Annals of Mathematics, **167** (2008), 575–599

The Poincaré inequality is an open ended condition

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Abstract

Let $p > 1$ and let (X, d, μ) be a complete metric measure space with μ Borel and doubling that admits a $(1, p)$-Poincaré inequality. Then there exists $\varepsilon > 0$ such that (X, d, μ) admits a $(1, q)$-Poincaré inequality for every $q > p-\varepsilon$, quantitatively.

1. Introduction

Metric spaces of homogeneous type, introduced by Coifman and Weiss [7], [8], have become a standard setting for harmonic analysis related to singular integrals and Hardy spaces. Such metric spaces are often referred to as a metric measure space with a doubling measure. An advantage of working with these spaces is the wide collection of examples (see [6], [47]). A second advantage is that many classical theorems from Euclidean space still remain true, including the Vitali covering theorem, the Lebesgue differentiation theorem, the Hardy-Littlewood maximal theorem, and the John-Nirenberg lemma; see [20], [47]. However, theorems that rely on methods beyond zero-order calculus are generally unavailable.

To move into the realm of first-order calculus requires limiting attention to fewer metric measure spaces, and is often achieved by requiring that a Poincaré inequality is admitted. Typically a metric measure space is said to admit a Poincaré inequality (or inequalities) if a significant collection of real-valued functions defined over the space observes Poincaré inequalities as in (2.2.1) in some uniform sense. There are many important examples of such spaces (see [28], [26]), and many classical first-order theorems from Euclidean space remain true in this setting. These include results from second-order partial differential equations, quasiconformal mappings, geometric measure theory, and Sobolev

*S.K. was partially supported by the Academy of Finland, project 53292, and the Australian Research Council. X.Z. was partially supported by the Academy of Finland, project 207288.

spaces (see [1], [19], [20]). As an example, Cheeger ([5]) showed that such spaces admit a fixed collection of coordinate functions with which Lipschitz functions can be differentiated almost everywhere; see also [28]. This result is akin to the Rademacher differentiation theorem in Euclidean space.

Poincaré inequalities and doubling measures have constants intrinsic to the underlying metric measure space. The best doubling constant corresponds to an upper bound for a dimension of the metric space. Similarly, the exponent $p \geq 1$ in the Poincaré inequality (2.2.1) describes the pervasive extent of the first-order calculus on the metric measure space, with a lower value for p corresponding to an *a priori* more restrictive condition. (Hölder's inequality states that any metric measure space that admits a $(1, p)$-Poincaré inequality, $p \geq 1$, also admits a $(1, q)$-Poincaré inequality for every $q \geq p$.) The values admitted by this parameter are important for all of the above mentioned areas of analysis — this topic is addressed later in the introduction. In this paper we show that the collection of values admitted by this parameter $p > 1$ is open ended on the left if the measure is doubling.

THEOREM 1.0.1. *Let p be > 1 and let (X, d, μ) be a complete metric measure space with μ Borel and doubling, that admits a $(1, p)$-Poincaré inequality. Then there exists $\varepsilon > 0$ such that (X, d, μ) admits a $(1, q)$-Poincaré inequality for every $q > p - \varepsilon$, quantitatively.*

Famous examples of an open ended property are Muckenhoupt A_p weights [9], and functions satisfying the reverse Hölder inequality [14]. These results concern the open property of the objects (weights and functions) defined on Euclidean space or metric measure spaces where the measure is doubling, and rely on at most zero-order calculus. In contrast, our result is first-order and in its most abstract setting concerns the open ended property of the metric measure space itself. As such, the proof relies on new methods in addition to classical methods from zero-order calculus.

The results of this paper are new not only in the abstract setting, but also in the case of measures on Euclidean space and Riemannian manifolds. For example, weights on Euclidean space that when integrated against give rise to doubling measures that support a $(1, p)$-Poincaré inequality, $p \geq 1$, are known as *p-admissible weights*, and are particularly pertinent in the study of the nonlinear potential theory of degenerate elliptic equations; see [21], [12]. The fact that the above definition for p-admissible weights coincides with the one given in [21] is proven in [18]. It is known that the A_p weights of Muckenhoupt are p-admissible for each $p \geq 1$ (see [21, Ch. 15]). However, the converse is not generally true for any $p \geq 1$ (see [21, p. 10], and also the discussion following [27, Th. 1.3.10]). Nonetheless, we see from the following corollary to Theorem 1.0.1, that p-admissible weights display the same open ended property of Muckenhoupt's A_p weights.

Corollary 1.0.2. *Let $p > 1$ and let w be a p-admissible weight in $\mathbf{R}^n$, $n \geq 1$. Then there exists $\varepsilon > 0$ such that w is q-admissible for every $q > p - \varepsilon$, quantitatively.*

For complete Riemannian manifolds, Saloff-Coste ([41], [42]) established that supporting a doubling measure and a $(1, 2)$-Poincaré inequality is equivalent to admitting the parabolic Harnack inequality, quantitatively (Grigor′yan [15] also independently established that the former implies the latter). The latter condition was further known to be equivalent to Gaussian-like estimates for the heat kernel, quantitatively (see for example [42]). Thus by Theorem 1.0.1, each of these conditions is also equivalent to supporting a doubling measure and a $(1, 2 - \varepsilon)$-Poincaré inequality for some $\varepsilon > 0$, quantitatively. Relations between $(1, 2)$-Poincaré inequalities, heat kernel estimates, and parabolic Harnack inequalities have been established in the setting of Alexandrov spaces by Kuwae, Machigashira, and Shioya ([37]), and in the setting of complete metric measure spaces that support a doubling Radon measure, by Sturm ([48]). Colding and Minicozzi II [10] proved that on complete noncompact Riemannian manifolds supporting a doubling measure and a $(1, 2)$-Poincaré inequality, the conjecture of Yau is true: the space of harmonic functions with polynomial growth of fixed rate is finite dimensional.

Heinonen and Koskela ([22], [23], [24], see also [20]) developed a notion of the Poincaré inequality and the Loewner condition for general metric measure spaces. The latter is a generalization of a condition proved by Loewner ([38]) for Euclidean space, that quantitatively describes metric measure spaces that are very well connected by rectifiable curves. Heinonen and Koskela demonstrated that quasiconformal homeomorphisms (the definition of which is given through an infinitesimal metric inequality) display certain global rigidity (that is, are quasisymmetric) when mapping between Loewner metric measure spaces with certain upper and lower measure growth restrictions on balls. They further showed that metric measure spaces with certain upper and lower measure growth restrictions on balls, specifically, Ahlfors α-regular metric measure spaces, $\alpha > 1$, are Loewner if and only if they admit a $(1, \alpha)$-Poincaré inequality, quantitatively. By Theorem 1.0.1 we see then that the following holds:

Theorem 1.0.3. *A complete Ahlfors α-regular metric measure space, $\alpha > 1$, is Loewner if and only if it supports a $(1, \alpha - \varepsilon)$-Poincaré inequality for some $\varepsilon > 0$, quantitatively.*

Theorem 1.0.1 has consequences in Gromov hyperbolic geometry. Laakso and the first author ([30]) demonstrated that complete Ahlfors α-regular metric measure spaces, $\alpha > 1$, cannot have their Assouad dimension lowered through quasisymmetric mappings if and only if they possess at least one weak-tangent that contains a collection of non-constant rectifiable curves with positive p-modulus, for some or any $p \geq 1$. There is no need here to pass to weak tangents for complete metric measure spaces that are sufficiently rich in symmetry. This result was used by Bonk and Kleiner ([4]) who subsequently showed that such metric measure spaces that arise as the boundary of a Gromov hyperbolic group, are Loewner. By Theorem 1.0.3 we see that such metric measure spaces further admit a $(1, \alpha - \varepsilon)$-Poincaré inequality for some $\varepsilon > 0$, quantitatively. One can then conclude rigidity type results for quasiconformal mappings between such spaces.

