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Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$

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Abstract

We obtain global well-posedness, scattering, and global $L^{10}_{t,x}$ spacetime bounds for energy-class solutions to the quintic defocusing Schrödinger equation in $\mathbb{R}^{1+3}$, which is energy-critical. In particular, this establishes global existence of classical solutions. Our work extends the results of Bourgain [4] and Grillakis [20], which handled the radial case. The method is similar in spirit to the induction-on-energy strategy of Bourgain [4], but we perform the induction analysis in both frequency space and physical space simultaneously, and replace the Morawetz inequality by an interaction variant (first used in [12], [13]). The principal advantage of the interaction Morawetz estimate is that it is not localized to the spatial origin and so is better able to handle nonradial solutions. In particular, this interaction estimate, together with an almost-conservation argument controlling the movement of L^2 mass in frequency space, rules out the possibility of energy concentration.

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1. Introduction

1.1. *Critical* NLS *and main result.* We consider the Cauchy problem for the quintic defocusing Schrödinger equation in $\mathbb{R}^{1+3}$

$$\begin{cases} iu_t + \Delta u = |u|^4 u \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u(t,x)$ is a complex-valued field in spacetime $\mathbb{R}_t \times \mathbb{R}_x^3$. This equation has as Hamiltonian,

$$E(u(t)) := \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 \, dx. \tag{1.2}$$

Since the Hamiltonian (1.2) is preserved by the flow (1.1) we shall often refer to it as the *energy* and write $E(u)$ for $E(u(t))$.

Semilinear Schrödinger equations - with and without potentials, and with various nonlinearities - arise as models for diverse physical phenomena, including Bose-Einstein condensates [23], [35] and as a description of the envelope dynamics of a general dispersive wave in a weakly nonlinear medium (see e.g.

the survey in [43], Chapter 1). Our interest here in the defocusing quintic equation (1.1) is motivated mainly, though, by the fact that the problem is critical with respect to the energy norm. Specifically, we map a solution to another solution through the scaling $u \mapsto u^\lambda$ defined by

$$u^\lambda(t,x) := \frac{1}{\lambda^{1/2}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \tag{1.3}$$

and this scaling leaves both terms in the energy invariant.

The Cauchy problem for this equation has been intensively studied ([9], [20], [4], [5],[18], [26]). It is known (see e.g. [10], [9]) that if the initial data $u_0(x)$ has finite energy, then the Cauchy problem is locally well-posed, in the sense that there exists a local-in-time solution to (1.1) which lies in $C_t^0 \dot{H}_x^1 \cap L_{t,x}^{10}$, and is unique in this class; furthermore the map from initial data to solution is locally Lipschitz continuous in these norms. If the energy is small, then the solution is known to exist globally in time, and scatters to a solution $u_\pm(t)$ to the free Schrödinger equation $(i\partial_t + \Delta)u_\pm = 0$, in the sense that $\|u(t) - u_\pm(t)\|_{\dot{H}^1(\mathbb{R}^3)} \to 0$ as $t \to \pm\infty$. For (1.1) with large initial data, the arguments in [10], [9] do not extend to yield global well-posedness, even with the conservation of the energy (1.2), because the time of existence given by the local theory depends on the profile of the data as well as on the energy.[1] For large finite energy data which is assumed to be in addition radially symmetric, Bourgain [4] proved global existence and scattering for (1.1) in $\dot{H}^1(\mathbb{R}^3)$. Subsequently Grillakis [20] gave a different argument which recovered part of [4] — namely, global existence from smooth, radial, finite energy data. For general large data — in particular, general smooth data — global existence and scattering were open.

Our main result is the following global well-posedness result for (1.1) in the energy class.

THEOREM 1.1. *For any u_0 with finite energy, $E(u_0) < \infty$, there exists a unique*[2] *global solution $u \in C_t^0(\dot{H}_x^1) \cap L_{t,x}^{10}$ to* (1.1) *such that*

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(t,x)|^{10} \, dxdt \leq C(E(u_0)). \tag{1.4}$$

for some constant $C(E(u_0))$ that depends only on the energy.

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[1] This is in constrast with sub-critical equations such as the cubic equation $iu_t + \Delta u = |u|^2 u$, for which one can use the local well-posedness theory to yield global well-posedness and scattering even for large energy data (see [17], and the surveys [7], [8]).

[2] In fact, uniqueness actually holds in the larger space $C_t^0(\dot{H}_x^1)$ (thus eliminating the constraint that $u \in L_{t,x}^{10}$), as one can show by adapting the arguments of [27], [15], [14]; see Section 16.

As is well-known (see e.g. [5], or [13] for the sub-critical analogue), the $L^{10}_{t,x}$ bound above also gives scattering, asymptotic completeness, and uniform regularity:

COROLLARY 1.2. *Let u_0 have finite energy. Then there exist finite energy solutions $u_\pm(t,x)$ to the free Schrödinger equation $(i\partial_t + \Delta)u_\pm = 0$ such that*

$$\|u_\pm(t) - u(t)\|_{\dot{H}^1} \to 0 \text{ as } t \to \pm\infty.$$

Furthermore, the maps $u_0 \mapsto u_\pm(0)$ are homeomorphisms from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$. Finally, if $u_0 \in H^s$ for some $s > 1$, then $u(t) \in H^s$ for all time t, and one has the uniform bounds

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} \leq C(E(u_0), s)\|u_0\|_{H^s}.$$

It is also fairly standard to show that the $L^{10}_{t,x}$ bound (1.4) implies further spacetime integrability on u. For instance u obeys all the Strichartz estimates that a free solution with the same regularity does (see, for example, Lemma 3.12 below).

The results here have analogs in previous work on second order wave equations on $\mathbb{R}^{3+1}$ with energy-critical (quintic) defocusing nonlinearities. Global-in-time existence for such equations from smooth data was shown by Grillakis [21], [22] (for radial data see Struwe [42], for small energy data see Rauch [36]); global-in-time solutions from finite energy data were shown in Kapitanski [25], Shatah-Struwe [39]. For an analog of the scattering statement in Corollary 1.2 for the critical wave equation; see Bahouri-Shatah [2], Bahouri-Gérard [1] for the scattering statement for Klein-Gordon equations see Nakanishi [30] (for radial data, see Ginibre-Soffer-Velo[16]). The existence results mentioned here all involve an argument showing that the solution's energy cannot concentrate. These energy nonconcentration proofs combine *Morawetz inequalities* (*a priori* estimates for the nonlinear equations which bound some quantity that scales like energy) with careful analysis that strengthens the Morawetz bound to control of energy. Besides the presence of infinite propagation speeds, a main difference between (1.1) and the hyperbolic analogs is that here time scales like λ^2, and as a consequence the quantity bounded by the Morawetz estimate is supercritical with respect to energy.

Section 4 below provides a fairly complete outline of the proof of Theorem 1.1. In this introduction we only briefly sketch some of the ideas involved: a suitable modification of the Morawetz inequality for (1.1), along with the frequency-localized L^2 almost-conservation law that we'll ultimately use to prohibit energy concentration.

A typical example of a Morawetz inequality for (1.1) is the following bound due to Lin and Strauss [33] who cite [34] as motivation,

$$\int_I \int_{\mathbb{R}^3} \frac{|u(t,x)|^6}{|x|}\, dxdt \lesssim \left(\sup_{t\in I} \|u(t)\|_{\dot{H}^{1/2}} \right)^2 \tag{1.5}$$

for arbitrary time intervals I. (The estimate (1.5) follows from a computation showing the quantity,

$$\int_{\mathbb{R}^3} \mathrm{Im}\left(\bar{u}\nabla u \cdot \frac{x}{|x|} \right) dx \tag{1.6}$$

is monotone in time.) Observe that the right-hand side of (1.5) will not grow in I if the H^1 and L^2 norms are bounded, and so this estimate gives a uniform bound on the left-hand side where I is any interval on which we know the solution exists. However, in the energy-critical problem (1.1) there are two drawbacks with this estimate. The first is that the right-hand side involves the $\dot{H}^{1/2}$ norm, instead of the energy E. This is troublesome since any Sobolev norm rougher than $\dot{H}^1$ is supercritical with respect to the scaling (1.3). Specifically, the right-hand side of (1.5) increases without bound when we simply scale given finite energy initial data according to (1.3) with λ large. The second difficulty is that the left-hand side is localized near the spatial origin $x = 0$ and does not convey as much information about the solution u away from this origin. To get around the first difficulty Bourgain [4] and Grillakis [20] introduced a localized variant of the above estimate:

$$\int_I \int_{|x|\lesssim |I|^{1/2}} \frac{|u(t,x)|^6}{|x|}\, dxdt \lesssim E(u)|I|^{1/2}. \tag{1.7}$$

As an example of the usefulness of (1.7), we observe that this estimate prohibits the existence of finite energy (stationary) pseudosoliton solutions to (1.1). By a (stationary) *pseudosoliton* we mean a solution such that $|u(t,x)| \sim 1$ for all $t \in \mathbb{R}$ and $|x| \lesssim 1$; this notion includes soliton and breather type solutions. Indeed, applying (1.7) to such a solution, we would see that the left-hand side grows by at least $|I|$, while the right-hand side is $O(|I|^{\frac{1}{2}})$, and so a pseudosoliton solution will lead to a contradiction for $|I|$ sufficiently large. A similar argument allows one to use (1.7) to prevent "sufficiently rapid" concentration of (potential) energy at the origin; for instance, (1.7) can also be used to rule out self-similar type blowup,[3], where the potential energy density $|u|^6$ concentrates in the ball $|x| < A|t - t_0|$ as $t \to t_0^-$ for some fixed $A > 0$. In [4], one main use of (1.7) was to show that for each fixed time interval I, there

[3]This is not the only type of self-similar blowup scenario; another type is when the energy concentrates in a ball $|x| \leq A|t - t_0|^{1/2}$ as $t \to t_0^-$. This type of blowup is consistent with the scaling (1.3) and is not directly ruled out by (1.7); however it can instead be ruled out by spatially local mass conservation estimates. See [4], [20]

exists at least one time $t_0 \in I$ for which the potential energy was dispersed at scale $|I|^{1/2}$ or greater (i.e. the potential energy could not concentrate on a ball $|x| \ll |I|^{1/2}$ for all times in I).

To summarize, the localized Morawetz estimate (1.7) is very good at preventing u from concentrating near the origin; this is especially useful in the case of radial solutions u, since the radial symmetry (combined with conservation of energy) enforces decay of u away from the origin, and so resolves the second difficulty with the Morawetz estimate mentioned earlier. However, the estimate is less useful when the solution is allowed to concentrate away from the origin. For instance, if we aim to preclude the existence of a moving pseudosoliton solution, in which $|u(t,x)| \sim 1$ when $|x - vt| \lesssim 1$ for some fixed velocity v, then the left-hand side of (1.7) only grows like $\log |I|$ and so one does not necessarily obtain a contradiction.[4]

It is thus of interest to remove the $1/|x|$ denominator in (1.5), (1.7), so that these estimates can more easily prevent concentration at arbitrary locations in spacetime. In [12], [13] this was achieved by translating the origin in the integrand of (1.6) to an arbitrary point y, and averaging against the L^1 mass density $|u(y)|^2\, dy$. In particular, the following *interaction Morawetz* estimate[5]

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^4 \, dxdt \lesssim \|u(0)\|_{L^2}^2 \left(\sup_{t\in I} \|u(t)\|_{\dot{H}^{1/2}} \right)^2 \tag{1.8}$$

was obtained. (We have since learned that this averaging argument has an analog in early work presenting and analyzing interaction functionals for one dimensional hyperbolic systems, e.g. [19], [38].) This $L^4_{t,x}$ estimate already gives a short proof of scattering in the energy class (and below!) for the cubic nonlinear Schrödinger equation (see [12], [13]); however, like (1.5), this estimate is not suitable for the critical problem because the right-hand side is not controlled by the energy $E(u)$. One could attempt to localize (1.8) as in (1.7), obtaining for instance a scale-invariant estimate such as

$$\int_I \int_{|x|\lesssim |I|^{1/2}} |u(t,x)|^4 \, dxdt \lesssim E(u)^2 |I|^{3/2}, \tag{1.9}$$

[4]At first glance it may appear that the global estimate (1.5) is still able to preclude the existence of such a pseudosoliton, since the right-hand side does not seem to grow much as I gets larger. This can be done in the cubic problem (see e.g. [17]) but in the critical problem one can lose control of the $\dot{H}^{1/2}$ norm, by adding some very low frequency components to the soliton solution u. One might object that one could use L^2 conservation to control the $H^{1/2}$ norm, however one can rescale the solution to make the L^2 norm (and hence the $\dot{H}^{1/2}$ norm) arbitrarily large.

[5]Strictly speaking, in [12], [13] this estimate was obtained for the *cubic* defocusing nonlinear Schrödinger equation instead of the quintic, but the argument in fact works for all nonlinear Schrödinger equations with a pure power defocusing nonlinearity, and even for a slightly more general class of repulsive nonlinearities satisfying a standard monotonicity condition. See [13] and Section 10 below for more discussion.

but this estimate, while true (in fact it follows immediately from Sobolev and Hölder), is useless for such purposes as prohibiting soliton-like behaviour, since the left-hand side grows like $|I|$ while the right-hand side grows like $|I|^{3/2}$. Nor is this estimate useful for preventing any sort of energy concentration.

Our solution to these difficulties proceeds in the context of an induction-on-energy argument as in [4]: assume for contradiction that Theorem 1.1 is false, and consider a solution of minimal energy among all those solutions with $L^{10}_{x,t}$ norm above some threshhold. We first show, without relying on any of the above Morawetz-type inequalities, that such a *minimal energy blowup solution* would have to be localized in both frequency and in space at all times. Second, we prove that this localized blowup solution satisfies Proposition 4.9, which localizes (1.8) in *frequency* rather than in space. Roughly speaking, the *frequency localized Morawetz inequality* of Proposition 4.9 states that after throwing away some small energy, low frequency portions of the blow-up solution, the remainder obeys good $L^4_{t,x}$ estimates. In principle, this estimate should follow simply by repeating the proof of (1.8) with u replaced by the high frequency portion of the solution, and then controlling error terms; however some of the error terms are rather difficult and the proof of the frequency-localized Morawetz inequality is quite technical. We emphasize that, unlike the estimates (1.5), (1.7), (1.8), the frequency-localized Morawetz inequality (4.19) is not an *a priori* estimate valid for all solutions of (1.1), but instead applies only to minimal energy blowup solutions; see Section 4 for further discussion and precise definitions.

The strategy is then to try to use Sobolev embedding to boost this $L^4_{t,x}$ control to $L^{10}_{t,x}$ control which would contradict the existence of the blow-up solution. There is, however, a remaining enemy, which is that the solution may shift its energy from low frequencies to high, possibly causing the $L^{10}_{t,x}$ norm to blow up while the $L^4_{t,x}$ norm stays bounded. To prevent this we look at what such a frequency evacuation would imply for the location — in frequency space — of the blow-up solution's L^2 mass. Specifically, we prove a frequency localized L^2 mass estimate that gives us information for longer time intervals than seem to be available from the *spatially* localized mass conservation laws used in the previous radial work ([4], [20]). By combining this frequency localized mass estimate with the $L^4_{t,x}$ bound and plenty of Strichartz estimate analysis, we can control the movement of energy and mass from one frequency range to another, and prevent the low-to-high cascade from occurring. The argument here is motivated by our previous low-regularity work involving almost conservation laws (e.g. [13]).

The remainder of the paper is organized as follows: Section 2 reviews some simple, classical conservation laws for Schrödinger equations which will be used througout, but especially in proving the frequency localized interaction Morawetz estimate. In Section 3 we recall some linear and multilinear

Strichartz estimates, along with the useful nonlinear perturbation statement of Lemma 3.10. Section 4 outlines in some detail the argument behind our main Theorem, leaving the proofs of each step to Sections 5–15 of the paper. Section 16 presents some miscellaneous remarks, including a proof of the unconditional uniqueness statement alluded to above.

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1.2. *Notation.* If X, Y are nonnegative quantities, we use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \leq CY$ for some C (which may depend on the critical energy E_{crit} (see Section 4) but not on any other parameter such as η), and $X \sim Y$ to denote the estimate $X \lesssim Y \lesssim X$. We use $X \ll Y$ to mean $X \leq cY$ for some small constant c (which is again allowed to depend on E_{crit}).

We use $C \gg 1$ to denote various large finite constants, and $0 < c \ll 1$ to denote various small constants.

The Fourier transform on $\mathbb{R}^3$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi} f(x)\, dx,$$

giving rise to the fractional differentiation operators $|\nabla|^s$, $\langle\nabla\rangle^s$ defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi); \quad \widehat{\langle\nabla\rangle^s f}(\xi) := \langle\xi\rangle^s \hat{f}(\xi)$$

where $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$. In particular, we will use ∇ to denote the spatial gradient ∇_x. This in turn defines the Sobolev norms

$$\|f\|_{\dot{H}^s(\mathbb{R}^3)} := \||\nabla|^s f\|_{L^2(\mathbb{R}^3)}; \quad \|f\|_{H^s(\mathbb{R}^3)} := \|\langle\nabla\rangle^s f\|_{L^2(\mathbb{R}^3)}.$$

More generally we define

$$\|f\|_{\dot{W}^{s,p}(\mathbb{R}^3)} := \||\nabla|^s f\|_{L^p(\mathbb{R}^3)}; \quad \|f\|_{W^{s,p}(\mathbb{R}^3)} := \|\langle\nabla\rangle^s f\|_{L^p(\mathbb{R}^3)}$$

for $s \in \mathbb{R}$ and $1 < p < \infty$.

We let $e^{it\Delta}$ be the free Schrödinger propagator; in terms of the Fourier transform, this is given by

$$\widehat{e^{it\Delta} f}(\xi) = e^{-4\pi^2 it|\xi|^2} \hat{f}(\xi) \tag{1.10}$$

while in physical space we have

$$e^{it\Delta}f(x) = \frac{1}{(4\pi it)^{3/2}} \int_{\mathbb{R}^3} e^{i|x-y|^2/4t} f(y)\, dy \tag{1.11}$$

for $t \neq 0$, using an appropriate branch cut to define the complex square root. In particular the propagator preserves all the Sobolev norms $H^s(\mathbb{R}^3)$ and $\dot{H}^s(\mathbb{R}^3)$, and also obeys the *dispersive inequality*

$$\|e^{it\Delta}f\|_{L^\infty_x(\mathbb{R}^3)} \lesssim |t|^{-3/2}\|f\|_{L^1_x(\mathbb{R}^3)}. \tag{1.12}$$

We also record *Duhamel's formula*

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_{t_0}^{t} e^{i(t-s)\Delta}(iu_t + \Delta u)(s)\, ds \tag{1.13}$$

for any Schwartz u and any times $t_0, t \in \mathbb{R}$, with the convention that $\int_{t_0}^{t} = -\int_{t}^{t_0}$ if $t < t_0$.

We use the notation $\mathcal{O}(X)$ to denote an expression which is *schematically* of the form X; this means that $\mathcal{O}(X)$ is a finite linear combination of expressions which look like X but with some factors possibly replaced by their complex conjugates. Thus for instance $3u^2\overline{v}^2|v|^2+9|u|^2|v|^4+3\overline{u}^2v^2|v|^2$ qualifies to be of the form $\mathcal{O}(u^2v^4)$, and similarly we have

$$|u+v|^6 = |u|^6 + |v|^6 + \sum_{j=1}^{5} \mathcal{O}(u^j v^{6-j}) \tag{1.14}$$

and

$$|u+v|^4(u+v) = |u|^4u + |v|^4v + \sum_{j=1}^{4} \mathcal{O}(u^j v^{5-j}). \tag{1.15}$$

We will sometimes denote partial derivatives using subscripts: $\partial_{x_j}u = \partial_j u = u_j$. We will also implicitly use the summation convention when indices are repeated in expressions below.

We shall need the following Littlewood-Paley projection operators. Let $\varphi(\xi)$ be a bump function adapted to the ball $\{\xi \in \mathbb{R}^3 : |\xi| \leq 2\}$ which equals 1 on the ball $\{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}$. Define a *dyadic number* to be any number $N \in 2^{\mathbb{Z}}$ of the form $N = 2^j$ where $j \in \mathbb{Z}$ is an integer. For each dyadic number N, we define the Fourier multipliers

$$\begin{aligned}
\widehat{P_{\leq N}f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi)\\
\widehat{P_{>N}f}(\xi) &:= (1-\varphi(\xi/N))\hat{f}(\xi)\\
\widehat{P_N f}(\xi) &:= (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi).
\end{aligned}$$

We similarly define $P_{<N}$ and $P_{\geq N}$. Note in particular the telescoping identities

$$P_{\leq N}f = \sum_{M\leq N} P_M f; \quad P_{>N}f = \sum_{M>N} P_M f; \quad f = \sum_M P_M f$$

for all Schwartz f, where M ranges over dyadic numbers. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

whenever $M \leq N$ are dyadic numbers. Similarly define $P_{M\leq\cdot\leq N}$, etc.

The symbol u shall always refer to a solution to the nonlinear Schrödinger equation (1.1). We shall use u_N to denote the frequency piece $u_N := P_N u$ of u, and similarly define $u_{\geq N} = P_{\geq N} u$, etc. While this may cause some confusion with the notation u_j used to denote derivatives of u, the meaning of the subscript should be clear from context.

The Littlewood-Paley operators commute with derivative operators (including $|\nabla|^s$ and $i\partial_t + \Delta$), the propagator $e^{it\Delta}$, and conjugation operations, are self-adjoint, and are bounded on every Lebesgue space L^p and Sobolev space $\dot{H}^s$ (if $1 \leq p \leq \infty$, of course). Furthermore, they obey the following easily verified Sobolev (and Bernstein) estimates for $\mathbb{R}^3$ with $s \geq 0$ and $1 \leq p \leq q \leq \infty$:

$$\|P_{\geq N} f\|_{L^p} \lesssim N^{-s} \||\nabla|^s P_{\geq N} f\|_{L^p}, \tag{1.16}$$

$$\|P_{\leq N} |\nabla|^s f\|_{L^p} \lesssim N^s \|P_{\leq N} f\|_{L^p}, \tag{1.17}$$

$$\|P_N |\nabla|^{\pm s} f\|_{L^p} \sim N^{\pm s} \|P_N f\|_{L^p}, \tag{1.18}$$

$$\|P_{\leq N} f\|_{L^q} \lesssim N^{\frac{3}{p}-\frac{3}{q}} \|P_{\leq N} f\|_{L^p}, \tag{1.19}$$

$$\|P_N f\|_{L^q} \lesssim N^{\frac{3}{p}-\frac{3}{q}} \|P_N f\|_{L^p}. \tag{1.20}$$

2. Local conservation laws

In this section we record some standard facts about the (non)conservation of mass, momentum and energy densities for general nonlinear Schrödinger equations of the form[6]

$$i\partial_t \phi + \Delta\phi = \mathcal{N} \tag{2.1}$$

on the spacetime slab $I_0 \times \mathbb{R}^d$ with I_0 a compact interval. Our primary interest is of course the quintic defocusing case (1.1) on $I_0 \times \mathbb{R}^3$ when $\mathcal{N} = |\phi|^4\phi$, but we will also discuss here the $U(1)$-gauge invariant Hamiltonian case, when $\mathcal{N} = F'(|\phi|^2)\phi$ with $\mathbb{R}$-valued F. Later on we will consider various truncated versions of (1.1) with non-Hamiltonian forcing terms. These local conservation laws will be used not only to imply the usual global conservation of mass and energy, but also derive "almost conservation" laws for various localized portions of mass, energy, and momentum, where the localization is either in physical space or frequency space. The localized momentum inequalities are closely

[6]We will use ϕ to denote general solutions to Schrödinger-type equations, reserving the symbol u for solutions to the quintic defocusing nonlinear Schrödinger equation (1.1).

related to virial identities, and will be used later to deduce an interaction Morawetz inequality which is crucial to our argument.

To avoid technicalities (and to justify all exchanges of derivatives and integrals), let us work purely with fields ϕ, $\mathcal{N}$ which are smooth, with all derivatives rapidly decreasing in space; in practice, we can then extend the formulae obtained here to more general situations by limiting arguments. We begin by introducing some notation which will be used to describe the mass and momentum (non)conservation properties of (2.1).

Definition 2.1. Given a (Schwartz) solution ϕ of (2.1) we define the *mass density*

$$T_{00}(t,x) := |\phi(t,x)|^2,$$

the *momentum density*

$$T_{0j}(t,x) := T_{j0}(t,x) := 2\mathrm{Im}(\overline{\phi}\phi_j),$$

and the (linear part of the) *momentum current*

$$L_{jk}(t,x) = L_{kj}(t,x) := -\partial_j\partial_k|\phi(t,x)|^2 + 4\mathrm{Re}(\overline{\phi_j}\phi_k).$$

Definition 2.2. Given any two (Schwartz) functions $f, g : \mathbb{R}^d \to \mathbb{C}$, we define the mass bracket

$$\{f,g\}_m := \mathrm{Im}(f\overline{g}) \tag{2.2}$$

and the momentum bracket

$$\{f,g\}_p := \mathrm{Re}(f\nabla\overline{g} - g\nabla\overline{f}). \tag{2.3}$$

Thus $\{f,g\}_m$ is a scalar-valued function, while $\{f,g\}_p$ defines a vector field on $\mathbb{R}^d$. We will denote the jth component of $\{f,g\}_p$ by $\{f,g\}_p^j$.

With these notions we can now express the mass and momentum (non)-conservation laws for (2.1), which can be validated with straightforward computations.

LEMMA 2.3 (Local conservation of mass and momentum). *If ϕ is a (Schwartz) solution to* (2.1) *then there exist the local mass conservation identity*

$$\partial_t T_{00} + \partial_j T_{0j} = 2\{\mathcal{N},\phi\}_m \tag{2.4}$$

and the local momentum conservation identity

$$\partial_t T_{0j} + \partial_k L_{kj} = 2\{\mathcal{N},\phi\}_p^j. \tag{2.5}$$

Here we adopt the usual[7] *summation conventions for the indices j, k.*

[7] Repeated Euclidean coordinate indices are summed. As the metric is Euclidean, we will not systematically match subscripts and superscripts.

Observe that the mass current coincides with the momentum density in (2.5), while the momentum current in (2.5) has some "positive definite" tendencies (think of $\Delta = \partial_k \partial_k$ as a negative definite operator, whereas the ∂_j will eventually be dealt with by integration by parts, reversing the sign). These two facts will underpin the interaction Morawetz estimate obtained in Section 10.

We now specialize to the gauge invariant Hamiltonian case, when $\mathcal{N} = F'(|\phi|^2)\phi$; note that (1.1) would correspond to the case $F(|\phi|^2) = \frac{1}{3}|\phi|^6$. Observe that

$$\{F'(|\phi|^2)\phi, \phi\}_m = 0 \tag{2.6}$$

and

$$\{F'(|\phi|^2)\phi, \phi\}_p = -\nabla G(|\phi|^2) \tag{2.7}$$

where $G(z) := zF'(z) - F(z)$. In particular, for the quintic case (1.1) we have

$$\{|\phi|^4\phi, \phi\}_p = -\frac{2}{3}\nabla|\phi|^6. \tag{2.8}$$

Thus, in the gauge invariant case we can re-express (2.5) as

$$\partial_t T_{0j} + \partial_k T_{jk} = 0 \tag{2.9}$$

where

$$T_{jk} := L_{jk} + 2\delta_{jk} G(|\phi|^2) \tag{2.10}$$

is the (linear and nonlinear) *momentum current*. Integrating (2.4) and (2.9) in space we see that the total mass

$$\int_{\mathbb{R}^d} T_{00}\ dx = \int_{\mathbb{R}^d} |\phi(t,x)|^2\ dx$$

and the total momentum

$$\int_{\mathbb{R}^d} T_{0j}\ dx = 2\int_{\mathbb{R}^d} \mathrm{Im}(\overline{\phi(t,x)}\partial_j \phi(t,x))\ dx$$

are both conserved quantities. In this Hamiltonian setting one can also verify the local energy conservation law

$$\partial_t \left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}F(|\phi|^2)\right] + \partial_j\left[\mathrm{Im}(\overline{\phi}_k \phi_{kj}) - F'(|\phi|^2)\mathrm{Im}(\phi\overline{\phi}_j)\right] = 0 \tag{2.11}$$

which implies conservation of total energy

$$\int_{\mathbb{R}^d} \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}F(|\phi|^2)\ dx.$$

Note also that (2.10) continues the tendency of the right-hand side of (2.5) to be "positive definite"; this is a manifestation of the defocusing nature of the equation. Later in our argument, however, we will be forced to deal with frequency-localized versions of the nonlinear Schrödinger equations, in which one does not have perfect conservation of mass and momentum, leading to a number of unpleasant error terms in our analysis.

3. Review of Strichartz theory in $\mathbb{R}^{1+3}$

In this section we review some standard (and some slightly less standard) Strichartz estimates in three dimensions, and their application to the well-posedness and regularity theory for (1.1). We use $L_t^q L_x^r$ to denote the spacetime norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} |u(t,x)|^r \, dx\right)^{q/r} \, dt\right)^{1/q},$$

with the usual modifications when q or r is equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^3$ is replaced by a smaller region of spacetime such as $I\times\mathbb{R}^3$. When $q=r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

3.1. *Linear Strichartz estimates.* We say that a pair (q,r) of exponents is *admissible* if $\frac{2}{q}+\frac{3}{r}=\frac{3}{2}$ and $2\le q,r\le\infty$; examples include $(q,r)=(\infty,2)$, $(10,30/13)$, $(5,30/11)$, $(4,3)$, $(10/3,10/3)$, and $(2,6)$.

Let $I\times\mathbb{R}^3$ be a spacetime slab. We define[8] the L^2 *Strichartz norm* $\dot{S}^0(I\times\mathbb{R}^3)$ by

$$\|u\|_{\dot{S}^0(I\times\mathbb{R}^3)} := \sup_{(q,r)\text{ admissible}} \Big(\sum_N \|P_N u\|^2_{L_t^q L_x^r(I\times\mathbb{R}^3)}\Big)^{1/2} \tag{3.1}$$

and for $k=1,2$ we then define the $\dot{H}^k$ *Strichartz norm* $\dot{S}^k(I\times\mathbb{R}^3)$ by

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^3)} := \|\nabla^k u\|_{\dot{S}^0(I\times\mathbb{R}^3)}.$$

We shall work primarily with the $\dot{H}^1$ Strichartz norm, but will need the L^2 and $\dot{H}^2$ norms to control high frequency and low frequency portions of the solution u respectively.

We observe the elementary inequality

$$\Big\|\Big(\sum_N |f_N|^2\Big)^{1/2}\Big\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} \le \Big(\sum_N \|f_N\|^2_{L_t^q L_x^r(I\times\mathbb{R}^3)}\Big)^{1/2} \tag{3.2}$$

for all $2\le q,r\le\infty$ and arbitrary functions f_N; this is easy to verify in the extreme cases $(q,r)=(2,2),(2,\infty),(\infty,2),(\infty,\infty)$, and the intermediate cases then follow by complex interpolation. In particular, (3.2) holds for all admissible exponents (q,r). From this and the Littlewood-Paley inequality

[8]The presence of the Littlewood-Paley projections here may seem unusual, but they are necessary in order to obtain a key $L_t^4 L_x^\infty$ endpoint Strichartz estimate below.

(see e.g. [40]) we have

$$\begin{aligned}\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} &\lesssim \Big\| \Big(\sum_N |P_N u|^2\Big)^{1/2} \Big\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} \\ &\lesssim \Big(\sum_N \|P_N u\|^2_{L_t^q L_x^r(I\times\mathbb{R}^3)}\Big)^{1/2} \\ &\lesssim \|u\|_{\dot S^0(I\times\mathbb{R}^3)}\end{aligned}$$

and hence

$$\|\nabla u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} \lesssim \|u\|_{\dot S^1(I\times\mathbb{R}^3)}. \tag{3.3}$$

Indeed, the $\dot S^1$ norm controls the following spacetime norms:

LEMMA 3.1 ([44]). *For any Schwartz function u on $I\times\mathbb{R}^3$,*

$$\begin{aligned}&\|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_t^{10} L_x^{30/13}} + \|\nabla u\|_{L_t^5 L_x^{30/11}} + \|\nabla u\|_{L_t^4 L_x^3} + \|\nabla u\|_{L_{t,x}^{10/3}} \\ &+ \|\nabla u\|_{L_t^2 L_x^6} + \|u\|_{L_t^4 L_x^\infty} + \|u\|_{L_t^6 L_x^{18}} + \|u\|_{L_{t,x}^{10}} + \|u\|_{L_t^\infty L_x^6} \lesssim \|u\|_{\dot S^1}.\end{aligned} \tag{3.4}$$

where all spacetime norms are on $I\times\mathbb{R}^3$.

Proof. All of these estimates follow from (3.3) and Sobolev embedding except for the $L_t^4 L_x^\infty$ norm, which is a little more delicate because endpoint Sobolev embedding does not work at L_x^∞. Write

$$c_N := \|P_N \nabla u\|_{L_t^2 L_x^6} + \|P_N \nabla u\|_{L_t^\infty L_x^2};$$

then by the definition of $\dot S^1$ we have

$$\Big(\sum_N c_N^2\Big)^{1/2} \lesssim \|u\|_{\dot S^1}.$$

On the other hand, for any dyadic frequency N we see from Bernstein's inequality (1.20) and (1.18) that

$$N^{\frac12}\|P_N u\|_{L_t^2 L_x^\infty} \lesssim c_N \text{ and } N^{-\frac12}\|P_N u\|_{L_t^\infty L_x^\infty} \lesssim c_N.$$

Thus, if $a_N(t) := \|P_N u(t)\|_{L_x^\infty}$, we have

$$\Big(\int_I a_N(t)^2\, dt\Big)^{1/2} \lesssim N^{-\frac12} c_N \tag{3.5}$$

and

$$\sup_{t\in I} a_N(t) \lesssim N^{\frac12} c_N. \tag{3.6}$$

Let us now compute

$$\|u\|^4_{L_t^4 L_x^\infty} \lesssim \int_I \Big(\sum_N a_N(t)\Big)^4\, dt.$$

Expanding this out and using symmetry, we have

$$\|u\|_{L_t^4 L_x^\infty}^4 \lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \int_I a_{N_1}(t) a_{N_2}(t) a_{N_3}(t) a_{N_4}(t)\, dt.$$

Estimating the two highest frequencies using (3.5) and the lowest two using (3.6), we can bound this by

$$\lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \frac{N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}}{N_1^{\frac{1}{2}} N_2^{\frac{1}{2}}} c_{N_1} c_{N_2} c_{N_3} c_{N_4}.$$

Let $\tilde{c}_N$ denote the quantity

$$\tilde{c}_N := \sum_{N'} \min(N/N', N'/N)^{1/10} c_{N'}.$$

Clearly we can bound the previous expression by

$$\lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \frac{N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}}{N_1^{\frac{1}{2}} N_2^{\frac{1}{2}}} \tilde{c}_{N_1} \tilde{c}_{N_2} \tilde{c}_{N_3} \tilde{c}_{N_4}.$$

But we have $\tilde{c}_{N_j} \lesssim (N_1/N_j)^{1/10} \tilde{c}_{N_1}$ for $j = 2, 3, 4$; hence we can bound the above by

$$\lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \frac{N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}}{N_1^{\frac{1}{2}} N_2^{\frac{1}{2}}} \tilde{c}_{N_1}^4 (N_1/N_2)^{1/10} (N_1/N_3)^{1/10} (N_1/N_4)^{1/10}.$$

Summing in N_4, then in N_3, then in N_2, we see that this is bounded by

$$\sum_{N_1} \tilde{c}_{N_1}^4 \lesssim \Big(\sum_N \tilde{c}_N^2 \Big)^2.$$

But by Young's inequality this is bounded by $\lesssim (\sum_N c_N^2)^2 \lesssim \|u\|_{\dot{S}^1}^4$, and the claim follows. □

We have the following standard Strichartz estimates:

LEMMA 3.2. *Let I be a compact time interval, and let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be a Schwartz solution to the forced Schrödinger equation*

$$iu_t + \Delta u = \sum_{m=1}^M F_m$$

for some Schwartz functions $F_1, \dots, F_M$. Then

$$\|u\|_{\dot{S}^k(I \times \mathbb{R}^3)} \lesssim \|u(t_0)\|_{\dot{H}^k(\mathbb{R}^3)} + C \sum_{m=1}^M \|\nabla^k F_m\|_{L_t^{q_m'} L_x^{r_m'}(I \times \mathbb{R}^3)} \tag{3.7}$$

for any integer $k \geq 0$, any time $t_0 \in I$, and any admissible exponents $(q_1, r_1), \dots$ $\dots, (q_m, r_m)$, where p' denotes the dual exponent to p; thus $1/p' + 1/p = 1$.

