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The classification of p-compact groups for p odd

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Abstract

A p-compact group, as defined by Dwyer and Wilkerson, is a purely homotopically defined p-local analog of a compact Lie group. It has long been the hope, and later the conjecture, that these objects should have a classification similar to the classification of compact Lie groups. In this paper we finish the proof of this conjecture, for p an odd prime, proving that there is a one-to-one correspondence between connected p-compact groups and finite reflection groups over the p-adic integers. We do this by providing the last, and rather intricate, piece, namely that the exceptional compact Lie groups are uniquely determined as p-compact groups by their Weyl groups seen as finite reflection groups over the p-adic integers. Our approach in fact gives a largely self-contained proof of the entire classification theorem for p odd.

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1. Introduction

It has been a central goal in homotopy theory for about half a century to single out the homotopy theoretical properties characterizing compact Lie groups, and obtain a corresponding classification, starting with the work of Hopf [75] and Serre [123, Ch. IV] on H-spaces and loop spaces. Materializing old dreams of Sullivan [134] and Rector [121], Dwyer and Wilkerson, in their seminal paper [56], introduced the notion of a p-compact group, as a p-complete loop space with finite mod p cohomology, and proved that p-compact groups have many Lie-like properties. Even before their introduction it has been the hope [120], and later the conjecture [59], [89], [48], that these objects should admit a classification much like the classification of compact connected Lie groups, and the work toward this has been carried out by many authors. The goal of this paper is to complete the proof of the classification theorem for p an odd prime, showing that there is a one-to-one correspondence between connected p-compact groups and finite reflection groups over the p-adic integers $\mathbf{Z}_p$. We do this by providing the last—and rather intricate—piece, namely that the p-completions of the exceptional compact connected Lie groups are uniquely determined as p-compact groups by their Weyl groups, seen as $\mathbf{Z}_p$-reflection groups. In fact our method of proof gives an essentially self-contained proof of the entire classification theorem for p odd.

We start by very briefly introducing p-compact groups and some objects associated to them, necessary to state the classification theorem—we will later in the introduction return to the history behind the various steps of the proof. We refer the reader to [56] for more details on p-compact groups and also recommend the overview articles [48], [89], and [95]. We point out that it is the technical advances on homotopy fixed points by Miller [94], Lannes [88], and others which make this theory possible.

A space X with a loop space structure, for short a *loop space*, is a triple (X, BX, e) where BX is a pointed connected space, called the *classifying space* of X, and $e : X \to \Omega BX$ is a homotopy equivalence. A *p-compact group* is a loop space with the two additional properties that $H^*(X; \mathbf{F}_p)$ is finite dimensional over $\mathbf{F}_p$ (to be thought of as 'compactness') and that BX is $\mathbf{F}_p$-local [21], [56, §11] (or, in this context, equivalently $\mathbf{F}_p$-complete [22, Def. I.5.1]). Often we refer to a loop space simply as X. When working with a loop space we shall only be concerned with its classifying space BX, since this determines the rest of the structure—indeed, we could instead have defined a p-compact group to be a space BX with the above properties. The loop space $(G\hat{_p}, BG\hat{_p}, e)$, corresponding to a pair (G, p) (where p is a prime, G a compact Lie group with component group a finite p-group, and $(\cdot)\hat{_p}$ denotes $\mathbf{F}_p$-completion [22, Def. I.4.2], [56, §11]) is a p-compact group. (Note however that a compact Lie group G is not uniquely determined by $BG\hat{_p}$, since we are only focusing on the structure 'visible at the prime p'; e.g., $B\,\mathrm{SO}(2n+1)\hat{_p} \simeq B\,\mathrm{Sp}(n)\hat{_p}$ if $p \neq 2$, as originally proved by Friedlander [66]; see Theorem 11.5 for a complete analysis.)

A *morphism* $X \to Y$ between loop spaces is a pointed map of spaces $BX \to BY$. We say that two morphisms are *conjugate* if the corresponding maps of classifying spaces are freely homotopic. A morphism $X \to Y$ is called an *isomorphism* (or *equivalence*) if it has an inverse up to conjugation, or in other words if $BX \to BY$ is a homotopy equivalence. If X and Y are p-compact groups, we call a morphism a *monomorphism* if the homotopy fiber Y/X of the map $BX \to BY$ is $\mathbf{F}_p$-finite.

The loop space corresponding to the $\mathbf{F}_p$-completed classifying space $BT = (B\mathrm{U}(1)^r)\hat{_p}$ is called a *p-compact torus* of rank r. A *maximal torus* in X is a monomorphism $i : T \to X$ such that the homotopy fiber of $BT \to BX$ has nonzero Euler characteristic. (We define the Euler characteristic as the alternating sum of the $\mathbf{F}_p$-dimensions of the $\mathbf{F}_p$-homology groups.) Fundamental to the theory of p-compact groups is the theorem of Dwyer-Wilkerson [56, Thm. 8.13] that, analogously to the classical situation, any p-compact group admits a maximal torus. It is unique in the sense that for any other maximal torus $i' : T' \to X$, there exists an isomorphism $\varphi : T \to T'$ such that $i'\varphi$ and i are conjugate. Note the slight difference from the classical formulation due to the fact that a maximal torus is defined to be a *map* and not a subgroup.

Fix a p-compact group X with maximal torus $i : T \to X$ of rank r. Replace the map $Bi : BT \to BX$ by an equivalent fibration, and define the *Weyl space* $\mathcal{W}_X(T)$ as the topological monoid of self-maps $BT \to BT$ over BX. The *Weyl group* is defined as $W_X(T) = \pi_0(\mathcal{W}_X(T))$ [56, Def. 9.6]. By [56, Prop. 9.5] $W_X(T)$ is a finite group of order $\chi(X/T)$. Furthermore, by [56, Pf. of Thm. 9.7], if X is connected then $W_X(T)$ identifies with the set of conjugacy classes of self-equivalences φ of T such that i and $i\varphi$ are conjugate. In other words, the canonical homomorphism $W_X(T) \to \mathrm{Aut}(\pi_1(T))$ is injective, so we can view $W_X(T)$ as a subgroup of $\mathrm{GL}_r(\mathbf{Z}_p)$, and this subgroup is independent of T up to conjugation in $\mathrm{GL}_r(\mathbf{Z}_p)$. We will therefore suppress T from the notation.

Now, by [56, Thm. 9.7] this exhibits $(W_X, \pi_1(T))$ as a finite reflection group over $\mathbf{Z}_p$. Finite reflection groups over $\mathbf{Z}_p$ have been classified for p odd by Notbohm [107] extending the classification over $\mathbf{Q}_p$ by Clark-Ewing [34] and Dwyer-Miller-Wilkerson [52] (which again builds on the classification over $\mathbf{C}$ by Shephard-Todd [126]); we recall this classification in Section 11 and extend Notbohm's result to all primes. Recall that a finite $\mathbf{Z}_p$-reflection group is a pair (W, L) where L is a finitely generated free $\mathbf{Z}_p$-module, and W is a finite subgroup of $\mathrm{Aut}(L)$ generated by elements α such that $1-\alpha$ has rank one. We say that two finite $\mathbf{Z}_p$-reflection groups (W, L) and (W', L') are *isomorphic*, if we can find a $\mathbf{Z}_p$-linear isomorphism $\varphi : L \to L'$ such that the group $\varphi W \varphi^{-1}$ equals W'.

Given any self-homotopy equivalence $Bf : BX \to BX$, there exists, by the uniqueness of maximal tori, a map $B\tilde{f} : BT \to BT$ such that $Bf \circ Bi$ is homotopy equivalent to $Bi \circ B\tilde{f}$. Furthermore, the homotopy class of $B\tilde{f}$ is unique up to the action of the Weyl group, as is easily seen from the definitions (cf. Lemma 4.1). This sets up a homomorphism $\Phi : \pi_0(\mathrm{Aut}(BX)) \to N_{\mathrm{GL}(L_X)}(W_X)/W_X$, where $\mathrm{Aut}(BX)$ is the space of self-homotopy equivalences of BX. (This map has precursors going back to Adams-Mahmud [2]; see Lemma 4.1 and Theorem 1.4 for a more elaborate version.) The group $N_{\mathrm{GL}(L_X)}(W_X)/W_X$ can be completely calculated; see Section 13.

The main classification theorem which we complete in this paper, is the following.

THEOREM 1.1. *Let p be an odd prime. The assignment that to each connected p-compact group X associates the pair (W_X, L_X) via the canonical action of W_X on $L_X = \pi_1(T)$ defines a bijection between the set of isomorphism classes of connected p-compact groups and the set of isomorphism classes of finite $\mathbf{Z}_p$-reflection groups.*

Furthermore, for each connected p-compact group X the map

$$\Phi : \pi_0(\mathrm{Aut}(BX)) \to N_{\mathrm{GL}(L_X)}(W_X)/W_X$$

is an isomorphism, i.e., the group of outer automorphisms of X is canonically isomorphic to the group of outer automorphisms of (W_X, L_X).

In particular this proves, for p odd, Conjecture 5.3 in [48] (see Theorem 1.4). The self-map part of the statement can be viewed as an extension to p-compact groups, p odd, of the main result of Jackowski-McClure-Oliver [82], [83]. Our method of proof via centralizers is 'dual', but logically independent, of the one in [82], [83] (see e.g. [47], [72]).

By [57] the identity component of $\mathrm{Aut}(BX)$ is the classifying space of a p-compact group $\mathcal{Z}X$, which is defined to be the *center* of X. We call X *center-free* if $\mathcal{Z}X$ is trivial. For p odd this is equivalent to (W_X, L_X) being *center-free*, i.e., $(L_X \otimes \mathbf{Z}/p^\infty)^{W_X} = 0$, by [57, Thm. 7.5]. Furthermore recall that a connected p-compact group X is called *simple* if $L_X \otimes \mathbf{Q}$ is an irreducible W-representation and X is called *exotic* if it is simple and (W_X, L_X) does not come from a $\mathbf{Z}$-reflection group (see Section 11). By inspection of the classification of finite $\mathbf{Z}_p$-reflection groups, Theorem 1.1 has as a corollary that the theory of p-compact groups on the level of objects splits in two parts, as has been conjectured (Conjectures 5.1 and 5.2 in [48]).

THEOREM 1.2. *Let X be a connected p-compact group, p odd. Then X can be written as a product of p-compact groups*

$$X \cong G_{\hat{p}} \times X'$$

where G is a compact connected Lie group, and X' is a direct product of exotic p-compact groups. Any exotic p-compact group is simply connected, center-free, and has torsion-free $\mathbf{Z}_p$-cohomology.

Theorem 1.1 has both an existence and a uniqueness part to it, the existence part being that all finite $\mathbf{Z}_p$-reflection groups are realized as Weyl groups of a connected p-compact group. The finite $\mathbf{Z}_p$-reflection groups which come from compact connected Lie groups are of course realizable, and the finite $\mathbf{Z}_p$-reflection groups where p does not divide the order of the group can also relatively easily be dealt with, as done by Sullivan [134, p. 166–167] and Clark-Ewing [34] long before p-compact groups were officially defined. The remaining cases were realized by Quillen [118, §10], Zabrodsky [146, 4.3], Aguadé [4], and Notbohm-Oliver [108], [110, Thm. 1.4]. The classification of finite $\mathbf{Z}_p$-reflection groups, Theorem 11.1, guarantees that the construction of these examples actually enables one to realize all finite $\mathbf{Z}_p$-reflection groups as Weyl groups of connected p-compact groups.

The work toward the uniqueness part, to show that a connected p-compact group is uniquely determined by its Weyl group, also predates the introduction of p-compact groups. The quest was initiated by Dwyer-Miller-Wilkerson [51], [52] (building on [3]) who proved the statement, using slightly different language, in the case where p is prime to the order of W_X as well as for $\mathrm{SU}(2)_{\hat{2}}$ and $\mathrm{SO}(3)_{\hat{2}}$. Notbohm [105] and Møller-Notbohm [101, Thm. 1.9] extended this to a uniqueness statement for all p-compact groups X where $\mathbf{Z}_p[L_X]^{W_X}$

(the ring of W_X-invariant polynomial functions on L_X) is a polynomial algebra and (W_X, L_X) comes from a finite $\mathbf{Z}$-reflection group. Notbohm [108], [110] subsequently also handled the cases where (W_X, L_X) does not come from a finite $\mathbf{Z}$-reflection group. It is worth mentioning that if X has torsion-free $\mathbf{Z}_p$-cohomology (or equivalently, if $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra), then it is straightforward to see that $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra (see Theorem 12.1). The reverse implication is also true, but the argument is more elaborate (see Remark 10.11 and also Theorem 1.8 and Remark 10.9); some of the papers quoted above in fact operate with the *a priori* more restrictive assumption on X.

To get general statements beyond the case where $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra, i.e., to attack the cases where there exists p-torsion in the cohomology ring, the first step is to reduce the classification to the case of simple, center-free p-compact groups. The results necessary to obtain this reduction were achieved by the splitting theorem of Dwyer-Wilkerson [58] and Notbohm [111] along with properties of the center of a p-compact group established by Dwyer-Wilkerson [57] and Møller-Notbohm [100]. We explain this reduction in Section 6; most of this reduction was already explained by the third-named author in [98] via different arguments.

An analysis of the classification of finite $\mathbf{Z}_p$-reflection groups together with explicit calculations (see [109] and Theorem 12.2) shows that, for p odd, $\mathbf{Z}_p[L]^W$ is a polynomial algebra for all irreducible finite $\mathbf{Z}_p$-reflection groups (W, L) that are center-free, except the reflection groups coming from the p-compact groups $\mathrm{PU}(n)\hat{_p}$, $(E_8)\hat{_5}$, $(F_4)\hat{_3}$, $(E_6)\hat{_3}$, $(E_7)\hat{_3}$, and $(E_8)\hat{_3}$. For exceptional compact connected Lie groups the notation E_6 etc. denotes their *adjoint* form.

The case $\mathrm{PU}(n)\hat{_p}$ was handled by Broto-Viruel [25], using a Bockstein spectral sequence argument to deduce it from the result for $\mathrm{SU}(n)$, generalizing earlier partial results of Broto-Viruel [24] and Møller [97]. The remaining step in the classification is therefore to handle the exceptional compact connected Lie groups, in particular the problematic E-family at the prime 3, and this is what is carried out in this paper. (The fourth named author has also given alternative proofs for $(F_4)\hat{_3}$ and $(E_8)\hat{_5}$ in [137] and [136].)