Specifically, Heinonen and Koskela ([24, Th. 7.11]) showed that the pullback measure of a quasisymmetric homeomorphism from a complete Ahlfors α-regular metric measure space that supports a p-Poincaré inequality, to a complete Ahlfors α-regular metric space, is an A_∞ weight in the sense of Muckenhoupt if $1 \leq p < \alpha$, quantitatively. This extended classical results of Bojarski ([3]) in $\mathbf{R}^2$ and Gehring ([14]) in $\mathbf{R}^n$, $n \geq 3$. For the critical case, that is, when $p = \alpha$, Heinonen, Koskela, Shanmugalingam, and Tyson ([25, Cor. 8.15]) showed that a quasisymmetric homeomorphism, from a complete Ahlfors α-regular Loewner metric measure space to a complete Ahlfors α-regular metric space, is absolutely continuous with respect to α-Hausdorff measure. This left open the question of whether the given quasisymmetric homeomorphism actually induces an A_∞ weight. Theorem 1.0.3 in conjunction with [24, Th. 7.11] gives an affirmative answer to this question.

THEOREM 1.0.4. *Let (X, d, μ) and (Y, l, ν) be complete Ahlfors α-regular metric measure spaces, $\alpha > 1$, with (X, d, μ) Loewner, and let $f : X \longrightarrow Y$ be a quasisymmetric homeomorphism. Then the the pullback $f^*\nu$ of ν by f is A_∞ related to μ, quantitatively. Consequently there exists a measurable function $w : X \longrightarrow [0, \infty)$ such that $df^*\nu = w d\mu$, and such that*

$$\left(\fint_B w^{1+\varepsilon} \, d\mu \right)^{1/(1+\varepsilon)} \leq C \fint_B w \, d\mu,$$

for every ball B in X, quantitatively.

There are several papers on the topic of nonlinear potential theory where the standing hypothesis is made that a given measure on $\mathbf{R}^n$ is q-admissible, or that a given metric measure space supports a doubling Borel regular measure and a q-Poincaré inequality, for some $1 < q < p$. Typically p is the "critical dimension" of analysis. These includes papers by Björn, MacManus, and Shanmugalingam ([2]), Kinnunen and Martio ([32], [33]), and Kinnunen and Shanmugalingam ([34]). It follows by Theorem 1.0.1 that in each of these cases, the standing assumption can be replaced by the *a priori* weaker assumption that the given metric measure space supports a doubling Borel regular measure and a p-Poincaré inequality. As an example, Kinnunen and Shanmugalingam

([34]) have shown in the setting of metric measure spaces that support a doubling Borel regular measure and a $(1, q)$-Poincaré inequality (in the sense of Heinonen and Koskela in [24]; see Section 1.1), that quasiminimizers of p-Dirichlet integrals satisfy Harnack's inequality, the strong maximum principle, and are locally Hölder continuous, if $1 < q < p$. This leads to the following result.

THEOREM 1.0.5. *Quasiminimizers of p-Dirichlet integrals on metric measure spaces that support a Borel doubling Borel regular measure and a $(1, p)$-Poincaré inequality, $p > 1$, satisfy Harnack's inequality, the strong maximum principle, and are locally Hölder continuous, quantitatively.*

Alternate definitions for Sobolev-type spaces on metric measure spaces have been introduced by a variety of authors. Here we consider the Sobolev space $H_{1,p}(X)$, $p \geq 1$, introduced by Cheeger in [5], the Newtonian space $N^{1,p}(X)$ introduced by Shanmugalingam in [46], and the Sobolev space $M^{1,p}(X)$ introduced by Hajłasz in [16]. (We have used the same notation as the respective authors, and refer the reader to the cited papers for the definitions of these Sobolev-type spaces.) It is known that generally this last Sobolev-type space does not always coincide with the former two ([46, Examples 6.9 and 6.10]). Nonetheless, Shanmugalingam has shown that $H_{1,p}(X)$, $p > 1$, is isometrically equivalent in the sense of Banach spaces to $N^{1,p}(X)$ whenever the underlying measure is Borel regular; and furthermore, that all of the above three spaces are isomorphic as Banach spaces whenever the given metric measure space X supports a doubling Borel regular measure and a $(1, q)$-Poincaré inequality for some $1 \leq q < p$ (in the sense of Heinonen and Koskela in [24]), quantitatively ([46, Ths. 4.9 and 4.10]). By Theorem 1.0.1 we see then that the following holds:

THEOREM 1.0.6. *Let X be a complete metric measure space that supports a doubling Borel regular measure and a $(1, p)$-Poincaré inequality, $p > 1$. Then $H_{1,p}(X)$, $M^{1,p}(X)$, and $N^{1,p}(X)$ are isomorphic, quantitatively.*

1.1. *A note on the various definitions of a Poincaré inequality.* There are various formulations for a Poincaré inequality on a metric measure space that might not necessarily hold for every metric measure space, but that still make sense for every metric measure space. This partly arises in this general setting because the notion of a gradient of a function is not always easily defined, and because it is not clear which class of functions the inequality should be required to hold for. These considerations are discussed by Semmes in [45, §2.3]. Nonetheless, most reasonable definitions coincide when the metric measure space is complete and supports a doubling Borel regular measure. In particular, the definitions of Heinonen and Koskela in [24], Semmes in [45,

§2.3], and several other definitions of the first author, including the definition adopted here (Definition 2.2.1), all coincide in this case. Some of this is shown by the first author in [29], [27], the rest is shown by Rajala and the first author in [31].

Theorem 1.0.1 would not generally be true if we removed the hypothesis that the given metric measure space is complete, although, this depends on which definition is used for the Poincaré inequality. In particular, it would not generally be true if one used the definition of Heinonen and Koskela in [24]. For each $p > 1$, an example demonstrating this is given by Koskela in [35], consisting of an open set Ω in Euclidean space endowed with the standard Euclidean metric and Lebesgue measure. The main reason that our proof fails in that setting (as it should) is that Lipschitz functions, and indeed any subspace of the Sobolev space $W^{1,p}(\Omega)$ contained in $W^{1,q}(\Omega)$, is not dense in $W^{1,q}(\Omega)$ for any $1 \leq q < p$. (Here $W^{1,r}(\Omega)$, $r \geq 1$, is the completion of the real-valued smooth functions defined on Ω, under the norm $\|\cdot\|_{1,r}$ given by $\|u\|_{1,r} = \|u\|_r + \|\,|\nabla u|\,\|_r$.) Indeed, our proof works at the level of functions in $W^{1,p}$, and to simplify the exposition we consider only Lipschitz functions. In the case when the metric measure space is complete and supports a doubling Borel regular measure, we can appeal to results of the Rajala and the first author([31]), and the first author ([29], [27]), to recover the improved Poincaré inequality for all functions.

The definition adopted in this paper for the Poincaré inequality (Definition 2.2.1) is preserved under taking the completion of the metric measure space, and still holds if one removes any null set with dense complement. Consequently, the assumption in Theorem 1.0.1 that the given metric measure space is complete, is superfluous, and was included for the sake of clarity when comparing against other papers that use a different definition for the Poincaré inequality.

Finally, the reader may be concerned that this paper is needlessly limited to only $(1,p)$-Poincaré inequalities, instead of (q,p)-Poincaré inequalities for $q > 1$ — inequalities where the L^1 average on the left is replaced by an L^q average (see Definition 2.2.1). Our justification for doing this comes from the fact, as proven by Hajłasz and Koskela [19], that a metric measure space that supports a doubling Borel regular measure and a $(1,p)$-Poincaré inequality, $p \geq 1$, also supports the *a priori* stronger (q,p)-Poincaré inequality, for some $q > p$, quantitatively.

