

# Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses

By KUO-CHANG CHEN

## Abstract

Poincaré made the first attempt in 1896 on applying variational calculus to the three-body problem and observed that collision orbits do not necessarily have higher values of action than classical solutions. Little progress had been made on resolving this difficulty until a recent breakthrough by Chenciner and Montgomery. Afterward, variational methods were successfully applied to the  $N$ -body problem to construct new classes of solutions. In order to avoid collisions, the problem is confined to symmetric path spaces and all new planar solutions were constructed under the assumption that some masses are equal. A question for the variational approach on planar problems naturally arises: Are minimizing methods useful only when some masses are identical?

This article addresses this question for the three-body problem. For various choices of masses, it is proved that there exist infinitely many solutions with a certain topological type, called retrograde orbits, that minimize the action functional on certain path spaces. Cases covered in our work include triple stars in retrograde motions, double stars with one outer planet, and some double stars with one planet orbiting around one primary mass. Our results largely complement the classical results by the Poincaré continuation method and Conley's geometric approach.

## 1. Introduction

Periodic and quasi-periodic solutions to the Newtonian three-body problem have been extensively studied for centuries. Until today, in general it is still a difficult task to prove the existence of solutions with prescribed topological types and masses.

Calculus of variations, in spite of its long history, should be considered a relatively new approach to the three-body problem. In 1896 Poincaré [23] made the first attempt to utilize minimizing methods to obtain solutions for the three-body problem, but found out the discouraging fact that existence of collisions does not necessarily cause a significant increment in the value of

the action functional. As a result solutions were obtained only for the strong-force potential, instead of the Newtonian case. In 1977 Gordon [13] proved a minimizing property for elliptical Keplerian orbits, including the degenerate case – collision-ejection orbit. It turns out that the actions of these orbits over one period depend only on the masses and the period, not on eccentricity. From this point of view the collision-ejection orbits and other elliptical orbits are not distinguishable. A common doubt at the time is: Are minimizing methods useful for the  $N$ -body problem? Concerning this question, Chenciner-Venturelli [8] constructed the “hip-hop” orbit for the four-body problem with equal masses and, a few months later, Chenciner-Montgomery [7] constructed the celebrated figure-8 orbit for the three-body problem with equal masses, a solution numerically discovered in [20]. Afterward, Marchal [16] found a class of solutions related to the figure-8 orbit and made important progress on excluding collision paths [17], [5]. Inspired by the discovery of the figure-8 orbit, a large number of new solutions [2], [3], [4], [11], [26] were proved to exist by variational methods. These discoveries attract much attention not only because they are not covered by classical approaches, but also due to the amusing symmetries they exhibit. On the other hand, these orbits were constructed under the assumption that some masses are equal. Except a class of nonplanar solutions constructed by varying planar relative equilibria in a direction perpendicular to the plane (see Chenciner [5], [6]), among the discoveries for the  $N$ -body problem, none of the new solutions constructed by variational methods can totally discard this constraint. A question for the variational approach, especially on planar problems, naturally arises: Are minimizing methods useful only when some masses are identical?

This article is concerned with variational methods on the existence of certain types of solutions to the planar three-body problem with various choices of masses. There is a natural way of classifying orbits by their topological types in the configuration space. From the terminology normally used in lunar theory, we call a solution *retrograde* if its homotopy type in the configuration space (with collision set removed) is the same as those retrograde orbits in the lunar theory. Detailed descriptions are left to Section 2 and 3. Our main theorem (Theorem 1) shows the existence of many periodic and quasi-periodic retrograde solutions to the three-body problem provided the mass ratios fall inside the white regions in Figure 1. The method used is a variational approach with a mixture of topological and symmetry constraints. The advantage of our approach, as Figure 1 indicates, is that it applies to a wide range of masses.

In sharp contrast with the results obtained from the classical Poincaré continuation method [22] (see [24], [18] and references therein) and Conley’s geometric approach [9], [10], our main theorem does not apply to Hill’s lunar theory and many satellite orbits, both of which treat the case with one dominant mass. It is worth mentioning that Hill’s lunar theory can also be

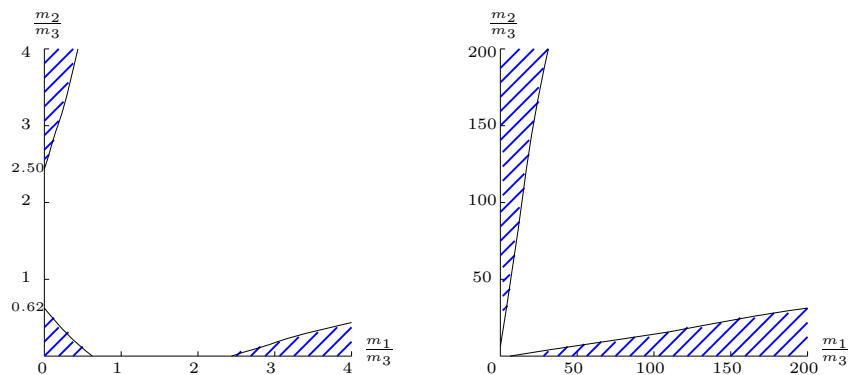


Figure 1: Admissible mass ratios (the white region) for the main theorem.

analyzed by variational methods; see Arioli-Gazzola-Terracini [1]. Cases we are able to cover include retrograde triple stars, double stars with one outer planet, and some double stars with one planet orbiting around one primary mass. See Section 2 and Figure 3 for details. Moreover, due to the minimizing properties the orbits we obtained do not contain tight binaries, and there are periodic ones with very short periods in the sense that the prime periods are small integral multiples of their prime relative periods. Classical approaches normally produce orbits with very long periods.

### 2. The Main Theorem

The planar three-body problem concerns the motion of three masses  $m_1, m_2, m_3 > 0$  moving in the complex plane  $\mathbb{C}$  in accordance with Newton’s law of gravitation:

$$(1) \quad m_k \ddot{x}_k = \frac{\partial}{\partial x_k} U(x), \quad k = 1, 2, 3$$

where  $x = (x_1, x_2, x_3), x_k \in \mathbb{C}$  is the position of  $m_k$ , and

$$U(x) = \frac{m_1 m_2}{|x_1 - x_2|} + \frac{m_2 m_3}{|x_2 - x_3|} + \frac{m_1 m_3}{|x_3 - x_1|},$$

is the potential energy (negative Newtonian potential). The kinetic energy is given by

$$K(\dot{x}) = \frac{1}{2} (m_1 |\dot{x}_1|^2 + m_2 |\dot{x}_2|^2 + m_3 |\dot{x}_3|^2) .$$

There is no loss of generality to assume that the mass center is at the origin; that is, assuming  $x$  stays inside the *configuration space*:

$$V := \{x \in \mathbb{C}^3 : m_1 x_1 + m_2 x_2 + m_3 x_3 = 0\} .$$

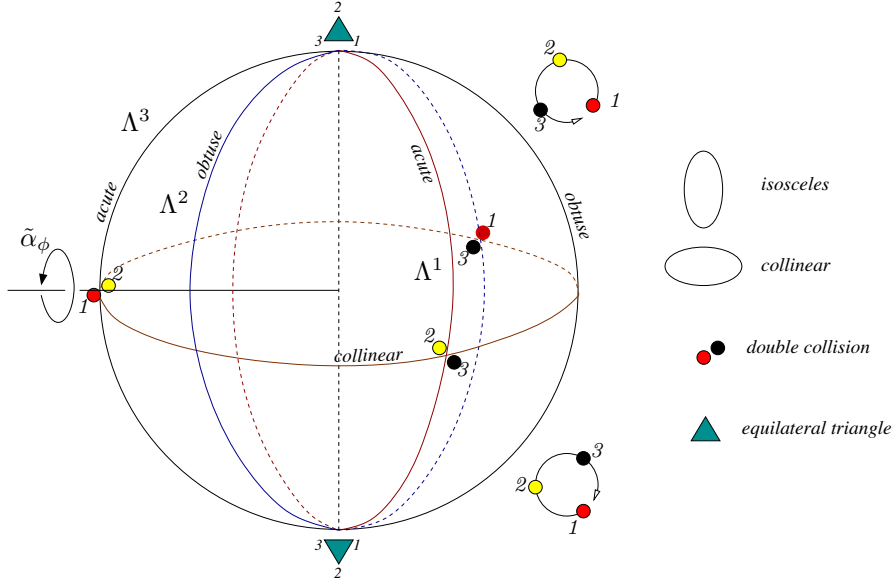


Figure 2: The unit shape sphere.

A preferred way of parametrizing  $V$  is to use Jacobi's coordinates:

$$(z_1, z_2) := \left( \sqrt{M_1}(x_2 - x_1), \sqrt{M_2}(x_3 - \hat{x}_{12}) \right),$$

where  $M_1 = \frac{m_1 m_2}{m_1 + m_2}$ ,  $M_2 = \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3}$ , and  $\hat{x}_{12} = \frac{1}{m_1 + m_2}(m_1 x_1 + m_2 x_2)$  is the mass center of the binary  $\{x_1, x_2\}$ . The *reduced configuration space*  $\tilde{V}$  is obtained by quotient out from  $V$  the rotational symmetry given by the  $SO(2)$ -action:  $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$ . The identification  $\tilde{V} = V/SO(2)$  is via the Hopf map

$$(2) \quad (u_1, u_2, u_3) := (|z_1|^2 - |z_2|^2, 2 \operatorname{Re}(\bar{z}_1 z_2), 2 \operatorname{Im}(\bar{z}_1 z_2)).$$

Each single point in  $\tilde{V}$  represents a congruence class of triangles formed by the three mass points, and each point on its unit sphere  $\{|u|^2 = 1\}$ , called the *unit shape sphere*, represents a similarity class of triangles. The signed area of the triangle is given by  $\frac{1}{2}u_3$ .

