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The Calabi-Yau conjectures for embedded surfaces

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0. Introduction

In this paper we will prove the Calabi-Yau conjectures for embedded surfaces (i.e., surfaces without self-intersection). In fact, we will prove considerably more. The heart of our argument is very general and should apply to a variety of situations, as will be more apparent once we describe the main steps of the proof later in the introduction.

The Calabi-Yau conjectures about surfaces date back to the 1960s. Much work has been done on them over the past four decades. In particular, examples of Jorge-Xavier from 1980 and Nadirashvili from 1996 showed that the immersed versions were false; we will show here that for embedded surfaces, i.e., injective immersions, they are in fact true.

Their original form was given in 1965 in [Ca] where E. Calabi made the following two conjectures about minimal surfaces (they were also promoted by S. S. Chern at the same time; see page 212 of [Ch]):

Conjecture 0.1. "*Prove that a complete minimal hypersurface in* $\mathbf{R}^n$ *must be unbounded*."

Calabi continued: "It is known that there are no compact minimal submanifolds of $\mathbf{R}^n$ (or of any simply connected complete Riemannian manifold with sectional curvature ≤ 0). A more ambitious conjecture is":

Conjecture 0.2. "*A complete* [*nonflat*] *minimal hypersurface in* $\mathbf{R}^n$ *has an unbounded projection in every* $(n-2)$*-dimensional flat subspace*."

These conjectures were revisited in S. T. Yau's 1982 problem list (see problem 91 in [Ya1]) by which time the Jorge-Xavier paper had appeared:

Question 0.3. "*Is there any complete minimal surface in* $\mathbf{R}^3$ *which is a subset of the unit ball*?"

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This was asked by Calabi, [Ca]. There is an example of a complete [nonflat] minimally immersed surface between two parallel planes due to L. Jorge and F. Xavier, [JXa2]. Calabi has also shown that such an example exists in $\mathbf{R}^4$. (One takes an algebraic curve in a compact complex surface covered by the ball and lifts it up.)"

The *immersed* versions of these conjectures turned out to be false. As mentioned above, Jorge and Xavier, [JXa2], constructed nonflat minimal immersions contained between two parallel planes in 1980, giving a counterexample to the immersed version of the more ambitious Conjecture 0.2; see also [RoT]. Another significant development came in 1996, when N. Nadirashvili, [Na1], constructed a complete immersion of a minimal disk into the unit ball in $\mathbf{R}^3$, showing that Conjecture 0.1 also failed for immersed surfaces; see [MaMo1], [LMaMo1], [LMaMo2], for other topological types than disks.

The conjectures were again revisited in Yau's 2000 millenium lecture (see page 360 in [Ya2]) where Yau stated:

QUESTION 0.4. "*It is known* [Na1] *that there are complete minimal surfaces properly immersed into the* [*open*] *ball. What is the geometry of these surfaces? Can they be embedded?...*"

As mentioned in the very beginning of the paper, we will in fact show considerably more than Calabi's conjectures. This is in part because the conjectures are closely related to properness. Recall that an immersed surface in an open subset Ω of Euclidean space $\mathbf{R}^3$ (where Ω is all of $\mathbf{R}^3$ unless stated otherwise) is *proper* if the pre-image of any compact subset of Ω is compact in the surface. A well-known question generalizing Calabi's first conjecture asks when is a complete *embedded* minimal surface proper? (See for instance question 4 in [MeP], or the "Properness Conjecture", Conjecture 5, in [Me], or question 5 in [CM7].)

Our main result is a chord arc bound[1] for intrinsic balls that implies properness. Obviously, intrinsic distances are larger than extrinsic distances, so the significance of a chord arc bound is the reverse inequality, i.e., a bound on intrinsic distances from above by extrinsic distances. This is accomplished in the next theorem:

THEOREM 0.5. *There exists a constant $C > 0$ so that if $\Sigma \subset \mathbf{R}^3$ is an embedded minimal disk, $\mathcal{B}_{2R} = \mathcal{B}_{2R}(0)$ is an intrinsic ball*[2] *in $\Sigma \setminus \partial \Sigma$ of radius $2R$, and if $\sup_{\mathcal{B}_{r_0}} |A|^2 > r_0^{-2}$ where $R > r_0$, then for $x \in \mathcal{B}_R$*

$$C \operatorname{dist}_\Sigma(x, 0) < |x| + r_0 \,. \tag{0.6}$$

[1]A chord arc bound is a bound from above and below for the ratio of intrinsic to extrinsic distances.

[2]Intrinsic balls will be denoted with script capital "b" like $\mathcal{B}_r(x)$ whereas extrinsic balls will be denoted by an ordinary capital "b" like $B_r(x)$.

The assumption of a lower bound for the supremum of the sum of the squares of the principal curvatures, i.e., $\sup_{\mathcal{B}_{r_0}} |A|^2 > r_0^{-2}$, in the theorem is a necessary normalization for a chord arc bound. This can easily be seen by rescaling and translating the helicoid. Equivalently this normalization can be expressed in terms of the curvature, since by the Gauss equation $-\frac{1}{2}|A|^2$ is equal to the curvature of the minimal surface.

Properness of a complete embedded minimal disk is an immediate consequence of Theorem 0.5. Namely, by (0.6), as intrinsic distances go to infinity, so do extrinsic distances. Precisely, if Σ is flat, and hence a plane, then obviously Σ is proper and if it is nonflat, then $\sup_{\mathcal{B}_{r_0}} |A|^2 > r_0^{-2}$ for some $r_0 > 0$ and hence Σ is proper by (0.6). In sum, we get the following corollary:

Corollary 0.7. *A complete embedded minimal disk in* $\mathbf{R}^3$ *must be proper.*

Corollary 0.7 in turn implies that the first of Calabi's conjectures is true for *embedded* minimal disks. In particular, Nadirashvili's examples cannot be embedded. We also get from it an answer to Yau's questions (Questions 0.3 and 0.4).

Another immediate consequence of Theorem 0.5 together with the one-sided curvature estimate of [CM6] (i.e., Theorem 0.2 in [CM6]) is the following version of that estimate for intrinsic balls; see question 3 in [CM7] where this was conjectured:

Corollary 0.8. *There exists* $\varepsilon > 0$, *so that if*

$$\Sigma \subset \{x_3 > 0\} \subset \mathbf{R}^3 \tag{0.9}$$

is an embedded minimal disk with intrinsic ball $\mathcal{B}_{2R}(x) \subset \Sigma \setminus \partial\Sigma$ *and* $|x| < \varepsilon R$, *then*

$$\sup_{\mathcal{B}_R(x)} |A_\Sigma|^2 \leq R^{-2}\,. \tag{0.10}$$

As a corollary of this intrinsic one-sided curvature estimate we get that the second, and "more ambitious", of Calabi's conjectures is also true for *embedded* minimal disks. In particular, Jorge-Xavier's examples cannot be embedded. Namely, letting $R \to \infty$ in Corollary 0.8 gives the following halfspace theorem:

Corollary 0.11. *The plane is the only complete embedded minimal disk in* $\mathbf{R}^3$ *in a halfspace.*

In the final section, we will see that our results for disks imply both of Calabi's conjectures and properness also for embedded surfaces with finite topology. Recall that a surface Σ is said to have finite topology if it is homeomorphic to a closed Riemann surface with a finite set of points removed or "punctures". Each puncture corresponds to an end of Σ.

The following generalization of the halfspace theorem gives Calabi's second, "more ambitious", conjecture for embedded surfaces with finite topology:

Corollary 0.12. *The plane is the only complete embedded minimal surface with finite topology in a halfspace of* $\mathbf{R}^3$.

Likewise, we get the properness of embedded surfaces with finite topology:

Corollary 0.13. *A complete embedded minimal surface with finite topology in* $\mathbf{R}^3$ *must be proper.*

Most of the classical theorems on minimal surfaces assume properness, or something which implies properness (such as finite total curvature). In particular, this assumption can now be removed from these theorems.

Before we recall in more detail some of the earlier work on these conjectures we will try to give the reader an idea of why these kinds of properness results should hold.

The proof that complete embedded minimal disks are proper, i.e., Corollary 0.7, consists roughly of the following three main steps:

(1) Show that if the surface is compact in a ball, then in this ball we have good chord arc bounds.

(2) Show that if each component of the intersection of each ball of a certain size is compact (so that by the first step we have good estimates), then each intersection with double the Euclidean balls is also compact, initially possible with a much worse constant but then by the first step with a good constant.

(3) Iterate the above two steps.

Step 1 above relies on our earlier results (see [CM3]–[CM6]; see also [CM9] for a survey) about *properly* embedded minimal disks. We will come back to this in the main body of the paper and instead here outline the proof of step 2 assuming step 1.

Suppose therefore that all intersections of the given disk with all Euclidean balls of radius r are compact and have good chord arc bounds. We will show the same for all Euclidean balls of radius $2r$.

If not; then there are two points x, $y \in B_{2r} \cap \Sigma$ in the same connected component of $B_{2r} \cap \Sigma$ but with $\text{dist}_\Sigma(x,y) \geq C\,r$ for some large constant C. Let γ be an intrinsic geodesic in $B_{2r} \cap \Sigma$ connecting x and y. By dividing γ into segments, we conclude that there must be a pair of points x_0 and y_0 on γ in B_{2r} where the balls are intrinsically far apart yet extrinsically close. We will start at these two points and build out showing that x_0 and y_0 could not connect in $B_{2r} \cap \Sigma$. This will be the desired contradiction.

By the assumption, each component of $B_r(x_0)\cap\Sigma$ is compact and by step 1 has good chord arc bounds; hence x_0 and y_0 must lie in different components. Thus we have two compact components of $B_r(x_0)\cap\Sigma$ which are extrinsically close near the center. Earlier results (the one-sided curvature estimate of [CM6]; see Theorem 0.2 there) show that half of each of these two components must have curvature bounds. Since this bound for the curvature is in terms of the size of the relevant balls, then it follows that a fixed fraction of these components must be almost flat - again relative to its size. In fact, it follows now easily that these two almost flat regions contains intrinsic balls centered at x_0 and y_0 and with radii a fixed fraction of r. We can therefore go to the boundary of these almost flat intrinsic balls and find two points x_1 and y_1; one point in each intrinsic ball so that the two points are extrinsically close yet intrinsically far apart.

Repeat the argument with x_1 and y_1 in place of x_0 and y_0 to get points x_2 and y_2. Iterating gives large regions in the surface centered at x_0 and y_0 with *a priori* curvature bounds. Once we have *a priori* curvature bounds then improvements involving stability show that even these large regions are almost flat and thus could not combine in B_{2r}. This is the desired contradiction and hence completes the outline of step 2 above of the proof that embedded minimal disks are proper.