Proof. We first observe that we may take $M = 1$, since the claim for general M then follows from the principle of superposition (exploiting the linearity of the operator $(i\partial_t + \Delta)$, or equivalently using the Duhamel formula (1.13)) and the triangle inequality. We may then take $k = 0$, since the estimate for higher k follows simply by applying ∇^k to both sides of the equation and noting that this operator commutes with $(i\partial_t + \Delta)$. The Littlewood-Paley projections P_N also commute with $(i\partial_t + \Delta)$, and so

$$(i\partial_t + \Delta)P_N u = P_N F_1$$

for each N. From the Strichartz estimates in [32] we obtain

$$\|P_N u\|_{L^q_t L^r_x(I\times\mathbb{R}^3)} \lesssim \|P_N u(t_0)\|_{L^2(\mathbb{R}^3)} + \|P_N F_1\|_{L_t^{q_1'} L_x^{r_1'}(I\times\mathbb{R}^3)}$$

for any admissible exponents (q, r), (q_1, r_1). Finally, we square, sum this in N and use the dual of (3.2) to obtain the result. $\square$

Remark 3.3. In practice we shall take $k = 0, 1, 2$ and $M = 1, 2$, and (q_m, r_m) to be either $(\infty, 2)$ or $(2, 6)$; i.e., we shall measure part of the inhomogeneity in $L^1_t \dot H^k_x$, and the other part in $L^2_t \dot W^{k,6/5}_x$.

3.2. *Bilinear Strichartz estimate.* It turns out that to control the interactions between very high frequency and very low frequency portions of the Schrödinger solution u, Strichartz estimates are insufficient, and we need the following bilinear refinement, which we state in arbitrary dimension (though we need it only in dimension $d = 3$).

LEMMA 3.4. *Let $d \ge 2$. For any spacetime slab $I_* \times \mathbb{R}^d$, any $t_0 \in I_*$, and for any $\delta > 0$,*

$$\begin{aligned}\|uv\|_{L^2_t L^2_x(I_*\times\mathbb{R}^d)} \le C(\delta)(&\|u(t_0)\|_{\dot H^{-1/2+\delta}} + \|(i\partial_t+\Delta)u\|_{L^1_t \dot H_x^{-1/2+\delta}}) \\ &\times (\|v(t_0)\|_{\dot H^{\frac{d-1}{2}-\delta}} + \|(i\partial_t+\Delta)v\|_{L^1_t \dot H_x^{\frac{d-1}{2}-\delta}}).\end{aligned} \tag{3.8}$$

This estimate is very useful when u is high frequency and v is low frequency, as it moves plenty of derivatives onto the low frequency term. This estimate shows in particular that there is little interaction between high and low frequencies; this heuristic will underlie many of our arguments to come, especially when we begin to control the movement of mass, momentum, and energy from high modes to low or vice versa. This estimate is essentially the refined Strichartz estimate of Bourgain in [3] (see also [5]). We make the trivial remark that the $L^2_{t,x}$ norm of uv is the same as that of $u\overline{v}$, $\overline{u}v$, or $\overline{uv}$, thus the above estimate also applies to expressions of the form $\mathcal{O}(uv)$.

Proof. We fix δ, and allow our implicit constants to depend on δ. We begin by addressing the homogeneous case, with $u(t) := e^{it\Delta}\zeta$ and $v(t) := e^{it\Delta}\psi$ and consider the more general problem of proving

$$\|uv\|_{L^2_{t,x}} \lesssim \|\zeta\|_{\dot H^{\alpha_1}}\|\psi\|_{\dot H^{\alpha_2}}. \tag{3.9}$$

Scaling invariance of this estimate demands that $\alpha_1+\alpha_2=\frac{d}{2}-1$. Our first goal is to prove this for $\alpha_1=-\frac{1}{2}+\delta$ and $\alpha_2=\frac{d-1}{2}-\delta$. The estimate (3.9) may be recast using duality and renormalization as

(3.10)
$$\int g(\xi_1+\xi_2,|\xi_1|^2+|\xi_2|^2)|\xi_1|^{-\alpha_1}\widehat{\zeta}(\xi_1)|\xi_2|^{-\alpha_2}\widehat{\psi}(\xi_2)d\xi_1 d\xi_2 \lesssim \|g\|_{L^2}\|\zeta\|_{L^2}\|\psi\|_{L^2}.$$

Since $\alpha_2\ge\alpha_1$, we may restrict attention to the interactions with $|\xi_1|\ge|\xi_2|$. Indeed, in the remaining case we can multiply by $(\frac{|\xi_2|}{|\xi_1|})^{\alpha_2-\alpha_1}\ge 1$ to return to the case under consideration. In fact, we may further restrict attention to the case where $|\xi_1|>4|\xi_2|$ since, in the other case, we can move the frequencies between the two factors and reduce to the case where $\alpha_1=\alpha_2$, which can be treated by $L^4_{t,x}$ Strichartz estimates[9] when $d\ge 2$. Next, we decompose $|\xi_1|$ dyadically and $|\xi_2|$ in dyadic multiples of the size of $|\xi_1|$ by rewriting the quantity to be controlled as (N,Λ dyadic):

$$\sum_N\sum_\Lambda \int\int g_N(\xi_1+\xi_2,|\xi_1|^2+|\xi_2|^2)|\xi_1|^{-\alpha_1}\widehat{\zeta_N}(\xi_1)|\xi_2|^{-\alpha_2}\widehat{\psi_{\Lambda N}}(\xi_2)d\xi_1 d\xi_2.$$

Note that subscripts on g,ζ,ψ have been inserted to evoke the localizations to $|\xi_1+\xi_2|\sim N, |\xi_1|\sim N, |\xi_2|\sim \Lambda N$, respectively. Note that in the situation we are considering here, namely $|\xi_1|\ge 4|\xi_2|$, we have that $|\xi_1+\xi_2|\sim|\xi_1|$ and this explains why g may be so localized.

By renaming components, we may assume that $|\xi_1^1|\sim|\xi_1|$ and $|\xi_2^1|\sim|\xi_2|$. Write $\xi_2=(\xi_2^1,\underline{\xi_2})$. We now change variables by writing $u=\xi_1+\xi_2,\ v=|\xi_1|^2+|\xi_2|^2$ and $dudv=Jd\xi_2^1 d\xi_1$. A calculation then shows that $J=|2(\xi_1^1\pm\xi_2^1)|\sim|\xi_1|$. Therefore, upon changing variables in the inner two integrals, we encounter

$$\sum_N N^{-\alpha_1}\sum_{\Lambda\le 1}(\Lambda N)^{-\alpha_2}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^d} g_N(u,v)H_{N,\Lambda}(u,v,\underline{\xi_2})dudvd\underline{\xi_2}$$

where

$$H_{N,\Lambda}(u,v,\underline{\xi_2})=\frac{\widehat{\zeta_N}(\xi_1)\widehat{\psi_{\Lambda N}}(\xi_2)}{J}.$$

We apply Cauchy-Schwarz on the u,v integration and change back to the original variables to obtain

$$\sum_N N^{-\alpha_1}\|g_N\|_{L^2}\sum_{\Lambda\le 1}(\Lambda N)^{-\alpha_2}\int_{\mathbb{R}^{d-1}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\frac{|\widehat{\zeta_N}(\xi_1)|^2|\widehat{\psi_{\Lambda N}}(\xi_2)|^2}{J}d\xi_1 d\xi_2^1\right]^{\frac{1}{2}}d\underline{\xi_2}.$$

[9]In one dimension $d=1$, Lemma 3.4 fails when u,v have comparable frequencies, but continues to hold when u,v have separated frequencies; see [11] for further discussion.

We recall that $J \sim N$ and use Cauchy-Schwarz in the $\underline{\xi_2}$ integration, keeping in mind the localization $|\xi_2| \sim \Lambda N$, to get

$$\sum_N N^{-\alpha_1-\frac{1}{2}}\|g_N\|_{L^2}\sum_{\Lambda\leq 1}(\Lambda N)^{-\alpha_2+\frac{d-1}{2}}\|\widehat{\zeta_N}\|_{L^2}\|\widehat{\psi_{\Lambda N}}\|_{L^2}.$$

Choose $\alpha_1 = -\frac{1}{2}+\delta$ and $\alpha_2 = \frac{d-1}{2}-\delta$ with $\delta > 0$ to obtain

$$\sum_N \|g_N\|_{L^2}\|\widehat{\zeta_N}\|_{L^2}\sum_{\Lambda\leq 1}\Lambda^{\delta}\|\widehat{\psi_{\Lambda N}}\|_{L^2}$$

which may be summed up to give the claimed homogeneous estimate.

We turn our attention to the inhomogeneous estimate (3.8). For simplicity we set $F := (i\partial_t+\Delta)u$ and $G := (i\partial_t+\Delta)v$. Then we use Duhamel's formula (1.13) to write

$$u = e^{i(t-t_0)\Delta}u(t_0)-i\int_{t_0}^{t}e^{i(t-t')\Delta}F(t')\,dt',\quad v = e^{i(t-t_0)\Delta}v(t_0)-i\int_{t_0}^{t}e^{i(t-t')\Delta}G(t').$$

We obtain[10]

$$\begin{aligned}\|uv\|_{L^2} &\lesssim \left\|e^{i(t-t_0)\Delta}u(t_0)e^{i(t-t_0)\Delta}v(t_0)\right\|_{L^2}\\ &+\left\|e^{i(t-t_0)\Delta}u(t_0)\int_{t_0}^{t}e^{i(t-t')\Delta}G(t')\,dt'\right\|_{L^2}\\ &+\left\|e^{i(t-t_0)\Delta}v(t_0)\int_{t_0}^{t}e^{i(t-t')\Delta}F(t')dt'\right\|_{L^2}\\ &+\left\|\int_{t_0}^{t}e^{i(t-t')\Delta}F(t')dt'\int_{t_0}^{t}e^{i(t-t'')\Delta}G(x,t'')\,dt''\right\|_{L^2}\\ &:= I_1+I_2+I_3+I_4.\end{aligned}$$

The first term was treated in the first part of the proof. The second and the third are similar and so we consider only I_2. By the Minkowski inequality,

$$I_2 \lesssim \int_{\mathbb{R}}\|e^{i(t-t_0)\Delta}u(t_0)e^{i(t-t')\Delta}G(t')\|_{L^2}\,dt',$$

and in this case the lemma follows from the homogeneous estimate proved above. Finally, again by Minkowski's inequality we have

$$I_4 \lesssim \int_{\mathbb{R}}\int_{\mathbb{R}}\|e^{i(t-t')\Delta}F(t')e^{i(t-t'')\Delta}G(t'')\|_{L^2_x}dt'dt'',$$

and the proof follows by inserting in the integrand the homogeneous estimate above. □

[10] Alternatively, one can absorb the homogeneous components $e^{i(t-t_0)\Delta}u(t_0)$, $e^{i(t-t_0)\Delta}v(t_0)$ into the inhomogeneous term by adding an artificial forcing term of $\delta(t-t_0)u(t_0)$ and $\delta(t-t_0)v(t_0)$ to F and G respectively, where δ is the Dirac delta.

Remark 3.5. In the situation where the initial data are dyadically localized in frequency space, the estimate (3.9) is valid [3] at the endpoint $\alpha_1 = -\frac{1}{2}, \alpha_2 = \frac{d-1}{2}$. Bourgain's argument also establishes the result with $\alpha_1 = -\frac{1}{2} + \delta, \alpha_2 = \frac{d-1}{2} + \delta$, which is not scale invariant. However, the full estimate fails at the endpoint. This can be seen by calculating the left and right sides of (3.10) in the situation where $\widehat{\zeta_1} = \chi_{R_1}$ with $R_1 = \{\xi : \xi_1 = Ne^1 + O(N^{\frac{1}{2}})\}$ (where e^1 denotes the first coordinate unit vector), $\widehat{\psi_2}(\xi_2) = |\xi_2|^{-\frac{d-1}{2}} \chi_{R_2}$ where $R_2 = \{\xi_2 : 1 \ll |\xi_2| \ll N^{\frac{1}{2}}, \xi_2 \cdot e^1 = O(1)\}$ and $g(u,v) = \chi_{R_0}(u,v)$ with $R_0 = \{(u,v) : u = Ne^1 + O(N^{\frac{1}{2}}), v = |u|^2 + O(N)\}$. A calculation then shows that the left side of (3.10) is of size $N^{\frac{d+1}{2}} \log N$ while the right side is of size $N^{\frac{d+1}{2}} (\log N)^{\frac{1}{2}}$. Note that the same counterexample shows that the estimate

$$\|u\overline{v}\|_{L^2_{t,x}} \lesssim \|\zeta\|_{\dot{H}^{\alpha}_1} \|\psi\|_{\dot{H}^{\alpha}_2},$$

where $u(t) = e^{it\Delta}\zeta,\ v(t) = e^{it\Delta}\psi$, also fails at the endpoint.

3.3. *Quintilinear Strichartz estimates.* We record the following useful inequality:

LEMMA 3.6. *For any $k = 0, 1, 2$ and any slab $I \times \mathbb{R}^3$, and any smooth functions $v_1, \dots, v_5$ on this slab,*

$$\begin{aligned} (3.11) \quad & \|\nabla^k \mathcal{O}(v_1 v_2 v_3 v_4 v_5)\|_{L^1_t L^2_x} \\ & \lesssim \sum_{\{a,b,c,d,e\} = \{1,2,3,4,5\}} \|v_a\|_{\dot{S}^1} \|v_b\|_{\dot{S}^1} \|v_c\|_{L^{10}_{x,t}} \|v_d\|_{L^{10}_{x,t}} \|v_e\|_{\dot{S}^k} \end{aligned}$$

where all the spacetime norms are on the slab $I \times \mathbb{R}^3$. In a similar spirit,

$$\|\nabla \mathcal{O}(v_1 v_2 v_3 v_4 v_5)\|_{L^2_t L^{6/5}_x} \lesssim \prod_{j=1}^{5} \|\nabla v_j\|_{L^{10}_t L^{30/13}_x} \leq \prod_{j=1}^{5} \|v_j\|_{\dot{S}^1}. \tag{3.12}$$

Proof. Consider, for example, the $k = 1$ case of (3.11). Applying the Leibnitz rule, we encounter various terms to control including

$$\|\mathcal{O}((\nabla v_1) v_2 v_3 v_4 v_5)\|_{L^1_t L^2_x} \lesssim \|\nabla v_1\|_{L^{\frac{10}{3}}_{t,x}} \|v_2\|_{L^4_t L^\infty_x} \|v_3\|_{L^4_t L^\infty_x} \|v_4\|_{L^{10}_{t,x}} \|v_5\|_{L^{10}_{t,x}}.$$

The claim follows then by (3.4). The $k = 2$ case of (3.11) follows similarly by estimates such as

$$\|\mathcal{O}((\nabla^2 v_1) v_2 v_3 v_4 v_5)\|_{L^1_t L^2_x} \lesssim \|\nabla^2 v_1\|_{L^{\frac{10}{3}}_{t,x}} \|v_2\|_{L^4_t L^\infty_x} \|v_3\|_{L^4_t L^\infty_x} \|v_4\|_{L^{10}_{t,x}} \|v_5\|_{L^{10}_{t,x}}$$

and

$$\|\mathcal{O}((\nabla v_1)(\nabla v_2) v_3 v_4 v_5)\|_{L^1_t L^2_x} \lesssim \|\nabla v_1\|_{L^{\frac{10}{3}}_{t,x}} \|\nabla v_2\|_{L^4_t L^\infty_x} \|v_3\|_{L^4_t L^\infty_x} \|v_4\|_{L^{10}_{t,x}} \|v_5\|_{L^{10}_{t,x}}.$$

The $k = 0$ case is similar (omit all the ∇s).

Finally, estimate (3.12) similarly follows from the Sobolev embedding $\|u\|_{L^{10}_{t,x}} \lesssim \|\nabla u\|_{L^{10}_t L^{30/13}_x}$, (3.4) and Hölder's inequality,

$$\|\mathcal{O}(\nabla v_1 v_2 v_3 v_4 v_5)\|_{L^2_t L^{6/5}_x} \lesssim \|\nabla v_1\|_{L^{10}_t L^{30/13}_x} \|v_2\|_{L^{10}_{t,x}} \|v_3\|_{L^{10}_{t,x}} \|v_4\|_{L^{10}_{t,x}} \|v_5\|_{L^{10}_{t,x}}. \quad \square$$

We need a variant of the above lemma which also exploits the bilinear Strichartz inequality in Lemma 3.4 to obtain a gain when some of the factors are "high frequency" and others are "low frequency".

LEMMA 3.7. *Suppose v_{hi}, v_{lo} are functions on $I \times \mathbb{R}^3$ such that*

$$\|v_{hi}\|_{\dot S^0} + \|(i\partial_t + \Delta) v_{hi}\|_{L^1_t L^2_x(I\times\mathbb{R}^3)} \lesssim \varepsilon K,$$
$$\|v_{hi}\|_{\dot S^1} + \|\nabla(i\partial_t + \Delta) v_{hi}\|_{L^1_t L^2_x(I\times\mathbb{R}^3)} \lesssim K,$$
$$\|v_{lo}\|_{\dot S^1} + \|\nabla(i\partial_t + \Delta) v_{lo}\|_{L^1_t L^2_x(I\times\mathbb{R}^3)} \lesssim K,$$
$$\|v_{lo}\|_{\dot S^2} + \|\nabla^2(i\partial_t + \Delta) v_{lo}\|_{L^1_t L^2_x(I\times\mathbb{R}^3)} \lesssim \varepsilon K$$

for some constants $K > 0$ and $0 < \varepsilon \ll 1$. Then for any $j = 1, 2, 3, 4$,

$$\|\nabla \mathcal{O}(v_{hi}^j v_{lo}^{5-j})\|_{L^2_t L^{6/5}_x(I\times\mathbb{R}^3)} \lesssim \varepsilon^{\frac{9}{10}} K^5.$$

Remark 3.8. The point here is the gain of $\varepsilon^{9/10}$, which cannot be obtained directly from the type of arguments used to prove Lemma 3.6. As the proof will reveal, one can replace the exponent 9/10 with any exponent less than one, though for our purposes all that matters is that the power of ε is positive. The $\dot S^0$ bound on v_{hi} effectively restricts v_{hi} to high frequencies (as the low and medium frequencies will then be very small in $\dot S^1$ norm); similarly, the $\dot S^2$ control on v_{lo} effectively restricts v_{lo} to low frequencies. This lemma is thus an assertion that the components of the nonlinearity in (1.1) arising from interactions between low and high frequencies are rather weak; this phenomenon underlies the important frequency localization result in Proposition 4.3, but the motif of controlling the interaction between low and high frequencies underlies many other parts of our argument also, notably in Proposition 4.9 and Proposition 4.15.

Proof. Throughout this proof all spacetime norms shall be on $I \times \mathbb{R}^3$. We may normalize $K := 1$. By the Leibnitz rule we have

$$\|\nabla \mathcal{O}(v_{hi}^j v_{lo}^{5-j})\|_{L^2_t L^{6/5}_x} \lesssim \|\mathcal{O}(v_{hi}^j v_{lo}^{4-j} \nabla v_{lo})\|_{L^2_t L^{6/5}_x} + \|\mathcal{O}(v_{hi}^{j-1} v_{lo}^{5-j} \nabla v_{hi})\|_{L^2_t L^{6/5}_x}.$$

Consider the ∇v_{lo} terms first, which are rather easy. By Hölder we have

$$\|\mathcal{O}(v_{hi}^j v_{lo}^{4-j} \nabla v_{lo})\|_{L^2_t L^{6/5}_x} \lesssim \|\nabla v_{lo}\|_{L^\infty_t L^6_x} \|v_{hi}\|_{L^\infty_t L^2_x} \|v_{lo}\|^{4-j}_{L^6_t L^{18}_x} \|v_{hi}\|^{j-1}_{L^6_t L^{18}_x}.$$

Applying (3.4), this is bounded by

$$\lesssim \|v_{lo}\|_{\dot S^2} \|v_{hi}\|_{\dot S^0} \|v_{lo}\|^{4-j}_{\dot S^1} \|v_{hi}\|^{j-1}_{\dot S^1} \lesssim \varepsilon^2$$

which is acceptable.

Now consider the ∇v_{hi} terms, which are more difficult. First consider the $j = 2, 3, 4$ cases. By Hölder we have

$$\|\mathcal{O}(v_{hi}^{j-1} v_{lo}^{5-j} \nabla v_{hi})\|_{L_t^2 L_x^{6/5}} \lesssim \|\nabla v_{hi}\|_{L_t^2 L_x^6} \|v_{lo}\|_{L_t^\infty L_x^\infty} \|v_{hi}\|_{L_t^\infty L_x^2}^{1/2} \|v_{lo}\|_{L_t^\infty L_x^6}^{4-j} \|v_{hi}\|_{L_t^\infty L_x^6}^{j-3/2}.$$

Now observe (for instance from (1.20), (1.18) and dyadic decomposition) that

$$\|v_{lo}\|_{L_t^\infty L_x^\infty} \lesssim \|v_{lo}\|_{L_t^\infty L_x^6}^{1/2} \|\nabla v_{lo}\|_{L_t^\infty L_x^6}^{1/2}.$$

Thus, by (3.4),

$$\|\mathcal{O}(v_{hi}^{j-1} v_{lo}^{5-j} \nabla v_{hi})\|_{L_t^2 L_x^{6/5}} \lesssim \|v_{hi}\|_{\dot{S}^1}^{j-1/2} \|v_{hi}\|_{\dot{S}^0}^{1/2} \|v_{lo}\|_{\dot{S}^2}^{1/2} \|v_{lo}\|_{\dot{S}^1}^{9/2-j}$$

which is $O(\varepsilon^{9/10})$, and is acceptable.

Finally consider the $j = 1$ term. For this term we must use dyadic decomposition, writing

$$\|\mathcal{O}(v_{lo}^4 \nabla v_{hi})\|_{L_t^2 L_x^{6/5}} \lesssim \sum_{N_1, N_2, N_3, N_4} \|\mathcal{O}((P_{N_1} v_{lo})(P_{N_2} v_{lo})(P_{N_3} v_{lo})(P_{N_4} v_{lo}) \nabla v_{hi})\|_{L_t^2 L_x^{6/5}}.$$

By symmetry we may take $N_1 \geq N_2 \geq N_3 \geq N_4$. We then estimate this using Hölder by

$$\sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \|\mathcal{O}(P_{N_1} v_{lo} \nabla v_{hi})\|_{L_t^2 L_x^2} \|P_{N_2} v_{lo}\|_{L_t^\infty L_x^6} \|P_{N_3} v_{lo}\|_{L_t^\infty L_x^6} \|P_{N_4} v_{lo}\|_{L_t^\infty L_x^\infty}.$$

The middle two factors can be estimated by $\|v_{lo}\|_{\dot{S}^1} = O(1)$. The last factor can be estimated using Bernstein (1.18) either as

$$\|P_{N_4} v_{lo}\|_{L_t^\infty L_x^\infty} \lesssim N_4^{1/2} \|P_{N_4} v_{lo}\|_{L_t^\infty L_x^6} \lesssim N_4^{1/2} \|v_{lo}\|_{\dot{S}^1} \lesssim N_4^{1/2}$$

or as

$$\|P_{N_4} v_{lo}\|_{L_t^\infty L_x^\infty} \lesssim N_4^{-1/2} \|\nabla P_{N_4} v_{lo}\|_{L_t^\infty L_x^6} \lesssim N_4^{-1/2} \|v_{lo}\|_{\dot{S}^2} \lesssim \varepsilon N_4^{-1/2}.$$

Meanwhile, the first factor can be estimated using (3.8) as

$$\begin{aligned}\|\mathcal{O}(P_{N_1} v_{lo} \nabla v_{hi})\|_{L_t^2 L_x^2} \lesssim & (\|\nabla v_{hi}(t_0)\|_{\dot{H}^{-1/2+\delta}} + \|(i\partial_t + \Delta) \nabla v_{hi}\|_{L_t^1 \dot{H}_x^{-1/2+\delta}}) \\ & \times (\|P_{N_1} v_{lo}(t_0)\|_{\dot{H}^{1-\delta}} + \|(i\partial_t + \Delta) P_{N_1} v_{lo}\|_{L_t^1 \dot{H}_x^{1-\delta}}),\end{aligned}$$

where $t_0 \in I$ is an arbitrary time and $0 < \delta < 1/2$ is an arbitrary exponent. From the hypotheses on v_{hi} and interpolation we see that

$$\|\nabla v_{hi}(t_0)\|_{\dot{H}^{-1/2+\delta}} + \|(i\partial_t + \Delta) \nabla v_{hi}\|_{L_t^1 \dot{H}_x^{-1/2+\delta}} \lesssim \varepsilon^{1/2-\delta}$$

while from the hypotheses on v_{lo} and (1.18),

$$\|P_{N_1} v_{lo}(t_0)\|_{\dot{H}^{1-\delta}} + \|(i\partial_t + \Delta) P_{N_1} v_{lo}\|_{L_t^1 \dot{H}_x^{1-\delta}}) \lesssim N_1^{-\delta}.$$

Putting this all together, we obtain

$$\|\mathcal{O}(v_{lo}^4 \nabla v_{hi})\|_{L_t^2 L_x^{6/5}} \lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \varepsilon^{1/2-\delta} N_1^{-\delta} \min(N_4^{1/2}, \varepsilon N_4^{-1/2}).$$

Performing the N_1 sum, then the N_2, then the N_3, then the N_4, we obtain the desired bound of $O(\varepsilon^{9/10})$, if δ is sufficiently small. □

3.4. *Local well-posedness and perturbation theory.* It is well known (see e.g. [5]) that the equation (1.1) is *locally* well-posed[11] in $\dot{H}^1(\mathbb{R}^3)$, and indeed that this well-posedness extends to any time interval on which one has a uniform bound on the $L_{t,x}^{10}$ norm; this can already be seen from Lemma 3.6 and (3.7) (see also Lemma 3.12 below). In this section we detail some variants of the local well-posedness argument which describe how we can perturb finite-energy solutions (or near-solutions) to (1.1) in the energy norm when we control the original solution in the $L_{t,x}^{10}$ norm and the error of near-solutions in a dual Strichartz space. The arguments we give are similar to those in previous work such as [5].

We begin with a preliminary result where the near solution, the error of the near-solution, and the free evolution of the perturbation are all assumed to be small in *spacetime* norms, but allowed to be large in energy norm.

Lemma 3.9 (Short-time perturbations). *Let I be a compact interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^3$ which is a near-solution to* (1.1) *in the sense that*

$$(i\partial_t + \Delta)\tilde{u} = |\tilde{u}|^4 \tilde{u} + e \tag{3.13}$$

for some function e. Suppose that we also have the energy bound

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^3)} \leq E$$

for some $E > 0$. Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}_x^1} \leq E' \tag{3.14}$$

for some $E' > 0$. Assume also that there exist the smallness conditions

$$\|\nabla \tilde{u}\|_{L_t^{10} L_x^{30/13}(I \times \mathbb{R}^3)} \leq \varepsilon_0, \tag{3.15}$$

$$\|\nabla e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L_t^{10} L_x^{30/13}(I \times \mathbb{R}^3)} \leq \varepsilon, \tag{3.16}$$

$$\|\nabla e\|_{L_t^2 L_x^{6/5}} \leq \varepsilon \tag{3.17}$$

for some $0 < \varepsilon < \varepsilon_0$, where ε_0 is some constant $\varepsilon_0 = \varepsilon_0(E, E') > 0$.

[11] In particular, we have uniqueness of this Cauchy problem, at least under the assumption that u lies in $L_{t,x}^{10} \cap C_t^0 \dot{H}_x^1$, and so whenever we construct a solution u to (1.1) with specified initial data $u(t_0)$, we will refer to it as *the* solution to (1.1) with these data.

Then there exists a solution u to (1.1) *on $I \times \mathbb{R}^3$ with the specified initial data $u(t_0)$ at t_0, and furthermore*

$$\|u - \tilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^3)} \lesssim E', \tag{3.18}$$

$$\|u\|_{\dot{S}^1(I\times\mathbb{R}^3)} \lesssim E' + E, \tag{3.19}$$

$$\|u - \tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim \|\nabla(u - \tilde{u})\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} \lesssim \varepsilon, \tag{3.20}$$

$$\|\nabla(i\partial_t + \Delta)(u - \tilde{u})\|_{L^2_t L^{6/5}_x(I\times\mathbb{R}^3)} \lesssim \varepsilon. \tag{3.21}$$

Note that $u(t_0) - \tilde{u}(t_0)$ is allowed to have large energy, albeit at the cost of forcing ε to be smaller, and worsening the bounds in (3.18). From the Strichartz estimate (3.7), (3.14) we see that the hypothesis (3.16) is redundant if one is willing to take $E' = O(\varepsilon)$.

Proof. By the well-posedness theory reviewed above, it suffices to prove (3.18)–(3.21) as *a priori* estimates.[12] We establish these bounds for $t \geq t_0$, since the corresponding bounds for the $t \leq t_0$ portion of I are proved similarly.

First note that the Strichartz estimate (Lemma 3.2), Lemma 3.6 and (3.17) give,

$$\|\tilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^3)} \lesssim E + \|\tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \cdot \|\tilde{u}\|^4_{\dot{S}^1(I\times\mathbb{R}^3)} + \varepsilon.$$

By (3.15) and Sobolev embedding we have $\|\tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim \varepsilon_0$. A standard continuity argument in I then gives (if ε_0 is sufficiently small depending on E)

$$\|\tilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^3)} \lesssim E. \tag{3.22}$$

Define $v := u - \tilde{u}$. For each $t \in I$ define the quantity

$$S(t) := \|\nabla(i\partial_t + \Delta)v\|_{L^2_t L^{6/5}_x([t_0,t]\times\mathbb{R}^3)}.$$

From using Lemma 3.1, Lemma 3.2, (3.16), we have

$$\begin{aligned} \|\nabla v\|_{L^{10}_t L^{30/13}_x([t_0,t]\times\mathbb{R}^3)} &\lesssim \|\nabla(v - e^{i(t-t_0)\Delta}v(t_0))\|_{L^{10}_t L^{30/13}_x([t_0,t]\times\mathbb{R}^3)} \\ &\quad + \|\nabla e^{i(t-t_0)\Delta}v(t_0)\|_{L^{10}_t L^{30/13}_x([t_0,t]\times\mathbb{R}^3)} \\ &\lesssim \|v - e^{i(t-t_0)\Delta}v(t_0)\|_{\dot{S}^1([t_0,t]\times\mathbb{R}^3)} + \varepsilon \end{aligned} \tag{3.23}$$

$$\lesssim S(t) + \varepsilon. \tag{3.24}$$

On the other hand, since v obeys the equation

$$(i\partial_t + \Delta)v = |\tilde{u} + v|^4(\tilde{u} + v) - |\tilde{u}|^4\tilde{u} - e = \sum_{j=1}^{5} \mathcal{O}(v^j \tilde{u}^{5-j}) - e$$

[12] That is, we may assume the solution u already exists and is smooth on the entire interval I.

by (1.15), we easily check using (3.12), (3.15), (3.17), (3.24) that

$$S(t) \lesssim \varepsilon + \sum_{j=1}^{5} (S(t)+\varepsilon)^j \varepsilon_0^{5-j}.$$

If ε_0 is sufficiently small, a standard continuity argument then yields the bound $S(t) \lesssim \varepsilon$ for all $t \in I$. This gives (3.21), and (3.20) follows from (3.24). Applying Lemma 3.2, (3.14) we then conclude (3.18) (if ε is sufficiently small), and then from (3.22) and the triangle inequality we conclude (3.19). $\square$

We will actually be more interested in iterating the above lemma[13] to deal with the more general situation of near-solutions with finite but arbitrarily large $L^{10}_{t,x}$ norms.

Lemma 3.10 (Long-time perturbations). *Let I be a compact interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^3$ which obeys the bounds*

$$\|\tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq M \tag{3.25}$$

and

$$\|\tilde{u}\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^3)} \leq E \tag{3.26}$$

for some $M, E > 0$. Suppose also that $\tilde{u}$ is a near-solution to (1.1) *in the sense that it solves* (3.13) *for some e. Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that*

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_x} \leq E'$$

for some $E' > 0$. Assume also the smallness conditions,

$$\begin{aligned} \|\nabla e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} &\leq \varepsilon, \\ \|\nabla e\|_{L^2_t L^{6/5}_x(I\times\mathbb{R}^3)} &\leq \varepsilon, \end{aligned} \tag{3.27}$$

for some $0 < \varepsilon < \varepsilon_1$, where ε_1 is some constant $\varepsilon_1 = \varepsilon_1(E, E', M) > 0$. Now there exists a solution u to (1.1) *on $I \times \mathbb{R}^3$ with the specified initial data $u(t_0)$ at t_0, and furthermore*

$$\begin{aligned} \|u - \tilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^3)} &\leq C(M, E, E'), \\ \|u\|_{\dot{S}^1(I\times\mathbb{R}^3)} &\leq C(M, E, E'), \\ \|u - \tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq \|\nabla(u - \tilde{u})\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} &\leq C(M, E, E')\varepsilon. \end{aligned}$$

Once again, the hypothesis (3.27) is redundant by the Strichartz estimate if one is willing to take $E' = O(\varepsilon)$; however it will be useful in our applications

[13]We are grateful to Monica Visan for pointing out an incorrect version of Lemma 3.10 in a previous version of this paper, and also in simplifying the proof of Lemma 3.9.

to know that this lemma can tolerate a perturbation which is large in the energy norm but whose free evolution is small in the $L^{10}_t \dot{W}^{1,30/13}_x$ norm.

This lemma is already useful in the $e = 0$ case, as it says that one has local well-posedness in the energy space whenever the $L^{10}_{t,x}$ norm is bounded; in fact one has locally Lipschitz dependence on the initial data. For similar perturbative results see [4], [5].

Proof. As in the previous proof, we may assume that t_0 is the lower bound of the interval I. Let $\varepsilon_0 = \varepsilon_0(E, 2E')$ be as in Lemma 3.9. (We need to replace E' by the slightly larger $2E'$ as the $\dot{H}^1$ norm of $u - \tilde{u}$ is going to grow slightly in time.)

The first step is to establish a $\dot{S}^1$ bound on $\tilde{u}$. Using (3.25) we may subdivide I into $C(M, \varepsilon_0)$ time intervals such that the $L^{10}_{t,x}$ norm of $\tilde{u}$ is at most ε_0 on each such interval. By using (3.26) and Lemmas 3.2, 3.6 as in the proof of (3.22) we see that the $\dot{S}^1$ norm of $\tilde{u}$ is $O(E)$ on each of these intervals. Summing up over all the intervals we conclude

$$\|\tilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^3)} \le C(M, E, \varepsilon_0)$$

and in particular by Lemma 3.1

$$\|\nabla\tilde{u}\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} \le C(M, E, \varepsilon_0).$$

We can then subdivide the interval I into $N \le C(M, E, \varepsilon_0)$ subintervals $I_j \equiv [T_j, T_{j+1}]$ so that on each I_j we have,

$$\|\nabla\tilde{u}\|_{L^{10}_t L^{30/13}_x(I_j\times\mathbb{R}^3)} \le \varepsilon_0.$$

We can then verify inductively using Lemma 3.9 for each j that if ε_1 is sufficiently small depending on ε_0, N, E, E', then

$$\begin{aligned}
\|u - \tilde{u}\|_{\dot{S}^1(I_j\times\mathbb{R}^3)} &\le C(j)E', \\
\|u\|_{\dot{S}^1(I_j\times\mathbb{R}^3)} &\le C(j)(E' + E), \\
\|\nabla(u - \tilde{u})\|_{L^{10}_t L^{30/13}_x(I_j\times\mathbb{R}^3)} &\le C(j)\varepsilon, \\
\|\nabla(i\partial_t + \Delta)(u - \tilde{u})\|_{L^2_t L^{6/5}_x(I_j\times\mathbb{R}^3)} &\le C(j)\varepsilon.
\end{aligned}$$

Hence by Strichartz (3.7) and (3.4) we have

$$\begin{aligned}
&\|\nabla e^{i(t-T_{j+1})\Delta}(u(T_{j+1}) - \tilde{u}(T_{j+1}))\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} \\
&\quad \le \|\nabla e^{i(t-T_j)\Delta}(u(T_j) - \tilde{u}(T_j))\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} + C(j)\varepsilon
\end{aligned}$$

and

$$\|u(T_{j+1}) - \tilde{u}(T_{j+1})\|_{\dot{H}^1} \le \|u(T_j) - \tilde{u}(T_j)\|_{\dot{H}^1} + C(j)\varepsilon,$$

allowing one to continue the induction (if ε_1 is sufficiently small depending on E, N, E', ε_0, then the quantity in (3.14) will not exceed $2E'$). The claim follows. $\square$

Remark 3.11. The value of ε_1 given by the above lemma deteriorates exponentially with M, or more precisely it behaves like $\exp(-M^C)$ in its dependence[14] on M. As this lemma is used quite often in our argument, this will cause the final bounds in Theorem 1.1 to grow extremely rapidly in E, although they will still of course be finite for each E.