Theorem 1.3. *Let X be a connected p-compact group, for p odd, with Weyl group equal to $(W_G, L_G \otimes \mathbf{Z}_p)$ for $(G, p) = (F_4, 3)$, $(E_8, 5)$, $(E_6, 3)$, $(E_7, 3)$, or $(E_8, 3)$. Then X is isomorphic, as a p-compact group, to the $\mathbf{F}_p$-completion of the corresponding exceptional group G.*

We will in fact give an essentially self-contained proof of the entire classification Theorem 1.1, since this comes rather naturally out of our inductive approach to the exceptional cases. We however still rely on the classification of finite $\mathbf{Z}_p$-reflection groups (see [107], [109] and Sections 11 and 12) as well as the above mentioned structural results from [56], [57], [100], [58], and [111].

We remark that we also need not assume known *a priori* that 'unstable Adams operations' [134], [141], [66] exist.

The main ingredient in handling the exceptional groups, once the right inductive setup is in place, is to get sufficiently detailed information about their many conjugacy classes of elementary abelian p-subgroups, and then to use this information to show that the relevant obstruction groups are trivial, using properties of Steinberg modules combined with formulas of Oliver [113] (see also [72]); we elaborate on this at the end of this introduction and in Section 2.

It is possible to formulate a more topological version of the uniqueness part of Theorem 1.1 which holds for all p-compact groups (p odd), not necessarily connected, which is however easily seen to be equivalent to the first one using [6, Thm. 1.2]. It should be viewed as a topological analog of Chevalley's isomorphism theorem for linear algebraic groups (see [76, §32], [133, Thm. 1.5] and [42], [116], [106]). To state it, we define the *maximal torus normalizer* $\mathcal{N}_X(T)$ to be the loop space such that $B\mathcal{N}_X(T)$ is the Borel construction of the canonical action of $\mathcal{W}_X(T)$ on BT. Note that by construction $\mathcal{N}_X(T)$ comes with a morphism $\mathcal{N}_X(T) \to X$. By [56, Prop. 9.5], $\mathcal{W}_X(T)$ is a discrete space, so $B\mathcal{N}_X(T)$ has only two nontrivial homotopy groups and fits into a fibration sequence $BT \to B\mathcal{N}_X(T) \to BW_X$. (Beware that in general $\mathcal{N}_X(T)$ will not be a p-compact group since its group of components W_X need not be a p-group.)

Theorem 1.4 (Topological isomorphism theorem for p-compact groups, p odd). *Let p be an odd prime and let X and X' be p-compact groups with maximal torus normalizers $\mathcal{N}_X$ and $\mathcal{N}_{X'}$. Then $X \cong X'$ if and only if $B\mathcal{N}_X \simeq B\mathcal{N}_{X'}$.*

Furthermore the spaces of self-homotopy equivalences $\operatorname{Aut}(BX)$ *and* $\operatorname{Aut}(B\mathcal{N}_X)$ *are equivalent as group-like topological monoids. Explicitly, turn $i : B\mathcal{N}_X \to BX$ into a fibration which we will again denote by i, and let* $\operatorname{Aut}(i)$ *denote the group-like topological monoid of self-homotopy equivalences of the map i. Then the following canonical zig-zag, given by restrictions, is a zig-zag of homotopy equivalences*:

$$B\operatorname{Aut}(BX) \xleftarrow{\simeq} B\operatorname{Aut}(i) \xrightarrow{\simeq} B\operatorname{Aut}(B\mathcal{N}_X).$$

In the above theorem, the fact that the evaluation map $\operatorname{Aut}(i) \to \operatorname{Aut}(BX)$ is an equivalence follows by a short general argument (Lemma 4.1), which gives a canonical homomorphism $\Phi : \operatorname{Aut}(BX) \xrightarrow{\cong} \operatorname{Aut}(i) \to \operatorname{Aut}(B\mathcal{N}_X)$, whereas the equivalence $\operatorname{Aut}(i) \to \operatorname{Aut}(B\mathcal{N}_X)$ requires a detailed case-by-case analysis.

We point out that the classification of course gives easy, although somewhat unsatisfactory, proofs that many theorems from Lie theory extend to p-compact groups, by using the fact that the theorem is known to be true in the Lie group case, and then checking the exotic cases. Since the classifying spaces of the exotic p-compact groups have cohomology ring a polynomial algebra, this can turn out to be rather straightforward. In this way one for instance sees that Bott's celebrated result about the structure of G/T [17] still holds true for p-compact groups, at least on cohomology.

THEOREM 1.5 (Bott's theorem for p-compact groups). *Let X be a connected p-compact group, p odd, with maximal torus T and Weyl group W_X. Then $H^*(X/T;\mathbf{Z}_p)$ is a free $\mathbf{Z}_p$-module of rank $|W_X|$, concentrated in even degrees.*

Likewise combining the classification with a case-by-case verification for the exotic p-compact groups by Castellana [29], [30], we obtain that the Peter-Weyl theorem holds for connected p-compact groups, p odd:

THEOREM 1.6 (Peter-Weyl theorem for connected p-compact groups). *Let X be a connected p-compact group, p odd. Then there exists a monomorphism $X \to \mathrm{U}(n)\hat{_p}$ for some n.*

We also still have the 'standard' formula for the fundamental group (the subscript denotes coinvariants).

THEOREM 1.7. *Let X be a connected p-compact group, p odd. Then*

$$\pi_1(X) = (L_X)_{W_X}.$$

The classification also gives a verification that results of Borel, Steinberg, Demazure, and Notbohm [110, Prop. 1.11] extend to p-compact groups, p odd. Recall that an elementary abelian p-subgroup of X is just a monomorphism $\nu : E \to X$, where $E \cong (\mathbf{Z}/p)^r$ for some r.

THEOREM 1.8. *Let X be a connected p-compact group, p odd. The following conditions are equivalent*:

(1) *X has torsion-free $\mathbf{Z}_p$-cohomology.*

(2) *BX has torsion-free $\mathbf{Z}_p$-cohomology.*

(3) *$\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra over $\mathbf{Z}_p$.*

(4) *All elementary abelian p-subgroups of X factor through a maximal torus.*

(See also Theorem 12.1 for equivalent formulations of condition (1).) Even in the Lie group case, the proof of the above theorem is still not entirely satisfactory despite much effort—see the comments surrounding our proof in Section 10 as well as Borel's comments [13, p. 775] and the references [11], [43], and [132]. The *centralizer* $\mathcal{C}_X(\nu)$ of an elementary abelian p-subgroup

$\nu : E \to X$ is defined as $\mathcal{C}_X(\nu) = \Omega \operatorname{map}(BE, BX)_{B\nu}$; cf. Section 2. The following related result from Lie theory also holds true.

THEOREM 1.9. *Let X be a connected p-compact group, p odd. Then the following conditions are equivalent*:

(1) $\pi_1(X)$ *is torsion-free.*

(2) *Every rank one elementary abelian p-subgroup $\nu : \mathbf{Z}/p \to X$ has connected centralizer $\mathcal{C}_X(\nu)$.*

(3) *Every rank two elementary abelian p-subgroup factors through a maximal torus.*

Results about p-compact groups can in general, via Sullivan's arithmetic square, be translated into results about finite loop spaces, and the last theorem in this introduction is an example of such a translation. (For another instance see [7].) Recall that a *finite loop space* is a loop space (X, BX, e), where X is a finite CW-complex. A maximal torus of a finite loop space is simply a map $B\mathrm{U}(1)^r \to BX$ for some r, such that the homotopy fiber is homotopy equivalent to a finite CW-complex of nonzero Euler characteristic. The classical maximal torus conjecture (stated in 1974 by Wilkerson [140, Conj. 1] as "a popular conjecture toward which the author is biased"), asserts that compact connected Lie groups are the *only* connected finite loop spaces which admit maximal tori. A slightly more elaborate version states that the classifying space functor should set up a bijection between isomorphism classes of compact connected Lie groups and isomorphism classes of connected finite loop spaces admitting a maximal torus, under which the outer automorphism group of the Lie group G equals the outer automorphism group of the corresponding loop space (G, BG, e). (The last part is known to be true by [83, Cor. 3.7].) It is well known that a proof of the conjectured classification of p-compact groups for all primes p would imply the maximal torus conjecture. Our results at least imply that the conjecture is true after inverting the single prime 2.

THEOREM 1.10. *Let X be a connected finite loop space with a maximal torus. Then there exists a compact connected Lie group G such that $BX[\frac{1}{2}]$ and $BG[\frac{1}{2}]$ are homotopy equivalent spaces, where $[\frac{1}{2}]$ indicates $\mathbf{Z}[\frac{1}{2}]$-localization.*

Relationship to the Lie group case and the conjectural picture for $p = 2$. We now state a common formulation of both the classification of compact connected Lie groups and the classification of connected p-compact groups for p odd, which conjecturally should also hold for $p = 2$. We have not encountered this—in our opinion quite natural—description before in the literature (compare [48] and [89]).

Let R be an integral domain and W a finite R-reflection group. For an *RW-lattice* L (i.e., an RW-module which is finitely generated and free as an R-module) define SL to be the sublattice of L generated by $(1-w)x$ where $w \in W$ and $x \in L$. Define an *R-reflection datum* to be a triple (W, L, L_0) where (W, L) is a finite R-reflection group and L_0 is an RW-lattice such that $SL \subseteq L_0 \subseteq L$ and L_0 is isomorphic to SL' for some RW-lattice L'. (If $R = \mathbf{Z}_p$, p odd, then 'S' is idempotent and $L_0 = SL$, since W is generated by elements of order prime to p so $H_1(W; L_W) = 0$.) Two reflection data (W, L, L_0) and (W', L', L_0') are said to be isomorphic if there exists an R-linear isomorphism $\varphi : L \to L'$ such that $\varphi W \varphi^{-1} = W'$ and $\varphi(L_0) = L_0'$.

If $\mathcal{D}$ is either the category of compact connected Lie groups or connected p-compact groups, then we can consider the assignment which to each object X in $\mathcal{D}$ associates the triple (W, L, L_0), where W is the Weyl group, $L = \pi_1(T)$ is the *integral lattice*, and $L_0 = \ker(\pi_1(T) \to \pi_1(X))$ is the *coroot lattice*.

Theorems 1.1 and 1.7 as well as the classification of compact connected Lie groups [20, §4, no. 9] can now be reformulated as follows:

THEOREM 1.11. *Let $\mathcal{D}$ be the category of compact connected Lie groups, $R = \mathbf{Z}$, or connected p-compact groups for p odd, $R = \mathbf{Z}_p$. For X in $\mathcal{D}$ the associated triple (W, L, L_0) is an R-reflection datum and this assignment sets up a bijection between the objects of $\mathcal{D}$ up to isomorphism and R-reflection data up to isomorphism. Furthermore the group of outer automorphisms of X equals the group of outer automorphisms of the corresponding R-reflection datum.*

CONJECTURE 1.12. *Theorem* 1.11 *is also true if $\mathcal{D}$ is the category of connected* 2*-compact groups.*

One can check that the conjecture on objects is equivalent to the conjecture given in [48] and [89], and the self-map statement would then follow from [83, Cor. 3.5] and [112, Thm. 3.5]. The role of the coroot lattice L_0 in the above theorem and conjecture is in fact only to be able to distinguish direct factors isomorphic to $\mathrm{SO}(2n+1)$ from direct factors isomorphic to $\mathrm{Sp}(n)$; cf. Theorem 11.5. Alternatively one can use the extension class $\gamma \in H^3(W; L)$ of the maximal torus normalizer (see Section 5) rather than L_0 but in that picture it is not *a priori* clear which triples (W, L, γ) are realizable. It would be desirable to have a 'topological' version of Theorem 1.11 and Conjecture 1.12, i.e., statements on the level of automorphism spaces like Theorem 1.4, but we do not know a general formulation which incorporates this feature.

Organization of the paper. The paper is organized around Section 2 which sets up the framework of the proof and gives an inductive proof of the main theorems, referring to the later sections of the paper for many key statements.

The remaining sections can be read in an almost arbitrary order. We now briefly sketch how these sections are used.

We first say a few words about Section 4–7, before describing Section 2 and the later sections in a little more detail. The short Sections 4 and 5 construct the map $\Phi : \mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N}_X)$ and give an algebraic description of the automorphisms of $B\mathcal{N}_X$. Section 6 contains the reduction to the case of simple, center-free, connected p-compact groups. In Section 7 we prove an integral version of a theorem of Nakajima, and show how this leads to an easy criterion for inductively constructing certain p-compact groups; this criterion will, in the setup of the induction, lead to a construction of the exotic p-compact groups and show that they have torsion-free $\mathbf{Z}_p$-cohomology.

Armed with this information let us now summarize Section 2. In the inductive framework of the main theorem the results in Section 7 guarantee that we have concrete models for conjecturally all p-compact groups, and that those coming from exotic finite $\mathbf{Z}_p$-reflection groups have torsion-free $\mathbf{Z}_p$-cohomology. Likewise, by the reduction theorems in Section 6, we are furthermore reduced to showing that if X' is an unknown connected center-free simple p-compact group with associated $\mathbf{Z}_p$-reflection group (W, L) then it agrees with our known model X realizing (W, L). We want, using the inductive assumption, to construct a map from the centralizers in X to X', and show that these maps glue together to give an isomorphism $X \to X'$. To be able to glue the maps together, we need to have a preferred choice on each centralizer and know that these agree on the intersection—this is why we also have to keep track of the automorphisms of p-compact groups in our inductive hypothesis.

If X has torsion-free $\mathbf{Z}_p$-cohomology, then every elementary abelian p-subgroup factors through the maximal torus, and it follows from our construction that our maps on the different centralizers of elementary abelian p-subgroups in X to X' match up, as maps in the *homotopy category*. This is not obvious in the case where X has torsion in its $\mathbf{Z}_p$-cohomology, and we develop tools in Section 3 which suffice to handle all the torsion cases, on a case-by-case basis. This step should be thought of as inductively showing that X and X' have the same (centralizer) fusion.