It is clear from the definition of proper that a proper minimal surface in $\mathbf{R}^3$ must be unbounded, so the examples of Nadirashvili are not proper. Much less obvious is that the plane is the only complete *proper* immersed minimal surface in a halfspace. This is however a consequence of the strong halfspace theorem of D. Hoffman and W. Meeks, [HoMe], and implies that also the examples of Jorge-Xavier are not proper.

There has been extensive work on both properness (as in Corollary 0.7) and the halfspace property (as in Corollary 0.11) assuming various *curvature bounds*. Jorge and Xavier, [JXa1] and [JXa2], showed that there cannot exist a complete immersed minimal surface with *bounded curvature* in $\cap_i\{x_i > 0\}$; later Xavier proved that the plane is the only such surface in a halfspace, [Xa]. Recently, G. P. Bessa, Jorge and G. Oliveira-Filho, [BJO], and H. Rosenberg, [Ro], have shown that if a complete embedded minimal surface has bounded curvature, then it must be proper. This properness was extended to embedded minimal surfaces with locally bounded curvature and finite topology by Meeks and Rosenberg in [MeRo]; finite topology was subsequently replaced by finite genus in [MePRs] by Meeks, J. Perez and A. Ros.

Inspired by Nadirashvili's examples, F. Martin and S. Morales constructed in [MaMo2] a complete bounded minimal immersion which is proper in the (open) unit ball. That is, the preimages of compact subsets of the (open) unit ball are compact in the surface and the image of the surface accumulates on the boundary of the unit ball. They extended this in [MaMo3] to show that any convex, possibly noncompact or nonsmooth, region of $\mathbf{R}^3$ admits a proper complete minimal immersion of the unit disk; cf. [Na2].

Finally, we note that Calabi and P. Jones, [Jo], have constructed bounded complete holomorphic (and hence minimal) *embeddings* in higher codimension. Jones' example is a graph and he used purely analytic methods (including the Fefferman-Stein duality theorem between H^1 and BMO) while, as mentioned in Question 0.3, Calabi's approach was algebraic: Calabi considered the lift of an algebraic curve in a complex surface covered by the unit ball.

Throughout this paper, we let x_1, x_2, x_3 be the standard coordinates on $\mathbf{R}^3$. For $y \in \Sigma \subset \mathbf{R}^3$ and $s > 0$, the extrinsic and intrinsic balls are $B_s(y)$ and $\mathcal{B}_s(y)$, respectively, and $\operatorname{dist}_\Sigma(\cdot, \cdot)$ is the intrinsic distance in Σ. We will use $\Sigma_{y,s}$ to denote the component of $B_s(y) \cap \Sigma$ containing y; see Figure 1. The two-dimensional disk $B_s(0) \cap \{x_3 = 0\}$ will be denoted by D_s. The sectional curvature of a smooth surface $\Sigma \subset \mathbf{R}^3$ is K_Σ and A_Σ will be its second fundamental form. When Σ is oriented, $\mathbf{n}_\Sigma$ is the unit normal.

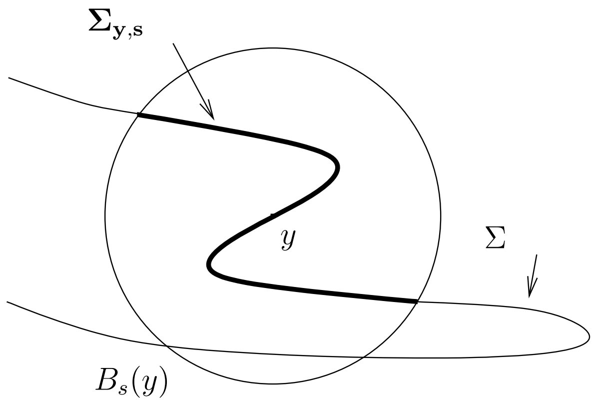

Figure 1: $\mathbf{\Sigma_{y,s}}$ denotes the component of $B_s(y) \cap \Sigma$ containing y.

We will use freely that each component of the intersection of a minimal disk with an extrinsic ball is also a disk (see, e.g., appendix C in [CM6]). This follows easily from the maximum principle since $|x|^2$ is subharmonic on a minimal surface.

In [CM9], the results of this paper as well as [CM3]–[CM6] are surveyed.

1. Theorem 0.5 and estimates for intrinsic balls

The main result of this paper (Theorem 0.5) will follow by combining the next proposition with a result from [CM6]. This next proposition gives a weak chord arc bound for an embedded minimal disk but, unlike Theorem 0.5, only for one component of a smaller extrinsic ball. The result from [CM6] will then be used to show that there is in fact only one component, giving the theorem.

PROPOSITION 1.1. *There exists* $\delta_1 > 0$ *so that if* $\Sigma \subset \mathbf{R}^3$ *is an embedded minimal disk, then for all intrinsic balls* $\mathcal{B}_R(x)$ *in* $\Sigma \setminus \partial\Sigma$, *the component* $\Sigma_{x,\delta_1 R}$ *of* $B_{\delta_1 R}(x) \cap \Sigma$ *containing* x *satisfies*

$$\Sigma_{x,\delta_1 R} \subset \mathcal{B}_{R/2}(x)\,. \tag{1.2}$$

The result that we need from [CM6] to show Theorem 0.5 is a consequence of the one-sided curvature estimate of [CM6]; it is Corollary 0.4 in [CM6]. This corollary says that if two disjoint embedded minimal disks with boundary in the boundary of a ball both come close to the center, then each has an interior curvature estimate. Precisely, this is the following result:

COROLLARY 1.3 ([CM6]). *There exist constants* $c > 1$ *and* $\varepsilon > 0$ *so that the following holds*: *Let* Σ_1 *and* Σ_2 *be disjoint embedded minimal surfaces in* $B_{cR} \subset \mathbf{R}^3$ *with* $\partial\Sigma_i \subset \partial B_{cR}$ *and* $B_{\varepsilon R} \cap \Sigma_i \neq \emptyset$. *If* Σ_1 *is a disk, then for all components* Σ_1' *of* $B_R \cap \Sigma_1$ *which intersect* $B_{\varepsilon R}$

$$\sup_{\Sigma_1'} |A|^2 \leq R^{-2}\,. \tag{1.4}$$

Using this corollary, we can now prove Theorem 0.5 assuming Proposition 1.1, whose proof will fill up the next two sections.

Proof of Theorem 0.5 *using Corollary* 1.3 *and assuming Proposition* 1.1. Let $c > 1$ and $\varepsilon > 0$ be given by Corollary 1.3 and $\delta_1 > 0$ by Proposition 1.1.

Let $x \in \mathcal{B}_R(0)$ be a fixed but arbitrary point and let Σ_0 and Σ_x be the components of

$$B_{\frac{c\,(|x|+r_0)}{\varepsilon}} \cap \Sigma \tag{1.5}$$

containing 0 and x, respectively. Here r_0 is given by the curvature assumption in the statement of the theorem. We will divide into two cases depending on whether or not we have the following inequality

$$\frac{2\,c\,(|x|+r_0)}{\delta_1\,\varepsilon} \leq R\,. \tag{1.6}$$

If (1.6) holds, then Proposition 1.1 (with radius equal to $\frac{2\,c\,(|x|+r_0)}{\delta_1\,\varepsilon}$) implies that

$$\Sigma_0 \subset \mathcal{B}_{\frac{c\,(|x|+r_0)}{\delta_1\,\varepsilon}}(0) \tag{1.7}$$

and also, since $B_{\frac{c\,(|x|+r_0)}{\varepsilon}} \subset B_{\frac{2\,c\,(|x|+r_0)}{\varepsilon}}(x)$ by the triangle inequality,

$$\Sigma_x \subset \mathcal{B}_{\frac{c\,(|x|+r_0)}{\delta_1\,\varepsilon}}(x)\,. \tag{1.8}$$

On the other hand, by definition, the embedded minimal disks Σ_0 and Σ_x are contained in $B_{\frac{c\,(|x|+r_0)}{\varepsilon}}$. Since 0 and x are in the smaller extrinsic ball

$B_{c\,(|x|+r_0)}$, then both Σ_0 and Σ_x intersect $B_{c\,(|x|+r_0)}$. Furthermore, (1.7) and (1.8) imply that Σ_0 and Σ_x are both compact and have boundary in $\partial B_{\frac{c\,(|x|+r_0)}{\varepsilon}}$. However, it follows from Corollary 1.3 and the lower curvature bound (i.e., $\sup_{\mathcal{B}_{r_0}} |A|^2 > r_0^{-2}$) that there can only be one component with all of these properties. Hence, we have $\Sigma_0 = \Sigma_x$ so that

$$\Sigma_x \subset \mathcal{B}_{\frac{c\,(|x|+r_0)}{\delta_1\,\varepsilon}}(0)\,, \tag{1.9}$$

giving the claim (0.6).

In the remaining case, where (1.6) does not hold, the claim (0.6) follows trivially. □

Before discussing the proof of Proposition 1.1, we conclude this section by noting some additional applications of Theorem 0.5. As alluded to in the introduction, an immediate consequence of Theorem 0.5 is that we get intrinsic versions of all of the results of [CM6]. For instance we get the following:

THEOREM 1.10. *Intrinsic balls in embedded minimal disks are part of properly embedded double spiral staircases. Moreover, a sequence of such disks with curvature blowing up converges to a lamination.*

For a precise statement of Theorem 1.10, see Theorem 0.1 of [CM6], with intrinsic balls instead of extrinsic balls.

A double spiral staircase consists of two multi-valued graphs (or spiral staircases) spiralling together around a common axis, without intersecting, so that the the flights of stairs alternate between the two staircases. Intuitively, an (embedded) multi-valued graph is a surface such that over each point of the annulus, the surface consists of N graphs; the actual definition is recalled in Appendix A.

2. Chord arc properties of *properly* embedded minimal disks

The proof of Proposition 1.1 will be divided into several steps over the next two sections. The first step is to prove the special case where we assume in addition that Σ is compact and has boundary in the boundary of an extrinsic ball. The advantage of this assumption is that the results of [CM3]–[CM6] can be applied directly.

2.1. *Properly embedded disks.* The next proposition gives a weak chord arc bound for a compact embedded minimal disk with boundary in the boundary of a ball. The fact that this bound is otherwise independent of Σ will be crucial later when we remove these assumptions.

PROPOSITION 2.1. *Let $\Sigma \subset \mathbf{R}^3$ be a compact embedded minimal disk. There exists a constant $\delta_2 > 0$ independent of Σ such that if $x \in \Sigma$ and $\Sigma \subset B_R(x)$ with $\partial \Sigma \subset \partial B_R(x)$, then the component $\Sigma_{x,\delta_2 R}$ of $B_{\delta_2 R}(x) \cap \Sigma$*

containing x satisfies

$$\Sigma_{x,\delta_2 R} \subset \mathcal{B}_{\frac{R}{2}}(x)\,. \tag{2.2}$$

The key ingredient in the proof of Proposition 2.1 is an effective version of the first main theorem in [CM6]. Before we can state this effective version, we need to recall two definitions from [CM6].