A related result involves persistence of L^2 or $\dot H^2$ regularity:

LEMMA 3.12 (Persistence of regularity). *Let $k = 0, 1, 2$, I be a compact time interval, and let u be a finite energy solution to* (1.1) *on $I \times \mathbb{R}^3$ obeying the bounds*

$$\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq M.$$

Then, if $t_0 \in I$ and $u(t_0) \in H^k$,

$$\|u\|_{\dot S^k(I\times\mathbb{R}^3)} \leq C(M, E(u))\|u(t_0)\|_{\dot H^k}. \tag{3.28}$$

In particular, once we control the $L^{10}_{t,x}$ norm of a finite energy solution, we in fact control all the Strichartz norms in $\dot S^1$, and can even control the $\dot S^2$ norm if the initial data are in $H^2(\mathbb{R}^3)$. From this and standard iteration arguments, one can in fact show that a Schwartz solution can be continued in time as long as the $L^{10}_{t,x}$ norm does not blow up to infinity.

Proof. By the local well-posedness theory it suffices to prove (3.28) as an *a priori* bound.

Applying Lemma 3.10 with $\tilde u := u$, $e := 0$, and $E' := 0$ we obtain the bound

$$\|u\|_{\dot S^1(I\times\mathbb{R}^3)} \lesssim C(M, E). \tag{3.29}$$

By (3.11) we also have

$$\|\nabla^k \mathcal{O}(u^5)\|_{L^1_t L^2_x} \lesssim \|u\|_{L^{10}_{x,t}} \|u\|_{\dot S^k} \|u\|^3_{\dot S^1}; \tag{3.30}$$

the main thing to observe here is the presence of one factor of $\|u\|_{L^{10}_{x,t}}$ on the right-hand side.

As in the proof of Lemma 3.10, divide the time interval I into $N \approx \left(1 + \frac{M}{\delta}\right)^{10}$ subintervals $I_j := [T_j, T_j + 1]$ on which

$$\|u\|_{L^{10}_{x,t}(I_j\times\mathbb{R}^3)} \leq \delta \tag{3.31}$$

[14]With respect to all its parameters, $\varepsilon_1(E, E', M) \approx \exp(-M^C\langle E\rangle^C\langle E'\rangle^C)$.

where δ will be chosen momentarily. We have on each I_j by the Strichartz estimates (Lemma 3.2) and (3.30),

$$\begin{aligned}\|u\|_{\dot{S}^k(I_j\times\mathbb{R}^3)} &\leq C\left(\|u(T_j)\|_{\dot{H}^k(\mathbb{R}^3)} + \|\nabla^k(|u|^4u)\|_{L^1_tL^2_x(I_j\times\mathbb{R}^3)}\right)\\ &\leq C\left(\|u(T_j)\|_{\dot{H}^k(\mathbb{R}^3)} + \|u\|_{L^{10}_{x,t}(I_j\times\mathbb{R}^3)}\cdot\|u\|_{\dot{S}^k(I\times\mathbb{R}^3)}\cdot\|u\|^3_{\dot{S}^1(I_j\times\mathbb{R}^3)}\right).\end{aligned}$$

Choosing $\delta \leq (2CC(M,E))^{-3}$ gives,

$$\|u\|_{\dot{S}^k(I_j\times\mathbb{R}^3)} \leq 2C\|u(T_j)\|_{\dot{H}^k(\mathbb{R}^3)}. \tag{3.32}$$

The bound (3.28) now follows by adding up the bounds (3.32) we have on each subinterval. □

4. Overview of proof of global spacetime bounds

We now outline the proof of Theorem 1.1, breaking it down into a number of smaller propositions.

4.1. *Zeroth stage*: *Induction on energy.* We say that a solution u to (1.1) is *Schwartz* on a slab $I\times\mathbb{R}^3$ if $u(t)$ is a Schwartz function for all $t\in I$; note that such solutions are then also smooth in time as well as space, thanks to (1.1).

The first observation is that in order to prove Theorem 1.1, it suffices to do so for Schwartz solutions. Indeed, once one obtains a uniform $L^{10}_{t,x}(I\times\mathbb{R}^3)$ bound for all Schwartz solutions and all compact I, one can then approximate arbitrary finite energy initial data by Schwartz initial data and use Lemma 3.10 to show that the corresponding sequence of solutions to (1.1) converges in $\dot{S}^1(I\times\mathbb{R}^3)$ to a finite energy solution to (1.1). We omit the standard details.

For every energy $E\geq 0$ we define the quantity $0\leq M(E)\leq+\infty$ by

$$M(E) := \sup\{\|u\|_{L^{10}_{t,x}(I_*\times\mathbb{R}^3)}\}$$

where $I_*\subset\mathbb{R}$ ranges over all compact time intervals, and u ranges over all Schwartz solutions to (1.1) on $I_*\times\mathbb{R}^3$ with $E(u)\leq E$. We shall adopt the convention that $M(E)=0$ for $E<0$. By the above discussion, it suffices to show that $M(E)$ is finite for all E.

In the argument of Bourgain [4] (see also [5]), the finiteness of $M(E)$ in the spherically symmetric case is obtained by an induction on the energy E; indeed a bound of the form

$$M(E)\leq C(E,\eta,M(E-\eta^4))$$

is obtained for some explicit $0<\eta=\eta(E)\ll 1$ which does not collapse to 0 for any finite E, and this easily implies via induction that $M(E)$ is finite for all E. Our argument will follow a similar induction on energy strategy; however it will be convenient to run this induction in the contrapositive, assuming for contradiction that $M(E)$ can be infinite. We study the minimal energy E_{crit} for which this is true, and then obtaining a contradiction using the "induction hypothesis" that $M(E)$ is finite for all $E < E_{\text{crit}}$. This will be more convenient for us, especially as we will require more than one small parameter η.

We turn to the details and assume for contradiction that $M(E)$ is not always finite. From Lemma 3.10 we see that the set $\{E : M(E) < \infty\}$ is open; clearly it is also connected and contains 0. By our contradiction hypothesis, there must therefore exist a *critical energy* $0 < E_{\text{crit}} < \infty$ such that $M(E_{\text{crit}}) = +\infty$, but $M(E) < \infty$ for all $E < E_{\text{crit}}$. One can think of E_{crit} as the minimal energy required to create a blowup solution. For instance, we have

LEMMA 4.1 (Induction on energy hypothesis). *Let $t_0 \in \mathbb{R}$, and let $v(t_0)$ be a Schwartz function such that $E(v(t_0)) \leq E_{\text{crit}} - \eta$ for some $\eta > 0$. Then there exists a Schwartz global solution $v : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}$ to* (1.1) *with initial data $v(t_0)$ at time $t = t_0$ such that $\|v\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \leq M(E_{\text{crit}} - \eta) = C(\eta)$. Furthermore we have $\|v\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C(\eta)$.*

Indeed, this lemma follows immediately from the definition of E_{crit}, the local well-posedness theory in $L^{10}_{t,x}$, and Lemma 3.12.

As in the argument in [4], we will need a small parameter $0 < \eta = \eta(E_{\text{crit}}) \ll 1$ depending on E_{crit}. In fact, our argument is somewhat lengthy and we will actually use *seven* such parameters

$$1 \gg \eta_0 \gg \eta_1 \gg \eta_2 \gg \eta_3 \gg \eta_4 \gg \eta_5 \gg \eta_6 > 0.$$

Specifically, we will need a small quantity $0 < \eta_0 = \eta_0(E_{\text{crit}}) \ll 1$ assumed to be sufficiently small depending on E_{crit}. Then we need a smaller quantity $0 < \eta_1 = \eta_1(\eta_0, E_{\text{crit}}) \ll 1$ assumed sufficiently small depending on E_{crit}, η_0 (in particular, it may be chosen smaller than positive quantities such as $M(E_{\text{crit}} - \eta_0^{100})^{-1}$). We continue in this fashion, choosing each $0 < \eta_j \ll 1$ to be sufficiently small depending on all previous quantities $\eta_0, \ldots, \eta_{j-1}$ and the energy E_{crit}, all the way down to η_6 which is extremely small, much smaller than any quantity depending on $E_{\text{crit}}, \eta_0, \ldots, \eta_5$ that will appear in our argument. We will always assume implicitly that each η_j has been chosen to be sufficiently small depending on the previous parameters. We will often display the dependence of constants on a parameter; e.g. $C(\eta)$ denotes a large constant depending on η, and $c(\eta)$ will denote a small constant depending on η. When $\eta_1 \gg \eta_2$, we will understand $c(\eta_1) \gg c(\eta_2)$ and $C(\eta_1) \ll C(\eta_2)$.

Since $M(E_{\text{crit}})$ is infinite, it is, in particular, larger than $1/\eta_6$. By definition of M, this means that we may find a compact interval $I_* \subset \mathbb{R}$ and a smooth solution $u : I_* \times \mathbb{R}^3 \to \mathbb{C}$ to (1.1) with $E_{\text{crit}}/2 \leq E(u) \leq E_{\text{crit}}$ so that u is ridiculously large in the sense that

$$\|u\|_{L^{10}_{t,x}(I_*\times\mathbb{R}^3)} > 1/\eta_6. \tag{4.1}$$

We will show that this leads to a contradiction.[15] Although u does not actually blow up (it is assumed smooth on all of the compact interval I_*), it is still convenient to think of u as almost[16] blowing up in $L^{10}_{t,x}$ in the sense of (4.1). We summarize the above discussion with the following:

Definition 4.2. A *minimal energy blowup solution* of (1.1) is a Schwartz solution on a time interval I_* with energy,[17]

$$\frac{1}{2}E_{\text{crit}} \leq E(u)(t) = \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 \, dx \leq E_{\text{crit}} \tag{4.2}$$

with the $L^{10}_{x,t}$ norm enormous in the sense of (4.1).

We remark that both conditions (4.1), (4.2) are invariant under the scaling (1.3) (though of course the interval I_* will be dilated by λ^2 under this scaling). Thus applying the scaling (1.3) to a minimal energy blowup solution produces another minimal energy blowup solution. Some of the proofs of the sub-propositions below will revolve around a specific frequency N; using this scale invariance, we can then normalize that frequency to equal 1 for the duration of that proof. (Different parts of the argument involve different key frequencies, but we will not run into problems because we will only normalize one frequency at a time).

Henceforth we will not mention the E_{crit} dependence of our constants explicitly, as all our constants will depend on E_{crit}. We shall need however to keep careful track of the dependence of our argument on $\eta_0, \dots, \eta_6$. Broadly speaking, we will start with the largest η, namely η_0, and slowly "retreat" to increasingly smaller values of η as the argument progresses (such a retreat will for instance usually be required whenever the induction hypothesis Lemma 4.1 is invoked). However we will only retreat as far as η_5, not η_6, so that (4.1) will eventually lead to a contradiction when we show that

$$\|u\|_{L^{10}_{t,x}(I^* \times \mathbb{R}^3)} \leq C(\eta_0, \dots, \eta_5).$$

Together with our assumption that we are considering a minimal energy blowup solution u as in Definition 4.2, Sobolev embedding implies the bounds

[15] Assuming, of course, that the parameters $\eta_0, \dots, \eta_6$ are each chosen to be sufficiently small depending on previous parameters. It is important to note however that the η_j cannot be chosen to be small depending on the interval I_* or the solution u; our estimates must be uniform with respect to these parameters.

[16] For instance, u might genuinely blow up at some time $T_* > 0$, but I_* is of the form $I_* = [0, T_* - \varepsilon]$ for some very small $0 < \varepsilon \ll 1$, and thus u remains Schwartz on $I_* \times \mathbb{R}^3$.

[17] We could modify our arguments below to allow the assumption here $E(u) = E_{\text{crit}}$. For example, the arguments in the proof of Proposition 4.3 below also show that the function $\tilde{M}(s) := \sup_{E(u)=s}\{\|u\|_{L^{10}_{x,t}}\}$ is a nondecreasing function of s. On first reading, the reader may imagine $E(u) = E_{\text{crit}}$ in Definition 4.2.

on kinetic energy

$$\|u\|_{L_t^\infty \dot H_x^1(I_* \times \mathbb{R}^3)} \sim 1 \tag{4.3}$$

and potential energy

$$\|u\|_{L_t^\infty L_x^6(I_* \times \mathbb{R}^3)} \lesssim 1 \tag{4.4}$$

(since our implicit constants are allowed to depend on E_{crit}). Note that we do not presently have any *lower* bounds on the potential energy, but see below.

Having displayed our preliminary bounds on the kinetic and potential energy, we briefly discuss the mass $\int_{\mathbb{R}^3} |u(t,x)|^2\, dx$, which is another conserved quantity. Because of our *a priori* assumption that u is Schwartz, we know that this mass is finite. However, we cannot obtain uniform control on this mass using our bounded energy assumption, because the very low frequencies of u may simultaneously have very small energy and very large mass. Furthermore it is dangerous to rely too much on this conserved mass for this energy-critical problem as the mass is not invariant under the natural scaling (1.3) of the equation (indeed, it is super-critical with respect to that scaling). On the other hand, from (4.3) and (1.16) we know that the *high frequencies* of u have small mass:

$$\|P_{>M} u\|_{L^2(\mathbb{R}^3)} \lesssim \frac{1}{M} \text{ for all } M \in 2^{\mathbb{Z}}. \tag{4.5}$$

Thus we will still be able to use the concept of mass in our estimates as long as we restrict our attention to sufficiently high frequencies.

4.2. *First stage*: *Localization control on u.* We aim to show that a minimal energy blowup solution as in Definition 4.2 does not exist. Intuitively, it seems reasonable to expect that a minimal-energy blowup solution should be "irreducible" in the sense that it cannot be decoupled into two or more components of strictly smaller energy that essentially do not interact with each other (i.e. each component also evolves via (1.1) modulo small errors), since one of the components must then also blow up, contradicting the minimal-energy hypothesis. In particular, we expect at every time that such a solution should be localized in both frequency and space.

The first main step in the proof of Theorem 1.1 is to make the above heuristics rigorous for our solution u. Roughly speaking, we would like to emphasize that at each time t, the solution $u(t)$ is localized in both space and frequency to the maximum extent allowable under the uncertainty principle (i.e. if the frequency is localized to $N(t)$, we would like to localize $u(t)$ spatially to the scale $1/N(t)$).

These sorts of localizations already appear for instance in the argument of Bourgain [4], [5], where the induction on energy argument is introduced.

Informally,[18] the reason that we can expect such localization is as follows. Suppose for contradiction that at some time t_0 the solution $u(t_0)$ can be split into two parts $u(t_0) = v(t_0) + w(t_0)$ which are widely separated in either space or frequency, and which each carry a nontrivial amount $O(\eta^C)$ of energy for some $\eta_5 \leq \eta \leq \eta_0$. Then by orthogonality we expect v and w to each have strictly smaller energy than u, e.g. $E(v(t_0)), E(w(t_0)) \leq E_{\text{crit}} - O(\eta^C)$. Thus by Lemma 4.1 we can extend $v(t)$ and $w(t)$ to all of $I_* \times \mathbb{R}^3$ by evolving the nonlinear Schrödinger equation (1.1) for v and w separately, and furthermore we have the bounds

$$\|v\|_{L^{10}_{t,x}(I_*\times\mathbb{R}^3)}, \|w\|_{L^{10}_{t,x}(I_*\times\mathbb{R}^3)} \leq M(E_{\text{crit}} - O(\eta^C)) \leq C(\eta).$$

Since v and w both solve (1.1) separately, and v and w were assumed to be widely separated, we thus expect $v + w$ to solve (1.1) *approximately*. The idea is then to use the perturbation theory from Section 3.4 to obtain a bound of the form $\|u\|_{L^{10}_{t,x}(I_*\times\mathbb{R}^3)} \leq C(\eta)$, which contradicts (4.1) if η_6 is sufficiently small.

A model example of this type of strategy occurs in Bourgain's argument [5], where substantial effort is invested in locating a "bubble" - a small localized pocket of energy - which is sufficiently isolated in physical space from the rest of the solution. One then removes this bubble, the remainder of the solution evolves, and then one uses perturbation theory, augmented with the additional information about the isolation of the bubble, to place the bubble back in. We will use arguments similar to these in the sequel, but first we need instead to show that a solution of (1.1) which is sufficiently delocalized in frequency space is globally spacetime bounded. More precisely, we have:

Proposition 4.3 (Frequency delocalization implies spacetime bound). *Let* $\eta > 0$, *and suppose there exists a dyadic frequency* $N_{lo} > 0$ *and a time* $t_0 \in I_*$ *such that the energy separation conditions hold*:

$$\|P_{\leq N_{lo}} u(t_0)\|_{\dot{H}^1(\mathbb{R}^3)} \geq \eta \tag{4.6}$$

and

$$\|P_{\geq K(\eta)N_{lo}} u(t_0)\|_{\dot{H}^1(\mathbb{R}^3)} \geq \eta. \tag{4.7}$$

If $K(\eta)$ *is sufficiently large depending on* η, *i.e.*

$$K(\eta) \geq C(\eta)$$

then,

$$\|u\|_{L^{10}_{t,x}(I_*\times\mathbb{R}^3)} \leq C(\eta). \tag{4.8}$$

[18]The heuristic that minimal energy blowup solutions should be strongly localized in both space and frequency has been employed in previous literature for a wide variety of nonlinear equations, including many of elliptic or parabolic type. Our formalizations of this heuristic, however, rely on the induction on energy methods of Bourgain and perturbation theory, as opposed to variational or compactness arguments.

We prove this in Section 5. The basic idea is as outlined in previous discussion; the main technical tool needed is the multilinear improvements to Strichartz' inequality in Section 3.3 to control the interaction between the two components and thus allow the resconstruction of the original solution u.

Clearly the conclusion of Proposition 4.3 is in conflict with the hypothesis (4.1), and so we should now expect the solution to be localized in frequency for every time t. This is indeed the case:

COROLLARY 4.4 (Frequency localization of energy at each time). *A minimal energy blowup solution of* (1.1) (*see Definition* 4.2) *satisfies*: *For every time* $t \in I_*$ *there exists a dyadic frequency* $N(t) \in 2^{\mathbb{Z}}$ *such that for every* $\eta_5 \leq \eta \leq \eta_0$ *we have small energy at frequencies* $\ll N(t)$,

$$\|P_{\leq c(\eta)N(t)} u(t)\|_{\dot{H}^1} \leq \eta, \tag{4.9}$$

small energy at frequencies $\gg N(t)$,

$$\|P_{\geq C(\eta)N(t)} u(t)\|_{\dot{H}^1} \leq \eta, \tag{4.10}$$

with the large energy at frequencies $\sim N(t)$,

$$\|P_{c(\eta)N(t) < \cdot < C(\eta)N(t)} u(t)\|_{\dot{H}^1} \sim 1. \tag{4.11}$$

Here $0 < c(\eta) \ll 1 \ll C(\eta) < \infty$ *are quantities depending on* η.

Informally, this corollary asserts that at every given time t the solution u is essentially concentrated at a single frequency $N(t)$. Note however that we do not presently have any information as to how $N(t)$ evolves in time; obtaining long-term control on $N(t)$ will be a key objective of later stages of the proof.

Proof. For each time $t \in I_*$, we define $N(t)$ as

$$N(t) := \sup\{N \in 2^{\mathbb{Z}} : \|P_{\leq N} u(t)\|_{\dot{H}^1} \leq \eta_0\}.$$

Since $u(t)$ is Schwartz, we see that $N(t)$ is strictly larger than zero; from the lower bound in (4.3) we see that $N(t)$ is finite. By definition of $N(t)$,

$$\|P_{\leq 2N(t)} u(t)\|_{\dot{H}^1} > \eta_0.$$

Now let $\eta_5 \leq \eta \leq \eta_0$. Observe that we now have (4.10) if $C(\eta)$ is chosen sufficiently large, because if (4.10) failed then Proposition 4.3 would imply that $\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq C(\eta)$, contradicting (4.1) if η_6 is sufficiently small. In particular we have (4.10) for $\eta = \eta_0$. Since we also have (4.9) for $\eta = \eta_0$ by construction of $N(t)$, we thus see from (4.3) that we have (4.11) for $\eta = \eta_0$, which of course then implies (again by (4.3)) the same bound for all $\eta_5 \leq \eta \leq \eta_0$. Finally, we obtain (4.9) for all $\eta_5 \leq \eta \leq \eta_0$ if $c(\eta)$ is chosen sufficiently small, since if (4.9) failed then by combining it with (4.11) and Proposition 4.3 we would once again imply that $\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq C(\eta)$, contradicting (4.1). $\square$

Having shown that any minimal energy blowup solution u must be localized in frequency at each time, we now turn to showing that such a u is also localized in physical space. This turns out to be somewhat more involved, although it still follows the same general strategy. We first borrow a useful trick from [4]; since u is Schwartz, we may divide the interval I_* into three consecutive pieces $I_* := I_- \cup I_0 \cup I_+$ where each of the three intervals contains a third of the $L^{10}_{t,x}$ density:

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dxdt = \frac{1}{3} \int_{I_*} \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dxdt \text{ for } I = I_-, I_0, I_+.$$

In particular from (4.1) we have

$$\|u\|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \gtrsim 1/\eta_6 \text{ for } I = I_-, I_0, I_+. \tag{4.12}$$

Thus to contradict (4.1) it suffices to obtain $L^{10}_{t,x}$ bounds on just one of the three intervals I_-, I_0, I_+.

It is in the middle interval I_0 that we can obtain physical space localization; this will be done in several stages. The first step is to ensure that the potential energy $\int_{\mathbb{R}^3} |u(t,x)|^6\, dx$ is bounded from below.

Proposition 4.5 (Potential energy bounded from below). *For any minimal energy blowup solution of* (1.1) (*see Definition* 4.2), *for all* $t \in I_0$,

$$\|u(t)\|_{L^6_x} \geq \eta_1. \tag{4.13}$$

This is proven in Section 6, and is inspired by a similar argument of Bourgain [4]. Using (4.13) and some simple Fourier analysis, we can thus establish the following concentration result:

Proposition 4.6 (Physical space concentration of energy at each time). *Any minimal energy blowup solution of* (1.1) *satisfies*: *For every* $t \in I_0$, *there exists an* $x(t) \in \mathbb{R}^3$ *such that*

$$\int_{|x - x(t)| \leq C(\eta_1)/N(t)} |\nabla u(t,x)|^2\, dx \gtrsim c(\eta_1) \tag{4.14}$$

and

$$\int_{|x - x(t)| \leq C(\eta_1)/N(t)} |u(t,x)|^p\, dx \gtrsim c(\eta_1) N(t)^{\frac{p}{2} - 3} \tag{4.15}$$

for all $1 < p < \infty$, *where the implicit constant can depend on* p. *In particular, we have*

$$\int_{|x - x(t)| \leq C(\eta_1)/N(t)} |u(t,x)|^6\, dx \gtrsim c(\eta_1). \tag{4.16}$$

This is proven in Section 7. Similar results were obtained in [4], [20] in the radial case; see also [3]. Informally, the above estimates emphasize that $u(t,x)$ is roughly of size $N(t)^{1/2}$ on the average when $|x - x(t)| \lesssim 1/N(t)$; observe that this is consistent with bounded energy (4.3) as well as with Corollary 4.4 and the uncertainty principle.

It turns out that in our argument, it is not enough to know that the energy concentrates at one location $x(t)$ at each time; we must also show that the energy is small at all other locations, where $|x-x(t)| \gg 1/N(t)$. The main tool for achieving this is

Proposition 4.7 (Physical space localization of energy at each time). *For any minimal energy blowup solution of* (1.1), *for every* $t \in I_0$

$$\int_{|x-x(t)|>1/(\eta_2 N(t))} |\nabla u(t,x)|^2 \, dx \lesssim \eta_1. \tag{4.17}$$

This is proven in Section 8. The proof follows a similar strategy to that used to prove Proposition 4.3; the main difference is that we now consider spatially separated components of u rather than frequency separated components, and instead of using multilinear Strichartz estimates to establish the decoupling of these components, we shall rely instead on approximate finite speed of propagation and on the pseudoconformal identity.

To summarize, at each time t we have a location $x(t)$, around which the kinetic and potential energy are large, and away from which the kinetic energy is small (and one can also show the potential energy is small, although we will not need this). From this and a little Fourier analysis we obtain an important conclusion:

Proposition 4.8 (Reverse Sobolev inequality). *When u is a minimal energy blowup solution* (*and hence* (4.2), (4.9)–(4.17) *hold*), *then for every* $t_0 \in I_0$, *any* $x_0 \in \mathbb{R}^3$, *and any* $R \geq 0$,

$$\int_{B(x_0,R)} |\nabla u(t_0,x)|^2 \, dx \lesssim \eta_1 + C(\eta_1,\eta_2) \int_{B(x_0,C(\eta_1,\eta_2)R)} |u(t_0,x)|^6 \, dx. \tag{4.18}$$

Thus, up to an error of η_1, we are able to control the kinetic energy locally by the potential energy.[19] This will be proved in Section 9. This fact will be crucial in the interaction Morawetz portion of our argument when we have an error term involving the kinetic energy, and control of a positive term which

[19]Note that this is a special property of the minimal energy blowup solution, reflecting the very strong physical space localization properties of such a solution; it is false in general, even for solutions to the free Schrödinger equation. Of course, Proposition 4.5 is similarly false in general; for instance, for solutions of the free Schrödinger equation, the L^6_x norm goes to zero as $t \to \pm\infty$.

involves the potential energy; the reverse Sobolev inequality is then used to control the former by the latter.

To summarize, the statements above tell us that any minimal energy blowup solution (Definition 4.2) to the equation (1.1) must be localized in both frequency and physical space at every time. We are still far from done: we have not yet precluded blowup in finite time (which would happen if $N(t) \to \infty$ as $t \to T_*$ for some finite time T_*), nor have we eliminated soliton or soliton-like solutions (which would correspond, roughly speaking, to $N(t)$ staying close to constant for all time t). To achieve this we need spacetime integrability bounds on u. Our main tool for this is a frequency-localized version of the interaction Morawetz estimate (1.8), to which we now turn.

4.3. *Second stage*: *Localized Morawetz estimate.* In order to localize the interaction Morawetz inequality, it turns out to be convenient to work at the "minimum" frequency attained by u.

From (1.18) we observe that

$$\|P_{c(\eta_0)N(t)<\cdot<C(\eta_0)N(t)}u(t)\|_{\dot H^1} \leq C(\eta_0)N(t)\|u\|_{L^\infty_t L^2_x}.$$

Comparing this with (4.11) we obtain the lower bound

$$N(t) \geq c(\eta_0)\|u\|^{-1}_{L^\infty_t L^2_x}$$

for $t \in I_0$. Since u is Schwartz, the right-hand side is nonzero, and thus the quantity

$$N_{\min} := \inf_{t\in I_0} N(t)$$

is strictly positive.

From (4.9) we see that the low frequency portion of the solution - where $|\xi| \leq c(\eta_0)N_{\min}$ - has small energy; one might then hope to use Strichartz estimates to obtain some spacetime control on these low frequencies. However, we do not yet have much control on the high frequencies $|\xi| \geq c(\eta_0)N_{\min}$, apart from the energy bounds (4.3) and (4.4) of course.

Our initial spacetime bound in the high frequencies is provided by the following interaction Morawetz estimate.

Proposition 4.9 (Frequency-localized interaction Morawetz estimate). *When u is a minimal energy blowup solution of* (1.1) (*and hence* (4.2), (4.9)–(4.18) *all hold*), *then for all* $N_* < c(\eta_3)N_{\min}$

$$\int_{I_0}\int |P_{\geq N_*}u(t,x)|^4 \, dxdt \lesssim \eta_1 N_*^{-3}. \tag{4.19}$$

Remark 4.10. The factor N_*^{-3} on the right-hand side of (4.19) is mandated by scale-invariance considerations (cf. (1.3)). The η_1 factor on the right side reflects our smallness assumption on N_*: if we think of N_* as being very small and then scale the solution so that $N_* = 1$, we are pushing the energy to very high frequencies so heuristically it's not unreasonable to expect the supercritical $L^4_{x,t}$ norm on the left-hand side to be small.

Regarding the size of N_*: write for the moment $\tilde{c}(\eta_3)$ as the constant appearing in Corollary 4.4 with $\eta = \eta_3$. The constant $c(\eta_3)$ appearing in Proposition 4.9 is chosen so that $c(\eta_3) \lesssim \tilde{c}(\eta_3) \cdot \eta_3$. Hence at all times we know there is very little energy at frequencies below $\frac{N_*}{\eta_3}$, and (ignoring factors of N_* which can be scaled to 1) above frequency N_* there is very little (at most η_3/N_*) L^2 mass.

This small η_1 factor will be used to close a bootstrap argument in the proof of the important estimate on the movement of energy to very low frequencies in Lemma 15.1 below.

If one already had Theorem 1.1, then this proposition would follow (but with η_1 replaced by $C(E_{\text{crit}})$) from Lemma 3.12, since the S^1 norm will control $\|\nabla u\|_{L^4_t L^3_x}$ and hence $\||\nabla|^{3/4} u\|_{L^4_t L^4_x}$ by Sobolev embedding. Of course, we will not prove Proposition 4.9 this way, as it would be circular. Instead, this proposition is based on the interaction Morawetz inequality developed in [12], [13] (see also a recent extension in [24]). The key thing about this estimate is that the right-hand side does not depend on I_0; thus for instance it is already useful in eliminating soliton or pseudosoliton solutions, at least for frequencies close to $N_{\min}$. (Frequencies much larger than $N_{\min}$ still cause difficulty, and will be dealt with later in the argument). Proposition 4.9 roughly corresponds to the localized Morawetz inequality used by Bourgain [4], [5] and Grillakis [20] in the radial case (see (1.7) above). The main advantage of (4.19) is that it is not localized to near the spatial origin, in contrast with the standard (1.5) and localized (1.7) Morawetz inequalities.

Although this proposition is based on the interaction Morawetz inequality developed in the references given above, there are significant technical difficulties in truncating that inequality to the high frequencies. As a consequence the proof of this proposition is somewhat involved and is given in Sections 10-14. Also, we caution the reader that the above proposition is not proved as an *a priori* estimate; indeed the proof relies crucially on the assumption that u is a minimal energy blowup solution in the sense of (4.1), and in particular verifies the reverse Sobolev inequality (4.18). See Section 10 for further remarks on the proof.

Combining Proposition 4.9 with Proposition 4.6 gives us the following integral bound on $N(t)$.

Corollary 4.11. *For any minimal energy blowup solution of* (1.1),

$$\int_{I_0} N(t)^{-1}\, dt \lesssim C(\eta_1, \eta_3) N_{\min}^{-3}. \tag{4.20}$$

Proof. Let $N_* := c(\eta_3)N_{\min}$ for some sufficiently small $c(\eta_3)$. Then from Proposition 4.9,

$$\int_{I_0}\int_{\mathbb{R}^3} |P_{\geq N_*}u(t,x)|^4\ dxdt \lesssim \eta_1 N_*^{-3} \lesssim C(\eta_1,\eta_3)N_{\min}^{-3}.$$

On the other hand, from Bernstein (1.19) and (4.4), for each $t \in I_0$ that

$$\int_{|x-x(t)|\leq C(\eta_1)/N(t)} |P_{<N_*}u(t,x)|^4\ dx \lesssim N(t)^{-3}\|P_{<N_*}u(t)\|_{L_x^\infty}^4 \lesssim C(\eta_1)N(t)^{-3}N_*^2.$$

So by (4.15) and the triangle inequality (since $N_* \leq c(\eta_3)N(t)$),

$$\int_{\mathbb{R}^3} |P_{\geq N_*}u(t,x)|^4\ dx \gtrsim c(\eta_1)N(t)^{-1}.$$

Comparing this with the previous estimate, the claim follows. □

Remark 4.12. The estimate (4.20) is scale-invariant under the natural scaling (1.3) (N has the units of length^{-1}, and t has the units of length^2). In the radial case, a somewhat similar estimate was obtained by Bourgain [4] and implicitly also by Grillakis [20]; in our notation, this bound would be the assertion that

$$\int_I N(t)\ dt \lesssim |I|^{1/2} \tag{4.21}$$

for all $I \subseteq I_0$; indeed in the radial case (when $x(t) = 0$) this bound easily follows from Proposition 4.6 and (1.7). Both estimates are equally good at estimating the amount of time for which $N(t)$ is comparable to $N_{\min}$, but Corollary 4.11 is much weaker than (4.21) when it comes to controlling the times for which $N(t) \gg N_{\min}$. Indeed if we could extend (4.21) to the nonradial case we could obtain a significantly shorter proof of Theorem 1.1. However we were unable to prove this bound directly,[20] although it can be deduced from Corollary 4.11 and Proposition 4.15 below).

This Corollary allows us to obtain some useful $L^{10}_{t,x}$ bounds in the case when $N(t)$ is bounded from above.

Corollary 4.13 (Nonconcentration implies spacetime bound). *Let $I \subseteq I_0$, and suppose there exists an $N_{\max} > 0$ such that $N(t) \leq N_{\max}$ for all $t \in I$.*

[20]Comparing (4.19) with (1.7) one also sees that our control on how often the solution concentrates is weaker than that in the radial arguments of Bourgain and Grillakis. Heuristically: (4.19) allows a long train in time of N^3 "bubbles" at frequency $N \gg 1$, with size $\sim N^{\frac{1}{2}}$, spatial extent N^{-1}, and individual duration $\sim N^{-2}$, so the total lifespan of the train is $\sim N$. The estimate (1.7), on the other hand, restricts such bubbles to a set of dimension less than $\frac{1}{2}$ in time. Our proof of Theorem 1.1 makes up for this weakness in its next stage, specifically the relatively strong frequency localized L^2 almost conservation estimate of Lemma (4.14).

Then for any localized minimal energy blowup solution of (1.1),

$$\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim C(\eta_1, \eta_3, N_{\max}/N_{\min})$$

and furthermore

$$\|u\|_{\dot S^1(I\times\mathbb{R}^3)} \lesssim C(\eta_1, \eta_3, N_{\max}/N_{\min}).$$

Proof. We may use scale invariance (1.3) to rescale $N_{\min} = 1$. From Corollary 4.11 we obtain the useful bound

$$|I| \lesssim C(\eta_1, \eta_3, N_{\max}).$$

Let $\delta = \delta(\eta_0, N_{\max}) > 0$ be a small number to be chosen later. We may partition I into $O(|I|/\delta)$ intervals $I_1, \dots, I_J$ of length at most δ. Let I_j be any of these intervals, and let t_j be any time in I_j. Observe from Corollary 4.4 and the hypothesis $N(t_j) \leq N_{\max}$ that

$$\|P_{\geq C(\eta_0)N_{\max}} u(t_j)\|_{\dot H^1} \leq \eta_0$$

(for instance). Now let $\tilde u(t) := e^{i(t-t_j)\Delta} P_{<C(\eta_0)N_{\max}} u(t_j)$ be the free evolution of the low and medium frequencies of u. The above estimate then becomes

$$\|u(t_j) - \tilde u(t_j)\|_{\dot H^1} \leq \eta_0.$$

On the other hand, from Bernstein (1.19) and (4.3) we have

$$\|\nabla \tilde u(t)\|_{L^{30/13}_x} \lesssim C(\eta_0, N_{\max}) \|\tilde u(t_j)\|_{\dot H^1} \lesssim C(\eta_0, N_{\max})$$

for all $t \in I_j$, and hence

$$\|\nabla \tilde u\|_{L^{10}_t L^{30/13}_x (I_j\times\mathbb{R}^3)} \lesssim C(\eta_0, N_{\max})\delta^{1/10}.$$

Similarly,

$$\begin{aligned}\|\nabla(|\tilde u(t)|^4 \tilde u(t))\|_{L^{6/5}_x} &\lesssim \|\nabla \tilde u(t)\|_{L^6_x} \|\tilde u(t)\|^4_{L^6_x} \\ &\lesssim C(\eta_0, N_{\max}) \|\tilde u(t_j)\|^5_{\dot H^1} \lesssim C(\eta_0, N_{\max})\end{aligned}$$

and hence

$$\|\nabla(|\tilde u(t)|^4 \tilde u(t))\|_{L^2_t L^{6/5}_x (I_j\times\mathbb{R}^3)} \lesssim C(\eta_0, N_{\max})\delta^{1/2}.$$

From these two estimates, the energy bound (4.3), and Lemma 3.9 with $e = -|\tilde u|^4 \tilde u$, we see (if δ is chosen sufficiently small) that

$$\|u\|_{L^{10}_{t,x}(I_j\times\mathbb{R}^3)} \lesssim 1$$

Summing this over each of the $O(|I|/\delta)$ intervals I_j we obtain the desired $L^{10}_{t,x}$ bound. The $\dot S^1$ bound then follows from Lemma 3.12. □

The corollary above gives the desired contradiction to (4.12) when $N_{\max}/N_{\min}$ is bounded; i.e., $N(t)$ stays in a bounded range.

4.4. *Third stage*: *Nonconcentration of energy.* Of course, any global well-posedness argument for (1.1) must eventually exclude a blowup scenario (self-similar or otherwise) where $N(t)$ goes to infinity in finite time, and indeed by Corollary 4.13 this is the only remaining possibility for a minimal energy blowup solution. Corollary 4.4 implies that in such a scenario the energy must almost entirely evacuate the frequencies near $N_{\min}$, and instead concentrate at frequencies much larger than $N_{\min}$. While this scenario is consistent with conservation of energy, it turns out to not be consistent with the time and frequency distribution of mass.