We now have to rigidify our maps on the centralizers from a consistent collection of maps in the homotopy category to a consistent collection map in the category of spaces. There is an obstruction theory for dealing with this issue. Again, in the case where X does not have torsion there is a general argument for showing that these obstruction groups vanish, whereas we in the case where X has torsion have to show this on a case-by-case basis. To deal with this we give in the purely algebraic Section 8 complete information about all nontoral elementary abelian p-subgroups of the projective unitary groups and the exceptional compact connected Lie groups, along with their Weyl groups and centralizers. This information is needed as input in Section 9 for showing that the obstruction groups vanish. Hence we get a map in the category of spaces from the centralizers in X to X', which then glues together to produce a map $X \to X'$ which then by our construction is easily seen to be an isomorphism. As a by-product of the analysis we also conclude that X has the right automorphism group. This proves the main theorems. Section 10 establishes the consequences of the main theorem, listed in the introduction.

There are three appendices: In Section 11 we give a concise classification of finite $\mathbf{Z}_p$-reflection groups generalizing Notbohm's classification to all primes. In Section 12 we recall Notbohm's results on invariant rings of finite $\mathbf{Z}_p$-reflection groups. These facts are all used multiple times in the proof. Finally in Section 13 we briefly calculate the outer automorphism groups of the finite $\mathbf{Z}_p$-reflection groups to make the automorphism statement in the main result more explicit.

Notation. We have tried to introduce the definitions relating to p-compact groups as they are used, but it is nevertheless probably helpful for the reader unfamiliar with p-compact groups to keep copies of the excellent papers [56] and [57] of Dwyer-Wilkerson (whose terminology we follow) within reach. As a technical term we say that a p-compact group X is *determined by* $\mathcal{N}_X$ if any p-compact group X' with the same maximal torus normalizer is isomorphic to X (which will be true for all p-compact groups, p odd, by Theorem 1.4).

We tacitly assume that any space in this paper has the homotopy type of a CW-complex, if necessary replacing a given space by the realization of its singular complex [93].

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2. Skeleton of the proof of the main Theorems 1.1 and 1.4

The purpose of this section is to give the skeleton of the proof of the main Theorems 1.1 and 1.4, but in the proofs referring forward to the remaining sections in the paper for the proof of many key statements, as explained in the organizational remarks in the introduction.

We start by explaining the proof in general terms, which is carried out via a grand induction—for simplicity we focus first on the uniqueness statement. Suppose that X is a known p-compact group and X' is another p-compact group with the same maximal torus normalizer. We want to construct an isomorphism $X \to X'$, by decomposing X in terms of centralizers of its nontrivial elementary abelian p-subgroups, as we will explain below. Using an inductive assumption we can construct a homomorphism from each of these centralizers to X', and we want to see that we can do this in a coherent way, so that they glue together to give the desired map $X \to X'$.

We first explain the centralizer decomposition. It is a theorem of Lannes [88, Thm. 3.1.5.1] and Dwyer-Zabrodsky [46] (see also [82, Thm. 3.2]), that for an elementary abelian p-group E and a compact Lie group G with component group a p-group, we have a homotopy equivalence

$$\coprod_{\nu \in \mathrm{Rep}(E,G)} BC_G(\nu(E))\hat{_p} \xrightarrow{\simeq} \mathrm{map}(BE, BG\hat{_p})$$

induced by the adjoint of the canonical map $BE \times BC_G(\nu(E)) \to BG$. Here $\mathrm{Rep}(E,G)$ denotes the set of homomorphisms $E \to G$, modulo conjugacy in G.

Generalizing this, one defines, for a p-compact group X, an *elementary abelian p-subgroup* of X to be a monomorphism $\nu : E \to X$, and its *centralizer* to be the p-compact group $\mathcal{C}_X(\nu)$ with classifying space $B\mathcal{C}_X(\nu) = \mathrm{map}(BE, BX)_{B\nu}$. By a theorem of Dwyer-Wilkerson [56, Props. 5.1 and 5.2] this actually is a p-compact group and the evaluation map to X is a monomorphism. Note however that $\mathcal{C}_X(\nu)$ is not defined as a subobject of X, i.e., the map to X is defined in terms of ν, unlike the Lie group case.

For a p-compact group X, let $\mathbf{A}(X)$ denote the Quillen category of X. The objects of $\mathbf{A}(X)$ are conjugacy classes of monomorphisms $\nu : E \to X$ of nontrivial elementary abelian p-groups E into X. The morphisms $(\nu : E \to X) \to (\nu' : E' \to X)$ of $\mathbf{A}(X)$ consists of all group monomorphisms $\rho : E \to E'$ such that ν and $\nu'\rho$ are conjugate.

The centralizer construction gives a functor

$$B\mathcal{C}_X : \mathbf{A}(X)^{\mathrm{op}} \to \mathrm{Spaces}$$

that takes the monomorphism $(\nu : E \to X) \in \mathrm{Ob}(\mathbf{A}(X))$ to its centralizer $B\mathcal{C}_X(\nu) = \mathrm{map}(BE, BX)_{B\nu}$ and a morphism ρ to composition with $B\rho : BE \to BE'$.

The centralizer decomposition theorem of Dwyer-Wilkerson [57, Thm. 8.1], generalizing a theorem for compact Lie groups by Jackowski-McClure [81, Thm. 1.3], says that the evaluation map

$$\mathrm{hocolim}_{\mathbf{A}(X)} B\mathcal{C}_X \to BX$$

induces an isomorphism on mod p homology. If X is connected and center-free, then for all ν, the centralizer $\mathcal{C}_X(\nu)$ is a p-compact group with smaller cohomological dimension, hence setting the stage for a proof by induction; cf. [57, §9]. (The *cohomological dimension* of a p-compact group X is defined as $\mathrm{cd}(X) = \max\{n | H^n(X; \mathbf{F}_p) \neq 0\}$; see [56, Def. 6.14] and [58, Lem. 3.8].)

To make use of this we need a way to construct a map from centralizers of elementary abelian p-subgroups in X to any other p-compact group X' with the same maximal torus normalizer $\mathcal{N}$. Let $\mathcal{N}$ be embedded via homomorphisms $j : \mathcal{N} \to X$ and $j' : \mathcal{N} \to X'$ respectively. If $\nu : E \to X$ can be factored through a maximal torus $i : T \to X$, i.e., if there exists $\mu : E \to T$ such that $i\mu = \nu$, then μ is unique up to conjugation as a map to $\mathcal{N}$ by [58, Prop. 3.4]. Furthermore by [57, Pf. of Thm. 7.6(1)], $\mathcal{C}_{\mathcal{N}}(\mu)$ is a maximal torus normalizer in $\mathcal{C}_X(\nu)$, where centralizers in $\mathcal{N}$ are defined in the same way as in a p-compact group. In this case $j'\mu$ will be an elementary abelian p-subgroup of X', which we have assigned without making any choices, and $\mathcal{C}_{X'}(j'\mu)$ will have maximal torus normalizer $\mathcal{C}_{\mathcal{N}}(\mu)$. Suppose that $\mathcal{C}_X(\nu)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\nu)}$ (i.e., any p-compact group with maximal torus normalizer isomorphic to $\mathcal{N}_{\mathcal{C}_X(\nu)}$ is isomorphic to $\mathcal{C}_X(\nu)$) and that the homomorphism $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\nu)) \to \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\nu)})$, defined after Theorem 1.4, is an equivalence. Since $\mathcal{C}_X(\nu)$ is determined by its maximal torus normalizer, surjectivity of $\pi_0(\Phi)$ implies that there exists an isomorphism h_ν making the diagram

(2.1)
$$\begin{tikzcd} & \mathcal{C}_{\mathcal{N}}(\mu) \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ \mathcal{C}_X(\nu) \arrow[rr, "h_\nu", "\cong"'] & & \mathcal{C}_{X'}(j'\mu) \end{tikzcd}$$

commute, and h_ν is unique up to conjugacy, by the injectivity of $\pi_0(\Phi)$. (In fact the space of such h_ν is contractible, since Φ is an equivalence.) This constructs the desired map $\varphi_\nu : \mathcal{C}_X(\nu) \xrightarrow{h_\nu} \mathcal{C}_{X'}(j'\mu) \to X'$ for elementary abelian p-subgroups $\nu : E \to X$ which factor through the maximal torus. An elementary abelian p-subgroup is called *toral* if it has this property, and *nontoral* if not.

We want to construct maps also for nontoral elementary abelian p-subgroups, by utilizing the centralizers of rank one elementary abelian p-subgroups, which are always toral by [56, Prop. 5.6] if X is connected. For this we need to recall the construction of adjoint maps.

Construction 2.1 (Adjoint maps). Let A be an abelian p-compact group (i.e., a p-compact group such that $\mathcal{Z}A \to A$ is an isomorphism), X a p-compact group, and $\nu : A \to X$ a homomorphism. Suppose that E is an elementary abelian p-subgroup of A and note that we have a canonical map

$$BA \times BE \xrightarrow{\text{mult}} BA \to BX$$

whose homotopy class only depends on the conjugacy class of ν. Since furthermore

$$\pi_0(\mathrm{map}(BA \times BE, BX)) \cong \coprod_{\xi \in [BE,BX]} \pi_0(\mathrm{map}(BA, \mathrm{map}(BE, BX)_\xi))$$

every homomorphism $\nu : A \to X$ gives rise to a homomorphism $\tilde{\nu} : A \to \mathcal{C}_X(\nu|_E)$ making the diagram

$$\begin{tikzcd} & \mathcal{C}_X(\nu|_E) \arrow[d, "\mathrm{ev}"] \\ A \arrow[ur, "\tilde{\nu}"] \arrow[r, "\nu"] & X \end{tikzcd}$$

commutative. Here $\tilde{\nu}$ is well-defined up to conjugacy in terms of the conjugacy class of ν. We will always use the notation $\widetilde{(\cdot)}$ for this construction.

Let $\nu : E \to X$ be an *arbitrary* nontrivial elementary abelian p-subgroup of a connected p-compact group X and let V be a rank one subgroup of E. Then $\nu|_V$ is toral by [56, Prop. 5.6]; i.e., it factors through T and the map $\mu : V \to T \to \mathcal{N}$ is unique up to conjugation in $\mathcal{N}$. Furthermore if $\mathcal{C}_X(\nu|_V)$ is determined by $\mathcal{C}_{\mathcal{N}}(\mu)$ and $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\nu|_V)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\nu|_V)})$ then $h_{\nu|_V}$ is defined as before, and we can look at the composite

$$\varphi_{\nu,V} : \mathcal{C}_X(\nu) \longrightarrow \mathcal{C}_X(\nu|_V) \xrightarrow[\cong]{h_{\nu|_V}} \mathcal{C}_{X'}(j'\mu) \longrightarrow X'.$$

This is the definition we will use in general. It is easy to see using adjoint maps that this construction generalizes the previous one in the case where ν is toral, under suitable inductive assumptions (cf. the proof of Theorem 2.2 below). However if ν is nontoral it is not obvious that this map is independent of the choice of subgroup V of E, which is needed in order to get a map (in the homotopy category) from the centralizer diagram of BX to BX'. Checking that this is the case basically amounts to inductively establishing that elementary abelian p-subgroups and their centralizers are conjugate in the same way in X and X', i.e., that they have the same fusion. Furthermore we want see that this diagram can be rigidified to a diagram in the category of spaces, to get an induced map from the homotopy colimit of the centralizer diagram. The next theorem states precisely what needs to be checked—the calculations to verify that these conditions are indeed satisfied for all simple center-free p-compact groups is essentially the content of the rest of the paper.

THEOREM 2.2. *Let X and X' be two connected p-compact groups with the same maximal torus normalizer $\mathcal{N}$ embedded via j and j' respectively. Assume that X satisfies the following inductive assumption*:

($\star$) *For all rank one elementary abelian p-subgroups $\nu : E \to X$ of X the centralizer $\mathcal{C}_X(\nu)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\nu)}$ and $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\nu)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\nu)})$ when ν is of rank one or two.*

Then:

(1) *Assume that for every rank two nontoral elementary abelian p-subgroup $\nu : E \to X$ the induced map $\varphi_{\nu,V}$ is independent of the choice of the rank one subgroup V of E. Then there exists a map in the homotopy category of spaces from the centralizer diagram of BX to BX' (seen as a constant diagram), i.e., an element in $\lim^0_{\nu \in \mathbf{A}(X)}[B\mathcal{C}_X(\nu), BX']$, given via the maps $\varphi_{\nu,V}$ described above.*

(2) *Assume furthermore that $\lim^i_{\nu \in \mathbf{A}(X)} \pi_j(B\mathcal{Z}\mathcal{C}_X(\nu)) = 0$ for $j = 1, 2$ and $i = j, j+1$. Then there is a lift of this element in $\lim^0$ to a map in the (diagram) category of spaces. This produces an isomorphism $f : X \to X'$ under $\mathcal{N}$, unique up to conjugacy, and $\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N})$.*

Proof. As explained before the theorem, if $\nu : E \to X$ has rank one then ν factors through T to give a map $\mu : E \to \mathcal{N}$, unique up to conjugation in $\mathcal{N}$; so the inductive assumption ($\star$) guarantees that we can construct a map $\mathcal{C}_X(\nu) \to X'$, under $C_{\mathcal{N}}(\mu)$, and this map is well-defined up to conjugation in X'.

We now want to see that in the case where E has rank two, the map $\varphi_{\nu,V}$ is in fact independent of the choice of the rank one subgroup V. Assume first that $\nu : E \to X$ is toral and let $\mu : E \to \mathcal{N}$ be a factorization of ν through T. By adjointness we have the following commutative diagram

(2.2)

$$\begin{tikzcd}
 & & \mathcal{C}_{\mathcal{N}}(\mu) \arrow[ddll, "j"'] \arrow[ddrr, "j'"] & & \\
 & & \mathcal{C}_{\mathcal{C}_{\mathcal{N}}(\mu|_V)}(\tilde{\mu}) \arrow[u, "\cong"'] \arrow[dl] \arrow[dr] & & \\
\mathcal{C}_X(\nu) & \mathcal{C}_{\mathcal{C}_X(\nu|_V)}(\tilde{\nu}) \arrow[l, "\cong"'] \arrow[rr, "\widetilde{h_{\nu|_V}}", "\cong"'] & & \mathcal{C}_{\mathcal{C}_{X'}(j'\mu|_V)}(\widetilde{j'\mu}) \arrow[r, "\cong"] & \mathcal{C}_{X'}(j'\mu)
\end{tikzcd}$$

where $\widetilde{h_{\nu|_V}}$ is the map induced from $h_{\nu|_V}$ on the centralizers. The rank two uniqueness assumption in ($\star$) now guarantees that the bottom left-to-right composite ψ is independent of the choice of V.