First, given a constant $\delta > 0$ and a point $z \in \mathbf{R}^3$, we denote by $\mathbf{C}_\delta(z)$ the (convex) cone with vertex z, cone angle $(\pi/2 - \arctan\delta)$, and axis parallel to the x_3-axis. That is,

$$\mathbf{C}_\delta(z) = \{x \in \mathbf{R}^3 \,|\, (x_3 - z_3)^2 \geq \delta^2\left((x_1 - z_1)^2 + (x_2 - z_2)^2\right)\}\,. \tag{2.3}$$

Second, recall from [CM6] that, roughly speaking, a blow-up pair (y, s) consists of a point y where the curvature is almost maximal in a (extrinsic) ball of radius roughly s. To be precise, fix a constant C_1, then a point y and a scale $s > 0$ is a *blow-up pair* (y, s) if

$$\sup_{B_{C_1\, s}(y)\cap\Sigma} |A|^2 \leq 4\, s^{-2} = 4\,|A|^2(y)\,. \tag{2.4}$$

The constant C_1 will be given by Theorem 0.7 in [CM6] that gives the existence of a multi-valued graph starting on the scale s.

We are now ready to state a local version of the first main theorem in [CM6]. This is Lemma 2.5 below and shows that a compact embedded minimal disk, with boundary in the boundary of an extrinsic ball, is part of a double spiral staircase. In particular, it consists of two multi-valued graphs spiralling together away from a collection of balls whose centers lie along a Lipschitz curve transverse to the graphs. (The centers y_i will be ordered by height around a "middle point" y_0; negative values of i should be thought of as points below y_0.)

Lemma 2.5. *Let $\Sigma \subset \mathbf{R}^3$ be a compact embedded minimal disk. There exist constants c_{in}, c_{out}, c_{dist}, $c_{\max}$, and $\delta > 0$ independent of Σ so that if $\Sigma \subset B_R$ with $\partial\Sigma \subset \partial B_R$ and*

$$\sup_{B_{R/c_{\max}}\cap\Sigma} |A|^2 \geq c_{\max}^2\, R^{-2}\,, \tag{2.6}$$

then there is a collection of blow-up pairs $\{(y_i, s_i)\}_i$ with $y_0 \in B_{R/(4c_{\text{out}})}$. In addition, after a rotation of $\mathbf{R}^3$, we have that:

(0) *For every i, we have $B_{C_1\, s_i}(y_i) \subset B_{6R/c_{\text{out}}}$.*

(1) *The extrinsic balls $B_{s_i}(y_i)$ are disjoint and the points $\{y_i\}$ lie in the intersections of the cones*

$$\cup_i\{y_i\} \subset \cap_i \mathbf{C}_\delta(y_i)\,. \tag{2.7}$$

(2) *The points* y_i *"string together" starting at* y_0*: For each* $i > 0$*, we have* $y_i \in B_{c_{\text{in}}\, s_i}(y_{i-1})$*; for each* $i < 0$*, we have* $y_i \in B_{c_{\text{in}}\, s_i}(y_{i+1})$*.*

(3) *The* y_i*'s go from top to bottom, i.e., there is a curve* $\tilde{\mathcal{S}} \subset B_{R/c_{\text{out}}} \cap \cup_i B_{c_{\text{in}}\, s_i}(y_i)$ *with*

$$\inf_{\tilde{\mathcal{S}}} x_3 \leq -\frac{\delta\, R}{2\, c_{\text{out}}} < \frac{\delta\, R}{2\, c_{\text{out}}} \leq \sup_{\tilde{\mathcal{S}}} x_3 \,. \tag{2.8}$$

(4) *"Graphical away from balls":* $B_{R/c_{\text{out}}} \cap \Sigma \setminus \cup_i B_{c_{\text{in}}\, s_i}(y_i)$ *consists of exactly two multi-valued graphs (which spiral together) with gradient* $\leq \delta/2$*.*

(5) *"Chord arc": For each* i*, we have* $B_{c_{\text{in}}\, s_i}(y_i) \cap \Sigma \subset \mathcal{B}_{c_{\text{dist}}\, s_i}(y_i)$*.*

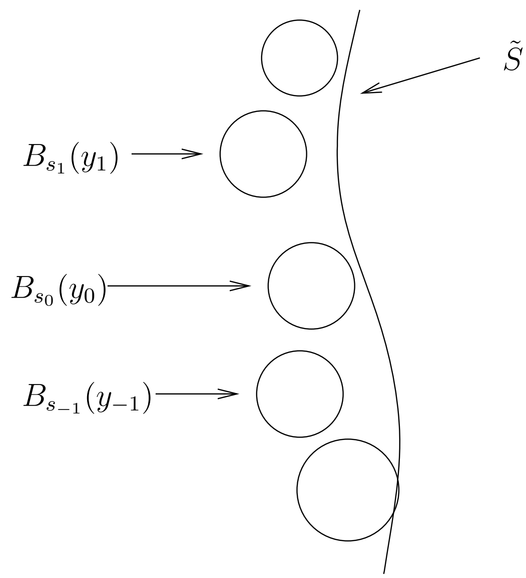

Figure 2: The balls $B_{s_i}(y_i)$ in the statement of Lemma 2.5 are disjoint, yet consecutive balls are not too far apart; cf. (2). In particular, the ratio of the radii of consecutive balls is bounded.

Note that (1)–(3) are the effective version of the fact that the singular set $\mathcal{S}$ in [CM6] is a Lipschitz graph over the x_3-axis. Property (4) says that the surface is graphical away from the balls $B_{c_{\text{in}}\, s_i}(y_i)$. Finally, (5) is a chord arc property showing that the extrinsic balls $B_{c_{\text{in}}\, s_i}(y_i)$ are contained in intrinsic balls $\mathcal{B}_{c_{\text{dist}}\, s_i}(y_i)$.

The proof of Lemma 2.5 is essentially contained in [CM6] but was not made explicit there. We will describe where to find properties (0)–(5) in [CM6], as well as the necessary modifications, over the next three subsections. The reader who wishes to take these six properties (0)–(5) for granted should jump to subsection 2.5.

2.2. *Results from* [CM6]. We will first recall a few of the results from [CM6] to be used. The first of these, Theorem 0.7 in [CM6], gives the existence of multi-valued graphs near a blow-up pair; cf. (2.4). The precise statement is the following:

LEMMA 2.9 ([CM6]). *Given $N \in \mathbf{Z}_+$ and $\varepsilon > 0$, there exist C_1 and $C_2 > 0$ so that the following holds: Let $\Sigma \subset \mathbf{R}^3$ be an embedded minimal disk with $0 \in \Sigma \subset B_R$ and $\partial\Sigma \subset \partial B_R$. If $(0, s)$ with $0 < s < R/C_1$ is a blow-up pair (i.e., satisfies (2.4) with $y = 0$ and this C_1), then there exists (after a rotation of $\mathbf{R}^3$) an N-valued graph*

$$\Sigma_g \subset \Sigma \cap \{x_3^2 \leq \varepsilon^2\,(x_1^2 + x_2^2)\} \tag{2.10}$$

over $D_{R/C_2} \setminus D_{C_1 s}$ with gradient $\leq \varepsilon$.

The second result that we will need to recall is the existence of blow-up pairs nearby a given blow-up pair. This will be used to show that the points y_i string together. This was a key ingredient in the proofs of both main theorems in [CM6] and is recorded in Proposition I.0.11 there (it was proven in Corollary III.3.5 in [CM5]). For clarity, we restate this next and give an elementary proof using [CM6]. Note, however, that we could not have used this elementary proof in [CM5] since [CM6] relies on [CM5].

LEMMA 2.11 ([CM5]). *Let N, ε, C_1, and C_2 be as in Lemma 2.9. Then there exists a constant $C_5 > 4\,C_1$ so that if*

(a) *$\Sigma \subset \mathbf{R}^3$ is an embedded minimal disk with $\Sigma \subset B_{C_5\,s}(y)$ and $\partial\Sigma \subset \partial B_{C_5\,s}(y)$;*

(b) *(y, s) is a blow-up pair,*

then we get two blow-up pairs (y_+, s_+) above y and (y_-, s_-) below y with

$$B_{C_1\,s_\pm}(y_\pm) \subset B_{C_5\,s}(y) \setminus B_{C_1\,s}(y)\,. \tag{2.12}$$

Proof. After rescaling and translating Σ, we can assume that $y = 0$ and $s = 1$. We will find the blow-up pair (y_+, s_+) above y (the other case is identical). Let Σ^+ denote the portion of Σ above 0 (i.e., above the multi-valued graph corresponding to this blow-up pair).

It is easy to see by a simple blow-up argument (Lemma 5.1 in [CM4]) that it suffices to show that

$$\sup_{z \in B_{C_5/2} \cap \Sigma^+ \setminus B_{4\,C_1}} |z|^{-2}\,|A|^2(z) \geq 4\,C_1\,. \tag{2.13}$$

We will argue by contradiction; suppose therefore that Σ_i is a sequence of embedded minimal disks satisfying (a) and (b) with $y = 0$, $s = 1$, and $C_5 = i$ but so that (2.13) fails for every i.

Rescaling the Σ_i's by a factor of $\sqrt{i}$, we get a new sequence $\tilde{\Sigma}_i$ with $\tilde{\Sigma}_i \subset B_{\sqrt{i}}$ and $\partial \tilde{\Sigma}_i \subset \partial B_{\sqrt{i}}$ and so that $|A|^2(0) \to \infty$. Hence, we can apply the first main theorem of [CM6] (Theorem 0.1 there) to get a subsequence $\tilde{\Sigma}_{i'}$ converging off of a Lipschitz curve $\mathcal{S}$ (where $|A| \to \infty$) to a foliation of $\mathbf{R}^3$ by parallel planes. Moreover, this Lipschitz curve goes through 0 and is transverse to the planes and consequently intersects every hemisphere above the plane through 0. However, this is a contradiction since (2.13) gives a scale-invariant curvature bound above this plane. □

Finally, we will need an easy consequence of the one-sided curvature estimate of [CM6] (this consequence is Corollary I.1.9 in [CM6]):

COROLLARY 2.14 ([CM6]). *There exists $\delta_0 > 0$ so that the following holds: Let $\Sigma \subset B_{2R_0}$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_{2R_0}$. If Σ contains a 2-valued graph $\Sigma_d \subset \{x_3^2 \leq \delta_0^2\,(x_1^2 + x_2^2)\}$ over $D_{R_0} \setminus D_{r_0}$ with gradient $\leq \delta_0$, then each component of*

$$B_{R_0/2} \cap \Sigma \setminus (\mathbf{C}_{\delta_0}(0) \cup B_{2r_0}) \tag{2.15}$$

is a multi-valued graph with gradient ≤ 1.

2.3. *Properties* (0)–(4) *in Lemma* 2.5. Properties (1) through (4) in Lemma 2.5 were implicit in [CM6] and we will describe below how to prove them using the results in [CM6].