More specifically, we know there is a $t_{\min} \in I_0$ so that for all $t \in I_0, N(t) \geq N(t_{\min}) := N_{\min} > 0$. By Corollary 4.4, at time $t_{\min}$ the solution has the bulk of its energy near the frequency $N_{\min}$, and hence the medium frequencies at that time have mass bounded below by,

$$\|P_{c(\eta_0)N_{\min} \leq \cdot \leq C(\eta_0)N_{\min}} u(t_{\min})\|_{L^2} \gtrsim c(\eta_0) N_{\min}^{-1}. \tag{4.22}$$

The idea is to prove the following approximate mass conservation law for these high frequencies,[21] which states that while some mass might slip to very low frequencies as the solution moves to high frequencies, it cannot all do so.

Lemma 4.14 (Some mass freezes away from low frequencies). *Suppose u is a minimal energy blowup solution of* (1.1), *and let* $[t_{\min}, t_{\text{evac}}] \subset I_0$ *be such that* $N(t_{\min}) = N_{\min}$ *and* $N(t_{\text{evac}})/N_{\min} \geq C(\eta_5)$. *Then for all* $t \in [t_{\min}, t_{\text{evac}}]$,

$$\|P_{\geq \eta_4^{100} N_{\min}} u(t)\|_{L^2} \gtrsim \eta_1 N_{\min}^{-1}. \tag{4.23}$$

Lemma 4.14 will quickly show that the evacuation scenario - wherein the solution cleanly concentrates energy to very high frequencies - cannot occur. Instead the solution always leaves a nontrivial amount of mass and energy behind at medium frequencies. This "littering" of the solution will serve (via Corollary 4.4) to keep $N(t)$ from escaping to infinity[22] and gives us,

[21] It is necessary to truncate to the high frequencies in order to exploit mass conservation because the low frequencies contain an unbounded amount of mass. This strategy of mollifying the solution in frequency space in order to exploit a conservation law that would otherwise be unbounded or useless is inspired by the "I-method" for sub-critical dispersive equations; see e.g. [13].

[22] It is interesting to note that one must exploit conservation of energy, conservation of mass, *and* conservation of momentum (via the Morawetz inequality) in order to prevent blowup for the equation (1.1); the same phenomenon occurs in the previous arguments [4], [20] in the radial case, even though the details of those arguments are in many ways quite different from those here.

Proposition 4.15 (Energy cannot evacuate from low frequencies). *For any minimal energy blowup solution of* (1.1),

$$N(t) \lesssim C(\eta_5) N_{\min} \tag{4.24}$$

for all $t \in I_0$.

We give the somewhat complicated proof of Lemma 4.14 and Proposition 4.15 in Section 15.

By combining Proposition 4.15 with Corollary 4.13, we encounter a contradiction to (4.12) which completes the proof of Theorem 1.1.

The proofs of the above claims occupy the remainder of this paper. Before moving to these proofs, we summarize the role of the parameters $\eta_i, i = 0, \dots, 5$ which have now all been introduced. The number η_1 represents the amount of potential energy that must be present at every time in a minimal energy blowup solution (Proposition 4.5); it also represents the extent of concentration of energy (on the scale of $1/N(t)$) that must occur in physical space at every time in a minimal energy blowup solution (Proposition 4.6). The number η_2 is introduced in Proposition 4.7, where $1/\eta_2$ represents the extent that there is localization (on the scale of $1/N(t)$) of energy in a minimal energy blowup solution. The number η_3 measures, on the scale of the quantity $N_{\min}$, what we mean by "high frequency" when we say Proposition 4.9 is an interaction Morawetz estimate localized to high frequencies. The number η_4 measures the frequency (on the scale of $N_{\min}$) below which the evolution can't move a certain portion (namely, η_1) of the L^2 mass. Finally, the number η_0 enters in Corollary 4.4 and various other points in the paper where we simply use its value as a small, universal constant.

5. Frequency delocalized at one time $\implies$ spacetime bounded

We now prove Proposition 4.3. Let $0 < \varepsilon = \varepsilon(\eta) \ll 1$ be a small number to be chosen later. If $K(\eta)$ is sufficiently large depending on ε, then one can find at least ε^{-2} disjoint intervals $[\varepsilon^2 N_j, N_j/\varepsilon^2]$, $j = 1, \dots, \varepsilon^{-2}$, contained inside $[N_{lo}, K(\eta) N_{lo}]$. From (4.3) and the pigeonhole principle, there must therefore exist an N_j so that the interval $[\varepsilon^2 N_j, N_j/\varepsilon^2]$ is mostly free of energy:

$$\|P_{\varepsilon^2 N_j \leq \cdot \leq N_j/\varepsilon^2} u(t_0)\|_{\dot{H}^1} \lesssim \varepsilon. \tag{5.1}$$

As the statement and conclusion of Proposition 4.3 is invariant under the scaling (1.3) we may set $N_j := 1$. Now define

$$u_{lo}(t_0) := P_{\leq \varepsilon} u(t_0); \quad u_{hi}(t_0) := P_{\geq 1/\varepsilon} u(t_0).$$

The functions $u_{lo}(t_0)$, $u_{hi}(t_0)$ have strictly smaller energy than u:

LEMMA 5.1. *If ε is sufficiently small depending on η, then*

$$E(u_{lo}(t_0)), E(u_{hi}(t_0)) \leq E_{\text{crit}} - c\eta^C.$$

Proof. We prove this for $E(u_{lo}(t_0))$; the claim for $E(u_{hi}(t_0))$ is similar. Define $u_{hi'}(t_0) := P_{>\varepsilon} u(t_0)$, so that $u(t_0) = u_{lo}(t_0) + u_{hi'}(t_0)$, and consider the quantity

$$|E(u(t_0)) - E(u_{lo}(t_0)) - E(u_{hi'}(t_0))|. \tag{5.2}$$

From (1.2) we can bound this by

(5.3)
$$(5.2) \lesssim |\langle \nabla u_{lo}(t_0), \nabla u_{hi'}(t_0)\rangle| + |\int |u(t_0)|^6 - |u_{lo}(t_0)|^6 - |u_{hi'}(t_0)|^6 \, dx|.$$

The functions u_{lo} and $u_{hi'}$ almost have disjoint supports, and their inner product is very close to zero. Indeed from Parseval and (5.1) we have

$$|\langle \nabla u_{lo}(t_0), \nabla u_{hi'}(t_0)\rangle| \lesssim \varepsilon^2.$$

Now we estimate the L^6-type terms. From the pointwise estimate

$$\big||u(t_0)|^6 - |u_{lo}(t_0)|^6 - |u_{hi'}(t_0)|^6\big| \lesssim |u_{lo}||u_{hi'}|(|u_{lo}| + |u_{hi'}|)^4$$

(cf. (1.14)) and Hölder's inequality, we can bound the second term in (5.3) by

$$\lesssim \|u_{lo}\|_\infty \|u_{hi'}\|_3 (\|u_{lo}\|_6 + \|u_{hi'}\|_6)^4.$$

From (4.3), (5.1), and Bernstein's inequality (1.20) we see that

$$\begin{aligned} \|u_{lo}\|_\infty &\lesssim \sum_{N \leq \varepsilon} \|P_N u\|_\infty \lesssim \sum_{N \leq \varepsilon} N^{1/2} \|P_N u\|_{\dot{H}^1} \\ &\lesssim \sum_{N \leq \varepsilon^2} N^{1/2} + \sum_{\varepsilon^2 < N \leq \varepsilon} N^{1/2} \varepsilon \lesssim \varepsilon \end{aligned}$$

and

$$\begin{aligned} \|u_{hi'}\|_3 &\lesssim \sum_{N \geq \varepsilon} \|P_N u\|_3 \lesssim \sum_{N \geq \varepsilon} N^{-1/2} \|P_N u\|_{\dot{H}^1} \\ &\lesssim \sum_{N \geq 1/\varepsilon} N^{-1/2} + \sum_{\varepsilon \leq N \leq \frac{1}{\varepsilon}} N^{-1/2} \varepsilon \lesssim \varepsilon^{1/2}. \end{aligned}$$

Thus from (4.4) we can bound the second term in (5.3) by $O(\varepsilon^{3/2})$. Combining this with the estimate obtained on the first piece of (5.3), we thus see that

$$|E(u) - E(u_{lo}(t_0)) - E(u_{hi'}(t_0))| \lesssim \varepsilon^{3/2}.$$

On the other hand, by hypothesis on u we have $E(u) \leq E_{\text{crit}}$, while from (4.7) and (1.2) we have $E(u_{hi'}(t_0)) \gtrsim \eta^C$. The claim follows if ε is chosen sufficiently small. $\square$

From Lemma 5.1 and Lemma 4.1 we know that there exist Schwartz solutions u_{lo}, u_{hi} on the slab $I_* \times \mathbb{R}^3$ with initial data $u_{lo}(t_0)$, $u_{hi}(t_0)$ at time t_0, and furthermore

$$\|u_{lo}\|_{\dot{S}^1(I_* \times \mathbb{R}^3)}, \|u_{hi}\|_{\dot{S}^1(I_* \times \mathbb{R}^3)} \lesssim C(\eta). \tag{5.4}$$

Let $\tilde{u} := u_{lo} + u_{hi}$. We now claim that $\tilde{u}$ is an approximate solution to (1.1):

Lemma 5.2. *We have*

$$i\tilde{u}_t + \Delta\tilde{u} = |\tilde{u}|^4\tilde{u} - e$$

where the error e obeys the bounds

$$\|\nabla e\|_{L^2_t L^{6/5}_x(I_* \times \mathbb{R}^3)} \lesssim C(\eta)\varepsilon^{1/2}. \tag{5.5}$$

Proof. We begin by establishing further estimates on u_{lo} and u_{hi}, beyond (5.4). For u_{hi}, we observe from (4.5) that $\|u_{hi}(t_0)\|_2 \lesssim \varepsilon$; so from Lemma 3.12,

$$\|u_{hi}\|_{\dot{S}^0(I_* \times \mathbb{R}^3)} \lesssim C(\eta)\varepsilon. \tag{5.6}$$

Similarly, from (4.3) and (1.17) we have $\|u_{lo}(t_0)\|_{\dot{H}^2} \lesssim C\varepsilon$, and so from Lemma 3.12 again we have

$$\|u_{lo}(t_0)\|_{\dot{S}^2(I_* \times \mathbb{R}^3)} \lesssim C(\eta)\varepsilon. \tag{5.7}$$

From Lemma 3.6 we also see that

$$\begin{aligned}
\||u_{hi}|^4 u_{hi}\|_{L^1_t L^2_x(I_* \times \mathbb{R}^3)} &\lesssim C(\eta)\varepsilon,\\
\|\nabla(|u_{hi}|^4 u_{hi})\|_{L^1_t L^2_x(I_* \times \mathbb{R}^3)} &\lesssim C(\eta),\\
\|\nabla(|u_{lo}|^4 u_{lo})\|_{L^1_t L^2_x(I_* \times \mathbb{R}^3)} &\lesssim C(\eta),\\
\|\nabla^2(|u_{lo}|^4 u_{lo})\|_{L^1_t L^2_x(I_* \times \mathbb{R}^3)} &\lesssim C(\eta)\varepsilon.
\end{aligned}$$

From Lemma 3.7 we thus have

$$\|\nabla\mathcal{O}(u_{hi}^j u_{lo}^{5-j})\|_{L^2_t L^{6/5}_x(I_* \times \mathbb{R}^3)} \lesssim C(\eta)\varepsilon^{1/2}$$

for $j = 1, 2, 3, 4$. Since $e = \sum_{j=1}^4 \mathcal{O}(u_{hi}^j u_{lo}^{5-j})$ by (1.15), the claim follows. □

We now pass from estimates on $\tilde{u}$ to estimates on u by perturbation theory. From (5.1) we have the perturbation bound

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1} \lesssim \varepsilon,$$

while from (5.4) we have

$$\|\tilde{u}\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \lesssim C(\eta).$$

Thus if ε is small enough depending on η, we may apply Lemma 3.10 and obtain the desired bound (4.8). The proof of Proposition 4.3 is now complete. □

Remark 5.3. The dependence of constants in Proposition 4.3 given by the above argument is quite poor. Specifically, the separation $K(\eta)$ needs to be as large as

$$K(\eta) \geq C \exp(C\eta^{-C} M(E_{\text{crit}} - \eta^C)^C)$$

(mainly in order for the pigeonhole argument to work) and the bound one obtains on the $L^{10}_{t,x}$ norm at the end has a similar size. This implies that the dependence of the constants $C(\eta_j)$, $c(\eta_j)$ in Corollary 4.4 is similarly similarly:

$$\begin{aligned} C(\eta_j) &\geq C \exp(C\eta_j^{-C} M(E_{\text{crit}} - \eta_j^C)^C), \\ c(\eta_j) &\leq (C(\eta_j))^{-1}. \end{aligned}$$

This will force us to select each η_{j+1} quite small depending on previous η_j; indeed in some cases the induction hypothesis is used more than once and so η_{j+1} is even smaller than the above expressions suggest. If one then runs the induction of energy argument in a direct way (rather than arguing by contradiction as we do here), this leads to very rapidly growing (but still finite) bound for $M(E)$ for each E, which can only be expressed in terms of multiply iterated towers of exponentials (the Ackermann hierarchy). More precisely, if we use $X \uparrow Y$ to denote exponentiation X^Y,

$$X \uparrow\uparrow Y := X \uparrow (X \uparrow \ldots \uparrow X)$$

to denote the tower formed by exponentiating Y copies of X,

$$X \uparrow\uparrow\uparrow Y := X \uparrow\uparrow (X \uparrow\uparrow \ldots \uparrow\uparrow X)$$

to denote the double tower formed by tower-exponentiating Y copies of X, and so forth, then we have computed our final bound for $M(E)$ for large E to essentially be

$$M(E) \leq C \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow (CE^C).$$

This rather Bunyanesque bound is mainly due to the large number of times we invoke the induction hypothesis Lemma 4.1, and is presumably not best possible. For instance, the best bound known[23] in the radial case is $M(E) \leq C \uparrow\uparrow (CE^C)$, where the induction hypothesis is used only once; see [4]. Finally, in the case of the subcritical cubic nonlinear Schrödinger equation, the bound for the analogue of $M(E)$ is polynomial, $M(E) \leq CE^C$; see [13].

6. Small L^6_x norm at one time $\implies$ spacetime bounded

We now prove Proposition 4.5. The argument here is similar to the induction on energy arguments in [4]. The point is that the linear evolution of the

[23]Note added in proof: a bound of $M(E) \leq C \uparrow (CE^C)$ in the radial case was recently obtained in [45].

solution must concentrate at some point (t_1, x_1) in spacetime (otherwise we could iterate using the small data theory). If the solution does not concentrate in L^6 at time $t = t_0$, then t_1 must be far away from t_0. The idea is then to remove the energy concentrating at (t_1, x_1) and induct on energy.

We turn to the details. Assume for contradiction that (4.13) failed for some time $t_0 \in I_0$, so that

$$\|u(t_0)\|_{L^6_x} \leq \eta_1. \tag{6.1}$$

Fix this t_0. By rescaling using (1.3) we may normalize $N(t_0) = 1$. Observe that if the linear solution $e^{i(t-t_0)\Delta}u(t_0)$ had small $L^{10}_{t,x}$ norm, then the standard small data well-posedness theory (based on Strichartz estimates and (3.11) or (3.12)) would already show that the nonlinear solution u had bounded L^{10} norm. Thus we may assume that

$$\|e^{i(t-t_0)\Delta}u(t_0)\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \gtrsim 1.$$

On the other hand, by Corollary 4.4 we have

$$\|P_{lo}u(t_0)\|_{\dot{H}^1_x} + \|P_{hi}u(t_0)\|_{\dot{H}^1_x} \lesssim \eta_0$$

where we define $P_{lo} := P_{<c(\eta_0)}$ and $P_{hi} := P_{>C(\eta_0)}$; so by Strichartz (Lemma 3.2)

$$\|e^{i(t-t_0)\Delta}P_{lo}u(t_0)\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} + \|e^{i(t-t_0)\Delta}P_{hi}u(t_0)\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \eta_0.$$

If we then define $P_{\text{med}} := 1 - P_{lo} - P_{hi}$, we must then have

$$\|e^{i(t-t_0)\Delta}P_{\text{med}}u(t_0)\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \sim 1.$$

On the other hand, by (4.3) we know that $P_{\text{med}}u(t_0)$ has bounded energy and has Fourier support in the region $c(\eta_0) \lesssim |\xi| \lesssim C(\eta_0)$. Thus by Strichartz (3.7) we have that

$$\|e^{i(t-t_0)\Delta}P_{\text{med}}u(t_0)\|_{L^{10/3}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim C(\eta_0)$$

(for instance). From these two estimates and Hölder we see that the $L^\infty_{t,x}$ norm cannot be too small:

$$\|e^{i(t-t_0)\Delta}P_{\text{med}}u(t_0)\|_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \gtrsim c(\eta_0).$$

In particular, there exist a time $t_1 \in \mathbb{R}$ and a point x_1 such that we have the concentration

$$|e^{i(t_1-t_0)\Delta}(P_{\text{med}}u(t_0))(x_1)| \gtrsim c(\eta_0).$$

By perturbing t_1 a little we may assume that $t_1 \neq t_0$; by time reversal symmetry we may take $t_1 < t_0$.

Let δ_{x_1} be the Dirac mass at x_1, and let $f(t_1) := P_{\text{med}}\delta_{x_1}$. We extend f to all of $\mathbb{R}\times\mathbb{R}^3$ by the free evolution, thus $f(t) := e^{i(t-t_1)\Delta}f(t_1)$. We record some explicit estimates on f:

LEMMA 6.1. *For any $t \in \mathbb{R}$ and any $1 \leq p \leq \infty$,*

$$\|f(t)\|_{L^p_x} \lesssim C(\eta_0)(1+|t-t_1|)^{3/p-3/2}.$$

Proof. We may translate $t_1 = x_1 = 0$. From the unitarity of $e^{it\Delta}$ and Bernstein (1.20) we have

$$\|f(t)\|_{L^\infty_x(\mathbb{R}^3)} \lesssim C(\eta_0)\|f(t)\|_{L^2_x(\mathbb{R}^3)} = C(\eta_0)\|P_{\mathrm{med}}\delta_{x_1}\|_{L^2_x(\mathbb{R}^3)} \lesssim C(\eta_0)$$

while from the dispersive inequality (1.12) we have

$$\|f(t)\|_{L^\infty_x(\mathbb{R}^3)} \lesssim |t|^{-3/2}\|P_{\mathrm{med}}\delta_{x_1}\|_{L^1_x(\mathbb{R}^3)} \lesssim |t|^{-3/2}.$$

Combining these estimates we obtain the lemma in the case $p=\infty$. To obtain the other cases we need some decay on $f(t,x)$ in the region $|x| \gg C(\eta_0)(1+|t|)$. For this we use the Fourier representation (1.10) to write

$$f(t,x) = \int_{\mathbb{R}^3} e^{2\pi i(x\cdot\xi - 2\pi t|\xi|^2)}\varphi_{\mathrm{med}}(\xi)\, d\xi$$

where φ_{med} is the Fourier multiplier corresponding to P_{med}. When $|x| \gg 1+|t|$, then the phase oscillates in ξ with a gradient comparable in magnitude to $|x|$. By repeated integration by parts (see e.g. [41]) we thus obtain a bound of the form $|f(t,x)| \lesssim |x|^{-100}$ in this region. Combining this with the previous L^∞ bounds we obtain the result. □

In particular, from (6.1) and Hölder we have

$$|\langle u(t_0), f(t_0)\rangle| \lesssim \eta_1\|f(t_0)\|_{L^{6/5}_x(\mathbb{R}^3)} \lesssim C(\eta_0)\eta_1(1+|t_0-t_1|).$$

On the other hand, we have

$$|\langle u(t_0), f(t_0)\rangle| = |\langle e^{i(t_1-t_0)\Delta}P_{\mathrm{med}}u(t_0), \delta_{x_1}\rangle| \gtrsim c(\eta_0).$$

Thus the concentration point t_1 must be far away from t_0 (recall that η_1 is much smaller than η_0):

$$|t_1-t_0| \gtrsim c(\eta_0)\eta_1^{-1}.$$

In particular, the smallness of η_1 pushes the concentration time far away from the time when L^6_x is small. Since $t_0 > t_1$ by hypothesis, we thus see from Lemma 6.1 and the frequency localization of f that ∇f has small $L^{10}_t L^{30/13}_x$ norm to the future of t_0:

$$\begin{aligned}\|\nabla f\|_{L^{10}_t L^{30/13}_x([t_0,+\infty)\times\mathbb{R}^3)} &\lesssim C(\eta_0)\|f\|_{L^{10}_t L^{30/13}([t_0,+\infty)\times\mathbb{R}^3)}\\ &\lesssim C(\eta_0)\|(1+|t-t_1|)^{-2/10}\|_{L^{10}_t([t_0,+\infty))}\\ &\lesssim C(\eta_0)|t_0-t_1|^{-1/10}\\ &\lesssim C(\eta_0)\eta_1^{1/10}.\end{aligned} \tag{6.2}$$

We now use the induction hypothesis (inspired by a similar argument in [4]). We split $u(t_0) := v(t_0)+w(t_0)$ where $w(t_0) := \delta e^{i\theta}\Delta^{-1}f(t_0)$, and $\delta = \delta(\eta_0) > 0$

is a small number to be chosen shortly, and θ is a phase to be chosen shortly. We now claim that if δ and θ are chosen correctly, then $v(t_0)$ has slightly smaller energy than u. Indeed, we have by integration by parts and definition of f that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v(t_0)|^2 &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u(t_0)-\nabla w(t_0)|^2\\ &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u(t_0)|^2\\ &\quad - \delta\mathrm{Re}\int_{\mathbb{R}^3} e^{-i\theta}\overline{\nabla\Delta^{-1}f(t_0)}\cdot\nabla u(t_0) + O(\delta^2\|\Delta^{-1}f(t_0)\|^2_{\dot{H}^1})\\ &\le E_{\mathrm{crit}} + \delta\mathrm{Re}e^{-i\theta}\langle u(t_0), f(t_0)\rangle + O(\delta^2 C(\eta_0)).\end{aligned}$$

Since $|\langle u(t_0), f(t_0)\rangle|$ was already shown to have magnitude at least $c(\eta_0)$, we see if $\delta = \delta(\eta_0)$ and θ are chosen correctly that we can ensure that

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v(t_0)|^2 \le E_{\mathrm{crit}} - c(\eta_0).$$

Meanwhile, another application of Lemma 6.1 shows that

$$\|w(t_0)\|_{L^6_x} \lesssim C(\eta_0)\|f(t_0)\|_{L^6_x} \lesssim C(\eta_0)\langle t_0 - t_1\rangle^{-1} \lesssim C(\eta_0)\eta_1.$$

So by (6.1) and the triangle inequality,

$$\int_{\mathbb{R}^3}|v(t_0)|^6 \lesssim C(\eta_0)\eta_1^6.$$

Thus if η_1 is sufficiently small depending on η_0, then

$$E(v(t_0)) \le E_{\mathrm{crit}} - c(\eta_0).$$

By Lemma 4.1 we may extend $v(t_0)$ into a solution of the nonlinear Schrödinger equation (1.1) on $[t_0, +\infty)$ such that

$$\|v\|_{L^{10}_{t,x}([t_0,+\infty)\times\mathbb{R}^3)} \lesssim M(E_{\mathrm{crit}} - c(\eta_0)) = C(\eta_0). \tag{6.3}$$

On the other hand, from (6.2) and the frequency localization we have

$$\|\nabla e^{i(t-t_0)\Delta}w(t_0)\|_{L^{10}_t L^{30/13}_x([t_0,+\infty)\times\mathbb{R}^3)} \lesssim C(\eta_0)\eta_1^{1/10}.$$

Thus if η_1 is sufficiently small depending on η_0 then we may use Lemma 3.10 (with $\tilde{u} := v$ and $e = 0$) to conclude that u extends to all of $[t_0, +\infty)$ with

$$\|u\|_{L^{10}_{t,x}([t_0,+\infty)\times\mathbb{R}^3)} \lesssim C(\eta_0, \eta_1).$$

Since the time interval $[t_0, +\infty)$ contains I_+, this contradicts (4.12). (If we had $t_0 < t_1$ instead, we would obtain a similar contradiction involving I_-.) This concludes the proof of Proposition 4.5. $\square$

7. Spatial concentration of energy at every time

We now prove Proposition 4.6. Fix t. By scaling using (1.3) we may take $N(t) = 1$. By Corollary 4.4 this implies that

$$\|P_{>C(\eta_1)}u(t)\|_{\dot{H}^1} + \|P_{<c(\eta_1)}u(t)\|_{\dot{H}^1} \lesssim \eta_1^{100} \tag{7.1}$$

(for instance). In particular by Sobolev we have

$$\|P_{>C(\eta_1)}u(t)\|_{L_x^6} + \|P_{<c(\eta_1)}u(t)\|_{L_x^6} \lesssim \eta_1^{100}$$

and hence by (4.13)

$$\|P_{\rm med}u(t)\|_{L_x^6} \gtrsim \eta_1$$

where $P_{\rm med} := P_{c(\eta_1)<\cdot<C(\eta_1)}$. On the other hand, from (4.3),

$$\|P_{\rm med}u(t)\|_{L_x^2} \lesssim C(\eta_1)$$

and thus by Hölder's inequality,

$$\|P_{\rm med}u(t)\|_{L_x^\infty} \gtrsim c(\eta_1).$$

Thus there exists $x(t) \in \mathbb{R}^3$ such that

$$c(\eta_1) \lesssim |P_{\rm med}u(t, x(t))|. \tag{7.2}$$

Let $K_{\rm med}$ denote the kernel associated to the operator $P_{\rm med}\nabla\Delta^{-1}$, and let $R > 0$ be a radius to be chosen later. Then (7.2) can be continued with

$$\begin{aligned}
c(\eta_1) &\lesssim |K_{\rm med} * \nabla u(t, x(t))| \\
&\lesssim \int |K_{\rm med}(x(t) - x)||\nabla u(t, x)|dx \\
&\sim \int_{|x-x(t)|<R)} |K_{\rm med}(x(t) - x)||\nabla u(t, x)|dx \\
&\quad + \int_{|x-x(t)|\geq R} |K_{\rm med}(x(t) - x)||\nabla u(t, x)|dx \\
&\lesssim C(\eta_1)\left(\int_{|x-x(t)|<R} |\nabla u(t, x)|^2 dx\right)^{\frac{1}{2}} \\
&\quad + C(\eta_1)\left(\int_{|x-x(t)|\geq R} \frac{|\nabla u(t, x)|}{|x - x(t)|^{100}} dx\right).
\end{aligned}$$

Here we used Cauchy-Schwarz and the fact that $K_{\rm med}$ is a Schwartz function. Using the fact that $(\int |\nabla u|^2 dx)^{1/2}$ is bounded uniformly, we obtain

$$c(\eta_1) \lesssim \left(\int_{|x-x(t)|<R} |\nabla u(t, x)|^2 dx\right)^{\frac{1}{2}} + C(\eta_1)R^{-10}$$

(say), which proves (4.14) after setting $R := C(\eta_1)$ sufficiently large. Similarly, writing $\tilde{K}_{\mathrm{med}}$ for the kernel associated to P_{med}, we have for all $1 < p < \infty$,

$$
\begin{aligned}
c(\eta_1) \lesssim & \int_{|x-x(t)|<R} |\tilde{K}_{\mathrm{med}}(x(t)-x)||u(t,x)|dx \\
& + \int_{|x-x(t)|\geq R} |\tilde{K}_{\mathrm{med}}(x(t)-x)||u(t,x)|dx \\
\lesssim & C(\eta_1)\left(\int_{|x-x(t)|\leq R} |u(t,x)|^p dx\right)^{\frac{1}{p}} + \int_{|x-x(t)|\geq R} \frac{|u(t,x)|}{|x-x(t)|^{100}} dx \\
\lesssim & C(\eta_1)\left(\int_{|x-x(t)|\leq R} |u(t,x)|^p dx\right)^{\frac{1}{p}} \\
& + \left(\int_{|x-x(t)|>R} \frac{1}{|x-x(t)|^{100\times\frac{6}{5}}} dx\right)^{\frac{5}{6}} \|u(t)\|_{L^6_x} \\
\lesssim & C(\eta_1)\left(\int_{|x-x(t)|\leq R} |u(t,x)|^p dx\right)^{\frac{1}{p}} + C(\eta_1)R^{-10},
\end{aligned}
$$

where we used (4.4). This proves (4.15) upon rescaling, if the radius $R = C(\eta_1)$ was chosen sufficiently large. $\square$

8. Spatial delocalized at one time $\Longrightarrow$ spacetime bounded

We now prove Proposition 4.7. This is the spatial analogue of the frequency delocalization result in Proposition 4.3. The role of the bilinear Strichartz estimate in that proposition will be played here by finite speed of propagation and pseudoconformal identity estimates. We follow the same basic strategy as in Proposition 4.3 (but now played out in the arena of physical space rather than frequency space). More specifically, we assume that there is a large amount of energy away from the concentration point, and then use approximate finite speed of propagation to decouple the solution into two nearly noninteracting components of strictly smaller energy, which can then be handled by the induction hypothesis and perturbation theory.

We turn to the details. We first need a number of large quantities. Specifically, we need a large integer $J = J(\eta_1) \gg 1$ to be chosen later, and then a large frequency[24] $N_0 = N_0(\eta_1, J) \gg 1$ to be chosen later, and then a large radius $R_0 = R_0(\eta_1, N_0, J) \gg 1$ to be chosen later.

[24]More precisely, this is a ratio of two frequencies, but as we will shortly normalize $N(0) = 1$, the distinction between a frequency and a frequency ratio becomes irrelevant. Similarly the radii given below should really be ratios of radii.

Suppose for contradiction that the proposition is false. Then there must exist a time $t_0 \in I_0$ such that

$$\int_{|x-x(t_0)|>1/(\eta_2 N(t_0))} |\nabla u(t_0,x)|^2 \, dx \gtrsim \eta_1.$$

Here $x(t)$ is the quantity constructed in Proposition 4.6 (see (7.2)). We may normalize $t_0 = x(t_0) = 0$, and rescale so that $N(0) = 1$; thus

$$\int_{|x|>1/\eta_2} |\nabla u(0,x)|^2 \, dx \gtrsim \eta_1. \tag{8.1}$$

On the other hand, if $R_0 = R_0(\eta_1)$ is chosen large enough then we see from Proposition 4.6 that

$$\int_{|x|<R_0} |\nabla u(0,x)|^2 \, dx \gtrsim c(\eta_1) \tag{8.2}$$

and

$$\int_{|x|<R_0} |u(0,x)|^6 \, dx \gtrsim c(\eta_1). \tag{8.3}$$

We then define the radii $R_0 \ll R_1 \ll \ldots \ll R_J$ recursively by $R_{j+1} := 100 R_j^{100}$.

The region $R_0 < |x| < R_J$ can be partitioned into J dyadic shells of the form $R_j < |x| < R_{j+1}$. By (4.3),(4.4) and the pigeonhole principle we may find $0 \le j < J$ such that

$$\int_{R_j<|x|<R_{j+1}} |\nabla u(0,x)|^2 + |u(0,x)|^6 \, dx \lesssim \frac{1}{J}. \tag{8.4}$$

Fixing this j, we now introduce cutoff functions χ_{inner}, χ_{outer}, where χ_{inner} is adapted to the ball $B(0,2R_j)$ and equals one on $B(0,R_j)$, whereas χ_{outer} is a bump function adapted to $B(0,R_{j+1})$ which equals one on $B(0,R_{j+1}/2)$. We then define $v(0)$, $w(0)$ as

(8.5)

$$v(0,x) := P_{1/N_0 \le \cdot \le N_0}(\chi_{\text{inner}} u(0)); \quad w(0) \; := \; P_{1/N_0 \le \cdot \le N_0}((1-\chi_{\text{outer}})u(0)).$$

By (8.4) we easily see that

$$\|P_{1/N_0 \le \cdot \le N_0} u(0) - v(0) - w(0)\|_{\dot H^1} \lesssim \|(\chi_{\text{outer}} - \chi_{\text{inner}})u(0)\|_{\dot H^1} \lesssim \frac{1}{J^{1/2}}.$$

Also, if N_0 is chosen sufficiently large depending on J we see from the normalization $N(0) = 1$ and Corollary 4.4 that

$$\|u(0) - P_{1/N_0 \le \cdot \le N_0} u(0)\|_{\dot H^1} \lesssim \frac{1}{J^{1/2}}$$

and thus

$$\|u(0) - v(0) - w(0)\|_{\dot H^1} \lesssim \frac{1}{J^{1/2}}. \tag{8.6}$$

We also know that v, w have slightly smaller energy than u:

LEMMA 8.1. *For the the functions* v, w *defined in* (8.5) *we have*

$$E(v(0)), E(w(0)) \leq E_{\text{crit}} - c(\eta_1).$$

Proof. The argument here is completely analogous to that in Lemma 5.1, except that now we work in physical space, exploiting the fact that v is mostly supported in the region $|x| < 3R_j$ and w is mostly supported in the region $|x| > R_{j+1}/2$.

We begin by estimating the quantity

$$|E(v(0) + w(0)) - E(v(0)) - E(w(0))| .$$

Expanding out the definition of energy, we can bound this by

$$\lesssim \int_{\mathbb{R}^3} |\nabla v(0,x)||\nabla w(0,x)| + |v(0,x)||w(0,x)|^5 + |v(0,x)|^5|w(0,x)| \; dx.$$

We subdivide $\mathbb{R}^3$ into the regions $|x| \leq R_{j+1}/2$ and $|x| > R_{j+1}/2$. Since v and w are bounded in $\dot{H}^1$ and hence in L^6, we can use Hölder's inequality to estimate the previous expression by

$$\lesssim \|\nabla v\|_{L^2(|x|>R_{j+1}/2)} + \|v\|_{L^6(|x|>R_{j+1}/2)} + \|\nabla w\|_{L^2(|x|\leq R_{j+1}/2)} + \|w\|_{L^6(|x|\leq R_{j+1}/2)}.$$

Consider for instance the quantity $\|\nabla v\|_{L^2(|x|>R_{j+1}/2)}$. Let K be the convolution kernel associated with $P_{1/N_0 \leq \cdot \leq N_0}$; then we have $\nabla v = (\nabla(\chi_{\text{inner}} u(0))) * K$. Since χ_{inner} is supported on the region $|x| \leq 2R_j \leq R_{j+1}/4$, we may restrict K to the region $|x| > R_{j+1}/4$. On this region, K decays rapidly and in fact has an L^1 norm of at most $C(N_0)/R_{j+1}^{100} \leq C(N_0)/R_0^{100}$ (for instance). Since $\nabla(\chi_{\text{inner}} u(0))$ is bounded in L^2, we thus have

$$\|\nabla v\|_{L^2(|x|>R_{j+1}/2)} \leq C(N_0)/R_0^{100}.$$

The other three terms above can be estimated similarly. Thus

$$|E(v(0) + w(0)) - E(v(0)) - E(w(0))| \leq C(N_0)/R_0^{100}.$$

On the other hand, from (8.6), the boundedness of $u(0), v(0), w(0)$ in $\dot{H}^1$ and L^6, and Hölder's inequality, we have

$$|E(u(0)) - E(v(0) + w(0))| \lesssim J^{-1/2}.$$

Thus we have

$$|E(u(0)) - E(v(0)) - E(w(0))| \lesssim J^{-1/2} + C(N_0)/R_0^{100}. \tag{8.7}$$

On the other hand, by (8.2) and (8.1) (choosing η_2 to be smaller than $1/R_J$), we know that $E(v)(0), E(w)(0) \gtrsim c(\eta_1)$. Together with (8.7), this yields the lemma when $J = J(\eta_1)$ and $R_0 = R_0(\eta_1, N_0, J)$ are chosen sufficiently large. $\square$

From the above lemma and Lemma 4.1 we may then extend v and w by the nonlinear Schrödinger equation (1.1) to all of $\mathbb{R} \times \mathbb{R}^3$, so that

$$\|v\|_{L^{10}_{t,x}} + \|w\|_{L^{10}_{t,x}} \lesssim M(E_{\text{crit}} - c(\eta_1)) = C(\eta_1). \tag{8.8}$$

From this, Lemma 3.12 and the frequency localization of v, w we thus obtain the Strichartz bounds

$$\|v\|_{\dot{S}^k} + \|w\|_{\dot{S}^k} \lesssim C(\eta_1, N_0) \tag{8.9}$$

for $k = 0, 1, 2$.

The idea is now to use our perturbation lemma to approximate u by $v+w$. To do this we need to ensure that v and w do not interact. This is the objective of the next two lemmas.