1.2. *Self-improvement for pairs of functions.* One might be tempted to hope that results analogous to Theorem 1.0.1 hold for pairs of functions that are linked by Poincaré type inequalities regardless of whether the given metric measure space supports a Poincaré inequality. Pairs of functions that satisfy similar relations have been extensively studied, see [39], [40]. Hajłasz and

Koskela [19, p. 19] have asked if given $u, g \in L^p(X)$ that satisfy a $(1,p)$-Poincaré inequality, where $p > 1$ and (X, d, μ) is a metric measure space with μ a doubling Borel regular measure, whether the pair u, g also satisfy a $(1,q)$-Poincaré inequality for some $1 \le q < p$. Here, a pair u, g is said to *satisfy a* $(1,q)$*-Poincaré inequality*, $q \ge 1$, if there exist $C, \lambda \ge 1$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le Cr \left(\fint_{B(x,\lambda r)} g^q \, d\mu \right)^{1/q}, \tag{1}$$

for every $x \in X$ and $r > 0$. The next proposition demonstrates that the answer to this question is no.

Proposition 1.2.1. *There exists an Ahlfors* 1*-regular metric measure space such that for every* $p > 1$, *there exists a pair of functions* $u, g \in L^p(X)$ *and constants* $C, \lambda \ge 1$ *such that* (1) *holds with* $q = p$ *for every* $x \in X$ *and* $r > 0$, *and such that there does not exist* $C, \lambda \ge 1$ *such that* (1) *holds with* $q < p$ *for every* $x \in X$ *and* $r > 0$.

Remark 1.2.2. In contrast to the above theorem, if a metric measure space (X, d, μ) admits a p-Poincaré inequality, $p > 1$, in the sense of Heinonen and Koskela, with μ doubling, then the following holds: there exists $\varepsilon > 0$, such that every pair of functions with $u, g \in L^p(X)$ that satisfies a p-Poincaré inequality in the sense of (1), further satisfies (1) for every $q \ge p - \varepsilon$, quantitatively. This is discussed further in Section 4.

1.3. *Outline.* In Section 2 we recall terminology and known results. The proof of Theorem 1.0.1 is contained in Section 3. Section 2 contains further discussion required for Remark 1.2.2 and Theorem 1.0.3, 1.0.4, 1.0.5 and 1.0.6, and the proof of Proposition 1.2.1.

1.4. *Acknowledgements.* Some of this research took place during a two-week stay in Autumn 2002, by the first author at the University of Jyväskylä. During this time the first author was employed by the University of Helsinki, and supported by both institutions. The first author would like to thank both institutions for their support and gracious hospitality during this time. The authors would also like to thank Juha Heinonen and Pekka Koskela for reading the paper and giving many valuable comments.

2. Terminology and standard lemmas

In this section we recall standard definitions and results needed for the proof of Theorem 1.0.1. With regard to language, when we say that a claim holds *quantitatively*, as in Theorem 1.0.1, we mean that the new parameters of the claim depend only on the previous parameters implicit in the hypotheses. For example, in Theorem 1.0.1 we mean that ε and the constants associated with the $(1,q)$-Poincaré inequality depend only on the constant p, the doubling constant of μ, and the constants associated with the assumed $(1,p)$-Poincaré inequality. When we say that two positive reals x, y are *comparable* with constant $C \geq 1$, we mean that $x/C \leq y \leq Cx$. We use $\chi|_W$ to denote the characteristic function on any set W.

2.1. *Metric measure spaces*, *doubling measures*, *and* Lip. In this paper (X, d, μ) denotes a metric measure space and μ is always Borel regular. We will use the notation $|E|$ and $\operatorname{diam} E$ to denote the μ-measure and the diameter of any measurable set $E \subset X$, respectively. The ball with center $x \in X$ and radius $r > 0$ is denoted by

$$B(x,r) = \{y \in X : d(x,y) < r\},$$

and we use the notation

$$tB(x,r) = \{y \in X : d(x,y) < tr\},$$

whenever $t > 0$. When we "fix a ball" it is implicitly meant that a center and radius have also been selected. We write $u_A = \frac{1}{|A|}\int_A u\, d\mu = \fint_A u\, d\mu$ for every $A \subset X$ and measurable function $u : X \longrightarrow [-\infty, \infty]$. The measure μ is said to be *doubling* if there is a constant $C \geq 1$ such that $|B(x, 2r)| \leq C|B(x,r)|$ for every $x \in X$ and $r > 0$.

LEMMA 2.1.1 ([20, pp. 103, 104]). *Let (X, d, μ) be a metric measure space with μ doubling. Then there exist constants $C, \alpha > 0$, that depend only on the doubling constant of μ, such that*

$$C\frac{|B(y,r)|}{|B(x,R)|} \geq \left(\frac{r}{R}\right)^{\alpha},$$

whenever $0 < r < R < \operatorname{diam} X$, $x \in X$, and $y \in B(x,R)$.

A function $u : X \longrightarrow \mathbf{R}$ is said to be *L-Lipschitz*, $L \geq 0$, if $|u(x) - u(y)| \leq Ld(x,y)$ for every $x, y \in X$. We often omit mention of the constant L and just describe such functions as being Lipschitz. Given a Lipschitz function $u : X \longrightarrow \mathbf{R}$ and $x \in X$, we let

$$\operatorname{Lip} u(x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x,y)}.$$

The following lemma can be easily deduced from Lemma 2.1.1; compare with the proof of [29, Prop. 3.2.3].

LEMMA 2.1.2. *Let (X, d, μ) be a metric measure space with μ doubling, and let f and g be real-valued Lipschitz functions defined on X. Then* $\operatorname{Lip} f = \operatorname{Lip} g$ *almost everywhere on the set where $f = g$.*

2.2. *The Poincaré inequality and geodesic metric spaces.* We can now state the definition for the Poincaré inequality on metric measure spaces to be used in this paper.

Definition 2.2.1. A metric measure space (X, d, μ) is said to admit a $(1, p)$-*Poincaré inequality*, $p \geq 1$, with constants $C \geq 1$ and $1 < t \leq 1$, if the following holds: Every ball contained in X has measure in $(0, \infty)$, and we have

$$\fint_{tB} |u - u_{tB}| \, d\mu \leq C(\operatorname{diam} B) \left(\fint_{B} (\operatorname{Lip} u)^p \, d\mu \right)^{1/p}, \tag{2}$$

for all balls $B \subset X$, and for every Lipschitz function $u : X \longrightarrow \mathbf{R}$.

If (X, d, μ) is complete with μ doubling and supports a $(1, p)$-Poincaré inequality, then (X, d, μ) is bi-Lipschitz to a geodesic metric space, quantitatively; see [27, Prop. 6.0.7]. We briefly recall what these words mean and refer to [20] for a more thorough discussion. A metric space is *geodesic* if every pair of distinct points can be connected by a path with length equal to the distance between the two points. A map $f : Y_1 \longrightarrow Y_2$ between metric spaces (Y_1, ρ_1) and (Y_2, ρ_2) is *L-bi-Lipschitz*, $L > 0$, if for every $x, y \in Y_1$ we have

$$\frac{1}{L} \rho_1(x, y) \leq \rho_2(f(x), f(y)) \leq L \rho_1(x, y).$$

Two metric spaces are said to be *L-bi-Lipschitz*, or just bi-Lipschitz, if there exists a surjective L-bi-Lipschitz map between them.

One advantage of working with geodesic metric spaces is that if (X, d, μ) is a geodesic metric space with μ doubling that admits a $(1, p)$-Poincaré inequality, $p \geq 1$, then (X, d, μ) admits a Poincaré inequality with $t = 1$ in (2), but possibly a different constant $C > 0$, quantitatively; see [20, Th. 9.5].

Another convenient property of geodesic metric spaces is that the measure of points sufficiently near the boundary of any ball is small. This claim is made precise by the following result that appears as Proposition 6.12 in [5], where it is accredited to Colding and Minicozzi II [11].

PROPOSITION 2.2.2. *Let (X, d, μ) be a geodesic metric measure space with μ doubling. Then there exists $\alpha > 0$ that depends only on the doubling constant of μ such that*

$$|B(x, r) \setminus B(x, (1 - \delta)r)| \leq \delta^{\alpha} |B(x, r)|,$$

for every $x \in X$ and $\delta, r > 0$.