We first describe how to get the blow-up points satisfying (0)–(3).

- **The slope δ and constant C_1**: Set $\delta = \delta_0$ from Corollary 2.14. Then let C_1 and C_2 be given by Lemma 2.9 with $N = 2$ and $\varepsilon = \delta/8$.
- **The initial multi-valued graph**: The lower curvature bound (2.6) and a simple blow-up argument (Lemma 5.1 in [CM4]) give a blow-up pair (y_0, s_0) with

$$B_{C_1 s_0}(y_0) \subset B_{C'\,R/c_{\max}}\,. \tag{2.16}$$

  Lemma 2.9 then gives an associated rotation of $\mathbf{R}^3$ and a 2-valued graph Σ_0 with gradient $\leq \delta/8$ over

$$D_{R/(2C_2)}(y_0) \setminus D_{C_1\,s_0}(y_0)\,. \tag{2.17}$$

  (Here we have used a slight abuse of notation since y_0 may not be in the plane $\{x_3 = 0\}$.)
- **Blow up pairs satisfying (0) are nearly parallel**: As long as c_{out} is sufficiently large, then any blow-up pair (y_i, s_i) satisfying (0) automatically has gradient $\leq \delta/3$. To see this, simply note that it has gradient $\leq \delta/8$ over *some* plane; embeddedness then forces this plane to be almost parallel to the plane $\{x_3 = 0\}$.

- **Nearby blow-up pairs satisfy (0)**: After possibly choosing $c_{\max}$ even larger, then (2.6) implies that any blow-up pair (y_i, s_i) with $y_i \in B_{2R/c_{\text{out}}}$ must have $C_1\, s_i \leq 4R/c_{\text{out}}$, i.e., must satisfy (0).
- **Blow up pairs satisfying (0), (1), and (2)**: We will iteratively apply Lemma 2.11 to blow-up pairs (y_i, s_i) satisfying (0)–(2). To get the first pair above y_0, apply Lemma 2.11 to get (y_1, s_1) above y_0 with

$$B_{C_1\, s_1}(y_1) \subset B_{C_5\, s_0}(y_0) \setminus B_{C_1\, s_0}(y_0)\,. \tag{2.18}$$

  Repeat this to find y_2, etc., until

$$B_{C_5\, s_i}(y_i) \cap \partial B_{2R/c_{\text{out}}} \neq \emptyset\,. \tag{2.19}$$

  The y_i's with $i < 0$ are constructed similarly. Note that every y_i is then contained in $B_{2R/c_{\text{out}}}$ so that (0) holds. Finally, the cone property (1) follows immediately from Corollary 2.14.
- **Property (3)**: Iteratively applying (1) directly gives (3). This is because (1) gives a lower bound for the slope of the line segment connecting consecutive y_i's.

We will next describe how to get (4) by combining (1)–(3) with results of [CM3]–[CM6]. Finally, we will establish (5) in the next subsection.

Observe first that Lemma 2.9 directly gives the gradient bound (4) on each of the corresponding 2-valued graphs. To extend this gradient bound to the rest of Σ, note that we can choose a constant C_2' so that each point

$$y \in B_{R/C_2'} \cap \Sigma \setminus \cup_i B_{C_2'\, s_i}(y_i) \tag{2.20}$$

satisfies a one-sided condition as in Corollary 1.3. Precisely, y is between the 2-valued graphs corresponding to some y_i and y_{i+1} and, furthermore, these graphs are themselves close enough together that we get two (in fact many) distinct components of

$$B_{|y-y_i|/2}(y) \cap \Sigma \tag{2.21}$$

which intersect the smaller concentric extrinsic ball

$$B_{\varepsilon\, |y-y_i|/(2c)}(y)\,. \tag{2.22}$$

Therefore, Corollary 1.3 gives a curvature estimate near y. Finally, the desired gradient bound (4) at y then follows from this curvature bound, the bound for the gradient of the 2-valued graphs y is pinched between, and the gradient estimate. The fact that there are exactly two of these multi-valued graphs was proven in Proposition II.1.3 in [CM6].

2.4. *The proof of* (5) *in Lemma* 2.5. The key to establishing (5) is to first prove a chord arc bound assuming bounded curvature (Lemma 2.23) and second to establish the curvature bound (Lemma 2.26). This chord arc bound is essentially Lemma II.2.1 of [CM6], but the statement there does not suffice for the application here. The statement that we need is the following:

LEMMA 2.23 (cf. Lemma II.2.1 in [CM6]). *There exists $C_d > 1$ so that given a constant C_a, we get another constant C_b such that the following holds: If $\Sigma \subset \mathbf{R}^3$ is an embedded minimal disk with $0 \in \Sigma \subset B_R$ and $\partial \Sigma \subset \partial B_R$ and in addition*

$$\sup_{B_R \cap \Sigma} |A|^2 \leq C_a \, R^{-2} \, , \tag{2.24}$$

then

$$\Sigma_{0, \frac{R}{C_d}} \subset \mathcal{B}_{C_b \, R}(0) \, . \tag{2.25}$$

Proof. See Appendix B. □

The second result from [CM6] that we will need is a curvature bound on a larger extrinsic ball $B_{C_3 s_i}(y_i)$ around a blow-up point (y_i, s_i). The proof of this curvature bound is essentially contained in the proof of Lemma I.1.10 in [CM6] but was not made explicit there. For completeness, we state and prove this bound below:

LEMMA 2.26 ([CM6]). *For every positive number C_3, there is a positive number C_4 with the following property. If*

(a) *$\Sigma \subset \mathbf{R}^3$ is an embedded minimal disk with $\Sigma \subset B_{C_4 \, s}(y)$ and $\partial \Sigma \subset \partial B_{C_4 \, s}(y)$,*

(b) *(y, s) is a blow-up pair,*

then we get the curvature bound

$$\sup_{B_{C_3 \, s}(y) \cap \Sigma} |A|^2 \leq C_4 \, s^{-2} \, . \tag{2.27}$$

Proof. After rescaling and translating Σ, we can assume that $y = 0$ and $s = 1$. We will argue by contradiction; suppose therefore that Σ_i is a sequence of embedded minimal disks satisfying (a) and (b) with $y = 0$, $s = 1$, and $C_4 = i$ but so that (2.27) fails for some fixed C_3.

Since both the radii i of the extrinsic balls go to infinity and

$$\sup_{B_{C_3}(0) \cap \Sigma_i} |A|^2 \to \infty \, , \tag{2.28}$$

we can apply the first main theorem of [CM6] (Theorem 0.1 there). Therefore, a subsequence $\Sigma_{i'}$ converges off of a Lipschitz curve $\mathcal{S}$ to a foliation of $\mathbf{R}^3$ by parallel planes. This convergence implies that the supremum of $|A|^2$ on each fixed extrinsic ball either goes to zero or infinity, depending on whether or not this ball intersects $\mathcal{S}$. However, this directly contradicts the assumption (b), thereby giving the lemma. □

To prove (5), we first use Lemma 2.26 to get a uniform curvature bound on larger extrinsic balls $B_{C_3' s_i}(y_i)$. Combining Lemma 2.23, and using the one-sided estimate (i.e., Corollary 1.3) to see that there is only such component, then gives (5).

2.5. *The proof of Proposition* 2.1. We will next see how properties (0)–(5) in Lemma 2.5 imply Proposition 2.1.

Proof (*of Proposition* 2.1). We will divide the proof into two cases, depending on whether or not the curvature is large, i.e., whether (2.6) holds.

Suppose first that (2.6) fails so that we have the curvature bound

$$\sup_{B_{R/c_{\max}}(x)\cap\Sigma} |A|^2 \leq c_{\max}^2 R^{-2} . \tag{2.29}$$

We can then apply Lemma 2.23 to get

$$\Sigma_{x,c_1' R} \subset \mathcal{B}_{c_1 R}(x) , \tag{2.30}$$

giving the proposition in this case.

In the second case, where (2.6) holds, the proposition will follow from Lemma 2.5. We do this in two steps.

First, for any point

$$z \in B_{\delta R/(4c_{\text{out}})}(x) \cap \Sigma , \tag{2.31}$$

we have

$$\text{dist}_\Sigma (z, \cup_i B_{c_{\text{in}} s_i}(y_i)) \leq C' R . \tag{2.32}$$

This follows immediately from the gradient bound for the multi-valued graphs given by (4) together with the fact that the points y_i go from top to bottom by (2) and (3).

Second, (1) and (5) imply a bound for the diameter of the union of the balls $B_{c_{\text{in}} s_i}(y_i)$. Namely, the balls $B_{s_i}(y_i)$ are disjoint and satisfy the cone property (1) and, therefore, we get a bound for the sum of the radii s_i of these balls

$$\sum_i s_i \leq C_0 R/c_{\text{in}} . \tag{2.33}$$

Combining this with the chord arc property (5) then gives a bound for the diameter of the union of these balls

$$\text{diam}_\Sigma (B_{R/c_{\text{out}}}(x) \cap \cup_i B_{c_{\text{in}} s_i}(y_i)) \leq C' R . \tag{2.34}$$

Combining the bounds (2.32) and (2.34), the triangle inequality gives the proposition in this case as well. □

3. The proof of Proposition 1.1

In this section, we will complete the proof of Proposition 1.1. To do this, we will first define a weak chord arc property for an intrinsic ball. This property requires that the intrinsic ball contains an entire component of Σ in a smaller extrinsic ball.

Throughout this section $\Sigma \subset \mathbf{R}^3$ is an embedded minimal disk, possibly noncompact, with boundary $\partial\Sigma$.

3.1. *Weakly chord arc.* To show Proposition 1.1, we need to prove that there is a constant $\delta_1 > 0$ so that for any intrinsic ball $\mathcal{B}_R(x) \subset \Sigma \setminus \partial\Sigma$ we have the inclusion

$$\Sigma_{x,\delta_1 R} \subset \mathcal{B}_{\frac{R}{2}}(x)\,, \tag{3.1}$$

where, as before, $\Sigma_{x,\delta_1 R}$ denotes the component of $B_{\delta_1 R}(x) \cap \Sigma$ containing x.

Since Σ is smooth, the inclusion (3.1) must hold for sufficiently small balls depending on Σ. The key step in the proof of Proposition 1.1 is to show that if (3.1) holds on one scale, then it also holds on five times the scale. (Here, when we say that it holds on a scale, we mean that it holds for all balls of this radius; cf. (A′) in the proof.) This will be done in Proposition 3.4 below. Proposition 1.1 will then follow by use of a blow-up argument (Lemma 3.39 below) to locate the largest scale where (3.1) holds and then application of Proposition 3.4 to see that (3.1) continues to hold on larger scales.