LEMMA 8.2. *Let $v(x,t), w(x,t)$ be the evolutions according to* (1.1) *of the functions defined in* (8.5). *For times $|t| \leq R_j^{10}$, there exists the "finite speed of propagation" estimate*

$$\int_{|x| \gtrsim R_j^{50}} |v(t,x)|^2 \; dx \lesssim C(\eta_1, N_0) R_j^{-20} \tag{8.10}$$

and for times $|t| \geq R_j^{10}$, the decay estimate

$$\int |v(t,x)|^6 \; dx \lesssim R_j^{-10}. \tag{8.11}$$

(*The powers of R_j are far from sharp.*) *Meanwhile, the mass density of w obeys the finite speed of propagation estimate*

$$\int_{|x| \lesssim R_j^{50}} |w(t,x)|^2 \; dx \lesssim C(\eta_1, N_0) R_j^{-20} \tag{8.12}$$

for all $|t| \leq R_j^{10}$ and, similarly, the energy density of w obeys the finite speed of propagation estimate

$$\int_{|x| \lesssim R_j^{50}} [\frac{1}{2}|\nabla w(t,x)|^2 + \frac{1}{6}|w(t,x)|^6] dx \lesssim R_j^{-20} C(\eta_1, N_0), \tag{8.13}$$

for all $|t| < R_j^{10}$. Also,

$$\int_{|x| \gtrsim R_j^{50}} [\frac{1}{2}|\nabla v(t,x)|^2 + \frac{1}{6}|v(t,x)|^6] dx \lesssim R_j^{-20} C(\eta_1, N_0). \tag{8.14}$$

Thus at short times $t = O(R_j^{10})$, v and w are separated in space, whereas at long times v has decayed (while w is still bounded in Strichartz norms).

Proof. The estimates (8.10) and (8.11) follow from the pseudoconformal law following arguments from [5]. Recall the pseudoconformal conservation law for sufficiently regular and decaying solutions of (1.1):

$$\|(x+2it\nabla)u(t)\|_{L^2_x}^2 + \frac{4}{3}t^2\|u(t)\|_{L^6_x}^6 = \||x|u_0\|_{L^2_x}^2 - \frac{16}{3}\int_0^t s\|u(s)\|_{L^6_x}^6 ds. \tag{8.15}$$

Thus, since v solves (1.1),

$$\int_{|x|\gtrsim R_j^{50}} |x|^2|v(t,x)|^2dx \lesssim t^2\|\nabla v(t)\|_{L^2_x}^2 + t^2\|v(t)\|_{L^6_x}^6 + \||x|v_0\|_{L^2_x}^2 + \int_0^t s\|v(s)\|_{L^6_x}^6 ds.$$

We restrict to times $|t| \le R_j^{10}$ and have

$$\begin{aligned}
&R_j^{100}\int_{|x|\gtrsim R_j^{50}} |v(t,x)|^2dx \\
&\quad\lesssim R_j^{20}\|\nabla v(t)\|_{L^\infty_{|t|\le R_j^{10}}L^2_x}^2 \quad + R_j^{20}\|v(t)\|_{L^\infty_{|t|\le R_j^{10}}L^6_x}^6 + R_j^2\|v_0\|_{L^2_x}{}^2 \\
&\quad\lesssim R_j^{20}\Big(\|\nabla v(t)\|_{L^\infty_{|t|\le R_j^{10}}L^2_x}^2 + \|v(t)\|_{L^\infty_{|t|\le R_j^{10}}L^6_x}^6\Big) + R_j^2N_0^2\|\nabla v_0\|_{L^2_x}^2 \\
&\quad\lesssim C(N_0)R_j^{20}E(u_0),
\end{aligned}$$

which proves (8.10).

From (8.15), we observe that

$$\|v(t)\|_{L^6_x}^6 \lesssim \frac{R_j^2N_0^2E(u_0)}{t^2},$$

so that, for times $|t| > R_j^{10}$, we obtain (8.11).

We control the L^2_x-mass of w in the ball $|x| < 1000R_j^{50}$ using a virial identity. Let ζ denote a nonnegative smooth bump function equaling 1 on $B(0, 1000R_j^{50})$ and supported on $B(0, 2000R_j^{50})$. Note that ζ has been chosen so that the support of $\nabla\zeta$ does not intersect the support of $\chi_{\text{inner}}(0)$ or the support of $(1-\chi_{\text{outer}})(0)$. From (2.4), (2.6) and integration by parts we have the identity

$$\partial_t\int \zeta(x)|w(t,x)|^2dx = -2\int \zeta_j(x)\operatorname{Im}(w\overline{w}_j)(t,x)dx.$$

Thus,

$$|\partial_t\int \zeta(x)|w(t,x)|^2dx| \lesssim R_j^{-50}\|\nabla w(t)\|_{L^2_x}\|w(t)\|_{L^2_x},$$

and we have, using the support properties and (4.3),

$$\begin{aligned}
\sup_{|t|<R_j^{10}}\int \zeta(x)|w(t,x)|^2dx &\lesssim \int \zeta(x)|w(0,x)|^2dx \\
&\qquad + R_j^{-40}\sup_{|t|<R_j^{10}}\|\nabla w(t)\|_{L^2_x}\|w(t)\|_{L^2_x} \\
&\lesssim R_j^{-40} + R_j^{-40}(E(u_0))^2N_0
\end{aligned}$$

(say), which proves (8.12).

We now control the energy density of w,

$$e(w)(t,x) := \frac{1}{2}|\nabla w(t,x)|^2 + \frac{1}{6}|w(t,x)|^6, \tag{8.16}$$

on the ball $|x| < R_j^{50}$ by a similar argument. From (2.11) and integration by parts we have

$$\frac{d}{dt}\int \zeta e(w)dx = \int \zeta_j[\operatorname{Im}(\overline{w}_k w_{kj}) - \delta_{jk}|w|^4 \operatorname{Im}(w\overline{w}_k)]dx,$$

which implies that

$$\begin{aligned}\int \zeta e(w)(T)dx \lesssim \int \zeta e(w)(0)dx &+ \int_0^T \int |\nabla\zeta||\nabla w||\nabla\nabla w|dxdt \\ &+ \int_0^T \int |\nabla\zeta||w|^5|\nabla w|dxdt.\end{aligned}$$

We will control the three terms on the right to obtain (8.13). The first term vanishes due to support properties of ζ and $1-\chi_{\text{outer}}$. The second term is crudely bounded using (4.3) by

$$R_j^{-50}R_j^{10}\|\nabla^2 w\|_{L^\infty_{|t|<R_j^{10}}L^2_x}.$$

By the induction hypothesis we have (8.9) and, in particular,

$$\|\nabla^2 w\|_{L^\infty_t L^2_x} \lesssim C(\eta_1, N_0).$$

The third term is bounded using Hölder by, say

$$\lesssim R_j^{-50}\|w\|^2_{L^{10}_{t,x}}\|w\|^3_{L^6_{t,x}}\|\nabla w\|_{L^{10/3}_{t,x}}.$$

Again, the global $L^{10}_{t,x}$ bound and Lemma 3.12 gave us (8.9) which, by interpolation, controls all the norms appearing here. This proves (8.13).

Replacing ζ by $1-\zeta$ and w by v in the discussion just completed establishes (8.14). □

Corollary 8.3. *For v, w as in Lemma* 8.2,

$$\begin{aligned}\|\nabla(|v+w|^4(v+w) - |v|^4 v - |w|^4 w)\|_{L^2_t L^{6/5}_x(\mathbb{R}\times\mathbb{R}^3)} \\ \lesssim C(\eta_1, N_0)R_j^{-5/6} \lesssim C(\eta_1, N_0)R_0^{-5/6}.\end{aligned}$$

Proof. By (1.15), the task is to control terms of the form $\mathcal{O}(v^j w^{4-j}\nabla w)$ and $\mathcal{O}(w^j v^{4-j}\nabla v)$, for $j = 1,2,3,4$, in $L^2_t L^{6/5}_x$. Separate the analysis into three spacetime regions based on the estimates in Lemma 8.2: (short time, near origin) $|t| < R_j^{10}$, $|x| < 2000R_j^{50}$; (short time, far from origin) $|t| < R_j^{10}$, $|x| \geq 2000R_j^{50}$; (long time) $|t| \geq R_j^{10}$. In all but one of these cases, an application of a variant of (3.12),

$$\|\nabla u_1 u_2 u_3 u_4 u_5\|_{L^2_t L^{6/5}_x} \lesssim \|\nabla u_1\|_{L^\infty_t L^2_x}\|u_2\|_{L^4_t L^\infty_x}\|u_3\|_{L^4_t L^\infty_x}\|u_4\|_{L^\infty_t L^6_x}\|u_5\|_{L^\infty_t L^6_x},$$

together with (8.9) and the decay properties in Lemma 8.2 establishes the claimed estimate controlling the interaction of v and w. The term $\mathcal{O}(w^4\nabla v)$ in the long time regime $|t| \geq R_j^{10}$ presents an exceptional case since we do not directly encounter the available long time decay estimate (8.11). This situation is treated separately with the following argument. By Hölder and interpolation, we have

$$\begin{aligned}
\|\mathcal{O}(w^4\nabla v)\|_{L_t^2 L_x^{6/5}} &\lesssim \|\nabla v\|_{L_t^\infty L_x^3} \|\mathcal{O}(w^4)\|_{L_t^2 L_x^2} \\
&\lesssim \|v\|_{L_t^\infty L_x^6}^{1/2} \|\nabla^2 v\|_{L_t^\infty L_x^2}^{1/2} \|\mathcal{O}(w^4)\|_{L_t^2 L_x^2} \\
&\lesssim \|v\|_{L_t^\infty L_x^6}^{1/2} \|v\|_{\dot{S}^2}^{1/2} \|\mathcal{O}(w^4)\|_{L_t^2 L_x^2}.
\end{aligned}$$

Note the appearance of (8.11) which contributes the decay $R_j^{-5/6}$. We complete the proof by estimating,

$$\begin{aligned}
\|\mathcal{O}(w^4)\|_{L_t^2 L_x^2} &\lesssim \|\mathcal{O}(w^3)\|_{L_t^2 L_x^6} \|w\|_{L_t^\infty L_x^3} \\
&\lesssim \|w\|_{L_t^6 L_x^{18}}^3 \|w\|_{L_t^\infty L_x^2}^{1/2} \|w\|_{L_t^\infty L_x^6}^{1/2} \lesssim \|w\|_{\dot{S}^1}^3 \|w(0)\|_{L_x^2}^{1/2} E^{1/4} \\
&\lesssim C(\eta_1, \eta_2) N_0^{1/2} \lesssim C(\eta_1, \eta_2, N_0).
\end{aligned}$$

□

In light of this corollary, (8.6), (8.8), and the observation that u, v, w all have bounded energy, we see from Lemma 3.10 (with $\tilde{u} := v + w$ and $e := |v+w|^4(v+w) - |v|^4 v - |w|^4 w$) that if J is sufficiently large depending on η_1, and R_0 sufficiently large depending on η_1, J, N_0, then

$$\|u\|_{L_{t,x}^{10}(I_* \times \mathbb{R}^3)} \lesssim C(\eta_1),$$

which contradicts (4.1). This proves Proposition 4.7. □

9. Reverse Sobolev inequality

We now prove Proposition 4.8. Fix t_0, x_0, R. We may normalize $x(t_0) = 0$ and $N(t_0) = 1$. Then by Proposition 4.7 we have

$$\int_{|x|>1/\eta_2} |\nabla u(t_0,x)|^2 \, dx \lesssim \eta_1. \tag{9.1}$$

Now suppose for contradiction that we have

$$\int_{B(x_0,R)} |\nabla u(t_0,x)|^2 \, dx \gg \eta_1 + K(\eta_1,\eta_2) \int_{B(x_0,K(\eta_1,\eta_2)R)} |u(t_0,x)|^6 \, dx \tag{9.2}$$

for some large $K(\eta_1, \eta_2)$ to be chosen later. From (9.2) and (9.1) we see that $B(x_0, R)$ cannot be completely contained inside the region $|x| > 1/\eta_2$; thus,

$$|x_0| \lesssim R + 1/\eta_2. \tag{9.3}$$

Next, we obtain a lower bound on R. Recall from the normalization $N(t_0) = 1$ and Corollary 4.4 that

$$\|P_{>C(\eta_1)} u(t_0)\|_{\dot{H}^1} \lesssim \eta_1.$$

On the other hand, from (9.2),

$$\int_{B(x_0,R)} |\nabla u(t_0,x)|^2 \, dx \gg \eta_1$$

and from the triangle inequality we see that

$$\int_{B(x_0,R)} |\nabla P_{\leq C(\eta_1)} u(t_0,x)|^2 \, dx \gg \eta_1.$$

But by Hölder, Bernstein (1.20), and (4.3) we can bound the left-hand side by

$$\lesssim R^3 \|\nabla P_{\leq C(\eta_1)} u(t_0)\|^2_{L^\infty_x} \lesssim C(\eta_1) R^3$$

and thus we have

$$R \gtrsim c(\eta_1).$$

Combining this with (9.3) we see that the ball $B(x_0, K(\eta_1,\eta_2)R)$ will contain $B(0, 1/\eta_2)$ (and hence any ball of the form $B(0, C(\eta_1))$) if the constant $K(\eta_1,\eta_2)$ is large enough. In particular, from Proposition 4.6 we have

$$\int_{B(x_0,K(\eta_1,\eta_2)R)} |u(t_0,x)|^6 \, dx \gtrsim c(\eta_1),$$

which inserted into (9.2) contradicts the energy bound (4.3) if $K(\eta_1,\eta_2)$ is chosen sufficiently large. This proves Proposition 4.8. $\square$

10. Interaction Morawetz: generalities

We shall shortly begin the proof of Proposition 4.9, which is a variant of the interaction Morawetz inequality (1.8). As noted above, this inequality cannot be applied directly to our situation because the right-hand side of (1.8) can be very large due to low frequency contributions to u. It is then natural (in light of (4.5)) to try to adapt the interaction Morawetz inequality to only deal with the high frequencies $u_{\geq 1}$, but this turns out to not quite be enough either. The trouble is that the inhomogeneous Schrödinger equation satisfied by $u_{\geq 1}$ is not Lagrangian - and in particular it no longer enjoys the usual L^2 conservation. Hence when we apply the argument from [13], [12] (which gave (1.8)) to this $u_{\geq 1}$ equation, we get new terms arising from the fact that the right side of (2.4) is no longer zero. We can find no appropriate bounds for these new terms. Our solution to this problem is to localize the previous interaction Morawetz arguments in space[25], yielding a much more complicated version (Theorem 11.1) of the inequality (1.8).

To summarize: the increased complexity on the right-hand side of (11.6) below is due to the fact that we have localized the argument in frequency (because (1.1) is critical) and space (because of the error terms introduced by the frequency localization). Of course, all of these extra terms will somehow have to be shown to be bounded, and to this end the second term on the *left* side of (11.6) is very important. An analogue of this term - where the x integration is taken over all of $\mathbb{R}^3$ - can also be included on the left side of (1.8) (see [13], [12]), but we had previously found no use for this term. In what follows, the second term on the left side of (11.6) will be used to absorb — via the reverse Sobolev inequality of Proposition 4.8 and an averaging argument — some of the most troublesome terms involving kinetic energy that appear on the right side of (11.6).[26]

The above argument will be carried out in the next section; in this section we prepare for our work involving the non-Lagrangian equation satisfied by $u_{\geq 1}$ by discussing interaction Morawetz inequalities in more general situations than the quintic NLS (1.1). In particular, we shall consider general solutions ϕ to the equation (2.1), where $\mathcal{N}$ is an arbitrary nonlinearity.

10.1. *Virial-type identity.* We introduce two related quantities which average the mass and momentum densities (see Definition 2.1) against a weight function $a(x)$.

Definition 10.1. Let $a(x)$ be a function[27] defined on the spacetime slab $I_0 \times \mathbb{R}^3$. We define the associated virial potential

$$V_a(t) = \int_{\mathbb{R}^3} a(x)|\phi(t,x)|^2 dx \tag{10.1}$$

and the associated Morawetz action

$$M_a(t) = \int_{\mathbb{R}^3} a_j 2\mathrm{Im}(\overline{\phi}\phi_j) dx. \tag{10.2}$$

A calculation using Lemma 2.3 shows that

$$\partial_t V_a = M_a + 2\int_{\mathbb{R}^3} a\{\mathcal{N}, \phi\}_m dx, \tag{10.3}$$

[25] See (10.5), where the novelty over the arguments from [13], [12] is now the presence of the spatial cut-off function.

[26] The discussion here gives another way to frame our regularity argument which was sketched in Section 4. We only bother to show that a minimal energy blowup solution must be localized in space in order that we can apply the reverse Sobolev inequality to such solutions. The reverse Sobolev inequality is needed in the proof of the frequency localized $L^4_{x,t}$ bound.

[27] In other contexts it's useful to consider also time dependent weight functions $a(t, x)$.

so that $M_a = \partial_t V_a$ when $\mathcal{N} = F'(|\phi|^2)\phi$. Using Lemma 2.3, a longer but similar calculation establishes,

LEMMA 10.2 (Virial-type Identity). *Let ϕ be a (Schwartz) solution of* (2.1). *Then*

$$\partial_t M_a = \int_{\mathbb{R}^3} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\mathrm{Re}(\overline{\phi_j}\phi_k) + 2a_j\{\mathcal{N},\phi\}_p^j dx. \tag{10.4}$$

We now infer a useful identity by choosing the weight function $a(x)$ above to be,

$$a(x) = |x|\chi(|x|), \tag{10.5}$$

where $\chi(r)$ denotes a smooth nonnegative bump function defined on $r \geq 0$, supported on $0 \leq r \leq 2$ and satisfying $\chi(r) = 1$ for $0 \leq r \leq 1$. We calculate,

$$\begin{aligned}
a_j(x) &= \frac{x^j}{|x|}\tilde{\chi}(|x|) \text{ where } \tilde{\chi}(r) = \chi(r) + r\chi'(r),\\
a_{jk}(x) &= \frac{1}{|x|}\left(\delta_{jk} - \frac{x^j}{|x|}\frac{x^k}{|x|}\right)\tilde{\chi}(|x|) + \frac{x^j}{|x|}\frac{x^k}{|x|}\tilde{\chi}'(|x|),\\
\Delta a(x) &= \frac{2}{|x|}\tilde{\chi}(|x|) + \tilde{\chi}'(|x|),\\
\Delta\Delta a(x) &= 2\Delta\left(\frac{1}{|x|}\right)\tilde{\chi}(|x|) + \psi(|x|),
\end{aligned}$$

where $\psi(|x|)$ is smooth and supported in $1 \leq |x| \leq 2$. Define now the notation $M^0 := M_a$ when $a(x)$ is chosen as in (10.5). (Later we will localize around a different fixed point $y \in \mathbb{R}^3$, in which case we'll write the Morawetz action as M^y. Note that the letter a is dropped completely now from the notation for the Morawetz action.) By the definition (10.2),

$$M^0(t) = 2\mathrm{Im}\int_{\mathbb{R}^3} \frac{x^j}{|x|}\tilde{\chi}(|x|)\phi_j(t,x)\overline{\phi}(t,x)dx. \tag{10.6}$$

Note that[28]

$$|M^0(t)| \leq 2\|\phi(t)\|_{L^2_x}\|\phi(t)\|_{\dot{H}^1}. \tag{10.7}$$

Inserting the calculation below (10.5) above into (10.4), and using the notation $\partial_{r(0)}$ to denote the radial part of the gradient with center at $0 \in \mathbb{R}^3$ (so that

[28]One can also show that $|M^0(t)| \lesssim \|\phi(t)\|^2_{\dot{H}^{1/2}}$; see Lemma 2.1 in [13].

$\partial_{r(0)} \equiv \frac{x}{|x|} \cdot \nabla$), we get

$$\begin{aligned}\partial_t M^0 = &-2 \int_{\mathbb{R}^3} \Delta(\frac{1}{|x|})|\phi(x)|^2 \tilde{\chi}(|x|)dx \\ &+ 4 \int_{\mathbb{R}^3} [|\nabla\phi(x)|^2 - |\partial_{r(0)}\phi(x)|^2] \frac{1}{|x|}\tilde{\chi}(|x|)dx \\ &+ 2 \int_{\mathbb{R}^3} \frac{(x)}{|x|} \cdot \{\mathcal{N}, \phi\}_p \tilde{\chi}(|x|)dx \\ &- \int_{\mathbb{R}^3} |\phi(x)|^2 \psi(|x|)dx + 4 \int_{\mathbb{R}^3} |\partial_{r(0)}\phi|^2 \tilde{\chi}'(|x|)dx.\end{aligned}$$

As remarked above, we translate the origin and choose, for fixed $y \in \mathbb{R}^3$

$$a(x) = |x - y|\chi(|x - y|), \tag{10.8}$$

instead of (10.5), in which case the Morawetz action (10.2) is written M^y. Then the preceding formula adjusts to give the following spatially localized virial-type identity, where we write $\tilde{\chi}$ for another bump function with the same properties as χ,

$$\partial_t M^y = \tag{10.9}$$

$$- 2 \int_{\mathbb{R}^3} \Delta(\frac{1}{|x-y|})|\phi(x)|^2 \tilde{\chi}(|x - y|)dx \tag{10.10}$$

$$+ 4 \int_{\mathbb{R}^3} |\nabla\!\!\!/_y \phi(x)|^2 \frac{1}{|x-y|}\tilde{\chi}(|x - y|)dx \tag{10.11}$$

$$+ 2 \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|} \cdot \{\mathcal{N}, \phi\}_p \tilde{\chi}(|x - y|)dx \tag{10.12}$$

$$+ O\left(\int_{\mathbb{R}^3} (|\phi(x)|^2 + |\partial_{r(y)}\phi(x)|^2)|\psi(x - y)|dx\right). \tag{10.13}$$

We have used here the notation $\partial_{r(y)}$ to denote the radial portion of the gradient centered at y and $\nabla\!\!\!/_y$ for the rest of the gradient. That is,

$$\partial_{r(y)} := \frac{x-y}{|x-y|} \cdot \nabla \qquad \text{and} \quad \nabla\!\!\!/_y := \nabla - \frac{x-y}{|x-y|}(\frac{x-y}{|x-y|} \cdot \nabla).$$

We have also taken the liberty to dismiss some of the structure in (10.13) using the fact that $\tilde{\chi}'$ has the same support properties as ψ.

10.2. *Interaction virial identity and general interaction Morawetz estimate for general equations.* When we choose $a(x) = |x|\chi(x)$ above, the virial potential reads $V_a(t) = \int_{\mathbb{R}^3} |\phi(x,t)|^2|x|\chi(x)dx$ and hence $M^0(t) := \frac{d}{dt}V_a(t)$ might be thought of as measuring the extent to which the mass of ϕ (near the origin at least) is moving away from the origin at time t. Similarly, for fixed $y \in \mathbb{R}^3$, $M^y(t)$ gives some measure of the mass movement away from the point y.

Since we are ultimately interested in global decay and scattering properties of ϕ, it's reasonable to look for some measure of how the mass is moving away from (or *interacting* with) *itself*. We might therefore sum over all $y \in \mathbb{R}^3$ the extent to which mass is moving away from y (that is, $M^y(t)$) multiplied by the amount of mass present at that point y (that is, $|\phi(y,t)|^2 dy$). The result is the following quantity which we'll call the *spatially localized Morawetz interaction potential*.

$$M^{\text{interact}}(t) = \int_{\mathbb{R}^3_y} |\phi(t,y)|^2 M^y(t) dy \tag{10.14}$$

$$= 2\text{Im} \int_{\mathbb{R}^3_y} \int_{\mathbb{R}^3_x} |\phi(t,y)|^2 \tilde{\chi}(|x-y|) \frac{(x-y)}{|x-y|} \cdot [\nabla \phi(t,x)] \overline{\phi}(t,x) dx dy. \tag{10.15}$$

Note that, using (10.7),

$$|M^{\text{interact}}(t)| \lesssim \|\phi(t)\|^3_{L^2_x} \|\phi(t)\|_{\dot{H}^1_x}. \tag{10.16}$$

We calculate, using (10.9) and (2.4),

$$\partial_t M^{\text{interact}} = \int_{\mathbb{R}^3_y} |\phi(y)|^2 \partial_t M^y dy \tag{10.17}$$

$$+ \int_{\mathbb{R}^3_y} [2\partial_{y^k} \text{Im}(\phi \overline{\phi}_k)(y) + 2\{\mathcal{N}, \phi\}_m] M^y(t) dy. \tag{10.18}$$

The ∂_{y^k} appearing in (10.18) will now be integrated by parts. Thus, using Lemma 2.3 and the fact that on $\mathbb{R}^3, \Delta \frac{1}{|x|} = -4\pi\delta$, we have our spatially localized interaction virial-type identity

$$\partial_t M^{\text{interact}} = \tag{10.19}$$

$$8\pi \int_{\mathbb{R}^3} |\phi(t,y)|^4 dy \tag{10.20}$$

$$+ 4 \int_{\mathbb{R}^3_y} \int_{\mathbb{R}^3_x} |\phi(t,y)|^2 \left[\frac{1}{|x-y|} \tilde{\chi}(|x-y|) \right] |\not\nabla_y \phi(t,x)|^2 dx dy \tag{10.21}$$

$$+ 2 \int_{\mathbb{R}^3_y} \int_{\mathbb{R}^3_x} |\phi(t,y)|^2 \left[\tilde{\chi}(|x-y|) \frac{(x-y)}{|x-y|} \right] \cdot \{\mathcal{N}, \phi\}_p(t,x) dx dy \tag{10.22}$$

$$- 4 \int_{\mathbb{R}^3_y} \int_{\mathbb{R}^3_x} \text{Im}(\phi \overline{\phi}_k)(t,y) \partial_{y^k} \left[\frac{(x-y)^j}{|x-y|} \tilde{\chi}(|x-y|) \right] \text{Im}(\phi_j \overline{\phi}(t,x)) dx dy \tag{10.23}$$

$$+ O\left(\int_{\mathbb{R}^3_y} \int_{\mathbb{R}^3_x} |\phi(t,y)|^2 |\psi(|x-y|)| [|\phi(t,x)|^2 + |\partial_{r(y)} \phi(t,x)|^2] dx dy \right) \tag{10.24}$$

$$+ 4 \int_{\mathbb{R}^3_y} \int_{\mathbb{R}^3_x} \{\mathcal{N}, \phi\}_m(t,y) \left[\tilde{\chi}(|x-y|) \frac{(x-y)}{|x-y|} \right] \cdot \text{Im}(\overline{\phi} \nabla \phi)(t,x) dx dy. \tag{10.25}$$

The ∂_{y^k} differentiation in (10.23) produces two terms. When ∂_{y^k} falls on $\tilde{\chi}(|x-y|)$, we encounter a term controlled by (10.24). Indeed, we get a term bounded by

$$\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x} |\phi(x)||\partial_{r(y)}\phi(x)||\phi(y)||\partial_{r(x)}\phi(y)|\tilde{\chi}'(|x-y|)dxdy.$$

Upon grouping the terms in the integrand as $[|\phi(x)||\partial_{r(x)}\phi(y)|][|\phi(y)||\partial_{r(y)}\phi(x)|]$ and using $|ab| \lesssim |a|^2 + |b|^2$, we find the second part of the expression (10.24). When the derivative falls on the unit vector, we encounter a term of indeterminate sign but which is bounded from below by (10.21). We present the details (which also appear in Proposition 2.2 of [13]). The term we are considering is (with t dependence supressed)

$$\begin{aligned}
&-4\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x} \mathrm{Im}(\phi\overline{\phi}_k)(y)\left[-\delta_{jk} + \frac{(x-y)^j(x-y)^k}{|x-y|^2}\right] \\
&\quad \cdot \mathrm{Im}(\phi_j\overline{\phi})(x)\left[\frac{1}{|x-y|}\tilde{\chi}(|x-y|)\right]dxdy \\
&\geq -4\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_y} \left|\mathrm{Im}(\phi\nabla\!\!\!/_x\overline{\phi})(y)\cdot \mathrm{Im}(\overline{\phi}\nabla\!\!\!/_y\phi)(x)\right|\left[\frac{1}{|x-y|}\tilde{\chi}(|x-y|)\right]dxdy \\
&\geq -4\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_y} |\phi(y)||\nabla\!\!\!/_x\phi(y)||\phi(x)||\nabla\!\!\!/_y\phi(x)|\left[\frac{1}{|x-y|}\tilde{\chi}(|x-y|)\right]dxdy \\
&\geq -2\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x} (|\phi(y)|^2|\nabla\!\!\!/_y\phi(x)|^2 + |\phi(x)|^2|\nabla\!\!\!/_x\phi(y)|^2)\left[\frac{1}{|x-y|}\tilde{\chi}(|x-y|)\right]dxdy \\
&\geq (10.21).
\end{aligned}$$

Thus, apart from another term of the form (10.24), we have shown that $-$(10.21) and (10.23) together contribute a nonnegative term.

Restricting the above calculations to the time interval I_0, we have the following useful estimate.

Proposition 10.3 (Spatially Localized Interaction Morawetz Inequality). *Let ϕ be a (Schwartz) solution to the equation* (2.1) *on a spacetime slab $I_0\times\mathbb{R}^3$ for some compact interval I_0. Then*

$$\begin{aligned}
&8\pi\int_{I_0}\int_{\mathbb{R}^3_y} |\phi(t,y)|^4 dydt \\
&\quad + 2\int_{I_0}\int_{R^3_y}\int_{R^3_x} |\phi(t,y)|^2\left[\tilde{\chi}(|x-y|)\frac{(x-y)}{|x-y|}\right]\cdot\{\mathcal{N},\phi\}_p(t,x)dxdydt \\
&\leq 2\|\phi\|^3_{L^\infty_t L^2_x(I_0\times\mathbb{R}^3)}\|\phi\|_{L^\infty_t \dot{H}^1_x(I_0\times\mathbb{R}^3)} \\
&\quad + 4\int_{I_0}\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x} |\{N,\phi\}_m(t,y)||\tilde{\chi}(|x-y|)||\phi(t,x)||\nabla\phi(t,x)|dxdydt \\
&\quad + O\left(\int_{I_0}\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x} |\phi(t,y)|^2|\psi(|x-y|)|[|\phi(t,x)|^2 + |\partial_{r(y)}\phi(t,x)|^2]dxdydt\right).
\end{aligned}$$

The proof follows directly by integrating (10.19) over the time interval I_0 using (10.16).

Remark 10.4. If we replace (10.8) by

$$a(x) = |x-y|\chi\left(\frac{|x-y|}{R}\right) \tag{10.26}$$

then there are adjustments to the inequality obtained in Proposition 10.3. Of course, $\tilde{\chi}(\cdot)$ is replaced by $\tilde{\chi}(\cdot/R)$. The annular cutoff ψ in the final term arises in the analysis above when derivatives fall on χ or $\tilde{\chi}$. A review of the derivation shows that this term adjusts under (10.26) into

(10.27)
$$O\left(\int_{I_0}\int_{R^3_y}\int_{R^3_x} |\phi(t,y)|^2 \left|\psi\left(\frac{|x-y|}{R}\right)\right| \left[\frac{1}{R^3}|\phi(t,x)|^2 + \frac{1}{R}|\partial_{r(y)}\phi(t,x)|^2\right] dxdydt\right).$$

If we send $R \to \infty$ and specialize to solutions of (1.1), then we can apply (2.6), (2.8) to obtain the bound

$$\begin{aligned}&\int_{I_0}\int_{\mathbb{R}^3_y} |u(t,y)|^4 dydt\\ &+ \int_{I_0}\int_{R^3_y}\int_{R^3_x} \frac{|u(t,y)|^2|u(t,x)|^6}{|x-y|} dxdydt \lesssim \|u\|^3_{L^\infty_t L^2_x(I_0\times\mathbb{R}^3)} \|u\|_{L^\infty_t \dot{H}^1_x(I_0\times\mathbb{R}^3)},\end{aligned}$$

which is basically (1.8) (see Footnote 10.1). However for the purposes of proving Proposition 4.9, it turns out not to be feasible to send $R \to \infty$, as one of the error terms (specifically, the portion of the mass bracket $\{\mathcal{N}, \phi\}_m$ which looks schematically like $u^5_{hi}u_{lo}$) becomes too difficult to estimate.

11. Interaction Morawetz: The setup and an averaging argument

Having discussed interaction Morawetz inequalities in general, we are now ready to begin the proof of Proposition 4.9.

From the invariance of this proposition under the scaling (1.3) we may normalize $N_* = 1$. Since we are assuming $1 = N_* < c(\eta_3)N_{\min}$, we have in particular that $1 < c(\eta_3)N(t)$ for all $t \in I_0$. From Corollary 4.4 and Sobolev we have the low frequency estimate

$$\|u_{<1/\eta_3}\|_{L^\infty_t \dot{H}^1_x(I_0\times\mathbb{R}^3)} + \|u_{<1/\eta_3}\|_{L^\infty_t L^6_x(I_0\times\mathbb{R}^3)} \lesssim \eta_3 \tag{11.1}$$

(for instance), if $c(\eta_3)$ was chosen sufficiently small. By (4.11) we thus see that our choice of $N_* = 1$ has forced $N_{\min} \geq c(\eta_0)\eta_3^{-1}$. Define $P_{hi} := P_{\geq 1}$ and $P_{lo} := P_{<1}$, and then define $u_{hi} := P_{hi}u$ and $u_{lo} := P_{lo}u$. Observe from (11.1) that u_{lo} has small energy,

$$\|u_{lo}\|_{L^\infty_t \dot{H}^1_x(I_0\times\mathbb{R}^3)} + \|u_{lo}\|_{L^\infty_t L^6_x(I_0\times\mathbb{R}^3)} \lesssim \eta_3. \tag{11.2}$$

while from (11.1), (1.16) and (4.5) we see that u_{hi} has small mass:

$$\|u_{hi}\|_{L_t^\infty L_x^2(I_0\times\mathbb{R}^3)} \lesssim \eta_3. \tag{11.3}$$

We wish to prove (4.19), or in other words

$$\|u_{hi}\|_{L_t^4 L_x^4(I_0\times\mathbb{R}^3)} \lesssim \eta_1^{1/4}. \tag{11.4}$$

By a standard continuity argument,[29] it will suffice to show this under the bootstrap hypothesis

$$\|u_{hi}\|_{L_t^4 L_x^4(I_0\times\mathbb{R}^3)} \leq (C_0\eta_1)^{1/4} \tag{11.5}$$

where $C_0 \gg 1$ is a large constant (depending only on the energy, and not on any of the η's). In practice we will overcome this loss of C_0 with a positive power of η_3 or η_1.

We now use Proposition 10.3 to obtain a Morawetz estimate for $\phi := u_{hi}$.

Theorem 11.1 (Spatially and frequency localized interaction Morawetz inequality). *Let the notation and assumptions be as above. Then for any* $R \geq 1$,

(11.6)
$$\int_{I_0}\int_{\mathbb{R}^3} |u_{hi}|^4 \, dxdt + \int_{I_0}\int\int_{|x-y|\leq 2R} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^6}{|x-y|} \, dxdydt \lesssim X_R,$$

where X_R *denotes the quantity*

(11.7)
$$X_R := \eta_3^3$$

(11.8)
$$+ \int_{I_0}\int\int_{|x-y|\leq 2R} \frac{|u_{hi}(t,y)|^2|u_{lo}(t,x)|^5|u_{hi}(t,x)|}{|x-y|} \, dxdydt$$

(11.9)
$$+ \sum_{j=0}^{4}\int_{I_0}\int\int_{|x-y|\leq 2R} |u_{hi}(t,y)||P_{hi}\mathcal{O}(u_{hi}^j u_{lo}^{5-j})(t,y)|$$
$$\cdot |u_{hi}(t,x)||\nabla u_{hi}(t,x)| \, dxdydt$$

(11.10)
$$+ \int_{I_0}\int\int_{|x-y|\leq 2R} |u_{hi}(t,y)||P_{lo}\mathcal{O}(u_{hi}^5)(t,y)||u_{hi}(t,x)||\nabla u_{hi}(t,x)| \, dxdydt$$

[29] Strictly speaking, one needs to prove that (11.5) implies (11.4) whenever I_0 is replaced by any subinterval I_1 of I_0, in order to run the continuity argument correctly, but it will be clear that the argument below works not only for I_0 but also for all subintervals of I_0.

$$+\eta_3^{1/10}\frac{1}{R}\int_{I_0}(\sup_{x\in\mathbb{R}^3}\int_{B(x,2R)}|u_{hi}(t,y)|^2\ dy)dt \tag{11.11}$$

$$+\frac{1}{R}\int_{I_0}\int\int_{|x-y|\le 2R}|u_{hi}(t,y)|^2(|\nabla u_{hi}(t,x)|^2+|u_{hi}(t,x)|^6)\ dxdydt. \tag{11.12}$$

Remark 11.2. This should be compared with (1.8). The terms (11.8)–(11.12) may look fearsome, but most of these terms are manageable, because of the spatial localization $|x-y|\le 2R$, and because there are not too many derivatives on the high-frequency term u_{hi}; the only truly difficult terms will be the last two (11.11), (11.12). Observe from (4.3), (4.4) that we could control (11.12) by (11.11) if we dropped the $\eta_3^{1/10}$ factor from (11.11); however this type of factor is indispensable in closing our bootstrap argument, and so we must treat (11.12) separately. The idea is to use the reverse Sobolev inequality, Proposition 4.8, to control (11.12) by the second term in (11.6), plus an error of the form (11.11). This can be done, but requires us to apply Theorem 11.1 not just for $R=1$ (which would be the most natural choice of R) but rather for a range of R and then average over such R; see the discussion after the proof of this theorem.