2.3. *Maximal type operators.* Given a Lipschitz function $u : X \longrightarrow \mathbf{R}$ and $x \in X$, we set

$$M^{\#} u(x) = \sup_{B} \frac{1}{\operatorname{diam} B} \fint_{B} |u - u_B| \, d\mu,$$

for every $x \in X$, where the supremum is taken over all balls B in X that contain x. This sharp fractional maximal operator should not be confused with the uncentered Hardy-Littlewood maximal operator which we denote by

$$Mu(x) = \sup_B \fint_B |u| \, d\mu,$$

for every $x \in X$, where the supremum is taken over all balls B that contain x. The following lemma is folklore; a similar proof to a similar fact can be found in [20, p. 73].

LEMMA 2.3.1. *Let (X, d, μ) be a metric measure space with μ doubling, and let $u : X \longrightarrow \mathbf{R}$ be Lipschitz. Then there exists $C > 0$ that depends only on the doubling constant of μ such that*

$$|u(x) - u_{B(y,r)}| \leq CrM^{\#}u(x),$$

whenever $r > 0$, $y \in X$, and $x \in B(y, r)$. Consequently, the restriction of u to

$$\{x \in X : M^{\#}u(x) \leq \lambda\}$$

is $2C\lambda$-Lipschitz.

3. Proof of Theorem 1.0.1

Theorem 1.0.1 is proved in this section. Let (X, d, μ) be a complete geodesic metric measure space with μ doubling that admits a $(1, p)$-Poincaré inequality, $p > 1$. The assumption that (X, d) is geodesic involves no loss of generality for the proof of Theorem 1.0.1 and is adopted to simplify the exposition. Indeed, the hypotheses and claim of Theorem 1.0.1 are invariant under bi-Lipschitz mappings. And as is explained in Section 2.2, the above remaining hypotheses ensure that (X, d) is bi-Lipschitz to a geodesic metric space.

In what follows we let $C > 1$ denote a *varying constant* that depends only on the data associated with the assumed $(1, p)$-Poincaré inequality, the doubling constant of μ, and p. This means that C denotes a positive variable whose value may vary between each usage, but is then fixed and depends only on the data outlined above.

3.1. *Local estimates.* Local weak L^1-type estimates for a sharp fractional maximal function are established in this section. Fix a ball X_1 in X and let $X_i = 2^{i-1}X_1$ for each $i \in \mathbf{N}$. Given a Lipschitz function $u : X_{i+1} \longrightarrow \mathbf{R}$, let

$$M_i^{\#}u(x) = \sup_B \frac{1}{\operatorname{diam} B} \fint_B |u - u_B| \, d\mu$$

for every $x \in X_{i+1}$, where the supremum is taken over all balls $B \subset X_{i+1}$ that contains x. Next, for the Lipschitz function u we define

$$U_\lambda = \{x \in X_4 : M_4^{\#}u(x) > \lambda\}$$

for every $\lambda > 0$.

The next proposition gives the local estimate of the level set of the fractional maximal function of u, and is the main result of this section.

PROPOSITION 3.1.1. *Let $\alpha \in \mathbf{N}$. There exists $k_1 \in \mathbf{N}$ that depends only on C and α such that for all integer $k \geq k_1$ and every $\lambda > 0$ with*

$$\frac{1}{\operatorname{diam} X_1} \fint_{X_1} |u - u_{X_1}|\, d\mu > \lambda, \tag{3}$$

we have

$$\begin{aligned} |X_1| \leq\ & 2^{kp-\alpha}|U_{2^k\lambda}| + 8^{kp-\alpha}|U_{8^k\lambda}| \\ &+ 8^{k(p+1)}|\{x \in X_5 : \operatorname{Lip} u(x) > 8^{-k}\lambda\}|. \end{aligned} \tag{4}$$

The above proposition is proved over the remainder of this section. For the sake of simplicity and without loss of generality we re-scale u by u/λ and so may assume $\lambda = 1$ in Proposition 3.1.1. Likewise, we re-scale the metric and the measure of (X, d, μ) so that X_1 has unit diameter and unit measure. Let $\alpha, k \in \mathbf{N}$, and suppose in order to achieve a contradiction that (4) does not hold with $\lambda = 1$. The assumed negation of (4) implies that

$$\begin{gathered} |U_{2^k}| < 2^{-kp+\alpha}, \quad |U_{8^k}| < 8^{-kp+\alpha}, \quad \text{and} \\ |\{x \in X_5 : \operatorname{Lip} u(x) > 8^{-k}\}| < 8^{-k(p+1)}. \end{gathered} \tag{5}$$

During the proof a fixed and finite number of lower bounds will be specified for k. These bounds are required for the proof to work, and depend only on C and α. To realize the contradiction at the end of the proof and thereby prove Proposition 3.1.1, we take k_1 to be equal to the maximum of this finite collection of lower bounds.

The next lemma demonstrates that u has some large scale oscillation outside U_{2^k}.

LEMMA 3.1.2. *We have*

$$\int_{X_2 \setminus U_{2^k}} |u - u_{X_2 \setminus U_{2^k}}|\, d\mu \geq 1/C.$$

Proof. We exploit (5). We can assume without loss of generality that $u_{X_2 \setminus U_{2^k}} = 0$ by an otherwise translating in the range of u. Let $\mathcal{G}$ be the collection of balls B in X_3 that intersect $U'_{2^k} := X_1 \cap U_{2^k}$ with

$$|B \setminus U_{2^k}| \geq |B|/4 \quad \text{and} \quad |B \cap U_{2^k}| \geq |B|/4. \tag{6}$$

For later use we observe that (5) together with Lemma 2.1.1 implies that each such B satisfies

$$\operatorname{diam} B \leq C^{-kp+\alpha}. \tag{7}$$

Therefore, as long as k is sufficiently large, we have $5B \subset X_2$.

Since (X, d) is geodesic, we claim by Proposition 2.2.2 that $\mathcal{G}$ is a cover of U'_{2^k}. Indeed, fix $x \in U'_{2^k}$, and define $h : (0, 1] \longrightarrow \mathbf{R}$ by

$$h(r) = \frac{|B(x, r) \cap U_{2^k}|}{|B(x, r)|}.$$

Since $M^{\#}$ is an uncentered maximal-type operator, we have U_{2^k} is open, and therefore $h(\delta) = 1$ for some $\delta > 0$. We also have $h(1) \leq |U_{2^k}| \leq 2^{-kp+\alpha}$, since $X_1 \subset B(x, 1)$ and X_1 has unit diameter and unit measure. Finally, Proposition 2.2.2 implies that h is continuous. Therefore there exists $r > 0$ such that $h(r) = 1/4$. This proves the claim. By a standard covering argument (see [20, Th. 1.2]), there exists a countable subcollection $\{B_i\}_{i \in J}$ of $\mathcal{G}$ consisting of mutually disjoint balls in X such that $U'_{2^k} \subset \cup_{i \in J} 5B_i$; here $J = \{1, 2, \dots\}$ is a possibly finite index set.

We now divide U_{2^k} amongst the members of $\{B_i\}_{i \in J}$. Let

$$E_i = 5B_i, \quad E_i^O = B_i \setminus U_{2^k}, \quad \text{and} \quad E_i^I = B_i \cap U_{2^k},$$

for each $i \in J$. Notice that by construction and by (6) we have that

$$|E_i| \leq C \min\{|E_i^O|, |E_i^I|\}, \tag{8}$$

that $\{E_i\}_{i \in J}$ is a cover of U'_{2^k}, and that $\{E_i^I\}_{i \in J}$ and $\{E_i^O\}_{i \in J}$ are collections of mutually disjoint measurable sets. Note that I and O stand for inside and outside, respectively.