However, for any particular choice of V we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_X(\nu) \arrow[r, "\psi", "\cong"'] \arrow[d] & \mathcal{C}_{X'}(j'\mu) \arrow[d] \arrow[dr] & \\
\mathcal{C}_X(\nu|_V) \arrow[r, "h_{\nu|_V}", "\cong"'] & \mathcal{C}_{X'}(j'\mu|_V) \arrow[r] & X'
\end{tikzcd}$$

and since ψ is independent of V this shows that $\varphi_{\nu,V}$ is independent of V as wanted. This handles the rank two toral case. For the rank two nontoral case we are simply assuming that $\varphi_{\nu,V}$ is independent of V. (Note that the problem which prevents the toral argument to carry over to the nontoral case is that we cannot choose a uniform $\mu : E \to \mathcal{N}$ such that $\mu|_V$ factors through T for all V, since this would imply that E itself was toral.)

The fact that $\varphi_{\nu,V}$ is independent of V when E is of rank two implies the statement in general: Let $\nu : E \to X$ be an elementary abelian p-subgroup of rank at least three. If V_1 and V_2 are two different rank one subgroups of E, we set $U = V_1 \oplus V_2$ and consider the following diagram

$$\begin{tikzcd}
 & \mathcal{C}_X(\nu|_{V_1}) \arrow[dr, "\varphi_{\nu|_{V_1}}"] & \\
\mathcal{C}_X(\nu) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{C}_X(\nu|_U) \arrow[u] \arrow[d] & X'. \\
 & \mathcal{C}_X(\nu|_{V_2}) \arrow[ur, "\varphi_{\nu|_{V_2}}"'] &
\end{tikzcd}$$

Here the left-hand side of the diagram is constructed by adjointness and hence commutes, and the right-hand side of the diagram commutes up to conjugation by the rank two assumption. This shows that the top left-to-right composite φ_{ν,V_1} is conjugate to the bottom left-to-right composite φ_{ν,V_2}, i.e., the map $\varphi_{\nu,V}$ is independent of the choice of rank one subgroup V in general. We hence drop the subscript V and denote this map by φ_ν.

With these preparations we can now easily finish the proof of part (1) of the theorem. We need to see that for an arbitrary morphism $\rho : (\nu : E \to X) \to (\nu' : E' \to X)$ in $\mathbf{A}(X)$ the diagram

$$\begin{tikzcd}
\mathcal{C}_X(\nu') \arrow[rr, "\mathcal{C}_X(\rho)"] \arrow[dr, "\varphi_{\nu'}"'] & & \mathcal{C}_X(\nu) \arrow[dl, "\varphi_\nu"] \\
 & X' &
\end{tikzcd}$$

commutes. Suppose first that E' has rank one, and let $\mu : E' \to T \to \mathcal{N}$ be the factorization of ν' through T. The statement follows here since the diagram

$$\begin{tikzcd}
\mathcal{C}_X(\nu') \arrow[r, "h_{\nu'}", "\cong"'] \arrow[d, "\mathcal{C}_X(\rho)"', "\cong"] & \mathcal{C}_{X'}(j'\mu) \arrow[r] \arrow[d, "\mathcal{C}_{X'}(\rho)"', "\cong"] & X' \\
\mathcal{C}_X(\nu'\rho) \arrow[r, "h_{\nu'\rho}", "\cong"'] & \mathcal{C}_{X'}(j'\mu\rho) \arrow[ur] &
\end{tikzcd}$$

commutes up to conjugation, by the uniqueness in the rank one case, since we can view the diagram of isomorphisms as taking place under $\mathcal{C}_\mathcal{N}(\mu) \to \mathcal{C}_\mathcal{N}(\mu\rho)$. The general case follows from the rank one case, by the independence of choice of rank one subgroup: If V is a rank one subgroup of E set $V' = \rho(V)$ and observe that by adjointness the diagram

$$\begin{tikzcd} \mathcal{C}_X(\nu') \arrow[r] \arrow[d, "\mathcal{C}_X(\rho)"] & \mathcal{C}_X(\nu'|_{V'}) \arrow[d, "\mathcal{C}_X(\rho|_V)"] \\ \mathcal{C}_X(\nu) \arrow[r] & \mathcal{C}_X(\nu|_V) \end{tikzcd}$$

commutes. Hence we have constructed a map up to conjugacy from the centralizer diagram of X to X' (seen as a constant diagram), or in other words we have defined an element

$$[\vartheta] \in \lim_{\nu \in \mathbf{A}(X)}{}^{0} \pi_0(\mathrm{map}(B\mathcal{C}_X(\nu), BX')).$$

This concludes the proof of part (1).

Using [59, Rem. after Def. 6.3], [57, Lem. 11.15] (which say that the centralizer diagram of a p-compact group is 'centric') it is easy to see that the map $\varphi_\nu : \mathcal{C}_X(\nu) \to X'$ induces a homotopy equivalence

$$\mathrm{map}(B\mathcal{C}_X(\nu), B\mathcal{C}_X(\nu))_1 \xrightarrow{\simeq} \mathrm{map}(B\mathcal{C}_X(\nu), BX')_{\varphi_\nu}$$

where the first term equals the classifying space of the center $B\mathcal{Z}\mathcal{C}_X(\nu)$ by definition [57]. Since this is natural it gives a canonical identification of the functor $\nu \mapsto \pi_i(\mathrm{map}(B\mathcal{C}_X(\nu), BX')_{[\vartheta]})$ with $\nu \mapsto \pi_i(B\mathcal{Z}\mathcal{C}_X(\nu))$.

By obstruction theory (see [143, Prop. 3], [84, Prop. 1.4]) the existence obstructions for lifting $[\vartheta]$ to an element in $\pi_0(\mathrm{holim}_{\mathbf{A}(X)} \mathrm{map}(B\mathcal{C}_X(\nu), BX')) \cong \pi_0(\mathrm{map}(BX, BX'))$ lie in

$$\lim_{\nu \in \mathbf{A}(X)}{}^{i+1} \pi_i(\mathrm{map}(B\mathcal{C}_X(\nu), BX')_{[\vartheta]}) \cong \lim_{\nu \in \mathbf{A}(X)}{}^{i+1} \pi_i(B\mathcal{Z}\mathcal{C}_X(\nu)),\ i \geq 1.$$

But by assumption all these groups are identically zero, so our element $[\vartheta]$ lifts to a map $Bf : BX \to BX'$.

We now want to see that the construction of f forces it to be an isomorphism. Let $\mathcal{N}_p$ denote a p-normalizer of T, i.e., the union of components in $\mathcal{N}$ corresponding to a Sylow p-subgroup of W. Since $\mathcal{N}_p$ has nontrivial center (by standard facts about p-groups), we can find a central rank one elementary abelian p-subgroup $\mu : V \to T \to \mathcal{N}_p$, and so we can view $\mathcal{N}_p$ as sitting inside $\mathcal{C}_{\mathcal{N}}(\mu)$. Hence by construction the diagram

$$\begin{tikzcd} & \mathcal{N}_p \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ X \arrow[rr, "f"] & & X' \end{tikzcd}$$

commutes up to conjugation, and in particular $fj : \mathcal{N}_p \to X'$ is a monomorphism. This implies that f is a monomorphism as well: $H^*(B\mathcal{N}_p; \mathbf{F}_p)$ is finitely generated over $H^*(BX'; \mathbf{F}_p)$ via $H^*(Bf \circ Bj; \mathbf{F}_p)$ by [56, Prop. 9.11]. By an application of the transfer [56, Thm. 9.13] the map $H^*(Bj; \mathbf{F}_p) : H^*(BX; \mathbf{F}_p) \to$

$H^*(B\mathcal{N}_p; \mathbf{F}_p)$ is a monomorphism, and since $H^*(BX'; \mathbf{F}_p)$ is noetherian by [56, Thm. 2.4] we conclude that $H^*(BX; \mathbf{F}_p)$ is finitely generated over $H^*(BX'; \mathbf{F}_p)$ as well. Hence $f : X \to X'$ is a monomorphism by another application of [56, Prop. 9.11]. Since we can identify the maximal tori of X' and X, the definition of the Weyl group produces a map between the Weyl groups $W_X \to W_{X'}$, which has to be injective since the Weyl groups act faithfully on T (by [56, Thm. 9.7]). But since we know that X and X' have the same maximal torus normalizer, the above map of Weyl groups is an isomorphism. By [57, Thm. 4.7] (or [100, Prop. 3.7] and [56, Thm. 9.7]) this means that f is indeed an isomorphism.

We now want to argue that f is a map under $\mathcal{N}$. By Lemma 4.1 we know that there exists $Bg \in \mathrm{Aut}(B\mathcal{N})$, unique up to conjugation, such that

$$\begin{CD} \mathcal{N} @>{g}>> \mathcal{N} \\ @V{j}VV @VV{j'}V \\ X @>{f}>> X' \end{CD}$$

commutes up to conjugation. By covering space theory and Sylow's theorem we can restrict g to a self-map g' making the diagram

$$\begin{CD} \mathcal{N}_p @>{g'}>> \mathcal{N}_p \\ @V{j}VV @VV{j'}V \\ X @>{f}>> X' \end{CD}$$

commute. Furthermore any other map $\mathcal{N}_p \to \mathcal{N}_p$ fitting in this diagram will be conjugate to g' in $\mathcal{N}$, by the proof of Lemma 4.1. However, by construction, f is a map under $\mathcal{N}_p$, so g' is conjugate in $\mathcal{N}$ to the identity map on $\mathcal{N}_p$. It follows from Propositions 5.1 and 5.2 that automorphisms of $\mathcal{N}$, up to conjugacy, are detected by their restriction to a maximal torus p-normalizer $\mathcal{N}_p$, so also g is conjugate to the identity, i.e., f is a map under $\mathcal{N}$. This also shows that $\Phi : \pi_0(\mathrm{Aut}(BX)) \to \pi_0(\mathrm{Aut}(B\mathcal{N}))$ is surjective, since for any automorphism $g : \mathcal{N} \to \mathcal{N}$, jg is also a maximal torus normalizer in X by [99, Thm. 1.2(3)].

Note that if the component of $\mathrm{Aut}(B\mathcal{N})$ of the identity map, $\mathrm{Aut}_1(B\mathcal{N})$, is not contractible we can find a rank one elementary abelian p-subgroup $\nu : V \to T$ such that $\mathcal{C}_{\mathcal{N}}(\nu) \xrightarrow{\cong} \mathcal{N}$ which by assumption means that $\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N})$. So we can assume that $\mathrm{Aut}_1(B\mathcal{N})$ is contractible in which case $\mathrm{Aut}_1(BX)$ is as well by [57, Thms. 1.3 and 7.5].

The only remaining claim in the theorem is that the map $\Phi : \pi_0(\mathrm{Aut}(BX)) \to \pi_0(\mathrm{Aut}(B\mathcal{N}))$ is injective under the additional assumption that

$$\lim_{\nu \in \mathbf{A}(X)}{}^{i} \pi_i(B\mathcal{Z}\mathcal{C}_X(\nu)) = 0, \qquad i \geq 1.$$

In other words we have to see that any self-equivalence f of X which, up to conjugacy, induces the identity on $\mathcal{N}$ is in fact conjugate to the identity. But if we examine the above argument with $X' = X$, the map on centralizers of rank one objects induced by f has to be the identity by the rank one uniqueness assumption. The maps for higher rank are centralizers of maps of rank one, so they as well have to be the identity. Hence f maps to the same element as the identity in $\lim^0_{\nu\in\mathbf{A}(X)} \pi_0(\mathrm{map}(B\mathcal{C}_X(\nu), BX))$, which means that f actually is the identity by the vanishing of the obstruction groups (again, e.g., by [143, Prop. 4] or [84, Prop. 1.4]). □

Remark 2.3. Note how the assumption of the theorem fails (as it should) for the group SO(3) at the prime 2, which is not determined by its maximal torus normalizer. In this case the element $\mathrm{diag}(-1,-1,1)$ in the maximal torus $\mathrm{SO}(2)\times 1$ is fixed under the Weyl group action and has centralizer equal to the maximal torus normalizer O(2).

Define the *cohomological dimension* $\mathrm{cd}(W,L)$ of a finite $\mathbf{Z}_p$-reflection group (W,L) to be $2\cdot$(the number of reflections in W) $+\,\mathrm{rk}\,L$, and note that it follows easily from [58, Lem. 3.8] and [10, Thm. 7.2.1] that for X a connected p-compact group $\mathrm{cd}(X) = \mathrm{cd}(W_X, L_X)$. We are now ready to give the proof of the main Theorems 1.1 and 1.4, referring forward to the rest of the paper—the statements we refer to can however easily be taken at face value and returned to later.

Proof of Theorems 1.1, *and* 1.4 *using Sections* 3–9, 11, *and* 12. We simultaneously show that Theorems 1.1 and 1.4 hold by an induction on the cohomological dimension of X and (W,L). We will furthermore add to the induction hypothesis the statement that if X is connected and $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial ring, then $H^*(BX;\mathbf{Z}_p)\cong H^*(BT;\mathbf{Z}_p)^{W_X}$.

By the Component Reduction Lemma 6.6, Theorem 1.4 holds for a p-compact group X if it holds for its identity component X_1, so we can assume that X is connected.

By a result of the first-named author [6, Thm. 1.2], if (W,L) is realized as the Weyl group of a p-compact group X, then $\mathcal{N}_X$ will be split, i.e., the unique possible k-invariant of $B\mathcal{N}_X$ is zero and $B\mathcal{N}_X \simeq (BT)_{hW}$. (See also [135], [63], and [103] for the Lie group case.) Furthermore, by the Component Group Formula (Lemma 6.4) we can read off the component group of X from $\mathcal{N}_X$. So, to prove Theorems 1.1 and 1.4 we have to show that given any finite $\mathbf{Z}_p$-reflection group (W,L) there exists a unique connected p-compact group X realizing (W,L), with self-maps satisfying $\Phi:\mathrm{Aut}(BX)\xrightarrow{\cong}\mathrm{Aut}(B\mathcal{N}_X)$, since this implies $\Phi:\pi_0(\mathrm{Aut}(BX))\xrightarrow{\cong}N_{\mathrm{GL}(L)}(W)/W$ by Propositions 5.1 and 5.2.