Proof. We apply P_{hi} to (1.1) to obtain

$$(i\partial_t+\Delta)u_{hi}=P_{hi}(|u|^4u),$$

and then apply Proposition 10.3 with $\phi:=u_{hi}$ and $F:=P_{hi}(|u|^4u)$, to obtain

$$\begin{aligned}
&c_1\int_{I_0}\int_{\mathbb{R}^3}|u_{hi}(t,x)|^4\ dxdt\\
&+c_2\int_{I_0}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|u_{hi}(t,y)|^2\tilde{\chi}(\frac{x-y}{R})\frac{x-y}{|x-y|}\cdot\{P_{hi}(|u|^4u),u_{hi}\}_p(t,x)\ dxdydt\\
&\lesssim \|u_{hi}\|^3_{L^\infty_t L^2_x(I_0\times\mathbb{R}^3)}\|u_{hi}\|_{L^\infty_t\dot H^1_x(I_0\times\mathbb{R}^3)}\\
&+\int_{I_0}\int\int_{|x-y|\le 2R}|\{P_{hi}(|u|^4u),u_{hi}\}_m(t,y)||u_{hi}(t,x)||\nabla u_{hi}(t,x)|\ dxdydt\\
&+\frac{1}{R}\int_{I_0}\int\int_{|x-y|\le 2R}|u_{hi}(t,y)|^2(\frac{1}{R^2}|u_{hi}(t,x)|^2+|\nabla u_{hi}(t,x)|^2)\ dxdydt.
\end{aligned}$$

We now estimate the right-hand side by X_R. Observe from (4.3), (11.3) that

$$\|u_{hi}\|^3_{L^\infty_t L^2_x(I_0\times\mathbb{R}^3)}\|u_{hi}\|_{L^\infty_t\dot H^1_x(I_0\times\mathbb{R}^3)}\lesssim\eta_3^3=(11.7)\le X_R,$$

while from (11.3) again we have

$$\int_{\mathbb{R}^3}\frac{1}{R^2}|u_{hi}(t,x)|^2\ dx\lesssim\eta_3^2/R^2\le\eta_3$$

and hence

$$\frac{1}{R}\int_{I_0}\int\int_{|x-y|\leq 2R}|u_{hi}(t,y)|^2\frac{1}{R^2}|u_{hi}(t,x)|^2\ dxdydt \lesssim (11.11) \leq X_R.$$

Similarly we have

$$\frac{1}{R}\int_{I_0}\int\int_{|x-y|\leq 2R}|u_{hi}(t,y)|^2|\nabla u_{hi}(t,x)|^2\ dxdydt \lesssim (11.12) \leq X_R.$$

Now we deal with the mass bracket term. We take advantage of the cancellation (2.6) to write

$$\{P_{hi}(|u|^4u), u_{hi}\}_m = \{P_{hi}(|u|^4u) - |u_{hi}|^4u_{hi}, u_{hi}\}_m.$$

We can write, using the notation (1.15),

$$\begin{aligned} P_{hi}(|u|^4u) - |u_{hi}|^4u_{hi} &= P_{hi}(|u|^4u - |u_{hi}|^4u_{hi}) - P_{lo}(|u_{hi}|^4u_{hi}) \\ &= \sum_{j=0}^{4} P_{hi}\mathcal{O}(u_{hi}^j u_{lo}^{5-j}) + P_{lo}\mathcal{O}(u_{hi}^5). \end{aligned}$$

Thus these terms can be bounded by $O((11.9) + (11.10)) = O(X_R)$ (where we take absolute values everywhere). To summarize so far, we have shown that

$$\begin{aligned} c_1\int_{I_0}\int_{\mathbb{R}^3}|u_{hi}(t,x)|^4\ dxdt + c_2\int_{I_0}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|u_{hi}(t,y)|^2\tilde{\chi}(\frac{x-y}{R})\frac{x-y}{|x-y|} \\ \cdot\{P_{hi}(|u|^4u), u_{hi}\}_p(t,x)\ dxdydt \lesssim X_R. \end{aligned}$$

We now deal with the momentum bracket term, which is a little more involved as it requires a little more integration by parts. We will need to exploit the positivity of one of the components of this term. In order to exploit the cancellation (2.8), we break up the momentum bracket into three pieces:

$$\begin{aligned} &\{P_{hi}(|u|^4u), u_{hi}\}_p \\ &= \{|u|^4u, u_{hi}\}_p - \{P_{lo}(|u|^4u), u_{hi}\}_p \\ &= \{|u|^4u, u\}_p - \{|u|^4u, u_{lo}\}_p - \{P_{lo}(|u|^4u), u_{hi}\}_p \\ &= \{|u|^4u, u\}_p - \{|u_{lo}|^4u_{lo}, u_{lo}\}_p - \{|u|^4u - |u_{lo}|^4u_{lo}, u_{lo}\}_p - \{P_{lo}(|u|^4u), u_{hi}\}_p \\ &= -\frac{2}{3}\nabla(|u|^6 - |u_{lo}|^6) - \{|u|^4u - |u_{lo}|^4u_{lo}, u_{lo}\}_p - \{P_{lo}(|u|^4u), u_{hi}\}_p. \end{aligned}$$

We first deal with $\{P_{lo}(|u|^4u), u_{hi}\}_p$ estimating the contribution of this term crudely in absolute values as

$$O\left(\int_{I_0}\int\int_{|x-y|\leq 2R}|u_{hi}(t,y)|^2|\{u_{hi}, P_{lo}(|u|^4u)\}_p(t,x)|\ dxdydt\right).$$

We wish to bound this term by $O((11.11)) = O(X_R)$, recalling that any positive power of η_3 overwhelms any loss in R powers, this follows from

LEMMA 11.3. $\int_{\mathbb{R}^3} |\{u_{hi}, P_{lo}(|u|^4 u)\}_p| \; dx \lesssim \eta_3^{1/2}$.

Proof. By (4.3), Hölder and Bernstein (1.19),

$$\begin{aligned}\int_{\mathbb{R}^3} |\{u_{hi}, P_{lo}(|u|^4 u)\}_p| \; dx &\lesssim \int_{\mathbb{R}^3} |\nabla u_{hi}||P_{lo}(|u|^4 u)| \; dx + \int_{\mathbb{R}^3} |u_{hi}||\nabla P_{lo}(|u|^4 u)| \; dx \\ &\lesssim \|\nabla u_{hi}\|_{L^2_x} \|P_{lo}(|u|^4 u)\|_{L^2_x} + \|u_{hi}\|_{L^2_x} \|\nabla P_{lo}(|u|^4 u)\|_{L^2_x} \\ &\lesssim \|P_{lo}(|u|^4 u)\|_{L^2_x}.\end{aligned}$$

Now decompose $u = u_{hi} + u_{lo}$, and use (1.15) to then decompose

$$P_{lo}(|u|^4 u) = \sum_{j=0}^{5} P_{lo}\mathcal{O}(u_{hi}^j u_{lo}^{5-j}).$$

The terms $j = 0, 1, 2, 3, 4$ can be estimated by using Bernstein (1.19) and then Hölder to estimate

$$\sum_{j=0}^{4} \|P_{lo}\mathcal{O}(u_{hi}^j u_{lo}^{5-j})\|_{L^2_x} \lesssim \sum_{j=0}^{4} \|\mathcal{O}(u_{hi}^j u_{lo}^{5-j})\|_{L^{6/5}_x} \lesssim \sum_{j=0}^{4} \|u_{hi}\|_{L^6_x}^j \|u_{lo}\|_{L^6_x}^{5-j} \lesssim \eta_3,$$

thanks to (4.4), (11.1). For the $j = 5$ term, we argue similarly; indeed we have

$$\|P_{lo}\mathcal{O}(u_{hi}^5)\|_{L^2_x} \lesssim \|\mathcal{O}(u_{hi}^5)\|_{L^1_x} \lesssim \|u_{hi}\|_{L^6_x}^{9/2} \|u_{hi}\|_{L^2_x}^{1/2} \lesssim \eta_3^{1/2}$$

by (4.3), (11.1). □

Now we deal with the second term in the momentum bracket, namely $\{|u|^4 u - |u_{lo}|^4 u_{lo}, u_{lo}\}_p$. We first move the derivative in (2.3) into a more favorable position, using the identity

$$\{f, g\}_p = \nabla\mathcal{O}(fg) + \mathcal{O}(f\nabla g)$$

and the identity $|u|^4 u - |u_{lo}|^4 u_{lo} = \sum_{j=1}^{5} \mathcal{O}(u_{hi}^j u_{lo}^{5-j})$ from (1.15) to write

$$\{|u|^4 u - |u_{lo}|^4 u_{lo}, u_{lo}\}_p = \sum_{j=1}^{5} (\nabla\mathcal{O}(u_{hi}^j u_{lo}^{6-j}) + \mathcal{O}(u_{hi}^j u_{lo}^{5-j}\nabla u_{lo})).$$

For the second term, we argue crudely again, estimating this contribution in absolute value as

(11.13)
$$O\Big(\int_{I_0} \sum_{j=1}^{5} \int \int_{|x-y|\le 2R} |u_{hi}(t,y)|^2 |u_{hi}(t,x)|^j |u_{lo}(t,x)|^{5-j} |\nabla u_{lo}(t,x)| \; dxdydt\Big).$$

But by (4.4), (11.1) and the low frequency localization of u_{lo} we have

$$\int_{\mathbb{R}^3} |u_{hi}|^j |u_{lo}|^{5-j} |\nabla u_{lo}| \; dx \lesssim \|u_{hi}\|_{L^6_x}^j \|u_{lo}\|_{L^6_x}^{5-j} \|\nabla u_{lo}\|_{L^6_x} \lesssim \|u_{hi}\|_{L^6_x}^j \|u_{lo}\|_{L^6_x}^{6-j} \lesssim \eta_3,$$

and so this term can also be bounded by $O((11.11)) = O(X_R)$. For the first term, we can integrate by parts and then take absolute values to estimate the contribution of this term by

$$\sum_{j=1}^{5} O\Big(\int_{I_0}\int\int_{|x-y|\le 2R} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^j|u_{lo}(t,x)|^{6-j}}{|x-y|}\,dxdydt\Big). \tag{11.14}$$

Note that the terms where the integration by parts hits the cutoff is of the same type, but with the $\frac{1}{|x-y|}$ factor replaced by $O(\frac{1}{R})$ on the region where $|x-y|\sim R$; it is clear that this term is dominated by (11.14) (or by a variant of (11.13), where we remove the ∇ from u_{lo}). We hold off on estimating this term for now, and turn to the first term in the momentum bracket: $-\frac{2}{3}\nabla(|u|^6 - |u_{lo}|^6)$. After an integration by parts, this term can be written as

$$\begin{aligned} c_3 \int_{I_0}\int\int_{|x-y|\le 2R} &\frac{|u_{hi}(t,y)|^2(|u(t,x)|^6 - |u_{lo}(t,x)|^6)}{|x-y|}\,dxdydt \\ &+ O\Big(\frac{1}{R}\int_{I_0}\int\int_{|x-y|\le 2R} |u_{hi}(t,y)|^2||u(t,x)|^6 - |u_{lo}(t,x)|^6|\,dxdydt\Big), \end{aligned}$$

for some explicit constant $c_3 > 0$; note that the error incurred by removing the cutoff $\chi(\frac{x-y}{R})$ by the cruder cutoff $|x-y| \le 2R$ can be controlled by (11.12) which is acceptable. To control the error term, we use (1.14) to split

$$|u(t,x)|^6 - |u_{lo}(t,x)|^6 = |u_{hi}(t,x)|^6 + \sum_{j=1}^{5} \mathcal{O}(u_{hi}^j(t,x)u_{lo}^{6-j}(t,x)). \tag{11.15}$$

The $|u_{hi}|^6$ term is of course bounded by $O((11.12)) = O(X_R)$. The remaining terms have an L^1_x norm of $O(\eta_3)$ by (4.4), (11.1), and Hölder, so that this error term is bounded by $O((11.11)) = O(X_R)$. For the main term, we again use (11.15) and observe that the contributions of the error terms are bounded by (11.14). Collecting all these estimates together, we have now shown that

$$\begin{aligned} c_1 \int_{I_0}\int_{\mathbb{R}^3} &|u_{hi}(t,x)|^4\,dxdt \\ &+ c_3 \int_{I_0}\int\int_{|x-y|\le 2R} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^6}{|x-y|} dxdydt \lesssim X_R \\ &+ \sum_{j=1}^{5} O\Big(\int_{I_0}\int\int_{|x-y|\le 2R} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^j|u_{lo}(t,x)|^{6-j}}{|x-y|} dxdydt\Big). \end{aligned}$$

We can eliminate the $j = 2, 3, 4, 5$ terms using the elementary inequality

$$|u_{hi}(t,x)|^j|u_{lo}(t,x)|^{6-j} \le \varepsilon|u_{hi}(t,x)|^6 + C(\varepsilon)|u_{hi}(t,x)||u_{lo}(t,x)|^5$$

for some small absolute constant ε; this allows us to control the $j = 2, 3, 4, 5$ terms by the $j = 1$ term, plus a small multiple of the $j = 6$ term which can then be absorbed by the second term on the left-hand side (shrinking c_3 slightly). The theorem follows. □

We would like to use Theorem 11.1 to prove (11.4). If it were not for the error terms (11.8)–(11.12) then this estimate would follow immediately from Theorem 11.1, since we can discard the second term in (11.6) as being positive. One would then hope to estimate all of the error terms (11.8)–(11.12) by $O(\eta_1)$ using the bootstrap hypothesis (11.5) and the other estimates available (e.g. (4.3), (4.4), (11.1), (11.3)). It turns out that this strategy works (with $R = 1$) for the first four error terms (11.8)–(11.11) but not for the fifth term (11.12). To estimate this term we need the reverse Sobolev inequality, which effectively replaces the $|\nabla u_{hi}|^2$ density here by $|u_{hi}|^6$, at which point one can hope to control this term by the second positive term in (11.6). But to do so it turns out that one cannot apply Theorem 11.1 for a single value of R, but must instead average over a range of R.

We turn to the details. We let $J = J(\eta_1, \eta_2) \gg 1$ be a large[30] integer, and apply Theorem 11.1 to all dyadic radii $R = 1, 2, \ldots, 2^J$ separately. We then average over R to obtain

$$\int_{I_0}\int_{\mathbb{R}^3} |u_{hi}|^4 \, dxdt + Y \lesssim X, \tag{11.16}$$

where Y is the positive quantity

$$Y := \frac{1}{J}\sum_{1\le R\le 2^J}\int_{I_0}\int\int_{|x-y|\le 2R}\frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^6}{|x-y|}\,dxdydt$$

(R is always understood to sum over dyadic numbers), and X is the quantity

(11.17)

$$X := \eta_3^3$$

(11.18)

$$+ \sup_{1\le R\le 2^J}\int_{I_0}\int\int_{|x-y|\le 2R}\frac{|u_{hi}(t,y)|^2|u_{lo}(t,x)|^5|u_{hi}(t,x)|}{|x-y|}\,dxdydt$$

(11.19)

$$+ \sup_{1\le R\le 2^J}\sum_{j=0}^{4}$$

$$\cdot \int_{I_0}\int\int_{|x-y|\le 2R}|u_{hi}(t,y)||P_{hi}\mathcal{O}(u_{hi}^j u_{lo}^{5-j})(t,y)||u_{hi}(t,x)||\nabla u_{hi}(t,x)|\,dxdydt$$

(11.20)

$$+ \sup_{1\le R\le 2^J}\int_{I_0}\int\int_{|x-y|\le 2R}|u_{hi}(t,y)||P_{lo}\mathcal{O}(u_{hi}^5)(t,y)||u_{hi}(t,x)||\nabla u_{hi}(t,x)|\,dxdydt$$

[30] But it is not *too* large; any factor of η_3 will easily be able to overcome any losses depending on J.

$$+ \eta_3^{1/10} \sup_{1 \leq R \leq 2^J} \frac{1}{R} \int_{I_0} \Big(\sup_{x \in \mathbb{R}^3} \int_{B(x,2R)} |u_{hi}(t,y)|^2 \ dy \Big) dt \tag{11.21}$$

$$+ \frac{1}{J} \sum_{1 \leq R \leq 2^J} \frac{1}{R} \int_{I_0} \int \int_{|x-y| \leq 2R} |u_{hi}(t,y)|^2 (|\nabla u_{hi}(t,x)|^2 + |u_{hi}(t,x)|^6) \ dxdydt, \tag{11.22}$$

where we have estimated the average $\frac{1}{J} \sum_{1 \leq R \leq 2^J}$ by the supremum in those terms for which the averaging is not important.[31]

The terms (11.17)–(11.22) are roughly arranged in increasing order of difficulty to estimate. But let us use the reverse Sobolev inequality already obtained in Proposition 4.8 to deal with the most difficult term (11.22), replacing it by easier terms.

LEMMA 11.4.

$$(11.21) + (11.22) \lesssim \eta_1^{1/100} (Y + W)$$

where

$$W := \sup_{1 \leq R \leq C(\eta_1, \eta_2) 2^J} \frac{1}{R} \int_{I_0} \Big(\sum_{x \in \frac{R}{100} \mathbb{Z}^3} \Big(\int_{B(x,3R)} |u_{hi}(t,y)|^2 \ dy \Big)^{100} \Big)^{1/100} dt, \tag{11.23}$$

and $\frac{R}{100}\mathbb{Z}^3$ *is the integer lattice* $\mathbb{Z}^3$ *dilated by* $R/100$.

Proof. We first handle (11.21). For every $x \in \mathbb{R}^3$ there exists $x' \in \frac{R}{100}\mathbb{Z}^3$ such that $B(x, 2R)$ is contained in $B(x', 3R)$. Thus

$$\frac{1}{R} \sup_{x \in \mathbb{R}^3} \int_{B(x,2R)} |u_{hi}(t,y)|^2 \ dy \lesssim \frac{1}{R} \Big(\sum_{x' \in \frac{R}{100}\mathbb{Z}^3} \Big(\int_{B(x',3R)} |u_{hi}(t,y)|^2 \ dy \Big)^{100} \Big)^{1/100},$$

and the claim follows by noting that $\eta_3^{1/10} \ll \eta_1^{1/100}$.

Now we consider (11.22), writing this term as

$$\frac{1}{J} \sum_{1 \leq R \leq 2^J} \frac{1}{R} \int_{I_0} \int_{\mathbb{R}^3} |u_{hi}(t,y)|^2 e_{hi}(t,y,2R) \ dydt,$$

where $e_{hi}(t,y,2R)$ is the local energy,

$$e_{hi}(t,y,2R) := \int_{B(y,2R)} |\nabla u_{hi}(t,x)|^2 + |u_{hi}(t,x)|^6 \ dx.$$

We will denote the same quantity with u_{hi} replaced with u by $e(t,y,2R)$. We split this term into the regions $e_{hi}(t,y,2R) \lesssim \eta_1$ and $e_{hi}(t,y,2R) \gg \eta_1$.

[31]Indeed, in those terms we will extract a gain of η_3 which will easily absorb any losses relating to $R = O(2^J)$, since J depends only on η_1 and η_2.

Large energy regions. Consider first the large energy regions $e_{hi}(t, y, 2R) \gg \eta_1$. By (11.2), the same lower bound holds for $e(t, y, 2R)$. For (t, y) in such regions, we apply Proposition 4.8 to conclude that

$$\int_{B(y,2R)} |\nabla u(t,x)|^2 dx \leq \frac{1}{2} e(t, y, 2R) + C(\eta_1, \eta_2) \int_{B(y,C(\eta_1,\eta_2)R)} |u(t,x)|^6 dx$$

which implies

$$\int_{B(y,2R)} |\nabla u(t,x)|^2 dx \lesssim C(\eta_1, \eta_2) \int_{B(y,C(\eta_1,\eta_2)R)} |u(t,x)|^6 dx.$$

The same estimate is valid for u_{hi} in light of (11.2). Thus we can bound the contribution to (11.22) of the large energy regions by

$$\lesssim C(\eta_1, \eta_2) \frac{1}{J} \sum_{1 \leq R \leq 2^J} \frac{1}{R} \int_{I_0} \int \int_{|x-y| \lesssim C(\eta_1,\eta_2)R} |u_{hi}(t,y)|^2 |u_{hi}(t,x)|^6 \, dxdydt.$$

Shifting R by $C(\eta_1, \eta_2)$, we can bound this by

$$\lesssim (C(\eta_1, \eta_2))^2 \frac{1}{J} \sum_{1 \leq R \leq C(\eta_1,\eta_2) 2^J} A_R \tag{11.24}$$

where

$$A_R := \frac{1}{R} \int_{I_0} \int \int_{|x-y| \leq R} |u_{hi}(t,y)|^2 |u_{hi}(t,x)|^6 \, dxdydt.$$

To bound this by $\eta_1^{1/100}(Y + W)$ (and not just by $O(Y)$) we exploit the averaging[32] in R. First observe that

$$\sum_{1 \leq R \leq R'} A_R \lesssim \int_{I_0} \int \int_{|x-y| \leq R'} \frac{|u_{hi}(t,y)|^2 |u_{hi}(t,x)|^6}{|x-y|} dxdydt$$

for all R'. Averaging this over all $1 \leq R' \leq 2^J$, we see that

$$\frac{1}{J} \sum_{1 \leq R' \leq 2^J} \sum_{1 \leq R \leq R'} A_R$$
$$\lesssim \frac{1}{J} \sum_{1 \leq R' \leq 2^J} \int_{I_0} \int \int_{|x-y| \leq R'} \frac{|u_{hi}(t,y)|^2 |u_{hi}(t,x)|^6}{|x-y|} dxdydt \lesssim Y.$$

Now let $1 < J_0 < J$ be a parameter depending on η_1, η_2 to be chosen shortly. Observe that for each $1 \leq R \leq 2^{J-J_0}$ there are at least J_0 values of R' which involve that value of R in the above sum. Thus we have

$$\frac{J_0}{J} \sum_{1 \leq R \leq 2^{J-J_0}} A_R \lesssim Y,$$

[32]The key point here is that while the A_R quantities have a factor of $1/R$, the quantities in Y have a larger factor of $1/|x-y|$. After averaging, the latter factor begins to dominate the former.

and thus the contribution to (11.24) of the terms where $R \leq 2^{J-J_0}$ is bounded by $O(\frac{(C(\eta_1,\eta_2))^2}{J_0}Y)$, which is acceptable if J_0 is chosen sufficiently large depending on η_1, η_2.

It remains to control the expression

$$\frac{(C(\eta_1,\eta_2))^2}{J} \sum_{2^{J-J_0} \leq R \leq C(\eta_1,\eta_2)2^J} \frac{1}{R} \int_{I_0} \int \int_{|x-y|\leq R} |u_{hi}(t,y)|^2 |u_{hi}(t,x)|^6 dxdydt.$$

This is bounded by

$$\frac{C(\eta_1,\eta_2,J_0)}{J} \sup_{1 \leq R \leq C(\eta_1,\eta_2)2^J} \frac{1}{R} \int_{I_0} \int \sup_{x \in \mathbb{R}^3} \Big(\int_{B(x,R)} |u_{hi}(t,y)|^2 dy \Big) |u_{hi}(t,x)|^6 dxdt.$$

Using the energy bound (4.4), we see that this is in turn bounded by

$$\begin{aligned}
&\lesssim \frac{C(\eta_1,\eta_2,J_0)}{J} \sup_{1 \leq R \leq C(\eta_1,\eta_2)2^J} \frac{1}{R} \left(\int_{I_0} \sup_{x \in \mathbb{R}^3} \Big(\int_{B(x,R)} |u_{hi}(t,y)|^2 dy \Big) dt \right) \\
&\lesssim \frac{1}{J} C(\eta_1,\eta_2,J_0) W \\
&\lesssim \eta_1^{1/100} W
\end{aligned}$$

when $J = J(\eta_1,\eta_2)$ is sufficiently large.

Small energy regions. Now we deal with the contribution of the low energy regions:

$$\frac{1}{J} \sum_{1 \leq R \leq 2^J} \frac{1}{R} \int_{I_0} \int_{e_{hi}(t,y,2R) \lesssim \eta_1} |u_{hi}(t,y)|^2 e_{hi}(t,y,2R) \; dydt.$$

Observe that if y is such that $e_{hi}(t,y,2R) \lesssim \eta_1$, then there exists $y' \in \frac{R}{100}\mathbb{Z}^3$ such that $y \in B(y',R) \subseteq B(y,2R) \subset B(y',3R)$, and thus $e_{hi}(t,y,R) \lesssim \min(\eta_1, e_{hi}(t,y',3R))$. Thus we can dominate the above expression by

$$\lesssim \frac{1}{J} \sum_{1 \leq R \leq 2^J} \frac{1}{R} \int_{I_0} \sum_{y' \in \frac{R}{100}\mathbb{Z}^3} \min(\eta_1, e_{hi}(t,y',3R)) \int_{B(y',3R)} |u_{hi}(t,y)|^2 \; dydt.$$

Now from (4.3) we have, using $\min(\eta_1, e_{hi}(\cdot))^{100/99} \leq \eta_1^{1/99} e_{hi}(\cdot)$, that

$$\begin{aligned}
&\left(\sum_{y' \in \frac{R}{100}\mathbb{Z}^3} \min(\eta_1, e_{hi}(t,y',3R))^{100/99} \right)^{99/100} \\
&\qquad\qquad \lesssim \eta_1^{1/100} \left(\sum_{y' \in \frac{R}{100}\mathbb{Z}^3} e_{hi}(t,y',3R) \right)^{99/100} \lesssim \eta_1^{1/100}.
\end{aligned}$$

Thus, by Hölder's inequality we can bound the previous expression by $O(\eta_1^{1/100} W)$ as desired. $\square$

In light of all the above estimates, we have thus shown that

$$\int_{I_0}\int_{\mathbb{R}^3} |u_{hi}|^4 \, dxdt \lesssim \eta_3^3 + (11.18) + (11.19) + (11.20) + \eta_1^{1/100} W, \tag{11.25}$$

since the Y term on the left can be used to absorb the $\eta_1^{1/100} Y$ term which would otherwise appear on the right. It thus suffices to show that all of the quantities (11.18)-(11.20) are $O(\eta_1)$, while the factor of $\eta_1^{1/100}$ allows the quantity W to be estimated using the weaker bound of $O(C_0\eta_1)$. (This is the main purpose of the reverse Sobolev inequality, Proposition 4.8, in our argument. The constant C_0 was defined in our bootstrap assumption (11.5).).

As mentioned earlier, the terms (11.18)–(11.20) are roughly arranged in increasing order of difficulty, and W is more difficult still. To estimate any of these expressions, we of course need good spacetime estimates on u_{hi} and u_{lo}. We do already have some estimates on these quantities (11.5), (4.3), (4.4), (11.2), (11.3), but it turns out that these are not directly sufficient to estimate (11.18)–(11.20) and W. Thus we shall first use the equation (1.1) and Strichartz estimates to bootstrap (11.5) to yield further spacetime integrability; this will be the purpose of the next section.

12. Interaction Morawetz: Strichartz control

In this section we establish the spacetime estimates we need in order to bound (11.18)–(11.20) and W. Ideally one would like to use (11.5) to show that u obeys the same estimates as a solution to the free Schrödinger equation; however the quantity (11.5) is supercritical (it has roughly the scaling of $\dot{H}^{1/4}$, instead of $\dot{H}^1$), and we must therefore accept some loss of derivatives in the high frequencies;[33] a model example of a function u obeying (11.5) to keep in mind is a pseudosoliton solution where u has magnitude $|u(t,x)| \sim N^{1/2}$ on the spacetime region $x = O(N^{-1}), t = O(N)$ and has Fourier transform supported near the frequency N for some large $N \gg 1$. We will however be able to show that u does behave like a solution to the free Schrödinger equation modulo a high frequency forcing term which is controlled in $L^2_t L^1_x$ but not in dual Strichartz spaces.

Recall that the constant C_0 which we use throughout these next three sections of the paper was defined in (11.5), our bootstrap hypothesis on the L^4 norm of u_{hi}. We begin by estimating the low frequency portion u_{lo} of the solution, which does behave like the free equation from the point of view of Strichartz estimates:

[33] Recall that u_{hi} and u_{lo} were defined at the beginning of Section 11.

Proposition 12.1 (Low frequency estimate). *For the functions* u_{hi}, u_{lo} *defined at the start of Section* 11,

$$\|\nabla u_{lo}\|_{L^2_t L^6_x(I_0\times\mathbb{R}^3)} \lesssim C_0^{1/2}\eta_1^{1/2} \tag{12.1}$$

and

$$\|u_{lo}\|_{L^4_t L^\infty_x(I_0\times\mathbb{R}^3)} \lesssim \eta_3^{1/2}. \tag{12.2}$$

In fact, there is the slightly more general statement that (12.1), (12.2) *hold for all* $u_{\leq N}$ *when* $N \sim 1$.

Observe that the $L^4_t L^\infty_x$ estimate gains a power of η_3, which will be very useful for us in overcoming certain losses of R in the sequel.

Proof. We may assume that $N \geq 1$, since the case when $N < 1$ of course then follows. Let Z denote the quantity

$$Z := \|\nabla u_{\leq N}\|_{L^2_t L^6_x(I_0\times\mathbb{R}^3)} + \|u_{\leq N}\|_{L^4_t L^\infty_x(I_0\times\mathbb{R}^3)}. \tag{12.3}$$

By Strichartz (3.7), (3.4) we have

$$Z \lesssim \|u_{\leq N}\|_{\dot{S}^1(I_0\times\mathbb{R}^3)} \lesssim \|u_{\leq N}(t_0)\|_{\dot{H}^1_x} + \|\nabla P_{\leq N}(|u|^4 u)\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)}.$$

By (11.1) we have

$$\|u_{\leq N}(t_0)\|_{\dot{H}^1_x} \lesssim \eta_3.$$

Now we consider the nonlinear term. We split $u = u_{\leq N} + u_{>N}$ and use (1.15) to write

$$\nabla P_{\leq N}(|u|^4 u) = \sum_{j=0}^{5} \nabla P_{\leq N}\mathcal{O}(u^j_{>N} u^{5-j}_{\leq N}).$$

Considering the $j = 0$ term first, we discard $P_{\leq N}$ and use the Leibnitz rule and Hölder to estimate

$$\begin{aligned}\|\nabla P_{\leq N}\mathcal{O}(u^5_{\leq N})\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)} &\lesssim \|\mathcal{O}(u^4_{\leq N}\nabla u_{\leq N})\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)}\\ &\lesssim \|u_{\leq N}\|^4_{L^\infty_t L^6_x(I_0\times\mathbb{R}^3)}\|\nabla u_{\leq N}\|_{L^2_t L^6_x(I_0\times\mathbb{R}^3)};\end{aligned}$$

by (11.1) and (12.3) this term is thus bounded by $O(\eta_3^4 Z)$.

Now consider the $j = 1$ term. We argue similarly to estimate this term as

$$\begin{aligned}&\|\nabla P_{\leq N}\mathcal{O}(u^4_{\leq N} u_{>N})\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)}\\ &\quad\lesssim \|\mathcal{O}(u^4_{\leq N}\nabla u_{>N})\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)}\\ &\quad+ \|\mathcal{O}(u^3_{\leq N} u_{>N}\nabla u_{\leq N})\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)}\\ &\quad\lesssim \|u_{\leq N}\|^2_{L^4_t L^\infty_x(I_0\times\mathbb{R}^3)}\|u_{\leq N}\|^2_{L^\infty_t L^6_x(I_0\times\mathbb{R}^3)}\|\nabla u_{>N}\|_{L^\infty_t L^2_x(I_0\times\mathbb{R}^3)}\\ &\qquad+ \|u_{\leq N}\|^3_{L^\infty_t L^6_x(I_0\times\mathbb{R}^3)}\|u_{>N}\|_{L^\infty_t L^6_x(I_0\times\mathbb{R}^3)}\|\nabla u_{\leq N}\|_{L^2_t L^6_x(I_0\times\mathbb{R}^3)};\end{aligned}$$

by (4.4), (11.1), (12.3) these terms are thus bounded by $O(\eta_3^2 Z^2 + \eta_3^3 Z)$.

Now consider the $j = 2, 3, 4, 5$ terms. This time we use Bernstein's inequality (1.20) (recalling that $N \sim 1$) and Hölder to estimate

$$\begin{aligned}
&\|\nabla P_{\leq N} \mathcal{O}(u_{\leq N}^{5-j} u_{>N}^{j})\|_{L_t^2 L_x^{6/5}(I_0 \times \mathbb{R}^3)} \\
&\quad \lesssim \|\mathcal{O}(u_{\leq N}^{5-j} u_{>N}^{j})\|_{L_t^2 L_x^1(I_0 \times \mathbb{R}^3)} \\
&\quad \lesssim \|u_{>N}\|_{L_t^4 L_x^4(I_0 \times \mathbb{R}^3)}^2 \|u_{>N}\|_{L_t^\infty L_x^6(I_0 \times \mathbb{R}^3)}^{j-2} \|u_{\leq N}\|_{L_t^\infty L_x^6(I_0 \times \mathbb{R}^3)}^{5-j}.
\end{aligned}$$

Applying (11.5), (4.4) we can bound these terms by $O(C_0^{1/2}\eta_1^{1/2})$ (we can do significantly better on the $j = 2, 3, 4$ terms but we will not exploit this). Combining all these estimates we see that

$$Z \lesssim \eta_3 + \eta_3^4 Z + \eta_3^2 Z^2 + \eta_3^3 Z + C_0^{1/2}\eta_1^{1/2};$$

by standard continuity arguments this then implies that $Z \lesssim C_0^{1/2}\eta_1^{1/2}$. This proves (12.1), but we did not achieve the η_3 gain in (12.2).

To obtain (12.2) we must refine the above analysis. Let $u_{\leq N}^0$ be the solution to the free Schrödinger equation with initial data $u_{\leq N}^0(t_0) = u_{\leq N}(t_0)$ for some $t_0 \in I_0$. Then by (11.1) and Lemma 3.1

$$\|u_{\leq N}^0\|_{L_t^4 L_x^\infty(I_0 \times \mathbb{R}^3)} \lesssim \eta_3.$$

Thus it suffices to prove that

$$\|u_{\leq N} - u_{\leq N}^0\|_{L_t^4 L_x^\infty(I_0 \times \mathbb{R}^3)} \lesssim \eta_3^{1/2}.$$

We estimate the left-hand side as

$$\sum_{N' \lesssim N} \|u_{N'} - u_{N'}^0\|_{L_t^4 L_x^\infty(I_0 \times \mathbb{R}^3)}.$$

By Bernstein (1.20) we may bound this by

$$\sum_{N' \lesssim N} N' \|u_{N'} - u_{N'}^0\|_{L_t^4 L_x^3(I_0 \times \mathbb{R}^3)}$$

which by interpolation can be bounded by

$$\sum_{N' \lesssim N} N' \|u_{N'} - u_{N'}^0\|_{L_t^2 L_x^6(I_0 \times \mathbb{R}^3)}^{1/2} \|u_{N'} - u_{N'}^0\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^3)}^{1/2}.$$

But from (11.1),

$$\|u_{N'} - u_{N'}^0\|_{L_t^\infty L_x^2(I_0 \times \mathbb{R}^3)} \lesssim (N')^{-1} \|u_{\leq N} - u_{\leq N}^0\|_{L_t^\infty \dot{H}^1(I_0 \times \mathbb{R}^3)} \lesssim (N')^{-1}\eta_3$$

so that

$$\|u_{\leq N} - u_{\leq N}^0\|_{L_t^4 L_x^\infty(I_0 \times \mathbb{R}^3)} \lesssim \eta_3^{1/2} \sum_{N' \leq N} (N')^{1/2} \|u_{N'} - u_{N'}^0\|_{L_t^2 L_x^6(I_0 \times \mathbb{R}^3)}^{1/2}.$$

Applying Strichartz (3.7) yields

$$\|u_{\leq N} - u^0_{\leq N}\|_{L^4_t L^\infty_x(I_0\times\mathbb{R}^3)} \lesssim \eta_3^{1/2} \sum_{N'\leq N} (N')^{1/2} \|P_{N'}(|u|^4 u)\|^{1/2}_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)}.$$

But the preceding analysis already showed that

$$\|P_{N'}(|u|^4 u)\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)} \lesssim \eta_3^4 Z + \eta_3^2 Z^2 + \eta_3^3 Z + C_0^{1/2}\eta_1^{1/2} \lesssim 1$$

(for instance), and the claim follows. $\square$

Now we estimate the high frequencies. We first need an estimate on the high-frequency portion of the nonlinearity $|u|^4 u$. It turns out that this quantity cannot be easily estimated in a single Strichartz norm, but must instead be decomposed into two pieces estimated using separate space-time Lebesgue norms (cf. the appearance of M in Lemma 3.2).

PROPOSITION 12.2. *We can decompose*

$$P_{hi}(|u|^4 u) = F + G$$

where F, G are Schwartz functions with Fourier support in the region $|\xi| \gtrsim 1$ and

$$\|\nabla F\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)} \lesssim \eta_3^{1/2}$$

and

$$\|G\|_{L^2_t L^1_x(I_0\times\mathbb{R}^3)} \lesssim C_0^{1/2}\eta_1^{1/2}.$$

Of the two pieces, F is by far the better term; indeed, if G were not present, then the Strichartz estimate (3.7) would be able to obtain $L^{10}_{t,x}$ bounds on u_{hi}. The reader may in fact assume as a first approximation that F is negligible, and that the nonlinearity $P_{hi}(|u|^4 u)$ is primarily in $L^2_t L^1_x$, which is not a dual $\dot{S}^1$ Lebesgue norm. Note also that the $\eta_1^{\frac{1}{2}}$ bound ultimately determines the η_1 on the right side of (4.19) at the end of Section 14.

