(S_\theta) = \mathrm{Gr}(S_\theta)/\operatorname{Tors}(S_\theta)$.

2.3. *Cohomological properties of coherent sheaves on* $\mathbb{P}_\theta(\overline{\mathbf{a}})$. In this section we discuss properties of categories of coherent sheaves on the noncommutative weighted projective spaces $\mathbb{P}_\theta(\overline{\mathbf{a}})$. Note that the usual commutative weighted projective space is a particular case of the noncommutative one, when θ is the matrix with all entries equal to 1.

All algebras $S_\theta(a_0, \dots, a_n)$ are noetherian. This follows from the fact that they are Ore extensions of commutative polynomial algebras (see for example [31]). For the same reason the algebras $S_\theta(a_0, \dots, a_n)$ have finite right (and left) global dimension, which is equal to $(n+1)$ (see [31, p. 273]). Recall that the global dimension of a ring A is the minimal number d (if it exists) such that for any two modules M and N we have $\mathrm{Ext}^{d+1}_A(M, N) = 0$.

The notion of a regular algebra was introduced in [1]. As we will see below, regular algebras have many good properties. More details can be found in [3].

Definition 2.5. A graded algebra A is called *regular of dimension* d if it satisfies the following conditions:

(1) A has global dimension d,

(2) A has polynomial growth, i.e. $\dim A_p \le cp^\delta$ for some $c, \delta \in \mathbb{R}$,

(3) A is Gorenstein, meaning that $\mathrm{Ext}^i_A(\mathbf{k}, A) = 0$ if $i \ne d$, and $\mathrm{Ext}^d_A(\mathbf{k}, A) = \mathbf{k}(l)$ for some l. The number l is called the *Gorenstein parameter.*

Here Ext_A stands for the Ext functor in the category of right modules $\mathrm{mod}(A)$.

PROPOSITION 2.6. *The algebra* $S_\theta(a_0, \dots, a_n)$ *is a noetherian regular algebra of global dimension* $n+1$. *The Gorenstein parameter* l *of this algebra is equal to the sum* $\sum_{i=0}^{n} a_i$.

Proof. Property (1) holds, as for all Ore extensions of commutative polynomial algebras. Property (2) holds because our algebras have the same growth as ordinary polynomial algebras. Property (3) follows from the following Koszul resolution of the right module $\mathbf{k}_{S_\theta}$

$$
\begin{aligned}
(2.4)\quad 0 \to S_\theta(-\sum_{i=0}^{n} a_i) \to \bigoplus_{i_0<\ldots<i_{n-1}} S_\theta(-\sum_{j=0}^{n-1} a_{i_j}) \to \cdots \\
\cdots \to \bigoplus_{i_0<i_1} S_\theta(-a_{i_0}-a_{i_1}) \to \bigoplus_{i=0}^{n} S_\theta(-a_i) \to S_\theta \to \mathbf{k}_{S_\theta} \to 0,
\end{aligned}
$$

and the fact that the transposed complex is a resolution of the left module ${}_{S_\theta}\mathbf{k}$, shifted to the degree $l = \sum a_i$. The explicit formula for the differentials in the complex (2.4) will be given later (see (2.8)). □

Denote by $\mathcal{O}(i)$ the object $\widetilde{S_\theta(i)}$ in the category $\mathrm{coh}(\mathbb{P}_\theta) = \mathrm{qgr}(S_\theta)$. Consider the sequence $\{\mathcal{O}(i)\}_{i\in\mathbb{Z}}$. The following properties hold true:

(a) For any coherent sheaf $\mathcal{F}$ there are integers $k_1, \ldots, k_s$ and an epimorphism
$$\overset{s}{\underset{i=1}{\oplus}} \mathcal{O}(-k_i) \to \mathcal{F}.$$

(b) For every epimorphism $\mathcal{F} \to \mathcal{G}$ the induced map $\mathrm{Hom}(\mathcal{O}(-n), \mathcal{F}) \to \mathrm{Hom}(\mathcal{O}(-n), \mathcal{G})$ is surjective for $n \gg 0$.

A sequence which satisfies such conditions will be called *ample*. It is proved in [3] that the sequence $\{\mathcal{O}(i)\}$ is ample in $\mathrm{qgr}(A)$ for any graded right noetherian $\mathbf{k}$-algebra A if it satisfies the extra condition:

$$(\chi_1): \quad \dim_{\mathbf{k}} \mathrm{Ext}^1_A(\mathbf{k}, M) < \infty$$

for any finitely generated, graded A-module M.

This condition can be verified for all noetherian regular algebras (see [3, Th. 8.1]). In particular, the sequence $\{\mathcal{O}(i)\}_{i\in\mathbb{Z}}$ in the category $\mathrm{coh}(\mathbb{P}_\theta)$ is ample.

For any sheaf $\mathcal{F} \in \mathrm{qgr}(A)$ we can define a graded module $\Gamma(\mathcal{F})$ by the rule:
$$\Gamma(\mathcal{F}) := \underset{i\geq 0}{\oplus} \mathrm{Hom}(\mathcal{O}(-i), \mathcal{F}).$$
It is proved in [3] that for any noetherian algebra A that satisfies the condition (χ_1) the correspondence Γ is a functor from $\mathrm{qgr}(A)$ to $\mathrm{gr}(A)$ and the composition of Γ with the natural projection $\pi : \mathrm{gr}(A) \longrightarrow \mathrm{qgr}(A)$ is isomorphic to the identity functor (see [3, § 3,4]).

We formulate next a result about the cohomology of sheaves on noncommutative, weighted projective spaces. This result is proved in [3, Th. 8.1] for a general regular algebra and parallels the commutative case.

PROPOSITION 2.7. *Let $S_\theta = S_\theta(a_0, \dots, a_n)$ be the algebra of the noncommutative weighted projective space $\mathbb{P}_\theta = \mathbb{P}_\theta(\overline{\mathbf{a}})$. Then*

1) *The cohomological dimension of the category* $\mathrm{coh}(\mathbb{P}_\theta(\overline{\mathbf{a}}))$ *is equal to n, i.e. for any two coherent sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{coh}(\mathbb{P}_\theta)$ the space* $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ *vanishes if $i > n$.*

2) *There are isomorphisms*

$$H^p(\mathbb{P}_\theta, \mathcal{O}(k)) = \begin{cases} (S_\theta)_k & \text{for } p = 0,\ k \geq 0 \\ (S_\theta)^*_{-k-l} & \text{for } p = n,\ k \leq -l \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

This proposition and the ampleness of the sequence $\{\mathcal{O}(i)\}$ imply the following corollary.

COROLLARY 2.8. *For any sheaf $\mathcal{F} \in \mathrm{coh}(\mathbb{P}_\theta)$ and for all sufficiently large $i \gg 0$ we have $H^k(\mathbb{P}_\theta, \mathcal{F}(i)) = 0$ for all $k > 0$.*

Proof. The group $H^k(\mathbb{P}_\theta, \mathcal{F}(i))$ coincides with $\mathrm{Ext}^k(\mathcal{O}(-i), \mathcal{F})$. Let k be the maximal integer (it exists because the global dimension is finite) such that for some $\mathcal{F}$ there exists arbitrarily large i such that $\mathrm{Ext}^k(\mathcal{O}(-i), \mathcal{F}) \neq 0$. Assume that $k \geq 1$. Choose an epimorphism $\overset{s}{\underset{j=1}{\oplus}} \mathcal{O}(-k_j) \to \mathcal{F}$. Let $\mathcal{F}_1$ denote its kernel. Then for $i > \max\{k_j\}$ we have $\mathrm{Ext}^{>0}(\mathcal{O}(-i), \overset{s}{\underset{j=1}{\oplus}} \mathcal{O}(-k_j)) = 0$, hence $\mathrm{Ext}^k(\mathcal{O}(-i), \mathcal{F}) \neq 0$ implies $\mathrm{Ext}^{k+1}(\mathcal{O}(-i), \mathcal{F}_1) \neq 0$. This contradicts the assumption of the maximality of k. □

One of the useful properties of commutative smooth projective varieties is the existence of the dualizing sheaf. Recall that a sheaf ω_X is called dualizing if for any $\mathcal{F} \in \mathrm{coh}(X)$ there are natural isomorphisms of **k**-vector spaces

$$H^i(X, \mathcal{F}) \cong \mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X)^*,$$

where $*$ denotes the **k**–dual space. The Serre duality theorem asserts the existence of a dualizing sheaf for smooth projective varieties. In this case the dualizing sheaf is a line bundle and coincides with the sheaf of differential forms Ω^n_X of top degree.

Since the definition of ω_X is given in abstract categorical terms, it can be extended to the noncommutative case as well. More precisely, we will say that $\mathrm{qgr}(A)$ satisfies classical *Serre duality* if there is an object $\omega \in \mathrm{qgr}(A)$ together with natural isomorphisms

$$\mathrm{Ext}^i(\mathcal{O}, -) \cong \mathrm{Ext}^{n-i}(-, \omega)^*.$$

Our noncommutative varieties $\mathbb{P}_\theta(\overline{\mathbf{a}})$ satisfy classical Serre duality, with dualizing sheaves being $\mathcal{O}(-l)$, where $l = \sum a_i$ is the Gorenstein parameter for $S_\theta(a_0, \dots, a_n)$. This follows from the paper [42], where the existence of a dualizing sheaf in $\mathrm{qgr}(A)$ has been proved for a class of algebras which includes all noetherian regular algebras. In addition, the authors of [42] showed that the dualizing sheaf coincides with $\widetilde{A}(-l)$, where l is the Gorenstein parameter for A.

There is a reformulation of Serre duality in terms of bounded derived categories [11]. A *Serre functor* in the bounded derived category $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ is by definition an exact auto-equivalence S of $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ such that for any objects $X, Y \in \mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ there is a bifunctorial isomorphism

$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(Y, SX)^*.$$

Serre duality can be reinterpreted as the existence of a Serre functor in the bounded derived category.

2.4. *Exceptional collection on* $\mathbb{P}_\theta(\overline{\mathbf{a}})$. For many reasons it is more natural to work not with the abelian category of coherent sheaves but with its bounded derived category $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$. The purpose of this section is to describe the bounded derived category of coherent sheaves on the noncommutative weighted projective spaces in the terms of exceptional collections.

First, we briefly recall the definition of the bounded derived category for an abelian category $\mathcal{A}$. We start with the category $\mathbf{C}^b(\mathcal{A})$ of bounded differential complexes

$$M^\bullet = (0 \longrightarrow \cdots \longrightarrow M^p \xrightarrow{d^p} M^{p+1} \xrightarrow{d^{p+1}} M^{p+2} \longrightarrow \cdots \longrightarrow 0),$$
$$M^p \in \mathcal{A}, \quad p \in \mathbb{Z}, \quad d^2 = 0.$$

A morphism of complexes $f : M^\bullet \longrightarrow N^\bullet$ is called null-homotopic if $f^p = d_N h^p + h^{p+1} d_M$ for all $p \in \mathbb{Z}$ and some family of morphisms $h^p : M^p \longrightarrow N^{p-1}$. Now the *homotopy category* $\mathbf{H}^b(\mathcal{A})$ is defined as a category with the same objects as $\mathbf{C}^b(\mathcal{A})$, whereas morphisms in $\mathbf{H}^b(\mathcal{A})$ are equivalence classes $\overline{f}$ of morphisms of complexes modulo null-homotopic morphisms. A morphism of complexes $s : N^\bullet \to M^\bullet$ is called a *quasi-isomorphism* if the induced morphisms $H^p s : H^p(N^\bullet) \to H^p(M^\bullet)$ are isomorphisms for all $p \in \mathbb{Z}$. Denote by Σ the class of all quasi-isomorphisms. The bounded derived category $\mathbf{D}^b(\mathcal{A})$ is now defined as the *localization* of $\mathbf{H}^b(\mathcal{A})$ with respect to the class Σ of all quasi-isomorphisms. This means that the derived category has the same objects as the homotopy category $\mathbf{H}^b(\mathcal{A})$, and that morphisms in the derived category are given by left fractions $s^{-1} \circ f$ with $s \in \Sigma$.

Remark 2.9. For any full subcategory $\mathcal{E} \subset \mathcal{A}$ one can construct the homotopy category $\mathbf{H}^b(\mathcal{E})$ and a functor $\mathbf{H}^b(\mathcal{E}) \to \mathbf{D}^b(\mathcal{A})$. In some cases, for example when $\mathcal{A}$ is the abelian category of modules over an algebra A of finite global dimension and $\mathcal{E}$ is the subcategory of projective modules, this functor $\mathbf{H}^b(\mathcal{E}) \to \mathbf{D}^b(\mathcal{A})$ is an equivalence of triangulated categories.

Second, we recall the notion of an exceptional collection.

Definition 2.10. An object E of a $\mathbf{k}$-linear triangulated category $\mathcal{D}$ is said to be *exceptional* if $\mathrm{Hom}(E, E[k]) = 0$ for all $k \neq 0$, and $\mathrm{Hom}(E, E) = \mathbf{k}$.

An ordered set of exceptional objects $\sigma = (E_0, \dots E_n)$ is called *an exceptional collection* if $\mathrm{Hom}(E_j, E_i[k]) = 0$ for $j > i$ and all k. The exceptional collection σ is called *strong* if it satisfies the additional condition $\mathrm{Hom}(E_j, E_i[k]) = 0$ for all i, j and for $k \neq 0$.

Definition 2.11. An exceptional collection $(E_0, \dots, E_n)$ in a category $\mathcal{D}$ is called *full* if it generates the category $\mathcal{D}$, i.e. the minimal triangulated subcategory of $\mathcal{D}$ containing all objects E_i coincides with $\mathcal{D}$. We write in this case

$$\mathcal{D} = \langle E_0, \dots, E_n \rangle .$$

Consider the bounded derived category of coherent sheaves $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$. We prove that this category has an exceptional collection which is strong and full. In this case we will say that the noncommutative weighted projective space $\mathbb{P}_\theta$ possesses a full strong exceptional collection.

THEOREM 2.12. *For any noncommutative weighted projective space* $\mathbb{P}_\theta(\overline{\mathbf{a}})$ *and for any* $k \in \mathbb{Z}$, *the ordered set* $\sigma(k) = (\mathcal{O}(k), \dots, O(k+l-1))$, *where* $l = \sum a_i$, *is the Gorenstein parameter of* S_θ, *forms a full strong exceptional collection in the category* $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$.

Proof. It follows directly from Proposition 2.7 that the collection $\sigma(k)$ is exceptional and strong. To prove that the collection is full let us consider the triangulated subcategory $\mathcal{D} \subset \mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ generated by the collection $\sigma(k)$. The exact sequence (2.4) induces the exact sequence

$$\begin{aligned} (2.6) \quad 0 \to \mathcal{O}(-\sum_{i=0}^{n} a_i) \to \bigoplus_{i_0<\dots<i_{n-1}} \mathcal{O}(-\sum_{j=0}^{n-1} a_{i_j}) \to \cdots \\ \cdots \to \bigoplus_{i_0<i_1} \mathcal{O}(-a_{i_0} - a_{i_1}) \to \bigoplus_{i=0}^{n} \mathcal{O}(-a_i) \to \mathcal{O} \to 0. \end{aligned}$$

Shifting this by $k+l$ we obtain that the object $\mathcal{O}(k+l)$ also belongs to $\mathcal{D}$ and repeating this procedure we deduce that $\mathcal{O}(i)$ for all i belongs to $\mathcal{D}$. Assume that $\mathcal{D}$ does not coincide with $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ and take an object U which does not belong to $\mathcal{D}$. It is proved in [10, Th. 3.2] that the subcategory $\mathcal{D}$ is *admissible*, i.e. the natural embedding functor $\mathcal{D} \hookrightarrow \mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ has right and left adjoint functors. Denote by j the right adjoint and complete the canonical map $jU \longrightarrow U$ to a distinguished triangle

$$jU \longrightarrow U \longrightarrow C \longrightarrow jU[1].$$

It follows from adjointness that for any object $V \in \mathcal{D}$ the space $\mathrm{Hom}(V, C)$ vanishes. The object C is a bounded complex of coherent sheaves. Denote by $H^k(C)$ the leftmost nontrivial cohomology of the complex C. The ampleness of the sequence $\{\mathcal{O}(i)\}_{i\in\mathbb{Z}}$ guarantees that for sufficiently large i the space $\mathrm{Hom}(\mathcal{O}(-i), H^k(C))$ is nontrivial. This implies that $\mathrm{Hom}(\mathcal{O}(-i)[-k], C)$ is nontrivial, which contradicts the fact that the object $\mathcal{O}(-i)[-k]$ belongs to $\mathcal{D}$. □

The strong exceptional collection on the ordinary projective space $\mathbb{P}^n$ was constructed by Beilinson in [9]. This question for the weighted projective spaces was considered in [7].

Definition 2.13. The *algebra of the strong exceptional collection* $(E_0, \dots$ $\dots, E_n)$ is the algebra of endomorphisms of the object $\overset{n}{\underset{i=0}{\oplus}} E_i$. Denote by $\mathcal{T}_\theta$ the sheaf $\overset{l-1}{\underset{i=0}{\oplus}} \mathcal{O}(i)$ and by B_θ the algebra of the collection $(\mathcal{O}, \dots, \mathcal{O}(l-1))$ on the noncommutative weighted projective space $\mathbb{P}_\theta$, i.e. $B_\theta = \mathrm{End}(\mathcal{T}_\theta)$.

The algebra B_θ is a finite dimensional algebra over $\mathbf{k}$. Denote by mod–B_θ the category of finitely generated right modules over B_θ. For any coherent sheaf $\mathcal{F} \in \mathrm{coh}(\mathbb{P}_\theta)$ the space $\mathrm{Hom}(\mathcal{T}_\theta, \mathcal{F})$ has the structure of a right B_θ-module. Denote by P_i the modules $\mathrm{Hom}(\mathcal{T}_\theta, \mathcal{O}(i))$ for $i = 0, \dots, (l-1)$. All these are projective B_θ-modules and $B_\theta = \overset{l-1}{\underset{i=0}{\oplus}} P_i$. The algebra B_θ has l primitive idempotents $e_i, i = 0, \dots, l-1$ such that $1_{B_\theta} = e_0 + \dots + e_{l-1}$ and $e_i e_j = 0$ if $i \neq j$. The right projective modules P_i coincide with $e_i B_\theta$. The morphisms between them can be easily described since

$$\mathrm{Hom}(P_i, P_j) = \mathrm{Hom}(e_i B_\theta, e_j B_\theta) \cong e_j B_\theta e_i \cong \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) = (S_\theta)_{j-i}.$$

Moreover, the algebra B_θ has finite global dimension. This follows from the fact that any right (and left) module M has a finite projective resolution consisting of the projective modules P_i. Indeed the map

$$\bigoplus_{i=0}^{l-1} \mathrm{Hom}(P_i, M) \otimes P_i \longrightarrow M$$

is surjective and there are no nontrivial homomorphisms from P_{l-1} to the kernel of this map. Iterating this procedure we get a finite resolution of M.

Sometimes it is useful to represent the algebra B_θ as a category $\mathfrak{B}_\theta$ which has l objects, say $v_0, \dots, v_{l-1}$, and morphisms defined by

$$\mathrm{Hom}(v_i, v_j) \cong \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \cong (S_\theta)_{j-i}$$

with the natural composition law. Thus $B_\theta = \bigoplus_{0\le i,j\le l-1} \mathrm{Hom}(v_i, v_j)$.

The algebra B_θ is a basis algebra. This means that the quotient of B_θ by the radical $\operatorname{rad}(B_\theta)$ is isomorphic to the direct sum of l copies of the field $\mathbf{k}$. The category mod–B_θ has l irreducible modules which will be denoted $Q_i, i = 0, \dots, l-1$, and $\overset{l-1}{\underset{i=0}{\oplus}} Q_i = B_\theta / \operatorname{rad}(B_\theta)$. The modules Q_i are chosen so that $\operatorname{Hom}(P_i, Q_j) \cong \delta_{i,j}\,\mathbf{k}$.

Our next topic is the notion of *mutation* in an exceptional collection. Let $\sigma = (E_0, \dots, E_n)$ be an exceptional collection in a triangulated category $\mathcal{D}$. Consider a pair (E_i, E_{i+1}) and the canonical maps

$$\operatorname{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i \longrightarrow E_{i+1} \qquad \text{and} \quad E_i \longrightarrow \operatorname{Hom}^\bullet(E_i, E_{i+1})^* \otimes E_{i+1},$$

where by definition

$$\operatorname{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i = \bigoplus_{k \in \mathbb{Z}} \operatorname{Hom}^k(E_i, E_{i+1}) \otimes E_i[-k],$$

$$\operatorname{Hom}^\bullet(E_i, E_{i+1})^* \otimes E_{i+1} = \bigoplus_{k \in \mathbb{Z}} \operatorname{Hom}^{-k}(E_i, E_{i+1}) \otimes E_{i+1}[-k]$$

(recall that the tensor product of a vector space V with an object X may be considered as the direct sum of $\dim V$ copies of the object X).

We define objects $\mathrm{L}E_{i+1}$ and $\mathrm{R}E_i$ as the objects obtained from the distinguished triangles

$$\mathrm{L}E_{i+1} \longrightarrow \operatorname{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i \longrightarrow E_{i+1},$$

$$E_i \longrightarrow \operatorname{Hom}^\bullet(E_i, E_{i+1})^* \otimes E_{i+1} \longrightarrow \mathrm{R}E_i.$$

The object $\mathrm{L}E_{i+1}$ (resp. $\mathrm{R}E_i$) is called by *left* (*right*) *mutation* of E_{i+1} (resp. E_i) in the collection σ. It can be checked that the objects $\mathrm{L}E_{i+1}$ and $\mathrm{R}E_i$ are exceptional and, moreover, the two collections

$$L_i\sigma = (E_0, \dots, E_{i-1}, \mathrm{L}E_{i+1}, E_i, E_{i+2}, \dots, E_n)\,,$$

$$R_i\sigma = (E_0, \dots, E_{i-1}, E_{i+1}, \mathrm{R}E_i, E_{i+2}, \dots, E_n)$$

are exceptional as well. These collections are called left and right mutations of the collection σ in the pair (E_i, E_{i+1}). Consider R_i and L_i as operations on the set of all exceptional collections in the category $\mathcal{D}$. It is easy to see that they are mutually inverse, i.e. $R_i L_i = 1$. Moreover, L_i (resp. R_i) satisfy the Artin braid group relations:

$$L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}, \qquad R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$$

(see [10], [19]).

Denote by $L^{(k)} E_i$ with $k \le i$ the result of k left mutations of the object E_i in the collection σ, analogously for right mutations.

Definition 2.14. The exceptional collection $(L^{(n)}E_n, L^{(n-1)}E_{n-1}, \dots E_0)$ is called the *left dual collection* for the collection $(E_0, \dots, E_n)$. Analogously, the *right dual collection* is defined as $(E_n, \mathrm{R}E_{n-1}, \dots, R^{(n)}E_0)$.

Example 2.15. For example, let us consider the full exceptional collection $(P_0, \dots, P_{l-1})$ in the category $\mathbf{D}^b(\mathrm{mod}-B_\theta)$, consisting of the projective B_θ-modules P_i. It can be shown (e.g. [10, Lemma 5.6]) that the irreducible modules $Q_i, 0 \le i < l$ can be expressed as

$$Q_i \cong L^{(i)}P_i[i].$$

Thus, the left dual for the exceptional collection $(P_0, \dots, P_{l-1})$ coincides with the collection $(Q_{l-1}[1-l], \dots, Q_0)$.

2.5. *A description of the derived categories of coherent sheaves on* $\mathbb{P}_\theta(\bar{\mathbf{a}})$. The natural isomorphisms $\mathrm{Hom}(P_i, P_j) \cong \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j))$, which are direct consequences of the construction of the algebra B_θ, allow us to construct a functor $\bar{F} : \mathbf{H}^b(\mathcal{P}) \longrightarrow \mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$, where $\mathcal{P}$ is the full subcategory of the category of right modules mod–B_θ consisting of finite direct sums of the projective modules $P_i, i = 0, \dots, l-1$. The functor $\bar{F}$ sends P_i to $\mathcal{O}(i)$ and any bounded complex of projective modules to the corresponding complex of $\mathcal{O}(i), i = 0, \dots, l-1$. It follows from Remark 2.9 that the functor $\bar{F}$ induces a functor

$$F : \mathbf{D}^b(\mathrm{mod}\text{–}B_\theta) \longrightarrow \mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta)).$$

THEOREM 2.16. *The functor* $F : \mathbf{D}^b(\mathrm{mod}\text{–}B_\theta) \longrightarrow \mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ *is an equivalence of the derived categories.*

Since the exceptional collection $(\mathcal{O}, \dots, \mathcal{O}(l-1))$ generates the category $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ it is sufficient to check that the functor F is fully faithful. We know that for any $0 \le i, j \le l-1$ and any k there are isomorphisms

$$\mathrm{Hom}(P_i, P_j[k]) \xrightarrow{\sim} \mathrm{Hom}(FP_i, FP_j[k]) = \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)[k]).$$

Since $P_i, i = 0, \dots, l-1$, generate $\mathbf{D}^b(\mathrm{mod}\text{–}B_\theta)$, the proof of the theorem is a consequence of the following lemma.