We first deal with the existence part. By the classification of finite $\mathbf{Z}_p$-reflection groups (Theorem 11.1), (W,L) can be written as a product of exotic finite $\mathbf{Z}_p$-reflection groups and a finite $\mathbf{Z}_p$-reflection group of the form $(W_G, L_G\otimes\mathbf{Z}_p)$ for some compact connected Lie group G. The factor

$(W_G, L_G \otimes \mathbf{Z}_p)$ can of course be realized by $G\hat{_p}$, and in this case $H^*(BG\hat{_p}; \mathbf{Z}_p) \cong H^*(BT; \mathbf{Z}_p)^{W_G}$ if and only if $\mathbf{Z}_p[L_G \otimes \mathbf{Z}_p]^{W_G}$ is a polynomial algebra by the invariant theory appendix (Theorems 12.2 and 12.1). If (W, L) is an exotic finite $\mathbf{Z}_p$-reflection group then $\mathbf{Z}_p[L]^W$ is a polynomial algebra by Theorem 12.2 and (W, L) satisfies $\breve{T}^W = 0$ by the classification of finite $\mathbf{Z}_p$-reflection groups Theorem 11.1, where $\breve{T} \cong L \otimes \mathbf{Z}/p^\infty$ is a discrete approximation to T. By our integral version of a theorem of Nakajima (Theorem 7.1), the subgroup W_V of W fixing a nontrivial elementary abelian p-subgroup V in $\breve{T}$ is again a $\mathbf{Z}_p$-reflection group, and since reflections in W_V are also reflections in W (and W_V is a proper subgroup of W), we see that (W_V, L) has smaller cohomological dimension than (W, L). Hence by the induction hypothesis, the assumptions of the Inductive Polynomial Realization Theorem 7.3 are satisfied. So, by this theorem there exists a (unique) connected p-compact group X with Weyl group (W, L) and this satisfies $H^*(BX; \mathbf{Z}_p) \cong H^*(BT; \mathbf{Z}_p)^{W_X}$.

We now want to show that X is uniquely determined by $(W, L) = (W_X, L_X)$ and that X satisfies $\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N})$, i.e., that X satisfies the conclusion of Theorem 1.4 (and hence that of Theorem 1.1). By the Center Reduction Lemma 6.8 we can assume that X is center-free. Likewise by the splitting theorem [58, Thms. 1.4 and 1.5] together with the Product Automorphism Lemma 6.1 we can assume that X is simple. By the classification of finite $\mathbf{Z}_p$-reflection groups (Theorem 11.1) and the invariant theory appendix (Theorem 12.2) either (W, L) has the property that $\mathbf{Z}_p[L]^W$ is a polynomial algebra, or (W, L) is one of the reflection groups $(W_{\mathrm{PU}(n)}, L_{\mathrm{PU}(n)} \otimes \mathbf{Z}_p)$ (with $p|n$), $(W_{E_8}, L_{E_8} \otimes \mathbf{Z}_5)$, $(W_{F_4}, L_{F_4} \otimes \mathbf{Z}_3)$, $(W_{E_6}, L_{E_6} \otimes \mathbf{Z}_3)$, $(W_{E_7}, L_{E_7} \otimes \mathbf{Z}_3)$, or $(W_{E_8}, L_{E_8} \otimes \mathbf{Z}_3)$.

We will go through these cases individually. We can assume that X is either constructed via the Inductive Polynomial Realization Theorem, or $X = G\hat{_p}$ for the relevant compact connected Lie group G. Let X' be a connected p-compact group with Weyl group (W, L). We want to see that the assumptions of Theorem 2.2 are satisfied. For this we use the calculation of the elementary abelian p-subgroups in Section 8 sometimes together with a specialized lemma from Section 3 to see that the assumption of Theorem 2.2(1) is satisfied. The assumption of Theorem 2.2(2) follows from the Obstruction Vanishing Theorem 9.1.

If $\mathbf{Z}_p[L]^W$ is a polynomial algebra, then by the Inductive Polynomial Realization Theorem, X satisfies $H^*(BX; \mathbf{Z}_p) \cong H^*(B^2L; \mathbf{Z}_p)^W$. Hence all elementary abelian p-subgroups of X are toral by an application of Lannes' T-functor (cf. Lemma 10.8). In particular X has no rank two nontoral elementary abelian p-subgroups, so the assumption of Theorem 2.2(1) is satisfied. By the Obstruction Vanishing Theorem 9.1 the assumption of Theorem 2.2(2) also holds, and hence Theorem 2.2 implies that there exists an isomorphism of p-compact groups $X \to X'$, and that X satisfies the conclusion of Theorem 1.4.

Now consider $(W, L) = (W_{\mathrm{PU}(n)}, L_{\mathrm{PU}(n)} \otimes \mathbf{Z}_p)$ where $p \mid n$. Theorem 8.5 says that $\mathrm{PU}(n)$ has exactly one conjugacy class of rank two nontoral elementary abelian p-subgroups E and gives its Weyl group and centralizer. We divide into two cases. If $n \neq p$, Lemma 3.3 implies that the assumption of Theorem 2.2(1) is satisfied. If $n = p$, Lemma 3.2 implies that again the assumption of Theorem 2.2(1) is satisfied. In both cases the assumption of Theorem 2.2(2) is satisfied by the Obstruction Vanishing Theorem 9.1, so Theorem 1.4 holds for X.

If $(W, L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for $(G, p) = (E_8, 5)$, $(F_4, 3)$, $(2E_7, 3)$, or $(E_8, 3)$ then G (and hence X) does not have any rank two nontoral elementary abelian p-subgroups by Theorem 8.2(3), so the assumption of Theorem 2.2(1) is vacuously satisfied. The assumption of Theorem 2.2(2) holds by the Obstruction Vanishing Theorem 9.1, so Theorem 1.4 holds also in these cases.

Finally, if $(W, L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for $(G, p) = (E_6, 3)$ there are by Theorem 8.10 two isomorphism classes of rank two nontoral elementary abelian 3-subgroups $E^{2a}_{E_6}$ and $E^{2b}_{E_6}$ in $\mathbf{A}(X)$, $X = G\hat{_p}$. These both satisfy the assumption of Theorem 2.2(1) by Lemma 3.3 and the information about the centralizers in Theorem 8.10. Since the assumption of Theorem 2.2(2) as usual is satisfied by the Obstruction Vanishing Theorem 9.1 we conclude by Theorem 2.2 that Theorem 1.4 holds for X as well. This concludes the proof of the main theorems. $\square$

Remark 2.4. Note that taking the case $(W_{E_6}, L_{E_6} \otimes \mathbf{Z}_3)$ last in the above theorem is a bit misleading, since groups with adjoint form E_6 appear as centralizers in E_7 and E_8, so a separate inductive proof of uniqueness in those cases would require knowing uniqueness of E_6.

Remark 2.5. The very careful reader might have noticed that the proof of the splitting result in [6], which we use in the above proof, refers to a uniqueness result in [24] in the case of $(W_{\mathrm{PU}(3)}, L_{\mathrm{PU}(3)} \otimes \mathbf{Z}_3)$. We now quickly sketch a more direct way to get the splitting in this case, which we were told by Dwyer-Wilkerson: We need to see that a 3-compact group with Weyl group $(W_{\mathrm{PU}(3)}, L_{\mathrm{PU}(3)} \otimes \mathbf{Z}_3)$ has to have split maximal torus normalizer $\mathcal{N}$. So, suppose that X is a hypothetical 3-compact group as above but with nonsplit maximal torus normalizer. By a transfer argument (cf. [56, Thm. 9.13]), $\mathcal{N}_3$ has to be nonsplit as well. Since every elementary abelian 3-subgroup in X can be conjugated into $\mathcal{N}_3$ (since $\chi(X/\mathcal{N}_p)$ is prime to p), this means that all elementary abelian 3-subgroups in X are toral. Furthermore by [58, Prop. 3.4] conjugation between toral elementary abelian p-subgroups is controlled by the Weyl group, so the Quillen category of X in fact agrees with the Quillen category of $\mathcal{N}$. The category has up to isomorphism one object of rank two and two objects of rank one. The centralizers $\mathcal{C}_{\mathcal{N}}(V)$ of these are respectively T, $T : \mathbf{Z}/2$, and $T \cdot \mathbf{Z}/3$. The unique 3-compact groups corresponding to these centralizers are in fact given by $B\mathcal{C}_{\mathcal{N}}(V)\hat{_3}$. Hence the map $B\mathcal{N} \to BX$ is an equivalence by the centralizer cohomology decomposition theorem [57, Thm. 8.1]. But since $\mathcal{N}$ is nonsplit, we can find a map $\mathbf{Z}/9 \to \mathcal{N}$, which is not conjugate in $\mathcal{N}$ to a map into T. Hence the corresponding map $\mathbf{Z}/9 \to X$ cannot be conjugated into T either, contradicting [56, Prop. 5.6].

3. Two lemmas used in Section 2

In this section we prove two lemmas which are used to verify the assumption in Theorem 2.2(1) for a nontoral elementary abelian p-subgroup of rank two—see the text preceding Theorem 2.2 for an explanation of this assumption; we continue with the notation of Section 2. We first need a proposition which establishes a bound on the Weyl group of a self-centralizing rank two nontoral elementary abelian p-subgroup of a connected p-compact group. (The *Weyl group* of an elementary abelian p-subgroup $\nu : E \to X$ of a p-compact group X is the subgroup of $\mathrm{GL}(E)$ consisting of elements α such that $\nu\alpha$ is homotopic to ν.) Let $\breve{\mathcal{N}}_X$ and $\breve{T}$ denote discrete approximations to $\mathcal{N}_X$ and T respectively; i.e., $\breve{T} \cong L \otimes \mathbf{Z}/p^\infty$ and $\breve{\mathcal{N}}_X$ is an extension of W_X by $\breve{T}$ such that $B\breve{\mathcal{N}}_X \to B\mathcal{N}_X$ is an $\mathbf{F}_p$-equivalence—we refer to [57, §3] for facts about discrete approximations.

PROPOSITION 3.1. *Let X be a connected p-compact group, and let $\nu : E \to X$ be a rank two elementary abelian p-subgroup with $\mathcal{C}_X(\nu) \cong E$. Then* $\mathrm{SL}(E) \subseteq W(\nu)$, *where $W(\nu)$ denotes the Weyl group of ν.*

Proof. Let V be an arbitrary rank one subgroup of E and consider the adjoint map $\tilde{\nu} : E \to \mathcal{C}_X(\nu|_V)$. Let $\breve{\mathcal{N}}_p$ denote a discrete approximation to the p-normalizer $\mathcal{N}_p$ of a maximal torus in $\mathcal{C}_X(\nu|_V)$, which has positive rank since X is assumed connected. Since $\chi(\mathcal{C}_X(\nu|_V)/\mathcal{N}_p)$ is not divisible by p we can factor $\tilde{\nu}$ through $\breve{\mathcal{N}}_p$ (see [57, Prop. 2.14(1)]), and by an elementary result about p-groups $N_{\breve{\mathcal{N}}_p}(E)$ contains a p-group strictly larger than E. By assumption $\mathcal{C}_X(\nu) \cong E$ so $\mathcal{C}_{\mathcal{C}_X(\nu|_V)}(\tilde{\nu}) \cong E$, and hence $C_{\breve{\mathcal{N}}}(E) = E$. Thus $N_{\breve{\mathcal{N}}}(E)/C_{\breve{\mathcal{N}}}(E) \subseteq W(\nu) \subseteq \mathrm{GL}(E)$ contains a subgroup of order p stabilizing V. Since V was arbitrary, this shows that $W(\nu)$ contains all Sylow p-subgroups in $\mathrm{SL}(E)$, and hence $\mathrm{SL}(E)$ itself; cf. [80, Satz II.6.7]. □

LEMMA 3.2. *Let X and X' be two connected p-compact groups with the same maximal torus normalizer $\mathcal{N}$ embedded via j and j' respectively. Assume that for all elementary abelian p-subgroups $\eta : E \to X$ of X of rank one the centralizer $\mathcal{C}_X(\eta)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\eta)}$ and $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\eta)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\eta)})$.*

If $\nu : E \to X$ is a rank two nontoral elementary abelian p-subgroup of X such that $\mathcal{C}_X(\nu) \cong E$ then the map $\varphi_{\nu,V} : \mathcal{C}_X(\nu) \to X'$ is independent of the *choice of the rank one subgroup V of E (i.e., the assumption of Theorem 2.2(1) is satisfied for ν).*

Proof. Fix a rank one subgroup $V \subseteq E$ and let $\mu : V \to T \to \mathcal{N}$ be the factorization of the toral elementary abelian p-subgroup $\nu|_V : V \to X$ through T, unique as a map to $\mathcal{N}$. Then $\varphi_{\nu,V} : E \cong \mathcal{C}_X(\nu) \to X'$ is an elementary abelian p-subgroup of X' and since we have an isomorphism $h_{\nu|_V} : \mathcal{C}_X(\nu|_V) \xrightarrow{\cong} \mathcal{C}_{X'}(j'\mu)$ by assumption, it follows by adjointness that $\mathcal{C}_{X'}(\varphi_{\nu,V}) \cong E$. By Proposition 3.1 we get $\mathrm{SL}(E) \subseteq W_X(\nu)$ and $\mathrm{SL}(E) \subseteq W_{X'}(\varphi_{\nu,V})$.