Figure 2, due to Moeckel [19], relates the configurations of the three bodies with points on the unit shape sphere. In the figure  $\Lambda^j$  represents isosceles triangles with  $j$ th mass equally distant from the other two. The equator ( $u_3 = 0$ ) represents collinear configurations. On the upper hemisphere ( $u_3 > 0$ ), triangles with vertices  $\{x_1, x_2, x_3\}$  are positively oriented; on the lower hemisphere they are negatively oriented. The poles correspond to equilateral triangles.

Let  $\Delta := \{x \in \mathbb{C}^3 : x_i = x_j \text{ for some } i \neq j\}$  be the variety of collision configurations. It is invariant under rotations and its projection  $\tilde{\Delta}$  in  $\tilde{V}$  is the union of three lines emanating from the origin (the triple collision). Each line represents a similarity class of one type of double collision. Let  $S^3$  be the unit

sphere in  $V$  and  $S^2$  be the unit shape sphere. The Hopf fibration (2) renders  $S^3 \setminus \Delta$  the structure of an  $SO(2)$ -bundle over  $S^2 \setminus \tilde{\Delta}$ , whose fundamental group is a free group with two generators. For  $\phi > 0$ , let  $\alpha_\phi$  be the following loop in  $V \setminus \Delta$ :

$$(3) \quad \alpha_\phi(t) := e^{\phi t i} \left( m_3(M - m_2) - m_2 M e^{-2\pi t i}, \right. \\ \left. m_3(M + m_1) + m_1 M e^{-2\pi t i}, -(m_1 + m_2)M \right),$$

where  $M = m_1 + m_2 + m_3$  is the total mass. The homotopy class of the projection  $\tilde{\alpha}_\phi$  of  $\alpha_\phi$  in  $\tilde{V} \setminus \tilde{\Delta}$  over  $t \in [0, 1]$  is one of the two generators for  $\pi_1(S^2 \setminus \tilde{\Delta})$ . The left side of Figure 2 depicts the path  $\tilde{\alpha}_\phi$  over  $t \in [0, 1]$ .

A solution  $x$  of (1) is called *relative periodic* if its projection  $\tilde{x}$  in the reduced configuration space  $\tilde{V}$  is periodic. The *prime relative period* of  $x$  is the prime period of  $\tilde{x}$ . Our major result concerns the existence of relative periodic solutions to the three-body problem that are homotopic to  $\alpha_\phi$  in  $V \setminus \Delta$  respecting the rotation and reflection symmetry of  $\alpha_\phi$ . A precise description is given in (9). These types of solutions, called *retrograde orbits*, are of special importance in the three-body problem. When  $0 < m_1, m_2 \ll m_3$ , the search for this type of solutions is an important problem in lunar theory. A typical example is the system Sun-Jupiter-Asteroid. When  $0 < m_3 \ll m_2, m_1$ , these types of solutions are sometimes called satellite orbits or comet orbits. If all masses are comparable in size and none of them stay far from the other two, then the system forms a triple star or triple planet. Another interesting case is  $0 < m_2 \ll m_1, m_3$ . The binary  $m_1, m_3$  form a double star (or double planet) and  $m_2$  is a planet (or satellite) orbiting around  $m_1$ . There is no evident borderline between these categories. The dash lines in Figure 3 make a rough sketch of the borders between them.

There is no loss of generality in assuming  $m_3 = 1$ . Let  $M = m_1 + m_2 + 1$  be the total mass. Define functions  $J : [0, 1] \rightarrow \mathbb{R}_+$  and  $F, G : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  by

$$(4) \quad J(s) := \int_0^1 \frac{1}{|1 - s e^{2\pi t i}|} dt,$$

$$(5) \quad F(m_1, m_2) := \frac{3}{2} \left[ \frac{2^{2/3} - 1}{\max\{m_i\}} + 1 - \left( \frac{M}{m_1 + m_2} \right)^{\frac{1}{3}} \right],$$

$$(6) \quad G(m_1, m_2) := \frac{1}{m_1} \left( J \left( \frac{m_1}{M^{1/3}(m_1 + m_2)^{2/3}} \right) - 1 \right) \\ + \frac{1}{m_2} \left( J \left( \frac{m_2}{M^{1/3}(m_1 + m_2)^{2/3}} \right) - 1 \right).$$

The following is our main theorem.

**THEOREM 1.** *Let  $m_3 = 1$ ,  $M = m_1 + m_2 + 1$  be the total mass, and let  $F, G$  be as in (5), (6). Then the three-body problem (1) has infinitely many*

*periodic and quasi-periodic retrograde orbits provided*

$$(7) \quad F(m_1, m_2) > G(m_1, m_2).$$

*Furthermore, there exists a periodic retrograde orbit whose prime period is twice its prime relative period.*

Theorem 1 applies to the complement of the shaded region in Figure 3. Following from a minimizing property described in Section 3, orbits given by Theorem 1 do not possess tight binaries. In Section 6 we will explain this and demonstrate a more general theorem. Classical results on retrograde orbits treat the case with one tight binary or with one dominant mass, including Hill’s lunar theory and some satellite orbits. From this point of view Theorem 1 largely complements classical results.

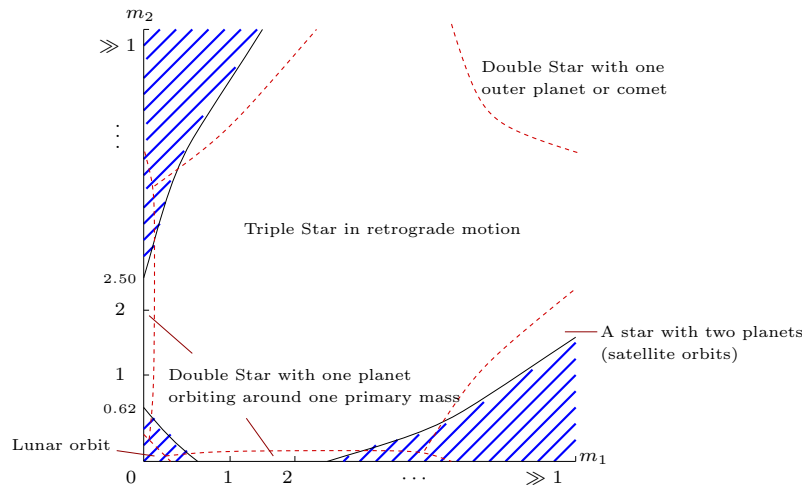


Figure 3: Theorem 1 applies to the complement of the shaded region.

### 3. A minimizing problem

In this section we set up a variational problem for which minimizers exist and which solves (1) with the claimed properties in Theorem 1.

Equation (1) and following are the Euler-Lagrange equations for the action functional  $\mathcal{A} : H_{loc}^1(\mathbb{R}, V) \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$\mathcal{A}(x) := \int_0^1 K(\dot{x}) + U(x) dt.$$

By choosing a sequence of motionless paths with greater and greater mutual distances, it is easy to see that the infimum of  $\mathcal{A}$  on  $H_{loc}^1(\mathbb{R}, V)$  is zero, which

is not attained. To ensure that the minimizing problem is solvable, we select the following ground space:

$$H_\phi := \{x \in H_{\text{loc}}^1(\mathbb{R}, V) : x(t) = e^{-\phi t} x(t+1)\},$$

where  $\phi \in (0, \pi]$  is some fixed constant. Any path  $x$  in  $H_\phi$  satisfies

$$\langle x(0), x(1) \rangle = \cos \phi |x(0)| \cdot |x(1)|.$$

Here  $\langle \cdot, \cdot \rangle$  represents the standard scalar product on  $(\mathbb{R}^2)^3$ . From this condition, the action functional  $\mathcal{A}$  restricted to  $H_\phi$  is coercive (see [3, Prop. 2], for instance). By using Fatou's lemma and the fact that any norm is weakly sequentially lower semicontinuous, it is an easy exercise to show that  $\mathcal{A}$  is weakly sequentially lower semicontinuous on  $H_\phi$ . Following a standard argument in the calculus of variations, the action functional  $\mathcal{A}$  attains its infimum on  $H_\phi$ .

Although it may appear as an easy fact, let us remark here that collision-free critical points of  $\mathcal{A}$  restricted to  $H_\phi$  are classical solutions to (1). If  $H_\phi^*$  is the space  $H_\phi$  except that the configuration space  $V$  is replaced by  $(\mathbb{R}^2)^3$ , then on  $H_\phi^*$  the fundamental lemmas for the calculus of variations are clearly applicable. Now if  $x$  is a collision-free critical point of  $\mathcal{A}$  restricted to  $H_\phi$ , from the first variation of  $\mathcal{A}$  constrained to  $H_\phi$ , at  $x$  we have

$$0 = \delta_h \mathcal{A}(x) = - \int_0^1 \sum_{k=1}^3 \left( m_k \ddot{x}_k - \frac{\partial U}{\partial x_k} \right) \cdot h_k dt$$

for any  $h = (h_1, h_2, h_3) \in C_0^\infty([0, 1], V)$ . Let  $y_k = m_k \ddot{x}_k - \frac{\partial U}{\partial x_k}$ , then

$$(y_1(t), y_2(t), y_3(t)) \in V^\perp$$

for any  $t$ . A basis for the subspace  $V^\perp$  of  $(\mathbb{R}^2)^3$  is

$$\{(m_1, 0, m_2, 0, m_3, 0), (0, m_1, 0, m_2, 0, m_3)\}.$$

Therefore  $y_i(t) = m_i \alpha(t)$  for some  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  and for each  $i$ . It can be easily verified that  $\sum_{k=1}^3 y_k(t) = 0$ , that is  $(m_1 + m_2 + m_3)\alpha(t) = 0$ . Then  $\alpha$  and hence every  $y_i$  is identically zero. This proves that  $x$  is indeed a classical solution of (1).