LEMMA 2.17. *Let* $\mathcal{A}$ *be an abelian category and* $\mathcal{D}$ *be a triangulated category. Let* $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{D}$ *be an exact functor and let* $\{E_i\}_{i\in I}$ *be a set of objects of* $\mathbf{D}^b(\mathcal{A})$ *which generates* $\mathbf{D}^b(\mathcal{A})$ (*i.e. the minimal full triangulated subcategory of* $\mathbf{D}^b(\mathcal{A})$ *containing all* E_i *coincides with* $\mathbf{D}^b(\mathcal{A})$). *Assume that the maps*

$$\mathrm{Hom}(E_i, E_j[k]) \longrightarrow \mathrm{Hom}(FE_i, FE_j[k])$$

are isomorphisms for all $i, j \in I$ *and any* $k \in \mathbb{Z}$. *Then the functor* F *is fully faithful.*

Proof. This lemma is known and results from dévissage (e.g. [21, 10.10], [25, 4.2]). We first consider the full subcategory $\mathcal{C} \in \mathbf{D}^b(\mathcal{A})$ which consists of all objects X such that the maps

$$\operatorname{Hom}(X, E_i[k]) \xrightarrow{\sim} \operatorname{Hom}(FX, FE_i[k])$$

are isomorphisms for all $i \in I$ and all $k \in \mathbb{Z}$. The category $\mathcal{C}$ is a triangulated subcategory, because it is closed with respect to the translation functor and, for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1],$$

if X and Y belong to $\mathcal{C}$, then Z belongs too. The last statement is a consequence of the five lemma, i.e., since the morphisms f_1, f_2, f_4, f_5 in the diagram

$$\begin{tikzcd}
\operatorname{Hom}(Y[1], E_i) \arrow[r] \arrow[d, "f_1"'] & \operatorname{Hom}(X[1], E_i) \arrow[r] \arrow[d, "f_2"'] & \operatorname{Hom}(Z, E_i) \arrow[r] \arrow[d, "f_3"] & {} \\
\operatorname{Hom}(FY[1], FE_i) \arrow[r] & \operatorname{Hom}(FX[1], FE_i) \arrow[r] & \operatorname{Hom}(FZ, FE_i) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \operatorname{Hom}(Y, E_i) \arrow[r] \arrow[d, "f_4"] & \operatorname{Hom}(X, E_i) \arrow[d, "f_5"] \\
{} \arrow[r] & \operatorname{Hom}(FY, FE_i) \arrow[r] & \operatorname{Hom}(FX, FE_i)
\end{tikzcd}$$

are isomorphisms, the morphism f_3 is an isomorphism too. The subcategory $\mathcal{C}$ contains the objects E_i and, hence, coincides with $\mathbf{D}^b(\mathcal{A})$. Now consider the full subcategory $\mathcal{B} \subset \mathbf{D}^b(\mathcal{A})$ consisting of all objects X such that the map

$$\operatorname{Hom}(Y, X[k]) \xrightarrow{\sim} \operatorname{Hom}(FY, FX[k])$$

is an isomorphism for every object $Y \in \mathbf{D}^b(\mathcal{A})$ and all $k \in \mathbb{Z}$. By the same argument as above the subcategory $\mathcal{B}$ is triangulated and contains all E_i. Therefore, it coincides with $\mathbf{D}^b(\mathcal{A})$. This proves the lemma and completes the proof of the theorem. □

There is also a right adjoint to F, namely a functor $G : \mathbf{D}^b(\operatorname{coh}(\mathbb{P}_\theta)) \longrightarrow \mathbf{D}^b(\operatorname{mod} -B_\theta)$. To construct it we have to consider the functor

$$\operatorname{Hom}(\mathcal{T}_\theta, -) : \operatorname{Qcoh}(\mathbb{P}_\theta) \longrightarrow \operatorname{Mod} -B_\theta$$

where $\operatorname{Mod} -B_\theta$ is the category of all right modules over B_θ. Since $\operatorname{Qcoh}(\mathbb{P}_\theta)$ has enough injectives and has finite global dimension there is a right derived functor

$$\mathbf{R}\operatorname{Hom}(\mathcal{T}_\theta, -) : \mathbf{D}^b(\operatorname{Qcoh}(\mathbb{P}_\theta)) \longrightarrow \mathbf{D}^b(\operatorname{Mod} -B_\theta).$$

$\mathbf{D}^b(\operatorname{coh}(\mathbb{P}_\theta))$ is equivalent to the full subcategory $\mathbf{D}^b_{\operatorname{coh}}(\operatorname{Qcoh}(\mathbb{P}_\theta))$ of $\mathbf{D}^b(\operatorname{Qcoh}(\mathbb{P}_\theta))$ whose objects are complexes with cohomologies in $\operatorname{coh}(\mathbb{P}_\theta)$.

Moreover, the functor $\mathbf{R}\operatorname{Hom}(\mathcal{T}_\theta, -)$ sends an object of $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Qcoh}(\mathbb{P}_\theta))$ to an object of the subcategory $\mathbf{D}^b_{\mathrm{mod}}(\mathrm{Mod}-B_\theta)$, which is also equivalent to $\mathbf{D}^b(\mathrm{mod}-B_\theta)$. This gives us a functor

$$G = \mathbf{R}\operatorname{Hom}(\mathcal{T}_\theta, -) : \mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta)) \longrightarrow \mathbf{D}^b(\mathrm{mod}-B_\theta).$$

The functor G is right adjoint to F, and it is an equivalence of categories as well.

In the end of this paragraph we describe an equivalence relation $\theta \sim \theta'$ on the space of all matrices θ with $\theta_{ij} \in \mathbf{k}^*$ for all i, j under which the noncommutative, weighted projective spaces $\mathbb{P}_\theta$ and $\mathbb{P}_{\theta'}$ have equivalent abelian categories of coherent sheaves. It was mentioned above that the graded algebras S_θ depend only on the matrix θ^{an} defined by the rule (2.3). However, it can also happen that two different algebras S_θ and $S_{\theta'}$ produce isomorphic algebras B_θ and $B_{\theta'}$.

Proposition 2.18. *Let $(m_0, \ldots, m_n) \in (\mathbf{k}^*)^{(n+1)}$ be any vector with nonzero entries. Suppose that two matrices $\theta, \theta' \in \mathrm{M}(n+1, \mathbf{k}^*)$ are related by the formula*

$$\theta'_{ij} = \theta_{ij} \cdot m_i^{a_j}. \tag{2.7}$$

Then the algebras $B_{\theta'}$ and B_θ are isomorphic.

Proof. Consider the category $\mathfrak{B}_{\theta'}$ and its auto-equivalence τ which acts by identity on the objects and acts on the spaces $\operatorname{Hom}(v_i, v_j)$ as the multiplication by $(m_i)^{(j-i)}$. There is a natural basis of the spaces $\operatorname{Hom}(v_i, v_j)$ which is induced by the monomial basis $x_{i_0} \cdots x_{i_k}$, $0 \le i_0 \le \cdots \le i_k \le n$ of $S_{\theta'}$. The transformation of this basis under the equivalence τ gives us a new basis in which the category $\mathfrak{B}_{\theta'}$ coincides with the category $\mathfrak{B}_\theta$ equipped with its natural basis coming from the monomial basis of S_θ. The equivalence of the categories $\mathfrak{B}_{\theta'}$ and $\mathfrak{B}_\theta$ implies an isomorphism of the algebras $B_{\theta'}$ and B_θ. $\square$

If now the algebras $B_{\theta'}$ and B_θ are isomorphic, then the composition of the functors

$$\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_{\theta'})) \xrightarrow{G_{\theta'}} \mathbf{D}^b(\mathrm{mod}-B_{\theta'}) \cong \mathbf{D}^b(\mathrm{mod}-B_\theta) \xrightarrow{F_\theta} \mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$$

is an equivalence of derived categories. This equivalence evidently takes a sheaf $\mathcal{O}(i)$, $0 \le i \le l-1$ on $\mathbb{P}_{\theta'}$ to the sheaf $\mathcal{O}(i)$ on $\mathbb{P}_\theta$. Using the resolution (2.6) it can be easily checked that this functor takes $\mathcal{O}(i)$ to $\mathcal{O}(i)$ for all $i \in \mathbb{Z}$. Now, it follows from the ampleness condition on $\{\mathcal{O}(i)\}$ and Corollary 2.8 that the functor sends the subcategory $\mathrm{coh}(\mathbb{P}_{\theta'})$ to $\mathrm{coh}(\mathbb{P}_\theta)$ and induces an equivalence $\mathrm{coh}(\mathbb{P}_{\theta'}) \cong \mathrm{coh}(\mathbb{P}_\theta)$. We just proved:

Corollary 2.19. *If the matrices θ' and θ are connected by the relation* (2.7) *then the noncommutative weighted projective spaces $\mathbb{P}_{\theta'}(\bar{\mathbf{a}})$ and $\mathbb{P}_\theta(\bar{\mathbf{a}})$ have equivalent abelian categories of coherent sheaves* $\mathrm{coh}(\mathbb{P}_{\theta'})$ *and* $\mathrm{coh}(\mathbb{P}_\theta)$.

In the case $n = 1$, it follows immediately that for any $\theta, \theta' \in M(2, \mathbf{k}^*)$ the categories $\mathrm{coh}(\mathbb{P}_\theta(a_0, a_1))$ and $\mathrm{coh}(\mathbb{P}_{\theta'}(a_0, a_1))$ are equivalent.

Next consider the case $n = 2$. For any matrix $\theta \in M(3, \mathbf{k}^*)$ denote the expression

$$(\theta_{01}^{an})^{a_2}(\theta_{12}^{an})^{a_0}(\theta_{20}^{an})^{a_1} = (\theta_{01})^{a_2}(\theta_{12})^{a_0}(\theta_{20})^{a_1}(\theta_{10})^{-a_2}(\theta_{21})^{-a_0}(\theta_{02})^{-a_1}$$

by $q(\theta)$. Now, the result of Proposition 2.18 can be written in the following form.

COROLLARY 2.20. *Let $n = 2$ and let θ' and θ be two matrices from* $\mathrm{M}(3, \mathbf{k}^*)$. *If $q(\theta') = q(\theta)$ then the abelian categories* $\mathrm{coh}(\mathbb{P}_{\theta'}(a_0, a_1, a_2))$ *and* $\mathrm{coh}(\mathbb{P}_\theta(a_0, a_1, a_2))$ *are equivalent.*

2.6. DG *algebras and Koszul duality.* The aim of this section is to give another description of the derived category $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$. It was shown above that this category is equivalent to the derived category $\mathbf{D}^b(\mathrm{mod}-B_\theta)$. We introduce a finite dimensional differential $\mathbb{Z}$-graded algebra (DG algebra) $C_\theta^\bullet$ and prove that the category $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ is equivalent to the derived category of $C_\theta^\bullet$.

This new description of the derived category in terms of the DG-algebra $C_\theta^\bullet$ naturally yields an exceptional collection (Corollary 2.27), which is essentially the (left) dual of the collection described in Theorem 2.12; cf. the discussion at the end of §2.4.

We recall here that a DG algebra over $\mathbf{k}$ is a graded associative $\mathbf{k}$–algebra

$$R = \bigoplus_{p \in \mathbb{Z}} R^p$$

with a differential d of degree $+1$ such that

$$d(rs) = (dr)s + (-1)^p r(ds)$$

for all $r \in R^p, s \in R$. We will suppose that R is noetherian as a graded algebra.

A right DG module over a DG algebra is a graded right R–module $M = \bigoplus_{p \in \mathbb{Z}} M^p$ with a differential ∇ of degree 1 such that

$$\nabla(mr) = (\nabla m)r + (-1)^p m dr$$

for all $m \in M^p$ and $r \in R$.

A morphism of DG R-modules $f : M \longrightarrow N$ is called null-homotopic if $f = d_N h + h d_M$, where $h : M \longrightarrow N$ is a morphism of the underlying graded R-modules which is homogeneous of degree -1. The homotopy category $\mathbf{H}^b(R)$ is defined as a category which has all finitely generated DG R-modules as objects, and whose morphisms are the equivalence classes $\overline{f}$ of morphisms of DG R-modules modulo null-homotopic morphisms. A morphism of DG R-modules $s : M \to N$ is called a quasi-isomorphism if the induced morphism

$H^*s : H^*(M) \to H^*(N)$ is an isomorphism of graded vector spaces. Now, by definition, the derived category $\mathbf{D}^b(R)$ is the localization

$$\mathbf{D}^b(R) := \mathbf{H}^b(R)\left[\Sigma^{-1}\right],$$

where Σ is the class of all quasi-isomorphisms. It can be checked that there are canonical isomorphisms

$$\mathrm{Hom}_{\mathbf{D}^b(R)}(R, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{H}^b(R)}(R, M) \xrightarrow{\sim} H^0 M$$

for each DG R-module M.

Any ordinary $\mathbf{k}$-algebra A can be considered as the DG algebra $A^\bullet$ with $A^0 = A$ and $A^p = 0$ for all $p \neq 0$. In this case the derived category of the DG algebra $\mathbf{D}^b(A^\bullet)$ identifies with the bounded derived category of finitely generated right A-modules; i.e., $\mathbf{D}^b(A^\bullet) \cong \mathbf{D}^b(\mathrm{mod}-A)$. For a detailed exposition of the facts about derived categories of DG algebras, see [25], [26].

Now denote by $B_{\theta s}$ the algebra $B_\theta/\operatorname{rad}(B_\theta)$ and consider it as a right B_θ-module, isomorphic to the sum $\overset{l-1}{\underset{i=0}{\oplus}} Q_i$ of all irreducibles. Introduce the finite dimensional DG algebra

$$\mathrm{Ext}^\bullet_{B_\theta}(B_{\theta s}, B_{\theta s}) = \underset{p\in\mathbb{Z}}{\oplus} \mathrm{Ext}^p_{B_\theta}(B_{\theta s}, B_{\theta s})$$

with the natural composition law and trivial differential. In what follows we give a precise description of this DG algebra and prove the existence of an equivalence

$$\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta)) \cong \mathbf{D}^b(\mathrm{Ext}^\bullet_{B_\theta}(B_{\theta s}, B_{\theta s})),$$

which gives the promised description of the category $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$.

Let us introduce a graded DG algebra $\Lambda^\bullet = \Lambda^\bullet(a_0, \dots, a_n)$. As a DG algebra it is the skew-symmetric algebra with trivial differential which is generated by skew-commutative elements y_i, $i = 0, \dots, l-1$ of degree 1, i.e.

$$\Lambda^\bullet = \bigoplus_{p=0}^{n+1} \Lambda^p,$$

where $y_i \in \Lambda^1$ with the relations $y_i y_j = -y_j y_i$ for all $0 \le i, j \le n$.

The additional grading on the DG algebra $\Lambda^\bullet(a_0, \dots, a_n)$ is defined by putting $y_i \in \Lambda^\bullet_{-a_i}$. Thus $\Lambda^\bullet(a_0, \dots, a_n)$ is just a bigraded skew-symmetric algebra

$$\Lambda^\bullet(a_0, \dots, a_n) = \bigoplus_{p,i\in\mathbb{Z}} \Lambda^p_i$$

with generators $y_i \in \Lambda^1_{-a_i}$. For any $(n+1) \times (n+1)$-matrix θ we also can define a graded DG algebra $\Lambda^\bullet_\theta(a_0, \dots, a_n)$ as the DG algebra which has trivial differential and is generated by elements $y_i \in (\Lambda_\theta)^1_{-a_i}, i = 0, \dots, n$ with the relations

$$\theta_{ij}y_iy_j + \theta_{ji}y_jy_i = 0$$

for all $0 \le i, j \le n$.

Consider the following complex $\mathrm{Com}^\bullet$ of right S_θ-modules

$$\mathrm{Com}^\bullet := 0 \to S_\theta(-\sum_{i=0}^{n} a_i) \to \bigoplus_{i_0<\ldots<i_{n-1}} S_\theta(-\sum_{j=0}^{n-1} a_{i_j}) \to \cdots$$
$$\cdots \to \bigoplus_{i_0<i_1} S_\theta(-a_{i_0} - a_{i_1}) \to \bigoplus_{i=0}^{n} S_\theta(-a_i) \to S_\theta \to 0, \tag{2.8}$$

in which the differentials are defined componentwise as follows: for any set $I = \{i_0, \ldots i_k\}$ the differential sends the generator of $S_\theta(-\sum_{i\in I} a_i)$ to the sum of the elements

$$(-1)^s \left(\prod_{i\in I} \theta_{ii_s}\right) x_{i_s}$$

of $S_\theta\big(-\sum_{i\in(I\setminus i_s)} a_i\big)$, for $0 \le s \le k$. With this we see that the complex $\mathrm{Com}^\bullet$ is a free resolution of the right S_θ-module $\mathbf{k}_{S_\theta}$.

Now we define a structure of left DG module over the DG algebra $\Lambda_\theta^\bullet$ on the complex $\mathrm{Com}^\bullet$, such that the element y_j takes the generator of $S_\theta(-\sum_{i\in I} a_i)$ to the generator of $S_\theta(-\sum_{i\in(I\setminus i_s)} a_i)$ with coefficient

$$(-1)^s \prod_{i\in I} \theta_{i_s i}$$

if $j = i_s \in I = \{i_0, \ldots, i_k\}$, and takes it to zero if $j \notin I$. It can be checked that this action is well defined and makes the complex $\mathrm{Com}^\bullet$ a DG $\Lambda_\theta^\bullet$-S_θ-bimodule.

Remark 2.21. It is not difficult to see that the complex $\mathrm{Com}^\bullet$ as a graded $\Lambda_\theta^\bullet$-S_θ-bimodule (i.e. without differential) is isomorphic to $(\Lambda_\theta^\bullet)^* \underset{\mathbf{k}}{\otimes} S_\theta$, where $(\Lambda_\theta^\bullet)^*$ is $\mathrm{Hom}_{\mathbf{k}}(\Lambda_\theta^\bullet, \mathbf{k})$.

Definition 2.22. Define a DG *category* $\mathfrak{C}_\theta$ (actually a *graded category*, because all differentials are trivial) as a DG *category with l objects*, say $w_0, \ldots$ $\ldots, w_{l-1}$, and the spaces of morphisms between which are the complexes

$$\mathrm{Hom}^\bullet(w_j, w_i) \cong (\Lambda_\theta^\bullet)_{i-j}$$

with the natural composition law induced by that of the DG algebra $\Lambda_\theta^\bullet$.

It follows from the definition of the DG algebra $\Lambda_\theta^\bullet$ that

$$\mathrm{Hom}^\bullet(w_j, w_i) = 0 \qquad \text{when} \quad j < i.$$

Definition 2.23. Define the DG *algebra* $C^\bullet_\theta$ as the DG *algebra of the* DG *category* $\mathfrak{C}_\theta$, i.e.

$$C^\bullet_\theta := \bigoplus_{0\le i,j\le l-1} \mathrm{Hom}^\bullet(w_j, w_i).$$

The quotient of this DG algebra by its radical is isomorphic to $\mathbf{k}^{\oplus l}$. In particular the DG algebra $C^\bullet_\theta$, similarly to the algebra B_θ, has l irreducible DG modules in degree 0. Moreover, as a right DG $C^\bullet_\theta$-module the algebra $C^\bullet_\theta$ is a direct sum

$$C^\bullet_\theta = \bigoplus_{i=0}^{l-1} H_i, \qquad \text{where} \quad H_i = \bigoplus_{0\le j\le l-1} \mathrm{Hom}^\bullet(w_j, w_i),$$

and the direct summands H_i are homotopically projective right DG $C^\bullet_\theta$-modules. Recall that a DG module H is called homotopically projective if, for any acyclic DG module N, $\mathrm{Hom}(H, N) = 0$ in the homotopy category (see e.g. [25], [26]).

Let us construct a DG $C^\bullet_\theta$-B_θ-bimodule $X^\bullet$, obtained from the DG $\Lambda^\bullet_\theta$-S_θ-bimodule $\mathrm{Com}^\bullet$ by the formula

$$X^\bullet = \bigoplus_{0\le i,j\le l-1} X^\bullet(i,j), \qquad \text{with} \quad X^\bullet(i,j) \cong \mathrm{Com}^\bullet_{j-i}$$

where $\mathrm{Com}^\bullet_{j-i}$ is the degree $(j-i)$ component of the graded complex $\mathrm{Com}^\bullet$. In particular, $X^\bullet(i,j) = 0$ when $i > j$ and $X^\bullet(i,i) \cong \mathbf{k}$ for all i. The structure of DG $C^\bullet_\theta$-B_θ-bimodule on $X^\bullet$ comes from the structure of DG $\Lambda^\bullet_\theta$-S_θ-bimodule on $\mathrm{Com}^\bullet$. The bimodule $X^\bullet$ is quasi-isomorphic to $\mathbf{k}^{\oplus l}$, and it is quasi-isomorphic to $B_\theta/\mathrm{rad}(B_\theta)$ as a right B_θ-module and to $C^\bullet_\theta/\mathrm{rad}(C^\bullet_\theta)$ as a left DG $C^\bullet_\theta$-module. This fact allows us to say that the DG algebra $C^\bullet_\theta$ is the *Koszul dual* to the algebra B_θ.

Remark 2.24. It follows from Remark 2.21 that $X^\bullet$ as a graded $C^\bullet_\theta$-B_θ-bimodule (i.e. without differential) is isomorphic to

$$\bigoplus_{i=0}^{l-1} H_i^* \otimes P_i,$$

where H_i^* are the left DG $C^\bullet_\theta$-modules $\mathrm{Hom}_{\mathbf{k}}(H_i, \mathbf{k})$. In other words, as a graded $C^\bullet_\theta$-B_θ-bimodule $X^\bullet$ is isomorphic to $C^{\bullet *}_\theta \otimes_{\mathbf{k}^{\oplus l}} B_\theta$.

For any right DG $C^\bullet_\theta$-module N, the tensor product $N \otimes_{\mathbf{k}} X^\bullet$ is naturally a complex of right B_θ-modules, in which the module structure is given by the action of B_θ on $X^\bullet$, and the grading and differential are given by

$$(N \otimes_{\mathbf{k}} X^\bullet)^k = \bigoplus_{p+q=k} N^p \otimes_{\mathbf{k}} X^q, \qquad d(n\otimes x) = (dn)\otimes x + (-1)^p n \otimes dx$$

for all $n \in N^p, x \in X^\bullet$. The $\mathbf{k}$-submodule generated by all differences $nc\otimes x - m\otimes cx$ is closed under the differential and under multiplication by any element of B_θ. So the quotient by this submodule, which we denote by $N \otimes_{C_\theta^\bullet} X^\bullet$, is a well-defined complex of right B_θ-modules.

For any complex M of right B_θ-modules we define a right DG $C_\theta^\bullet$-module

$$\mathcal{H}om_{B_\theta}(X^\bullet, M)^k = \prod_{p-q=k} \mathrm{Hom}_{B_\theta}(X^q, M^p),$$
$$(df)(x) = d(f(x)) - (-1)^n f(dx).$$

In this way we get a pair of adjoint functors $(-) \otimes_{C_\theta^\bullet} X^\bullet$ and $\mathcal{H}om_{B_\theta}(X^\bullet, -)$ between homotopy categories, which induce a pair of adjoint functors on the level of derived categories as well:

$$\overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet : \mathbf{D}^b(C_\theta^\bullet) \longrightarrow \mathbf{D}^b(\mathrm{mod}\text{-}B_\theta),$$
$$\mathbf{R}\,\mathrm{Hom}_{B_\theta}(X^\bullet, -) : \mathbf{D}^b(\mathrm{mod}\text{-}B_\theta) \longrightarrow \mathbf{D}^b(C_\theta^\bullet).$$

Moreover, since $X^\bullet$ is a projective finitely generated right B_θ-module and a flat left $C_\theta^\bullet$-module, both functors $(-) \otimes_{C_\theta^\bullet} X^\bullet$ and $\mathcal{H}om_{B_\theta}(X^\bullet, -)$ between homotopy categories preserve acyclicity. Hence, the derived functors in this case are defined by the same formulas. For more information about derived functors see e.g. [25].

THEOREM 2.25. *The functors* $\overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet$ *and* $\mathbf{R}\,\mathrm{Hom}_{B_\theta}(X^\bullet, -)$ *are equivalences of triangulated categories.*

Proof. It is evident that the first functor $\overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet$ takes $C_\theta^\bullet$ as a right DG $C_\theta^\bullet$-module to $X^\bullet$ as a right B_θ-module which is isomorphic to $B_{\theta s} = \bigoplus_{i=0}^{l-1} Q_i$ in the derived category $\mathbf{D}^b(\mathrm{mod}\text{-}B_\theta)$. On the other hand, it follows from Remark 2.24 and the equalities $\mathrm{Hom}_{B_\theta}(P_i, Q_j) = \delta_j^i \mathbf{k}$ that the latter functor, $\mathbf{R}\,\mathrm{Hom}_{B_\theta}(X^\bullet, -)$, takes the module $B_{\theta s} = \bigoplus_{i=0}^{l-1} Q_i$ to the free DG module $C_\theta^\bullet = \bigoplus_{i=0}^{l-1} H_i$ and takes Q_i to H_i for any $0 \le i \le l-1$. Thus, the composition functor $\mathbf{R}\,\mathrm{Hom}_{B_\theta}(X^\bullet, -) \overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet$ sends B_s to itself and it also sends all direct summands Q_i to Q_i. The adjunction maps

$$\mathbf{R}\,\mathrm{Hom}_{B_\theta}(X^\bullet, Q_i) \overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet \longrightarrow Q_i$$

cannot be trivial, hence they are isomorphisms for all i. Therefore, we obtain isomorphisms

$$\mathrm{Hom}_{B_\theta}(Q_i, Q_j[k]) \xrightarrow{\sim} \mathrm{Hom}_{C_\theta^\bullet}(\mathbf{R}\,\mathrm{Hom}_{B_\theta}(X^\bullet, Q_i),$$
$$\mathbf{R}\,\mathrm{Hom}_{B_\theta}(X^\bullet, Q_j)[k]) \cong \mathrm{Hom}(H_i, H_j[k])$$

for any $0 \le i, j \le l-1$ and all $k \in \mathbb{Z}$.

Since $Q_i, i = 0, \dots, l-1$ generate the derived category $\mathbf{D}^b(\text{mod-}B_\theta)$, Lemma 2.17 implies that the functor

$$\mathbf{R}\operatorname{Hom}_{B_\theta}(X^\bullet, -) : \mathbf{D}^b(\text{mod-}B_\theta) \longrightarrow \mathbf{D}^b(C_\theta^\bullet)$$

is fully faithful.

Consider the triangulated subcategory $\mathcal{D}$ of $\mathbf{D}^b(C_\theta^\bullet)$ generated by H_i, $i = 0, \dots, l-1$. By Remark 2.24 $X^\bullet$ as a graded $C_\theta^\bullet$-B_θ-bimodule is isomorphic to $\bigoplus_{i=0}^{l-1} H_i^* \otimes P_i$, and hence, the dual to $X^\bullet$ over $\mathbf{k}$ gives a resolution of $C_\theta^\bullet / \operatorname{rad}(C_\theta^\bullet)$ in terms of H_i. Therefore, the subcategory $\mathcal{D}$ contains all irreducible DG modules and coincides with the whole $\mathbf{D}^b(C_\theta^\bullet)$. Thus, H_i, $i = 0, \dots, l-1$ generate the category $\mathbf{D}^b(C_\theta^\bullet)$, and the functor $\mathbf{R}\operatorname{Hom}_{B_\theta}(X^\bullet, -)$ is an equivalence of the derived categories. $\square$

Corollary 2.26. *There is an isomorphism of* DG *algebras*

$$C_\theta^\bullet \cong \bigoplus_{0 \le i,j \le l-1} \operatorname{Ext}^\bullet(Q_i, Q_j).$$

The assertion of the corollary is clear now, because the functor $\overset{\mathbf{L}}{\otimes}$, which is an equivalence, sends $C_\theta^\bullet$ to $B_{\theta s} = \overset{l-1}{\underset{i=0}{\oplus}} Q_i$.

Corollary 2.27. *The derived category of coherent sheaves* $\mathbf{D}^b(\operatorname{coh}(\mathbb{P}_\theta))$ *on the noncommutative weighted space* $\mathbb{P}_\theta$ *is equivalent to the derived category* $\mathbf{D}^b(C_\theta^\bullet)$.

2.7. *Hirzebruch surfaces* $\mathbb{F}_n$. The surfaces $\mathbb{F}_n$ are minimal rational surfaces defined as the projectivizations $\mathbf{Proj}\,(\mathcal{O} \oplus \mathcal{O}(-n))$ of the vector bundles $\mathcal{O} \oplus \mathcal{O}(-n)$ over $\mathbb{P}^1$. The surface $\mathbb{F}_n$ has a $(-n)$-section that will be denoted by s. There is a simple connection between $\mathbb{F}_n$ and the weighted projective plane $\mathbf{P}(1,1,n)$; namely, the latter can be obtained from $\mathbb{F}_n$ by contracting the $(-n)$-section s. In this way $\mathbb{F}_n$ is a resolution of the singularity of $\mathbf{P}(1,1,n)$. Thus, we have two different resolutions of the singularity of $\mathbf{P}(1,1,n)$:

$$\begin{tikzcd}
\mathbb{F}_n \arrow[dr] & & \mathbb{P}(1,1,n) \arrow[dl] \\
& \mathbf{P}(1,1,n) &
\end{tikzcd}$$

For this reason the derived categories of coherent sheaves on $\mathbb{F}_n$ and on $\mathbb{P}(1,1,n)$ are closely related to each other. We will show that for $n \ge 2$ there is a fully faithful functor

$$MK_n : \mathbf{D}^b(\operatorname{coh}(\mathbb{F}_n)) \longrightarrow \mathbf{D}^b(\operatorname{coh}(\mathbb{P}(1,1,n)))$$

and will give its description.