Now let $\alpha \in \mathrm{SL}(E) \subseteq W_X(\nu)$. Then $\alpha(V) \xrightarrow{\alpha^{-1}} V \xrightarrow{\mu} \mathcal{N}$ is the factorization of $(\nu \circ \alpha^{-1})|_{\alpha(V)} \cong \nu|_{\alpha(V)}$ through T, unique as a map to $\mathcal{N}$. Now consider the diagram

$$\begin{tikzcd}
E \arrow[r, "\cong"] \arrow[d, "\alpha"] & \mathcal{C}_X(\nu) \arrow[r] & \mathcal{C}_X(\nu|_V) \arrow[r, "h_{\nu|_V}", "\cong"'] \arrow[d, "-\circ B\alpha^{-1}"] & \mathcal{C}_{X'}(j'\mu) \arrow[r] \arrow[d, "-\circ B\alpha^{-1}"] & X' \arrow[d, equal] \\
E \arrow[r, "\cong"] & \mathcal{C}_X(\nu) \arrow[r] & \mathcal{C}_X(\nu|_{\alpha(V)}) \arrow[r, "h_{\nu|_{\alpha(V)}}", "\cong"'] & \mathcal{C}_{X'}(j'\mu \circ (\alpha^{-1}|_{\alpha(V)})) \arrow[r] & X'.
\end{tikzcd}$$

The left-hand and right-hand squares obviously commute and the middle square commutes by our assumption on rank one subgroups. We thus conclude that $\varphi_{\nu,\alpha(V)} \circ \alpha$ is conjugate to $\varphi_{\nu,V}$ for all $\alpha \in \mathrm{SL}(E)$. Since $W_{X'}(\varphi_{\nu,V})$ contains $\mathrm{SL}(E)$ and $\mathrm{SL}(E)$ acts transitively on the rank one subgroups of E it follows that $\varphi_{\nu,V}$ is independent of the choice of the rank one subgroup V of E as desired. □

LEMMA 3.3. *Let X and X' be two connected p-compact groups with the same maximal torus normalizer $\mathcal{N}$ embedded via j and j' respectively. Assume the inductive hypothesis $(\star)$ of Theorem 2.2, i.e., that for all elementary abelian p-subgroups $\eta : E \to X$ of X the centralizer $\mathcal{C}_X(\eta)$ is determined by $\mathcal{N}_{\mathcal{C}_X(\eta)}$ when η has rank one and that $\Phi : \mathrm{Aut}(B\mathcal{C}_X(\eta)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{\mathcal{C}_X(\eta)})$ when η has rank one or two.*

If $\nu : E \to X$ is a rank two nontoral elementary abelian p-subgroup of X such that $\mathcal{C}_X(\nu)_1$ is nontrivial then the map $\varphi_{\nu,V} : \mathcal{C}_X(\nu) \to X'$ is independent of the choice of the rank one subgroup V of E (i.e., the assumption of Theorem 2.2(1) is satisfied for ν).

Proof. Choose a rank one elementary abelian p-subgroup $\xi : U = \mathbf{Z}/p \to \mathcal{C}_X(\nu)_1$ in the center of the p-normalizer of a maximal torus in $\mathcal{C}_X(\nu)$, which is always possible since the action of a finite p-group on a nontrivial p-discrete torus has a nontrivial fixed point. Let $\xi \times \nu : U \times E \to X$ be the map defined by adjointness. For any rank one subgroup V of E, consider the map $\xi \times \nu|_V : U \times V \to X$ obtained by restriction. By construction $\xi \times \nu|_V$ is the adjoint of the composite $U \xrightarrow{\xi} \mathcal{C}_X(\nu)_1 \xrightarrow{\mathrm{res}} \mathcal{C}_X(\nu|_V)_1$, so $\xi \times \nu|_V : U \times V \to X$

factors through a maximal torus in X, since every rank one elementary abelian p-subgroup in a connected p-compact group factors through a maximal torus by [56, Prop. 5.6]. We want to see that $\xi \times \nu$ is a monomorphism, using the theory of kernels [56, §7]: If $\xi \times \nu$ was not a monomorphism then it would have a rank one kernel K, which by the choice of ξ cannot be equal to U. But this would mean that, for some rank one subgroup V' of E, both ν and $\xi \times \nu|_{V'}$ would be monomorphisms of rank two and factor through the monomorphism $\overline{(\xi \times \nu)} : (U \times E)/K \to X$ of rank two. But this is a contradiction since $\xi \times \nu|_{V'}$ is toral and ν is not.

Now consider the following diagram

$$\begin{tikzcd}
& \mathcal{C}_X(\nu|_V) \arrow[dr, "\varphi_{\nu|_V}"] & \\
U \times E \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{C}_X(\xi \times \nu|_V) \arrow[u] \arrow[d] & X'. \\
& \mathcal{C}_X(\xi) \arrow[ur, "\varphi_\xi"'] &
\end{tikzcd}$$

Here the left-hand side of the diagram is constructed by taking adjoints of $\xi \times \nu$ and hence it commutes. The right-hand side is also forced to commute by our inductive assumption, as explained in the beginning of the proof of Theorem 2.2, since $\xi \times \nu|_V$ is toral of rank two. We can hence without ambiguity define $(\xi \times \nu)'$ as either the top left-to-right composite (for some rank one subgroup $V \subseteq E$) or the bottom left-to-right composite. We let ν' denote the restriction of $(\xi \times \nu)'$ to E.

Finally consider the diagram

$$\begin{tikzcd}
\mathcal{C}_X(\xi \times \nu) \arrow[r, "\widetilde{h_{(\xi\times\nu)|_V}}", "\cong"'] \arrow[d] & \mathcal{C}_{X'}((\xi \times \nu)') \arrow[d] \\
\mathcal{C}_X(\nu) \arrow[r, "\widetilde{h_{\nu|_V}}", "\cong"'] & \mathcal{C}_{X'}(\nu')
\end{tikzcd}$$

and note that as before the inductive assumption guarantees that $\widetilde{h_{(\xi\times\nu)|_V}} = \widetilde{h_{(\xi\times\nu)|_U}}$, since $\xi \times \nu|_V$ is toral, and in particular $\widetilde{h_{(\xi\times\nu)|_V}}$ is independent of the choice of V.

We want to see that this forces the same to be true for the bottom map $\widetilde{h_{\nu|_V}}$. By our induction hypothesis, an automorphism of $\mathcal{C}_X(\nu)$ is determined by the induced map on a maximal torus normalizer. Furthermore, in general, for a p-compact group Y, an automorphism $\varphi : \mathcal{N}_Y \to \mathcal{N}_Y$ is determined up to conjugacy by the restriction $\mathcal{N}_{p,Y} \to \mathcal{N}_Y \xrightarrow{\varphi} \mathcal{N}_Y$ to a p-normalizer $\mathcal{N}_{p,Y}$: For Y connected this follows directly from Propositions 5.1 and 5.2, since elements in $H^1(W_Y; \breve{T}_Y)$ are determined by their restriction to a Sylow

p-subgroup of W_Y; for general Y the same argument works, once we note that W_Y is generated by W_{Y_1} and the image of $\breve{\mathcal{N}}_{p,Y}$ in W_Y. Now, by our choice of ξ, the centralizer $\mathcal{C}_X(\xi \times \nu)$ contains a p-normalizer of a maximal torus in $\mathcal{C}_X(\nu)$, so the above shows that $\widetilde{h_{\nu|_V}}$ is independent of V as wanted. Hence $\varphi_{\nu,V} : \mathcal{C}_X(\nu) \xrightarrow{\widetilde{h_{\nu|_V}}} \mathcal{C}_{X'}(\nu') \xrightarrow{\text{ev}} X'$ is independent of V. $\square$

4. The map $\Phi : \mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N}_X)$

The purpose of this very short section is to construct the map $\Phi : \mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N}_X)$ which we will prove is an equivalence. We have been unable to find this description in the literature.

For a fibration $f : \mathcal{E} \to \mathcal{B}$ we let $\mathrm{Aut}(f)$ denote the space of commutative diagrams

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "f"'] & \mathcal{E} \arrow[d, "f"] \\ \mathcal{B} \arrow[r] & \mathcal{B} \end{tikzcd}$$

such that the horizontal maps are homotopy equivalences. (This is a subspace of $\mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{B})$.)

LEMMA 4.1 (Adams-Mahmud lifting). *Let X be a p-compact group with maximal torus normalizer $\mathcal{N}_X$. Turn the inclusion of the maximal torus normalizer into a fibration $i : B\mathcal{N}_X \to BX$. Then the restriction map $\mathrm{Aut}(i) \to \mathrm{Aut}(BX)$ is an equivalence of group-like topological monoids.*

In particular any self-homotopy equivalence of BX lifts to a self-homotopy equivalence of $B\mathcal{N}_X$, which is unique in the strong sense that the space of lifts is contractible. Choosing a homotopy inverse to the homotopy equivalence $B\,\mathrm{Aut}(i) \to B\,\mathrm{Aut}(BX)$, we get a canonical map

$$\Phi : B\,\mathrm{Aut}(BX) \xrightarrow{\simeq} B\,\mathrm{Aut}(i) \to B\,\mathrm{Aut}(B\mathcal{N}_X).$$

Proof. For any $\varphi \in \mathrm{Aut}(BX)$, there exists, e.g. by [99, Thm. 1.2(3)], a map $\psi \in \mathrm{Aut}(B\mathcal{N}_X)$ such that φi is homotopic to $i\psi$. Since i is assumed to be a fibration, ψ can furthermore be modified such that the equality is strict. This shows that the evaluation map $\mathrm{Aut}(i) \to \mathrm{Aut}(BX)$ is surjective on components. This map of group-like topological monoids is furthermore easily seen to have the homotopy lifting property. To see that it is a homotopy equivalence we hence just have to verify that the fiber $\mathrm{Aut}_{BX}(B\mathcal{N}_X)$ over the identity map is contractible. First observe that, by [56, Prop. 8.11] and the definitions, there is a unique map $BT \to B\mathcal{N}_X$ over BX, up to homotopy. This shows that the homotopy fixed point space $(X/\mathcal{N}_X)^{hT}$ is at least connected. (We refer to [56, §10] for basic facts and definitions about homotopy actions.)

Consider the following diagram in which the rows and columns are fibrations

$$\begin{tikzcd}
\mathcal{W}_X \arrow[r] \arrow[d, equal] & X/T \arrow[r] \arrow[d] & X/\mathcal{N}_X \arrow[d] \\
\mathcal{W}_X \arrow[r] \arrow[d] & BT \arrow[r] \arrow[d] & B\mathcal{N}_X \arrow[d] \\
* \arrow[r] & BX \arrow[r, equal] & BX.
\end{tikzcd}$$

Take homotopy T-fixed points of the top row, which by definition equals taking spaces of liftings of the map from BT to the bottom row to the corresponding term in the middle row. This produces an induced fibration sequence $\mathcal{W}_X \to (X/T)^{hT} \to (X/\mathcal{N}_X)^{hT}$ since $(X/\mathcal{N}_X)^{hT}$ is connected. However the map $\mathcal{W}_X \to (X/T)^{hT}$ is the identity, so $(X/\mathcal{N}_X)^{hT}$ is in fact contractible. By [56, Lem. 10.5 and Rem. 10.9] we can rewrite $(X/\mathcal{N}_X)^{h\mathcal{N}_X} \simeq ((X/\mathcal{N}_X)^{hT})^{hW_X}$, which shows that $(X/\mathcal{N}_X)^{h\mathcal{N}_X}$ is contractible as well. Hence any self-map of $B\mathcal{N}_X$ over BX is an equivalence, and $\mathrm{Aut}_{BX}(B\mathcal{N}_X)$ is contractible as wanted. $\square$

5. Automorphisms of maximal torus normalizers

The aim of this short section is to establish some easy facts about automorphisms of maximal torus normalizers which are needed to carry out the reduction to connected, center-free simple p-compact groups in Section 6. At the same time the section serves to make the automorphism statement of Theorem 1.1 more explicit.

Recall that an *extended p-compact torus* is a loop space $\mathcal{N}$ such that $W = \pi_0(\mathcal{N})$ is a finite group and the identity component $\mathcal{N}_1$ of $\mathcal{N}$ is a p-compact torus T. Let $\breve{\mathcal{N}}$ be the discrete approximation to $\mathcal{N}$ (see [57, 3.12]), and recall that $\breve{\mathcal{N}}$ will have a unique largest p-divisible subgroup $\breve{T}$, which will be a discrete approximation to T.

Proposition 5.1. *For an extended p-compact torus $\mathcal{N}$, the obvious map associating to a self-homotopy equivalence of $B\breve{\mathcal{N}}$ a self-homotopy equivalence of $B\mathcal{N}$ via fiber-wise $\mathbf{F}_p$-completion* [22, Ch. I, §8] *induces an equivalence of aspherical group-like topological monoids*

$$\mathrm{Aut}(B\breve{\mathcal{N}})^\wedge_p \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}).$$

If $\pi_0(\mathcal{N})$ acts faithfully on $\pi_1(\mathcal{N}_1)$ then $\mathrm{Aut}_1(B\breve{\mathcal{N}})$, the component of $\mathrm{Aut}(B\breve{\mathcal{N}})$ of the identity map, has the homotopy type of $B(\breve{T}^W)$ where $\breve{T}$ is a discrete approximation to T.