The conventional definition of inner product on the Sobolev space  $H^1([0, 1], V)$  defines an inner product on  $H_\phi$  as well:

$$\langle x, y \rangle_\phi := \int_0^1 \langle x(t), y(t) \rangle + \langle \dot{x}(t), \dot{y}(t) \rangle dt.$$

Critical points of  $\mathcal{A}$  on  $H_\phi$  are critical points of  $\mathcal{A}$  on  $H^1([0, 1], V)$ . One can easily verify that, for any  $x \in H_\phi$  and  $\tau \in \mathbb{R}$ ,

$$\begin{aligned} \mathcal{A}(x) &= \int_\tau^{1+\tau} K(\dot{x}) + U(x) dt, \\ \langle x, y \rangle_\phi &= \int_\tau^{1+\tau} \langle x(t), y(t) \rangle + \langle \dot{x}(t), \dot{y}(t) \rangle dt. \end{aligned}$$

From these observations, any critical point  $x$  of  $\mathcal{A}$  on  $H_\phi$  is a solution of (1), but possibly with collisions. If we can show that  $x$  has no collision on  $[0, 1]$ , then there is no collision at all and  $x$  indeed solves (1) for any  $t \in \mathbb{R}$ . Moreover,  $x$  is periodic if  $\frac{\phi}{\pi}$  is rational; it is quasi-periodic if  $\frac{\phi}{\pi}$  is irrational.

Consider a linear transformation  $g$  on  $H_\phi$  defined by

$$(8) \quad (g \cdot x)(t) := \overline{x(-t)}.$$

The space of  $g$ -invariant paths in  $H_\phi$  is denoted by  $H_\phi^g$ . That is,

$$H_\phi^g := \{x \in H_\phi : g \cdot x = x\}.$$

Observe that  $g$  is an isometry of order 2, and the action functional  $\mathcal{A}$  defined on  $H_\phi$  is  $g$ -invariant. By Palais' principle of symmetric criticality [21], any collision-free critical point of  $\mathcal{A}$  while restricted to  $H_\phi^g$  is also a collision-free critical point of  $\mathcal{A}$  on  $H_\phi$ , and hence solves (1).

Let  $\alpha_\phi$  be as in (3). The space  $\mathcal{X}_\phi$  of retrograde paths in  $H_\phi^g$  is defined as the path-component of collision-free paths in  $H_\phi^g$  containing  $\alpha_\phi$ . In other words,

$$(9) \quad \mathcal{X}_\phi := \left\{ x \in H_\phi^g : \begin{array}{l} x(t) \notin \Delta \text{ for any } t, \text{ } x \text{ is homotopic to } \alpha_\phi \text{ in } V \setminus \Delta \\ \text{within the class of collision-free paths in } H_\phi^g \end{array} \right\}.$$

The set  $\mathcal{X}_\phi$  is an open subset of  $H_\phi^g$ . Therefore, critical points of  $\mathcal{A}$  in  $\mathcal{X}_\phi$ , if they exist, are retrograde orbits. Now we consider the following minimizing problem:

$$(10) \quad \inf_{x \in \mathcal{X}_\phi} \mathcal{A}(x).$$

As noted before, the action functional  $\mathcal{A}$  is coercive and hence attains its infimum on the weak closure of  $\mathcal{X}_\phi$ . The boundary  $\partial\mathcal{X}_\phi$  of  $\mathcal{X}_\phi$  consists of paths in  $H_\phi^g$  that have nonempty intersection with the collision set  $\Delta$ . The next two sections are devoted to proving the inequality

$$\inf_{x \in \mathcal{X}_\phi} \mathcal{A}(x) < \inf_{x \in \partial\mathcal{X}_\phi} \mathcal{A}(x)$$

for  $\phi \in (0, \pi]$  sufficiently close to  $\pi$ , under the assumptions in Theorem 1.

#### 4. Upper bound estimates for the action functional $\mathcal{A}$

This section is devoted to providing an upper bound estimate for (10).

Assume  $m_3 = 1$ ,  $\phi \in (0, \pi]$ , and  $M = m_1 + m_2 + 1$ . Let

$$Q(t) := \frac{1}{(M\phi)^{2/3}} e^{\phi ti},$$

$$R(t) := \frac{1}{(m_1 + m_2)^{2/3} (2\pi - \phi)^{2/3}} e^{(\phi - 2\pi)ti},$$



and

$$\begin{aligned} x^{(\phi)}(t) &= (x_1^{(\phi)}, x_2^{(\phi)}, x_3^{(\phi)}) \\ &:= (Q(t) - m_2 R(t), Q(t) + m_1 R(t), -(m_1 + m_2) Q(t)). \end{aligned}$$

It is routine to verify that  $x^{(\phi)} \in \mathcal{X}_\phi$ . See Figure 4 for the retrograde path  $x^{(\phi)}$ .

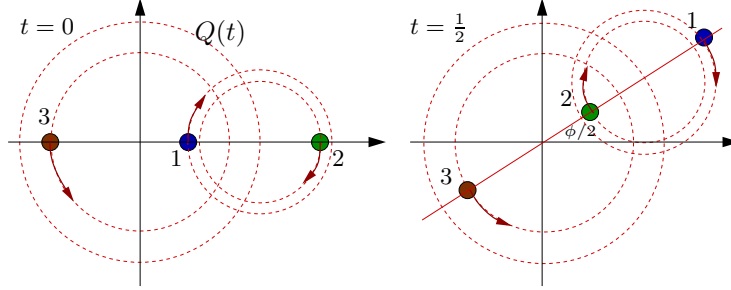


Figure 4: The retrograde path  $x^{(\phi)}$ .

The calculation for  $K(\dot{x}^{(\phi)})$  is simple:

$$\begin{aligned} |\dot{x}_1^{(\phi)}|^2 &= \frac{\phi^{2/3}}{M^{4/3}} + m_2^2 \frac{(2\pi - \phi)^{2/3}}{(m_1 + m_2)^{4/3}} + 2m_2 \frac{\phi^{1/3}(2\pi - \phi)^{1/3}}{M^{2/3}(m_1 + m_2)^{2/3}} \cos(2\pi t) \\ |\dot{x}_2^{(\phi)}|^2 &= \frac{\phi^{2/3}}{M^{4/3}} + m_1^2 \frac{(2\pi - \phi)^{2/3}}{(m_1 + m_2)^{4/3}} - 2m_1 \frac{\phi^{1/3}(2\pi - \phi)^{1/3}}{M^{2/3}(m_1 + m_2)^{2/3}} \cos(2\pi t) \\ |\dot{x}_3^{(\phi)}|^2 &= (m_1 + m_2)^2 \frac{\phi^{2/3}}{M^{4/3}} \\ K(\dot{x}^{(\phi)}) &= \frac{1}{2} \left[ (m_1 + m_2) \frac{\phi^{2/3}}{M^{1/3}} + \frac{m_1 m_2 (2\pi - \phi)^{2/3}}{(m_1 + m_2)^{1/3}} \right]. \end{aligned}$$

Note that  $K(\dot{x}^{(\phi)})$  is independent of time. Define

$$(11) \quad \xi = \xi(m_1, m_2, \phi) := \frac{1}{M^{1/3}(m_1 + m_2)^{2/3}} \left( \frac{\phi}{2\pi - \phi} \right)^{\frac{2}{3}},$$

$$(12) \quad \xi_\pi := \xi(m_1, m_2, \pi) = \frac{1}{M^{1/3}(m_1 + m_2)^{2/3}}.$$

Let  $J(s)$  be as in (4). In terms of  $J$  and  $\xi$ , the contribution of  $U(x^{(\phi)})$  to the total action can be written

$$\begin{aligned} \int_0^1 U(x^{(\phi)}) dt &= \int_0^1 \frac{m_1 m_2}{|x_1^{(\phi)} - x_2^{(\phi)}|} + \frac{m_1}{|x_1^{(\phi)} - x_3^{(\phi)}|} + \frac{m_2}{|x_2^{(\phi)} - x_3^{(\phi)}|} dt \\ &= \int_0^1 \frac{m_1 m_2 (2\pi - \phi)^{2/3}}{(m_1 + m_2)^{1/3}} + \left(\frac{\phi^{2/3}}{M^{1/3}}\right) \frac{m_1}{|1 - m_2 \xi e^{-2\pi t i}|} \\ &\quad + \left(\frac{\phi^{2/3}}{M^{1/3}}\right) \frac{m_2}{|1 - m_1 \xi e^{-2\pi t i}|} dt \\ &= \frac{m_1 m_2 (2\pi - \phi)^{2/3}}{(m_1 + m_2)^{1/3}} + \left(\frac{\phi^2}{M}\right)^{\frac{1}{3}} \left(m_1 J(m_2 \xi) + m_2 J(m_1 \xi)\right). \end{aligned}$$

Combining this with  $K(\dot{x}^{(\phi)})$ , we have proved

LEMMA 2. Assume  $m_3 = 1$ . Let  $J, \xi, \xi_\pi$  be as in (4), (11), (12). Then

$$\begin{aligned} \inf_{x \in \mathcal{X}_\phi} \mathcal{A}(x) &\leq \frac{3m_1 m_2}{2} \frac{(2\pi - \phi)^{2/3}}{(m_1 + m_2)^{1/3}} \\ &\quad + \left(\frac{\phi^2}{M}\right)^{\frac{1}{3}} \left[\frac{m_1 + m_2}{2} + m_1 J(m_2 \xi) + m_2 J(m_1 \xi)\right]. \end{aligned}$$

In particular, when  $\phi = \pi$ ,

$$\begin{aligned} \inf_{x \in \mathcal{X}_\pi} \mathcal{A}(x) &\leq \frac{3m_1 m_2 \pi^{2/3}}{2(m_1 + m_2)^{1/3}} \\ &\quad + \frac{\pi^{2/3}}{M^{1/3}} \left[\frac{m_1 + m_2}{2} + m_1 J(m_2 \xi_\pi) + m_2 J(m_1 \xi_\pi)\right]. \end{aligned}$$

### 5. Lower bound estimates for $\mathcal{A}$ on collision paths

Let  $x = (x_1, x_2, x_3)$  be any path in  $H_{\text{loc}}^1(\mathbb{R}, V)$ . From the assumption on the center of mass the action functional  $\mathcal{A}$  can be written

$$(13) \quad \mathcal{A}(x) = \frac{1}{M} \sum_{i < j} m_i m_j \int_0^1 \frac{1}{2} |\dot{x}_i - \dot{x}_j|^2 + \frac{M}{|x_i - x_j|} dt.$$

This formulation has been used to construct Lagrange’s equilateral solutions by Venturelli [25] and Zhang-Zhou [27]. Each integral in this expression will be estimated by the formula in the first subsection below. In the second subsection, we will provide lower bound estimates for collision paths in  $\partial\mathcal{X}_\phi$ .