Denote by f the class of the fiber of $\mathbb{F}_n$ in the Picard group. Since $\mathbb{F}_n$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ the derived category of coherent sheaves on $\mathbb{F}_n$ has an exceptional collection of length 4 (see [32]). More precisely, we have

PROPOSITION 2.28. *The collection*

$$\sigma = (\mathcal{O}, \mathcal{O}(f),\ \mathcal{O}(s+nf),\ \mathcal{O}(s+(n+1)f))$$

is a full, strong, exceptional collection on $\mathbb{F}_n$. *The derived category* $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_n))$ *is equivalent to the derived category* $\mathbf{D}^b(\mathrm{mod}\text{-}F(n))$, *where* $F(n)$ *is the algebra of the exceptional collection* σ.

Denote by U the two dimensional vector space $H^0(\mathbb{F}_n, \mathcal{O}(f))$. From the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(s+nf) \longrightarrow \mathcal{O}_s \longrightarrow 0$$

we find that $H^0(\mathbb{F}_n, \mathcal{O}(s+nf))$ is the direct sum of the space S^nU and a one-dimensional space. Analogously, we can check that $H^0(\mathbb{F}_n, \mathcal{O}(s+(n+1)f))$ is isomorphic to $S^nU \oplus U$.

On the other hand, we know that the weighted projective plane $\mathbb{P}(1,1,n)$ has an exceptional collection

$$(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n), \mathcal{O}(n+1))\,.$$

Denote the algebra of this exceptional collection by $B(1,1,n)$. It follows from Proposition 2.7 that the space $H^0(\mathbb{P}(1,1,n), \mathcal{O}(1))$ is isomorphic to U, $H^0(\mathbb{P}(1,1,n), \mathcal{O}(n))$ is isomorphic to the direct sum of S^nU and a one-dimensional space, and $H^0(\mathbb{P}(1,1,n), \mathcal{O}(n+1))$ is isomorphic to $S^nU \oplus U$. This implies that the algebra of the exceptional collection $(\mathcal{O}, \mathcal{O}(f), \mathcal{O}(s+nf), \mathcal{O}(s+(n+1)f))$ on $\mathbb{F}_n$ is isomorphic to the algebra of the exceptional collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(n), \mathcal{O}(n+1))$ on $\mathbb{P}(1,1,n)$.

Thus, the algebra of endomorphisms of the projective $B(1,1,n)$-module

$$M = P_0 \oplus P_1 \oplus P_n \oplus P_{n+1}$$

coincides with $F(n)$, which makes M an $F(n)$-$B(1,1,n)$-bimodule. The natural functor

$$(-) \overset{\mathbf{L}}{\otimes}_{F(n)} M : \mathbf{D}^b(\mathrm{mod}\text{-}F(n)) \longrightarrow \mathbf{D}^b(\mathrm{mod}\text{-}B(1,1,n))$$

takes the free module $F(n)$ to M, and there are isomorphisms

$$\mathrm{Hom}_{F(n)}(F(n), F(n)[k]) \xrightarrow{\sim} \mathrm{Hom}_{B(1,1,n)}(M, M[k]).$$

Since the direct summands of $F(n)$ generate the derived category $\mathbf{D}^b(\mathrm{mod}\text{-}F(n))$, Lemma 2.17 guarantees that the functor $(-) \overset{\mathbf{L}}{\otimes}_{F(n)} M$ is fully faithful. Using the descriptions of the derived categories of coherent sheaves on $\mathbb{F}_n$ and $\mathbb{P}(1,1,n)$ in terms of the exceptional collections, we obtain the following theorem.

THEOREM 2.29. *The functor*

$$MK_n : \mathbf{D}^b(\mathrm{coh}(\mathbb{F}_n)) \longrightarrow \mathbf{D}^b(\mathrm{coh}(\mathbb{P}(1,1,n)))$$

induced by $(-) \overset{\mathbf{L}}{\otimes}_{F(n)} M$ *is fully faithful.*

3. Categories of Lagrangian vanishing cycles

3.1. *The category of vanishing cycles of an affine Lefschetz fibration.* We begin this section by briefly reviewing Seidel's construction of a Fukaya-type A_∞-category associated to a symplectic Lefschetz fibration [34], [35], [36], following a proposal of Kontsevich [28]. For an account of the underlying physics, the reader is referred to the work of Hori et al. [22].

Let (X, ω) be an open symplectic manifold, and let $f : X \to \mathbb{C}$ be a symplectic Lefschetz fibration, i.e. a C^∞ complex-valued function with isolated nondegenerate critical points $p_1, \dots, p_r$ near which f is given in local complex coordinates by $f(z_1, \dots, z_n) = f(p_i) + z_1^2 + \cdots + z_n^2$, where the fibers of f are symplectic submanifolds of X. Fix a regular value λ_0 of f, and consider an arc $\gamma \subset \mathbb{C}$ joining λ_0 to a critical value $\lambda_i = f(p_i)$. Using the horizontal distribution given by the symplectic orthogonal to the fibers of f, we can transport the Lagrangian vanishing cycle at p_i along the arc γ to obtain a Lagrangian disc $D_\gamma \subset X$ fibered above γ, whose boundary is an embedded Lagrangian sphere L_γ in the fiber $\Sigma_0 = f^{-1}(\lambda_0)$. When the fibers of f are noncompact, parallel transport along the horizontal distribution is not always well-defined; we will always assume that the symplectic form ω satisfies the conditions required to make the construction valid. The Lagrangian disc D_γ is called the *Lefschetz thimble* over γ, and its boundary L_γ is the vanishing cycle associated to the critical point p_i and to the arc γ.

Let $\gamma_1, \dots, \gamma_r$ be a collection of arcs in $\mathbb{C}$ joining the reference point λ_0 to the various critical values of f, intersecting each other only at λ_0, and ordered in the clockwise direction around p_0. Each arc γ_i gives rise to a Lefschetz thimble $D_i \subset X$, whose boundary is a Lagrangian sphere $L_i \subset \Sigma_0$. After a small perturbation we can always assume that these spheres intersect each other transversely inside Σ_0.

Definition 3.1 (Seidel). The *directed category of vanishing cycles* $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ is an A_∞-category (over a coefficient ring R) with r objects $L_1, \dots, L_r$ corresponding to the vanishing cycles (or more accurately to the thimbles); the morphisms between the objects are given by

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j; R) = R^{|L_i \cap L_j|} & \text{if } i < j \\ R \cdot \mathrm{id} & \text{if } i = j \\ 0 & \text{if } i > j; \end{cases}$$

and the differential m_1, composition m_2 and higher order products m_k are defined in terms of Lagrangian Floer homology inside Σ_0. More precisely,

$$m_k : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \mathrm{Hom}(L_{i_{k-1}}, L_{i_k}) \to \mathrm{Hom}(L_{i_0}, L_{i_k})[2-k]$$

is trivial when the inequality $i_0 < i_1 < \cdots < i_k$ fails to hold (i.e. it is always zero in this case, except for m_2 where composition with an identity morphism is given by the obvious formula). When $i_0 < \cdots < i_k$, m_k is defined by fixing a generic ω-compatible almost-complex structure on Σ_0 and counting pseudo-holomorphic maps from a disc with $k+1$ cyclically ordered marked points on its boundary to Σ_0, mapping the marked points to the given intersection points between vanishing cycles, and the portions of boundary between them to $L_{i_0}, \ldots, L_{i_k}$ respectively.

While the general definition of Lagrangian Floer homology is a very delicate task [16], we will only consider cases where most of the technical considerations can be skipped. For example, Seidel considers the case where the symplectic form ω is exact ($\omega = d\theta$ for some 1-form θ) and the L_i are exact Lagrangian submanifolds in Σ_0 (i.e. $\theta_{|L_i} = dg_i$ is also exact). Here, we assume instead that the restricted symplectic form $\omega_{|\Sigma_0}$ is exact and that the homotopy groups $\pi_2(\Sigma_0)$ and $\pi_2(\Sigma_0, L_i)$ are trivial. The first condition prevents the bubbling of pseudo-holomorphic spheres, while the second one prevents the bubbling of pseudo-holomorphic discs in the definition of Lagrangian Floer homology. Therefore, the moduli spaces of pseudo-holomorphic maps involved in the definition of $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ have well-defined fundamental classes.

Another assumption that we will make concerns the Maslov class, which we will assume to vanish over L_i. In fact, we will restrict ourselves to the case where X and Σ_0 are affine Calabi-Yau manifolds, and the spheres L_i can be lifted to graded Lagrangian submanifolds of Σ_0, e.g. by fixing a holomorphic volume form on Σ_0 and choosing a real lift of the phase $\exp(i\phi) = \Omega_{|L_i}/vol_{L_i} : L_i \to S^1$. This makes it possible to define a $\mathbb{Z}$-grading (by Maslov index) on the Floer complexes $CF^*(L_i, L_j; R)$, as will be discussed later (see also [34]).

For simplicity, Seidel uses $R = \mathbb{Z}/2$ as a coefficient ring in his definition; however the moduli spaces considered below are orientable, so that it is possible to assign a sign ± 1 to each pseudo-holomorphic curve and hence define Floer homology over $\mathbb{Z}$. We will further extend the coefficient ring to $R = \mathbb{C}$, and count the contribution of each pseudo-holomorphic disc $u : (D^2, \partial D^2) \to (\Sigma_0, \bigcup L_i)$ in the moduli space with a coefficient of the form $\pm \exp(-2\pi \int_{D^2} u^*\omega)$. Weighting by area is irrelevant in the case of exact Lagrangian vanishing cycles considered by Seidel, where it does not affect at all the structure of the category: indeed, the symplectic areas can then be expressed in terms of the primitives g_i of θ over L_i, and can be eliminated from the description simply by a rescaling of the chosen bases of the Floer complexes (considering the basis $\{\exp(g_i(p) - g_j(p))\, p,\ p \in L_i \cap L_j\}$ of $CF^*(L_i, L_j)$). On the contrary, in the nonexact case it is important to incorporate this weighting by area into the definition.

Hence, given two intersection points $p \in L_i \cap L_j$, $q \in L_j \cap L_k$ $(i < j < k)$, we have by definition

$$m_2(p,q) = \sum_{\substack{r \in L_i \cap L_k \\ \deg r = \deg p + \deg q}} \left(\sum_{[u] \in \mathcal{M}(p,q,r)} \pm \exp(-2\pi \int_{D^2} u^*\omega) \right) r$$

where $\mathcal{M}(p,q,r)$ is the moduli space of pseudo-holomorphic maps u from the unit disc to M (equipped with a generic ω-compatible almost-complex structure) such that $u(1) = p$, $u(\mathrm{j}) = q$, $u(\mathrm{j}^2) = r$ (where $\mathrm{j} = \exp(\frac{2i\pi}{3})$), where we map the portions of the unit circle $[1, \mathrm{j}]$, $[\mathrm{j}, \mathrm{j}^2]$, $[\mathrm{j}^2, 1]$ to L_i, L_j and L_k respectively. The other products are defined similarly.

It is worth mentioning that this definition of Floer homology over the complex numbers is in fact essentially equivalent to the use of coefficients in a Novikov ring, since in both cases the main goal is to keep track of (relative) homology classes.

Although the category $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ depends on the chosen ordered collection of arcs $\{\gamma_i\}$, Seidel has obtained the following result [34]:

THEOREM 3.2 (Seidel). *If the ordered collection $\{\gamma_i\}$ is replaced by another one, $\{\gamma_i'\}$, then the categories $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ and $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i'\})$ differ by a sequence of mutations.*

Hence, the category naturally associated to the Lefschetz fibration f is not the finite directed category defined above, but rather a (bounded) *derived* category, obtained from $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ by consideration of twisted complexes of formal direct sums of Lagrangian vanishing cycles, and addition of idempotent splittings and formal inverses of quasi-isomorphisms. It is a classical result that if two categories are related by mutations, then their derived categories are equivalent; hence the derived category $D(\mathrm{Lag}_{\mathrm{vc}}(f))$ only depends on the Lefschetz fibration f rather than on the choice of an ordered system of arcs [34].

We finish this overview with a couple of remarks. In "usual" Fukaya categories, objects are pairs consisting of a compact Lagrangian submanifold and a flat connection on some complex vector bundle defined over it. In the case of the category associated to a Lefschetz fibration, the objects are vanishing cycles, or perhaps more accurately, the Lefschetz thimbles bounded by the vanishing cycles. Since the thimbles are contractible, all flat vector bundles over them are trivial, which eliminates the need to consider Floer homology with twisted coefficients. This ceases to be true in presence of a nontrivial B-field, but even then the equivalence class of the connection is entirely determined by the thimble. Another related issue is the choice of a spin structure on the vanishing cycles in order to fix the orientation on the moduli spaces: in the one-dimensional case that will be of interest to us, each vanishing cycle admits two distinct spin structures ($H^1(S^1, \mathbb{Z}/2) = \mathbb{Z}/2$). However we must always consider the spin structure which extends to the thimble, i.e. the nontrivial one.

The reader is referred to Seidel's papers [34], [35] for various examples – we will focus specifically on the Landau-Ginzburg models' mirror to weighted projective spaces and Hirzebruch surfaces.

3.2. *Structure of the proof of Theorem* 1.2. Derived categories of coherent sheaves on the weighted projective planes $\mathbb{P}^2(a,b,c)$ and their noncommutative deformations $\mathbb{P}^2_\theta(a,b,c)$ were described in Chapter 2. Hence, to prove Theorem 1.2, we need to find a similar description of the derived categories of Lagrangian vanishing cycles on the mirror Landau-Ginzburg models.

Recall that the mirror to $\mathbb{P}^2_\theta(a,b,c)$ is (X, W), where X is the affine hypersurface $\{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, equipped with an exact (for the commutative case) or nonexact (for the noncommutative case) symplectic form, and the superpotential $W = x + y + z$.

By construction, categories of Lagrangian vanishing cycles for Lefschetz fibrations always admit full exceptional collections. Indeed, for any choice of arcs $\{\gamma_i\}$ the objects L_i of $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ form a generating exceptional collection of the derived category. Hence, in view of Theorem 2.12 and Corollary 2.27, all we need to do is exhibit a set of arcs $\{\gamma_i\}$ for which $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is equivalent to one of the categories $\mathfrak{B}_\theta$ or $\mathfrak{C}_\theta$ introduced in Section 2 (it turns out that the latter choice is slightly easier to achieve).

Recall from Corollary 2.27 that $\mathbf{D}^b(\mathrm{coh}(\mathbb{P}^2_\theta(a,b,c)))$ is equivalent to the derived category of the DG-algebra $C^\bullet_\theta$ associated to the finite DG-category $\mathfrak{C}_\theta$ which has $l = a + b + c$ objects $w_0, \ldots, w_{l-1}$, with morphisms between them given by the complexes

$$\mathrm{Hom}^\bullet(w_j, w_i) \cong (\Lambda^\bullet_\theta)_{i-j}$$

with the natural composition law induced by that of the deformed exterior algebra $\Lambda^\bullet_\theta$ on three generators of degrees $-a, -b, -c$, with relations of the form $\theta_{ij} y_i y_j + \theta_{ji} y_j y_i$ where $\theta \in M(3, \mathbb{C}^*)$ (see §2.6). Moreover, by Corollary 2.20, this category depends only on the quantity

$$q(\theta) = (\theta_{01})^c (\theta_{12})^a (\theta_{20})^b (\theta_{10})^{-c} (\theta_{21})^{-a} (\theta_{02})^{-b}.$$

From a practical viewpoint, the cyclic group $\mathbb{Z}/(a+b+c)$ acts by diagonal multiplication on X, and the superpotential $W = x + y + z$ is equivariant with respect to this action. The $(a+b+c)$ critical values of W form a single orbit under this action (see §4.2). In order to exploit this symmetry, it is therefore natural to choose the smooth fiber $\Sigma_0 = W^{-1}(0)$ as our reference fiber, and an ordered system of arcs $\gamma_i \subset \mathbb{C}$ ($i = 0, \dots, a+b+c-1$) consisting of straight line segments from the origin to the various critical values λ_i.

With this understood, Theorem 1.2 follows immediately from Corollary 2.27 and the following statement:

THEOREM 3.3. $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ *is a* DG *category, and it is equivalent to* $\mathfrak{C}_\theta$ *for any* $\theta \in \mathrm{M}(3, \mathbf{k}^*)$ *such that* $q(\theta) = \exp(2\pi i[B+i\omega]\cdot[T])$, *where* $[B+i\omega] \in H^2(X, \mathbb{C})$ *is the complexified Kähler class, and* $[T]$ *is the generator of* $H_2(X, \mathbb{Z})$.

The proof of Theorem 3.3 consists of several steps, carried out in the various subsections of §4. First, as a prerequisite to the determination of the vanishing cycles, one needs a convenient description of the reference fiber Σ_0. This is done by considering the projection to the first coordinate axis, $\pi_x : \Sigma_0 \to \mathbb{C}^*$, which makes Σ_0 a $(b+c)$-fold covering of $\mathbb{C}^*$ branched in $(a+b+c)$ points (Lemma 4.1). With this understood, it becomes fairly easy to identify the vanishing cycles associated to the arcs γ_j, at least in the special case where the symplectic form is anti-invariant under complex conjugation (which implies its exactness). Indeed, this assumption implies that the vanishing cycles L_j are *Hamiltonian isotopic* (and hence equivalent from the point of view of Floer theory) to the double lifts via π_x of certain arcs $\delta_j \subset \mathbb{C}^*$ (Lemma 4.2) which can be described explicitly (Fig. 5).

With an explicit description of the vanishing cycles at hand, it becomes possible to understand the Floer complexes $CF^*(L_i, L_j)$, by studying the intersections between L_i and L_j for all $0 \le i < j < a+b+c$. By use of the projection to the first coordinate, these correspond to certain specific intersections between the arcs δ_i and δ_j in $\mathbb{C}^*$, as dictated by the combinatorics of the branched covering π_x. Such a description is given by Lemma 4.3, from which it follows readily that $CF^*(L_i, L_j) \simeq (\Lambda_\theta^\bullet)_{i-j}$ for all i, j.

The next step is to study the Floer differentials and products in $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ by counting pseudo-holomorphic maps from $(D^2, \partial D^2)$ to $(\Sigma_0, \bigcup L_i)$. This is done by searching for immersed polygonal regions in Σ_0 with boundary contained in $\bigcup L_i$, or equivalently, images of such regions under the projection π_x (see §4.4). In our case, it turns out that the only possible contributions come from triangular regions in Σ_0; hence, the Floer differential m_1 and the higher compositions $(m_k)_{k\ge 3}$ are identically zero (Lemmas 4.3 and 4.4) for purely topological reasons, while the Floer product m_2 has a particularly simple structure (Lemma 4.5). In particular, the A_∞-category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is actually a DG category with trivial differential.

The grading in $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is determined by the Maslov indices of intersection points. Since the Maslov class vanishes, each L_i can be lifted to a *graded Lagrangian* submanifold of Σ_0 by choosing a real lift of its phase function (see §4.5). The degree of a given intersection point $p \in L_i \cap L_j$ is then determined by the difference between the phases of L_i and L_j at p. Although the determination of phases is the most technical part of the argument, it actually presents little conceptual difficulty, and after some calculations one readily checks that the grading of morphisms in $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is the expected one. Namely, the "generating" morphisms corresponding to the generators of the deformed exterior algebra $\Lambda_\theta^\bullet$ have degree 1, and their pairwise products have degree 2 (cf. Lemma 4.7).

The argument is then completed by determining more precisely the structure coefficients, for the Floer product m_2, which depend on the symplectic areas of the various pseudo-holomorphic discs and on the choice of consistent orientations of the moduli spaces (see §4.6). In the case where the symplectic form is anti-invariant under complex conjugation, the argument is greatly simplified by symmetry considerations, and the Floer products obey the anti-commutation rules of an (undeformed) exterior algebra (Lemma 4.8) – recall that complex conjugation anti-invariance implies exactness of the symplectic form. In the nonexact case or in presence of a nonzero B-field, there is no simple method for determining the symplectic areas of the various pseudo-holomorphic discs involved in the definition of m_2. However the deformation of the category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is governed by a single parameter (analogous to the quantity $q(\theta)$ introduced in Corollary 2.20), for which a simple topological interpretation can be found, involving only the evaluation of $[B + i\omega]$ on the generator of $H_2(X, \mathbb{Z})$ (Lemmas 4.9 and 4.10).

This provides the desired characterization of the category of Lagrangian vanishing cycles, and Theorem 3.3 becomes an easy corollary of Lemmas 4.3–4.10. The only subtle point is that the objects of the category $\mathfrak{C}_\theta$ are numbered "backward" (because the generators of $\Lambda_\theta^\bullet$ are assigned *negative* degrees), so the equivalence of categories actually takes the objects $L_0, \dots, L_{a+b+c-1}$ of $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ to the objects $w_{a+b+c-1}, \dots, w_0$ of $\mathfrak{C}_\theta$.

3.3. *Mirrors of weighted projective lines.* As a warm-up example, we prove HMS for the weighted projective lines $\mathbb{CP}^1(a, b)$, where a, b are mutually prime positive integers (see also [35] and [39]). The argument is an extremely simplified version of that outlined in Section 3.2. Indeed, the mirror Landau-Ginzburg model is the curve $X = \{x^a y^b = 1\} \subset (\mathbb{C}^*)^2$ equipped with the superpotential $W = x + y$, whose generic fiber is just a finite set of $a + b$ points; so most of the considerations that arise in the case of weighted projective planes are irrelevant here (in particular the symplectic structure on X plays no role whatsoever, which is consistent with the fact that the category $\mathrm{coh}(\mathbb{P}_\theta(a, b))$ does not depend on θ).

More precisely, the fiber of W above a point $\lambda \in \mathbb{C}$ is

$$W^{-1}(\lambda) = \{(x, \lambda - x) \in (\mathbb{C}^*)^2,\ x^a(\lambda - x)^b = 1\},$$

which consists of $a + b$ distinct points, unless $P(x) = x^a(\lambda - x)^b - 1$ has a double root. Since

$$P'(x) = \Big(\frac{a}{x} - \frac{b}{\lambda - x}\Big)(P(x)+1),$$

a root of P is a double root if and only if $x = \frac{a}{a+b}\lambda$; hence a double root exists if and only if $P(\frac{a}{a+b}\lambda) = 0$, i.e.

$$\lambda^{a+b} = \frac{(a+b)^{a+b}}{a^a b^b}. \tag{3.1}$$

Let λ_0 be the positive real root of this equation, and let $\lambda_j = \lambda_0 \zeta^{-j}$ where $\zeta = \exp(\frac{2\pi i}{a+b})$: then the critical values of W are exactly $\lambda_0, \dots, \lambda_{a+b-1}$. We choose $\Sigma_0 = W^{-1}(0)$ as our reference fiber, and consider the ordered system of arcs $\gamma_0, \dots, \gamma_{a+b-1}$, where $\gamma_j \subset \mathbb{C}$ is a straight line segment joining the origin to λ_j. With this understood, we have the following result, which implies that HMS holds for $\mathbb{CP}^1(a,b)$:

THEOREM 3.4. $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ *is a* DG *category, equivalent to* $\mathfrak{C}_\theta$ *for any* $\theta \in \mathrm{M}(2, \mathbf{k}^*)$.

In order to prove Theorem 3.4, we study the vanishing cycles of the superpotential W and their intersection properties. To start with, observe that W is equivariant with respect to the diagonal action of the cyclic group $\mathbb{Z}/(a+b)$. Therefore, the vanishing cycles $L_j \subset \Sigma_0$ (which are Lagrangian 0-spheres, i.e. pairs of points) form a single $\mathbb{Z}/(a+b)$-orbit, and $L_j = \zeta^{-j} \cdot L_0$.

In order to determine L_0, we study how the fiber $W^{-1}(\lambda)$ varies as λ increases along the positive real axis (see Fig. 1). For $\lambda = 0$, the fiber Σ_0 consists of $a+b$ points whose first coordinates are the roots of the equation $x^{a+b} = (-1)^b$ (these form a $\mathbb{Z}/(a+b)$-orbit; hence the points of Σ_0 can naturally be identified with the elements of $\mathbb{Z}/(a+b)$ up to a translation). As λ increases towards λ_0, two complex conjugate points of the fiber converge towards each other, and become real points for $\lambda > \lambda_0$. By consideration of the situation for $\lambda \to +\infty$, where the solutions of $x^a(\lambda - x)^b = 1$ split into two groups, one consisting of a roots near the origin, and the other consisting of b roots near λ, one easily checks that the vanishing cycle L_0 consists of the two points of Σ_0 with first coordinate $x = \exp(\pm\frac{i\pi\, b}{a+b})$.

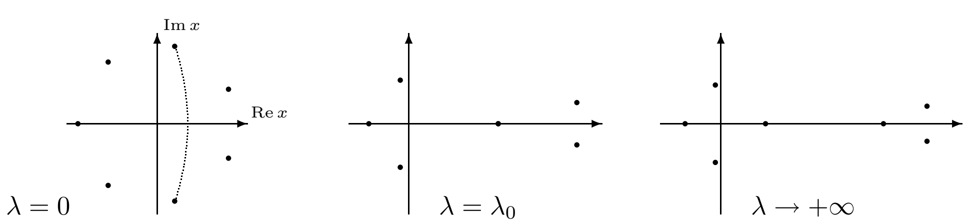

Figure 1: The fiber of W for $\lambda \in \mathbb{R}_+$ $((a,b) = (4,3))$

Hence, for a suitable identification of the fiber Σ_0 with $\mathbb{Z}/(a+b)$, the vanishing cycle associated to the arc $\gamma_0 = [0, \lambda_0]$ is $L_0 = \{0, b\}$. It follows immediately that $L_j = \zeta^{-j} \cdot L_0 = \{-j, b-j\}$ for all $j = 0, 1, \dots, a+b-1$.

Given $0 \leq i < j < a+b$, the vanishing cycles L_i and L_j intersect if and only if the subsets $\{-i, b-i\}$ and $\{-j, b-j\}$ of $\mathbb{Z}/(a+b)$ have nonempty intersection, i.e. if $j = i+a$ or $j = i+b$. Therefore, we have:

LEMMA 3.5. *The direct sum* $\bigoplus_{i<j} CF^*(L_i, L_j)$ *is a free module of total rank* $(a+b)$ *over the coefficient ring, generated by the intersection points*

$$x_i \in CF^*(L_i, L_{i+a}) \quad (0 \leq i < b) \quad \text{and} \quad y_i \in CF^*(L_i, L_{i+b}) \quad (0 \leq i < a).$$

Because Σ_0 is a discrete set, all pseudo-holomorphic curves in Σ_0 must be constant maps. However, each point of Σ_0 occurs exactly once as an intersection between two vanishing cycles (there are no triple intersections), which implies that the Floer differentials and products are trivial. Therefore, we have:

LEMMA 3.6. *The differentials and products* m_k, $k \geq 1$ *in the* A_∞*-category* $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ *are all identically zero, with the exception of the obvious ones* $m_2(\cdot, \mathrm{id})$ *and* $m_2(\mathrm{id}, \cdot)$.