It follows from these just stated properties and (3) that

$$1 < \int_{X_1} |u - u_{X_1}| \, d\mu \leq 2 \int_{X_1} |u| \, d\mu \leq 2 \int_{X_1 \setminus U_{2^k}} |u| \, d\mu + 2 \sum_{i \in J} \int_{E_i} |u| \, d\mu,$$

whereas

$$\begin{aligned} \sum_{i \in J} \int_{E_i} |u| \, d\mu &\leq \sum_{i \in J} |E_i| \, |u_{E_i^O}| + \sum_{i \in J} \int_{E_i} |u - u_{E_i^O}| \, d\mu \\ &\leq C \int_{X_2 \setminus U_{2^k}} |u| \, d\mu + C \sum_{i \in J} \int_{E_i} |u - u_{E_i^O}| \, d\mu, \end{aligned}$$

and therefore

$$1 \leq C \int_{X_2 \setminus U_{2^k}} |u| \, d\mu + C \sum_{i \in J} \int_{E_i} |u - u_{E_i^O}| \, d\mu. \tag{9}$$

Consequently, to complete the proof we need to show that for sufficiently large $k \in \mathbf{N}$, that depends only on C and α, that the right-hand most term in (9) is less than $1/2$. We use (8), and then the fact that E_i intersects the complement of U_{2^k}, to obtain

$$\fint_{E_i} |u - u_{E_i^O}| \, d\mu \leq C \fint_{E_i} |u - u_{E_i}| \, d\mu \leq C 2^k \operatorname{diam}(E_i),$$

for every $i \in J$. Thus, the right-hand most term of (9) is bounded by

$$C2^k \left(\sup_{i\in J} \operatorname{diam} B_i\right) \sum_{i\in J} |E_i|.$$

We now apply (7) and (8) to bound the above sum by

$$C2^k C^{-kp+\alpha} \sum_{i\in J} |E_i^I| \leq C2^k C^{-kp+\alpha} |U_{2^k}| \leq C2^{(1-p)k} C^{-kp+\alpha}.$$

We conclude that for sufficiently large $k \in \mathbf{N}$ that depends only on C and α, that the right-hand most term in (9) is less than $1/2$. This completes the proof. Note that this part of the proof did not really require the fact that $p > 1$. □

By argument as in Lemma 2.3.1, we have

$$|u(x) - u_{B(y,r)}| \leq CrM_i^{\#}u(x),$$

whenever $y \in X_i$ and $0 < r < \operatorname{dist}(y, X \setminus X_i)$, and also $x \in B(y,r)$. This with the fact that (X,d) is geodesic implies that the restriction of u to the set

$$\{x \in X_i : M_i^{\#}u(x) \leq \lambda\}$$

is $2C\lambda$-Lipschitz. We use this to remove the small scale oscillation from u while still preserving the large scale oscillation as follows.

Lemma 3.1.3. *There exists a $C8^k$-Lipschitz extension f of $u|_{X_3\setminus U_{8^k}}$ to X_3 such that*

$$M_2^{\#}f(x) \leq CM_4^{\#}u(x) \tag{10}$$

for every $x \in X_2 \setminus U_{8^k}$.

Proof. By Lemma 2.3.1, we have that $u|_{X_3\setminus U_{8^k}}$ is $C8^k$-Lipschitz. We could now extend u to X using the McShane extension (see [20, Th. 6.2]). However, it is not clear that this would then satisfy (10). Instead we use another standard extension technique based on a Whitney-like decomposition of U_{8^k}; similar methods of extension also appear in [44], [36], [17], [43]. The novelty here is not the extension, but rather that there is a Lipschitz extension that satisfies (10).

Observe that because $M^{\#}$ is uncentered, we have U_{8^k} is open. We can then apply a standard covering argument ([20, Th. 1.2]) to the collection

$$\{B(x, \operatorname{dist}(x, X \setminus U_{8^k})/4) : x \in U_{8^k}\},$$

and so obtain a countable subcollection $\mathcal{F} = \{B_i\}_{i\in I}$, where $I = \{1, 2, \dots\}$ is a possibly finite index set, such that $U_{8^k} = \cup_{i\in\mathbf{N}} B_i$, and such that $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \emptyset$ for $i, j \in I$ with $i \neq j$. It then follows from the fact that μ is doubling that

$$\sum_{i\in I} \chi|_{2B_i} \leq C, \tag{11}$$

where we use $\chi|_W$ to denote the characteristic function on any set W.

We now construct a partition of unity subordinate to this collection of balls. For each $i \in I$, let $\hat{\psi}_i : X_4 \longrightarrow \mathbf{R}$ be a $C \operatorname{dist}(B_i, X \setminus U_{8^k})^{-1}$-Lipschitz function with $\hat{\psi}_i = 1$ on B_i and $\hat{\psi}_i = 0$ on $X \setminus 2B_i$. Then let

$$\psi_i = \frac{\hat{\psi}_i}{\sum_{j \in I} \hat{\psi}_j}.$$

As usual the sum in the denominator is well-defined at each point in X_4, because of (11), as all but a finite number of terms in the sum are non-zero. Next define $f : X_4 \longrightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} \sum_{i \in I} u_{B_i} \psi_i(x) & \text{if } x \in U_{8^k}, \\ u(x) & \text{if } x \in X_4 \setminus U_{8^k}. \end{cases}$$

We now show that $f|_{X_3}$ is $C8^k$-Lipschitz, that is, we show that

$$|f(x) - f(y)| \leq C8^k d(x, y) \tag{12}$$

for every $x, y \in X_3$. By Lemma 2.3.1 (actually by the proof of Lemma 2.3.1, as we explained before), we have (12) holds whenever $x, y \in X_3 \setminus U_{8^k}$. Next consider the case when $x \in X_3 \cap U_{8^k}$ and $y \in X_3 \setminus U_{8^k}$. By the triangle inequality, and the case considered two sentences back, we can further suppose that

$$d(x, y) \leq 2 \operatorname{dist}(x, X \setminus U_{8^k}).$$

Let B be a ball in $\mathcal{F}$ that contains x. Then $B = B(w, r)$ for some $w \in U_{8^k} \subset X_4$ and $r > 0$ with r, $d(x, y)$ and $d(w, y)$ comparable with constant C. By (5), $|U_{8^k}|$ is small for sufficiently k, so $B = B(w, r) \subset X_4$ and $B' = B(w, 2d(w, y)) \subset X_4$. We can then use Lemma 2.3.1 and the doubling property of μ, to deduce that

$$\begin{aligned} |u(y) - u_B| &\leq |u(y) - u_{B'}| + |u_B - u_{B'}| \leq CrM_4^{\#}u(y) + C\fint_{B'} |u - u_{B'}| \, d\mu \\ &\leq CrM_4^{\#}u(y) \leq C8^k d(x, y). \end{aligned}$$

The estimate (12) then follows from the definition of f.

Finally we consider the case when $x, y \in X_3 \cap U_{8^k}$. Due to the last two cases considered, we can further suppose that $r = \operatorname{dist}(x, X \setminus U_{8^k})$ is comparable to $d(y, X \setminus U_{8^k})$ with comparability constant C, and that $d(x, y) \leq r$. Let $B = B(x, 5r)$. Again (5) implies that if k is sufficently large, we have $B \subset X_4$. Observe that if $B_i = B(z, s)$ is a ball in $\mathcal{F}$, for some $i \in I$, $z \in X_4$ and $s > 0$, such that $\{x, y\} \cap B(z, 2s) \neq \emptyset$, then r and s are comparable with comparability constant C, the function ψ_i is Cr^{-1}-Lipschitz, and $B(z, s) \subset B$. Consequently, we have

$$|u_{B_i} - u_B| \leq C\fint_B |u - u_B| \, d\mu \leq CrM_4^{\#}u(w) \leq C8^k r,$$

for some $w \in B \cap (X_4 \setminus U_{8^k})$. It then follows from the fact that $\{\psi_i\}_{i\in I}$ is a partition of unity on U_{8^k}, that

$$\begin{aligned} |f(x) - f(y)| &= \left| \sum_{i\in I} \psi_i(x) u_{B_i} - \sum_{i\in I} \psi_i(x) u_B + \sum_{i\in I} \psi_i(y) u_B - \sum_{i\in I} \psi_i(y) u_{B_i} \right| \\ &= \left| \sum_{i\in I} (\psi_i(x) - \psi_i(y))(u_{B_i} - u_B) \right| \le C 8^k d(x,y), \end{aligned}$$

as desired. This completes the demonstration that f is $C8^k$-Lipschitz.