5.1. *An estimate for the Keplerian action functional.* Given any  $\phi \in (0, \pi]$ ,  $T > 0$ , consider the following path space:

$$\begin{aligned} \Gamma_{T,\phi} &:= \{\mathbf{r} \in H^1([0, T], \mathbb{C}) : \langle \mathbf{r}(0), \mathbf{r}(T) \rangle = |\mathbf{r}(0)| |\mathbf{r}(T)| \cos \phi\}, \\ \Gamma_{T,\phi}^* &:= \{\mathbf{r} \in \Gamma_{T,\phi} : \mathbf{r}(t) = 0 \text{ for some } t \in [0, T]\}. \end{aligned}$$

The symbol  $\langle \cdot, \cdot \rangle$  stands for the standard scalar product in  $\mathbb{R}^2 \cong \mathbb{C}$ . Let  $\mu, \alpha$  be positive constants. Define a functional  $I_{\mu,\alpha,T} : H^1([0, T], \mathbb{C}) \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$I_{\mu,\alpha,T}(\mathbf{r}) := \int_0^T \frac{\mu}{2} |\dot{\mathbf{r}}|^2 + \frac{\alpha}{|\mathbf{r}|} dt.$$

In terms of polar coordinates,  $\mathbf{r} = re^{i\theta}$ ; then

$$I_{\mu,\alpha,T}(\mathbf{r}) = \int_0^T \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{\alpha}{r} dt.$$

This is actually the action functional for the Kepler problem with reduced mass  $\mu$  and some suitable gravitation constant, under the assumption that the mass center is at rest. Each integral in (13) is of this form. In this sense, expression (13) is essentially treating the system as three Kepler problems. The proposition below is an extension of a result in [4, Th. 3.1]. It concerns the minimizing problem for  $I_{\mu,\alpha,T}$  over  $\Gamma_{T,\phi}$  and  $\Gamma_{T,\phi}^*$ . We reproduce it here because (15) is not contained in [4], and the proof below is shorter and makes no use of Marchal’s theorem [17], [5].

PROPOSITION 3. *Let  $\phi \in (0, \pi]$ ,  $T > 0$ ,  $\mu > 0$ ,  $\alpha > 0$  be constants. Then*

$$(14) \quad \inf_{\mathbf{r} \in \Gamma_{T,\phi}} I_{\mu,\alpha,T}(\mathbf{r}) = \frac{3}{2} (\mu \alpha^2 \phi^2)^{\frac{1}{3}} T^{\frac{1}{3}},$$

$$(15) \quad \inf_{\mathbf{r} \in \Gamma_{T,\phi}^*} I_{\mu,\alpha,T}(\mathbf{r}) = \frac{3}{2} (\mu \alpha^2 \pi^2)^{\frac{1}{3}} T^{\frac{1}{3}}.$$

*Proof.* Consider the following subset of  $\Gamma_{T,\phi}$ ,

$$\Delta_{T,\phi} = \{ \mathbf{r} = re^{i\theta} \in H^1([0, T], \mathbb{C}) : \theta(0) = \theta(T) - \phi = 0 \},$$

which consists of paths that start from the positive real axis and end on  $\{re^{i\phi} : r \geq 0\}$ . Let

$$\Delta_{T,\phi}^* = \{ \mathbf{r} = re^{i\theta} \in \Delta_{T,\phi} : r(t) = 0 \text{ for some } t \in [0, T] \}.$$

It is easy to show that both  $\Delta_{T,\phi}$  and  $\Delta_{T,\phi}^*$  are weakly closed.

Given any  $\mathbf{r} \in \Gamma_{T,\phi}$  (resp.  $\Gamma_{T,\phi}^*$ ), there is an  $A \in \mathbf{O}(2)$  and  $\tilde{\mathbf{r}} \in \Delta_{T,\phi}$  (resp.  $\Delta_{T,\phi}^*$ ) such that  $\tilde{\mathbf{r}} = A\mathbf{r}$  and  $I_{\mu,\alpha,T}(\mathbf{r}) = I_{\mu,\alpha,T}(\tilde{\mathbf{r}})$ . This is because the space  $\Gamma_{T,\phi}$  (resp.  $\Gamma_{T,\phi}^*$ ) is actually the image of  $\mathbf{O}(2)$  acting on  $\Delta_{T,\phi}$  (resp.  $\Delta_{T,\phi}^*$ ). Therefore, we may just consider the minimizing problem over  $\Delta_{T,\phi}$  and  $\Delta_{T,\phi}^*$ .

Let  $\mathbf{r}_\phi \in \Delta_{T,\phi}^*$  be a minimizer of  $I_{\mu,\alpha,T}$  on  $\Delta_{T,\phi}^*$ . Suppose  $\xi_1 = \mathbf{r}_\phi(0)$ ,  $\xi_2 = \mathbf{r}_\phi(T)$ , then clearly  $\mathbf{r}_\phi$  also minimizes  $I_{\mu,\alpha,T}$  over paths with fixed ends  $\xi_1, \xi_2$ . In particular, this implies  $\mathbf{r}_\phi$  is a Keplerian orbit with collision(s), and thus has zero angular momentum almost everywhere. Now we recall a result by Gordon [13, Lemma 2.1] that implies such a path with lowest possible action

is the collision (or ejection) orbit that begins (or ends) with zero velocity, whereby

$$\inf_{\Delta_{T,\phi}^*} I_{\mu,\alpha,T} = \frac{3}{2}(\mu\alpha^2\pi^2)^{\frac{1}{3}}T^{\frac{1}{3}}.$$

This proves (15).

When  $\phi = \pi$ , the path  $\mathbf{r}_\pi$  can be extended to a loop by concatenating  $\mathbf{r}_\pi$  with its complex conjugate; that is,

$$\mathbf{R}(t) = \begin{cases} \mathbf{r}_\pi(t) & \text{for } t \in [0, T] \\ \overline{\mathbf{r}_\pi(2T-t)} & \text{for } t \in (T, 2T]. \end{cases}$$

By Gordon's theorem [13],

$$I_{\mu,\alpha,T}(\mathbf{r}_\pi) = \frac{1}{2} \int_0^{2T} \frac{\mu}{2} |\dot{\mathbf{R}}|^2 + \frac{\alpha}{|\mathbf{R}|} dt \geq \frac{3}{2}(\mu\alpha^2\pi^2)^{\frac{1}{3}}T^{\frac{1}{3}}.$$

The lower bound on the right-hand side is achieved when and only when  $\mathbf{r}_\pi$  is half of an elliptical Keplerian orbit (including collision-ejection orbits) with prime period  $2T$ . This proves (14) for the case  $\phi = \pi$ .

Now suppose  $\mathbf{r}_\phi \in \Delta_{T,\phi}$  minimizes  $I_{\mu,\alpha,T}$  over  $\Delta_{T,\phi}$  for  $\phi \in (0, \pi)$ . Consider the circular Keplerian orbit with prime period  $\frac{2\pi T}{\phi}$ :

$$\tilde{\mathbf{r}}_\phi(t) = \left( \frac{\alpha T^2}{\mu \phi^2} \right)^{\frac{1}{3}} e^{\frac{t}{T}\phi i}.$$

The calculation for  $I_{\mu,\alpha,T}(\tilde{\mathbf{r}}_\phi)$  is easy:

$$\begin{aligned} I_{\mu,\alpha,T}(\tilde{\mathbf{r}}_\phi) &= \frac{\phi}{2\pi} \int_0^{\frac{2\pi T}{\phi}} \frac{\mu}{2} |\dot{\tilde{\mathbf{r}}}_\phi|^2 + \frac{\alpha}{|\tilde{\mathbf{r}}_\phi|} dt = \frac{3\phi}{2\pi} \left( \frac{\mu\alpha^2\pi^2}{2} \right)^{\frac{1}{3}} \left( \frac{2\pi T}{\phi} \right)^{\frac{1}{3}} \\ &= \frac{3}{2} (\mu\alpha^2\phi^2)^{\frac{1}{3}} T^{\frac{1}{3}} < \inf_{\Delta_{T,\phi}^*} I_{\mu,\alpha,T}. \end{aligned}$$

The value of  $I_{\mu,\alpha,T}(\tilde{\mathbf{r}}_\phi)$  is indeed the right-hand side of (14). The last inequality shows that  $\mathbf{r}_\phi$  has no collision at all, and therefore it is a Keplerian orbit with nonzero angular momentum. Note that any other circular Keplerian orbits in  $\Delta_{T,\phi}$  that wind around the origin by an angle  $2k\pi + \phi$ ,  $k \in \mathbb{Z} \setminus \{0\}$  have higher action than  $\tilde{\mathbf{r}}_\phi$ . Now it remains to show that  $\mathbf{r}_\phi$  is circular.