This of course greatly simplifies the argument, eliminating the need for many of the arguments required in the case of higher-dimensional weighted projective spaces. At this point, our only remaining task is to determine the Maslov indices of the various intersection points, by choosing graded Lagrangian lifts of the vanishing cycles. A word of warning is in order here: because we are actually dealing with graded Lagrangian submanifolds in a Calabi-Yau 0-fold, the argument is very specific (see §2 of [35] for a discussion of graded Lagrangian submanifolds of 0-dimensional symplectic manifolds) and does not provide clear understanding of the higher-dimensional case.

LEMMA 3.7. *There exists a natural choice of gradings for which* $\deg(x_i) = \deg(y_i) = 1$.

Proof. View the curve $X = \{x^a y^b = 1\} \subset (\mathbb{C}^*)^2$ as a complex manifold. The holomorphic volume form $d\log x \wedge d\log y$ on $(\mathbb{C}^*)^2$ induces a $(1,0)$-form Ω on X, characterized by the property that it is the restriction to X of a 1-form (which we also call Ω) such that $\Omega \wedge d(x^a y^b) = d\log x \wedge d\log y$. That is, using the fact that $x^a y^b = 1$ along X, we have

$$\Omega \wedge \left(\frac{a}{x}\,dx + \frac{b}{y}\,dy\right) = \frac{dx \wedge dy}{xy}.$$

Outside of the branch points of W, the 1-form Ω can be expressed as $\Theta\,dW$, for some meromorphic function Θ with simple poles at the branch points. The above equation becomes $\Theta(\frac{b}{y} - \frac{a}{x}) = \frac{1}{xy}$, i.e.

$$\Theta = (bx - ay)^{-1} = ((a+b)x - aW)^{-1}.$$

In particular, near $\Sigma_0 = W^{-1}(0)$, we have $\arg\Theta = -\arg x$.

The complex-valued function Θ is (up to scaling by a positive real factor) the natural holomorphic volume form induced by Ω on the 0-dimensional manifold $\Sigma_0 = W^{-1}(0)$. Let $L_0 = \{p_-, p_+\}$, where the x-coordinate of $p_\pm$ is $x_\pm = \exp(\pm\frac{i\pi b}{a+b})$. The *phase* of L_0 is the function $\phi_{L_0} : L_0 \to \mathbb{R}/\pi\mathbb{Z}$ defined by

$$\phi_{L_0}(p_\pm) = \arg\Theta(p_\pm) = \mp\frac{\pi b}{a+b}.$$

Note that an orientation on L_0 determines a lift of ϕ_{L_0} to an $\mathbb{R}/2\pi\mathbb{Z}$-valued function; in order to define the Maslov index, we need to view L_0 as a *graded Lagrangian* submanifold, i.e. to choose a real lift $\tilde\phi_{L_0} : L_0 \to \mathbb{R}$ of the phase function. Although there is *a priori* a $\mathbb{Z}^2$-space of such choices, one has to restrict oneself to only those lifts which are compatible with a graded Lagrangian lift of the Lefschetz thimble D_0 (which reduces the space of choices to $\mathbb{Z}$, as expected since vanishing cycles are only defined up to shifts). If we orient D_0 from p_- towards p_+, then the phase of D_0 (the function $\phi_{D_0} : D_0 \to \mathbb{R}/2\pi\mathbb{Z}$ defined by $\phi_{D_0}(p) = \arg\Omega(v)$ for any $p \in D_0$ and $v \in T_pD_0 - \{0\}$ compatible with the orientation) has the property that

$$\phi_{D_0}(p_-) = \frac{\pi b}{a+b} \quad\text{and}\quad \phi_{D_0}(p_+) = \frac{\pi a}{a+b}.$$

Moreover, it is easy to check that $\phi_{D_0}(p) \in (0,\pi)$ for all $p \in D_0$ (because $\Omega = \frac{1}{b}d\log x$, and $\arg x$ is monotonically increasing along D_0). Hence, there exists a graded Lagrangian lift of D_0 for which the phase function takes values in $(0,\pi)$, which means that we can choose a graded lift of L_0 by setting

$$\tilde\phi_{L_0}(p_-) = \frac{\pi b}{a+b} \quad\text{and}\quad \tilde\phi_{L_0}(p_+) = \frac{\pi a}{a+b}.$$

Arguing similarly for the other vanishing cycles (or using the $\mathbb{Z}/(a+b)$-equivariance), we can choose graded lifts of $L_j = \{p_{j,-}, p_{j,+}\}$ (where $\arg x_{j,\pm} = \frac{1}{a+b}(\pm\pi b - 2\pi j)$) by setting

$$\tilde\phi_{L_j}(p_{j,-}) = \frac{\pi(b+2j)}{a+b} \quad\text{and}\quad \tilde\phi_{L_j}(p_{j,+}) = \frac{\pi(a+2j)}{a+b}.$$

Now, the degree of the morphism x_j, corresponding to $p_{j,+} = p_{j+a,-} \in L_j \cap L_{j+a}$, is given by the difference of phases:

$$\deg x_j = \frac{1}{\pi}(\tilde\phi_{L_{j+a}}(p_{j+a,-}) - \tilde\phi_{L_j}(p_{j,+})) = \frac{b+2(j+a)}{a+b} - \frac{a+2j}{a+b} = 1.$$

Similarly for y_j:

$$\deg y_j = \frac{1}{\pi}(\tilde\phi_{L_{j+b}}(p_{j+b,+}) - \tilde\phi_{L_j}(p_{j,-})) = \frac{a+2(j+b)}{a+b} - \frac{b+2j}{a+b} = 1. \quad \square$$

Theorem 3.4 now follows immediately from Lemmas 3.5–3.7; as in the case of weighted projective planes, the only difference between the DG-categories $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ and $\mathfrak{C}_\theta$ is that the objects of $\mathfrak{C}_\theta$ are numbered "backward", so that the equivalence of categories takes the objects $L_0, \dots, L_{a+b-1}$ of $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ to the objects $w_{a+b-1}, \dots, w_0$ of $\mathfrak{C}_\theta$.

4. Mirrors of weighted projective planes

4.1. *The mirror Landau-Ginzburg model and its fiber* Σ_0. The mirror to the weighted projective plane $\mathbb{CP}^2(a, b, c)$ is the affine hypersurface $X = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, equipped with the superpotential $W = x + y + z$, and a symplectic form ω that we leave unspecified for the moment. During most of the argument, we will assume ω to be anti-invariant under complex conjugation (which implies exactness) and invariant under the diagonal action of the cyclic group $\mathbb{Z}/(a + b + c)$, but these assumptions will be weakened at the end. Of course, since X is noncompact, we also need to choose ω in such a way as to ensure that the Lefschetz thimbles and vanishing cycles considered below are well-defined. It is easy to check that, among many other possibilities, a symplectic form such as

$$\omega = i \sum_{i,j=1}^{3} a_{ij} \frac{dz_i}{z_i} \wedge \frac{d\bar{z}_j}{\bar{z}_j}$$

(where (a_{ij}) is a positive definite Hermitian matrix, with real coefficients if we require complex conjugation anti-invariance) generates a horizontal distribution for which parallel transport is well-defined, because, with respect to the induced Kähler metric, X is complete and the gradient vector of W has norm bounded from below outside of a compact set.

Topologically, X is just a complex torus $(\mathbb{C}^*)^2$, at least if $\delta = \gcd(a, b, c) = 1$; otherwise X is disconnected, and each of its δ components is a complex torus.

For each $\lambda \in \mathbb{C}$, the fiber $\Sigma_\lambda = W^{-1}(\lambda) \subset X$ is an affine curve given by the equation $x^a y^b (\lambda - x - y)^c = 1$; this curve is smooth unless λ is one of the $a + b + c$ critical values of W. We will view Σ_λ as a branched covering of $\mathbb{C}^*$, by projecting to the x axis (this choice is arbitrary, and we will occasionally use the symmetry between the variables x, y, z in the argument). For a generic value of $x \in \mathbb{C}^*$, the polynomial $x^a y^b (\lambda - x - y)^c - 1$ of degree $b+c$ in the variable y admits $b + c$ distinct simple roots; therefore, the projection $\pi_x : \Sigma_\lambda \to \mathbb{C}^*$ is a $(b+c)$-fold covering. The branch points of π_x are those values of x for which there is a double root, i.e. a value of y such that $P(y) = x^a y^b (\lambda - x - y)^c = 1$ and $P'(y) = 0$. Since

$$\frac{P'(y)}{P(y)} = \frac{b}{y} - \frac{c}{\lambda - x - y},$$

the condition $P'(y) = 0$ implies that $cy = b(\lambda - x - y)$, i.e. $y = \frac{b}{b+c}(\lambda - x)$. Substituting into the equation of Σ_λ, we obtain the equation

$$x^a(\lambda - x)^{b+c} = \frac{(b+c)^{b+c}}{b^b\, c^c} \tag{4.1}$$

for the branch points of π_x. Since this is a polynomial equation of degree $a+b+c$, for a generic value of λ there are $a+b+c$ distinct branch points, all of which are simple (i.e. isolated nondegenerate critical points of π_x).

In the remainder of this section, we set $\lambda = 0$, and describe the curve Σ_0 in detail, by computing the monodromy of the $(b+c)$-fold branched covering $\pi_x : \Sigma_0 \to \mathbb{C}^*$ around the origin and around its $a+b+c$ branch points.

Lemma 4.1. *The fiber of $\pi_x : \Sigma_0 \to \mathbb{C}^*$ can be identified with $\mathbb{Z}/(b+c)$ in such a way that the monodromy of π_x around the origin in $\mathbb{C}^*$ is given by $q \mapsto q - a$, and the monodromies around the $a+b+c$ branch points are given by the transpositions $(j, j+b)$, $0 \le j < a+b+c$ (see Fig. 2).*

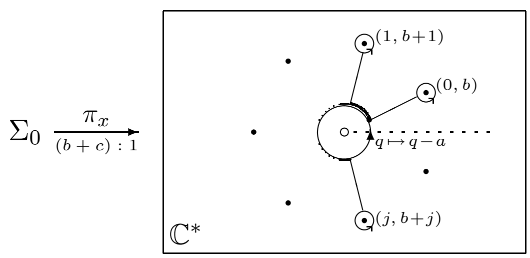

Figure 2: The projection $\pi_x : \Sigma_0 \to \mathbb{C}^*$ (of degree $b+c$, with $a+b+c$ branch points).

To understand this statement, first observe that, when $x = \varepsilon e^{i\theta}$ is close to 0, the $b+c$ roots of the equation

$$x^a y^b(-x-y)^c = 1 \tag{4.2}$$

lie close to those of the equation

$$(-1)^c y^{b+c} = \varepsilon^{-a} e^{-ia\theta}.$$

Hence, we can choose an identification of the fiber of π_x above a small real positive value $x = \varepsilon$ (or any other $\varepsilon e^{i\theta}$ fixed in advance) with the cyclic group $\mathbb{Z}/(b+c)$ in a manner compatible with the cyclic ordering of the points. Moreover, varying θ from 0 to 2π, we obtain that the monodromy of π_x around the origin is given by the translation $q \mapsto q - a$ in $\mathbb{Z}/(b+c)$ (i.e., the permutation sending the root y_q of $x^a y^b(-x-y)^c = 1$ to y_{q-a}).

Next, consider a critical value of π_x, i.e. a root x_0 of (4.1) for $\lambda = 0$, and the radial half-line ℓ through x_0, i.e. the set of all $x \in \mathbb{C}^*$ with argument equal to $\theta_0 = \arg x_0$. Moving x along ℓ starting from a point $x_* = \varepsilon e^{i\theta_0}$ close to the origin, two of the $b+c$ roots of (4.2) become equal to each other as x approaches x_0; this determines the monodromy of π_x around x_0, namely a transposition in the symmetric group S_{b+c} acting on a fiber of π_x. We claim that, identifying the fiber $\pi_x^{-1}(x_*)$ with $\mathbb{Z}/(b+c)$ as above, this transposition exchanges two elements q_0 and $q_0 + b$. This can be seen as follows.

Assume for simplicity that $b+c$ is even and that x_0 is the positive real root of (4.1) for $\lambda = 0$; the general case is handled similarly, inserting factors $e^{i\theta_0}$ where needed. For $x \to 0$, as explained above, the $b+c$ roots of (4.2) are close to those of

$$y^{b+c} = (-1)^c x^{-a},$$

i.e. $b+c$ evenly spaced points on a circle (Fig. 3, left). As x increases, two complex conjugate roots $y, \bar{y}$ approach the real axis and eventually become equal for $x = x_0$ (Fig. 3, center), so that there are two additional real roots for $x > x_0$. As $x \to +\infty$, the roots of (4.2) are divided into two groups, b roots close to the origin, approximated by those of

$$y^b = (-1)^c x^{-(a+c)},$$

and c roots close to $-x$, corresponding to values of $z = -x - y$ close to the origin and approximated by the roots of

$$z^c = (-1)^b x^{-(a+b)}$$

(Fig. 3, right). Hence, when we identify the fiber of π_x for x small with $\mathbb{Z}/(b+c)$ in a manner compatible with the cyclic ordering, the two points which merge for $x = x_0$ (the vanishing cycle of π_x at x_0) differ from each other by exactly b (this can also be checked by numerical experimentation).

The above argument shows that the monodromy around one of the branch points x_0 of π_x, e.g. the branch point located on the positive real axis or immediately above it, is a transposition (q_0, q_0+b); changing the identification between the reference fiber of π_x above x_* and the cyclic group $\mathbb{Z}/(b+c)$ if necessary, we can assume that $q_0 = 0$.

We now find the monodromy around the other branch points of π_x. For this purpose, observe that the group $G = \mathbb{Z}/(a+b+c)$ acts on X by $(x, y, z) \mapsto$

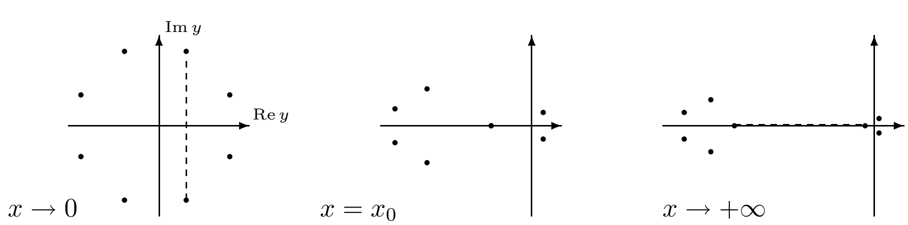

Figure 3: The roots of $x^a y^b (-x-y)^c = 1$ for $x \in \mathbb{R}_+$ $((a, b, c) = (1, 3, 5))$

$(x\zeta^j, y\zeta^j, z\zeta^j)$, where $\zeta = \exp(\frac{2\pi i}{a+b+c})$, and that this action preserves Σ_0, mapping the fiber of π_x above x to the fiber above $x\zeta^j$. Hence, denoting by y', y'' the two points of the fiber above $x_* = \varepsilon e^{i\theta_0}$ which converge to each other as x moves radially outward to x_0 (those labelled 0 and b), we know that the two points of the fiber above $x_*\zeta^j$ which converge to each other as x moves radially outward to $x_0\zeta^j$ are $y'\zeta^j$ and $y''\zeta^j$. We now transport these two values of y from the fiber $\pi_x^{-1}(x_*\zeta^j)$ to $\pi_x^{-1}(x_*)$ along the arc $x(t) = x_* e^{2\pi i t}$ for $t \in [0, \frac{j}{a+b+c}]$. Approximating the $b+c$ points of $\pi_x^{-1}(\varepsilon e^{i\theta})$ by the roots of $(-1)^c y^{b+c} = \varepsilon^{-a} e^{-ia\theta}$, the parallel transport along the considered arc induces a multiplication by $\exp(2\pi i \frac{a}{b+c}\frac{j}{a+b+c})$. Observing that

$$\zeta^j \exp(2\pi i \tfrac{j\,a}{(b+c)(a+b+c)}) = \exp(2\pi i \tfrac{j}{b+c}),$$

we obtain that the two points of $\pi_x^{-1}(x_*)$ which become equal as x is moved first counterclockwise around the origin and then radially outward to $x_0\zeta^j$ are those which correspond to the elements j and $b+j$ of $\mathbb{Z}/(b+c)$. Hence, the monodromy of π_x around $x_0\zeta^j$ (joining x_* to $x_0\zeta^j$ in the prescribed way) is the transposition $(j, b+j)$, which completes the proof of Lemma 4.1. By the way, note that the comparison between the values $j = 0$ and $j = a+b+c$ is consistent with our determination of the monodromy around $x = 0$.

4.2. *The vanishing cycles.* Now that the fiber Σ_0 is well-understood, we compute the vanishing cycles of the Lefschetz fibration $W : X \to \mathbb{C}$ by studying the degeneration of Σ_λ as λ approaches a critical value of W.

The curve Σ_λ becomes singular when two branch points of the projection $\pi_x : \Sigma_\lambda \to \mathbb{C}^*$ merge with each other, giving rise to a nodal point. This occurs whenever (4.1) admits a double root. Considering the logarithmic derivative of the left-hand side, we obtain the relation $\frac{a}{x} - \frac{b+c}{\lambda - x} = 0$, which leads to $x = \frac{a}{a+b+c}\lambda$ for a double root of (4.1), and substituting we obtain the equation

$$\lambda^{a+b+c} = \frac{(a+b+c)^{a+b+c}}{a^a\, b^b\, c^c} \tag{4.3}$$

for the $a+b+c$ critical values of W (this equation can also be obtained directly).

For symmetry and for simplicity, we will choose the smooth curve $\Sigma_0 = W^{-1}(0)$ as our reference fiber of the Lefschetz fibration $W : X \to \mathbb{C}$, and we will choose straight line segments for the arcs γ_j joining the origin to the various critical values $\lambda_j = \lambda_0 \zeta^{-j}$ of W $(0 \le j < a+b+c)$, where λ_0 is the real positive root of (4.3) and $\zeta = \exp(\frac{2\pi i}{a+b+c})$. Hence, in order to construct the category of Lagrangian vanishing cycles of W, we need to understand how the smooth fiber Σ_0 above the reference point 0 degenerates to the nodal curve Σ_{λ_j} when λ moves radially from 0 to λ_j.

We first consider the motion of the branch points of π_x as λ increases along the positive real axis from 0 to the critical value λ_0. For each value of

λ, the $a+b+c$ branch points are given by the roots of (4.1). When $\lambda = 0$, they all lie on a circle centered at the origin, as represented in Figure 2. As $\lambda \to \lambda_0$, two complex conjugate branch points converge to each other, so that for $\lambda = \lambda_0$ the equation (4.1) has a double root $x = \frac{a}{a+b+c}\lambda_0$ on the positive real axis (Fig. 4, center). Finally, for $\lambda \to +\infty$, the roots of (4.1) split into two groups, one of a points close to the origin that can be approximated by the roots of $x^a = K_{b,c}\lambda^{-(b+c)}$ (where $K_{b,c} = b^{-b}c^{-c}(b+c)^{b+c}$), and one of $b+c$ points close to λ for which $\xi = \lambda - x$ can be approximated by the roots of $\xi^{b+c} = K_{b,c}\lambda^{-a}$ (Fig. 4, right). Hence, it can be checked that the two branch points of $\pi_x : \Sigma_0 \to \mathbb{C}^*$ which merge for $\lambda \to \lambda_0$ are those with argument $\arg x = \pm\frac{b+c}{a+b+c}\pi$, and that the projection to $\mathbb{C}^*$ of the corresponding vanishing cycle is an arc δ_0 which is symmetric with respect to the real axis, intersects it only once in its positive part, and remains everywhere inside the circle containing the critical values of π_x (Fig. 4, left).

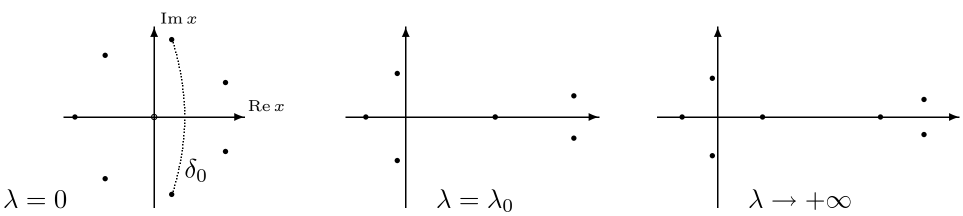

Figure 4: The branch points of π_x for $\lambda \in \mathbb{R}_+$ $((a,b,c) = (4,2,1))$

More precisely, the above discussion gives us a *topological* description of the vanishing cycle $L_0 \subset \Sigma_0$, up to homotopy. Namely, two of the $b+c$ lifts to Σ_0 of the arc $\delta_0 \subset \mathbb{C}^*$ have common end points (the ramification points of π_x lying above the end points of δ_0), and their union forms a closed loop L_0' in Σ_0. This loop is a topological vanishing cycle; i.e., it shrinks to a point in Σ_λ when $\lambda \to \lambda_0$, but *a priori* it is only homotopic to the symplectic vanishing cycle L_0 (obtained by parallel transport using the symplectic connection).

The actual position of the vanishing cycle L_0 inside Σ_0 depends on the choice of the symplectic form ω on X; for a given ω it can be calculated numerically (and it can be checked that for "reasonable" choices of ω, L_0 and L_0' intersect all other vanishing cycles in the same manner). However, this calculation is unnecessary for our purposes. Indeed, if we endow X with a symplectic form that is anti-invariant by complex conjugation, then the vanishing cycle L_0 is invariant by complex conjugation, i.e. complex conjugation maps L_0 to itself in an orientation-preserving manner, and the same is true of L_0'. Since L_0 and L_0' are homotopic to each other in Σ_0, their (oriented) invariance under complex conjugation is sufficient to imply that they are *Hamiltonian isotopic*, which means that for the purpose of determining categories of vanishing cycles, L_0 and L_0' are interchangeable.

If we deform ω to a nonexact form, complex conjugation invariance is lost. The intersection patterns between vanishing cycles remain the same for small deformations (and can be forced to remain the same even for large deformations by performing suitable Hamiltonian isotopies), but the calculation of the coefficient assigned to a given pseudo-holomorphic curve involves its symplectic area and hence requires one to work with the actual vanishing cycles rather than their topological approximations. Hence, we may obtain nontrivial deformations of the category of vanishing cycles; however, these deformations only amount to modifications of the structure constants of the products m_k, rather than changes in the Floer complexes themselves or in the types of pseudo-holomorphic curves that may arise.

In any case, except at the very end of the argument, we will always consider symplectic forms that are anti-invariant under complex conjugation, in which case the approximation of L_0 by L'_0 is legitimate.

We now consider the other vanishing cycles L_j of the Lefschetz fibration W. Recall that the group $G = \mathbb{Z}/(a+b+c)$ acts on X, in a manner that preserves Σ_0; moreover, $W : X \to \mathbb{C}$ is G-equivariant. If we assume the symplectic form ω to be G-invariant, the symplectic connection and the associated parallel transport will also be G-equivariant. Therefore, since the arc $\gamma_j \subset \mathbb{C}$ joining the origin to $\lambda_j = \lambda_0\zeta^{-j}$ is the image of γ_0 by the action of ζ^{-j} (where $\zeta = \exp(\frac{2\pi i}{a+b+c})$), the same is true of the corresponding Lefschetz thimbles, and hence of the vanishing cycles in Σ_0. This gives us a description of L_j for all values of j. As in the case of L_0, we will consider, rather than L_j itself, a loop $L'_j \subset \Sigma_0$ which is homotopic to L_j and can be obtained as a double lift via $\pi_x : \Sigma_0 \to \mathbb{C}^*$ of an embedded arc $\delta_j \subset \mathbb{C}^*$. The loop L'_j is defined to be the image of L'_0 by the action of ζ'_j, which means that δ_j is the image of δ_0 by a rotation of angle $-\frac{2\pi j}{a+b+c}$. If, in addition to its G-invariance, ω is assumed to be anti-invariant under complex conjugation, then L'_j is Hamiltonian isotopic to L_j, so that we can work with L'_j instead of L_j.

Hence, to summarize the above discussion, we have the following lemma:

LEMMA 4.2. *The vanishing cycles* $L_j \subset \Sigma_0$ $(0 \leq j < a+b+c)$ *are homotopic* (*and*, *if* ω *is invariant under the action of* $\mathbb{Z}/(a+b+c)$ *and anti-invariant under complex conjugation*, *Hamiltonian isotopic*) *to closed loops* $L'_j \subset \Sigma_0$ *which project by* π_x *to arcs* $\delta_j \subset \mathbb{C}^*$ *as represented in Figure* 5 (*the end points of* δ_j *are the branch points of* π_x *for which* $\arg x = -2\pi\frac{j}{a+b+c} \pm \pi\frac{b+c}{a+b+c}$).

In the following sections, we assume that ω is $\mathbb{Z}/(a+b+c)$-invariant and anti-invariant under complex conjugation, and we implicitly identify L_j with L'_j.

4.3. *The Floer complexes.* The objects of the category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_j\})$ are described by Lemma 4.2; we now determine its morphisms by studying the

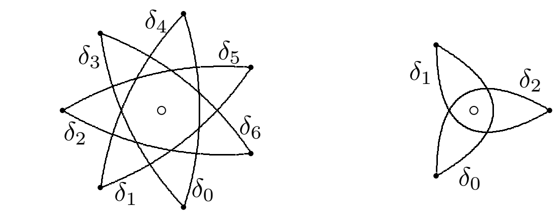

Figure 5: The vanishing cycles $L_j \subset \Sigma_0$

intersections between the closed loops $L_j \subset \Sigma_0$. This simply involves looking carefully at Figures 2 and 5 in order to determine, among the intersections between δ_i and δ_j, which ones lift to intersections between L_i and L_j.

Lemma 4.3. *The direct sum* $\bigoplus_{i<j} CF^*(L_i, L_j)$ *is a free module of total rank* $3(a+b+c)$ *over the coefficient ring, generated by the following intersection points*:

$$\begin{array}{llll} x_i \in CF^*(L_i, L_{i+a}) & (0 \le i < b+c), & \bar{x}_i \in CF^*(L_i, L_{i+b+c}) & (0 \le i < a), \\ y_i \in CF^*(L_i, L_{i+b}) & (0 \le i < a+c), & \bar{y}_i \in CF^*(L_i, L_{i+a+c}) & (0 \le i < b), \\ z_i \in CF^*(L_i, L_{i+c}) & (0 \le i < a+b), & \bar{z}_i \in CF^*(L_i, L_{i+a+b}) & (0 \le i < c). \end{array}$$

Moreover, the Floer differential is trivial, i.e. $m_1 = 0$.

To determine $CF^*(L_i, L_j)$ for given $0 \le i < j < a+b+c$, one must look for intersection points between the projected arcs δ_i and δ_j. The arcs δ_i and δ_j intersect only if $j-i \le b+c$ or $j-i \ge a$; in all other cases, $\delta_i \cap \delta_j = \emptyset$ and hence $CF^*(L_i, L_j) = 0$. More precisely, $\delta_i \cap \delta_j$ contains one point if $j-i \le b+c$, and one point if $j-i \ge a$; if both conditions hold simultaneously, then $|\delta_i \cap \delta_j| = 2$ (see Lemma 4.2 and Figure 5). Moreover, if equality holds ($j-i = b+c$ or $j-i = a$), then the corresponding intersection occurs at an end point of δ_i and δ_j, i.e. a branch point of π_x. In this case, the intersection of δ_i and δ_j always lifts to a transverse intersection of L_i and L_j, at the corresponding critical point of π_x; this accounts for the generators x_i and $\bar{x}_i$ mentioned in the statement of Lemma 4.3.