Proof. We split $u = u_{lo} + u_{hi}$, and then use (1.15) to split

$$P_{hi}(|u|^4 u) = \sum_{j=0}^{5} P_{hi}\mathcal{O}(u^j_{hi} u^{5-j}_{lo}).$$

Considering the $j = 0$ term first, we have

$$\begin{aligned}
&\|\nabla P_{hi}\mathcal{O}(u^5_{lo})\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)} \\
&\qquad \lesssim \|\mathcal{O}(u^4_{lo}\nabla u_{lo})\|_{L^2_t L^{6/5}_x(I_0\times\mathbb{R}^3)} \\
&\qquad \lesssim \|u_{lo}\|^2_{L^4_t L^\infty_x(I_0\times\mathbb{R}^3)} \|u_{lo}\|^2_{L^\infty_t L^6_x(I_0\times\mathbb{R}^3)} \|\nabla u_{lo}\|_{L^\infty_t L^2_x(I_0\times\mathbb{R}^3)}
\end{aligned}$$

which is $O(\eta_3^{1/2})$ (for instance) by Proposition 12.1 and (4.4), (4.3). Thus this term may be placed as part of F.

Now consider the $j = 1$ term. We have

$$\begin{aligned}
&\|\nabla P_{hi}\mathcal{O}(u_{lo}^4 u_{hi})\|_{L_t^2 L_x^{6/5}(I_0\times\mathbb{R}^3)} \\
&\qquad\lesssim \|\mathcal{O}(u_{lo}^4 \nabla u_{hi})\|_{L_t^2 L_x^{6/5}(I_0\times\mathbb{R}^3)} \\
&\qquad+ \|\mathcal{O}(u_{lo}^3 u_{hi}\nabla u_{lo})\|_{L_t^2 L_x^{6/5}(I_0\times\mathbb{R}^3)} \\
&\qquad\lesssim \|u_{lo}\|^2_{L_t^4 L_x^\infty(I_0\times\mathbb{R}^3)}\|u_{lo}\|^2_{L_t^\infty L_x^6(I_0\times\mathbb{R}^3)}\|\nabla u_{hi}\|_{L_t^\infty L_x^2(I_0\times\mathbb{R}^3)} \\
&\qquad+ \|u_{lo}\|^3_{L_t^\infty L_x^6(I_0\times\mathbb{R}^3)}\|u_{hi}\|_{L_t^\infty L_x^6(I_0\times\mathbb{R}^3)}\|\nabla u_{lo}\|_{L_t^2 L_x^6(I_0\times\mathbb{R}^3)}.
\end{aligned}$$

Applying Proposition 12.1, (4.4), (4.3), and (11.1) this expression is $O(\eta_3^3)$ and so this term may also be placed as part of F.

Now consider the $j = 2, 3, 4, 5$ terms. We estimate

$$\begin{aligned}
&\|P_{hi}\mathcal{O}(u_{hi}^j u_{lo}^{5-j})\|_{L_t^2 L_x^1(I_0\times\mathbb{R}^3)} \\
&\qquad\qquad\lesssim \|u_{hi}\|^2_{L_t^4 L_x^4(I_0\times\mathbb{R}^3)}\|u_{hi}\|^{j-2}_{L_t^\infty L_x^6(I_0\times\mathbb{R}^3)}\|u_{lo}\|^{5-j}_{L_t^\infty L_x^6(I_0\times\mathbb{R}^3)}
\end{aligned}$$

which is $O(C_0^{1/2}\eta_1^{1/2})$ by (11.5) and (4.4). Thus we may place this term as part of G. □

Corollary 12.3. *For every $N \geq 1$,*

$$\|u_N\|_{L_t^2 L_x^6(I_0\times\mathbb{R}^3)} \lesssim C_0^{1/2} N^{1/2}\eta_1^{1/2}. \tag{12.4}$$

Proof. From Strichartz (3.7) and Proposition 12.2,

$$\|u_N\|_{L_t^2 L_x^6(I_0\times\mathbb{R}^3)} \lesssim \|u_N(t_0)\|_{L_x^2(\mathbb{R}^3)} + \|P_N F\|_{L_t^2 L_x^{6/5}(I_0\times\mathbb{R}^3)} + \|P_N G\|_{L_t^2 L_x^{6/5}(I_0\times\mathbb{R}^3)}.$$

The first term is certainly acceptable by (11.3). The second term is $O(\eta_3^{1/2}N^{-1})$ by Proposition 12.2, and the third term is $O(C_0^{1/2}N^{1/2}\eta_1^{1/2})$ by Bernstein (1.20) and Proposition 12.2. The claim follows. □

13. Interaction Morawetz: Error estimates

We now show that the comparatively easy terms (11.18), (11.19), (11.20) are indeed controlled by $O(\eta_1)$, which is in turn bounded by the right-hand side of (11.4). The term W is however significantly harder and will be deferred to the next section.

- *The estimation of* (11.18). We have to show that

$$\int_{I_0}\int\int_{|x-y|\leq 2R} \frac{|u_{hi}(t,y)|^2|u_{lo}(t,x)|^5|u_{hi}(t,x)|}{|x-y|}\, dxdydt \lesssim \eta_1$$

for all $1 \leq R \leq 2^J$. This term will be fairly easy because of the localization $|x-y| \leq 2R$. Observe that the kernel $\frac{1}{|x|}$ has an L_x^1 norm of $O(R^2) \leq O(2^{2J})$

on the ball $B(0, 2R)$. Thus by Young's inequality and Cauchy-Schwarz we see that

$$\int\int_{|x-y|\leq 2R} \frac{F(x)G(y)}{|x-y|}\ dxdy \lesssim R^2\|F\|_{L^2_x}\|G\|_{L^2_x}$$

for any functions F, G. In particular the expression to be estimated is bounded by

$$\lesssim 2^{2J}\int_{I_0} \|u_{hi}(t)\|^2_{L^4_x}\||u_{lo}|^5|u_{hi}|(t)\|_{L^2_x}\ dt.$$

We use Hölder and (11.3) to estimate

$$\||u_{lo}|^5|u_{hi}|(t)\|_{L^2_x} \lesssim \|u_{lo}(t)\|^5_{L^\infty_x}\|u_{hi}(t)\|_{L^2_x} \lesssim \eta_3\|u_{lo}(t)\|^5_{L^\infty_x}.$$

We dispose of three of the five factors of $\|u_{lo}(t)\|^5_{L^\infty_x}$ by observing from (4.4) and Bernstein's inequality (1.20) that $\|u_{lo}(t)\|^3_{L^\infty_x} \lesssim \eta_3^3 \lesssim 1$. Combining all these estimates and then using Hölder in time, we thus can bound the expression to be estimated by

$$2^{2J}\eta_3\|u_{hi}\|^2_{L^4_tL^4_x(I_0\times\mathbb{R}^3)}\|u_{lo}\|^2_{L^4_tL^\infty_x(I_0\times\mathbb{R}^3)},$$

which is bounded by the right-hand side of (11.4) using (11.5) and Proposition 12.1 (note that η_3 will absorb 2^{2J} since J depends only on η_1 and η_2). This concludes the treatment of (11.18).

- *The estimation of* (11.19). We now handle (11.19). We have to show that

$$\int_{I_0}\int\int_{|x-y|\leq 2R} |u_{hi}(t,y)||P_{hi}\mathcal{O}(u^j_{hi}u^{5-j}_{lo})(t,y)||u_{hi}(t,x)||\nabla u_{hi}(t,x)|\ dxdydt \lesssim \eta_1$$

for $j = 0, 1, 2, 3, 4$ and $1 \leq R \leq 2^J$.

We begin by considering the cases $j = 1, 2, 3, 4$. We observe from Hölder and (4.3) that

$$\begin{aligned}\int_{B(y,2R)} |u_{hi}(t,x)||\nabla u_{hi}(t,x)|\ dx &\lesssim R^{3/4}\|u_{hi}(t)\|_{L^4_x}\|\nabla u_{hi}(t)\|_{L^2_x}\\ &\lesssim R^{3/4}\|u_{hi}(t)\|_{L^4_x},\end{aligned}$$

and hence it suffices to show that

$$R^{\frac{3}{4}}\int_{I_0}\|u_{hi}(t)\|_{L^4_x}\int_{\mathbb{R}^3}|u_{hi}(t,y)||P_{hi}\mathcal{O}(u^j_{hi}u^{5-j}_{lo})(t,y)|\ dydt \lesssim \eta_1.$$

We use Hölder (and the hypothesis $j = 1, 2, 3, 4$) to estimate this as

$$\lesssim R^{\frac{3}{4}}\|u_{hi}\|^3_{L^4_tL^4_x(I_0\times\mathbb{R}^3)}\|u_{lo}\|_{L^4_tL^\infty_x(I_0\times\mathbb{R}^3)}\|u_{hi}\|^{j-1}_{L^\infty_tL^6_x(I_0\times\mathbb{R}^3)}\|u_{lo}\|^{4-j}_{L^\infty_tL^6_x(I_0\times\mathbb{R}^3)}.$$

The $L^\infty_tL^6_x(I_0\times\mathbb{R}^3)$ factors are bounded by (4.4), leaving us to prove

$$R^{\frac{3}{4}}\|u_{hi}\|^3_{L^4_tL^4_x(I_0\times\mathbb{R}^3)}\|u_{lo}\|_{L^4_tL^\infty_x(I_0\times\mathbb{R}^3)} \lesssim \eta_1.$$

But this follows from (11.5) and Proposition 12.1, again using η_3 to wallop a positive power of R.

Finally, we consider the $j = 0$ case of (11.19), where we have to prove

$$\int_{I_0}\int\int_{|x-y|\leq 2R} |u_{hi}(t,y)||P_{hi}\mathcal{O}(u_{lo}^5)(t,y)||u_{hi}(t,x)||\nabla u_{hi}(t,x)|\ dxdydt \lesssim \eta_1.$$

Here we use Cauchy-Schwarz and (11.3) to crudely bound

$$\int_{\mathbb{R}^3} |u_{hi}(t,x)||\nabla u_{hi}(t,x)|\ dx \lesssim \eta_3$$

and then use Hölder to reduce to showing that

$$\eta_3 \|u_{hi}\|_{L_t^\infty L_x^2(I_0\times\mathbb{R}^3)} \|P_{hi}\mathcal{O}(u_{lo}^5)\|_{L_t^1 L_x^2(I_0\times\mathbb{R}^3)} \lesssim \eta_1.$$

The factor $\|u_{hi}\|_{L_t^\infty L_x^2(I_0\times\mathbb{R}^3)}$ is $O(\eta_3)$ by (11.3). For the second factor, we take advantage of the high frequency localization (using (1.16)) to write

$$\begin{aligned}
\|P_{hi}\mathcal{O}(u_{lo}^5)\|_{L_t^1 L_x^2(I_0\times\mathbb{R}^3)} &\lesssim \|\nabla\mathcal{O}(u_{lo}^5)\|_{L_t^1 L_x^2(I_0\times\mathbb{R}^3)}\\
&\lesssim \|\mathcal{O}(u_{lo}^4\nabla u_{lo})\|_{L_t^1 L_x^2(I_0\times\mathbb{R}^3)}\\
&\lesssim \|\nabla u_{lo}\|_{L_t^\infty L_x^2(I_0\times\mathbb{R}^3)} \|u_{lo}\|^4_{L_t^4 L_x^\infty(I_0\times\mathbb{R}^3)}.
\end{aligned}$$

The claim then follows from (4.3) and Proposition 12.1.

- *The estimation of* (11.20). We now handle (11.20). We have to show that

(13.1)
$$\int_{I_0}\int\int_{|x-y|\leq 2R} |u_{hi}(t,y)||P_{lo}\mathcal{O}(u_{hi}^5)(t,y)||u_{hi}(t,x)||\nabla u_{hi}(t,x)|\ dxdydt \lesssim \eta_1.$$

We begin by using Hölder to write

$$\int_{B(y,2R)} |u_{hi}(t,x)||\nabla u_{hi}(t,x)|\ dx \lesssim R^{1/2}\|u_{hi}(t)\|_{L_x^3}\|\nabla u_{hi}(t)\|_{L_x^2}$$

and apply (4.3) to estimate the left-hand side of (13.1) by

$$\lesssim R^{\frac{1}{2}} \int_{I_0} \|u_{hi}(t)\|_{L_x^3} \int |u_{hi}(t,y)||P_{lo}\mathcal{O}(u_{hi}^5)(t,y)|\ dy.$$

By Hölder and Bernstein (1.20) we have

$$\begin{aligned}
\int |u_{hi}(t,y)||P_{lo}\mathcal{O}(u_{hi}^5)(t,y)|\ dy &\lesssim \|u_{hi}(t)\|_{L_x^3}\|P_{lo}\mathcal{O}(u_{hi}(t)^5)\|_{L_x^{3/2}}\\
&\lesssim \|u_{hi}(t)\|_{L_x^3}\|\mathcal{O}(u_{hi}(t)^5)\|_{L_x^1}\\
&\lesssim \|u_{hi}(t)\|_{L_x^3}\|u_{hi}(t)\|^2_{L_x^4}\|u_{hi}(t)\|^3_{L_x^6}.
\end{aligned}$$

The terms $\|u_{hi}(t)\|_{L_x^6}^3$ are bounded by (4.4), so by a Hölder in time and (11.5) we have

$$\begin{aligned}(11.20) &\lesssim R^{\frac{1}{2}} \|u_{hi}\|^2_{L_t^4 L_x^3(I_0\times\mathbb{R}^3)} \|u_{hi}\|^2_{L_t^4 L_x^4(I_0\times\mathbb{R}^3)} \\ &\lesssim R^{1/2} C_0^{1/2} \eta_1^{1/2} \|u_{hi}\|^2_{L_t^4 L_x^3(I_0\times\mathbb{R}^3)}.\end{aligned} \tag{13.2}$$

From the triangle inequality and Hölder,

$$\begin{aligned}\|u_{hi}\|_{L_t^4 L_x^3(I_0\times\mathbb{R}^3)} &\lesssim \sum_{N\geq 1} \|u_N\|_{L_t^4 L_x^3(I_0\times\mathbb{R}^3)} \\ &\lesssim \sum_{N\geq 1} \|u_N\|^{1/2}_{L_t^\infty L_x^2(I_0\times\mathbb{R}^3)} \|u_N\|^{1/2}_{L_t^2 L_x^6(I_0\times\mathbb{R}^3)}.\end{aligned}$$

From (11.3), (4.5) we have $\|u_N\|_{L_t^\infty L_x^2(I_0\times\mathbb{R}^3)} \lesssim \min(\eta_3, N^{-1})$. Applying (12.4) we thus have

$$\|u_{hi}\|_{L_t^4 L_x^3(I_0\times\mathbb{R}^3)} \lesssim \sum_{N\geq 1} \min(\eta_3, N^{-1})^{1/2} C_0^{1/4} N^{1/4} \eta_1^{1/4} \lesssim \eta_3^{1/2} C_0^{1/4} \eta_1^{1/4}.$$

Inserting this estimate into (13.2) we see that (11.20) is acceptable (again, the power of η_3 counteracts the loss in C_0 and the presence of the power $R^{1/2}$).

Note that the first four factors on the right side of (11.25) have all in fact been shown to be controlled with a positive power of η_3.

14. Interaction Morawetz: A double Duhamel trick

To conclude the proof of Proposition 4.9, we have to show that

$$W \lesssim C_0 \eta_1,$$

or in other words that

$$\frac{1}{R} \int_{I_0} \Big(\sum_{x\in \frac{R}{100}\mathbb{Z}^3} \Big(\int_{B(x,3R)} |u_{hi}(t,y)|^2\, dy \Big)^{100} \Big)^{1/100} dt \lesssim C_0 \eta_1$$

for all $1 \leq R \leq C(\eta_1, \eta_2) 2^J$. By duality we have

$$\Big(\sum_{x\in \frac{R}{100}\mathbb{Z}^3} \Big(\int_{B(x,3R)} |u_{hi}(t,y)|^2\, dy \Big)^{100} \Big)^{1/100} = \sum_{x\in \frac{R}{100}\mathbb{Z}^3} c(t,x) \int_{B(x,3R)} |u_{hi}(t,y)|^2\, dy$$

where $c(t,x) > 0$ are a collection of numbers which are almost summable in the sense that

$$\sum_{x\in \frac{R}{100}\mathbb{Z}^3} c(t,x)^{100/99} = 1 \tag{14.1}$$

for all t. Thus it suffices to show that

$$\frac{1}{R} \int_{I_0} \sum_{x\in \frac{R}{100}\mathbb{Z}^3} c(t,x) \int_{B(x,3R)} |u_{hi}(t,y)|^2\, dy dt \lesssim C_0 \eta_1.$$

Let ψ be a bump function adapted to $B(0,5)$ which equals 1 on $B(0,3)$. Since

$$\int_{B(x,3R)} |u_{hi}(t,y)|^2 \, dy \lesssim \int |u_{hi}(t,y)|^2 \psi\left(\frac{y-x}{R}\right) \, dy,$$

it suffices to show that

$$\frac{1}{R}\int_{I_0} \sum_{x \in \frac{R}{100}\mathbb{Z}^3} c(t,x) \int_{\mathbb{R}^3} |u_{hi}(t,y)|^2 \psi\left(\frac{y-x}{R}\right) \, dydt \lesssim C_0 \eta_1. \tag{14.2}$$

In the proof of Lemma 11.4, we obtained spacetime control on u_{hi} by using the (forward-in-time) Duhamel formula (1.13) followed by Strichartz estimates. This seems to be insufficient to prove (14.2) (the best argument available seems to lose a logarithm of the derivative); instead, we rely on both the forward-in-time and backward-in-time Duhamel formulae (1.13), and argue using the fundamental solution (1.11). This will only lose a factor of C_0 (see (11.5)), which is acceptable because of the gain of $\eta_1^{1/100}$ which was obtained earlier in Lemma 11.4.

Let us write $I_0 = [t_-, t_+]$ for some times $-\infty < t_- < t_+ < \infty$. We use Proposition 12.2 to decompose $P_{hi}(|u|^4 u) = F + G$. We define the functions $u_{hi}^{\pm}$ to solve the Cauchy problem

$$(i\partial_t + \Delta) u_{hi}^{\pm} = F; \quad u_{hi}^{\pm}(t_\pm) = u_{hi}(t_\pm);$$

thus this is the same equation as u_{hi} satisfies but without the term G. From (1.13) observe the forward-in-time Duhamel formula

$$u_{hi}(t) = u_{hi}^{-}(t) - i \int_{t_- < s_- < t} e^{i(t-s_-)\Delta} G(s_-) \, ds_-$$

and the backward-in-time Duhamel formula

$$u_{hi}(t) = u_{hi}^{+}(t) + i \int_{t < s_+ < t_+} e^{i(t-s_+)\Delta} G(s_+) \, ds_+.$$

Let us see how we would prove (14.2) if u_{hi} were replaced by $u_{hi}^{\pm}$. From Strichartz (3.7), (11.3), and Proposition 12.2 (discarding a derivative) we see that

$$\|u_{hi}^{\pm}\|_{L_t^2 L_x^6(I_0 \times \mathbb{R}^3)} \lesssim \eta_3^{1/2}.$$

But from Hölder we have

$$\begin{aligned}
&\frac{1}{R} \sum_{x \in \frac{R}{100}\mathbb{Z}^3} c(t,x) \int_{\mathbb{R}^3} |u_{hi}^{\pm}(t,y)|^2 \psi\left(\frac{y-x}{R}\right)\, dy \\
&\quad \lesssim \frac{1}{R} \sum_{x \in \frac{R}{100}\mathbb{Z}^3} c(t,x) R^2 \left(\int_{\mathbb{R}^3} |u_{hi}^{\pm}(t,y)|^6 \psi\left(\frac{y-x}{R}\right)\, dy \right)^{1/3} \\
&\quad \lesssim R \left(\sum_{x \in \frac{R}{100}\mathbb{Z}^3} c(t,x)^{3/2} \right)^{2/3} \left(\sum_{x \in \frac{R}{100}\mathbb{Z}^3} \int_{\mathbb{R}^3} |u_{hi}^{\pm}(t,y)|^6 \psi\left(\frac{y-x}{R}\right)\, dy \right)^{1/3} \\
&\quad \lesssim R \left(\sum_{x \in \frac{R}{100}\mathbb{Z}^3} c(t,x)^{100/99} \right)^{99/100} \|u_{hi}^{\pm}\|_{L_x^6}^2 \\
&\quad = R \|u_{hi}^{\pm}\|_{L_x^6}^2
\end{aligned}$$

and hence

$$\frac{1}{R} \int_{I_0} \sum_{x \in \frac{R}{100}\mathbb{Z}^3} c(t,x) \int_{\mathbb{R}^3} |u_{hi}(t,y)|^2 \psi\left(\frac{y-x}{R}\right)\, dydt \lesssim R\eta_3$$

which is acceptable if η_3 is sufficiently small (recall that $R \leq C(\eta_1,\eta_2)2^J$ and J depends only on η_1, η_2). Thus we see that (14.2) would be easy to prove if u_{hi} were replaced by $u_{hi}^{\pm}$.

It is now natural to use one of the two Duhamel formulae listed above, and attempt to prove (14.2) for the integral term. This however turns out to be rather difficult. It will be significantly easier if we use *both* formulae simultaneously. More precisely, we re-arrange the above Duhamel formulae as

$$-i \int_{t_- < s_- < t} e^{i(t-s_-)\Delta} G(s_-)\, ds_- = u_{hi}(t) - u_{hi}^-(t)$$

and

$$i \int_{t < s_+ < t_+} e^{i(t-s_+)\Delta} G(s_+)\, ds_+ = u_{hi}(t) - u_{hi}^+(t).$$

Then we multiply the first identity by the conjugate of the second to obtain

$$\begin{aligned}
&- \int\int_{t_- < s_- < t < s_+ < t_+} (e^{i(t-s_-)\Delta} G(s_-)) \overline{(e^{i(t-s_+)\Delta} G(s_+))}\, ds_+\, ds_- \\
&\qquad = |u_{hi}(t)|^2 - u_{hi}^-(t)\overline{u_{hi}(t)} - u_{hi}(t)\overline{u_{hi}^+(t)} + u_{hi}^-(t)\overline{u_{hi}^+(t)}.
\end{aligned}$$

From the elementary pointwise estimates

$$\begin{aligned}
|u_{hi}^-(t)\overline{u_{hi}(t)}| &\leq \frac{1}{4}|u_{hi}(t)|^2 + O(|u_{hi}^-(t)|^2), \\
|u_{hi}(t)\overline{u_{hi}^+(t)}| &\leq \frac{1}{4}|u_{hi}(t)|^2 + O(|u_{hi}^+(t)|^2), \\
|u_{hi}^-(t)\overline{u_{hi}^+(t)}| &\leq O(|u_{hi}^-(t)|^2) + O(|u_{hi}^+(t)|^2)
\end{aligned}$$

we thus have the pointwise inequality

$$(14.3)\quad |u_{hi}(t)|^2 \lesssim \Big| \int\int_{t_-<s_-<t<s_+<t_+} e^{i(t-s_-)\Delta}G(s_-)\overline{e^{i(t-s_+)\Delta}G(s_+)}\ ds_-ds_+\Big| + |u_{hi}^-(t)|^2 + |u_{hi}^+(t)|^2.$$

This should be compared with what one would obtain with a single Duhamel formula (1.13), namely something like

$$|u_{hi}(t)|^2 \lesssim \Big| \int\int_{t_-<s,s'<t} e^{i(t-s)\Delta}G(s)\overline{e^{i(t-s')\Delta}G(s')}\ dsds'\Big| + |u_{hi}^-(t)|^2.$$

This turns out to be an inferior formulation; the basic problem is that the integral $\int_{t_-<s,s'<t} \frac{dsds'}{|s-s'|}$ is logarithmically divergent, whereas the integral $\int_{t_-<s_-<t<s_+<t_+} \frac{ds_-ds_+}{|s_--s_+|}$ is not.

We now return to (14.2), and insert (14.3). The latter two terms were already shown to be acceptable. So we are left to prove that

$$(14.4)\quad \frac{1}{R}\Big| \int\int\int_{t_-<s_-<t<s_+<t_+} \sum_{x\in\frac{R}{100}\mathbb{Z}^3} c(t,x)\int_{\mathbb{R}^3} e^{i(t-s_-)\Delta}G(s_-)(y)\overline{e^{i(t-s_+)\Delta}G(s_+)}(y) \cdot\psi(\frac{y-x}{R})\ dyds_-ds_+dt\Big| \lesssim C_0\eta_1.$$

To compute the y integral, we use the following stationary phase estimate.

Lemma 14.1. *For any $t_- < s_- < t < s_+ < t_+$, and any Schwartz functions $f_-(x), f_+(x)$,*

$$\Big| \sum_{x\in\frac{R}{100}\mathbb{Z}^3} c(t,x)\int_{\mathbb{R}^3} e^{i(t-s_-)\Delta}f_-(y)\overline{e^{i(t-s_+)\Delta}f_+(y)}\psi(\frac{y-x}{R})\ dy\Big| \lesssim |s_+-s_-|^{-3/2}\min(\theta^{-3/2+3/100},1)\|f_-\|_{L^1_x}\|f_+\|_{L^1_x}$$

where $\theta := \frac{|t-s_+||t-s_-|}{R^2|s_+-s_-|}$.

Proof. Fix t. We use the explicit formula for the fundamental solution (1.11) to estimate the left-hand side as

$$\lesssim \frac{1}{|t-s_+|^{3/2}|t-s_-|^{3/2}} \sum_{x\in\frac{R}{100}\mathbb{Z}^3} c(t,x)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} e^{i(|y-x_-|^2/(t-s_-)-|y-x_+|^2/(t-s_+))}\psi\left(\frac{y-x}{R}\right) dy\right) f_+(x_+)\overline{f_-(x_-)}\ dx_+dx_-,$$

so it suffices to show that

$$\Big| \sum_{x \in \frac{R}{100}\mathbb{Z}^3} c(t,x) \int_{\mathbb{R}^3} e^{i(|y-x_-|^2/(t-s_-) - |y-x_+|^2/(t-s_+))} \psi(\frac{y-x}{R}) dy \Big|$$
$$\lesssim \frac{|t-s_+|^{3/2}|t-s_-|^{3/2}}{|s_+-s_-|^{3/2}} \min(\theta^{-3/2+3/100}, 1)$$
$$= R^3 \min(\theta^{3/100}, \theta^{3/2})$$

for all x_-, x_+. With the change of variables $y = x + Rz$, this becomes

$$\Big| \sum_{x \in \frac{R}{100}\mathbb{Z}^3} c(t,x) I(x) \Big| \lesssim \min(\theta^{3/100}, \theta^{3/2}),$$

where the integrals $I(x)$ are defined by

$$I(x) := \int_{\mathbb{R}^3} e^{i\Phi_x(z)} \psi(z) dz$$

and $\Phi_x = \Phi_{x,R,x_-,x_+,s_-,s_+,t}$ is the phase

$$\Phi_x(z) := |x - x_- + Rz|^2/(t-s_-) - |x - x_+ + Rz|^2/(t-s_+).$$

By the normalization of $c(t,x)$, it thus suffices to prove the bounds

$$\Big(\sum_{x \in \frac{R}{100}\mathbb{Z}^3} I(x)^{100} \Big)^{1/100} \lesssim \min(\theta^{3/100}, \theta^{3/2}).$$

We now divide into two cases, depending on the size of θ. First suppose that $\theta \gg 1$. Observe that the gradient of the phase Φ_x in z is

$$\begin{aligned} \nabla_z \Phi_x(z) &= 2R(x - x_- + Rz)/(t-s_-) - 2R(x - x_+ + Rz)/(t-s_+) \\ &= 2R(x + Rz - x_*) \frac{s_- - s_+}{(t-s_-)(t-s_+)} \end{aligned}$$

where $x_* = x_*(s_-, s_+, t, x_-, x_+, R)$ is a quantity not depending on x or z. In particular, in the region where

$$|x - x_*| \gg \frac{|t-s_-||t-s_+|}{R|s_+-s_-|} = R\theta \gg R$$

we can obtain an extremely good bound from the principle of nonstationary phase,[34] namely that

$$|I(x)| \lesssim \Big(\frac{|x-x_*|}{R\theta} \Big)^{-100}. \tag{14.5}$$

[34] In other words, one can use repeated integration by parts; see [41]. An alternate approach in this lemma is to use a Gaussian cutoff $e^{-\pi|x|^2}$ instead of ψ, and then compute all the integrals explicitly by contour integration (or equivalently by using the "Gaussian beam" solutions of the Schrödinger equation).

In the remaining cases where

$$|x - x_*| \lesssim R\theta$$

(note that there are $O(\theta^3)$ such cases), we use the crude bound

$$|I(x)| \lesssim 1,$$

and obtain the final estimate

$$\Big(\sum_{x \in \frac{R}{100}\mathbb{Z}^3} I(x)^{100} \Big)^{1/100} \lesssim \theta^{3/100}$$

as desired. Now consider the case $\theta \lesssim 1$. In the region

$$|x - x_*| \gg R$$

one can get very good bounds from nonstationary phase again, namely (14.5). There are only $O(1)$ remaining values of x. For each of these values we observe that the double derivative $\nabla_z^2 \Phi_x$ is nondegenerate, indeed it is equal to $2R^2 \frac{s_- - s_+}{(t-s_-)(t-s_+)} = \frac{2}{\theta}$ times the identity matrix. Thus by the principle of stationary phase (see [41]) we have

$$|I(x)| \lesssim \theta^{3/2}$$

in these cases. Upon summing we obtain

$$\Big(\sum_{x \in \frac{R}{100}\mathbb{Z}^3} I(x)^{100} \Big)^{1/100} \lesssim \theta^{3/2}$$

as claimed. □

Using Lemma 14.1, we can estimate the left-hand side of (14.4) as

(14.6)
$$\begin{aligned} &\lesssim \int\int\int_{t_- < s_- < t < s_+ < t_+} \frac{1}{R} |s_+ - s_-|^{-3/2} \\ &\quad \cdot \min\left(\left(\frac{|t - s_+||t - s_-|}{R^2 |s_+ - s_-|} \right)^{-3/2 + 3/100}, 1 \right) \|G(s_-)\|_{L^1_x} \|G(s_+)\|_{L^1_x} \, ds_- ds_+ dt. \end{aligned}$$

Now we use the crucial time ordering $s_- < t < s_+$. An elementary computation (treating the cases $s_+ - s_- < R^2$ and $s_+ - s_- \geq R^2$ separately) shows that

$$\begin{aligned} \frac{1}{R} \int_{s_- < t < s_+} |s_+ - s_-|^{-3/2} \min\left(\left(\frac{|t - s_+||t - s_-|}{R^2 |s_+ - s_-|} \right)^{-3/2 + 3/100}, 1 \right) \, dt \\ \lesssim \min(R |s_+ - s_-|^{-3/2}, R^{-1} |s_+ - s_-|^{-1/2}). \end{aligned}$$

The kernel $\min(R|s|^{-3/2}, R^{-1}|s|^{-1/2})$ has an L^1_s norm of[35] $O(1)$. Thus by Young's inequality in time (and Cauchy-Schwarz in time), we can bound (14.6) by

$$\|G\|^2_{L^2_t L^1_x(I_0 \times \mathbb{R}^3)}$$

which is $O(C_0\eta_1)$ by Proposition 12.2, as desired. This (finally!) concludes the proof of Proposition 4.9. □

15. Preventing energy evacuation

We now prove Proposition 4.15. By the scaling (1.3) we may take $N_{\min} = 1$.

15.1. *The setup and contradiction argument.* Since the $N(t)$ can only take values in a discrete set (the integer powers of 2), there thus exists a time $t_{\min} \in I_0$ such that

$$N(t_{\min}) = N_{\min} = 1.$$

At this time $t = t_{\min}$, we see from (4.11) and (1.18) that we have a substantial amount of mass[36] (and energy) at medium frequencies:

$$\|P_{c(\eta_0) \leq \cdot \leq C(\eta_0)} u(t_{\min})\|_{L^2(\mathbb{R}^3)} \geq c(\eta_0). \tag{15.1}$$

This should be contrasted with (4.5), which shows that there is not much mass at the frequencies much higher than $C(\eta_0)$.

Our task is to prove (4.24). Suppose for contradiction that this estimate failed; then there exists a time $t_{\text{evac}} \in I_0$ for which $N(t_{\text{evac}}) \gg C(\eta_5)$. If $C(\eta_5)$ is sufficiently large, we then see from Corollary 4.4 that energy has been almost entirely evacuated from low and medium frequencies at time t_{evac}:

$$\|P_{<1/\eta_5} u(t_{\text{evac}})\|_{\dot{H}^1(\mathbb{R}^3)} \leq \eta_5. \tag{15.2}$$

Under the intuition that the L^2 mass density does not rapidly adjust during the NLS evolution, (15.2) will be contradicted if the low frequency L^2 mass (15.1) sticks around $N_{\min}$ until t_{evac}. We will validate this slow L^2 mass motion intuition by proving a frequency localized L^2 mass almost conservation law which leads to the contradiction, provided that the η_j are chosen small enough.

[35] It is crucial to note here that the powers of R have cancelled out. This seems to be a consequence of dimensional analysis, although the presence of the frequency localization $|\xi| \gtrsim 1$ makes this analysis heuristic rather than rigorous.

[36] Note that we are *not* using the assumption that u is Schwartz (and thus has finite L^2 norm) to obtain these estimates; the bounds here are independent of the global L^2 norm of u, which may be very large even for fixed energy E because the very low frequencies can simultaneously have small energy and large mass, and are also not preserved by the scale-invariance which we have exploited to normalize $N_{\min} = 1$.

Having surveyed the argument, we carry out the details. Fix t_{evac}; by time reversal symmetry we may assume that $t_{\text{evac}} > t_{\min}$. (From (15.1) it is clear that t_{evac} cannot equal $t_{\min}$).

From (15.1) we have

$$\|u_{>c(\eta_0)}(t_{\min})\|_{L^2(\mathbb{R}^3)} \geq \eta_1$$

(for instance). In particular, if

$$P_{hi} := P_{\geq \eta_4^{100}}, P_{lo} := P_{<\eta_4^{100}}, u_{hi} := P_{hi}u, u_{lo} := P_{lo}u,$$

then we have

$$\|u_{hi}(t_{\min})\|_{L^2(\mathbb{R}^3)} \geq \eta_1. \tag{15.3}$$

Suppose we could show that

$$\|u_{hi}(t_{\text{evac}})\|_{L^2(\mathbb{R}^3)} \geq \frac{1}{2}\eta_1. \tag{15.4}$$

From (4.3) and (4.5) we would thus have

$$\|P_{\leq C(\eta_1)}u_{hi}(t_{\text{evac}})\|_{L^2(\mathbb{R}^3)} \geq \frac{1}{4}\eta_1.$$

By (1.16) this implies that

$$\|P_{\leq C(\eta_1)}u(t_{\text{evac}})\|_{\dot{H}^1(\mathbb{R}^3)} \gtrsim c(\eta_1, \eta_4).$$

But then Corollary 4.4 implies that $N(t_{\text{evac}}) \lesssim C(\eta_1, \eta_4)$, which contradicts (15.2) if η_5 is chosen sufficiently small.

It remains to prove (15.4). We use the continuity method. Suppose we have a time $t_{\min} \leq t_* \leq t_{\text{evac}}$ for which

$$\inf_{t_{\min} \leq t \leq t_*} \|u_{hi}(t)\|_{L^2(\mathbb{R}^3)} \geq \frac{1}{2}\eta_1. \tag{15.5}$$

We will show that (15.5) can be bootstrapped to

$$\inf_{t_{\min} \leq t \leq t_*} \|u_{hi}(t)\|_{L^2(\mathbb{R}^3)} \geq \frac{3}{4}\eta_1. \tag{15.6}$$

This implies that the set of times t_* for which (15.5) holds is both open and closed in $[t_{\min}, t_{\text{evac}}]$, which will imply (15.4) as desired. Note that the introduction of $\eta_4^{100} \ll N_{\min}$ allows for the L^2 mass near $N_{\min}$ at time $t_{\min}$ to move toward low frequencies but the estimate (15.6) shows a portion of the mass stays above η_4^{100} for $t \in [t_{\min}, t_{\text{evac}}]$. Note also that upon proving (15.6), we will have established Lemma 4.14.