From the first variation of  $I_{\mu,\alpha,T}$  with respect to  $r$ , it is easy to see that  $\dot{r}(0) = \dot{r}(T) = 0$ . Since  $\mathbf{r}_\phi = r e^{i\theta} \in \Delta_{T,\phi}$  is a nondegenerate conic section, there are constants  $\mathfrak{p} > 0$ ,  $\epsilon \geq 0$ ,  $\theta_0 \in [0, 2\pi)$  such that

$$\frac{\mathfrak{p}}{r} = 1 + \epsilon \cos(\theta - \theta_0).$$

Differentiating the identity with respect to  $t = 0$  and  $T$ , this yields

$$-\epsilon \sin(-\theta_0) \cdot \dot{\theta}(0) = 0 = -\epsilon \sin(\phi - \theta_0) \cdot \dot{\theta}(T).$$

The only possibility is  $\epsilon = 0$  because  $\phi \in (0, \pi)$  and the angular momentum is nonzero. This shows the minimizing orbit  $\mathbf{r}_\phi$  is a circular Keplerian orbit, completing the proof.  $\square$

5.2. *Lower bound estimates for collision paths.* First note that  $[0, \frac{1}{2}]$  is a fundamental domain of the action  $g$  defined in (8). Let  $x \in \partial\mathcal{X}_\phi$ ; then  $x_i(t) = x_j(t)$  for some  $t \in [0, \frac{1}{2}]$  and  $i \neq j$ . Assume for now  $i = 1, j = 2$ . According to  $g$ -invariance and the definition of  $H_\phi$ , all masses are aligned on the real axis at  $t = 0$ , and

$$e^{-\frac{\phi}{2}i}x\left(\frac{1}{2}\right) = e^{-\frac{\phi}{2}i}\overline{x\left(-\frac{1}{2}\right)} = \overline{e^{\frac{\phi}{2}i}x\left(-\frac{1}{2}\right)} = \overline{e^{-\frac{\phi}{2}i}x\left(\frac{1}{2}\right)}.$$

This says all masses will be aligned on the line  $\{re^{\frac{\phi}{2}i} : r \in \mathbb{R}\}$  at  $t = \frac{1}{2}$ . Therefore,

$$\begin{aligned} x_1 - x_2 &\in \Gamma_{\frac{1}{2}, \frac{\phi}{2}}^* \text{ or } \Gamma_{\frac{1}{2}, \pi - \frac{\phi}{2}}^*, \\ x_1 - x_3, x_2 - x_3 &\in \Gamma_{\frac{1}{2}, \frac{\phi}{2}} \text{ or } \Gamma_{\frac{1}{2}, \pi - \frac{\phi}{2}}. \end{aligned}$$

Suppose both  $x_1 - x_3$  and  $x_2 - x_3$  belong to  $\Gamma_{\frac{1}{2}, \frac{\phi}{2}}$ . By (13), (14), and (15),

$$\begin{aligned} \mathcal{A}(x) &= \frac{2}{M} \sum_{i < j} m_i m_j \int_0^{\frac{1}{2}} \frac{1}{2} |\dot{x}_i - \dot{x}_j|^2 + \frac{M}{|x_i - x_j|} dt \\ &\geq \frac{2}{M} \left[ \frac{3}{2} m_1 m_2 (M\pi)^{\frac{2}{3}} \left(\frac{1}{2}\right)^{\frac{1}{3}} + \frac{3}{2} (m_1 m_3 + m_2 m_3) \left(\frac{M\phi}{2}\right)^{\frac{2}{3}} \left(\frac{1}{2}\right)^{\frac{1}{3}} \right]. \end{aligned}$$

If either  $x_1 - x_3$  or  $x_2 - x_3$ , say  $x_1 - x_3$ , belongs to  $\Gamma_{\frac{1}{2}, \pi - \frac{\phi}{2}}$ , then the term involving  $m_1 m_3$  becomes

$$\frac{3}{M} m_1 m_3 \left( M \left( \pi - \frac{\phi}{2} \right) \right)^{\frac{2}{3}} \left( \frac{1}{2} \right)^{\frac{1}{3}}.$$

Since  $\phi \in (0, \pi]$ , this results in a larger lower bound estimate than the one we obtained.

The estimates for other cases,  $x_1, x_3$  collide or  $x_2, x_3$  collide, are similar. To summarize, we have proved the following lemma.

LEMMA 4. *Let  $S_3$  be the permutation group for  $\{1, 2, 3\}$ . Then*

$$\inf_{x \in \partial\mathcal{X}_\phi} \mathcal{A}(x) \geq \frac{3}{2M^{1/3}} \min_{\sigma \in S_3} \left[ m_{\sigma_1} m_{\sigma_2} (2\pi)^{\frac{2}{3}} + (m_{\sigma_1} m_{\sigma_3} + m_{\sigma_2} m_{\sigma_3}) \phi^{\frac{2}{3}} \right].$$

*In particular, when  $\phi = \pi$ ,*

$$\inf_{x \in \partial\mathcal{X}_\pi} \mathcal{A}(x) \geq \frac{3\pi^{2/3}}{2M^{1/3}} \left[ \frac{(2^{2/3} - 1)m_1 m_2 m_3}{\max\{m_i\}} + m_1 m_2 + m_2 m_3 + m_1 m_3 \right].$$

## 6. Proof of Main Theorems

We begin with the proof of Theorem 1:

*Proof of Theorem 1.* Assume  $m_3 = 1$ ,  $M = m_1 + m_2 + 1$ . Let  $F(m_1, m_2)$ ,  $G(m_1, m_2)$  be as in (5), (6). Suppose  $F(m_1, m_2) > G(m_1, m_2)$ . Then

$$\begin{aligned} 0 &< \frac{m_1 m_2}{M^{1/3}} \left( F(m_1, m_2) - G(m_1, m_2) \right) \\ &= \frac{3}{2M^{1/3}} \left[ \frac{(2^{2/3} - 1)m_1 m_2}{\max\{m_i\}} + m_1 m_2 - m_1 m_2 \left( \frac{M}{m_1 + m_2} \right)^{\frac{1}{3}} \right] \\ &\quad - \frac{1}{M^{1/3}} [m_1 (J(m_2 \xi_\pi) - 1) + m_2 (J(m_1 \xi_\pi) - 1)] \\ &= \frac{3}{2M^{1/3}} \left[ \frac{(2^{2/3} - 1)m_1 m_2}{\max\{m_i\}} + m_1 m_2 + m_1 + m_2 \right] - \frac{3m_1 m_2}{2(m_1 + m_2)^{1/3}} \\ &\quad - \frac{1}{M^{1/3}} \left[ \frac{m_1 + m_2}{2} + m_1 J(m_2 \xi_\pi) + m_2 J(m_1 \xi_\pi) \right]. \end{aligned}$$

Thus, by Lemma 2 and Lemma 4,

$$\inf_{x \in \mathcal{X}_\pi} \mathcal{A}(x) < \inf_{x \in \partial \mathcal{X}_\pi} \mathcal{A}(x).$$

By continuity of the bounds in Lemmas 2 and 4 with respect to  $\phi$ , there is some  $\epsilon > 0$  such that

$$\inf_{x \in \mathcal{X}_\phi} \mathcal{A}(x) < \inf_{x \in \partial \mathcal{X}_\phi} \mathcal{A}(x)$$

for any  $\phi \in (\pi - \epsilon, \pi]$ . This proves the existence of infinitely many periodic and quasi-periodic retrograde orbits for the three-body problem (1) under the assumption (7). By the construction of  $\mathcal{X}_\phi$ , the prime period of any minimizer for the case  $\phi = \pi$  is twice its prime relative period. This completes the proof for Theorem 1.  $\square$

Boundary curves of the shaded regions in Figure 1 are implicitly defined by  $F(m_1, m_2) = G(m_1, m_2)$ . Clearly the simple criterion stated in Theorem 1 can be generalized to a more precise but complicated one by comparing the estimates in Lemmas 2 and 4 for general  $\phi$ . In Section 2, we ventured that solutions given by Theorem 1 do not possess tight binaries. This can be seen from the following generalization of Theorem 1.

**THEOREM 5.** *Let  $m_3 = 1$ ,  $M = m_1 + m_2 + 1$  be the total mass, and let  $J$ ,  $\mathcal{X}_\phi$ ,  $\xi$  be as in (4), (9), (11).*

- (a) Given any  $\phi \in (0, \pi]$ , the three-body problem (1) has a retrograde solution that minimize the action functional  $\mathcal{A}$  in  $\mathcal{X}_\phi$  provided

$$\begin{aligned} & \frac{3m_1m_2}{2} \frac{(2\pi - \phi)^{2/3}}{(m_1 + m_2)^{1/3}} + \left(\frac{\phi^2}{M}\right)^{\frac{1}{3}} \left[ \frac{m_1 + m_2}{2} + m_1J(m_2\xi) + m_2J(m_1\xi) \right] \\ & < \frac{3}{2M^{1/3}} \min_{\sigma \in \mathcal{S}_3} \left[ m_{\sigma_1}m_{\sigma_2} (2\pi)^{\frac{2}{3}} + (m_{\sigma_1}m_{\sigma_3} + m_{\sigma_2}m_{\sigma_3})\phi^{\frac{2}{3}} \right]. \end{aligned}$$

- (b) Let  $x \in \mathcal{X}_\phi$  be an action-minimizing retrograde solution described in (a), and let

$$\bar{r}_{ij} = \max_{t \in [0,1]} |x_i(t) - x_j(t)|, \quad \underline{r}_{ij} = \min_{t \in [0,1]} |x_i(t) - x_j(t)|.$$

Then

$$(16) \quad \mathcal{A}(x) \geq \frac{1}{M} \sum_{i < j} m_i m_j \left[ \frac{1}{2}(\bar{r}_{ij} - \underline{r}_{ij})^2 + \frac{1}{2}\underline{r}_{ij}^2\phi^2 + \frac{M}{\bar{r}_{ij}} \right].$$

*Proof.* Part (a) follows directly from the estimates in Lemma 2 and Lemma 4.