When $j-i < b+c$ or $j-i > a$, we need to consider the structure of the branched covering π_x in order to determine whether intersections between δ_i and δ_j lift to intersections between L_i and L_j. Call p_i the branch point of π_x with argument $\arg x = -2\pi\frac{j}{a+b+c} - \pi\frac{b+c}{a+b+c}$, which is an end point of δ_i, and define similarly p_j. When $j-i < b+c$, consider the corresponding intersection point $q \in \delta_i \cap \delta_j$, and use the arcs joining p_j to q in δ_j and q to p_i in δ_i to define an arc $\eta \subset \mathbb{C}^*$ joining p_j to p_i, with a rotation angle of $2\pi\frac{j-i}{a+b+c}$ around the origin. It follows from Lemma 4.1 (cf. also Fig. 2) that, over a neighborhood of η, we can consistently label the sheets of the covering π_x by elements of $\mathbb{Z}/(b+c)$, in such a way that the monodromies around the branch points p_i and p_j are transpositions of the form (k_i, k_i+b) and (k_j, k_j+b), with $k_i - k_j = j - i$. Hence, near the point q, the vanishing cycle L_i lies in the two sheets of π_x labelled k_i and k_i+b, and similarly for L_j; the intersections of L_i with L_j above q correspond to the elements of $\{k_i, k_i+b\} \cap \{k_j, k_j+b\}$. Since $0 < k_i - k_j = j - i < b + c$, this intersection is empty unless $k_i = k_j + b \bmod b+c$, i.e. $j - i = b$, which corresponds to the generator y_i of the Floer complex, or $k_j = k_i + b \bmod b+c$, i.e. $j-i=c$, which corresponds to the generator z_i. When $j - i > a$, one proceeds similarly, introducing an arc in $\mathbb{C}^*$ joining p_j to p_i through the relevant intersection point q' of δ_i with δ_j, with a rotation angle of $2\pi(\frac{j-i}{a+b+c} - 1)$ around the origin. The sheets of π_x containing L_i and L_j above the intersection point q' are now labelled k'_i, $k'_i + b$ and $k'_j, k'_j + b$, with k'_i and k'_j two constants in $\mathbb{Z}/(b+c)$ such that $k'_i - k'_j = j - i - (a+b+c) = j - i - a \bmod b+c$. Therefore, the two cases where L_i and L_j intersect above q' are when $i + j = a + b$, which corresponds to the generator z'_i of the Floer complex, and when $i+j = a+c$, which corresponds to y'_i.

At this point it is worth observing that, for generic values of (a,b,c), each Floer complex $CF^*(L_i, L_j)$ has total rank at most one, so that the Floer differential is necessarily zero. However, for specific values of (a,b,c) we may have numerical coincidences leading to more than one intersection between two vanishing cycles; the most striking example is that of the usual projective plane, $(a,b,c) = (1,1,1)$, for which $|L_i \cap L_j| = 3\ \forall i < j$ (cf. Fig. 5). Nonetheless, even in these cases, the Floer differential vanishes, because L_i and L_j always realize the minimal geometric intersection number between closed loops in their homotopy classes, as can be checked by enumerating the various possible cases. This minimality of intersection implies that Σ_0 contains no nonconstant immersed disc with boundary in $L_i \cup L_j$, and hence that the Floer differential vanishes.

Another way to prove the vanishing of the Floer differential is to endow Σ_0 and $\mathbb{C}^*$ with almost-complex structures which make the projection π_x holomorphic, and to observe that the projection to $\mathbb{C}^*$ of a pseudo-holomorphic disc in Σ_0 with boundary in $L_i \cup L_j$ is a pseudo-holomorphic disc in $\mathbb{C}^*$ with boundary in $\delta_i \cup \delta_j$. If $|\delta_i \cap \delta_j| = 1$, the maximum principle implies that the projected pseudo-holomorphic disc is a constant map, and hence that the disc in Σ_0 is contained in a fiber of π_x, which implies that it is also constant. If $|\delta_i \cap \delta_j| = 2$, one reaches the same conclusion by observing the respective positions of the two intersection points in $\mathbb{C}^*$ (a nonconstant disc would have to pass through the origin). As before, one concludes that the absence of nontrivial pseudo-holomorphic discs makes the Floer differential identically zero, which completes the proof of Lemma 4.3.

4.4. *The product structures.* The aim of this section is to prove the following results concerning the category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_j\})$:

LEMMA 4.4. *The higher products m_k $(k \geq 3)$ are all identically zero.*

LEMMA 4.5. *There exist nonzero constants $\alpha_{uv,i}$ such that*

$$\begin{aligned} m_2(x_i, y_{i+a}) &= \alpha_{xy,i}\, \bar{z}_i, & m_2(x_i, z_{i+a}) &= \alpha_{xz,i}\, \bar{y}_i, \\ m_2(y_i, z_{i+b}) &= \alpha_{yz,i}\, \bar{x}_i, & m_2(y_i, x_{i+b}) &= \alpha_{yx,i}\, \bar{z}_i, \\ m_2(z_i, x_{i+c}) &= \alpha_{zx,i}\, \bar{y}_i, & m_2(z_i, y_{i+c}) &= \alpha_{zy,i}\, \bar{x}_i. \end{aligned}$$

All other compositions (*except those involving identity morphisms*) *vanish.*

These results follow from careful observation of the boundary structure of a pseudo-holomorphic disc in Σ_0 with boundary in $\bigcup L_j$. Endow Σ_0 with any almost-complex structure, and let $u : D^2 \to \Sigma_0$ be a pseudo-holomorphic map from the disc with $k+1 \geq 3$ marked points on its boundary to Σ_0, mapping each segment on the boundary to an arc in one of the Lagrangian submanifolds L_j. Each "corner" of the image of u corresponds to an intersection point between two of the vanishing cycles, and as such it corresponds to a generator of the Floer complex.

According to Lemma 4.3, we can classify the generators of the Floer complex into three families, those of *type* x (corresponding to generators $x_i, \bar{x}_i$), those of *type* y (generators $y_i, \bar{y}_i$), and those of *type* z (generators $z_i, \bar{z}_i$). Moreover, observe that the total intersection of each L_i with all other vanishing cycles consists of 6 points, two of each type: depending on the value of i, L_i is either the source of the morphism x_i or the target of $\bar{x}_{i-b-c}$, and it is either the source of $\bar{x}_i$ or the target of x_{i-a}; similarly for types y and z.

The manner in which these points are arranged along the loop L_i can be seen easily by looking at Figure 5 and recalling the discussion in the previous section. Recall that L_i passes through two branch points of π_x, which split it into two halves (lifts of δ_i lying in different sheets of π_x). One of these branch points corresponds to x_i or $\bar{x}_{i-b-c}$, while the other corresponds to $\bar{x}_i$ or x_{i-a}. In between them, we have, on one half of L_i, one intersection of type y (either y_i or $\bar{y}_{i-a-c}$) and one of type z (either $\bar{z}_i$ or z_{i-c}); on the other half of L_i, we have similarly one intersection of type y (either $\bar{y}_i$ or y_{i-b}) and one of type z (either z_i or $\bar{z}_{i-a-b}$). This structure is summarized in Figure 6.

An important property is that, for every one of the six portions of L_i delimited by these intersection points, one of the two immediately adjacent components of $\Sigma_0 - \bigcup L_j$ (on either side of L_i) is unbounded (it is denoted by 0 or ∞ in Figure 6 depending on whether its image under π_x contains the origin or the point at infinity in $\mathbb{C}^*$). These unbounded components form an alternating pattern around L_i, changing sides (left or right) every time one of the intersection points is crossed.

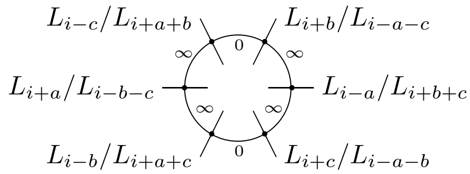

Figure 6: The intersections of L_i with the other vanishing cycles

On the other hand, the image of the pseudo-holomorphic map u cannot intersect any of the unbounded components of $\Sigma_0 - \bigcup L_j$, because otherwise the maximum principle would imply that the image of u is unbounded. This imposes very strong constraints on the behavior of u along the boundary of the disc. Namely, consider two consecutive marked points ("corners"), such that the portion of boundary ("edge") in between them is mapped to an arc η (oriented according to the boundary orientation of the unit disc) contained in the vanishing cycle L_i. Then, η is exactly one of the six portions of L_i delimited by its intersections with the other vanishing cycles, and its orientation is determined by the requirement that the component of $\Sigma_0 - \bigcup L_j$ immediately to the left of η be bounded (see Fig. 6). Moreover, the local behavior of u at an end point p of η is "convex", i.e. u locally maps into only one of the four regions delimited locally by the two vanishing cycles meeting at p. In other words, the boundary of $\mathrm{Im}(u)$ is an oriented piecewise smooth curve $\theta \subset \bigcup L_j$ which always turns left at every intersection point it encounters. This boundary behavior has several important consequences.

Lemma 4.6. *Among any three consecutive corners of the image of u, there is always exactly one of each type x, y, z.*

Proof. Observe that two consecutive corners of the image of u are necessarily of different types (because two adjacent intersections of L_i with other vanishing cycles are always of different types). Let p, q, r be three consecutive corners of the image of u, such that the edge from p to q lies in a vanishing cycle L_i and the edge from q to r lies in a vanishing cycle L_j. The knowledge of the types of the points p and q completely determines them, which in turn determines the type of r. For example, if p is of type y and q is of type z, then on the diagram of Figure 6 the edge joining them is the lowermost portion of L_i; in particular the edge from p to q is adjacent to an unbounded component of Σ_0 whose image under π_x contains the origin. When we consider the intersection diagram for L_j (similar to Figure 6), the point q can be located by comparison with the diagram for L_i (in our example, q is the point to the upper left of the diagram). Moreover, the direction from which θ reaches q can be determined by identifying the unbounded component to which it is adjacent (in our example, the component whose image under π_x contains the origin, so that θ reaches q from the innermost side of the diagram); since θ turns left at q, this determines the edge from q to r and hence the type of r (in our example, r is the left-most point on the intersection diagram, and hence of type x). It can be checked easily that in all six cases, the type of r is different from those of p and q. $\square$

Next, recall that by definition the successive edges of the image of u lie inside vanishing cycles $L_{i_0}, L_{i_1}, \dots, L_{i_k}$ with $i_0 < i_1 < \cdots < i_k$ (see Def. 3.1), and observe that following θ at a corner of u leads from a vanishing cycle L_i to another vanishing cycle L_j, with $i < j$ if and only if the intersection point is x_i, y_i or z_i, and $i > j$ if and only if the intersection point is $\bar{x}_j$, $\bar{y}_j$ or $\bar{z}_j$ (see Fig. 6). Therefore, all corners of u but one correspond to generators of the Floer complexes among $\{x_i, y_i, z_i\}$, while the last corner (between the edge on L_{i_k} and the edge on L_{i_0}) corresponds to a generator among $\{\bar{x}_i, \bar{y}_i, \bar{z}_i\}$.

With this observation, Lemma 4.4 follows immediately from Lemma 4.6. Indeed, assume that there exists a pseudo-holomorphic map u from a disc with $k+1$ marked points to Σ_0, with edges lying in vanishing cycles $L_{i_0}, L_{i_1}, \dots, L_{i_k}$ $(0 \le i_0 < i_1 < \cdots < i_k < a+b+c)$, contributing to the product m_k, for some $k \ge 3$. Among the first three corners of u, one is among the generators x_i, one is among the y_i, and one is among the z_i. Therefore, $i_3 = i_0 + a + b + c$, which contradicts the inequality $i_3 < a + b + c$. Hence the moduli spaces of pseudo-holomorphic curves involved in the definition of m_k are all empty for $k \ge 3$, which implies that $m_k = 0$.

Lemma 4.5 also follows immediately at this point: in the case of a pseudo-holomorphic map u from a disc with three marked points, the three corners p, q, r are all of different types (by Lemma 4.6), and the first two corners p, q correspond to generators among $\{x_i, y_i, z_i\}$ while the last one r corresponds to a generator among $\{\bar{x}_i, \bar{y}_i, \bar{z}_i\}$. Therefore, p and q completely determine r, and moreover it is easy to check from the above discussion and from Figures 5 and 6 that the image of the pseudo-holomorphic map u is also uniquely determined by the pair (p, q). For example, if p is of type x and q is of type y, then necessarily there exists $i < c$ such that $p = x_i$, $q = y_{i+a}$, and $r = \bar{z}_i$; moreover, it is easy to check (see Lemma 4.2 and Fig. 5) that the moduli space determining the coefficient of $\bar{z}_i$ in $m_2(x_i, y_{i+a})$ consists of a single curve, regular, whose image $T_{xy,i}$ is the triangular region of Σ_0 delimited by arcs joining p, q, r in the vanishing cycles L_i, L_{i+a}, L_{i+a+b}. Therefore, we have $m_2(x_i, y_{i+a}) = \alpha_{xy,i}\, \bar{z}_i$, where $\alpha_{xy,i} = \pm \exp(-\mathrm{Area}(T_{xy,i}))$. The situation is the same in all other cases.

Remark. The $a+b+c$ triangles $T_{xy,i}$ $(i < c)$, $T_{yz,i}$ $(i < a)$, $T_{zx,i}$ $(i < b)$ are all related to each other via the action of the cyclic group $\mathbb{Z}/(a+b+c)$. Indeed, the diagonal multiplication by a power of $\zeta = \exp(\frac{2\pi i}{a+b+c})$ induces a permutation of the vanishing cycles and of the intersection points, preserving the cyclic ordering of the L_i and the types of their intersection points, and hence mapping every triangle in Σ_0 with boundary in $\bigcup L_i$ to another such triangle. A similar description holds for the triangles $T_{yx,i}$, $T_{zy,i}$, $T_{xz,i}$.

4.5. *Maslov index and grading.* The aim of this section is to define a $\mathbb{Z}$-grading on the Floer complexes $CF^*(L_i, L_j)$, and to compute the degree of the various generators. Using the triviality of the canonical bundles of Σ_0 and X, we prove easily (by considering the Lefschetz thimbles) that the Maslov class of L_i is trivial, and hence that it is possible to lift each vanishing cycle to a *graded Lagrangian* submanifold of Σ_0, denoted again by L_i. This lets us associate a degree to each generator of the Floer complex.

LEMMA 4.7. *There exists a natural choice of gradings, for which* $\deg(x_i) = \deg(y_i) = \deg(z_i) = 1$ *and* $\deg(\bar{x}_i) = \deg(\bar{y}_i) = \deg(\bar{z}_i) = 2$.

Assume for simplicity that the symplectic form ω is compatible with the standard complex structure of Σ_0 inherited from that of $(\mathbb{C}^*)^3$, which allows us to define explicitly a holomorphic volume form Ω on Σ_0 (i.e., a nonvanishing holomorphic 1-form). Then, given an oriented Lagrangian submanifold $L \subset \Sigma_0$, the *phase* of L is the function $\phi_L : L \to \mathbb{R}/2\pi\mathbb{Z}$ whose value at every point is the argument of the (nonzero) complex number obtained by evaluating Ω on an oriented volume element in L (in the 1-dimensional case, $\phi_L(x) = \arg\Omega(v)$ for v a tangent vector to L at x defining the orientation of L). The Maslov class is the 1-cocycle representing the obstruction to lift ϕ_L to a real-valued function; if it vanishes, then L can be lifted to a *graded Lagrangian* submanifold; i.e., we can choose a real-valued lift of the phase, $\tilde{\phi}_L : L \to \mathbb{R}$. In the 1-dimensional case, the relationship between Maslov index and phase is very simple: given a transverse intersection point p between two graded Lagrangians $L, L' \subset \Sigma_0$, the Maslov index of $p \in CF^*(L, L')$ is equal to the smallest integer greater than $\frac{1}{\pi}(\phi_{L'}(p) - \phi_L(p))$.

The holomorphic volume form Ω on Σ_0 can be defined from the standard holomorphic volume form $\Omega_0 = d\log x \wedge d\log y \wedge d\log z$ on $(\mathbb{C}^*)^3$ by taking residues first along the hypersurface X of equation $x^a y^b z^c = 1$ and then along the level set $W = 0$. We can characterize Ω as follows: Ω is the restriction to Σ_0 of a 1-form (denoted again by Ω) such that $\Omega \wedge dW \wedge d(x^a y^b z^c) = \Omega_0$; i.e., (by the fact that $x^a y^b z^c = 1$ along X)

$$\Omega \wedge (dx + dy + dz) \wedge (\frac{a}{x}\, dx + \frac{b}{y}\, dy + \frac{c}{z}\, dz) = \frac{dx \wedge dy \wedge dz}{xyz}.$$

(In fact the 1-form Ω determined in this way may differ from the "usual" one by a real positive factor, irrelevant for our purposes). At this point it is easy to see why the Maslov class of L_i is trivial: indeed, $\Omega \wedge dW$ extends to a nonvanishing $(2,0)$-form on X, whose phase over the Lefschetz thimble D_i admits a real lift; because W maps D_i to an embedded arc, the phase of $\Omega \wedge dW$

over the boundary of D_i and the phase of Ω over L_i differ by a constant term, so that the latter also admits a real lift.

At every point of Σ_0 except for the branch points of π_x, the 1-form Ω can be expressed as $\Theta\, dx$, for some meromorphic function Θ over Σ_0 (with simple poles at the branch points of π_x). The above equation becomes: $\Theta(\frac{c}{z} - \frac{b}{y}) = \frac{1}{xyz}$, which determines Θ. At this point, the most direct method of determination of the phases of the vanishing cycles L_i at their intersection points (and hence of the corresponding Maslov indices) involves computer calculations; however we will attempt to give a sketch of a geometric argument.

If we restrict ourselves to the domain where x is very small, then $y \simeq -z$, so that $\Theta \simeq \frac{1}{(b+c)xy}$. Therefore, $\arg\Theta \simeq -\arg x - \arg y$ in this region of Σ_0. Hence, the calculations are simplified if we can deform the vanishing cycles L_i in such a way that the intersection points of a given type (y or z) occur close to the origin in $\mathbb{C}^*$. Of course this process preserves gradings and Maslov indices only if the intersection pattern between the relevant vanishing cycles is not affected by the deformation. We consider a deformation where L_i is replaced by a loop $\tilde{L}_i \subset \Sigma_0$, obtained as a double lift of a piecewise smooth arc $\tilde{\delta}_i \subset \mathbb{C}^*$ joining two branch points of π_x (a deformation of δ_i with fixed end points). The arc $\tilde{\delta}_0$ consists of three line segments, two joining the end points $p, \bar{p} \in \mathrm{crit}(\pi_x)$ to two complex conjugate points $q, \bar{q}$ very close to the origin, and such that $0 < \mathrm{Re}\, q \ll \mathrm{Im}\, q \ll 1$. The other arcs $\tilde{\delta}_i$ are obtained from $\tilde{\delta}_0$ by the action of $\mathbb{Z}/(a+b+c)$ (see Fig. 7).

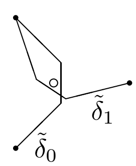

Figure 7: The deformed cycles $\tilde{L}_j$ ($(a,b,c) = (1,1,1)$)

Assuming that $b < a + c$, this deformation can be carried out for intersections of type y without affecting the intersection pattern between L_i and L_{i+b} or L_{i+a+c}, and in such a way that the intersection occurs in the central portion of $\tilde{\delta}_i$ (see Fig. 7). The same is true for intersections of type z when $c < a + b$. If we choose $a \geq b \geq c$ then these two assumptions hold, so we can use this method to determine the degrees of $y_i, z_i, \bar{y}_i, \bar{z}_i$.

We start by considering the portion of $\tilde{L}_0$ lying above the central segment in $\tilde{\delta}_0$ (joining q to $\bar{q}$). Recall that, for x small, the $b+c$ sheets of the covering π_x (i.e. the $b+c$ roots of $x^a y^b(-x-y)^c = 1$) can be approximated by the roots of $y^{b+c} = (-1)^c x^{-a}$. Hence, the possible values for the argument of y are $\arg y \simeq -\frac{a}{b+c}\arg x + \pi\frac{c}{b+c} \bmod \frac{2\pi}{b+c}$. It follows from Lemma 4.1 that the two sheets of π_x containing $\tilde{L}_0$ are those where $\arg y \simeq -\frac{a}{b+c}\arg x + \varepsilon\pi\frac{c}{b+c}$, for $\varepsilon = \pm 1$. Hence, we have $\arg \Theta \simeq \frac{a-b-c}{b+c} \arg x - \varepsilon\pi \frac{c}{b+c}$. We choose to orient $\tilde{L}_0$ in such a way that its projection goes counterclockwise around the origin in the sheet corresponding to $\varepsilon = 1$, and clockwise in the sheet corresponding to $\varepsilon = -1$. With this understood, since the projection of the oriented tangent vector to $\tilde{L}_0$ is positively proportional to εi, we obtain the following formula for the phase of the central portion of $\tilde{L}_0$, modulo 2π:

$$\phi(\tilde{L}_0) \simeq \frac{a-b-c}{b+c} \arg x + \varepsilon\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big). \tag{4.4}$$

We choose a lift of $\tilde{L}_0$ (and hence also L_0 via the isotopy between them) as a graded Lagrangian by setting the (real-valued) phase of $\tilde{L}_0$ to be given by (4.4), selecting $\arg x$ with the smallest absolute value; checking that the choices made in the two portions of $\tilde{L}_0$ corresponding to $\varepsilon = \pm 1$ are consistent with each other is a tedious task, best left to a computer program.

The phase of $\tilde{L}_j = \zeta^{-j} \cdot \tilde{L}_0$ is easily deduced from the above calculations for $\tilde{L}_0$. Indeed, the above formula for Θ implies that the value of $\arg\Theta$ at the point $\zeta^{-j} \cdot p$ differs from that at the point p by $4\pi\frac{j}{a+b+c}$. On the other hand, the argument of the x component of the tangent vector to $\tilde{L}_j$ at $\zeta^{-j} \cdot p$ differs from that of the tangent vector to $\tilde{L}_0$ at p by $-2\pi\frac{j}{a+b+c}$. Therefore, (4.4) implies that

$$\phi(\tilde{L}_j) \simeq \frac{a-b-c}{b+c}\Big(\arg x + \frac{2\pi j}{a+b+c}\Big) + \varepsilon\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big) + \frac{2\pi j}{a+b+c},$$

or equivalently

$$\phi(\tilde{L}_j) \simeq \frac{a-b-c}{b+c} \arg x + \varepsilon\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big) + \frac{2\pi j\, a}{(a+b+c)(b+c)}. \tag{4.5}$$

This formula can also be obtained directly by observing that the two sheets of π_x containing $\tilde{L}_j$ are those where $\arg y \simeq -\frac{a}{b+c}\arg x - 2\pi\frac{j}{b+c} + \varepsilon\pi\frac{c}{b+c}$, for $\varepsilon = \pm 1$, by Lemmas 4.1 and 4.2. As in the case of $\tilde{L}_0$, we choose a lift of $\tilde{L}_j$ whose (real-valued) phase is given by (4.5), using the determination of $\arg x$ closest to $-2\pi\frac{j}{a+b+c}$.

We are now in a position to compare the phases of two vanishing cycles at one of their intersection points. Consider an intersection point between $\tilde{L}_i$ and $\tilde{L}_{i+b}$, corresponding to the intersection y_i between L_i and L_{i+b}. Comparing the values of $\arg y$ on both vanishing cycles, we see easily that the intersection occurs in the $\varepsilon = 1$ part of L_i and in the $\varepsilon = -1$ part of L_{i+b}. Therefore, (4.5) yields that, at the intersection point,

$$\phi(\tilde{L}_{i+b}) - \phi(\tilde{L}_i) \simeq -2\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big) + \frac{2\pi\, b\, a}{(a+b+c)(b+c)} = \pi - \frac{2\pi\, b}{a+b+c},$$

which is between 0 and π since we have assumed that $b < a + c$. Therefore, we have $\deg y_i = 1$. Similarly, the intersection between $\tilde{L}_i$ and $\tilde{L}_{i+c}$ corresponding to z_i occurs in the $\varepsilon=-1$ part of $\tilde{L}_i$ and the $\varepsilon=1$ part of $\tilde{L}_{i+c}$, so that (4.5) yields

$$\phi(\tilde{L}_{i+c})-\phi(\tilde{L}_i)\simeq 2\Big(\frac{\pi}{2}-\frac{\pi\, c}{b+c}\Big)+\frac{2\pi\, c\, a}{(a+b+c)(b+c)}=\pi-\frac{2\pi\, c}{a+b+c},$$

which is also between 0 and π since $c<a+b$. Therefore, $\deg z_i=1$. In the case of $\bar{y}_i$, things are similar, but with one new subtlety: in accordance with the above prescriptions, the determinations of $\arg x$ at the intersection point to be used for $\tilde{L}_i$ and $\tilde{L}_{i+a+c}$ differ by 2π. Therefore, from (4.5) we now get (taking $\varepsilon=-1$ for $\tilde{L}_i$ and $+1$ for $\tilde{L}_{i+a+c}$)

$$\begin{aligned}\phi(\tilde{L}_{i+a+c})-\phi(\tilde{L}_i)\simeq &-2\pi\,\frac{a-b-c}{b+c}+2\Big(\frac{\pi}{2}-\frac{\pi\, c}{b+c}\Big)\\ &+\frac{2\pi\,(a+c)\, a}{(a+b+c)(b+c)}=\pi+\frac{2\pi\, b}{a+b+c},\end{aligned}$$

which is between π and 2π; therefore, $\deg\bar{y}_i=2$. Similarly, for $\bar{z}_i$ one finds that

$$\begin{aligned}\phi(\tilde{L}_{i+a+b})-\phi(\tilde{L}_i)\simeq &-2\pi\,\frac{a-b-c}{b+c}-2\Big(\frac{\pi}{2}-\frac{\pi\, c}{b+c}\Big)\\ &+\frac{2\pi\,(a+b)\, a}{(a+b+c)(b+c)}=\pi+\frac{2\pi\, c}{a+b+c},\end{aligned}$$

which is also between π and 2π, so that $\deg\bar{z}_i=2$.

Finally, the degrees of x_i and $\bar{x}_i$ can be deduced from those of the intersections of types y and z by consideration of e.g. the triangles $T_{xy,i}$, which gives that $\deg x_i+\deg y_{i+a}=\deg\bar{z}_i$, and hence $\deg x_i=1$, and $T_{yz,i}$, which gives that $\deg y_i+\deg z_{i+b}=\deg\bar{x}_i$, and hence $\deg\bar{x}_i=2$. This completes the proof of Lemma 4.7.

4.6. *The exterior algebra structure.* The aim of this section is to determine the coefficients appearing in Lemma 4.5, by studying the orientations of the moduli spaces of pseudo-holomorphic curves and the symplectic areas of their images ($T_{xy,i},\dots$).