We will say that an intrinsic ball where we have the inclusion (3.1) is weakly chord arc; namely, we make the following definition:

Definition 3.2 (weakly chord arc). An intrinsic ball $\mathcal{B}_s(x) \subset \Sigma \setminus \partial\Sigma$ is said to be *δ-weakly chord arc* for some $\delta > 0$ if (3.1) holds with $R = s$ and $\delta = \delta_1$. Note that (3.1) is only possible if $\delta \leq 1/2$.

It will be important later that subballs of a weakly chord arc ball are themselves weakly chord arc. While this does not follow directly from (3.1), we do directly get that the intersections with smaller extrinsic balls are compact and have boundary in the boundary of the smaller ball. In particular, these properties will allow us to apply Proposition 2.1 to conclude that the smaller balls are themselves δ_2-weakly chord arc; this will be done in the beginning of the proof of Proposition 1.1 when we replace (A) with (A′) there.

It will be convenient to introduce notation for the largest radius of a weakly chord arc ball about a given point. We will do this next.

Given a constant δ and a point $x \in \Sigma \setminus \partial\Sigma$, we let $R_\delta(x)$ denote the largest radius where $\mathcal{B}_{R_\delta(x)}(x)$ is δ-weakly chord arc, i.e.,

$$R_\delta(x) = \sup\{R \,|\, \mathcal{B}_R(x) \subset \Sigma \setminus \partial\Sigma \text{ is } \delta\text{-weakly chord arc}\}\,. \tag{3.3}$$

Since Σ is a smooth surface, we obviously have $R_\delta(x) > 0$ for every x and any $\delta < 1/2$.

We can now state the key proposition which shows that if all intrinsic balls of radius R_0 near a point y are δ_2-weakly chord arc, then so is the five-times ball $\mathcal{B}_{5\,R_0}(y)$ about y. The constant δ_2 in the proposition is given by Proposition 2.1.

PROPOSITION 3.4. *Let $\Sigma \subset \mathbf{R}^3$ be an embedded minimal disk. There exists a constant $C_b > 1$ independent of Σ so that if $\mathcal{B}_{C_b\,R_0}(y) \subset \Sigma \setminus \partial\Sigma$ is an intrinsic ball and*

(A′) *every intrinsic subball $\mathcal{B}_{R_0}(z) \subset \mathcal{B}_{C_b\,R_0}(y)$ is δ_2-weakly chord arc,*

then, for every $s \leq 5\,R_0$, the intrinsic ball $\mathcal{B}_s(y)$ is δ_2-weakly chord arc.

3.2. *Extrinsically close yet intrinsically far apart.* In this subsection, we recall from [CM2] and [CM4] several important properties of embedded minimal surfaces with bounded curvature. The basic point is that nearby, but disjoint, minimal surfaces with bounded curvature can be written as graphs over each other of a positive function u which satisfies a useful second order elliptic equation. We will focus here on two consequences of this. The first is a chord arc result assuming an *a priori* curvature bound (see Lemma 3.6 below). The second is that this elliptic equation for u implies a Harnack inequality for u that bounds the rate at which the two disjoint surfaces can pull apart.

We will need the notion of 1/2-stability. Recall from [CM4] that a domain $\Omega \subset \Sigma$ is said to be 1/2-stable if, for all Lipschitz functions ϕ with compact support in Ω, we have the 1/2-stability inequality:

$$\frac{1}{2} \int |A|^2\,\phi^2 \leq \int |\nabla \phi|^2\,. \tag{3.5}$$

Loosely speaking, the next elementary lemma shows that if two disjoint intrinsic balls are extrinsically close (see (3.8)) and have *a priori* curvature bounds (see (3.7)), then smaller concentric intrinsic balls are almost flat and thus in particular their boundaries are far away from their centers (see the conclusion (3.9)). Since it is only this last conclusion that we need, and not the stronger statement that they are almost flat, we only state this.

LEMMA 3.6. *There exists $C_0 > 1$ so that for every $C_a > 0$, there exists $\tau > 0$ such that if $\mathcal{B}_{C_0}(x_1)$ and $\mathcal{B}_{C_0}(x_2)$ are disjoint intrinsic balls in $\Sigma \setminus \partial\Sigma$ with*

$$\sup_{\mathcal{B}_{C_0}(x_1) \cup \mathcal{B}_{C_0}(x_2)} |A|^2 \leq C_a\,, \tag{3.7}$$

$$|z_1 - z_2| < \tau\,, \tag{3.8}$$

then for $i = 1, 2$

$$B_{10}(x_i) \cap \partial \mathcal{B}_{11}(x_i) = \emptyset \, . \tag{3.9}$$

Proof. Using the argument of [CM2] (i.e., curvature estimates for 1/2-stable surfaces) we get a constant $C_0 > 1$ so that if $\mathcal{B}_{C_0/2}(z) \subset \Sigma \setminus \partial \Sigma$ is 1/2-stable, then $\mathcal{B}_{11}(z)$ is a graph with

$$B_{10}(z) \cap \partial \mathcal{B}_{11}(z) = \emptyset \, . \tag{3.10}$$

Corollary 2.13 in [CM4] gives $\tau = \tau(C_a) > 0$ so that if $|z_1 - z_2| < \tau$ and $|A|^2 \leq C_a$ on (the disjoint balls) $\mathcal{B}_{C_0}(z_i)$, then each subball

$$\mathcal{B}_{\frac{C_0}{2}}(z_i) \subset \Sigma \tag{3.11}$$

is 1/2-stable. □

As mentioned above, one of the key points in the proof of the previous lemma was that nearby, but disjoint, embedded minimal surfaces with bounded curvature can be written as graphs over each other of a positive function u. Furthermore, standard calculations show that this function u satisfies a second order elliptic equation resembling the Jacobi equation (for the Jacobi equation, the functions a_{ij}, b_j, c in (3.14) vanish). These standard, but very useful, calculations were summarized in Lemma 2.4 of [CM4] which we recall next.

Lemma 3.12 ([CM4]). *There exists* $\delta_g > 0$ *so that if* Σ *is minimal and if* u *is a positive solution of the minimal graph equation over* Σ (*i.e.*, $\{x + u(x)\, \mathbf{n}_\Sigma(x) \,|\, x \in \Sigma\}$ *is minimal*) *with*

$$|\nabla u| + |u|\, |A| \leq \delta_g \, , \tag{3.13}$$

then u *satisfies on* Σ

$$\Delta u = \operatorname{div}(a \nabla u) + \langle b, \nabla u \rangle + (c-1)|A|^2 \, u \, , \tag{3.14}$$

for functions a_{ij}, b_j, c *on* Σ *with* $|a|, |c| \leq 3\, |A|\, |u| + |\nabla u|$ *and* $|b| \leq 2\, |A|\, |\nabla u|$.

Equation (3.14) implies a uniform Harnack inequality for u which bounds the supremum of u on a compact subset of $\Sigma \setminus \partial \Sigma$ by a multiple of the infimum; see, for instance, Theorem 8.20 in [GiTr]. We will use this in the next subsection to show that two nearby, but disjoint, components of Σ with bounded curvature pull apart very slowly.

3.3. *Extending weakly chord arc to a larger scale*: *The proof of Proposition* 3.4. We are now prepared to prove Proposition 3.4, i.e., to show that if all intrinsic balls of radius R_0 near a point y are weakly chord arc, then so is the five-times ball $\mathcal{B}_{5\, R_0}(y)$ about y. To do this, we first show that $\mathcal{B}_{5\, R_0}(y)$ is still weakly chord arc, but with a worse constant. We then use Proposition 2.1 to improve the constant, i.e., to see that it is in fact δ_2-weakly chord arc.

The reader may find it helpful to compare the proof below with the simpler proof of the special case where Σ has bounded curvature, i.e., with the proof of Lemma 2.23 given in Appendix B. The difference is that here the one-sided curvature estimate is used, while there, we simply assume an *a priori* bound on the curvature.

Proof of Proposition 3.4. After rescaling and translating Σ, we can assume that $R_0 = 1$ and $y = 0$.

The proposition follows from the next claim: There exists n so that

$$\Sigma_{0,\,5} \subset \mathcal{B}_{(6n+3)\,C_0}(0)\,, \tag{3.15}$$

where $C_0 > 1$ is as given by Lemma 3.6. The proposition will follow immediately from (3.15) by applying Proposition 2.1 to $\Sigma_{0,\,5}$. Namely, (3.15) implies that the embedded minimal disk $\Sigma_{0,\,5}$ is compact and

$$\partial \Sigma_{0,\,5} \subset \partial B_5\,. \tag{3.16}$$

We can therefore apply Proposition 2.1 for any $t \leq 5$ to get that

$$\Sigma_{0,\,\delta_2\,t} \subset \mathcal{B}_{t/2}(0)\,, \tag{3.17}$$

giving the proposition.

We will prove the claim (i.e., (3.15)) by arguing by contradiction; so suppose that (3.15) fails for some large n. Consequently, we get a curve

$$\sigma \subset \Sigma_{0,\,5} \subset B_5 \tag{3.18}$$

from 0 to a point in $\partial \mathcal{B}_{(6n+3)\,C_0}(0)$. For $i = 1, \dots, n$, fix points

$$z_i \in \partial \mathcal{B}_{6i\,C_0}(0) \cap \sigma\,. \tag{3.19}$$

It follows that the intrinsic balls $\mathcal{B}_{3\,C_0}(z_i)$:

- Are disjoint.
- Have centers in $B_5 \subset \mathbf{R}^3$.

Since the n points $\{z_i\}$ are all in the Euclidean ball $B_5 \subset \mathbf{R}^3$, there exist integers i_1 and i_2 with

$$0 < |z_{i_1} - z_{i_2}| < C'\,n^{-1/3}\,. \tag{3.20}$$

Furthermore, since each intrinsic ball of radius one about any z_i is δ-weakly chord arc by (A$'$), we have that each embedded minimal disk $\Sigma_{z_i,\delta}$ is compact and has

$$\partial \Sigma_{z_i,\delta} \subset \partial B_\delta(z_i)\,. \tag{3.21}$$

Consequently, for n large enough, (3.20) implies that the components Σ_1 and Σ_2 of

$$B_{\frac{\delta}{2}}(z_{i_1}) \cap \Sigma \tag{3.22}$$

containing z_{i_1} and z_{i_2}, respectively, are compact and have

$$\partial \Sigma_i \subset \partial B_{\frac{\delta}{2}}(z_{i_1}) \, . \tag{3.23}$$

Note that the center of this extrinsic ball is the same for Σ_1 and Σ_2. Let $c > 1$ be given by Corollary 1.3. For n sufficiently large, (3.20) implies that Σ_2 intersects the smaller concentric extrinsic ball $B_{\frac{\delta}{2c}}(z_{i_1})$ and, since Σ_1 contains the center of this ball, then it follows that for both $j = 1$ and $j = 2$,

$$B_{\frac{\delta}{2c}}(z_{i_1}) \cap \Sigma_j \neq \emptyset \, . \tag{3.24}$$

Combining (3.23) and (3.24), Corollary 1.3 gives the curvature bound for $j = 1, 2$

$$\sup_{\mathcal{B}_{\frac{\delta}{2c}}(z_{i_j})} |A|^2 \leq \left(\frac{\delta}{2c} \right)^{-2} \, . \tag{3.25}$$

By Lemma 2.11 of [CM4], the curvature bound (3.25) gives a constant $r' = r'(\delta, c)$ so that if n is sufficiently large, then $\mathcal{B}_{3\,r'}(z_{i_2})$ can be written as a normal exponential graph of a function u over a domain Ω, where:

(i) The function u satisfies (3.13).