It remains to derive (15.6) from (15.5). The idea is to treat the L^2 norm of $u_{hi}(t)$; i.e.,

$$L(t) := \int_{\mathbb{R}^3} |u_{hi}(t,x)|^2 \, dx,$$

as an almost conserved quantity. From (15.3) we have

$$L(t_{\min}) \geq \eta_1^2,$$

and so by the Fundamental Theorem of Calculus it will suffice to show that

$$\int_{t_{\min}}^{t_*} |\partial_t L(t)| \; dt \leq \frac{1}{100}\eta_1^2.$$

From (2.4) and (2.6) we have

$$\begin{aligned} \partial_t L(t) &= 2\int \{P_{hi}(|u|^4 u), u_{hi}\}_m \; dx \\ &= 2\int \{P_{hi}(|u|^4 u) - |u_{hi}|^4 u_{hi}, u_{hi}\}_m \; dx. \end{aligned}$$

Thus it will suffice to show that

$$\int_{t_{\min}}^{t_*} \left| \int \{P_{hi}(|u|^4 u) - |u_{hi}|^4 u_{hi}, u_{hi}\}_m \; dx \right| dt \leq \frac{1}{100}\eta_1^2. \tag{15.7}$$

The proof of (15.7) is accomplished using a quintilinear analysis of various interactions using three inputs: $\dot{S}^1$ Strichartz estimates on low frequencies, L^4_{xt} estimates via frequency localized interaction Morawetz and Bernstein estimates on medium and higher frequencies, and $\dot{S}^1$ Strichartz estimates on very high frequencies.

15.2. *Spacetime estimates for high, medium, and low frequencies.* To prove (15.7) we need a number of spacetime bounds on u, which we now discuss in this subsection. Observe by the discussion following (15.4) that the hypothesis (15.5) implies in particular that

$$N(t) \leq C(\eta_1, \eta_4) \text{ for all } t_{\min} \leq t \leq t_*.$$

This in turn can be combined with Corollary 4.13 to obtain the useful Strichartz bounds

$$\|u\|_{\dot{S}^1([t_{\min}, t_*] \times \mathbb{R}^3)} \leq C(\eta_1, \eta_3, \eta_4). \tag{15.8}$$

Because of the dependence of the right-hand side on η_4, this bound is only useful for us when there is a power of η_5 present. In all other circumstances, we resort instead to Proposition 4.9, which gives the $L^4_{t,x}$ bounds[37]

$$\int_{t_{\min}}^{t_{\text{evac}}} \int |P_{\geq N} u(t,x)|^4 \; dxdt \lesssim \eta_1 N^{-3} \tag{15.9}$$

[37] It is important here to note that while the L^2 control on u_{hi} only extends from $t_{\min}$ up to t_*, the $L^4_{t,x}$ control on $u_{\geq N}$ extends all the way up to t_{evac}. This allows us to access the evacuation hypothesis (15.2) to provide useful new control, especially at low frequencies, in the time interval $[t_{\min}, t_*]$. This additional control will be crucial for us to obtain the desired almost conservation law on the mass of u_{hi}, thus closing the bootstrap and allowing t_* to extend all the way up to t_{evac}, at which point we can declare a contradiction.

whenever $N < c(\eta_3)$. Roughly speaking, the bound (15.9) is better than (15.8) for medium frequencies, but (15.8) is superior for very high frequencies, and Lemma 15.1 below will be superior for very low frequencies.

It turns out that we also need some bounds on the low frequencies such as $u_{\leq \eta_4}$; the estimate (15.9) is inappropriate for this purpose because N^{-3} diverges as $N \to 0$. One can modify Proposition 12.1 to obtain some reasonable control, but it turns out that the constants obtained by that estimate are not strong enough to counteract the losses arising from (15.8). We need a stronger version of Proposition 12.1 which takes advantage of the evacuation hypothesis (15.2), which asserts among other things that $u_{\leq \eta_4}$ has extremely small energy at time t_{evac}. Because of this hypothesis we expect $u_{\leq \eta_4}$ to have extremely small energy at all other times in $[t_{\min}, t_{\mathrm{evac}}]$ (i.e. we expect bounds gaining an η_5 instead of just an η_3). Of course there is a little bit of energy leaking from the high frequencies to the low, but fortunately the $L^4_{t,x}$ bound on the high frequencies will limit[38] how far the high frequency energy can penetrate to the very low modes. This intuition is made rigorous as follows:

Lemma 15.1. *With the above notation and assumptions* (*in particular, assuming* (15.2) *and* (15.9)) *we have*

$$\|P_{\leq N} u\|_{\dot{S}^1([t_{\min}, t_{\mathrm{evac}}] \times \mathbb{R}^3)} \lesssim \eta_5 + \eta_4^{-3/2} N^{3/2} \tag{15.10}$$

for all $N \leq \eta_4$.

One should think of the $C\eta_5$ term on the right-hand side of (15.10) as the energy coming from the low modes of $u(t_{\mathrm{evac}})$, while the $\eta_4^{-3/2} N^{3/2}$ term comes from the nonlinear corrections generated by the high modes of $u(t)$ for $t_{\min} \leq t \leq t_{\mathrm{evac}}$. Note the very strong decay of $N^{3/2}$ as $N \to 0$; this means that the high modes cannot project their energy very far into the low modes. This estimate should be compared with Proposition 12.1. It begins to deteriorate if N gets too close to η_4; we avoid this problem by making the high-low frequency decomposition $u = u_{hi} + u_{lo}$ at η_4^{100} instead of η_4. Note that this bound is superior to either (15.8) or (15.9) at low frequencies.

Proof. As usual, we rely on the continuity method, although now we will evolve[39] *backwards* in time from t_{evac}, rather than forwards from $t_{\min}$. Let C_0

[38]It may seem surprising that the $L^4_{t,x}$ bound, which is supercritical, can lead to control on critical quantities such as the energy. The point is that once one localizes in frequency, the distinctions between subcritical, critical, and supercritical quantities become less relevant, as one can already see from Bernstein's inequality (1.20). In this section the entire analysis is localized around the frequency $N_{\min} = 1$, so that supercritical norms (such as $L^4_{t,x}$ or $L^\infty_t L^2_x$) can begin to play a useful role.

[39]The arguments in this section seem to rely in an essential way on evolving both forwards and backwards in time simultaneously; compare with the "double Duhamel trick" in Section 14.

be a large absolute constant (not depending on any of the η_j) to be chosen later. Let $\Omega \subseteq [t_{\min}, t_{\text{evac}})$ denote the set of all times $t_{\min} \leq t < t_{\text{evac}}$ such that we have the bounds

$$\|P_{\leq N} u\|_{\dot{S}^1([t,t_{\text{evac}}]\times\mathbb{R}^3)} \leq C_0\eta_5 + \eta_0\eta_4^{-3/2}N^{3/2} \tag{15.11}$$

for all $N \leq \eta_4$. To show (15.10), it will clearly suffice to show that $t_{\min} \in \Omega$ (the additional factor of η_0 is useful for the continuity method but will be discarded for the final estimate (15.10)). In particular, we observe from (15.11) that we have

$$\|P_{\leq N} u\|_{\dot{S}^1([t,t_{\text{evac}}]\times\mathbb{R}^3)} \lesssim \eta_0 \tag{15.12}$$

for all $N \leq \eta_4$.

First we show that $t \in \Omega$ for t sufficiently close to t_{evac}. We use (3.1), Hölder and Sobolev to estimate

$$\begin{aligned}\|P_{\leq N} u\|_{\dot{S}^1([t,t_{\text{evac}}]\times\mathbb{R}^3)} &\lesssim \|\nabla P_{\leq N} u\|_{L_t^\infty L_x^2([t,t_{\text{evac}}]\times\mathbb{R}^3)} + \|\nabla u\|_{L_t^2 L_x^6([t,t_{\text{evac}}]\times\mathbb{R}^3)} \\ &\lesssim \|\nabla P_{\leq N} u(t_{\text{evac}})\|_{L^2} + C|t_{\text{evac}} - t|\|\nabla\partial_t u\|_{L_t^\infty L_x^2(I_0\times\mathbb{R}^3)} \\ &\quad + |t_{\text{evac}} - t|^{1/2}\|\nabla u\|_{L_t^\infty L_x^6(I_0\times\mathbb{R}^3)}.\end{aligned}$$

Since u is Schwartz, the latter two norms are finite (though perhaps very large). By (15.2) we thus have the estimate

$$\|P_{\leq N} u\|_{\dot{S}^1([t,t_{\text{evac}}]\times\mathbb{R}^3)} \lesssim \eta_5 + C(I_0,u)|t_{\text{evac}} - t| + C(I_0,u)|t_{\text{evac}} - t|^{1/2}.$$

We now see that the bound (15.11) holds for t sufficiently close to t_{evac}, if C_0 is chosen large enough (but not depending on I_0, u or any of the η_j.)

Now suppose that $t \in \Omega$, so that (15.11) holds for all $N \leq \eta_4$. We shall bootstrap (15.11) to

$$\|P_{\leq N} u\|_{\dot{S}^1([t,t_{\text{evac}}]\times\mathbb{R}^3)} \leq \frac{1}{2}C_0\eta_5 + \frac{1}{2}\eta_0\eta_4^{-3/2}N^{3/2} \tag{15.13}$$

for all $N \leq \eta_4$. If this claim is true, this would imply (since u is Schwartz) that Ω is both open and closed, and so we have $t_{\min} \in \Omega$ as desired.

It remains to deduce (15.13) from (15.11). For the rest of the proof, all spacetime norms will be restricted to $[t, t_{\text{evac}}] \times \mathbb{R}^3$.

Fix $N \leq \eta_4$. By (1.1) and Lemma 3.2,

$$\|P_{\leq N} u\|_{\dot{S}^1([t,t_{\text{evac}}]\times\mathbb{R}^3)} \lesssim \|P_{\leq N} u(t_{\text{evac}})\|_{\dot{H}^1(\mathbb{R}^3)} + C\sum_{m=1}^{M} \|\nabla F_m\|_{L_t^{q'_m} L_x^{r'_m}([t,t_{\text{evac}}]\times\mathbb{R}^3)}$$

for some dual L^2-admissible exponents (q'_m, r'_m), and some decomposition $P_{\leq N}(|u|^4 u) = \sum_{m=1}^M F_m$ which we will give shortly.

From (15.2) we have

$$\|P_{\leq N} u(t_{\text{evac}})\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \eta_5,$$

which is acceptable for (15.13) if C_0 is large enough.

Now consider the nonlinear term $P_{\leq N}(|u|^4 u)$. We split u into high and low frequencies $u = u_{\leq\eta_4} + u_{>\eta_4}$, where of course $u_{\leq\eta_4} := P_{\leq\eta_4} u$ and $u_{>\eta_4} := P_{>\eta_4} u$, and use (1.15) to decompose

$$P_{\leq N}(|u|^4 u) = \sum_{j=0}^{5} F_j,$$

where $F_j := P_{\leq N}\mathcal{O}(u_{>\eta_4}^j u_{\leq\eta_4}^{5-j})$. We now treat the various terms separately.

- *Case* 1. *Estimation of* F_2, F_3, F_4, F_5. We estimate these terms in $L_t^2 L_x^{6/5}$ norm. We use Bernstein's inequality (1.19) to bound these terms by

$$CN^{3/2}\|\mathcal{O}(u_{>\eta_4}^j u_{\leq\eta_4}^{5-j})\|_{L_t^2 L_x^1([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)},$$

which by Hölder can be estimated by

$$CN^{3/2}\|u_{>\eta_4}\|_{L_t^\infty L_x^6([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}^{j-2}\|u_{\leq\eta_4}\|_{L_t^\infty L_x^6([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}^{5-j}\|u_{>\eta_4}\|_{L_t^4 L_x^4([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}^2.$$

Applying (4.4) and Sobolev, as well as (15.9), we obtain a bound of

$$C\eta_1^{\frac{1}{2}} N^{3/2}\eta_4^{-3/2},$$

which is acceptable for (15.13) if η_1 is sufficiently small. It is at this step that the small constant η_1 appearing in 4.19 is used to close the bootstrap.

- *Case* 2a. *Estimation of* F_1 *when* $N \ll \eta_4$. Now consider F_1 term

$$\|\nabla P_{\leq N}\mathcal{O}(u_{>\eta_4} u_{\leq\eta_4}^4)\|_{L_t^{q_1'} L_x^{r_1'}}. \tag{15.14}$$

Suppose first that $N < c\eta_4$. Then this expression vanishes unless at least one of the four $u_{\leq\eta_4}$ factors has frequency $> c\eta_4$. Thus we can essentially write (15.14) as[40]

$$\|\nabla P_{\leq N}\mathcal{O}(u_{\leq\eta_4}^3 (P_{>c\eta_4} u_{\leq\eta_4}) u_{>\eta_4})\|_{L_t^2 L_x^{6/5}([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)},$$

where we have chosen $(q_1', r_1') = (2, 6/5)$. By (15.9), the function $P_{\geq c\eta_4} u_{\leq\eta_4}$ obeys essentially the same $L_{t,x}^4$ estimates as $u_{>\eta_4}$, and so this term is acceptable when we repeat the arguments used to deal with F_2, F_3, F_4, F_5.

- *Case* 2b. *Estimation of* F_1 *when* $N \sim \eta_4$. Considering the case when $N \geq c\eta_4$, we choose $(q_1', r_1') = (1, 2)$ and use (1.18) to bound (15.14) by

$$C\eta_4\|\mathcal{O}(u_{>\eta_4} u_{\leq\eta_4}^4)\|_{L_t^1 L_x^2([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}$$

[40]Strictly speaking, one can only write (15.14) as a sum of such terms, where some of the $u_{\leq\eta_4}$ factors in $u_{\leq\eta_4}^3$ must be replaced by either $P_{>c\eta_4} u_{\leq\eta_4}$ or $P_{\leq c\eta_4} u_{\leq\eta_4}$. As these projections are bounded on every space under consideration we ignore this technicality. Similarly for other such decompositions in this lemma.

which we estimate using Hölder by

$$C\eta_4 \|u_{\leq \eta_4}\|^4_{L^4_t L^\infty_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} \|u_{>\eta_4}\|_{L^\infty_t L^2_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}.$$

From (4.3) we obtain

$$\|u_{>\eta_4}\|_{L^\infty_t L^2_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} \leq C\eta_4^{-1},$$

while from (3.4) and (15.12) we obtain

$$\|u_{\leq \eta_4}\|_{L^4_t L^\infty_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} \lesssim \|u_{\leq \eta_4}\|_{\dot{S}^1([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} \lesssim \eta_0.$$

Thus we can bound (15.14) by $O(\eta_0^4)$, which is acceptable for (15.13) since $N \sim \eta_4$. This concludes the estimation of F_1.

- *Case* 3. *Estimation of* F_0. It remains to consider the F_0 term; we set $(q_0', r_0') = (1,2)$ and estimate

$$\|\nabla P_{\leq N} \mathcal{O}(u^5_{\leq \eta_4})\|_{L^1_t L^2_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}.$$

We split $u_{\leq \eta_4} = u_{<\eta_5} + u_{\eta_5 \leq \cdot \leq \eta_4}$, and consider any term which involves the very low frequencies $u_{<\eta_5}$; schematically, this is

$$\|\nabla P_{\leq N} \mathcal{O}(u^4_{\leq \eta_4} u_{<\eta_5})\|_{L^1_t L^2_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}.$$

For this case we discard the $P_{\leq N}$, and apply Lemma 3.6 to estimate this term by

$$\lesssim \|u_{\leq \eta_4}\|^4_{\dot{S}^1([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} \|u_{<\eta_5}\|_{\dot{S}^1([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}.$$

Applying (15.11) we can bound this by

$$\lesssim (C_0\eta_5 + \eta_0)^4 (C_0 \eta_5 + \eta_0 \eta_4^{-3/2} \eta_5^{3/2}) \lesssim C_0 \eta_0^4 \eta_5$$

which is acceptable.

We can thus discard all the terms involving $u_{<\eta_5}$, and reduce to estimating

$$\|\nabla P_{\leq N} \mathcal{O}(u^5_{\eta_5 \leq \cdot \leq \eta_4})\|_{L^1_t L^2_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}. \tag{15.15}$$

By Bernstein (1.19), (1.17) we may estimate this by

$$\lesssim N^{3/2} \|\mathcal{O}(u^5_{\eta_5 \leq \cdot \leq \eta_4})\|_{L^1_t L^{3/2}_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} = N^{3/2} \|u_{\eta_5 \leq \cdot \leq \eta_4}\|^5_{L^5_t L^{15/2}_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)}.$$

But from (3.4), Bernstein (1.20), (1.18), and (15.11) we have

$$\begin{aligned}
\|u_{\eta_5 \leq \cdot \leq \eta_4}\|_{L^5_t L^{15/2}_x} &\leq \sum_{\eta_5 \leq N' \leq \eta_4} \|P_{N'} u\|_{L^5_t L^{15/2}_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} \\
&\lesssim \sum_{\eta_5 \leq N' \leq \eta_4} (N')^{-3/10} \|\nabla P_{N'} u\|_{L^5_t L^{30/11}_x([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} \\
&\lesssim \sum_{\eta_5 \leq N' \leq \eta_4} (N')^{-3/10} \|P_{N'} u\|_{\dot{S}^1([t,t_{\mathrm{evac}}]\times\mathbb{R}^3)} \\
&\lesssim \sum_{\eta_5 \leq N' \leq \eta_4} (N')^{-3/10} (C_0 \eta_5 + \eta_0 \eta_4^{-3/2} (N')^{3/2}) \\
&\lesssim \eta_0 \eta_4^{-3/10},
\end{aligned}$$

so that (15.15) is estimated by $O(\eta_0^5 \eta_4^{-3/2} N^{3/2})$ which is acceptable if η_0 is small enough. This proves (15.13), which closes the bootstrap. $\square$

15.3. *Controlling the localized L^2 mass increment.* Now we have enough estimates to prove (15.7). We first rewrite

$$P_{hi}(|u|^4 u) - |u_{hi}|^4 u_{hi} = P_{hi}(|u|^4 u - |u_{hi}|^4 u_{hi} - |u_{lo}|^4 u_{lo}) + P_{hi}(|u_{lo}|^4 u_{lo}) - P_{lo}(|u_{hi}|^4 u_{hi}).$$

Now, it will suffice to consider the three quantities

$$\int_{t_{\min}}^{t_*} \left| \int \overline{u_{hi}} P_{hi}(|u|^4 u - |u_{hi}|^4 u_{hi} - |u_{lo}|^4 u_{lo}) \, dx \right| dt, \tag{15.16}$$

$$\int_{t_{\min}}^{t_*} \left| \int \overline{u_{hi}} P_{hi}(|u_{lo}|^4 u_{lo}) \, dx \right| dt, \tag{15.17}$$

$$\int_{t_{\min}}^{t_*} \left| \int \overline{u_{hi}} P_{lo}(|u_{hi}|^4 u_{hi}) \, dx \right| dt, \tag{15.18}$$

which we shall estimate below: (15.18).

- *Case* 1. *Estimation of* (15.16). We move the self-adjoint operator P_{hi} onto $\overline{u_{hi}}$, and then apply the pointwise estimate (cf. (1.15))

$$\big||u|^4 u - |u_{hi}|^4 u_{hi} - |u_{lo}|^4 u_{lo}\big| \lesssim |u_{hi}|^4 |u_{lo}| + |u_{hi}||u_{lo}|^4$$

to bound (15.16) by

$$(15.16) \lesssim \int_{t_{\min}}^{t_*} \int |P_{hi} u_{hi}|(|u_{hi}|^4 |u_{lo}| + |u_{hi}||u_{lo}|^4) \, dxdt.$$

For notational convenience, we will ignore the P_{hi} projection and write $P_{hi} u_{hi}$ as u_{hi}; strictly speaking this is not quite accurate but as $P_{hi} u_{hi}$ obeys all the same estimates as u_{hi}, and we have already placed absolute values everywhere, this is a harmless modification. We can now write our bound for (15.16) as

$$(15.16) \lesssim \int_{t_{\min}}^{t_*} \int |u_{hi}|^5 |u_{lo}| + |u_{hi}|^2 |u_{lo}|^4 \, dxdt. \tag{15.19}$$

- *Case* 1a. *Contribution of* $|u_{hi}|^2 |u_{lo}|^4$. Consider first the contribution of $|u_{hi}|^2 |u_{lo}|^4$; we have to show that

$$\int_{t_{\min}}^{t_*} \int |u_{hi}|^2 |u_{lo}|^4 dxdt \ll \eta_1^2. \tag{15.20}$$

By Hölder we can bound this contribution by

$$(15.20) \lesssim \|u_{hi}\|^2_{L_t^\infty L_x^2([t_{\min}, t_*] \times \mathbb{R}^3)} \|u_{lo}\|^4_{L_t^4 L_x^\infty([t_{\min}, t_*] \times \mathbb{R}^3)}.$$

From (15.10) we have

$$\|u_{lo}\|_{L_t^4 L_x^\infty([t_{\min}, t_*] \times \mathbb{R}^3)} \lesssim \eta_4^{3/2}$$

while from (4.3), (15.10) we have

$$\begin{aligned}
\|u_{hi}\|_{L_t^\infty L_x^2([t_{\min},t_*]\times\mathbb{R}^3)} &\leq \|P_{\geq\eta_4} u_{hi}\|_{L_t^\infty L_x^2([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\quad + \sum_{\eta_4^{100}\leq N\leq\eta_4} \|P_N u_{hi}\|_{L_t^\infty L_x^2([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\lesssim \eta_4^{-1}\|u\|_{L_t^\infty \dot H_x^1([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\quad + \sum_{\eta_4^{100}\leq N\leq C\eta_4} N^{-1}\|P_N \nabla u_{hi}\|_{L_t^\infty L_x^2([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\lesssim \eta_4^{-1} + \sum_{\eta_4^{100}\leq N\leq\eta_4} N^{-1}\|P_N u_{hi}\|_{\dot S^1([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\lesssim \eta_4^{-1} + \sum_{\eta_4^{100}\leq N\leq\eta_4} N^{-1}(\eta_5 + \eta_4^{-3/2}N^{3/2}) \\
&\lesssim \eta_4^{-1} + \eta_5\eta_4^{-100} + \eta_4^{-1} \\
&\lesssim \eta_4^{-1}.
\end{aligned}$$

We thus obtain a bound of $O(\eta_4^4)$, which is acceptable.

- *Case* 1b. *Contribution of* $|u_{hi}|^5|u_{lo}|$. It remains to control the contribution of $|u_{hi}|^5|u_{lo}|$; in other words, we need to show

$$\int_{t_{\min}}^{t_*}\int |u_{hi}|^5|u_{lo}|\,dxdt \ll \eta_1^2. \tag{15.21}$$

This estimate will also be useful in controlling (15.18).

We will split u_{lo} further into somewhat low, and very low, frequency pieces:

$$u_{lo} = P_{>\eta_5^{1/2}} u_{lo} + P_{\leq\eta_5^{1/2}} u.$$

The contribution of the very low frequency piece to (15.21) can be bounded by Sobolev embedding (1.19) and (4.4) as

$$\begin{aligned}
&\int_{t_{\min}}^{t_*}\int |u_{hi}|^5|u_{\leq\eta_5^{1/2}}|\,dxdt \\
&\quad\lesssim \|u_{hi}\|^5_{L_t^5L_x^5([t_{\min},t_*]\times\mathbb{R}^3)}\|u_{\leq\eta_5^{1/2}}\|_{L_t^\infty L_x^\infty([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\quad\lesssim C(\eta_4)\|\nabla u\|^5_{L_t^5L_x^{30/11}([t_{\min},t_*]\times\mathbb{R}^3)}\eta_5^{1/4}\|\nabla u\|_{L_t^\infty L_x^2([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\quad\lesssim C(\eta_4)\eta_5^{1/4}\|u\|^5_{\dot S^1([t_{\min},t_*]\times\mathbb{R}^3)},
\end{aligned}$$

which is acceptable by (15.8). Hence we only need to consider the somewhat low frequencies $P_{>\eta_5^{1/2}} u_{lo}$. By Hölder, we obtain the bound

$$\begin{aligned}
&\||u_{hi}|^5|P_{>\eta_5^{1/2}}u_{lo}|\|_{L^1_{t,x}([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\qquad\leq C\|u_{hi}\|^5_{L_t^{10}L_x^5([t_{\min},t_*]\times\mathbb{R}^3)}\|P_{>\eta_5^{1/2}}u_{lo}\|_{L_t^2L_x^\infty([t_{\min},t_*]\times\mathbb{R}^3)}.
\end{aligned} \tag{15.22}$$

From Bernstein (1.20) and (15.10) we have

$$\begin{aligned}
\|P_{>\eta_5^{1/2}} u_{lo}\|_{L_t^2 L_x^\infty([t_{\min},t_*]\times\mathbb{R}^3)} &\leq \sum_{\eta_5^{1/2}<N\leq\eta_4^{100}} \|P_N u\|_{L_t^2 L_x^\infty([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\leq \sum_{\eta_5^{1/2}<N\leq\eta_4^{100}} N^{-1/2}\|\nabla P_N u\|_{L_t^2 L_x^6([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\leq \sum_{\eta_5^{1/2}<N\leq\eta_4^{100}} N^{-1/2}\|P_N u\|_{\dot{S}^1([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\leq \sum_{\eta_5^{1/2}<N\leq\eta_4^{100}} N^{-1/2}(\eta_5 + \eta_4^{-3/2}N^{3/2}) \\
&\leq C\eta_4^{-3/2}\eta_4^{100}.
\end{aligned}$$

To estimate $\|u_{hi}\|_{L_t^{10}L_x^5([t_{\min},t_*]\times\mathbb{R}^3)}$, we split u_{hi} into the higher frequencies $u_{>\eta_4}$ and the medium frequencies $u_{\eta_4^{100}\leq\cdot\leq\eta_4}$. For the higher frequencies we use[41] (15.9), (4.4), and Hölder to obtain

$$\begin{aligned}
\|u_{>\eta_4}\|_{L_t^{10}L_x^5([t_{\min},t_*]\times\mathbb{R}^3)} &\lesssim \|u_{>\eta_4}\|_{L_t^4 L_x^4([t_{\min},t_*]\times\mathbb{R}^3)}^{2/5} \|u_{>\eta_4}\|_{L_t^\infty L_x^6([t_{\min},t_*]\times\mathbb{R}^3)}^{3/5} \\
&\lesssim \eta_4^{-3/10},
\end{aligned}$$

while for the medium frequencies we instead use Bernstein (1.20), (1.18), (3.4) and Lemma 15.1 to estimate

$$\begin{aligned}
\|u_{\eta_4^{100}\leq\cdot\leq\eta_4}\|_{L_t^{10}L_x^5([t_{\min},t_*]\times\mathbb{R}^3)} &\lesssim \sum_{\eta_4^{100}\leq N\leq\eta_4} \|u_N\|_{L_t^{10}L_x^5([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\lesssim \sum_{\eta_4^{100}\leq N\leq\eta_4} N^{-3/10}\|\nabla u_N\|_{L_t^{10}L_x^{30/13}([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\lesssim \sum_{\eta_4^{100}\leq N\leq\eta_4} N^{-3/10}\|u_N\|_{\dot{S}^1([t_{\min},t_*]\times\mathbb{R}^3)} \\
&\lesssim \sum_{\eta_4^{100}\leq N\leq\eta_4} N^{-3/10}(\eta_5 + \eta_4^{-3/2}N^{3/2}) \\
&\lesssim \eta_4^{-3/10}.
\end{aligned}$$

Inserting these bounds into (15.22) we obtain a bound of $\eta_4^{-3}\eta_4^{100}$ for (15.21), which is acceptable.

- *Case* 2. *Estimation of* (15.17). Because of the presence of the P_{hi} projection, one of the u_{lo} terms must have frequency $\geq c\eta_4^{100}$. We then move P_{hi} over to the u_{hi}, bounding (15.17) as a sum of terms which are essentially

[41] Note that this application of (15.9) does not require the small constant η_1.

of the form[42]

$$\int_{t_{\min}}^{t_*} \int |P_{hi} u_{hi}| |P_{\geq c\eta_4^{100}} u_{lo}| |u_{lo}|^4 \, dx dt.$$

Now observe that $P_{hi} u_{hi}$ and $P_{\geq c\eta_4^{100}} u_{lo} = P_{lo} u_{\geq c\eta_4^{100}}$ satisfy essentially the same estimates as u_{hi}, so that this expression can be shown to be acceptable by a minor modification of (15.20).

- *Case* 3. *Estimation of* (15.18). We move projections around, using the identity $P_{lo} u_{hi} = P_{hi} u_{lo}$, to write (15.18) as

$$\int_{t_{\min}}^{t_*} |\int \overline{P_{hi} u_{lo}} |u_{hi}|^4 u_{hi} dx| dt. \tag{15.23}$$

Thus, we are concerned here with a term involving five u_{hi} factors and one $P_{hi} u_{lo}$ factor. But this is basically (15.21), which has already been shown to be acceptable. (We have $P_{hi} u_{lo}$ instead of u_{lo} but the reader may verify that the P_{hi} is harmless since it does not destroy any of the estimates of u_{lo}).

This proves (15.7), and the proof of Proposition 4.15 is complete. This (finally!) concludes the proof of Theorem 1.1. □

16. Remarks

We make here some miscellaneous remarks concerning certain variants of Theorem 1.1.

The global well-posedness result in Theorem 1.1 was asserted with regard to finite energy solutions u in the class $C_t^0 \dot{H}_x^1 \cap L_{t,x}^{10}$, in that the solution existed and was unique in this class, and depended continuously on the initial data (cf. Lemma 3.10). However, the uniqueness result can be strengthened, in the sense that the solution constructed by Theorem 1.1 is in fact the only such solution in the class $C_t^0 \dot{H}_x^1$ (without the assumption of finite $L_{t,x}^{10}$ norm). This type of "unconditional well-posedness" result was first obtained in [26], [27] (see also [15], [14]); the result in [26], [27] was phrased for the sub-critical Schrödinger equation but can be extended to the critical setting thanks to the endpoint Strichartz estimates in [32] (or Lemma 3.2). For the convenience of the reader we sketch here the ideas of this argument, which are essentially in [27], [15], [14], we are indebted to Thierry Cazenave for pointing out the relevance of the endpoint Strichartz estimate to the $\dot{H}^1$-critical uniqueness problem.

Let u_0 be finite energy initial data, and let $u \in C_t^0 \dot{H}_x^1 \cap L_{t,x}^{10}$ be the (global) solution to (1.1) constructed in Theorem 1.1 with these initial data; thus $u(0) = u_0$. Suppose for contradiction that we have another (local or global) solution

[42]Actually, some of the u_{lo} factors in $|u_{lo}|^4$ may have to be replaced by either $P_{\geq c\eta_4^{100}} u_{lo}$ or $P_{< c\eta_4^{100}} u_{lo}$, but this will make no difference to the estimates.

$v \in C_t^0 \dot{H}_x^1$ to (1.1) with initial u_0, in the sense that v verifies the (Duhamel) integral formulation of (1.1),

$$v(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta}(|v|^4 v(s))\, ds.$$

Note that $v \in C_t^0 \dot{H}_x^1 \subseteq C_t^0 L_x^6$ by Sobolev embedding, so that, in particular, the nonlinearity $|v|^4 v$ is locally integrable, and the right-hand side of the above formula makes sense distributionally at least. We now claim that $u \equiv v$ on the entire time interval for which v is defined. Actually, we shall just show that $u \equiv v$ for all times t in a sufficiently small neighborhood I of 0; one can then extend this to the whole time interval by a continuity argument and time translation invariance.

To prove the claim, we write $v = u + w$ and observe that w obeys a difference equation, which we write in integral form as

$$w(t) = -i \int_0^t e^{i(t-s)\Delta}(|u+w|^4(u+w)(s) - |u|^4 u(s))\, ds.$$

Let $\varepsilon > 0$ be a small number to be chosen shortly. Note that $w \in C_t^0 \dot{H}_x^1 \subseteq C_t^0 L_x^6$ and $w(0) = 0$, so that in particular we can ensure that $\|w\|_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \leq \varepsilon$ by choosing I sufficiently small. Also, from the Strichartz analysis u has finite $\dot{S}^1$ norm, and in particular it has finite $L_t^8 L_x^{12}$ norm. Thus we can also ensure that $\|u\|_{L_t^8 L_x^{12}(I\times\mathbb{R}^3)} \leq \varepsilon$ by choosing I sufficiently small. Now we use (1.15) to write the equation for w as

$$w(t) = \int_0^t e^{i(t-s)\Delta}(O(|w(s)|^5) + O(|u(s)|^4|w(s)|))\, ds.$$

We apply Lemma 3.2 with $k = 0$ to conclude in particular that

$$\|w\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)} \leq C\||w|^5\|_{L_t^2 L_x^{6/5}(I\times\mathbb{R}^3)} + C\||u|^4|w|\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)}.$$

From our choice of I and Hölder's inequality we see in particular that

$$\|w\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)} \leq C\varepsilon^4 \|w\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)}.$$

Note that the $L_t^2 L_x^6(I \times \mathbb{R}^3)$ norm of w is finite since $w \in C_t^0 L_x^6$. If we choose ε sufficiently small, we then conclude that w vanishes identically on $I \times \mathbb{R}^3$. One can then extend this vanishing to the entire time interval for which v is defined by a standard continuity argument which we omit.

We now briefly discuss possible extensions to Theorem 1.1. One obvious extension to study would be the natural analogue of Theorem 1.1 in higher dimensions $n > 3$, with the equation (1.1) replaced by its higher-dimensional energy-critical counterpart

$$iu_t + \Delta u = |u|^{\frac{4}{n-2}} u.$$

The four-dimensional case $n = 4$ seems particularly tractable since the nonlinearity is cubic.[43] In higher dimensions $n \geq 5$ one no longer expects a regularity result since the nonlinearity is not smooth when u vanishes. However one might still hope for a global well-posedness result in the energy space (especially since this is already known to be true for small energies; see [9]). In the radial case, such a result was obtained in four dimensions in [4], [5] and more recently in general dimension in [45], and so it is reasonable to conjecture that one in fact has global well-posedness in the energy space for all dimensions $n \geq 3$ and all finite energy data, in analogy with Theorem 1.1. However, extending our arguments here to the higher dimensional setting is far from automatic, even in the four-dimensional case; all the Strichartz numerology changes, of course, but also the interaction Morawetz inequality behave in a somewhat different manner in higher dimensions (since the quantity $\Delta \frac{1}{|x|}$ is no longer a Dirac mass, but instead a fractional integral potential). However, it seems that other parts of the argument, such as the induction on energy machinery, the localization of minimal-energy blowup solutions, and the energy evacuation arguments based on frequency-localized approximate mass conservation laws, do have a good chance of extending to this setting. We will not pursue these matters in detail here.

Another natural extension would be to add a lower order nonlinearity to (1.1), for instance combining the pure power quintic nonlinearity $|u|^4 u$ with a pure power cubic nonlinearity $|u|^2 u$. Heuristically, we do not expect such lower order terms to affect the global well-posedness and regularity of the equation (especially if those terms have the same defocusing sign as the top order term), although they may cause some difficulty in obtaining a scattering result (especially if one adds a nonlinearity of the form $|u|^{p-1} u$ for very low p, such as $p \leq 1 + \frac{4}{n} = \frac{7}{3}$ or $p \leq 1 + \frac{2}{n} = \frac{5}{3}$). However, these lower order terms do create some nontrivial difficulties in our argument, which relies heavily on scale-invariance. One may need to add some lower order terms (such as the L^2 mass) to the energy E, or to the definition of the quantity $M(E)$, in order to salvage the induction on energy argument in this setting. Again, we will not pursue these matters here.[44]

As remarked in Remark 5.3, our final bound $M(E)$ for the global $L^{10}_{t,x}$ norm of u in terms of the energy E is extremely bad; this is due to our extremely heavy reliance on the induction on energy hypothesis (Lemma 4.1) in

[43] Note added in proof: The four-dimensional case has been handled by a very recent preprint of Ryckman and Visan [37], using a modification of the methods here. The case of dimensions five and higher has also been very recently settled (Visan, personal communication).

[44] Note added in proof: the lower order terms have been successfully treated by Xiaoyi Zhang (personal communication), relying on this result and perturbation theory.

our argument.[45] We do not expect our bounds to be anywhere close to best possible. Indeed, any simplification of this argument would almost surely lead to less use of the induction hypothesis, and consequently to a better bound on $M(E)$. For recent progress in this direction in the radial case (in which no induction hypothesis is used at all, leading to a bound on $M(E)$ which is merely exponential in E), see [45].

The global existence and scattering result obtained here has analogs for the critical nonlinear Klein-Gordon equation $-\frac{1}{2c^2}u_{tt} + \Delta u = -|u|^4 u + \frac{m^2c^2}{2}u$ (see introduction for references). As we remarked earlier, there are some important differences between the methods employed for the Klein-Gordon equation and those we use here. In particular, it is not at all clear how our arguments might help show that the space-time bounds for the nonlinear Klein-Gordon equation are uniform in the nonrelativistic limit $c \to \infty$, even though one heuristically expects the nonlinear Klein-Gordon equation to converge in some sense to the nonlinear Schrödinger equation in this regimen with suitable normalizations and assumptions on the data. One major difficulty in extending our arguments to the relativistic case is that we have no analogue of the interaction Morawetz inequality (1.8) (or any localized variants) for the Klein-Gordon equation. For small energy data, uniform bounds on the solution are available in the nonrelativistic limit (see remarks in [31]), but for general solutions such bounds do not seem available. (See also [29] and references therein for further results on the subcritical problem.)

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