LEMMA 4.8. *If the symplectic form ω is anti-invariant under complex conjugation and invariant under the action of $\mathbb{Z}/(a+b+c)$, then there exists a constant $\alpha\in\mathbb{C}^*$ such that $\alpha_{xy,i}=\alpha_{yz,i}=\alpha_{zx,i}=\alpha$ and $\alpha_{yx,i}=\alpha_{zy,i}=\alpha_{xz,i}=-\alpha$ for all i. Therefore, $m_2(x_i,y_{i+a})=-m_2(y_i,x_{i+b})$, $m_2(y_i,z_{i+b})=-m_2(z_i,y_{i+c})$, and $m_2(z_i,x_{i+c})=-m_2(x_i,z_{i+a})$.*

The coefficients $\alpha_{xy,i},\dots$ are determined up to sign by the symplectic areas of the triangular regions $T_{xy,i},\dots$ inside Σ_0. To simplify notation, define

$$T_i=\begin{cases}T_{xy,i} & \text{if } 0\le i<c,\\ T_{zx,i-c} & \text{if } c\le i<b+c,\\ T_{yz,i-b-c} & \text{if } b+c\le i<a+b+c,\end{cases}$$

and

$$T_i' = \begin{cases} T_{xz,i} & \text{if } 0 \le i < b, \\ T_{yx,i-b} & \text{if } b \le i < b+c, \\ T_{zy,i-b-c} & \text{if } b+c \le i < a+b+c, \end{cases}$$

so that T_i and T_i' are the two triangles having either x_i or $\bar{x}_{i-b-c}$ as one of their vertices. We similarly define α_i and α_i' to be the coefficients associated to T_i and T_i' in the formula giving m_2, namely $\alpha_i = \pm\exp(-\mathrm{Area}(T_i))$ and $\alpha_i' = \pm\exp(-\mathrm{Area}(T_i'))$. Then, as observed at the end of §4.4, the invariance properties of ω imply that the $a+b+c$ triangles T_i form a single orbit under the action of $\mathbb{Z}/(a+b+c)$, with $\zeta^{-q}\cdot T_i = T_{i+q}$, and similarly for the other triangles T_i', with $\zeta^{-q}\cdot T_i' = T_{i+q}'$. Moreover, complex conjugation exchanges these two families of triangular regions, by mapping T_i to T_{b+c-i}' (see Fig. 5). It follows that all of these triangles have the same symplectic area, and therefore that the various constants α_i and α_i' are all equal up to sign.

In order to identify the signs, one needs to orient the relevant moduli spaces of pseudo-holomorphic discs in some consistent way, which requires the choice of a spin structure over each Lagrangian L_i. As explained at the end of §3.1, we need to endow each L_i with the spin structure which extends to the corresponding thimble, i.e. the nontrivial one.

We now describe a convenient rule for determining the correct signs in the one-dimensional case, due to Seidel [36]. We start with the case of trivial spin structures. Then to each intersection point $p \in L_i \cap L_j$ $(i<j)$ one can associate an orientation line $\mathcal{O}_p$. This orientation line is canonically trivial when $\deg p$ is even, whereas in the odd degree case, a choice of trivialization of $\mathcal{O}_p$ is equivalent to a choice of orientation of the line T_pL_j. If one considers a pseudo-holomorphic map $u : D^2 \to \Sigma_0$ contributing to m_k, whose image is a polygonal region with $k+1$ vertices $p_0,\dots,p_k$, then the corresponding sign factor is actually an element of the tensor product $\Lambda = \mathcal{O}_{p_0}\otimes\cdots\otimes\mathcal{O}_{p_k}$.

We can define a preferred trivialization of Λ by choosing, at each vertex of odd degree, the orientation of the vanishing cycle which agrees with the positive orientation on the boundary of the image of u. The sign factor associated to u is then equal to $+1$ with respect to this trivialization of Λ (or -1 with respect to the other trivialization). In the presence of nontrivial spin structures, this rule needs to be modified as follows: fix a marked point on each L_i carrying a nontrivial spin structure (distinct from its intersection points with the other vanishing cycles); then the sign associated to u is affected by a factor of -1 for each marked point that the boundary of u passes through [36].

It is worth mentioning that, while it is clear from the above construction that the individual sign factors fail to be canonical and depend on some choices, the various possibilities yield equivalent categories, since the coefficients of Floer homology and Floer products simply differ by the conjugation action of some diagonal matrix with ±1 coefficients.

In our case, we choose trivializations of the orientation lines as follows: for every intersection point $p \in L_i \cap L_j$ of degree 1 (i.e., one of x_i, y_i, z_i), we orient T_pL_j consistently with the boundary orientation of the single triangular region among $T_0, \dots, T_{a+b+c-1}$ having p among its vertices. If we consider trivial spin structures, then with this convention the sign factor associated to each triangle T_i is by definition equal to $+1$. In the case of T'_i, at each of the two vertices of degree 1 the chosen trivialization of T_pL_j disagrees with the boundary orientation of the triangular region, so that for trivial spin structures we get a sign factor of $(-1)^2 = +1$ again. Since we need to consider nontrivial spin structures, we must introduce a marked point on each L_i; for example, we choose this marked point in the portion of L_i that corresponds to the top-most edge in Figure 6. With this choice, the boundary of each T'_i passes through exactly one marked point (between the vertex of type z and that of type y), while the boundary of T_i does not meet any marked point. Therefore, with these conventions, the sign factors are $+1$ for all T_i and -1 for all T'_i; this completes the proof of Lemma 4.8.

4.7. *Nonexact symplectic forms and noncommutative deformations.* The purpose of this section is to describe the effect on the category of Lagrangian vanishing cycles of W of relaxing the assumptions made above on the symplectic form, losing in particular its exactness. In order to make the vanishing cycle construction well-defined, we will continue assuming that ω induces a complete Kähler metric on X and that the gradient of W with respect to this metric is bounded from below outside of a compact set. For example, choosing a 3×3 positive definite Hermitian matrix (a_{ij}), we can endow X with the symplectic form

$$\omega = i \sum_{i,j=1}^{3} a_{ij} \frac{dz_i}{z_i} \wedge \frac{d\bar{z}_j}{\bar{z}_j}.$$

Observe that $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}$ is generated by the torus $T = \{(x, y, z) \in X,\ |x| = |y| = |z| = 1\}$ (for simplicity we assume $\gcd(a, b, c) = 1$). An easy calculation shows that

$$[\omega] \cdot [T] = 4\pi^2 i \,(a\,(a_{23} - a_{32}) + b\,(a_{31} - a_{13}) + c\,(a_{12} - a_{21})). \tag{4.6}$$

Many other choices of symplectic forms are equally acceptable, and it is important to mention that the most sensible course of action in the presence of a nonexplicit symplectic form is to search for a *topological* interpretation of the category of Lagrangian vanishing cycles, involving only topological quantities such as the cohomology class of ω.

In comparison to the restrictive situation considered above, the vanishing cycles L_j remain in the same smooth isotopy classes, because one can continuously deform from one symplectic structure to the other. Hence, the vanishing cycles are smoothly isotopic to the loops $L'_j \subset \Sigma_0$ introduced in Section 4.2, but not necessarily Hamiltonian isotopic to them. Nonetheless, because the ends of the noncompact Riemann surface Σ_0 all have infinite volume, we can easily deform L'_j into loops $L''_j \subset \Sigma_0$ that are Hamiltonian isotopic to the vanishing cycles, without modifying the pattern of the intersections between them. More precisely, recall from §4.2 that each L'_j is the double lift via $\pi_x : \Sigma_0 \to \mathbb{C}^*$ of an arc joining two branch points of π_x. Then, by "pulling" a suitable portion of one of the two lifts towards an end of Σ_0 (either towards infinity or towards zero in the x-axis projection), we can make L'_j sweep through an arbitrarily large symplectic area to obtain the desired L''_j, without affecting the intersection points with the other vanishing cycles.

Since the vanishing cycles are Hamiltonian isotopic to the loops L''_j, we may use L''_j instead of the actual vanishing cycles in order to determine the category $D(\mathrm{Lag}_{\mathrm{vc}}(W))$. Hence, the symplectic deformation does not affect in any way the generators of the Floer complexes and the types of pseudo-holomorphic maps to be considered. The only significant change has to do with the coefficients assigned to the various pseudo-holomorphic discs appearing in the definition of m_2, as the symplectic areas of the various triangular regions T_i and T'_i $(i = 0, \dots, a+b+c-1)$ inside Σ_0 may now take more or less arbitrary values instead of all being equal to each other. Because the description of ω and of the vanishing cycles is not explicit, it is hopeless (and useless) to calculate the individual coefficients α_i and α'_i. However, we can state the following result:

LEMMA 4.9. *Lemmas* 4.3–4.7 *remain valid in the more general case of an arbitrary symplectic form inducing a complete Kähler metric on X for which $|\nabla W|$ is bounded from below at infinity. Moreover, the structure constants for the composition m_2 are related by the identity*

$$\frac{\prod_{i=0}^{a+b+c-1} \alpha_i}{\prod_{i=0}^{a+b+c-1} \alpha'_i} = \frac{\prod \alpha_{xy,i} \prod \alpha_{yz,i} \prod \alpha_{zx,i}}{\prod \alpha_{yx,i} \prod \alpha_{zy,i} \prod \alpha_{xz,i}} = (-1)^{a+b+c} \exp(-2\pi[\omega]\cdot[T]).$$

The assumption of completeness of the induced Kähler metric can be dropped if we have some other way of ensuring that the vanishing cycles are well-defined and that the deformation from L'_j to L''_j can be carried out without introducing new intersection points. In fact, the invariance of Floer homology under Hamiltonian isotopies essentially implies that the introduction of new intersection points in the deformation does not have any particular impact on the derived category, so the only thing that matters is actually the well-definedness of the vanishing cycles.

Although Lemma 4.9 seems to give only very partial information about the constants α_i and α'_i, it actually completely determines the category $D(\mathrm{Lag}_{\mathrm{vc}}(W))$. Indeed, simply by rescaling the generators of the Floer complexes we can modify the coefficients α_i and α'_i almost at will: for example, replacing x_i with λx_i has the effect of simultaneously multiplying α_i and α'_i by λ^{-1}; similarly, rescaling the generator y_i simultaneously affects α_{i-a} (or α_{i+b+c}) and α'_{i+b}. Still assuming $\gcd(a,b,c)=1$, it is not hard to check that the only quantity left invariant by all rescalings of the generators is the ratio $\prod \alpha_i / \prod \alpha'_i$, which is therefore sufficient to characterize the derived category. This observation that the symplectic deformations of $D(\mathrm{Lag_{vc}}(W))$ are governed by a single parameter is naturally related to the fact that the second Betti number of X is equal to 1.

Proof of Lemma 4.9. The key observation to be made here is that the boundary of the 2-chain $C=\sum T_i - \sum T'_i \subset \Sigma_0$ is exactly $\partial C = -\sum L_i$ (for a suitable choice of orientation of the L_i). Indeed, looking at Figure 6, we see that each of the six portions of L_i arises exactly once as an edge of one of the triangular regions, and the boundary orientation of the triangular region is the "clockwise" orientation of L_i in the case of $T_0,\dots,T_{a+b+c-1}$, and the "counterclockwise" orientation in the case of $T'_0,\dots,T'_{a+b+c-1}$. Recalling that each vanishing cycle L_i bounds a Lefschetz thimble D_i in X, we can build a 2-cycle $\tilde{C}\subset X$ by capping C with these $a+b+c$ Lagrangian discs. Next, observe that the sign factors arising from the orientations of the moduli spaces remain the same as in Section 4.6, and that $\int_{D_i}\omega = 0$, so that

$$\begin{aligned}\frac{\prod \alpha_i}{\prod \alpha'_i} &= (-1)^{a+b+c}\,\frac{\prod \exp(-2\pi\int_{T_i}\omega)}{\prod \exp(-2\pi\int_{T'_i}\omega)}\\ &= (-1)^{a+b+c}\exp(-2\pi\int_C \omega) = (-1)^{a+b+c}\exp(-2\pi[\omega]\cdot[\tilde{C}]).\end{aligned}$$

Hence, the last step in the proof is to show that $[\tilde{C}]$ and $[T]$ are the same elements of $H_2(X,\mathbb{Z})\simeq\mathbb{Z}$. A simple way to achieve this is to compute the intersection pairing of $\tilde{C}$ with the relative cycle $R=\{(x,y,z)\in X,\ x,y,z\in\mathbb{R}^+\}$, which intersects T transversely once at the point $(1,1,1)$.

To understand how R intersects $\tilde{C}$, we compare the values of W over R and over $\tilde{C}$. By construction, $\tilde{C}$ is the union of the 2-chain $C\subset\Sigma_0$, over which W vanishes identically, and the various Lefschetz thimbles D_j, which W maps to straight line segments joining the origin to the critical values λ_j. On the other hand, the restriction to R of $W=x+y+z$ is a proper function which takes real positive values. With respect to the standard complex structure, R is totally real and W is holomorphic, so any critical point of $W_{|R}$ is also a critical point of W, and in particular the minimum of W over R is a critical value of W. Indeed, a simple computation shows that the minimum of W over R is exactly $(a+b+c)(a^a b^b c^c)^{-1/(a+b+c)} = \lambda_0$, achieved at the critical point p_0 of W corresponding to the critical value λ_0.

It follows that the only point where $\tilde{C}$ and R intersect is p_0. Moreover, by considering the local model near p_0, we can check easily that this intersection is transverse, since the Hessian of W at p_0 restricts to the tangent space $T_{p_0}D_0$

as a negative definite real quadratic form, and to $T_{p_0}R$ as a positive definite real quadratic form. Therefore the intersection number between $\tilde{C}$ and R is equal to 1 (for a suitable choice of orientation that we will not discuss here), and it follows that $[\tilde{C}] = [T]$ in $H_2(X, \mathbb{Z})$. ☐

4.8. B-*fields and complexified deformations.* So far we have identified a real one-parameter family of deformations of the category of Lagrangian vanishing cycles of W. To extend this to a complex family of deformations, we need to introduce a nontrivial B-field, i.e. a closed 2-form $B \in \Omega^2(X, \mathbb{R})$. The presence of a B-field affects Fukaya categories by modifying the nature of the objects to be considered: namely, one should consider pairs consisting of a Lagrangian submanifold and a vector bundle over it equipped with a *projectively flat* (rather than flat) connection with curvature equal to $-2\pi i B \otimes \mathrm{Id}$ (depending on conventions, the factor of 2π is sometimes omitted).

In our case, we are considering Lagrangian vanishing cycles $L_j \simeq S^1$ arising as boundaries of the Lefschetz thimbles D_j. Since $\dim L_j = 1$, over L_j every bundle is trivial and every connection is flat; moreover, we can safely restrict ourselves to the case of line bundles. However, the presence of the B-field results in a nontrivial holonomy. By Stokes' theorem, if a U(1)-connection $\nabla_j = d + i\alpha_j$ is the restriction to L_j of a U(1)-connection with curvature $-2\pi i B$ over D_j, then the holonomy of ∇_j around L_j is given by $\mathrm{hol}_{\nabla_j}(L_j) = \exp(\int_{L_j} i\alpha_j) = \exp(\int_{D_j} i\, d\alpha_j) = \exp(-2\pi i \int_{D_j} B)$. Since this property characterizes the connection ∇_j uniquely up to gauge, we can drop the line bundle and the connection from the notation when considering the objects (L_j, E_j, ∇_j) of $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_j\})$.

However, we do need to take the holonomy of ∇_j into account when computing the twisted Floer differential and compositions m_k, since the weight attributed to a given pseudo-holomorphic disc $u : (D^2, \partial D^2) \to (\Sigma_0, \bigcup L_j)$ is modified by a factor corresponding to the holonomy along its boundary, and becomes $\pm\,\mathrm{hol}(u(\partial D^2)) \exp(2\pi i \int_{D^2} u^*(B + i\omega))$. More precisely, for each intersection point $p \in L_i \cap L_j$ we need to fix an isomorphism between the fibers $(E_i)_{|p}$ and $(E_j)_{|p}$; then it becomes possible to define the holonomy along the closed loop $u(\partial D^2)$ using the parallel transport induced by ∇_j from one "corner" of u to the next one, and the chosen isomorphism at each corner.

In this context, we now have the following result characterizing $D(\mathrm{Lag}_{\mathrm{vc}}(W))$:

LEMMA 4.10. *Lemmas* 4.3–4.7 *remain valid for an arbitrary symplectic form inducing a complete Kähler metric on X for which $|\nabla W|$ is bounded from below at infinity, and an arbitrary* B-*field. Moreover, the structure constants for the composition m_2 are related by the identity*

$$\frac{\prod_{i=0}^{a+b+c-1} \alpha_i}{\prod_{i=0}^{a+b+c-1} \alpha'_i} = \frac{\prod \alpha_{xy,i} \prod \alpha_{yz,i} \prod \alpha_{zx,i}}{\prod \alpha_{yx,i} \prod \alpha_{zy,i} \prod \alpha_{xz,i}} = (-1)^{a+b+c} \exp(2\pi i [B + i\omega] \cdot [T]).$$

Proof. We again consider the 2-chain $C = \sum T_i - \sum T'_i \subset \Sigma_0$, with boundary $\partial C = -\sum L_j$, and the 2-cycle $\tilde{C} \subset X$ obtained by capping C with the Lagrangian discs D_j. We now have:

$$\begin{aligned}
\frac{\prod \alpha_i}{\prod \alpha'_i} &= \frac{(-1)^{a+b+c}}{\prod \mathrm{hol}_{\nabla_j}(L_j)} \frac{\prod \exp(2\pi i \int_{T_i} B + i\omega)}{\prod \exp(2\pi i \int_{T'_i} B + i\omega)} \\
&= \frac{(-1)^{a+b+c}}{\prod \exp(\int_{D_j} -2\pi i B)} \exp\left(2\pi i \int_C B + i\omega\right) \\
&= (-1)^{a+b+c} \exp(2\pi i [B + i\omega] \cdot [\tilde{C}]).
\end{aligned}$$

This completes the proof since $[\tilde{C}] = [T]$. □

It is interesting to observe that this statement reinterpretes the quantity $\prod \alpha_i / \prod \alpha'_i$ in purely topological terms, thus avoiding the pitfall of having to compute the individual coefficients attached to the various pseudo-holomorphic discs in Σ_0. This outcome is rather unsurprising since, whereas the individual coefficients α_i and α'_i are heavily dependent on a number of arbitrary choices, the underlying derived category of Lagrangian vanishing cycles is expected to depend only on the meaningful parameters – in our case, the cohomology class $[B + i\omega]$.

We would like to suggest that this feature reflects a general *principle*. Namely, the various structure coefficients of the Floer differentials and products involved in the definition of the category $\mathrm{Lag}_{\mathrm{vc}}(W)$ depend on many choices and have no precise meaning in general. However, different sets of values of the structure coefficients may become equivalent after a suitable rescaling of the generators of the Floer complexes or other similarly benign operations. Hence, we can reduce to a much smaller set of parameters (certain combinations of the individual Floer coefficients) that actually govern the structure of the category. Then, we expect the following statement to hold in much greater generality than the examples studied here:

PROPERTY 4.11. *The structure of the derived category of Lagrangian vanishing cycles is governed by deformation parameters which are all of the form* $\exp(2\pi i [B + i\omega] \cdot [C_j])$ *for suitable* 2*-cycles* $C_j \subset X$.

This is of course ultimately related to the fact that Floer homology and Floer products can be defined over *Novikov rings*, counting pseudo-holomorphic discs with coefficients that reflect relative homology classes rather than actual symplectic areas; the version with complex coefficients that we used here is then recovered from the version with Novikov ring coefficients by evaluation at the point $[B + i\omega]$.

5. Hirzebruch surfaces

We now consider the case of Hirzebruch surfaces $\mathbb{F}_n$, for which the mirror Landau-Ginzburg model consists of $X = (\mathbb{C}^*)^2$ equipped with a superpotential of the form

$$W = x + y + \frac{a}{x} + \frac{b}{x^n y}$$

for some nonzero constants a, b. For simplicity we will only consider the case of an exact symplectic form. Since different values of the constants a, b lead to mutually isotopic exact symplectic Lefschetz fibrations, the actual choices do not matter (we can e.g. assume $a = b = 1$ or any other convenient choice).

5.1. *The case of* $\mathbb{F}_0$ *and* $\mathbb{F}_1$. The first two Hirzebruch surfaces $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{F}_1$ (i.e., $\mathbb{CP}^2$ blown up at one point) need to be considered separately.

Proposition 5.1. *When* $n = 0$, *there exists a system of arcs* $\{\gamma_i\}$ *such that* $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ *is equivalent to the full subcategory of* $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_0))$ *whose objects are* $\mathcal{O}$, $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$, $\mathcal{O}(1,1)$. *Therefore*, $D(\mathrm{Lag}_{\mathrm{vc}}(W)) \simeq \mathbf{D}^b(\mathrm{coh}(\mathbb{F}_0))$.

Proof. The four critical values of $W = x + y + \frac{a}{x} + \frac{b}{y}$ are $\pm 2a^{1/2} \pm 2b^{1/2}$. Up to an exact deformation which does not affect the category of Lagrangian vanishing cycles, we can choose $a > b > 0$, and assume the symplectic form to be anti-invariant under reflection about the imaginary axis $(x, y) \mapsto (-\bar{x}, -\bar{y})$. We choose $\Sigma_0 = W^{-1}(0)$ as our reference fiber, and join it to the singular fibers by considering arcs γ_i that pass below the real axis in $\mathbb{C}$, so that the clockwise ordering of the critical values agrees with their natural ordering $-2a^{1/2} - 2b^{1/2} < -2a^{1/2} + 2b^{1/2} < 2a^{1/2} - 2b^{1/2} < 2a^{1/2} + 2b^{1/2}$. The projection π_x to the x variable realizes Σ_0 as a double cover of $\mathbb{C}^*$ branched at four points, and the vanishing cycles L_i can be represented as double lifts of the arcs $\delta_i \subset \mathbb{C}^*$ shown in Figure 8.

It follows that $\mathrm{Hom}(L_1, L_2) = 0$, while $\mathrm{Hom}(L_0, L_1)$, $\mathrm{Hom}(L_2, L_3)$, $\mathrm{Hom}(L_0, L_2)$, and $\mathrm{Hom}(L_1, L_3)$ are two-dimensional; label the corresponding intersection points $L_0 \cap L_1 = \{s, t\}$, $L_2 \cap L_3 = \{s', t'\}$, $L_0 \cap L_2 = \{u, v\}$, $L_1 \cap L_3 = \{u', v'\}$. Finally, $\mathrm{Hom}(L_0, L_3)$ has rank 4. By considering the triangular regions delimited by the vanishing cycles in Σ_0, and using the symmetry

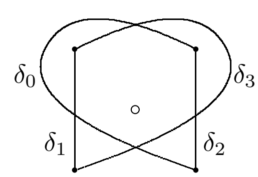

Figure 8: The vanishing cycles for $\mathbb{F}_0$

of the configuration with respect to $(x,y) \mapsto (-\bar{x},-\bar{y})$, we can easily show that $m_2(s,u') = m_2(s',u)$, $m_2(t,u') = m_2(t',u)$, $m_2(s,v') = m_2(s',v)$, and $m_2(t,v') = m_2(t',v)$; these four elements of $\mathrm{Hom}(L_0,L_3)$ are proportional to the generators. All other products vanish ($m_k = 0$ for $k \neq 2$). Finally, gradings can be chosen so that all morphisms have degree 0 (the verification is left to the reader).

Therefore, the category $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_i\})$ is indeed equivalent to the full subcategory of $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_0))$ whose objects are $\mathcal{O}$, $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$, $\mathcal{O}(1,1)$, as can be seen by thinking of (s,t) and (u,v) as homogeneous coordinates on the two factors of $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. Since these four line bundles form a full strong exceptional collection generating $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_0))$, the result follows. □

Alternatively, Proposition 5.1 can also be obtained as a direct corollary of a general product formula for categories of Lagrangian vanishing cycles of Lefschetz fibrations of the form $(X_1 \times X_2, W_1 + W_2)$ ([6]; cf. also §6.3).

Proposition 5.2. *When $n = 1$, there exists a system of arcs $\{\gamma_i\}$ such that $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_i\})$ is equivalent to the full subcategory of $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_1))$ whose objects are $\mathcal{O}$, $\pi^*(T_{\mathbb{P}^2}(-1))$, $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, $\mathcal{O}_E$ (where E is the exceptional curve and $\pi : \mathbb{F}_1 \to \mathbb{CP}^2$ is the blow-up map). Therefore, $D(\mathrm{Lag}_{\mathrm{vc}}(W)) \simeq \mathbf{D}^b(\mathrm{coh}(\mathbb{F}_1))$.*

Proof. We choose $a = b = 1$, and equip X with a symplectic form that is anti-invariant under complex conjugation. Let $(\lambda_i)_{0 \leq i \leq 3}$ be the four critical values of $W = x + y + \frac{1}{x} + \frac{1}{xy}$, ordered clockwise around the origin so that $\mathrm{Im}(\lambda_0) > 0$, $\lambda_1 \in \mathbb{R}_+$, $\mathrm{Im}(\lambda_2) < 0$, and $\lambda_3 \in \mathbb{R}_-$. We choose $\Sigma_0 = W^{-1}(0)$ as reference fiber, and choose the arcs γ_i joining 0 to λ_i to be straight lines. The projection π_x to the x variable realizes Σ_0 as a double cover of $\mathbb{C}^*$ branched at four points, and the vanishing cycles L_i can be represented as double lifts of the arcs $\delta_i \subset \mathbb{C}^*$ shown in Figure 9.

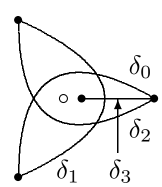

Figure 9: The vanishing cycles for $\mathbb{F}_1$

The corresponding category of vanishing cycles can then be studied explicitly. In fact, much of the work has already been carried out in Section 4, since the situation for L_0, L_1, L_2 is rigorously identical (including grading and orientation issues) to that previously considered for the three vanishing cycles of the Lefschetz fibration mirror to $\mathbb{CP}^2$. While the choice of grading used in Section 4 yields morphisms in degrees 1 and 2, a different choice of gradings (shifting L_1 by 1 and L_2 by 2) ensures that all morphisms between L_0, L_1, L_2 have degree 0. This readily implies that a category equivalent to the derived category of $\mathbb{CP}^2$ can be realized inside $D(\mathrm{Lag}_{\mathrm{vc}}(W))$ as a full subcategory, with the exceptional collection L_0, L_1, L_2 corresponding to the exceptional collection $\mathcal{O}, T_{\mathbb{P}^2}(-1), \mathcal{O}(1)$ dual to the standard one. (This claim can of course also be verified "by hand" following the same outline of argument as in §4.)

From Figure 9 it is clear that $\mathrm{Hom}(L_0, L_3)$ and $\mathrm{Hom}(L_2, L_3)$ are one-dimensional (call their generators p_0 and p_2), while $\mathrm{Hom}(L_1, L_3)$ has rank 2 (call its generators q and q'). To be consistent with the notation of §4, call x_0, y_0, z_0 (resp. x_1, y_1, z_1; resp. $\bar{x}, \bar{y}, \bar{z}$) the generators of $\mathrm{Hom}(L_0, L_1)$ (resp. $\mathrm{Hom}(L_1, L_2)$; resp. $\mathrm{Hom}(L_0, L_2)$). Then, looking at the various pseudo-holomorphic discs in Σ_0 (including a constant one at the triple intersection of L_0, L_2, L_3), we have: $m_2(x_0, q) = m_2(x_0, q') = 0$, $m_2(y_0, q) = \alpha\, p_0$, $m_2(y_0, q') = 0$, $m_2(z_0, q) = 0$, $m_2(z_0, q') = \alpha'\, p_0$, $m_2(x_1, p_2) = 0$, $m_2(y_1, p_2) = -\alpha\, q'$, $m_2(z_1, p_2) = \alpha'\, q$, $m_2(\bar{x}, p_2) = p_0$, $m_2(\bar{y}, p_2) = m_2(\bar{z}, p_2) = 0$ (for some nonzero constants α, α'). Moreover, for a suitable choice of grading of L_3 it can be checked that all morphisms have degree 0.