(ii) The domain Ω contains, and is contained in, concentric intrinsic balls as follows:

$$\mathcal{B}_{2\,r'}(z_{i_1}) \subset \Omega \subset \mathcal{B}_{4\,r'}(z_{i_1}) \, . \tag{3.26}$$

(To see this, first use the curvature bound to write each component locally as a graph and then use embeddedness to see that these graphs must be roughly parallel.) By Lemma 3.12 (and (3.20)), we can apply the Harnack inequality to u to get

$$\sup_{\mathcal{B}_{r'}(z_{i_1})} u \leq \tilde{C} \, |z_{i_2} - z_{i_1}| \leq \tilde{C}' \, n^{-1/3} \, . \tag{3.27}$$

As long as n is large enough, (3.27) allows us to repeat the argument with a point in the boundary $\partial \mathcal{B}_{r'}(z_{i_1})$ in place of z_{i_1}. Therefore, for n large enough, we can repeatedly combine Corollary 1.3 and the Harnack inequality to extend the curvature bound (3.25) to the larger intrinsic balls

$$\mathcal{B}_{C_0}(z_{i_j}) \text{ for } j = 1, \, 2 \, . \tag{3.28}$$

Now that we have a uniform curvature bound on the disjoint intrinsic balls (3.28) and the centers of these balls are extrinsically close by (3.20), we can apply Lemma 3.6 to get that

$$B_5 \cap \partial \mathcal{B}_{11}(z_{i_j}) = \emptyset \, . \tag{3.29}$$

(Here we used that $B_5 \subset B_{10}(z_{i_j})$ because $z_{i_j} \in B_5$.) Since the curve σ must intersect $\partial \mathcal{B}_{11}(z_{i_j})$, this contradicts the fact that the curve σ is contained in the ball B_5. This contradiction proves (3.15) and gives the proposition. □

The previous proposition is the key step in the proof of Proposition 1.1. To complete the proof, we will use a simple blow-up argument to find some small initial scale which is weakly chord arc and then apply Proposition 1.1 to get that so are larger scales. As is often the case in this type of blow-up argument, the existence of such an initial scale is complicated slightly by the fact that Σ has nonempty boundary.

To incorporate the boundary, we let a_δ be the supremum of the ratio of the distance to $\partial\Sigma$ to the largest radius of an intrinsic ball which is δ-weakly chord arc; i.e., we set

$$a_\delta = \sup_{z \in \Sigma} \frac{\mathrm{dist}_\Sigma(z, \partial\Sigma)}{R_\delta(z)} \, , \tag{3.30}$$

where $R_\delta(z)$ is given by (3.3).

3.4. *Upper bounds for a_δ.* Suppose for a moment that Σ is compact and smooth up to the boundary $\partial\Sigma$ and $\delta < 1/2$. We will, in the proof of Lemma 3.39 below, use that

$$a_\delta < \infty \, . \tag{3.31}$$

To see (3.31), observe that compactness and smoothness give uniform bounds on $|A|^2$ and the geodesic curvature of $\partial\Sigma$. Given any constant $\varepsilon > 0$, the bound on $|A|^2$ gives a constant $r_0 > 0$ so that if $s \leq r_0$ and $\mathcal{B}_s(z) \subset \Sigma \setminus \partial\Sigma$, then $\mathcal{B}_s(z)$ is a graph over some plane of a function with gradient $\leq \varepsilon$. In particular, the intrinsic ball $\mathcal{B}_s(z)$ is δ-weakly chord arc for ε sufficiently small. Furthermore, the bound on the geodesic curvature of $\partial\Sigma$ gives a constant $r_1 > 0$ so that if

$$d_z = \mathrm{dist}_\Sigma(z, \partial\Sigma) \leq r_1 \, , \tag{3.32}$$

then $\mathcal{B}_{d_z}(z) \subset \Sigma \setminus \partial\Sigma$. We can now establish (3.31) by considering two cases depending on the distance to the boundary. If

$$d_z = \mathrm{dist}_\Sigma(z, \partial\Sigma) \leq \min\{r_0, r_1\} \, , \tag{3.33}$$

then $\mathcal{B}_{d_z}(z)$ is δ-weakly chord arc so that

$$R_\delta(z) = \mathrm{dist}_\Sigma(z, \partial\Sigma) \, . \tag{3.34}$$

On the other hand, when (3.33) fails, then $\mathcal{B}_{r_2}(z)$ is δ-weakly chord arc where $r_2 = \min\{r_0, r_1\}$ and hence

$$\frac{\mathrm{dist}_\Sigma(z, \partial\Sigma)}{R_\delta(z)} \leq \frac{\mathrm{diam}\,(\Sigma)}{r_2}\,. \tag{3.35}$$

This shows that $a_\delta < \infty$ if Σ is compact and smooth.

Let us return to Proposition 1.1. It is not hard to see that the proposition is equivalent to an upper bound (independent of Σ) for a_δ for a fixed $\delta > 0$. Namely, suppose that $\mathcal{B}_R(x) \subset \Sigma \setminus \partial\Sigma$ is as in the proposition and we have an upper bound for a_δ

$$a_\delta \leq c < \infty\,. \tag{3.36}$$

Since $\mathcal{B}_R(x) \subset \Sigma \setminus \partial\Sigma$, then (3.30) implies that

$$R \leq \mathrm{dist}_\Sigma(x, \partial\Sigma) \leq c\, R_\delta(x)\,. \tag{3.37}$$

Consequently, by the definition (3.3) of $R_\delta(x)$, there exists a radius $s > \frac{R}{2\,c}$ so that $\mathcal{B}_s(x)$ is δ-weakly chord arc and hence

$$\Sigma_{x, \frac{\delta\, R}{4\, c}} \subset \Sigma_{x, \frac{\delta\, s}{2}} \subset \mathcal{B}_{\frac{R}{2}}(x)\,. \tag{3.38}$$

Equation (3.38) would then give Proposition 1.1.

3.5. *Locating the smallest scale which is not weakly chord arc.* We will first need to locate a smallest scale on which Σ is not δ-weakly chord arc. We do this in the next lemma with a simple blow-up argument. The Σ in this lemma is assumed to be compact and smooth up to the boundary so that $a_\delta < \infty$ by (3.31).

LEMMA 3.39. *Given Σ compact and smooth up to the boundary and a constant δ with $0 < \delta < 1/2$, there exists $y \in \Sigma$ and $R_0 > 0$ so that*:

(A) *$R_\delta(x) > R_0$ for every $x \in \mathcal{B}_{a_\delta\, R_0}(y)$, where $R_\delta(x)$ is given by* (3.3).

(B) *The intrinsic ball $\mathcal{B}_{5\, R_0}(y)$ is* not *δ-weakly chord arc.*

Proof. Define a function G on Σ by setting

$$G(x) = \frac{\mathrm{dist}_\Sigma(x, \partial\Sigma)}{R_\delta(x)}\,. \tag{3.40}$$

Since Σ is smooth and compact, (3.31) and the definitions of G and a_δ give that

$$a_\delta = \sup G < \infty\,. \tag{3.41}$$

We can therefore choose y so that $G(y)$ is greater than half the supremum a_δ of G on Σ:

$$\frac{\mathrm{dist}_\Sigma(y, \partial\Sigma)}{R_\delta(y)} = G(y) > \frac{\sup G}{2} = \frac{a_\delta}{2}\,. \tag{3.42}$$

We will see that (3.42) implies (A) and (B) with $R_0 = R_\delta(y)/4$.

Set $d_\partial = \text{dist}_\Sigma(y, \partial\Sigma)$ so that if $x \in \mathcal{B}_{d_\partial/2}(y)$, then by the triangle inequality

$$\text{dist}_\Sigma(x, \partial\Sigma) > \frac{d_\partial}{2} . \tag{3.43}$$

Combining (3.42) and (3.43) gives for $x \in \mathcal{B}_{d_\partial/2}(y)$ that

$$\frac{d_\partial}{2\, R_\delta(x)} < G(x) < 2\, G(y) = \frac{2\, d_\partial}{R_\delta(y)} , \tag{3.44}$$

and thus

$$R_\delta(x) > \frac{R_\delta(y)}{4} = R_0 . \tag{3.45}$$

From (3.42), we see that $2\, a_\delta\, R_0 < d_\partial$ and hence

$$\mathcal{B}_{a_\delta R_0}(y) \subset \mathcal{B}_{\frac{d_\partial}{2}}(y) . \tag{3.46}$$

Combining (3.45) and (3.46) gives (A). We get (B) immediately from the maximality of $R_\delta(y)$. □

3.6. *The proof of Proposition* 1.1: *Bounding* a_δ. We are now prepared to prove Proposition 1.1, i.e., to show that sufficiently small intrinsic balls in Σ are weakly chord arc. As mentioned above, this is equivalent to giving a uniform upper bound for the constant a_δ defined in (3.30) for some fixed $\delta > 0$ (the constant δ will be given by Proposition 2.1). In the actual proof, we will first use Lemma 3.39 to find the smallest scale which is not δ-weakly chord arc. To bound a_δ, it suffices to give a lower bound for this scale in terms of the distance to the boundary $\partial\Sigma$. This is precisely the content of Proposition 3.4.

Proof (*of Proposition* 1.1). Let the constant $\delta = \delta_2$ be given by Proposition 2.1. As we have seen in (3.38), the proposition follows from a uniform upper bound for the constant a_δ defined in (3.30). The rest of the proof is to establish such a bound.

Apply first Lemma 3.39 to locate the smallest scale which is not δ-weakly chord arc. This gives a point y in Σ and an intrinsic ball $\mathcal{B}_{a_\delta\, R_0}(y)$ so that:

(A) $R_\delta(z) > R_0$ for every $z \in \mathcal{B}_{a_\delta\, R_0}(y)$.

(B) $\mathcal{B}_{5R_0}(y)$ is *not* δ-weakly chord arc.

The condition (A) implies that each point $z \in \mathcal{B}_{a_\delta\, R_0}(y)$ is the center of some δ-weakly chord arc intrinsic ball of radius greater than R_0. However, Proposition 2.1 then easily gives that $\mathcal{B}_{R_0}(z)$ is in fact δ-weakly chord arc (here we use that δ is given by that proposition). Namely, (A) can be replaced by:

(A′) Every intrinsic ball $\mathcal{B}_{R_0}(z)$ with $z \in \mathcal{B}_{a_\delta\, R_0}(y)$ is δ-weakly chord arc.