Writing  $x_i - x_j$  in polar form  $r_{ij}e^{i\theta_{ij}}$ , we see that  $\theta_{ij}(1) - \theta_{ij}(0) = \phi$  and

$$\begin{aligned} \int_0^1 \frac{1}{2} |\dot{x}_i - \dot{x}_j|^2 + \frac{M}{|x_i - x_j|} dt &= \int_0^1 \frac{1}{2} (\dot{r}_{ij}^2 + r_{ij}^2 \dot{\theta}_{ij}^2) + \frac{M}{r_{ij}} dt \\ &\geq \frac{1}{2} \left( \int_0^1 |\dot{r}_{ij}| dt \right)^2 + \frac{1}{2}\underline{r}_{ij}^2 \left( \int_0^1 |\dot{\theta}_{ij}| dt \right)^2 + \frac{M}{\bar{r}_{ij}} \\ &\geq \frac{1}{2}(\bar{r}_{ij} - \underline{r}_{ij})^2 + \frac{1}{2}\underline{r}_{ij}^2\phi^2 + \frac{M}{\bar{r}_{ij}}. \end{aligned}$$

Part (b) follows easily from this observation and identity (13). □

Now let us see how (16) implies action-minimizers have no tight binaries. Firstly, Lemma 2 provides a precise upper bound estimate for the value of  $\mathcal{A}(x)$  in (16). In (16), the term  $\frac{M}{\bar{r}_{ij}}$  gives a positive lower bound  $C_1$  for  $\bar{r}_{ij}$ ,  $\frac{1}{2}\underline{r}_{ij}^2\phi^2$  gives an upper bound  $C_2$  for  $\underline{r}_{ij}$ , and  $\frac{1}{2}(\bar{r}_{ij} - \underline{r}_{ij})^2$  gives an upper bound  $C_3$  for  $\bar{r}_{ij} - \underline{r}_{ij}$ . Combining all these, we have

$$C_1 \leq \bar{r}_{ij} \leq C_2 + C_3.$$

We may choose  $C_1, C_2, C_3$  so that these inequalities hold for each pair of  $i < j$ . The ratio  $\bar{r}_{ij}/\bar{r}_{ik}$  is then bounded by  $\frac{C_2+C_3}{C_1}$  for any choice of  $i, j, k$ , which means no binary is “tight” relative to other binaries.

### 7. Some examples

The three examples below demonstrate how the admissible masses in Figure 1 can be obtained by direct calculations. Regions not included in these examples can be analyzed in the same fashion. As we shall see, the usefulness of Theorem 1 indeed relies on several nice features of the function  $J(s)$ . Most importantly,  $J(s)$  is strictly increasing on  $[0, 1)$ ,  $J'(0) = 0$ , and its value is considerably closer to 1 when  $s$  is away from 1. See the appendix.

*Example 6.* Consider the case  $1 = m_3 \leq m_1 = m$ ,  $m_2 = \lambda m$ ,  $\lambda \geq 1$ . Then

$$F(m, \lambda m) = \frac{3}{2} \left[ \frac{2^{2/3} - 1}{\lambda m} + 1 - \left( 1 + \frac{1}{(1 + \lambda)m} \right)^{\frac{1}{3}} \right].$$

$$mF(m, \lambda m) = \frac{3}{2} \left[ \frac{2^{2/3} - 1}{\lambda} - \frac{1}{(1 + \lambda) \left( 1 + \left( 1 + \frac{1}{(1 + \lambda)m} \right)^{1/3} + \left( 1 + \frac{1}{(1 + \lambda)m} \right)^{2/3} \right)} \right]$$

$$\geq \frac{3}{2} \left[ \frac{2^{2/3} - 1}{\lambda} - \frac{1}{3(1 + \lambda)} \right] =: a(\lambda).$$

Note that  $a(\lambda)$  is decreasing. Using the fact that  $J(s)$  is strictly increasing on  $[0, 1)$  (see (20)),

$$mG(m, \lambda m) = J \left( \frac{m^{1/3}}{(1 + (1 + \lambda)m)^{1/3} (1 + \lambda)^{2/3}} \right) - 1$$

$$+ \frac{1}{\lambda} \left( J \left( \frac{\lambda m^{1/3}}{(1 + (1 + \lambda)m)^{1/3} (1 + \lambda)^{2/3}} \right) - 1 \right)$$

$$< J \left( \frac{1}{1 + \lambda} \right) - 1 + \frac{1}{\lambda} \left( J \left( \frac{\lambda}{1 + \lambda} \right) - 1 \right) =: b(\lambda).$$

The function  $J(s)$  can be approximated by (17) with any desired precision. One simple way of finding those  $\lambda$  satisfying  $a(\lambda) > b(\lambda)$  is the following. Using the the monotonicity of  $a(\lambda)$  and  $J(s)$ , on any interval of the form  $[\bar{\lambda}, \bar{\lambda} + 1]$ , we have

$$a(\lambda) \geq \frac{3}{2} \left[ \frac{2^{2/3} - 1}{1 + \bar{\lambda}} - \frac{1}{2(2 + \bar{\lambda})} \right],$$

$$b(\lambda) < J \left( \frac{1}{1 + \bar{\lambda}} \right) - 1 + \frac{1}{\bar{\lambda}} \left( J \left( \frac{1 + \bar{\lambda}}{2 + \bar{\lambda}} \right) - 1 \right)$$

for any  $\lambda \in [\bar{\lambda}, \bar{\lambda} + 1]$ . For  $\bar{\lambda} = 1, 2, 3, 4, 5$ , the above lower bound for  $a(\lambda)$  is greater than the above upper bound for  $b(\lambda)$ . This implies  $a(\lambda) > b(\lambda)$ , and hence  $F(m, \lambda m) > G(m, \lambda m)$ , for any  $\lambda \in [1, 6]$ . The estimates for the case



$1 = m_3 \leq m_2 = m$ ,  $m_1 = \lambda m$ ,  $\lambda \geq 1$ , are identical. Theorem 1 applies to regions  $A1$  and  $A2$  in Figure 5. In comparison with Figure 3, this example covers triple stars in retrograde motions and double stars with one retrograde planet or comet. Some of those orbits are shown in Figure 6 and 7. The first orbit in Figure 6 satisfies  $(m_1, m_2, m_3) = (\pi - 1, 1, 3.03 \times 10^{-6})$ ,  $\phi = \pi$ , and the initial conditions are approximately

$$x(0) = (1, 1 - \pi, -3.4812), \quad \dot{x}(0) = (-0.3183i, 0.6817i, -1.1330i).$$

The other orbit is similar but with  $\phi < \pi$ . The upper left orbit in Figure 7 has equal masses. It was first numerically discovered by Hénon [14] (see also Moore [20]). The upper right orbit has masses  $(m_1, m_2, m_3) = (8, 8, 1)$  and initial conditions

$$x(0) = (0.6525, -0.5009, -1.2122), \quad \dot{x}(0) = (-1.6412i, 2.1361i, -3.9587i).$$

The other two orbits have  $\phi = \pi$  and masses  $(m_1, m_2, m_3) = (1.5, 7, 1)$ ,  $(3, 5, 1)$ . Their initial conditions are approximately

$$\begin{aligned} x(0) &= (-0.6142, 0.2677, -0.9526), & \dot{x}(0) &= (3.1288i, -0.2365i, -3.0377i); \\ x(0) &= (0.6822, -0.2282, -0.9055), & \dot{x}(0) &= (-1.5006i, 1.4974i, -2.9852i). \end{aligned}$$

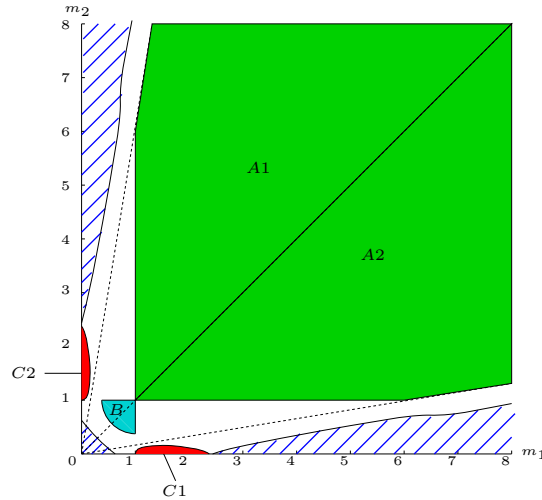


Figure 5: Admissible masses shown in Examples 6, 7, 8.