It is then easy to check that these formulas correspond exactly to the composition formulas in the full subcategory of $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_1))$ whose objects are the pull-backs $\mathcal{O}$, $\pi^*(T_{\mathbb{P}^2}(-1))$, $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, and the structure sheaf $\mathcal{O}_E$ of the exceptional curve (If one follows the analogy suggested by the notation between the morphisms from L_0 to L_2 and the homogeneous coordinates on $\mathbb{CP}^2$, then the blow-up point is located at $(1:0:0)$). The result follows. □

5.2. *Other Hirzebruch surfaces.* For larger values of n, the situation becomes different:

LEMMA 5.3. *If $n \geq 2$, then the Lefschetz fibrations over $(\mathbb{C}^*)^2$ defined by $W = x + y + \frac{1}{x} + \frac{1}{x^n y}$ and $\tilde{W} = x + y + \frac{1}{x^n y}$ are isotopic. Therefore, $D(\mathrm{Lag}_{\mathrm{vc}}(W)) \simeq D(\mathrm{Lag}_{\mathrm{vc}}(\tilde{W})) \simeq \mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$.*

Proof. Consider the maps $W_a = x + y + \frac{a}{x} + \frac{1}{x^n y}$ for $a \in [0,1]$. The key observation is that the $n+2$ critical points of W_a remain distinct and stay in a compact subset of $(\mathbb{C}^*)^2$. Indeed, the critical points of W_a are the solutions of

$$\begin{cases} 1 - \frac{a}{x^2} - \frac{n}{x^{n+1}y} = 0 \\ 1 - \frac{1}{x^n y^2} = 0; \end{cases}$$

i.e.,

$$y = n x^{1-n}(x^2 - a)^{-1}, \text{ and } x^{n-2}(x^2 - a)^2 - n^2 = 0.$$

It is easy to check that for $|a| \leq 1$ the roots of this equation satisfy $1 \leq |x| \leq \sqrt{n+1}$. It follows that $|x^2 - a| = n|x|^{1-\frac{n}{2}}$ is bounded between two positive constants, and hence that $y = n x^{1-n}(x^2 - a)^{-1} = (x^2 - a)/nx$ is also bounded between two positive constants independently of a. Hence the critical points of W_a remain inside a compact subset of $(\mathbb{C}^*)^2$. Moreover, the polynomial $P(x) = x^{n-2}(x^2-a)^2 - n^2$ always has simple roots when $|a| \leq 1$, since the roots of $P'(x) = x^{n-3}(x^2-a)((n+2)x^2-(n-2)a)$ are 0, $\pm\sqrt{a}$, and $\pm\sqrt{\frac{n-2}{n+2}a}$, where P never vanishes. In fact, even though this is not necessary for the argument, the critical values of W_a also remain distinct throughout the deformation, since at a critical point we have $W_a = \frac{n+2}{n}x + \frac{n-2}{n}\frac{a}{x}$, which as a function of x is injective over $\{|x| \geq 1\}$.

Therefore, W_a defines an exact symplectic Lefschetz fibration on $(\mathbb{C}^*)^2$ for all $a \in [0,1]$, which allows us to match the vanishing cycles of $W_1 = W$ with those of $W_0 = \tilde{W}$. The resulting categories of vanishing cycles differ at most by a deformation of the structure coefficients of the compositions m_2, but since the isotopy is through *exact* Lagrangian vanishing cycles, we need not worry about those (see also the argument for Lemma 4.9).

We can therefore conclude that $D(\mathrm{Lag_{vc}}(W)) \simeq D(\mathrm{Lag_{vc}}(\tilde{W}))$. Since $((\mathbb{C}^*)^2, \tilde{W})$ is exactly the mirror to $\mathbb{CP}^2(n,1,1)$ studied at length in Section 4, our result for weighted projective planes implies that this category is also equivalent to $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$. □

For $n=2$, it is well-known that $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_2)) \simeq \mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(2,1,1)))$, so we get the expected result. However, for $n \geq 3$ this is no longer true. Namely, the fully faithful functor MK_n constructed in Section 2.7 allows us to view the category $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_n))$ as a full subcategory of $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$, generated by the exceptional collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(n), \mathcal{O}(n+1))$. It is therefore a natural question to ask whether this subcategory can be singled out on the mirror side, by selecting 4 of the $n+2$ critical points of W. It turns out that this is indeed the case. Our first result in this direction is the following:

LEMMA 5.4. *For $n \geq 3$, in the limit $b \to 0$, $n-2$ of the critical values of the superpotentials $W_b = x + y + \frac{1}{x} + \frac{b}{x^n y}$ go to infinity, while the remaining four critical points stay in a bounded region.*

Proof. The x coordinates of the critical points of W_b are the solutions of
$$x^{n-2}(x^2-1)^2 - n^2 b = 0.$$
As $b \to 0$, four roots of this equation converge to ± 1, while the remaining $n-2$ converge to 0. Since at a critical point we also have $y = nbx^{1-n}(x^2-1)^{-1} = \frac{1}{n}(x - \frac{1}{x})$ and $W_b = \frac{n+2}{n}x + \frac{n-2}{n}\frac{1}{x}$, we conclude that four critical points of W_b converge to $(\pm 1, 0)$, with the corresponding critical values converging to ± 2, while the others escape to infinity. □

This suggests that the deformation $b \to 0$ singles out a subcategory of $D(\mathrm{Lag_{vc}}(W_b))$, obtained by restricting oneself to the preimage of a disc containing only four critical values of W_b. We start by describing the case $n = 3$.

For $n = 3$, we can study explicitly the deformation process as b changes from 1 to a value close to 0. For $b = 1$ the five critical values of W_b form a pentagon roughly centered at the origin (and can for all practical purposes be identified with the critical values of the superpotential mirror to $\mathbb{CP}^2(3,1,1)$). As b decreases along the real axis, two things happen: first, the two complex conjugate critical points with $\mathrm{Re}(W_b) > 0$ merge and turn into two real critical points; then, one of these two real critical points escapes to infinity as $b \to 0$. The process is easier to visualize if one avoids the two values of b in the interval $(0,1)$ for which two critical values of W_b coincide, by considering e.g. a deformation from $b = 1$ to $b = 0$ where the imaginary part of b is kept positive. It is then easy to check that, as $b \to 0$, two critical values converge to 2 and two others converge to -2, while the fifth one escapes to infinity in the manner represented in Figure 10.

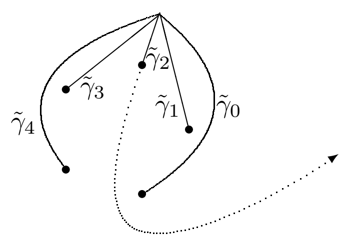

Figure 10: The deformation $b \to 0$ for $n = 3$

Therefore, if we consider the category of Lagrangian vanishing cycles associated to the system of arcs $\tilde\gamma_0, \dots, \tilde\gamma_4$ represented in Figure 10, the deformation $b \to 0$ singles out the full subcategory generated by the four vanishing cycles $\tilde L_0, \tilde L_1, \tilde L_3, \tilde L_4$ (where $\tilde L_i$ is the vanishing cycle associated to $\tilde\gamma_i$). The collection of arcs $\{\tilde\gamma_i\}$ looks very different from the collection $\{\gamma_i\}$ considered in Section 4, but they are related to each other by a sequence of elementary sliding transformations performed on consecutive arcs (see Fig. 11).

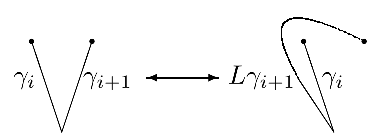

Figure 11: The (left) sliding operation $(\gamma_i, \gamma_{i+1}) \longleftrightarrow (L\gamma_{i+1}, \gamma_i)$

It follows immediately from Definition 3.1 that every ordered collection of arcs yields a full exceptional collection generating $D(\mathrm{Lag}_{\mathrm{vc}}(W))$; it was shown by Seidel that (left or right) sliding operations on collections of arcs correspond to (left or right) mutations of the corresponding exceptional collections [34]. With this is mind, and identifying implicitly the critical points of W_1 with those of the superpotential mirror to $\mathbb{CP}^2(3,1,1)$, we check easily that the left dual to the exceptional collection $(\tilde{L}_0, \dots, \tilde{L}_4)$ associated to the arcs $\{\tilde{\gamma}_i\}$ is equivalent (up to some shifts) to the exceptional collection associated to the arcs $(\gamma_2, \gamma_3, \gamma_4, \gamma_0, \gamma_1)$. Moreover, using $\mathbb{Z}/5$-equivariance for $\mathbb{CP}^2(3,1,1)$, we have an auto-equivalence of $D(\mathrm{Lag_{vc}}(W_1))$ which maps this exceptional collection to the one associated to the collection of arcs $(\gamma_0, \dots, \gamma_4)$ considered in Section 4.

Recall that the two exceptional collections for $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(3,1,1)))$ presented in Section 2 are mutually dual (cf. Example 2.15), and that Theorem 3.3 identifies the exceptional collection associated to the arcs $(\gamma_0, \dots, \gamma_4)$ with that given by Corollary 2.27. Therefore, there is an equivalence of categories which maps the exceptional collection $(\tilde{L}_0, \dots, \tilde{L}_4)$ for $D(\mathrm{Lag_{vc}}(W_1))$ to the exceptional collection $(\mathcal{O}, \dots, \mathcal{O}(4))$ for $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(3,1,1)))$. The full subcategory of $D(\mathrm{Lag_{vc}}(W_1))$ singled out by the deformation $b \to 0$ is that generated by the exceptional collection $(\tilde{L}_0, \tilde{L}_1, \tilde{L}_3, \tilde{L}_4)$, which corresponds under the above identification to the full subcategory of $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(3,1,1)))$ generated by the exceptional collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(3), \mathcal{O}(4))$, which is in turn known to be equivalent to the derived category of the Hirzebruch surface $\mathbb{F}_3$ (see §2.7).

A similar analysis of the deformation $b \to 0$ can be carried out for all values of n, and leads to the following result:

PROPOSITION 5.5. *Given any $n \geq 3$ and $R \gg 2$, and with the assumption that b is sufficiently close to 0, the full subcategory of $D(\mathrm{Lag_{vc}}(W_b))$ arising from restriction to the open domain $\{|W_b| < R\}$ is equivalent to $\mathbf{D}^b(\mathrm{coh}(\mathbb{F}_n))$.*

In order to prove this proposition we need a lemma about mutations in the standard full exceptional collection $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n+1))$ on the weighted projective plane $\mathbb{CP}^2(n,1,1)$. Let us fix a pair $(\mathcal{O}(k), \mathcal{O}(k+1))$ with $2 < k < n$. Denote by F_{k+2} the mutation of the object $\mathcal{O}(k+2)$ to the left through $\mathcal{O}(k), \mathcal{O}(k+1)$, i.e., $F_{k+2} \cong L^{(2)}\mathcal{O}(k+2)$. Performing the same mutations on $\mathcal{O}(k+3), \dots, \mathcal{O}(n+1)$ we obtain exceptional objects $F_i = L^{(2)}\mathcal{O}(i)$ for $k+2 \leq i \leq n+1$ and a new exceptional collection

$$(\mathcal{O}, \dots, \mathcal{O}(k-1), F_{k+2}, \dots, F_{n+1}, \mathcal{O}(k), \mathcal{O}(k+1))\,.$$

Denote by G_k, G_{k+1} the left mutations of $\mathcal{O}(k), \mathcal{O}(k+1)$ respectively through all F_i. We get an exceptional collection

$$(\mathcal{O}, \dots, \mathcal{O}(k-1), G_k, G_{k+1}, F_{k+2}, \dots, F_{n+1})\,.$$

Denote by $\mathcal{D}$ the triangulated subcategory of the category $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$ generated by the collection $(\mathcal{O}, \mathcal{O}(1), G_k, G_{k+1})$.

LEMMA 5.6. *The triangulated subcategory $\mathcal{D}$ coincides with the subcategory*

$$\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(n), \mathcal{O}(n+1) \rangle.$$

Proof. This lemma is equivalent to the statement that the subcategory $\langle G_k, G_{k+1}\rangle$ coincides with the subcategory $\langle \mathcal{O}(n), \mathcal{O}(n+1)\rangle$. First, let us show that $\mathcal{O}(n)$ and $\mathcal{O}(n+1)$ belong to $\langle G_k, G_{k+1}\rangle$. Since $\mathrm{Hom}(\mathcal{O}(l), \mathcal{O}(s)) = 0$ for $l = n, n+1$ and $0 \le s < k$, we can immediately conclude that $\mathcal{O}(n)$ and $\mathcal{O}(n+1)$ belong to $\langle G_k, G_{k+1}, F_{k+2}, \dots, F_{n+1}\rangle$. Therefore, it is sufficient to check that

$$\mathrm{Hom}^\bullet(F_i, \mathcal{O}(n)) = 0, \qquad \mathrm{Hom}^\bullet(F_i, \mathcal{O}(n+1)) = 0$$

for all $k+2 \le i \le n+1$.

By definition of F_i there are distinguished triangles

$$T_i \longrightarrow V_i \otimes \mathcal{O}(k+1) \longrightarrow \mathcal{O}(i), \tag{5.1}$$

$$F_i \longrightarrow W_i \otimes \mathcal{O}(k) \longrightarrow T_i, \tag{5.2}$$

with $V_i = \mathrm{Hom}(\mathcal{O}(k+1), \mathcal{O}(i))$ and $W_i = \mathrm{Hom}(\mathcal{O}(k), T_i)$. It is clear that $V_i \cong S^{i-k-1}U$, where U is the two-dimensional vector space $H^0(\mathbb{CP}^2(n,1,1), \mathcal{O}(1))$. By consideration of the sequence of Hom's from $\mathcal{O}(k)$ to the triangle (5.1), it is easy to check that $W_i \cong S^{i-k-2}U$ (we use an isomorphism $\Lambda^2 U \cong \mathbf{k}$).

We have isomorphisms

$$\mathrm{Hom}(V_i \otimes \mathcal{O}(k+1), \mathcal{O}(n+1)) = S^{i-k-1}U^* \otimes S^{n-k}U \cong \bigoplus_{j=0}^{i-k-1} S^{n-i+1+2j}U,$$

which implies that

$$\mathrm{Hom}(T_i, \mathcal{O}(n+1)) \cong \bigoplus_{j=1}^{i-k-1} S^{n-i+1+2j}U.$$

On the other hand, there are isomorphisms

$$\mathrm{Hom}(W_i \otimes \mathcal{O}(k), \mathcal{O}(n+1)) = S^{i-k-2}U^* \otimes S^{n-k+1}U \cong \bigoplus_{j=1}^{i-k-1} S^{n-i+1+2j}U,$$

and, moreover, it can be checked that the natural morphism $\mathrm{Hom}(T_i, \mathcal{O}(n+1)) \to \mathrm{Hom}(W_i \otimes \mathcal{O}(k), \mathcal{O}(n+1))$ is an isomorphism. Hence, $\mathrm{Hom}^\bullet(F_i, \mathcal{O}(n+1)) = 0$ for all $k+2 \le i \le n+1$. By the same reasons $\mathrm{Hom}^\bullet(F_i, \mathcal{O}(n)) = 0$ for all $k+2 \le i \le n+1$. Thus the subcategory $\langle \mathcal{O}(n), \mathcal{O}(n+1)\rangle$ is contained in $\langle G_k, G_{k+1}\rangle$.

Since $\mathrm{Hom}(G_k, G_{k+1}) \cong U \cong \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(n+1))$, these two categories are both equivalent to the derived category of representations of the quiver with two vertices and two arrows $\bullet \rightrightarrows \bullet$, and, as a consequence, it can be easily shown that they are equivalent. □

Proof of Proposition 5.5. The argument is similar to the case $n = 3$: in the initial configuration, for $b = 1$, the $n+2$ critical values of W_b approximate a regular polygon, and can essentially be identified with the critical values of the superpotential mirror to $\mathbb{CP}^2(n,1,1)$. We label these critical values by integers from 0 to $n+1$, with 0 corresponding to the positive real critical value, and continuing counterclockwise. As the value of b is decreased towards 0, pairs of complex conjugate critical values of W_b (those labelled k and $n+2-k$, for $1 \leq k \leq \frac{n}{2}$), successively converge towards each other. For $2 \leq k < \frac{n}{2}$, the corresponding vanishing cycles are disjoint, and the two complex conjugate critical values essentially exchange their positions before escaping to infinity (with complex arguments close to $\mp\frac{k-1}{n-2}\,2\pi$) for $b \to 0$. On the other hand, for $k=1$ the two complex conjugate critical points labelled 1 and $n+1$ merge and turn into two real critical points, one of which escapes to infinity as $b \to 0$; similarly for $k = \frac{n}{2}$ if n is even.

If instead of following the real axis we carry out the deformation $b \to 0$ with $\mathrm{Im}(b)$ small positive, then we can avoid all the values of b for which two critical values of W_b coincide, which allows us to keep track of the manner in which $n-2$ of the critical values escape to infinity. This is represented in Figure 12 (left).

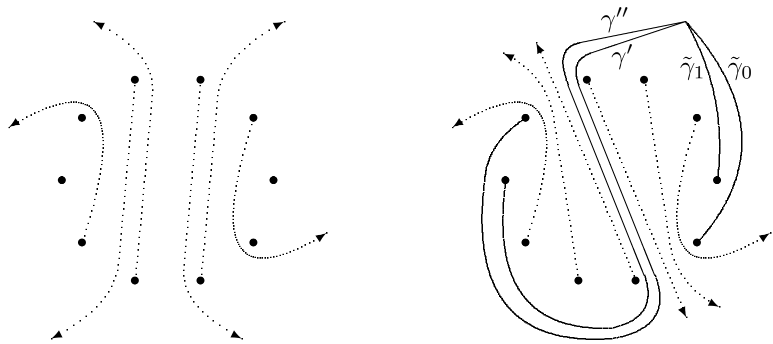

Figure 12: The deformation $b \to 0$ ($n=8$)

Observe that the vanishing cycles at the critical points corresponding to labels in the range $1 \leq k < \frac{n}{2}$ are disjoint from those at the critical points with labels in the range $\frac{n}{2}+2 \leq k \leq n$. Therefore, for the purposes of determining the remaining vanishing cycles as $b \to 0$, the family of Lefschetz fibrations W_b is equivalent to one where the various critical values escape to infinity in a slightly different manner, with the critical values coming from the $\mathrm{Im}\,W < 0$ half-plane staying "to the left" (towards the negative real axis) of those coming from the $\mathrm{Im}\,W > 0$ half-plane, as pictured in Figure 12 (right).

Therefore, if we consider the category of Lagrangian vanishing cycles associated to a system of arcs containing the four arcs $\tilde{\gamma}_0, \tilde{\gamma}_1, \gamma', \gamma''$ represented in Figure 12 right, then the full subcategory singled out by the deformation $b \to 0$ is that generated by the four vanishing cycles $\tilde{L}_0, \tilde{L}_1, L', L''$ associated to these arcs. A suitable collection of arcs can be built by a sequence of sliding operations, starting from a collection $\{\tilde{\gamma}_i,\ 0 \le i \le n+1\}$ where $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are as pictured, and all the $\tilde{\gamma}_i$ remain outside of the unit disc. Identify implicitly the critical points of W_1 with those of the superpotential mirror to $\mathbb{CP}^2(n,1,1)$, and recall that sliding operations correspond to mutations. Then the left dual to the exceptional collection $(\tilde{L}_0, \dots, \tilde{L}_{n+1})$ associated to the arcs $\{\tilde{\gamma}_i\}$ is equivalent (up to some shifts) to the exceptional collection associated to the arcs $(\gamma_2, \gamma_3, \dots, \gamma_{n+1}, \gamma_0, \gamma_1)$ (by the notation of §4). Using $\mathbb{Z}/(n+2)$-equivariance, we see that the latter is equivalent to the exceptional collection associated to the system of arcs $(\gamma_0, \dots, \gamma_{n+1})$ considered in Section 4.

Recall that the two exceptional collections for $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$ presented in Section 2 are mutually dual (cf. Example 2.15), and that Theorem 3.3 identifies the exceptional collection associated to the arcs $(\gamma_0, \dots, \gamma_{n+1})$ with that given by Corollary 2.27. Therefore, there is an equivalence of categories which maps the exceptional collection $(\tilde{L}_0, \dots, \tilde{L}_{n+1})$ for $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$ to the exceptional collection $(\mathcal{O}, \dots, \mathcal{O}(n+1))$ for $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$.

Next, let $k = \lfloor \frac{n+3}{2} \rfloor$, so that γ' and γ'' have the same endpoints as $\tilde{\gamma}_k$ and $\tilde{\gamma}_{k+1}$ respectively. First slide $\tilde{\gamma}_{k+2}, \dots, \tilde{\gamma}_{n+1}$ to the left of $\tilde{\gamma}_k$ and $\tilde{\gamma}_{k+1}$ to obtain another system of arcs $(\tilde{\gamma}_0, \dots, \tilde{\gamma}_{k-1}, \eta_{k+2}, \dots, \eta_{n+1}, \tilde{\gamma}_k, \tilde{\gamma}_{k+1})$. Then the arcs obtained by sliding $\tilde{\gamma}_k$ and $\tilde{\gamma}_{k+1}$ to the left of $\eta_{k+2}, \dots, \eta_{n+1}$ are homotopic to γ' and γ''. This gives us a new system of arcs $(\tilde{\gamma}_0, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{k-1}, \gamma', \gamma'', \eta_{k+2}, \dots \dots, \eta_{n+1})$, which determines a full exceptional collection $(\tilde{L}_0, \tilde{L}_1, \dots, \tilde{L}_{k-1}, L', L'', \Lambda_{k+2}, \dots, \Lambda_{n+1})$ in $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$.

By construction, the full subcategory $\langle \tilde{L}_0, \tilde{L}_1, L', L'' \rangle$ of the category $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$ is equivalent to the triangulated subcategory $\langle \mathcal{O}, \mathcal{O}(1), G_k, G_{k+1} \rangle$ of $\mathbf{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$, which by Lemma 5.6 coincides with $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(n), \mathcal{O}(n+1) \rangle$. As seen in Section 2.7 this category is equivalent to the derived category of the Hirzebruch surface $\mathbb{F}_n$, which completes the proof. $\square$

It is also possible to prove Proposition 5.5 by a direct calculation involving the monodromy of W_1, instead of Lemma 5.6. Starting from the description of the vanishing cycles associated to the arcs γ_i in Section 4, one can determine first the vanishing cycles $\tilde{L}_i$ associated to $\tilde{\gamma}_i$ for all i, and then those associated to γ' and γ''. It is then possible to check that, although the vanishing cycles associated to γ' and γ'' do not quite correspond to $\tilde{L}_n$ and $\tilde{L}_{n+1}$, after sliding γ' and γ'' around each other a certain number of times one obtains two vanishing cycles that are Hamiltonian isotopic to $\tilde{L}_n$ and $\tilde{L}_{n+1}$.

6. Further remarks

6.1. *Higher-dimensional weighted projective spaces.* Many of the arguments in Section 4 extend to higher-dimensional weighted projective spaces, where we worked by induction on dimension in a manner similar to the ideas in Section 5 of [4]. Indeed, the mirror to the weighted projective space $\mathbb{CP}^n(a_0, \dots, a_n)$ is the affine hypersurface $X = \{x_0^{a_0} \dots x_n^{a_n} = 1\} \subset (\mathbb{C}^*)^{n+1}$, equipped with the superpotential $W = x_0 + \cdots + x_n$ and an exact symplectic form ω that we can choose to be invariant under the diagonal action of $\mathbb{Z}/(a_0 + \cdots + a_n)$ and anti-invariant under complex conjugation for simplicity. It is easy to check that W has $a_0 + \cdots + a_n$ critical points over X, all isolated and nondegenerate; the corresponding critical values are the roots λ_j of

$$\lambda^{a_0+\cdots+a_n} = \frac{(a_0 + \cdots + a_n)^{a_0+\cdots+a_n}}{a_0^{a_0} \dots a_n^{a_n}}.$$

As in the two-dimensional case we use $\Sigma_0 = W^{-1}(0)$ as our reference fiber, and join it to the singular fibers of W via straight line segments $\gamma_j \subset \mathbb{C}$ joining the origin to λ_j.

In order to understand the vanishing cycles $L_j \subset \Sigma_0$, we consider as before the projection to one of the coordinate axes, for example $\pi_0 : (x_0, \dots, x_n) \mapsto x_0$. For generic values of λ, the map $\pi_0 : \Sigma_\lambda \to \mathbb{C}^*$ defines an affine Lefschetz fibration on $\Sigma_\lambda = W^{-1}(\lambda)$, with $a_0 + \cdots + a_n$ singular fibers. These singular fibers are the preimages of the critical values of π_0 over Σ_λ, which are the roots of

$$x_0^{a_0} (\lambda - x_0)^{a_1+\cdots+a_n} = \frac{(a_1 + \cdots + a_n)^{a_1+\cdots+a_n}}{a_1^{a_1} \dots a_n^{a_n}} \tag{6.1}$$

(compare with (4.1)). This equation acquires a double root whenever λ is one of the λ_j; the manner in which two of the roots approach each other as one moves from $\lambda = 0$ to $\lambda = \lambda_j$ along the arc γ_j defines an arc $\delta_j \subset \mathbb{C}^*$, which is a *matching path* for the Lefschetz fibration $\pi_0 : \Sigma_0 \to \mathbb{C}^*$. As in the two-dimensional case, the Lagrangian vanishing cycle $L_j \subset \Sigma_0$ is isotopic to a sphere L'_j which lies above the arc δ_j; the generic fiber of $\pi_{0|L'_j} : L'_j \to \delta_j \subset \mathbb{C}^*$ is now a Lagrangian $(n-2)$-sphere inside the fiber of π_0.

Because of the similarity between equations (6.1) and (4.1), it is easy to check that Lemma 4.2 extends almost verbatim to the higher-dimensional case, with the substitution of a_0 for a and $a_1 + \cdots + a_n$ for $b + c$.