The proposition now follows from Proposition 3.4. Namely, Proposition 3.4 gives a constant C_b so that if

$$a_\delta \geq C_b \,, \tag{3.47}$$

then (A′) implies that the five times intrinsic ball $\mathcal{B}_{5R_0}(y)$ is δ-weakly chord arc. Since this would contradict (B), we conclude that (3.47) cannot hold and the proposition follows. □

4. Finite topology: The proofs of Corollaries 0.12 and 0.13

In this section, we prove both of Calabi's conjectures and properness for complete embedded minimal surfaces with finite topology. Recall that a surface Σ is said to have finite topology if it is homeomorphic to a closed Riemann surface of genus g with a finite set of punctures. Each puncture corresponds to an end of Σ and thus the ends can be represented by punctured disks, i.e., each end is homeomorphic to the set

$$\{z \in \mathbf{C} \,|\, 0 < |z| \leq 1\} \,. \tag{4.1}$$

4.1. *Simply connected outside a compact set.* The key point for extending our results to surfaces with finite topology is to show that intrinsic balls are eventually simply connected so that our results for disks can be applied. This is made precise in the next lemma.

Lemma 4.2. *Let Γ be a complete noncompact embedded minimal annulus which contains one compact component γ of $\partial\Gamma$; the other boundary is at infinity. There is a constant $\bar{R}$ (depending on Γ) so that the following holds*:

(4.3) *If $d_x = \text{dist}_\Gamma(x, \gamma) > \bar{R}$, then the intrinsic ball $\mathcal{B}_{d_x/2}(x)$ is a disk.*

Proof. Suppose that (4.3) fails for every $\bar{R}$. It will follow from the fact that Γ is an annulus with nonpositive curvature that Γ has finite total curvature. Namely, if (4.3) fails, we get a sequence $x_i \in \Gamma$ with

$$d_i = \text{dist}_\Gamma(x_i, \gamma) \to \infty \tag{4.4}$$

so that the exponential map from x_i is not injective into $\mathcal{B}_{d_i/2}(x_i)$. In particular, there are distinct geodesics γ_i^a and γ_i^b in $\mathcal{B}_{d_i/2}(x_i)$ from x_i to a point $y_i \in \mathcal{B}_{d_i/2}(x_i)$ and the closed curve

$$\gamma_i = \gamma_i^a \cup \gamma_i^b \tag{4.5}$$

is homologous to the compact boundary component γ. Let Γ_i be the bounded component of $\Gamma \setminus \gamma_i$; so Γ_i is topologically an annulus bounded by γ and the piecewise smooth closed geodesic γ_i with breaks at x_i and y_i. Write $\int_\gamma k_g$ and

$\int_{\gamma_i} k_g$ for the two boundary terms in the Gauss-Bonnet theorem for the annulus Γ_i (both are uniformly bounded; $\int_{\gamma_i} k_g$ is after all just the angle contribution at x_i and y_i). It follows that

$$\int_{\Gamma_i} |A|^2 = -2 \int_{\Gamma_i} K_\Gamma = 2 \int_\gamma k_g + 2 \int_{\gamma_i} k_g \leq C \,. \tag{4.6}$$

Moreover, by the triangle inequality, we have that $\text{dist}_\Gamma(\gamma, \gamma_i) \geq d_i/2$ and hence Γ_i contains the intrinsic $(d_i/2)$-tubular neighborhood of γ. Since $d_i \to \infty$, the Γ_i's exhaust Γ, i.e., $\Gamma \subset \cup_i \Gamma_i$, and thus (4.6) implies that Γ has finite total curvature.

Finally, we will show that (4.3) must hold when Γ has finite total curvature. To see this, note that since Γ is an embedded annulus with finite total curvature, it is asymptotic to either a plane or half of a catenoid (see, e.g., [Sc2]). In either case, (4.3) must hold for points sufficiently far from the interior boundary γ. This completes the proof of the lemma. □

4.2. *Compact embedded annuli in a halfspace.* We will next bound the total curvature for a compact embedded minimal annulus in a halfspace. In the next lemma, we will use $\Gamma_{\gamma,R}$ to denote the component of $B_R \cap \Gamma$ containing the boundary component γ.

LEMMA 4.7. *Let Γ be as in Lemma* 4.2. *There exist constants $\varepsilon > 0$ and $\hat{R}$ so that if $R > \hat{R}$, the component $\Gamma_{\gamma,2R}$ is compact, and*

$$\Gamma_{\gamma,2R} \subset \{x_3 > -\varepsilon\, R\} \,, \tag{4.8}$$

$\Gamma_{\gamma,R}$ has bounded total curvature

$$\int_{\Gamma_{\gamma,R}} |A|^2 \leq 2 \int_\gamma k_g + 8\pi \,. \tag{4.9}$$

Proof. The bound (4.9) follows immediately from the Gauss-Bonnet theorem and the following two claims:

(C1) There is a constant $\varepsilon > 0$ so that if $\Gamma_{\gamma,2R} \subset \{x_3 > -\varepsilon\, R\}$ and $\partial \Gamma_{\gamma,R} \setminus \gamma$ intersects $\{x_3 < \varepsilon\, R\}$, then $\partial \Gamma_{\gamma,R} \setminus \gamma$ is a graph over (a curve in) $\{x_3 = 0\}$ and

$$\int_{\partial \Gamma_{\gamma,R} \setminus \gamma} k_g < 4\pi \,. \tag{4.10}$$

(C2) For any $\varepsilon > 0$, if R is sufficiently large, then $\partial \Gamma_{\gamma,R} \setminus \gamma$ intersects $\{x_3 < \varepsilon\, R\}$.

Since the statement is scale invariant, we can normalize so that $\gamma \subset B_1$. We will take R much larger than the constant $\bar{R}$ given by Lemma 4.2 so that (4.3) holds for $R/2 - 1$.

The key point for proving (C1) is that the intrinsic one-sided curvature estimate, Corollary 0.8, gives a constant $\mu > 0$ so that if $\varepsilon < \mu$ and $y \in \{|x_3| < \mu\, R\} \cap \partial \Gamma_{\gamma,R} \setminus \gamma$, then

$$\sup_{\mathcal{B}_{R/4}(y)} |A|^2 \leq C'\, \mu^2\, R^{-2}\,. \tag{4.11}$$

Note that to apply Corollary 0.8 here, we used Lemma 4.2 to see that $\mathcal{B}_{R/2}(y)$ is a topological disk. The claim (C1) follows easily from (4.11). Namely, first choose a point $y_0 \in \{x_3 < \varepsilon\, R\} \cap \partial \Gamma_{\gamma,R} \setminus \gamma$ and observe that the curvature bound (4.11) allows us to apply the gradient estimate to the positive harmonic function $x_3 + \varepsilon\, R$ on $\mathcal{B}_{R/4}(y_0)$ to get

$$\sup_{\mathcal{B}_{R/8}(y_0)} |\nabla_\Gamma x_3| \leq C\, \varepsilon\,. \tag{4.12}$$

The bound (4.12) implies that the ball $\mathcal{B}_{R/8}(y_0)$ is graphical and moreover is contained in the slab $\{|x_3| \leq C\, \varepsilon\, R\}$. In particular, for $\varepsilon > 0$ sufficiently small, we can repeat this process to get a chain of balls $\mathcal{B}_{R/8}(y_i)$ with $y_i \in \partial \Gamma_{\gamma,R} \cap \{|x_3| < \mu\, R\}$ and so that $\cup_i \mathcal{B}_{R/8}(y_i)$ forms a graph which circles the x_3-axis. The intersection of this graph with the cylinder $\{x_1^2 + x_2^2 = R^2\}$ contains a graph over the circle ∂D_R. Since $\Gamma_{\gamma,2R}$ is compact, the graph

$$\{|x_3| < \mu\, R\} \cap \{x_1^2 + x_2^2 = R^2\} \cap \Gamma_{\gamma,2R} \tag{4.13}$$

cannot spiral forever and, hence, closes up. Finally, the curvature bound (4.11) and the gradient bound for the graph imply a pointwise bound for the geodesic curvature of $\partial \Gamma_{\gamma,R}$; integrating this pointwise bound gives (4.10).

To prove the second claim (C2), we will use catenoid barriers and the strong maximum principle to argue by contradiction. Suppose therefore that $\varepsilon > 0$ and

$$\partial \Gamma_{\gamma,R} \setminus \gamma \subset \{x_3 > \varepsilon\, R\}\,. \tag{4.14}$$

Let Cat denote the standard catenoid ($\text{Cat} = \{\cosh^2(x_3) = x_1^2 + x_2^2\}$) so that

$$\{x_1^2 + x_2^2 \leq 3R\} \cap \text{Cat} \subset \{|x_3| \leq \cosh^{-1}(3R)\}\,. \tag{4.15}$$

Consider the one-parameter family of vertically translated catenoids $\text{Cat}_t = \text{Cat} + (0,0,t)$ and observe that $\text{Cat}_{-2R} \cap \Gamma_{\gamma,R} = \emptyset$. Furthermore, when R is large, (4.14) and (4.15) imply that $\text{Cat}_t \cap \partial \Gamma_{\gamma,R} = \emptyset$ for every $t \leq 5\, \cosh^{-1}(3R)$. Here we used (4.14) to deal with the outer boundary while the inner boundary γ came for free since it is contained in B_1. By the strong maximum principle, there cannot be a first $t \leq 5\, \cosh^{-1}(3R)$ where Cat_t intersects $\Gamma_{\gamma,R}$ and hence for $t \leq 5\, \cosh^{-1}(3R)$ we have

$$\text{Cat}_t \cap \Gamma_{\gamma,R} = \emptyset\,. \tag{4.16}$$

Arguing similarly give that a horizontal translation of $\text{Cat}_{3\, \cosh^{-1}(3R)}$ by a distance $2R$ cannot intersect $\Gamma_{\gamma,R}$. However, this horizontally translated catenoid separates B_1 and $\{x_3 > \varepsilon R\}$ in B_R and hence separates the components of $\partial \Gamma_{\gamma,R}$, giving the desired contradiction. □

4.3. *The proof of Corollary* 0.12. Both Corollary 0.12 and Corollary 0.13 will use the following weak chord arc property for annuli (cf.. Proposition 1.1):

LEMMA 4.17. *Let* Γ *be as in Lemma* 4.2. *There exist constants* $\tilde{R}$ *and* $\delta > 0$ *so that for all intrinsic tubular neighborhoods* $T_R(\gamma)$ *of* γ *in* Γ *with* $R \geq \tilde{R}$, *the component* $\Gamma_{\gamma,\delta R}$ *of* $B_{\delta R} \cap \Gamma$ *containing* γ *satisfies*

$$\Gamma_{\gamma,\delta R} \subset T_{R/2}(\gamma)\,. \tag{4.18}$$

Here $\tilde{R}$ *depends on* Γ *but* δ *does not.*

Proof. Let $\bar{R}$ be the constant given by Lemma 4.2 so that (4.3) holds. We can now directly follow the proof of claim (3.15) in the proof of Proposition 3.4 to get (4.18). This requires one modification to get that intrinsic subballs are weakly chord arc. Namely, rather than using condition (A$'$) there, we use (4.3) to first see that the intrinsic subballs are disks and then apply Proposition 1.1 to these disks. □