It remains to establish (10). Fix $x \in X_2 \setminus U_{8^k}$ and suppose that $M_2^{\#} f(x) > \delta$ for some $\delta > 0$. Thus there exists a ball $B = B(y,r) \subset X_3$ containing x, for some $y \in X_3$ and $r > 0$, such that

$$\frac{1}{\operatorname{diam} B} \fint_B |f - f_B| \, d\mu > \delta. \tag{13}$$

We would like to show that

$$\frac{1}{\operatorname{diam} B} \fint_{4B} |u - u_B| \, d\mu > \delta / C. \tag{14}$$

Since $4B \subset X_5$, this will then imply (10).

Observe that the above two estimates are invariant under a translation in the range of u and f. Furthermore, the construction of f from u is also invariant under a translation in the range of u and f. By this we mean that if u is replaced by $u + \beta$ for some $\beta \in \mathbf{R}$, then the construction above gives $f + \beta$ in place of f. Thus without loss of generality, by making such a translation, we can assume that $u_B = 0$. Since $u = f$ on $X_4 \setminus U_{8^k}$, we can also assume that $B \cap U_{8^k} \neq \emptyset$; otherwise (14) follows trivially from (13).

It then follows directly from the construction of $\mathcal{F}$, that if $B(z,s) \in \mathcal{F}$ for some $z \in U_{8^k}$ and $s > 0$, then $B(z, 2s) \cap B \neq \emptyset$ implies $s \le r$. Thus $B(z,s) \subset 4B$. Therefore

$$\begin{aligned} \int_B |f - f_B| \, d\mu &\le 2 \int_B |f| \, d\mu \\ &\le 2 \int_{B \setminus U_{8^k}} |f| \, d\mu + \sum_{i\in I} \int_B |\psi_i u_{B_i}| \, d\mu \\ &\le 2 \int_{B \setminus U_{8^k}} |u| \, d\mu + C \sum_{\substack{i\in I \\ 2B_i \cap B \neq \emptyset}} \int_{B_i} |u| \, d\mu \le C \int_{4B} |u| \, d\mu. \end{aligned}$$

It follows from this last estimate, the doubling property of μ, and our assumption that $u_B = 0$, that (14) holds. This proves (10) and so completes the proof of the lemma. $\square$

The function f can be viewed as a smoothed version of u, that is, with small scale oscillations removed, and large scale oscillations preserved. The following two lemmas utilize the previous estimates on the oscillation of u and f. Let

$$F_s = \{x \in X_2 : M_2^{\#} f(x) > s\}$$

for every $s > 0$.

LEMMA 3.1.4. *We have*

$$\int_{X_3 \setminus U_{8^k}} (\operatorname{Lip} f)^p \, d\mu \leq C 8^{-k}, \tag{15}$$

and

$$|F_s| \leq C s^{-p}, \tag{16}$$

for every $s > 0$.

Proof. We first prove (15). By Lemma 3.1.3 we have $f = u$ on $X_3 \setminus U_{8^k}$, and so Lemma 2.1.2 implies that $\operatorname{Lip} f = \operatorname{Lip} u$ almost everywhere on $X_3 \setminus U_{8^k}$. Since f is $C8^k$-Lipschitz, we therefore have $\operatorname{Lip} u \leq C8^k$ almost everywhere on $X_3 \setminus U_{8^k}$. It follows that

$$\begin{aligned} \int_{X_3 \setminus U_{8^k}} (\operatorname{Lip} f)^p \, d\mu &= \int_{X_3 \setminus U_{8^k}} (\operatorname{Lip} u)^p \, d\mu \\ &\leq C 8^{kp} \, |\{x \in X_3 : \operatorname{Lip} u(x) > 8^{-k}\}| + C 8^{-kp}. \end{aligned}$$

The estimate (15) then follows from (5).

We now prove (16). From (5) and (15),

$$\int_{X_3} (\operatorname{Lip} f)^p \, d\mu \leq 8^{kp} |U_{8^k}| + \int_{X_3 \setminus U_{8^k}} (\operatorname{Lip} f)^p \, d\mu \leq C.$$

Now, by the $(1, p)$-Poincaré inequality we have

$$\left(M_2^{\#} f \right)^p (x) \leq CM \left(\chi|_{X_3} \, (\operatorname{Lip} f)^p \right) (x),$$

for every $x \in X_2$. Here M denotes the uncentered Hardy-Littlewood maximal operator, and $\chi|_{X_3}$ the characteristic function on X_3. Therefore by the weak-L^1 bound for the uncentered Hardy-Littlewood maximal operator (see [20, Th. 2.2] in this setting), we get the desired estimate:

$$\begin{aligned} |F_s| &\leq |\{x \in X_2 : M \left(\chi|_{X_3} (\operatorname{Lip} f)^p \right) (x) > C s^p\}| \\ &\leq C s^{-p} \int_{X_3} (\operatorname{Lip} f)^p \, d\mu \leq C s^{-p}, \end{aligned}$$

for every $s > 0$. This proves (16), and completes the proof of the lemma. $\square$

Observe by Lemma 2.3.1, that for every $s > 0$, the restricted function $f|_{X_2 \setminus F_s}$ is Cs-Lipschitz. For each $j \in \mathbf{N}$ we let f_j be the McShane extension of $f|_{X_2 \setminus F_{2^j}}$ to a $C2^j$-Lipschitz function f_j on X; see [20, Th. 6.2]. (We are not fussy about the sort of Lipschitz extension used here; any decent one will do.) Next let

$$h = \frac{1}{k} \sum_{j=2k}^{3k-1} f_j.$$

LEMMA 3.1.5. *We have*

$$\int_{X_2} (\operatorname{Lip} h)^p \, d\mu \geq 1/C, \tag{17}$$

and

$$\operatorname{Lip} h(x) \leq \chi|_{X_2 \setminus U_{8^k}}(x) \operatorname{Lip} f(x) + \frac{C}{k} \sum_{j=2k}^{3k-1} 2^j \chi|_{U_{8^k} \cup F_{2^j}}(x), \tag{18}$$

for almost every $x \in X_2$.

Proof. We first prove (17). We require that $k \in \mathbf{N}$ be sufficiently large as determined by C, so that (10) implies $F_{4^k} \subset U_{2^k}$. We then have $f_j = u$ almost everywhere on $X_2 \setminus U_{2^k}$ for every $2k \leq j \leq 3k$. It then follows from the definition of h, that $h = u$ on $X_2 \setminus U_{2^k}$. Consequently, we can deduce from Lemma 3.1.2 that

$$\int_{X_2} |h - h_{X_2}| \geq 1/C.$$

Since h is Lipschitz we can apply the $(1, p)$-Poincaré inequality to conclude that (17) holds.

We now prove (18). Fix $j \in \mathbf{N}$. Observe that $f_j = f$ on $X_2 \setminus F_{2^j}$, and therefore Lemma 2.1.2 implies that $\operatorname{Lip} f_j = \operatorname{Lip} f$ almost everywhere on $X_2 \setminus F_{2^j}$. This and the fact that f_j is $C2^j$-Lipschitz, implies that

$$\operatorname{Lip} f_j(x) \leq \chi|_{X_2 \setminus U_{8^k}}(x) \operatorname{Lip} f(x) + C2^j \chi|_{U_{8^k} \cup F_{2^j}}(x),$$

for almost every $x \in X_2$. The estimate (18) now follows directly from the definition of h. This completes the proof. $\square$

Observe that $F_s \subset F_t$ whenever $0 \leq t \leq s$. This property with (16) and (5) implies that

$$\begin{aligned}\int_{X_2} \left(\frac{1}{k} \sum_{j=2k}^{3k-1} 2^j \chi|_{U_{8^k} \cup F_{2^j}} \right)^p d\mu &\leq \frac{1}{k^p} \int_{X_2} \sum_{j=2k}^{3k-1} \left(\sum_{i=2k}^{j} 2^i \right)^p \chi|_{U_{8^k} \cup F_{2^j}} \, d\mu \\ &\leq \frac{C}{k^p} \sum_{j=2k}^{3k-1} 2^{(j+1)p} 2^{-jp} = Ck^{1-p}.\end{aligned}$$