In order to determine the Floer complexes $CF^*(L_i, L_j)$, or equivalently $CF^*(L'_i, L'_j)$, we need to understand, for each point of $\delta_i \cap \delta_j$, how L'_i and L'_j intersect each other inside the corresponding fiber of π_0. Because L'_i and L'_j each arise from matching pairs of vanishing cycles of the Lefschetz fibration π_0, this can be done by studying in more detail the topology of the fiber of $\pi_0 : \Sigma_0 \to \mathbb{C}^*$ and the manner in which it degenerates as one moves from a generic value of x_0 to one of the critical values. In fact, we can use the same approach to study the vanishing cycles of $\pi_0 : \Sigma_0 \to \mathbb{C}^*$ as in the case of $W : X \to \mathbb{C}$, namely project the fiber $F_\mu = \pi_0^{-1}(\mu)$ to one of the coordinates, e.g. x_1. This gives rise to a map $\pi_1 : F_\mu \to \mathbb{C}^*$, which is again a Lefschetz fibration (whose fibers are now $(n-3)$-dimensional), with $a_1+\cdots+a_n$ singular fibers corresponding to values of x_1 that solve the equation

$$\mu^{a_0}x_1^{a_1}(-\mu-x_1)^{a_2+\cdots+a_n}=\frac{(a_2+\cdots+a_n)^{a_2+\cdots+a_n}}{a_2^{a_2}\dots a_n^{a_n}},$$

which presents a double root precisely when μ is a solution of (6.1) (for $\lambda=0$). The process can go on similarly, considering successive restrictions to fibers and coordinate projections until we reach the easily understood case of 0-dimensional fibers; once this process is completed, it becomes possible to describe explicitly $CF^*(L'_i,L'_j)$ in terms of the available combinatorial data. The final result is the following:

Proposition 6.1. *For $i<j$, the vanishing cycles L'_i and L'_j intersect transversely, and*

$$|L'_i\cap L'_j|=\#\{I\subset\{0,\dots,n\},\ \sum_{k\in I}a_k=j-i\}.$$

Hence the Floer complex $CF^*(L'_i,L'_j)$ is naturally isomorphic to the degree $j-i$ part of the exterior algebra on $n+1$ generators of respective degrees $a_0,\dots,a_n$. Moreover, the Floer differential is trivial, i.e. $m_1=0$.

Instead of providing a complete proof, we simply illustrate Proposition 6.1 by considering the example of the projective space $\mathbb{CP}^3$. In that case, Σ_0 is an affine K3 surface, and $\pi_0:\Sigma_0\to\mathbb{C}^*$ is a fibration by affine elliptic curves, with four singular fibers. The four vanishing cycles $L'_j\subset\Sigma_0$ project to arcs $\delta_j\subset\mathbb{C}^*$ as shown in Figure 13 (left).

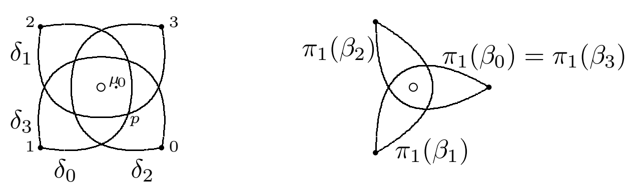

Figure 13: The case of $\mathbb{CP}^3$: images by π_0 of the vanishing cycles $L'_j\subset\Sigma_0$ of W (left), and images by π_1 of the vanishing cycles $\beta_j\subset F_{\mu_0}$ of π_0 (right)

Using the projection π_1 to the second coordinate, we can view each of the fibers of $\pi_0:\Sigma\to\mathbb{C}^*$ as a double cover of $\mathbb{C}^*$ branched in three points Fig. 13, right). To describe the monodromy of the elliptic fibration π_0, we choose a reference fiber $F_{\mu_0}=\pi_0^{-1}(\mu_0)$ for some $\mu_0\in\mathbb{C}^*$ close to 0 on the positive real axis. The monodromy of π_0 around the origin is the diffeomorphism of F_{μ_0} obtained by rotating the three branch points of π_1 counterclockwise by $2\pi/3$. To describe the four vanishing cycles of π_0, we join the regular value μ_0 of π_0 to each of the four critical values by considering arcs which start at μ_0, rotate clockwise around the origin from $\arg\mu=0$ to $\arg\mu=-\frac{\pi}{4}-j\frac{\pi}{2}$ $(0\le j\le 3)$, and then go radially outward to the corresponding critical values of π_0. The vanishing cycles $\beta_0, \dots, \beta_3$ obtained in this way are isotopic to the double lifts via $\pi_1 : F_{\mu_0} \to \mathbb{C}^*$ of the arcs shown in Figure 13 (right).

Now that the monodromy of π_0 is well-understood, it is not hard to visualize the Lagrangian spheres $L'_j \subset \Sigma_0$ lying above the arcs δ_j, and in particular their intersections. For example, $L'_0 \cap L'_1$ consists of four points, one of which is the critical point of π_0 with $\arg x_0 = \frac{3\pi}{4}$ (lying above the common end point of δ_0 and δ_1), while the three others lie in the fiber above the other point p of $\delta_0 \cap \delta_1$ (with $\arg x_0 = -\frac{\pi}{4}$), and correspond (under a suitable parallel transport operation) to the three intersections between β_1 and β_2 in F_{μ_0}. Similarly, $L'_0 \cap L'_2$ consists of six points (three above each point of $\delta_0 \cap \delta_2$), and so on.

Finally, we observe that there cannot be any contributions to the Floer differential m_1, for purely topological reasons. Indeed, if we consider any two intersection points $p, q \in L'_i \cap L'_j$ for some pair (i,j), and any two arcs $\gamma \subset L'_i$ and $\gamma' \subset L'_j$ joining p to q, then γ and γ' are never homotopic inside Σ_0, as easily seen by considering either $\pi_0(\gamma)$ and $\pi_0(\gamma')$ (if $\pi_0(p) \neq \pi_0(q)$), or $\pi_1(\gamma)$ and $\pi_1(\gamma')$ (if $\pi_0(p) = \pi_0(q)$).

The proof of Proposition 6.1 is essentially a careful induction on successive slices and coordinate projections, where one manages to understand the structure of the intersections between vanishing cycles by starting with a 1-dimensional slice of Σ_0 and then adding one extra dimension at a time; the main difficulty resides in setting up the induction properly and in choosing manageable notation for the many objects that appear in the proof, rather than in the actual calculations which are essentially always the same.

The next step towards understanding the category of vanishing cycles of the Lefschetz fibration $W : X \to \mathbb{C}$ would be to study the moduli spaces of pseudo-holomorphic maps from a disc with three or more marked points to Σ_0 with boundary on $\bigcup L'_j$, something which falls beyond the scope of this paper. Nonetheless, a careful observation suggests that the main features observed in the two-dimensional case, namely the vanishing of m_k for $k \geq 3$ and the exterior algebra structure underlying m_2, should extend to the higher-dimensional case.

For example, in the case of $\mathbb{CP}^3$, we can study $m_2 : \mathrm{Hom}(L'_0, L'_1) \otimes \mathrm{Hom}(L'_1, L'_2) \to \mathrm{Hom}(L'_0, L'_2)$ by looking carefully at Figure 13. Let α_0 (resp. β_0) be the morphism from L'_0 to L'_1 (resp. from L'_1 to L'_2) which corresponds to their intersection at a critical point of π_0, and let $\alpha_1, \alpha_2, \alpha_3$ (resp. $\beta_1, \beta_2, \beta_3$) be the three other morphisms between these two vanishing cycles (labelling them in a consistent way with respect to the other coordinate projections). When Σ_0 is equipped with an almost-complex structure for which the projection π_0 is holomorphic, pseudo-holomorphic discs project to immersed triangular regions in $\mathbb{C}^*$ with boundary on $\delta_0 \cup \delta_1 \cup \delta_2$; there are three such regions (to the upper-left, to the upper-right, and to the bottom of Figure 13 left). To start with, it is immediate from an observation of Figure 13 that

$m_2(\alpha_0, \beta_0) = 0$. Next, by deforming the arcs δ_0 and δ_1 to make them lie very close to each other near their common end point, we can shrink the upper-left region to a very thin triangular sector, in which case exactly one pseudo-holomorphic map contributes to the composition of α_0 with each of $\beta_1, \beta_2, \beta_3$. It is then easy to see that composition with α_0 induces an isomorphism from $\mathrm{span}(\beta_1, \beta_2, \beta_3) \subset \mathrm{Hom}(L'_1, L'_2)$ to the subspace of $\mathrm{Hom}(L'_0, L'_2)$ spanned by the three intersections for which $\arg x_0 = \frac{\pi}{2}$. Considering the upper-right triangular region delimited by $\delta_0, \delta_1, \delta_2$ in Figure 13 left, we can conclude that the same is true for the compositions of $\alpha_1, \alpha_2, \alpha_3$ with β_0; arguing by symmetry we can check that $m_2(\alpha_0, \beta_i) = \pm m_2(\alpha_i, \beta_0)$ for $i = 1, 2, 3$ (and, hopefully, a careful study of orientations should allow one to conclude that the signs are all negative).

By a similar argument, we can study $m_2(\alpha_i, \beta_j)$ for $1 \leq i, j \leq 3$ by shrinking the lower triangular region of Figure 13 left to a single point, which allows us to localize all the relevant intersection points and pseudo-holomorphic discs into a single fiber of π_0. The intersection pattern inside that fiber of π_0 is then described by Figure 13 right, so that things become essentially identical to the discussion carried out in the previous section for the Lefschetz fibration mirror to $\mathbb{CP}^2$ (observe the similarity between Figures 13 right and 5 right). Hence, the same argument as in the two-dimensional case shows in particular that $m_2(\alpha_i, \beta_i) = 0$ for $1 \leq i \leq 3$ and $m_2(\alpha_i, \beta_j) = \pm m_2(\alpha_j, \beta_i)$ for $1 \leq i \neq j \leq 3$.

6.2. *Noncommutative deformations of* $\mathbb{CP}^2$. As mentioned in the introduction, in the general case one expects the mirror to be obtained by partial (fiberwise) compactification of the Landau-Ginzburg model given by the toric mirror ansatz. While not required in the toric Fano case considered here, this fiberwise compactification allows for more freedom of deformation, since it enlarges $H^2(X, \mathbb{C})$; this sometimes makes it possible to recover more general (nontoric) noncommutative deformations of the Fano manifold. We now illustrate this by briefly discussing the case of $\mathbb{CP}^2$ (see [5] for more details and additional examples). We will show the following:

PROPOSITION 6.2. *Nonexact symplectic deformations of the fiberwise compactified Landau-Ginzburg model* $(\bar{X}, \bar{W})$ *correspond to general noncommutative deformations of the projective plane.*

Moreover, we expect that there is a simple relation between the cohomology class of the symplectic form on $\bar{X}$ and the noncommutative deformation parameters for $\mathbb{CP}^2$.

Recall that a general noncommutative projective plane is defined by a graded regular algebra which is presented by three generators of degree one and three quadratic relations. All these noncommutative planes were described in the papers [2], [1], and with another point of view in [12]. It was proved in [2] that isomorphism classes of regular graded algebras of dimension 3 generated by three elements of degree 1 are in bijective correspondence with isomorphism classes of regular triples $\mathcal{T} = (E, \sigma, L)$, where one of the following holds:

1) $E = \mathbb{P}^2$, σ is an automorphism of $\mathbb{P}^2$, and $L = \mathcal{O}(1)$;
2) E is a divisor of degree 3 in $\mathbb{P}^2$, L is the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$, and σ is an automorphism of E such that $(\sigma^* L)^2 \cong L \otimes \sigma^{2*} L, \quad \sigma^* L \ncong L$.

The triples (and the algebras) of the first type are related to the ordinary commutative $\mathbb{P}^2$ in the sense that the category qgr of such an algebra is equivalent to the category $\mathrm{coh}(\mathbb{P}^2)$, whereas the triples of the second type are related to the nontrivial noncommutative projective planes. For example, the toric noncommutative deformations of $\mathbb{P}^2$, which were discussed above, correspond to the triples with E isomorphic to a triangle (union of three lines).

Consider now the noncommutative projective planes which correspond to triples with E isomorphic to a smooth elliptic curve. We know that sometimes the categories qgr of two different graded algebras can be equivalent. In particular, with this point of view any triple with smooth E is equivalent to a triple with the same E and such that σ is a translation by a point of E (see §8 of [12]). On the other hand, according to [1, 10.14], the equations defining a generic regular graded algebra, which corresponds to a triple (E, σ, L) with E a smooth elliptic curve and σ a translation, can be put into the form

$$\begin{aligned} f_1 &= cx^2 + byz + azy = 0, \\ f_2 &= axz + cy^2 + bzx = 0, \\ f_3 &= bxy + ayx + cz^2 = 0. \end{aligned}$$

This means that the DG category $\mathfrak{C}$ for these noncommutative projective planes can be described in the following way. It has three objects, say l_0, l_1, l_2, and for $i < j$ the spaces of morphisms $\mathrm{Hom}(l_i, l_j)$ are 3-dimensional, with all elements of degree $(j - i)$. There are bases $x_0, y_0, z_0 \in \mathrm{Hom}(l_0, l_1)$, $x_1, y_1, z_1 \in \mathrm{Hom}(l_1, l_2)$, $\bar{x}, \bar{y}, \bar{z} \in \mathrm{Hom}(l_0, l_2)$ in which the nontrivial compositions are given by the following formulas:

$$\begin{array}{lll} m_2(x_0, y_1) = a\bar{z}, & m_2(x_0, z_1) = b\bar{y}, & m_2(x_0, x_1) = c\bar{x}, \\ m_2(y_0, z_1) = a\bar{x}, & m_2(y_0, x_1) = b\bar{z}, & m_2(y_0, y_1) = c\bar{y}, \\ m_2(z_0, x_1) = a\bar{y}, & m_2(z_0, y_1) = b\bar{x}, & m_2(z_0, z_1) = c\bar{z}. \end{array}$$

All other compositions (except those involving identity morphisms) vanish.

Recall from Section 4 that the mirror of $\mathbb{CP}^2$ is an elliptic fibration with three singular fibers. In the affine setting, the generic fibers of $W = x+y+z$ on $X = \{xyz = 1\}$ are tori with three punctures, but it is possible to compactify X partially into an elliptic fibration $\bar{W} : \bar{X} \to \mathbb{C}$ whose fibers are closed curves;

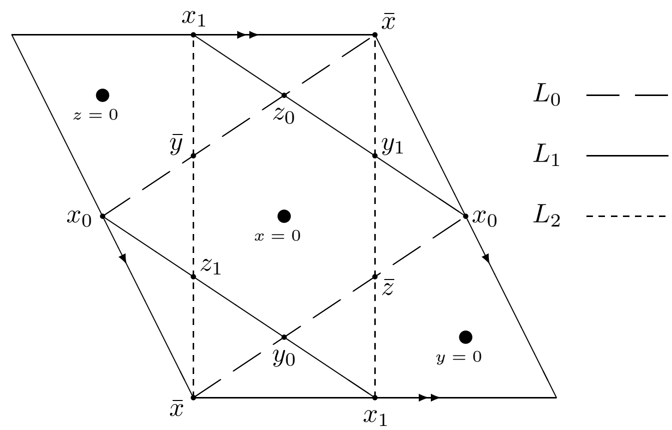

Figure 14: The vanishing cycles of the compactified mirror of $\mathbb{CP}^2$

unlike what happens in more complicated (nontoric) examples, this does not introduce any extra critical points.

The generic fiber of $\bar{W}$ and the three vanishing cycles are as represented in Figure 14 (compare with Figure 5 right, which represents the images by π_x of the same vanishing cycles; see also Figure 2 of [35]); the bold dots represent the intersections of the fiber with the compactification divisor.

While it is easy to see that m_k remains trivial for $k \neq 2$, the compactification modifies the product m_2 in the category $\mathrm{Lag}_{\mathrm{vc}}(\bar{W}, \{\gamma_i\})$ by introducing an infinite number of immersed triangular regions with boundary in $L_0 \cup L_1 \cup L_2$. This induces a deformation of the product structure, and the uncompactified case considered in Section 4 now arises as a limiting situation in which the areas of the hexagonal regions containing the intersections with the compactification divisor tend to infinity.

For example, the product $m_2(x_0, y_1)$ remains a multiple of $\bar{z}$, but the relevant coefficient is now a sum of infinitely many contributions, corresponding to immersed triangles in which the edge joining x_0 to y_1 is an arbitrary immersed arc between these two points inside L_1. The convergence of the series $\sum_i \pm \exp(-2\pi\,\mathrm{area}(T_i))$ follows directly from the fact that the area grows quadratically with the number of times that the $x_0 y_1$ edge wraps around L_1. Similarly, $m_2(y_0, x_1)$ is a multiple of $\bar{z}$ as in the uncompactified case, but with a coefficient now given by the sum of an infinite series of contributions; and similarly for $m_2(y_0, z_1)$ and $m_2(y_1, z_0)$, which remain multiples of $\bar{x}$, and for $m_2(z_0, x_1)$ and $m_2(x_0, z_1)$, which are proportional to $\bar{y}$.

The important new feature of the compactified Landau-Ginzburg model is that $m_2(x_0, x_1)$ is now a multiple of $\bar{x}$ (with a coefficient that may be zero or nonzero depending on the choice of the cohomology class of the symplectic form); since there are again infinitely many immersed triangular regions with vertices $x_0, x_1, \bar{x}$ (the smallest two of which are embedded and easily visible in Figure 14), the relevant coefficient is the sum of an infinite series.

Observe that the two embedded triangles are to be counted with opposite signs (the differences in orientations at the two vertices of degree 1 cancel each other, while the nontriviality of the spin structures and the complementarity of the sides result in a total of three sign changes, see §4.6); hence, in the "symmetric" case where the six triangular regions delimited by $L_0 \cup L_1 \cup L_2$ have equal areas, these two contributions cancel each other. The same is true of the other (immersed) triangles with vertices $x_0, x_1, \bar{x}$, which arise in similarly cancelling pairs. Hence, in the symmetric situation, we end up having $m_2(x_0, x_1) = 0$ as in Section 4; however in the general case $m_2(x_0, x_1)$ can still be a nonzero multiple of $\bar{x}$. There are similar statements for $m_2(y_0, y_1)$ and $m_2(z_0, z_1)$, which are multiples of $\bar{y}$ and $\bar{z}$ respectively (and also vanish in the symmetric case).

6.3. *HMS for products.* Let $W_1 : X_1 \to \mathbb{C}$ and $W_2 : X_2 \to \mathbb{C}$ be two Lefschetz fibrations, with critical points respectively p_i, $1 \le i \le r$, and q_j, $1 \le j \le s$, and associated critical values $\lambda_i = W_1(p_i)$ and $\mu_j = W_2(q_j)$. Then $W = W_1 + W_2 : X_1 \times X_2 \to \mathbb{C}$ is a Lefschetz fibration with rs critical points (p_i, q_j), and associated critical values $W(p_i, q_j) = \lambda_i + \mu_j$ (we will assume that these are pairwise distinct and nonzero).

For all $t \in \mathbb{C}$, the fiber $M_t = W^{-1}(t) \subset X_1 \times X_2$ can be viewed as the total space of a fibration $\phi_t : M_t \to \mathbb{C}$ given by $\phi_t(p, q) = W_1(p)$, with fiber $\phi_t^{-1}(\lambda) = W_1^{-1}(\lambda) \times W_2^{-1}(t - \lambda)$. The $r + s$ critical values of ϕ_t are $\lambda_1, \dots, \lambda_r$ and $t - \mu_1, \dots, t - \mu_s$. If t varies along an arc γ joining 0 to $\lambda_i + \mu_j$, the critical value $t - \mu_j$ of ϕ_t converges to the critical value λ_i by following the arc $\gamma - \mu_j$. Hence, the vanishing cycle $L_\gamma \subset M_0$ associated to the arc γ is a *fibered* Lagrangian sphere, mapped by ϕ_0 to the arc $\tilde{\gamma} = \gamma - \mu_j$ joining the critical values $-\mu_j$ and λ_i of ϕ_0.

More precisely, the fiber of ϕ_0 above an interior point of $\tilde{\gamma}$ is symplectomorphic to the product $\Sigma_1 \times \Sigma_2$ of the smooth fibers of W_1 and W_2, and its intersection with the vanishing cycle L_γ is a product of two Lagrangian spheres $S_i \times T_j \subset \Sigma_1 \times \Sigma_2$, where S_i and T_j correspond to vanishing cycles of W_1 and W_2 associated to the critical values λ_i and μ_j respectively. Above the end points of $\tilde{\gamma}$, the product $S_i \times T_j$ collapses to either $\{p_i\} \times T_j$ (above $\tilde{\gamma}(1) = \lambda_i$) or $S_i \times \{q_j\}$ (above $\tilde{\gamma}(0) = -\mu_j$). Denoting by n_i the complex dimension of X_i, a model for the topology of the restriction of ϕ_0 to L_γ is given by the map $\phi : S^{n_1+n_2-1} \to [0,1]$ defined over the unit sphere in $\mathbb{R}^{n_1+n_2}$ by $(x_1, \dots, x_{n_1}, x_{n_1+1}, \dots, x_{n_1+n_2}) \mapsto x_1^2 + \dots + x_{n_1}^2$.

Up to a suitable isotopy we can assume that the critical values λ_i all have the same imaginary part, and $0 < \mathrm{Im}(\lambda_i) \ll \mathrm{Re}(\lambda_1) \ll \dots \ll \mathrm{Re}(\lambda_r)$ (so that line segments joining the origin to λ_i form an ordered collection that can be used to define $\mathrm{Lag_{vc}}(W_1)$). Similarly, assume that μ_j all have the same real part, and $0 < \mathrm{Re}(\mu_j) \ll \mathrm{Im}(\mu_s) \ll \cdots \ll \mathrm{Im}(\mu_1)$. Then there is a natural way to choose arcs γ_{ij}, $1 \le i \le r$, $1 \le j \le s$, joining the origin to $\lambda_i + \mu_j$, with both real and imaginary parts monotonically increasing, in such a way that the lexicographic ordering of the labels ij coincides with the clockwise ordering of the arcs γ_{ij} around the origin. The arcs $\tilde{\gamma}_{ij}$ to which the vanishing cycles $L_{ij} \subset M_0$ project under ϕ_0 are then as shown in Figure 15.

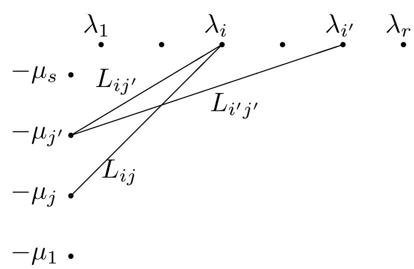

Figure 15: The vanishing cycles of $W = W_1 + W_2 : X_1 \times X_2 \to \mathbb{C}$

In this situation, we have the following result, which gives supporting evidence for Conjecture 1.3:

PROPOSITION 6.3. *The vanishing cycles L_{ij} of W are in one-to-one correspondence with pairs of vanishing cycles (S_i, T_j) of W_1 and W_2, and*

$$\mathrm{Hom}_{\mathrm{Lag_{vc}}(W_1+W_2)}(L_{ij}, L_{i'j'}) \simeq \mathrm{Hom}_{\mathrm{Lag_{vc}}(W_1)}(S_i, S_{i'}) \otimes \mathrm{Hom}_{\mathrm{Lag_{vc}}(W_2)}(T_j, T_{j'}).$$

Sketch of proof. For $i < i'$ and $j < j'$, the intersections between L_{ij} and $L_{i'j'}$ localize into a single smooth fiber of ϕ_0, whose intersection with L_{ij} is $S_i \times T_j$ while the intersection with $L_{i'j'}$ is $S_{i'} \times T_{j'}$ (up to isotopy in general, but by suitably modifying the fibrations W_1 and W_2 to make them trivial over large open subsets and by choosing the arcs γ_{ij} carefully we can make this hold strictly). Therefore, in this case intersection points between L_{ij} and $L_{i'j'}$ correspond to pairs of intersections between S_i and $S_{i'}$ and between T_j and $T_{j'}$, so that $\mathrm{Hom}(L_{ij}, L_{i'j'}) \simeq \mathrm{Hom}(S_i, S_{i'}) \otimes \mathrm{Hom}(T_j, T_{j'})$. After choosing suitable trivializations of the canonical bundles (so that the phase of L_{ij} at an intersection point can easily be compared with the sums of the phases of S_i and T_j), it becomes easy to check that this isomorphism is compatible with gradings.

When $i = i'$ and $j < j'$ the intersections between L_{ij} and $L_{ij'}$ lie in a singular fiber of ϕ_0 (of the form $W_1^{-1}(\lambda_i) \times \Sigma_2$), inside which L_{ij} and $L_{ij'}$ identify with $\{p_i\} \times S_j$ and $\{p_i\} \times S_{j'}$ respectively (see Fig. 15); recalling that $\mathrm{Hom}(S_i, S_i) = \mathbb{C}$ by definition, we obtain the desired formula, similarly for $L_{ij} \cap L_{i'j}$ when $i < i'$ and $j = j'$. Finally, the case $i = i'$ and $j = j'$ is trivial.

In all other cases, there are no morphisms from L_{ij} to $L_{i'j'}$. Indeed, if either $i > i'$ or $i = i'$ and $j > j'$ then (i,j) follows (i',j') in the lexicographic ordering, so that there are no morphisms from L_{ij} to $L_{i'j'}$. The only remaining case is when $i < i'$ and $j > j'$; in that case the triviality of $\mathrm{Hom}(L_{ij}, L_{i'j'})$ follows from the fact $L_{ij} \cap L_{i'j'} = \emptyset$ (because the projections $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}_{i'j'}$ are disjoint). □

In order to prove Conjecture 1.3, one needs to achieve a better understanding of pseudo-holomorphic discs in M_0 with boundary in $\bigcup L_{ij}$. This is most easily done in the case of low-dimensional examples such as the mirror to $\mathbb{CP}^1 \times \mathbb{CP}^1$ (already studied in a different manner in Section 5.1), or more generally any situation where the fibers are 0-dimensional, because the description then becomes purely combinatorial. Another piece of supporting evidence is the following:

LEMMA 6.4. *When $i < i' < i''$ and $j < j' < j''$, the composition m_2 : $\mathrm{Hom}(L_{ij}, L_{i'j'}) \otimes \mathrm{Hom}(L_{i'j'}, L_{i''j''}) \to \mathrm{Hom}(L_{ij}, L_{i''j''})$ is expressed (up to homotopy) in terms of compositions in $\mathrm{Lag}_{\mathrm{vc}}(W_1)$ and $\mathrm{Lag}_{\mathrm{vc}}(W_2)$ by the formula $m_2(s \otimes t, s' \otimes t') = m_2(s, s') \otimes m_2(t, t')$.*

Sketch of proof. After deforming the fibrations W_1 and W_2 and the arcs γ_{ij}, $\gamma_{i'j'}$, $\gamma_{i''j''}$ (hence "up to homotopy" in the statement), we can assume that all intersections between L_{ij}, $L_{i'j'}$ and $L_{i''j''}$ occur in a portion of M_0 where the fibration ϕ_0 is trivial. Choose an almost-complex structure which is locally a product in $\phi_0^{-1}(U) \simeq U \times \Sigma_1 \times \Sigma_2 \subset M_0$. Then every pseudo-holomorphic disc with boundary in $L_{ij} \cup L_{i'j'} \cup L_{i''j''}$ contributing to m_2 projects under ϕ_0 to the same triangular region in U (the unique triangular region with boundary in $\tilde{\gamma}_{ij} \cup \tilde{\gamma}_{i'j'} \cup \tilde{\gamma}_{i''j''}$, which we can assume to be arbitrarily small), while the projections to the factors Σ_1 and Σ_2 are exactly those pseudo-holomorphic discs which contribute to $m_2 : \mathrm{Hom}(S_i, S_{i'}) \otimes \mathrm{Hom}(S_{i'}, S_{i''}) \to \mathrm{Hom}(S_i, S_{i''})$ and $m_2 : \mathrm{Hom}(T_j, T_{j'}) \otimes \mathrm{Hom}(T_{j'}, T_{j''}) \to \mathrm{Hom}(T_j, T_{j''})$. □

Other parts of Conjecture 1.3 are also accessible to similar methods. However, the general situation is quite subtle, partly because the definition of higher compositions in a product of two A_∞-categories is more complicated than one might think, but also because one has to deal with more complicated moduli spaces of pseudo-holomorphic discs.

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