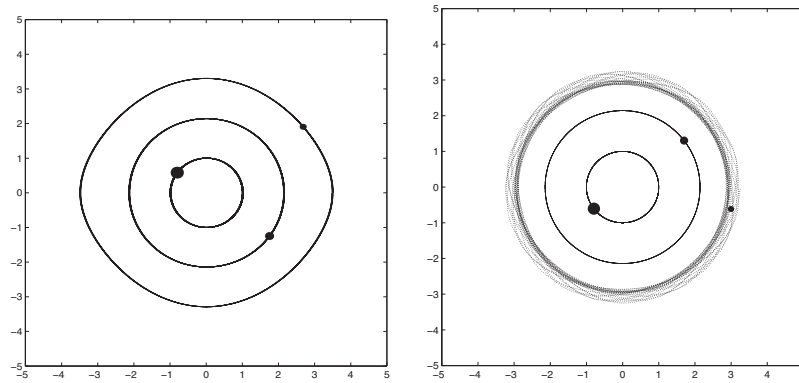
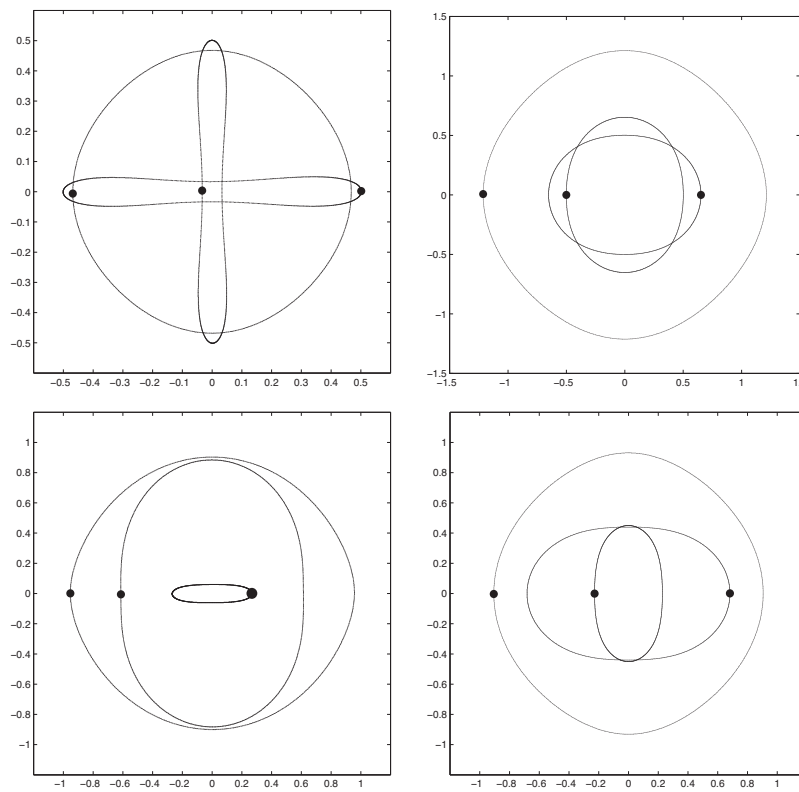


Figure 6: Double stars with retrograde planets

Figure 7: Retrograde triple stars with  $(m_1, m_2)$  in region A

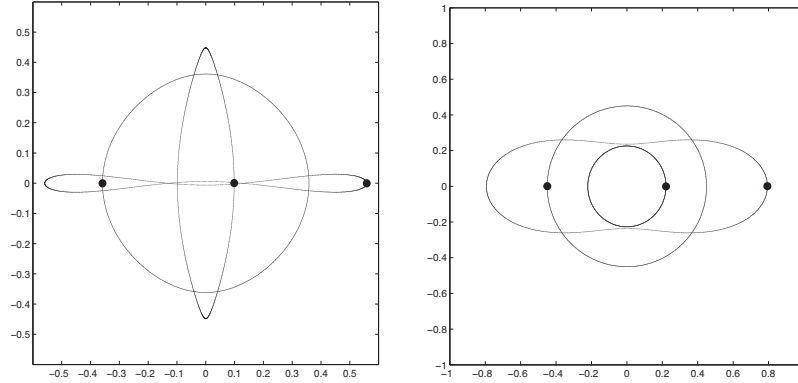


Figure 8: Retrograde triple stars with  $(m_1, m_2)$  in regions B, C

*Example 7.* Consider another type of triple star in retrograde motions:  $0 < m_1, m_2 \leq m_3 = 1, m_1 m_2 = \alpha^2, \alpha > \frac{1}{2}$ . Then  $2\alpha \leq m_1 + m_2 \leq 1 + \alpha^2$  and

$$\begin{aligned}
 F(m_1, m_2) &= \frac{3}{2} \left[ 2^{2/3} - \left( 1 + \frac{1}{m_1 + m_2} \right)^{\frac{1}{3}} \right] \\
 &\geq \frac{3}{2} \left[ 2^{2/3} - \left( 1 + \frac{1}{2\alpha} \right)^{\frac{1}{3}} \right] =: c(\alpha).
 \end{aligned}$$

Again, use the fact that  $J(s)$  is strictly increasing on  $[0, 1)$ ,

$$\begin{aligned}
 G(m_1, m_2) &\leq \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \left( J \left( \frac{1}{(1 + m_1 + m_2)^{1/3} (m_1 + m_2)^{2/3}} \right) - 1 \right) \\
 &\leq \frac{1 + \alpha^2}{\alpha^2} \left( J \left( \frac{1}{(1 + 2\alpha)^{1/3} (2\alpha)^{2/3}} \right) - 1 \right) =: d(\alpha).
 \end{aligned}$$

It is not hard to see that  $c(\alpha)$  is increasing and  $d(\alpha)$  is decreasing. One can verify, by using (17), that  $0.5543 \approx c(0.62) > d(0.62) \approx 0.5374$ . Thus  $c(\alpha) > d(\alpha)$ , and hence  $F(m_1, m_2) > G(m_1, m_2)$ , for any  $\alpha \in [0.62, 1]$ . Theorem 1 applies to the region B in Figure 5. A typical example is the first orbit in Figure 8, where the masses are  $(m_1, m_2, m_3) = (0.5, 0.8, 1)$  and the initial conditions are approximately

$$x(0) = (0.5589, 0.09859, -0.3583), \quad \dot{x}(0) = (-0.2211i, 1.5881i, -1.1599i).$$

*Example 8.* Consider  $0 < m_3 = 1 \leq m_1 = m, m_2 = \epsilon \approx 0$ . This case covers double stars with one planet orbiting around the heaviest mass. Numerically, the inferior mass of the action minimizer encircles the heaviest mass

along a peanut-shaped loop.

$$F(m, \epsilon) = \frac{3}{2} \left[ \frac{2^{2/3} - 1}{m} + 1 - \left( 1 + \frac{1}{m + \epsilon} \right)^{\frac{1}{3}} \right],$$

$$mF(m, \epsilon) = \frac{3}{2} \left[ 2^{2/3} - 1 - \left( \frac{m}{m + \epsilon} \right) \frac{1}{1 + \left( 1 + \frac{1}{m + \epsilon} \right)^{1/3} + \left( 1 + \frac{1}{m + \epsilon} \right)^{2/3}} \right]$$

$$> \frac{3}{2} \left[ 2^{2/3} - 1 - \frac{1}{1 + \left( 1 + \frac{1}{m} \right)^{1/3} + \left( 1 + \frac{1}{m} \right)^{2/3}} \right] + o(1) \text{ as } \epsilon \rightarrow 0.$$

Define the function in the last line without  $o(1)$  by  $e(m)$ . By (18), as  $\epsilon \rightarrow 0$ ,

$$mG(m, \epsilon) = J \left( \frac{m}{(1 + m + \epsilon)^{1/3} (m + \epsilon)^{2/3}} \right) - 1 + o(1)$$

$$\leq J \left( \frac{m^{1/3}}{(1 + m)^{1/3}} \right) - 1 + o(1).$$

Define the function in the last line without  $o(1)$  by  $f(m)$ . The function  $e(m)$  is decreasing and  $f(m)$  is increasing. By using (17), we obtain  $0.4371 \approx e(2.44) > f(2.44) \approx 0.4365$ . This implies  $F(m, \epsilon) > G(m, \epsilon)$  (and hence  $F(\epsilon, m) > G(\epsilon, m)$ ) for  $m \in [1, 2.44]$  and  $\epsilon$  sufficiently small. The regions of admissible masses are  $C1$  and  $C2$ . A typical example is the second orbit in Figure 8, where the masses are  $(m_1, m_2, m_3) = (2, 0.01, 1)$  and the initial conditions are approximately

$$x(0) = (0.2209, 0.7934, -0.4498), \quad \dot{x}(0) = (0.7127i, -1.0786i, -1.4146i).$$

The discussion for the case  $m_1 < m_3, m_2 \approx 0$  (or  $m_2 < m_3, m_1 \approx 0$ ) is similar. In this case the inferior mass for the action minimizer penetrates, without bias, across the nearly circular orbits formed by the primaries. Figure 9 shows two such solutions. Both of them have masses  $(m_1, m_2, m_3) = (10^{-7}, 0.7, 1)$ . The angle  $\phi$  is  $\pi$  for the first case and less than  $\pi$  in the second. Initial conditions for these orbits are approximately

$$x(0) = (2.1618, 1, -0.7), \quad \dot{x}(0) = (-0.2052i, 0.588235i, -0.411764i);$$

$$x(0) = (2.1128, 1, -0.7), \quad \dot{x}(0) = (-0.2227i, 0.588235i, -0.411764i).$$

Suppose the two primaries form a double star; then we call the inferior mass a *wagging planet* for the binary. Numerically, many wagging planets are stable. Since it is commonly believed that a large proportion of the star systems in the cosmos are double stars, there are good chances such wagging planets do exist somewhere.