This with Lemma 3.1.5 and (15) implies that

$$\begin{aligned}1/C &\leq \int_{X_2} (\operatorname{Lip} h)^p \, d\mu \\ &\leq C \int_{X_2 \setminus U_{8^k}} (\operatorname{Lip} f)^p \, d\mu + C \int_{X_2} \left(\frac{1}{k} \sum_{j=2k}^{3k-1} 2^j \chi|_{U_{8^k} \cup F_{2^j}} \right)^p d\mu \\ &\leq C 8^{-k} + C k^{1-p}.\end{aligned}$$

Since $p > 1$, we achieve a contradiction when $k \in \mathbf{N}$ is sufficiently large as determined by C. This completes the proof of Proposition 3.1.1. □

3.2. *Global estimates.* In this section the previously established local estimates are used to prove global estimates for a constrained sharp fractional maximal function. Fix a ball $\tilde{B}$ in X and a Lipschitz function $u : X \longrightarrow \mathbf{R}$. For $t \geq 1$, we define the constrained sharp fractional maximal operator

$$M_t^{\#*} u(x) = \sup_B \frac{1}{\operatorname{diam} B} \fint_B |u - u_B| \, d\mu, \tag{19}$$

for every $x \in \tilde{B}$, where the supremum is taken over all balls B such that $tB \subset \tilde{B}$ and $x \in B$. Consider

$$U_\lambda^* = \{x \in \tilde{B} : M_{40}^{\#*} u(x) > \lambda\}$$

and

$$U_\lambda^{**} = \{x \in \tilde{B} : M_2^{\#*} u(x) > \lambda\},$$

for every $\lambda > 0$. The number 40 here is not specific; any large number will do.

Lemma 3.2.1. *We have $|U_\lambda^{**}| \leq C |U_{\lambda/C}^*|$ for every $\lambda > 0$.*

Proof. Let $\mathcal{F}$ be the collection of balls B such that $2B \subset \tilde{B}$ and

$$\frac{1}{\operatorname{diam} B} \fint_B |u - u_B| \, d\mu > \lambda. \tag{20}$$

Then $\mathcal{F}$ is a cover of U_λ^{**}. By a standard covering argument [20, Th. 1.2], there exists a countable subcollection $\{B_i\}_{i \in I}$ of $\mathcal{F}$, with $2B_i \cap 2B_j = \emptyset$ for every $i, j \in I$ with $i \neq j$, and such that $U_\lambda^{**} \subset \cup_{i \in I} 10 B_i$; here $I = \{1, 2, ...\}$ is a possibly finite index set. Then

$$|U_\lambda^{**}| \leq C \sum_{i \in I} |B_i|.$$

For each $i \in I$, we claim that

$$|B_i| \leq C |U_{\lambda/C}^* \cap 2B_i|, \tag{21}$$

for a constant $C > 1$ depending only on the data. This then completes the proof of the lemma. To prove (21), we fix $i \in I$ and let $B_i = B(x, r)$. Let $\mathcal{F}'$

be the collection of balls centered in $B(x,r)$ with radius $r/80$. We may assume that for every $B' \in \mathcal{F}'$ we have

$$\fint_{2B'} |u - u_{2B'}|\, d\mu \le \delta\lambda r, \tag{22}$$

for some constant $0 < \delta < 1$ depending only on C. Since, otherwise, if there is one ball $B' \in \mathcal{F}'$ such that the above inequality (22) is not true, then $2B' \subset U^*_{\lambda/C} \cap 2B_i$ by definition, and (21) follows from the doubling property of μ.

We can suppose without loss of generality that $u_{B_0} = 0$ where $B_0 = B(x, r/80)$, by otherwise translating in the range of u. We will show that

$$\fint_{B'} |u|\, d\mu \le C\delta\lambda r \tag{23}$$

for every $B' \in \mathcal{F}'$. Indeed, fix one such ball and let $B' = B(y, r/80)$ for some $y \in B(x,r)$. Let γ be a geodesic from x to y. Then there exists a collection of points $(x_j)_{j=1}^n \subset \gamma$ with $n \le C$, such that $x_0 = x$ and $x_n = y$, and $d(x_j, x_{j+1}) \le r/100$ for $j = 0, 1, ..., n-1$; and therefore $|B_j| \le C|B_j \cap B_{j+1}|$ and $B_{j+1} \subset 2B_j$, where $B_j = B(x_j, r/80) \in \mathcal{F}'$. This implies by (22) that

$$|u_{B_j} - u_{B_{j+1}}| \le C \fint_{2B_j} |u - u_{2B_j}|\, d\mu \le C\delta\lambda r$$

for each $j = 0, 1, ..., n-1$. Since $u_{B_0} = 0$, it follows from the triangle inequality that $|u_{B'}| = |u_{B_n}| \le C\delta\lambda r$. This with (22) implies (23). Now (23) implies that

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C\delta\lambda r,$$

which is a contradiction to (20) if we choose δ small enough. Thus (22) is not true and (21) follows as we explained. This completes the proof of the claim, and the proof of the lemma. □

The next proposition gives the global estimate for the level set of the constrained sharp fractional maximal function of u.

Proposition 3.2.2. *Let $\alpha \in \mathbf{N}$. There exists $k_2 \in \mathbf{N}$ that depends only on C and α such that for all integers $k \ge k_2$ and every $\lambda > 0$,*

$$\begin{aligned} |U^*_\lambda| \le\ & 2^{kp-\alpha}|U^*_{2^k\lambda}| + 8^{kp-\alpha}|U^*_{8^k\lambda}| \\ & + 10^{kp}|\{x \in \tilde{B} : \operatorname{Lip} u(x) > 10^{-k}\lambda\}|. \end{aligned} \tag{24}$$

Proof. Let $\mathcal{F}$ be the collection of balls B with $40B \subset \tilde{B}$, such that

$$\frac{1}{\operatorname{diam} B} \fint_B |u - u_B|\, d\mu > \lambda.$$

Then $\mathcal{F}$ is a cover of U^*_λ. By a standard covering argument ([20, Th. 1.2]), there exists a countable subcollection $\{B_i\}_{i\in I}$ of $\mathcal{F}$, with $40B_i \cap 40B_j = \emptyset$ for every $i, j \in I$ with $i \neq j$, and such that $U^*_\lambda \subset \cup_{i\in I} 200B_i$; here $I = \{1, 2, ...\}$ is a possibly finite index set. Then

$$|U^*_\lambda| \le C \sum_{i\in I} |B_i|. \tag{25}$$

Now for each $i \in I$, we require $k \ge k_1 + 1$ and apply Proposition 3.1.1 with $X_1 = B_i$ to obtain

$$\begin{aligned} |B_i| \le 2^{kp-\alpha}|U^{**}_{2^k\lambda} \cap 40B_i| &+ 8^{kp-\alpha}|U^{**}_{8^k\lambda} \cap 40B_i| \\ &+ 8^{k(p+1)}|\{x \in 40B_i : \operatorname{Lip} u(x) > 8^{-k}\lambda\}|. \end{aligned} \tag{26}$$