Sketch of proof. The statement on the level of component groups follows directly from [57, Prop. 3.1]. (The point is that the homotopy fiber of $B\breve{\mathcal{N}} \to B\mathcal{N}$ will have homotopy type $K(V,1)$ for a $\mathbf{Q}_p$-vector space V, and hence the existence and uniqueness obstructions to lifting a map $B\breve{\mathcal{N}} \to B\breve{\mathcal{N}}$ to $B\mathcal{N}$ lie in $H^n(\breve{\mathcal{N}};V)$ where $n = 2,1$ which are easily seen to be zero.) It is likewise easy to see that both spaces are aspherical and that we get a homotopy equivalence of the identity components. The last statement is also obvious. $\square$

Let L be a finitely generated free $\mathbf{Z}_p$-module and suppose that $W \subseteq \operatorname{Aut}(\breve{T})$, where we set $\breve{T} = L \otimes \mathbf{Z}/p^\infty$. Consider the second cohomology group $H^2(W;\breve{T})$ which classifies extensions of W by $\breve{T}$ with the fixed action of W on $\breve{T}$. Given an isomorphism $\alpha : L \to L'$ sending $W \subseteq \mathrm{GL}(L)$ to $W' \subseteq \mathrm{GL}(L')$ we get an isomorphism of cohomology groups $H^2(W;\breve{T}) \to H^2(W';\breve{T}')$ by sending an extension $\breve{T} \xrightarrow{i} \breve{\mathcal{N}} \xrightarrow{\pi} W$ to the extension $\breve{T}' \xrightarrow{i\circ\alpha^{-1}} \breve{\mathcal{N}} \xrightarrow{c_\alpha\circ\pi} W'$, where c_α denotes conjugation by α. An isomorphism between two triples (W, L, γ) and (W', L', γ'), where γ and γ' are extension classes, is an isomorphism $L \to L'$ sending W to W' and γ to γ'. The automorphism group of a triple (W, L, γ) thus identifies with

$$^{\gamma}N_{\operatorname{Aut}(\breve{T})}(W) = \{\alpha \in N_{\operatorname{Aut}(\breve{T})}(W) \,|\, \alpha(\gamma) = \gamma \in H^2(W;\breve{T})\}.$$

It follows directly from the definition (since $\breve{T}$ is characteristic in $\breve{\mathcal{N}}$) that two triples as above are isomorphic if and only if the associated groups $\breve{\mathcal{N}}$ and $\breve{\mathcal{N}}'$ are isomorphic, where $\breve{\mathcal{N}}$ is obtained from the extension $1 \to \breve{T} \to \breve{\mathcal{N}} \to W \to 1$ given by γ, and analogously for γ'. However, $\breve{\mathcal{N}}$ and (W, L, γ) in general have slightly different automorphism groups, as described in the following lemma (see also [139]):

PROPOSITION 5.2. *In the notation above, for any exact sequence* $1 \to \breve{T} \to \breve{\mathcal{N}} \xrightarrow{\pi} W \to 1$ *with extension class* γ *there is a canonical exact sequence*

$$1 \to \operatorname{Der}(W,\breve{T}) \to \operatorname{Aut}(\breve{\mathcal{N}}) \to {}^{\gamma}N_{\operatorname{Aut}(\breve{T})}(W) \to 1 \tag{5.1}$$

where we embed the derivations $\operatorname{Der}(W,\breve{T})$ *in* $\operatorname{Aut}(\breve{\mathcal{N}})$ *by sending a derivation* s *to the automorphism given by* $x \mapsto s(\pi(x))x$*, and the map* $\operatorname{Aut}(\breve{\mathcal{N}}) \to {}^{\gamma}N_{\operatorname{Aut}(\breve{T})}(W)$ *is given by restricting an automorphism* $\varphi \in \operatorname{Aut}(\breve{\mathcal{N}})$ *to* $\breve{T}$.

This exact sequence has an exact subsequence $1 \to \breve{T}/\breve{T}^W \to \breve{\mathcal{N}}/Z\breve{\mathcal{N}} \to W \to 1$ *and the quotient exact sequence is*

$$1 \to H^1(W;\breve{T}) \to \operatorname{Out}(\breve{\mathcal{N}}) \to {}^{\gamma}N_{\operatorname{Aut}(\breve{T})}(W)/W \to 1.$$

In particular if (W, L) *is a finite* $\mathbf{Z}_p$*-reflection group and* p *is odd then* $H^1(W;\breve{T}) = 0$ *by* [6, Thm. 3.3], [82, Pf. of Prop. 3.5]*, so there is an isomorphism* $\operatorname{Out}(\breve{\mathcal{N}}) \xrightarrow{\cong} {}^{\gamma}N_{\operatorname{Aut}(\breve{T})}(W)/W$.

Proof. Let $\varphi \in \mathrm{Aut}(\breve{\mathcal{N}})$, and consider the restriction map $\varphi \mapsto \varphi|_{\breve{T}} \in \mathrm{Aut}(\breve{T})$. Note that for all $x \in \breve{\mathcal{N}}$, $l \in \breve{T}$ we have

$$(\varphi \circ c_x)(l) = \varphi(xlx^{-1}) = \varphi(x)\varphi(l)\varphi(x)^{-1} = (c_{\varphi(x)} \circ \varphi)(l),$$

so $\varphi|_{\breve{T}} \in N_{\mathrm{Aut}(\breve{T})}(W)$. That the image equals the set of elements which fix the extension class follows easily from the definitions: The diagram

$$\begin{tikzcd}
\breve{T} \arrow[r, "i\circ\varphi^{-1}"] \arrow[d, equal] & \breve{\mathcal{N}} \arrow[r, "c_\varphi\circ\pi"] \arrow[d, "\varphi"] & W \arrow[d, equal] \\
\breve{T} \arrow[r, "i"] & \breve{\mathcal{N}} \arrow[r, "\pi"] & W
\end{tikzcd}$$

shows that φ leaves γ invariant. Likewise, to see that the right-hand map in (5.1) is surjective let $\psi \in {}^{\gamma}N_{\mathrm{Aut}(\breve{T})}(W)$ and let $\breve{T} \to \tilde{N} \to W$ be the extension obtained by first pushing forward along $\psi : \breve{T} \to \breve{T}$ and then pulling back along $\psi^{-1}(-)\psi : W \to W$. Since ψ fixes γ there exists an isomorphism $\tilde{N} \to \breve{\mathcal{N}}$ making the following diagram commute:

$$\begin{tikzcd}
\breve{T} \arrow[r, "\psi"] \arrow[d] & \breve{T} \arrow[r, equal] \arrow[d] & \breve{T} \arrow[d] \\
\breve{\mathcal{N}} \arrow[r] \arrow[d] & \tilde{N} \arrow[r] \arrow[d] & \breve{\mathcal{N}} \arrow[d] \\
W \arrow[r, "\psi(-)\psi^{-1}"] & W \arrow[r, equal] & W.
\end{tikzcd}$$

This shows that $\mathrm{Aut}(\breve{\mathcal{N}}) \to {}^{\gamma}N_{\mathrm{Aut}(\breve{T})}(W)$ is surjective.

Now suppose $\varphi \in \mathrm{Aut}(\breve{\mathcal{N}})$ restricts to the identity on $\breve{T}$. For $x \in \breve{\mathcal{N}}$ and $l \in \breve{T}$ we have

$$\varphi(x)l\varphi(x^{-1}) = \varphi(x)\varphi(l)\varphi(x^{-1}) = \varphi(xlx^{-1}) = xlx^{-1},$$

so the induced map $\varphi : W \to W$ is the identity since W acts faithfully on $\breve{T}$. This means that we can define a map $s : W \to \breve{T}$ by $s(w) = \varphi(\tilde{w})\tilde{w}^{-1}$ where $\tilde{w}$ is a lift of w, and this is easily seen to be a derivation. Furthermore taking the automorphism of $\breve{\mathcal{N}}$ associated to s gives back φ, which establishes exactness in the middle, and we have proved the existence of the first exact sequence.

The existence of the short exact subsequence is clear, when we note that $Z\breve{\mathcal{N}} = \breve{T}^W$ (since W acts faithfully on $\breve{T}$) and that $\breve{T}/\breve{T}^W$ embeds in $\mathrm{Der}(W, \breve{T})$ as the principal derivations by sending l to the derivation $w \mapsto l(w \cdot l)^{-1}$. The last exact sequence is now obvious. □

Remark 5.3. See [73, Thm. 1.2] for a related exact sequence for compact connected Lie groups, fitting with the conjectured classification of connected p-compact groups for $p = 2$.

PROPOSITION 5.4. *Suppose* $\{(W_i, L_i, \gamma_i)\}_{i=0}^k$ *is a collection of pairwise nonisomorphic triples where* L_i *is a finitely generated free* $\mathbf{Z}_p$*-module,* W_i *is a finite subgroup of* $\mathrm{GL}(L_i)$ *such that* $L_i \otimes \mathbf{Q}$ *is an irreducible* W_i*-module,* $\gamma_i \in H^2(W_i; \breve{T}_i)$ *and* $(W_0, L_0, \gamma_0) = (1, \mathbf{Z}_p, 0)$*. Let* $(W, L, \gamma) = \prod_{i=0}^k (W_i, L_i, \gamma_i)^{m_i}$ *denote the product. Then*

$$\mathrm{GL}_{m_0}(\mathbf{Z}_p) \times \left(\prod_{i=1}^k \left({}^{\gamma_i}N_{\mathrm{GL}(L_i)}(W_i)/W_i \right) \wr \Sigma_{m_i} \right) \xrightarrow{\cong} {}^{\gamma}N_{\mathrm{GL}(L)}(W)/W.$$

Proof. The map of the proposition is injective by definition, and we have to see that it is surjective. To lessen confusion write

$$L = (L_{0,1} \oplus \cdots \oplus L_{0,m_0}) \oplus (L_{1,1} \oplus \cdots \oplus L_{1,m_1}) \oplus \cdots \oplus (L_{k,1} \oplus \cdots \oplus L_{k,m_k}),$$

which we consider as a $W = 1 \times (W_{1,1} \times \cdots \times W_{1,m_1}) \times \cdots \times (W_{k,1} \times \cdots W_{k,m_k})$-module, where $(W_{i,j}, L_{i,j})$ is isomorphic to (W_i, L_i) as $\mathbf{Z}_p$-reflection groups.

Consider $\varphi \in {}^{\gamma}N_{\mathrm{GL}(L)}(W)$; we need to see that this has the prescribed form. First note that for every $w \in W$ there exists a unique $\tilde{w} \in W$ such that

$$\varphi(wx) = \tilde{w}\varphi(x) \text{ for all } x \in L.$$

Let α denote the corresponding element in $\mathrm{Aut}(W)$ given by $w \mapsto \tilde{w}$. Note that the above splitting of L induces a splitting of ${}^{\alpha}L$, where the superscript means that we consider L as a W-module through α. Let M and N be indecomposable summands of L. By definition of α the canonical map

$$\varphi_{MN} : M \to L \xrightarrow{\varphi} {}^{\alpha}L \to {}^{\alpha}N$$

is W-equivariant. Therefore this map, after tensoring with $\mathbf{Q}$, has to be either an isomorphism or zero. Since all the nontrivial summands of $L \otimes \mathbf{Q}$ and ${}^{\alpha}L \otimes \mathbf{Q}$ occur with multiplicity one there is for each nontrivial M at most one N for which the map can be nonzero, and this N is necessarily nontrivial. Since φ is an isomorphism there is exactly one such N, and the map φ_{MN} has to be an isomorphism. Note furthermore that since φ_{MN} gives an isomorphism between $M = L_{i,j}$ and ${}^{\alpha}N = {}^{\alpha}L_{k,l}$ as $W_{i,j}$-modules, (α, φ) induces an isomorphism between the reflection groups $(W_{i,j}, L_{i,j})$ and $(W \cap \mathrm{GL}(L_{k,l}), L_{k,l})$, which by assumption has to send γ_i to γ_k, so $i = k$. This shows that φ is of the required form. □

6. Reduction to connected, center-free simple p-compact groups

In this section we prove some lemmas, which, together with the splitting theorems of Dwyer-Wilkerson [58] and Notbohm [111], reduce the proof of Theorem 1.4 to the case of connected, center-free simple p-compact groups. This reduction is known and most of it appears in [98] (relying on earlier work of that author). We here provide a self-contained and a bit more direct proof using [57].

LEMMA 6.1 (Product Automorphism Lemma). *Let X and X' be p-compact groups with maximal torus normalizers $\mathcal{N}$ and $\mathcal{N}'$. Then $\mathcal{N} \times \mathcal{N}'$ is a maximal torus normalizer for $X \times X'$ and the following statements hold*:

(1) $\mathrm{Aut}_1(BX) \times \mathrm{Aut}_1(BX') \xrightarrow{\cong} \mathrm{Aut}_1(BX \times BX')$ *and* $\mathrm{Aut}_1(B\mathcal{N}) \times \mathrm{Aut}_1(B\mathcal{N}') \xrightarrow{\cong} \mathrm{Aut}_1(B\mathcal{N} \times B\mathcal{N}')$, *where* Aut_1 *denotes the set of homotopy equivalences homotopic to the identity.*

(2) *If* $\Phi : \mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N})$ *and* $\Phi : \mathrm{Aut}(BX') \to \mathrm{Aut}(B\mathcal{N}')$ *are injective on* π_0, *then so is* $\Phi : \mathrm{Aut}(B(X \times X')) \to \mathrm{Aut}(B(\mathcal{N} \times \mathcal{N}'))$.

(3) *Suppose that p is odd and that X is connected with $X = X_1 \times \cdots \times X_k$ such that each X_i is simple and determined by its maximal torus normalizer. If, for each i,* $\Phi : \mathrm{Aut}(BX_i) \to \mathrm{Aut}(B\mathcal{N}_{X_i})$ *is surjective on* π_0 *then so is* $\Phi : \mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N})$.

Proof. Recall that the map $\Phi : \mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N})$ was described in Section 4. To see (1) first note that

$$\text{(6.1)}\quad \mathrm{map}(BX \times BX', BX \times BX') \simeq \mathrm{map}(BX, \mathrm{map}(BX', BX)) \times \mathrm{map}(BX', \mathrm{map}(BX, BX')).$$

The evaluation map $\mathrm{map}(BX', BX)_0 \to BX$ is an equivalence by the Sullivan conjecture for p-compact groups [57, Thm. 9.3 and Prop. 10.1], where the subscript 0 denotes the component of the constant map. Since the component of the identity map on the left-hand side of (6.1) is sent to the component of the constant map in $\mathrm{map}(BX', BX)$ this shows that

$$\mathrm{map}(BX \times BX', BX \times BX')_1 \simeq \mathrm{map}(BX, BX)_1 \times \mathrm{map}(BX', BX')_1$$

as wanted. (The statement just says that the center of a product of p-compact groups is the product of the centers, which of course also follows from the equivalence of the different definitions of the center from [57].)

To see (2) suppose that φ is a self-equivalence of $BX \times BX'$ such that its restriction to a self-equivalence of $B(\mathcal{N} \times \mathcal{N}')$ becomes homotopic to the identity. The restriction $\varphi|_{BX \times *}$ composed with the projection onto BX' becomes null homotopic upon restriction to $B\mathcal{N}$, which, e.g. by [96, Cor. 6.6], implies that it is null homotopic. Likewise the projection of $\varphi|_{* \times BX'}$ onto BX' becomes homotopic to the identity map upon restriction to $B\mathcal{N}$, which by assumption means that the projection of $\varphi|_{* \times BX'}$ onto BX' is the identity. But by adjointness, repeating the argument of the first claim, this implies that φ composed with the projection onto BX' is homotopic to the projection map onto BX' (this is [57, Lem. 5.3]). By symmetry this holds for the projection onto BX as well, and we conclude that φ is homotopic to the identity as wanted.