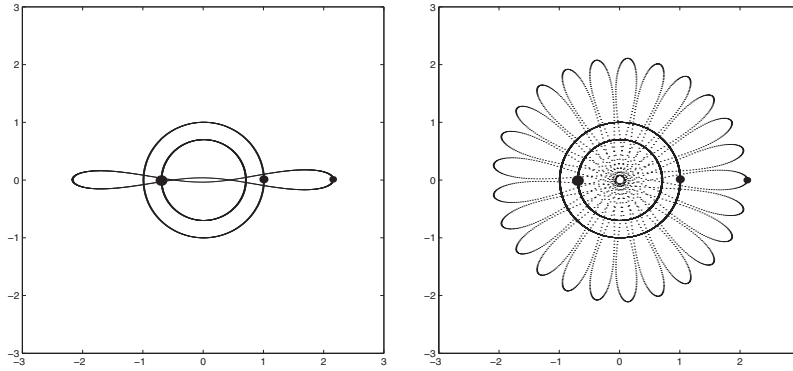


Figure 9: Double stars with wagging planets

**Appendix: Some properties of  $J(s)$**

Let  $J(s)$  be as in (4). In terms of a power series in  $\frac{4s}{(1+s)^2}$ ,  $J(s)$  can be written

$$(17) \quad J(s) = \frac{1}{1+s} \sum_{k=0}^{\infty} \left( \frac{(2k)!}{4^k (k!)^2} \right)^2 \left( \frac{4s}{(1+s)^2} \right)^k$$

for any  $s \in (0, 1)$ . Clearly  $J(0) = 1$  and  $J(1) = \infty$ . This series can be obtained by substituting  $u = \cos^2(\pi t)$ , resulting in a term

$$\frac{1}{\sqrt{(1+s)^2 - 4su}} = \frac{1}{1+s} \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} \left( \frac{4s}{(1+s)^2} \right)^k u^k$$

in the integrand, so that (4) can be expressed as

$$\begin{aligned} \int_0^1 \frac{1}{|1 - s e^{2\pi ti}|} dt &= 2 \int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1+s)^2 - 4s \cos^2(\pi t)}} dt \\ &= \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{u(1-u)}} \frac{1}{\sqrt{(1+s)^2 - 4su}} du \\ &= \frac{1}{(1+s)\pi} \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} \left( \frac{4s}{(1+s)^2} \right)^k B\left(\frac{1}{2} + k, \frac{1}{2}\right), \end{aligned}$$

where  $B$  is the beta function. Equation (17) follows easily from the last identity.

The power series (17) serves to acquire a rigorous approximation for  $J(s)$  without appealing to numerical integration. From (17),

$$(18) \quad \frac{1}{s} (J(s) - 1) = \frac{-1}{1+s} + \frac{1}{(1+s)^3} + \frac{9s}{4(1+s)^5} + \frac{25s^2}{4(1+s)^7} + \frac{1225s^3}{64(1+s)^9} + \dots$$

In particular,  $J'(0) = 0$ . Observe that  $J(s)$  is very close to 1 when  $s$  is away from 1. See Figure 10. As can be seen from Examples 6, 7, 8, this observation is quite crucial. If we use, for instance, the naïve estimate  $J(s) \leq \frac{1}{1-s}$  from the definition of  $J(s)$ , then the regions of admissible masses in Figure 5 would completely diminish.

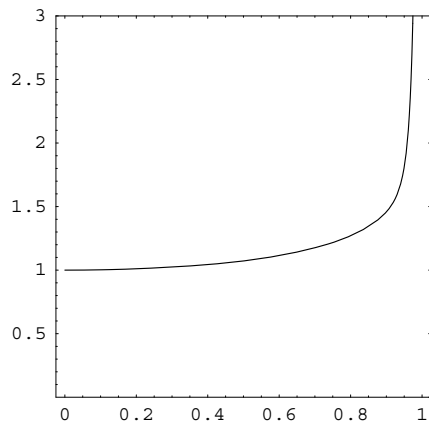


Figure 10: The graph of the function  $J(s)$ .

The function  $J(s)$  can be viewed as the potential at  $(1, 0) \in \mathbb{R}^2$  of a circular ring centered at the origin with radius  $s$  and uniform density. Moreover,

$$J(s) = \int_0^1 \frac{1}{|1 - se^{2\pi ti}|} dt = \int_0^1 \frac{1}{|e^{2\pi ti} - s|} dt.$$

Therefore  $J(s)$  can be also viewed as the potential at  $(s, 0) \in \mathbb{R}^2$  of a unit circular ring with uniform density.

This type of potential was first analyzed by Gauss [12], who provided an iterative algorithm to approximate  $J(s)$  instead of using the series (17). He observed that the value of  $J(s)$  can be obtained by computing what he called the *arithmetico-geometric mean* of  $1 + s$  and  $1 - s$ . A formula he derived is quite useful (see, for instance, [15, III.4]):

$$(19) \quad J(s) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - s^2 \sin^2 \psi}} d\psi.$$

It is not easy to see monotonicity of  $J(s)$  from (4) or (17), but from (19) it becomes apparent that  $J(s)$  is strictly increasing on  $[0, 1)$ :

$$(20) \quad \frac{d}{ds} J(s) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{s \sin^2 \psi}{(1 - s^2 \sin^2 \psi)^{3/2}} d\psi.$$

*Note added.* This article was completed around the same time as [28], in which the authors independently obtained similar results for the cases  $\phi = \pi$ ,  $\frac{2\pi}{3}$  and for certain masses.

*Acknowledgement.* I am most grateful to Rick Moeckel, Alain Chenciner, and the referees for valuable comments. Many thanks to Don Wang and Maciej Wojtkowski for enlightening conversations and their hospitality during my visit to the University of Arizona. The research work is partly supported by the National Science Council and the National Center for Theoretical Sciences in Taiwan.

NATIONAL TSING HUA UNIVERSITY, HSINCHU 300, TAIWAN  
*E-mail address:* kchen@math.nthu.edu.tw

## REFERENCES

- [1] G. ARIOLI, F. GAZZOLA, and S. TERRACINI, Minimization properties of Hill's orbits and applications to some  $N$ -body problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17** (2000), 617–650.
- [2] K.-C. CHEN, Action-minimizing orbits in the parallelogram four-body problem with equal masses, *Arch. Ration. Mech. Anal.* **158** (2001), 293–318.
- [3] ———, Binary decompositions for the planar  $N$ -body problem and symmetric periodic solutions, *Arch. Ration. Mech. Anal.* **170** (2003), 247–276.
- [4] ———, Variational methods on periodic and quasi-periodic solutions for the  $N$ -body problem, *Ergodic Theory Dynam. Systems* **23** (2003), 1691–1715.
- [5] A. CHENCINER, Action minimizing solutions in the Newtonian  $n$ -body problem: from homology to symmetry, *Proc. Internat. Congress of Mathematicians* (Beijing, 2002), Vol III, 279–294; Errata. *Proc. of the Internat. Congress of Mathematicians* (Beijing, 2002), Vol I, 651–653.
- [6] A. CHENCINER, Simple nonplanar periodic solutions of the  $n$ -body problem, *Proc. of NDDS Conference* (Kyoto, 2002); Available at [http://www.imcce.fr/Equipes/ASD/person/chenciner/chen\\_preprint.html](http://www.imcce.fr/Equipes/ASD/person/chenciner/chen_preprint.html).
- [7] A. CHENCINER and R. MONTGOMERY, A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math.* **152** (2000), 881–901.
- [8] A. CHENCINER and A. VENTURELLI, Minima de l'intégrale d'action du Problème newtonien de 4 corps de masses égales dans  $\mathbb{R}^3$ : orbites “hip-hop”, *Celestial Mech. Dynam. Astronomy* **77** (2000), 139–152.
- [9] C. CONLEY, On some new long periodic solutions of the plane restricted three body problem, *Comm. Pure Appl. Math.* **16** (1963) 449–467.
- [10] ———, The retrograde circular solutions of the restricted three-body problem via a submanifold convex to the flow, *SIAM J. Appl. Math.* **16** (1968), 620–625.
- [11] D. FERRARIO and S. TERRACINI, On the existence of collisionless equivariant minimizers for the classical  $n$ -body problem, *Invent. Math.* **155** (2004), 305–362.
- [12] C. F. GAUSS, *Werke*, Band III (1818), 331–355.
- [13] W. GORDON, A minimizing property of Keplerian orbits, *Amer. J. Math.* **99** (1977), 961–971.

- [14] M. HÉNON, A family of periodic solutions of the planar three-body problem, and their stability, *Celestial Mech.* **13** (1976), 267–285.
- [15] O. D. KELLOGG, *Foundations of potential theory*, Die Grundlehren der Mathematischen Wissenschaften, Band 31, Springer-Verlag, New York, 1967.
- [16] C. MARCHAL, The family  $P_{12}$  of the three-body problem—the simplest family of periodic orbits, with twelve symmetries per period, *Celestial Mech. Dynam. Astronomy* **78** (2000), 279–298.
- [17] ———, How the method of minimization of action avoids singularities, *Celestial Mech. Dynam. Astronomy* **83** (2002), 325–353.
- [18] K. R. MEYER, *Periodic Solutions of the  $N$ -body Problem*, *Lecture Notes in Math.* **1719**, Springer-Verlag, New York, 1999.
- [19] R. MOECKEL, Some qualitative features of the three-body problem, *Contemporary Math.* **81** (1988), 363–376.
- [20] C. MOORE, Braids in classical dynamics, *Phy. Rev. Letters* **70** (1993), 3675–3679.
- [21] R. PALAIS, The principle of symmetric criticality, *Comm. Math. Phys.* **69** (1979), 19–30.
- [22] H. POINCARÉ, *Les méthodes nouvelles de la mécanique céleste*, Vol. 1, Paris, 1892.
- [23] H. POINCARÉ, Sur les solutions périodiques et le principe de moindre action, *C. R. Acad. Sci. Paris.* **123** (1896), 915–918.
- [24] C. L. SIEGEL and J. K. MOSER, *Lectures on celestial mechanics*, Die Grundlehren der mathematischen Wissenschaften, Band 187, Springer-Verlag, New York, 1971.
- [25] A. VENTURELLI, Une caractérisation variationnelle des solutions de Lagrange du problème plan des trois corps, *C. R. Acad. Sci. Paris Sér. I Math.* **332** (2001), 641–644.
- [26] ———, Application de la minimisation de l'action au problème des  $N$  corps dans le plan et dans l'espace, *Thesis*, Université de Paris 7, 2002.
- [27] S. ZHANG and Q. ZHOU, A minimizing property of Lagrangian solutions, *Acta Math. Sinica* **17** (2001), 497–500.
- [28] S. ZHANG, Q. ZHOU, and Y. LIU, New periodic solutions for 3-body problems, *Celestial Mech. Dynam. Astronom.* **88** (2004), 365–378.

(Received April 22, 2003)

(Revised October 10, 2005)