The weak chord arc property given by Lemma 4.17 implies the necessary compactness needed to apply Lemma 4.7 and gives that embedded minimal annuli in a halfspace have finite total curvature:

COROLLARY 4.19. *Let* Γ *be as in Lemma* 4.2. *If* Γ *is contained in a halfspace, then* Γ *has finite total curvature. It follows that* Γ *is asymptotic to a plane or half of a catenoid.*

Proof. Lemma 4.17 implies that, for every R, the component $\Gamma_{\gamma,2R}$ of $B_{2R} \cap \Gamma$ containing γ is compact. Hence, we can apply Lemma 4.7 to $\Gamma_{\gamma,2R}$ for R sufficiently large to get

$$\int_{\Gamma_{\gamma,R}} |A|^2 \leq 2 \int_{\gamma} k_g + 8\pi\,. \tag{4.20}$$

As R goes to infinity, the $\Gamma_{\gamma,R}$'s exhaust Γ and hence (4.20) bounds the total curvature of Γ. The second statement follows since the annulus Γ is also embedded (see, e.g., [Sc2]). □

Corollary 0.12, and hence Calabi's conjectures for surfaces with finite topology, now follow easily from Corollary 4.19:

Proof of Corollary 0.12. Observe first that an embedded minimal surface Σ with finite topology in a halfspace has finite total curvature. This is because such a Σ can be written as the union of a compact piece Σ_0 which may have nonzero genus and a finite collection of noncompact annuli $\Gamma_1, \ldots, \Gamma_k$ each of which contains one of its boundary components. Clearly, Σ_0 has finite total curvature since it is compact. Furthermore, each Γ_i has finite total curvature by Corollary 4.19, so we conclude that Σ itself has finite total curvature.

Finally, since Σ has finite total curvature, [Hu] implies that Σ is parabolic (in the sense that any positive harmonic function is constant). Therefore the positive harmonic function x_3 is constant on Σ and Σ must be a plane as claimed. □

4.4. *The proof of Corollary* 0.13: *Properness.* The properness of embedded minimal surfaces with finite topology will be an almost immediate consequence of properness of embedded annuli that we will show next. As in the case of disks, the weak chord arc property given by Lemma 4.17 applies only to one component and therefore does not directly give properness.

PROPOSITION 4.21. *Let Γ be as in Lemma* 4.2. *Then Γ must be proper.*

Proof. The proposition follows from the following claim: For every radius $R > 0$, there is a constant $S_R > R$ (depending on both R and Γ) so that

$$B_R \cap \Gamma \subset \Gamma_{\gamma, S_R} . \tag{4.22}$$

Here, as in Lemma 4.17, Γ_{γ,S_R} denotes the component of $B_{S_R} \cap \Gamma$ containing γ. To get the proposition from (4.22), simply apply Lemma 4.17 (for R large) to get

$$\Gamma_{\gamma,S_R} \subset T_{S_R/(2\delta)}(\gamma) , \tag{4.23}$$

and observe that the (closure of the) intrinsic tubular neighborhood $T_{S_R/(2\delta)}(\gamma)$ is compact.

The rest of the proof is to establish (4.22). We will do this by contradiction; suppose therefore that $R > 0$ is fixed, $\gamma \subset B_R$, and y_i is a sequence of points in $B_R \cap \Gamma$ with

$$y_i \notin \Gamma_{\gamma, i\,R} . \tag{4.24}$$

We will show that (4.24) implies that Γ has finite total curvature and then get a contradiction from this.

The first step is to find large graphical regions in Γ. Observe that, by the triangle inequality,

$$d_i = \text{dist}_\Gamma(y_i, \gamma) \geq (i-1)\, R . \tag{4.25}$$

Since $i \to \infty$, it follows from (4.25) that for any J we can choose indices i_1 and i_2 so that

$$d_{i_1} > 2J \text{ and } d_{i_2} > 2J , \tag{4.26}$$

$$\text{dist}_\Gamma(y_{i_1}, y_{i_2}) > 2J . \tag{4.27}$$

When J is large, Lemma 4.2 and (4.26) imply that the intrinsic balls $\mathcal{B}_J(y_{i_1})$ and $\mathcal{B}_J(y_{i_2})$ are topological disks; and disjoint by (4.27). The one-sided curvature estimate now implies that $\mathcal{B}_J(y_{i_1})$ and $\mathcal{B}_J(y_{i_2})$ contains a graph Γ_1 and Γ_2, respectively, over a disk of radius $c\,J$ with small gradient $\leq \tau$ and $y_{i_j} \in \Gamma_j$ (where c depends on τ). To prove this, first apply Proposition 1.1 to see that the intrinsic balls are weakly chord arc and then apply Corollary 1.3 to get a curvature bound.

The second step is to use the large graphical region to show that Γ has finite total curvature. Namely, for J large, Lemma 4.17 implies that the component $\Gamma_{\gamma,cJ}$ of $B_{cJ} \cap \Gamma$ containing γ is compact. Moreover, since Γ is embedded, the graph Γ_1 forces $\Gamma_{\gamma,cJ}$ to be contained in a halfspace

$$\Gamma_{\gamma,cJ} \subset \{x_3 > -R - c\,\tau\,J\}\,. \tag{4.28}$$

(Here we have assumed that Γ_1 is beneath γ; this can be arranged after possibly reflecting across $\{x_3 = 0\}$.) For $\tau > 0$ small, we can apply Lemma 4.7 to get a bound for the total curvature of $\Gamma_{\gamma,cJ/2}$ which is independent of J. It follows that Γ has finite total curvature since the $\Gamma_{\gamma,cJ/2}$'s exhaust Γ as $J \to \infty$.

Finally, as in the proof of Lemma 4.2, we conclude that Γ is asymptotic to either a plane or half of a catenoid since it has finite total curvature. However, in either case, (4.22) clearly holds. This contradiction establishes the claim (4.22) and thus completes the proof. $\square$

The properness of embedded minimal surfaces with finite topology now follows easily:

Proof of Corollary 0.13. Write the embedded minimal surface with finite topology Σ as the union of a compact piece Σ_0 and a finite collection of noncompact annuli $\Gamma_1, \dots, \Gamma_k$ each of which contains one of its boundary components. Proposition 4.21 implies that each annulus Γ_i is proper and hence so is Σ. $\square$

Appendix A. Multi-valued graphs

To make the notion of multi-valued graphs precise, let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global polar coordinates (ρ, θ) so that $\rho > 0$ and $\theta \in \mathbf{R}$. An *N-valued graph* on the annulus $D_s \setminus D_r$ is a single valued graph of a function u over

$$\{(\rho, \theta) \,|\, r < \rho \leq s\,,\, |\theta| \leq N\,\pi\}\,. \tag{A.1}$$

For working purposes, we generally think of the intuitive picture of a multi-sheeted surface in $\mathbf{R}^3$, and we identify the single-valued graph over the universal cover with its multi-valued image in $\mathbf{R}^3$.

The multi-valued graphs considered in this paper will all be embedded, which corresponds to a nonvanishing separation between the sheets. Here the *separation* is the function

$$w(\rho,\theta) = u(\rho,\theta+2\pi) - u(\rho,\theta)\,. \tag{A.2}$$

If Σ is the helicoid (i.e., Σ can be parametrized by $(s\cos t, s\sin t, t)$ where $s, t \in \mathbf{R}$], then $\Sigma \setminus \{x_3 - \text{axis}\} = \Sigma_1 \cup \Sigma_2$, where Σ_1, Σ_2 are ∞-valued graphs on $\mathbf{C} \setminus \{0\}$. Now, Σ_1 is the graph of the function $u_1(\rho,\theta) = \theta$ and Σ_2 is the graph of the function $u_2(\rho,\theta) = \theta + \pi$. ($\Sigma_1$ is the subset where $s > 0$ and Σ_2 the subset where $s < 0$.) In either case the separation $w = 2\,\pi$.

Appendix B. The proof of Lemma 2.23

We will next include the proof of Lemma 2.23. This lemma is modelled on Lemma II.2.1 in [CM6]. The proof follows that of Lemma II.2.1 in [CM6] with very minor changes, but we include it here for completeness.

Proof of Lemma 2.23. Let $C_0 > 2$ be given by Lemma 3.6. We will show that there exists n depending on C_a so that

$$\Sigma_{0,\frac{R}{C_0}} \subset \mathcal{B}_{n\,R}(0)\,. \tag{B.1}$$

To prove this, we will argue by contradiction; so suppose that (B.1) fails for some large n. Consequently, we get a curve

$$\sigma \subset \Sigma_{0,\frac{R}{C_0}} \subset B_{\frac{R}{C_0}}(0) \tag{B.2}$$

from 0 to a point in $\partial\mathcal{B}_{n\,R}(0)$. For $i = 1, \ldots, n$, fix points

$$z_i \in \partial\mathcal{B}_{i\,R}(0) \cap \sigma\,. \tag{B.3}$$

It follows that the intrinsic balls $\mathcal{B}_{R/2}(z_i)$:

- Are disjoint.
- Have centers in $B_{\frac{R}{C_0}}(0)$.
- Do not intersect $\partial\Sigma$.

Since the n points $\{z_i\}$ are all in the Euclidean ball $B_{\frac{R}{C_0}}(0) \subset \mathbf{R}^3$, there exist integers i_1 and i_2 with

$$0 < |z_{i_1} - z_{i_2}| < C\,n^{-1/3}\,R\,. \tag{B.4}$$

Note that (2.24) gives a uniform curvature bound on the balls $\mathcal{B}_{R/2}(z_{i_1})$ and $\mathcal{B}_{R/2}(z_{i_2})$. Therefore, Lemma 3.6 implies that, for n sufficiently large (so the centers z_{i_1} and z_{i_2} are extrinsically close), we get for $j = 1, 2$ that

$$B_{\frac{R}{C_0}}(0) \cap \partial\mathcal{B}_{\frac{11\,R}{2C_0}}(z_{i_j}) = \emptyset\,. \tag{B.5}$$

(Here we used that $B_{R/C_0}(0) \subset B_{5R/C_0}(z_{i_j})$ because $z_{i_j} \in B_{R/C_0}(0)$.) Since the curve σ must intersect $\partial \mathcal{B}_{11R/(2C_0)}(z_{i_j})$, (B.5) contradicts the fact that the curve σ is contained in the ball $B_{R/C_0}(0)$. This contradiction proves (B.1) and consequently gives the lemma. □

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