This with (25) and Lemma 3.2.1 shows that

$$\begin{aligned} |U^*_\lambda| \le C2^{kp-\alpha}|U^*_{2^k\lambda/C}| &+ C8^{kp-\alpha}|U^*_{8^k\lambda/C}| \\ &+ 8^{k(p+1)}|\{x \in \tilde{B} : \operatorname{Lip} u(x) > 8^{-k}\lambda\}|, \end{aligned} \tag{27}$$

which proves the result by choice of suitable α and k. □

Proof of Theorem 1.0.1. We now show that

$$⨍_{t\tilde{B}} |u - u_{t\tilde{B}}|\, d\mu \le C(\operatorname{diam} \tilde{B}) \left(⨍_{\tilde{B}} g^{p-\varepsilon}\, d\mu \right)^{1/(p-\varepsilon)} \tag{28}$$

holds for some $\varepsilon > 0$, quantitatively. By generality this then proves Theorem 1.0.1. Fix $\alpha = 3$ and then let $k = k_2 + 1$, where k_2 is as given by Proposition 3.2.2. Choose $0 < \varepsilon < p - 1$ so that $8^{k\varepsilon} < 2$. Now integrate (24) against the measure $d\lambda^{p-\varepsilon}$ and over the range $(0, \infty)$ to obtain

$$\begin{aligned} \int_0^\infty |U^*_\lambda|\, d\lambda^{p-\varepsilon} \le 2^{k\varepsilon-3} \int_0^\infty |U^*_{2^k\lambda}|\, d(2^k\lambda)^{p-\varepsilon} + 8^{k\varepsilon-3} \int_0^\infty |U^*_{8^k\lambda}|\, d(8^k\lambda)^{p-\varepsilon} \\ + 10^{kp} \int_0^\infty |\{x \in \tilde{B} : \operatorname{Lip} u(x) \ge 10^{-k}\lambda\}|\, d\lambda^{p-\varepsilon}. \end{aligned}$$

It follows that

$$\int_{\tilde{B}} (M^{\#*}_{40} u)^{p-\varepsilon}\, d\mu \le \frac{8^{k\varepsilon}}{3} \int_{\tilde{B}} (M^{\#*}_{40} u)^{p-\varepsilon}\, d\mu + C \int_{\tilde{B}} (\operatorname{Lip} u)^{p-\varepsilon}\, d\mu,$$

and therefore by the choice of ε, that

$$\int_{\tilde{B}} (M^{\#*}_{40} u)^{p-\varepsilon}\, d\mu \le C \int_{\tilde{B}} (\operatorname{Lip} u)^{p-\varepsilon}\, d\mu.$$

This then implies (28) with $t = 1/40$. To see this observe that

$$M^{\#*}_{40} u(x) \ge \frac{1}{\operatorname{diam} B'} ⨍_{B'} |u - u_{B'}|\, d\mu$$

for every $x \in B' = \frac{1}{40}\tilde{B}$. This completes the proof. □

4. Proof of Remark 1.2.2, Theorems 1.0.3 to 1.0.6, and Proposition 1.2.1

4.1. *Proof of Remark* 1.2.2. Remark 1.2.2 can be inferred from a careful reading of the proof of Theorem 1.0.1 or indirectly as follows. Observe by [27, Th. 2] that the hypotheses given in Remark 1.2.2 imply that the completion of (X, d, μ) admits a $(1, p)$-Poincaré inequality as per Definition 2.2.1. Hence by Theorem 1.0.1 there exists $\varepsilon > 0$ such that the completion of (X, d, μ) admits a p-Poincaré inequality for each $q > p - \varepsilon$. Now by results in [46], or the correspondingly derived Corollary 1.0.6 of the present paper, we have g is a *p-weak-upper gradient* for u as defined in [46]. Since these considerations are local it involves no loss of generality to suppose that u and g have bounded support. Therefore g is a *q-weak-upper gradient* for u for every $1 \leq q \leq p$, and the conclusion follows.

4.2. *Proof of Theorems* 1.0.3 *to* 1.0.6. Theorem 1.0.4 and 1.0.5 can be easily deduced from Theorem 1.0.1 together with [24, Th. 7.11], or the main results of [34], respectively. (It is intentional here that the hypotheses of Theorem 1.0.5 do not require that the given space be complete; see Remark 1.2.2.) Similarly, Theorem 1.0.6 is easily deduced from [46, Th. 4.9 and 4.10]. To see that Theorem 1.0.3 follows from Theorem 1.0.1 and [24, Th. 5.13] (see also [20, Th. 9.6, and Theorem 9.8]), we need to recall that complete Ahlfors regular spaces (defined below) are proper, and that complete metric measure spaces that support a doubling measure and a Poincaré inequality are quasi-convex, quantitatively. These results are stated in [27, Prop. 6.0.7] and the discussion that follows. The meaning of these words, and the words used in the statement of Theorems 1.0.3, 1.0.4, 1.0.5, and 1.0.6, can be found in the respective references given above.

4.3. *Proof of Proposition* 1.2.1. Before proving Proposition 1.2.1, we recall that a metric measure space (X, d, μ) is *Ahlfors α-regular*, $\alpha > 0$, if μ is Borel regular and there exists $C \geq 1$ such that

$$\frac{1}{C} r^{\alpha} \leq \mu(B(x, r)) \leq C r^{\alpha},$$

for every $x \in X$ and $0 < r \leq \operatorname{diam} X$.

Proof of Proposition 1.2.1. Let X be the cantor set, which we identify with the collection of all sequences (a_n) where $a_n = 0$ or 1 for every $n \in \mathbf{N}$. Define a metric d on X by $d((a_n), (b_n)) = 2^{-k}$ for any $(a_n), (b_n) \in X$, where if $a_1 \neq b_1$ then we set $k = 0$, and otherwise we let k be the greatest integer such that $a_i = b_i$ for each $1 \leq i \leq k$. For every $x \in X$ and $r > 0$, let

$$Q(x, r) = \{y \in X : d(x, y) \leq r\},$$

and call such sets *cubes*. We further let

$$\lambda Q(x,r) = \{y \in X : d(x,y) \le \lambda r\},$$

for every $\lambda > 0$. Next let μ be the Borel measure on X determined by the condition that $\mu(Q(x, 2^{-k})) = 2^{-k}$ for every $x \in X$ and $k \in \mathbf{N}$; this can be defined using Carathéodory's construction, see [13, Th. 2.10.1]. Then (X, d, μ) is an Ahlfors 1-regular metric measure space.

For each $n \in \mathbf{N}$, let $Q_n = Q(x_n, 2^{-n})$ where $x_n \in X$ is the sequence consisting of $n-1$ zeroes followed by a one, and then followed by zeroes. Notice that (Q_n) is a sequence of mutually disjoint sets, with union equal to X. It is now easy to construct a Borel function $g : X \longrightarrow \mathbf{R}$ such that

$$\int_{Q_n} g^p \, d\mu = 2^{-n} \quad \text{and} \quad \int_{Q_n} g^{p-1/n} \, d\mu = 4^{-n},$$

for every $n \in \mathbf{N}$. Observe that $g \in L^p(X)$. Moreover, we have

$$\fint_{Q_n} g^p \, d\mu = 1, \tag{29}$$

for every $n \in \mathbf{N}$, and

$$\lim_{m \to \infty} \fint_{\lambda Q_m} g^{p-1/m} \, d\mu = 0, \tag{30}$$

for every $\lambda > 0$.

Define $u : X \longrightarrow \mathbf{R}$ by the condition $u(x) = 2^{-n}$ whenever $x \in X$ satisfies $x \in Q_n$ for some $n \in \mathbf{N}$. As required we have $u \in L^p(X)$. We claim that there exists $C, \lambda \ge 1$, such that (1) holds, with $q = p$, for every $x \in X$ and $r > 0$. It suffices to show that there exists $C \ge 1$ such that

$$\frac{1}{\operatorname{diam} R} \fint_R |u - u_R| \, d\mu \le C \left(\fint_R g^p \, d\mu \right)^{1/p}, \tag{31}$$

for every cube R in X. Fix a cube R in X. If $R \subset Q_n$ for some $n \in \mathbf{N}$, then u is constant on R and (31) is trivially true. Otherwise, we have $R = 2Q_n$ for some $n \in \mathbf{N}$, and therefore

$$2^{-4} \le \frac{1}{\operatorname{diam} R} \fint_R |u - u_R| \, d\mu \le 1. \tag{32}$$

It follows from this and (29), that (31) holds with $\lambda = 1$ and $C = 2^p$. This proves the above claim. Furthermore, since (32) holds with $R = 2Q_n$ for every $n \in \mathbf{N}$, we deduce from (30) that there does not exist $1 \le q < p$ and $C, \lambda \ge 1$, such that (1) holds for all $x \in X$ and $r > 0$. This completes the proof. $\square$

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(Received June 8, 2005)