Finally, to see (3), note that Propositions 5.1, 5.2, and 5.4 give a complete description of $\pi_0(\mathrm{Aut}(B\mathcal{N})) \cong \mathrm{Out}(\breve{\mathcal{N}})$. The assumption that each X_i is determined by its maximal torus normalizer means by definition that if $\mathcal{N}_{X_i}$ is isomorphic to $\mathcal{N}_{X_j}$ then X_i is isomorphic to X_j. It is now clear from the description of $\mathrm{Out}(\breve{\mathcal{N}})$ and the assumptions on the X_i's, that all elements in $\mathrm{Out}(\breve{\mathcal{N}})$ can be realized by self-equivalences of BX. $\square$

Remark 6.2. Part (3) of the above lemma is in general false for $p = 2$. For instance if $X = \mathrm{SO}(3)\hat{_2}$ then it is easy to calculate directly (or appeal to [83, Cor. 3.5]) that for both $Y = X$ and $Y = X \times X$ we have $\Phi : \pi_0(\mathrm{Aut}(BY)) \xrightarrow{\cong} N_{\mathrm{GL}(L_Y)}(W_Y)/W_Y$. But for $Y = X \times X$ we have $H^1(W_Y; \breve{T}_Y) \cong \mathbf{Z}/2 \times \mathbf{Z}/2$, so $B\mathcal{N}_Y$ has nontrivial automorphisms which restrict to the identity on BT_Y (see Proposition 5.2).

Remark 6.3. Note that the assumption that the factors X_i in (3) are determined by $\mathcal{N}_i$ of course appears for a good reason. The statement that $\Phi : \pi_0(\mathrm{Aut}(BX)) \to \pi_0(\mathrm{Aut}(B\mathcal{N}))$ is surjective for all p-compact groups X implies that all p-compact groups are determined by their maximal torus normalizer, as is seen by taking products. Hence the first part of Theorem 1.4 in fact follows from the second part.

Recall the observation that for p odd the component group of X is determined by W_X:

Lemma 6.4 (Component Group Formula). *Let X be a p-compact group for p odd, with maximal torus normalizer $j : \mathcal{N} \to X$. The map $\pi_0(j) : W_X = \pi_0(\mathcal{N}) \to \pi_0(X)$ is surjective and the kernel equals $O^p(W_X)$, the subgroup generated by elements of order prime to p. The kernel can also be identified with the Weyl group of the identity component X_1 of X, and is the largest $\mathbf{Z}_p$-reflection subgroup of W_X.*

Proof. By [57, Rem. 2.11] $\pi_0(j)$ is surjective with kernel the Weyl group of the identity component of X. Since $\pi_0(X)$ is a p-group, $O^p(\pi_0(\mathcal{N}))$ is contained in the kernel. On the other hand the Weyl group of X_1 is generated by elements of order prime to p, since it is a $\mathbf{Z}_p$-reflection group and p is odd, so equality has to hold. $\square$

Remark 6.5. For $p = 2$ the component group of X cannot be read off from $\mathcal{N}_X$, and one would have to remember $\pi_0(X)$ as part of the data. For instance the 2-compact groups $\mathrm{SO}(3)\hat{_2}$ and $\mathrm{O}(2)\hat{_2}$ have the same maximal torus normalizers, namely $\mathrm{O}(2)\hat{_2}$. Note however that if X is the centralizer of a toral abelian subgroup A of a connected p-compact group Y, then the component group of X *can* be read off from A and $\mathcal{N}_Y$ (see [57, Thm. 7.6]), a case of frequent interest.

Before proceeding recall that by [50] (see also [57, Prop. 11.9]) we have, for a fibration $\mathcal{F} \to \mathcal{E} \xrightarrow{f} \mathcal{B}$, a fibration sequence

$$\operatorname{map}(\mathcal{B}, B\operatorname{Aut}(\mathcal{F}))_{C(f)} \to B\operatorname{Aut}(f) \to B\operatorname{Aut}(\mathcal{B}).$$

Here $C(f)$ denotes the components corresponding to the orbit of the $\pi_0(\operatorname{Aut}(\mathcal{B}))$-action on the class in $[\mathcal{B}, B\operatorname{Aut}(\mathcal{F})]$ classifying the fibration.

We are interested in when the map of group-like topological monoids $\operatorname{Aut}(f) \to \operatorname{Aut}(\mathcal{E})$ is a homotopy equivalence. This will follow if we can see that $\operatorname{Aut}_1(f) \to \operatorname{Aut}_1(\mathcal{E})$ and $\pi_0(\operatorname{Aut}(f)) \to \pi_0(\operatorname{Aut}(\mathcal{E}))$ are equivalences. By an easy general argument given in [57, Prop. 11.10] the statement about the identity components follows if $\mathcal{B} \to \operatorname{map}(\mathcal{F}, \mathcal{B})_0$ is an equivalence, where the subscript 0 denotes the component of the constant map.

Lemma 6.6 (Component Reduction Lemma). *Let X be a p-compact group with maximal torus normalizer $\mathcal{N}$, and assume that p is odd (so that $\pi_0(X)$ can be read off from $\mathcal{N}$). Let $\mathcal{N}_1$ denote the kernel of the map $\mathcal{N} \to \pi_0(X)$, which is a maximal torus normalizer for X_1.*

If $\Phi : \operatorname{Aut}(BX_1) \xrightarrow{\cong} \operatorname{Aut}(B\mathcal{N}_1)$, then $\Phi : \operatorname{Aut}(BX) \xrightarrow{\cong} \operatorname{Aut}(B\mathcal{N})$. If furthermore BX_1 is determined by $B\mathcal{N}_1$ then BX is determined by $B\mathcal{N}$.

Proof. First note that by an inspection of Euler characteristics and using [99, Thm. 1.2(3)], $\mathcal{N}_1$ is indeed a maximal torus normalizer in X_1. Set $\pi = \pi_0(X)$ for short. We want to apply the setup described before the lemma to the fibrations $BX_1 \to BX \to B\pi$ and $B\mathcal{N}_1 \to B\mathcal{N} \to B\pi$ and to see that in both cases the map of monoids $\operatorname{Aut}(f) \to \operatorname{Aut}(\mathcal{E})$ are homotopy equivalences. By the remarks above this follows if it is an isomorphism on π_0 and $\mathcal{B} \to \operatorname{map}(\mathcal{F}, \mathcal{B})_0$ is an equivalence. The statement about π_0 is true in both cases since a self-map of $\mathcal{E}$ determines a unique self-map of $B\pi$. Likewise it is easy to see that $B\pi \xrightarrow{\simeq} \operatorname{map}(BX_1, B\pi)_0$ and that $B\pi \xrightarrow{\simeq} \operatorname{map}(B\mathcal{N}_1, B\pi)_0$. This means that our map $B\operatorname{Aut}(BX) \to B\operatorname{Aut}(B\mathcal{N})$ (from Lemma 4.1) fits in a map of fibration sequences

$$\begin{tikzcd}
\operatorname{map}(B\pi, B\operatorname{Aut}(BX_1))_{C(f)} \arrow[r] \arrow[d] & B\operatorname{Aut}(BX) \arrow[r] \arrow[d] & B\operatorname{Aut}(B\pi) \arrow[d, equal] \\
\operatorname{map}(B\pi, B\operatorname{Aut}(B\mathcal{N}_1))_{C(f)} \arrow[r] & B\operatorname{Aut}(B\mathcal{N}) \arrow[r] & B\operatorname{Aut}(B\pi).
\end{tikzcd}$$

Here the maps between the fibers and base spaces are homotopy equivalences by assumption, so the map between the total spaces is a homotopy equivalence as well.

Now assume furthermore that X_1 is determined by $\mathcal{N}_1$, and let X' be another p-compact group with maximal torus normalizer $\mathcal{N}$. By Lemma 6.4, $\pi = \pi_0(X) \cong \pi_0(X')$ and $\mathcal{N}_1$ is also a maximal torus normalizer in X'_1.

We want to show that the two fibrations $BX \to B\pi$ and $BX' \to B\pi$ are equivalent as fibrations over $B\pi$, or equivalently that the π-spaces BX_1 and BX_1' are $h\pi$-equivalent, i.e., that we can find a zig-zag of π-maps which are nonequivariant equivalences connecting the two (see e.g., [45] where this equivalence relation is called equivariant weak homotopy equivalence).

By the assumptions on X_1 we can choose a homotopy equivalence $Bf : BX_1 \to BX_1'$ such that

$$\begin{tikzcd} & B\mathcal{N}_1 \arrow[dl, "Bj"'] \arrow[dr, "Bj'"] & \\ BX_1 \arrow[rr, "Bf"] & & BX_1' \end{tikzcd}$$

commutes up to homotopy, and Bf is unique up to homotopy.

We now want to see that we can change Bf so that it becomes a π-map. For this, consider the π-map given by restriction

$$\mathrm{map}(BX_1, BX_1') \to \mathrm{map}(B\mathcal{N}_1, BX_1').$$

By the assumption on $\mathrm{Aut}(BX_1)$ this map sends distinct components of $\mathrm{map}(BX_1, BX_1')$ corresponding to homotopy equivalences to distinct components of $\mathrm{map}(B\mathcal{N}_1, BX_1')$. Moreover, by the proof of Lemma 4.1, we have a homotopy equivalence $\mathrm{map}(BX_1, BX_1')_{Bf} \simeq \mathrm{map}(B\mathcal{N}_1, BX_1')_{Bf\circ Bj}$. In particular the component $\mathrm{map}(BX_1, BX_1')_{Bf}$ is preserved under the π-action, since this obviously is so for $\mathrm{map}(B\mathcal{N}_1, BX_1')_{Bj'}$. Furthermore since $\mathrm{map}(B\mathcal{N}_1, BX_1')^{\pi}_{Bj'}$ contains Bj' we see that

$$\mathrm{map}(BX_1, BX_1')^{h\pi}_{Bf} \simeq \mathrm{map}(B\mathcal{N}_1, BX_1')^{h\pi}_{Bj'}$$

is nonempty, and so there exists a π-map $E\pi \times BX_1 \to BX_1'$ which is a homotopy equivalence. This shows that BX_1 and BX_1' are $h\pi$-homotopy equivalent as wanted. □

Remark 6.7. If X is a connected p-compact group, and p is odd, then it follows from [57, Thm. 7.5] that $Z(\breve{\mathcal{N}})$ is a discrete approximation to the center of X. The proof of the above lemma extends this to X nonconnected *provided* we know that self-equivalences of X_1 are detected by their restriction to $\mathcal{N}_1$, which will be a consequence of Theorem 1.4. Having to appeal to this is a bit unfortunate but seems unavoidable. The point is that if there existed a connected p-compact group X and a self-equivalence σ of finite p-power order which is not detected by $\mathcal{N}$, then we could form $X \rtimes \langle\sigma\rangle$, where σ would be central in the normalizer but not in the whole group. (See also Lemma 9.2.)

Lemma 6.8 (Center Reduction Lemma). *Let X be a connected p-compact group with center $\mathcal{Z}$. Then*:

(1) *If* $\Phi : \pi_0(\mathrm{Aut}(BX/\mathcal{Z})) \to \pi_0(\mathrm{Aut}(B\mathcal{N}/\mathcal{Z}))$ *is surjective and* $X/\mathcal{Z}$ *is determined by* $\mathcal{N}/\mathcal{Z}$ *then* X *is determined by* $\mathcal{N}$.

(2) *If* p *is odd and* $\Phi : \mathrm{Aut}(BX/\mathcal{Z}) \to \mathrm{Aut}(B\mathcal{N}/\mathcal{Z})$ *is a homotopy equivalence then* $\Phi : \mathrm{Aut}(BX) \to \mathrm{Aut}(B\mathcal{N})$ *is as well.*

Proof. To prove the first statement, suppose that X and X' have the same maximal torus normalizer $\mathcal{N}$, choose fixed inclusions $j : \mathcal{N} \to X$ and $j' : \mathcal{N} \to X'$, and let $\mathcal{Z}$ be the center of X, which we can view as a subgroup $\mathcal{N}$ via an inclusion $i : \mathcal{Z} \to \mathcal{N}$. We claim that $\mathcal{Z}$ is also central in X'. It is central in the identity component X'_1 by the formula for the center in [57, Thm. 7.5]. Furthermore, $\pi = \pi_0(X') \cong W_X/W_{X'_1}$ acts trivially on $\mathcal{Z}$, so the lift of $j'i$ to a map $k : \mathcal{Z} \to X'_1$ is unique up to conjugacy. Therefore we have fibration sequences

$$\begin{tikzcd}
\mathrm{map}(B\mathcal{Z}, BX'_1)_k \arrow[r] \arrow[d, "\simeq"] & \mathrm{map}(B\mathcal{Z}, BX')_{j'i} \arrow[r] \arrow[d] & \mathrm{map}(B\mathcal{Z}, B\pi)_0 \arrow[d, "\simeq"] \\
BX'_1 \arrow[r] & BX' \arrow[r] & B\pi
\end{tikzcd}$$

where the left vertical map is an equivalence since $\mathcal{Z}$ is central in X'_1. Hence the middle map in the above diagram is an equivalence as well, and $\mathcal{Z}$ is central in X' as claimed. Now assume that $X/\mathcal{Z}$ is isomorphic to $X'/\mathcal{Z}$. If $\Phi : \pi_0(\mathrm{Aut}(BX/\mathcal{Z})) \to \pi_0(\mathrm{Aut}(B\mathcal{N}/\mathcal{Z}))$ is surjective we can furthermore choose the homotopy equivalence $BX/\mathcal{Z} \to BX'/\mathcal{Z}$ in such a way that

$$\begin{tikzcd}
& B\mathcal{N}/\mathcal{Z} \arrow[dl, "j/\mathcal{Z}"'] \arrow[dr, "j'/\mathcal{Z}"] & \\
BX/\mathcal{Z} \arrow[rr] & & BX'/\mathcal{Z}
\end{tikzcd}$$

commutes up to homotopy.

We have canonical maps $BX/\mathcal{Z} \to B^2\mathcal{Z}$ and $BX'/\mathcal{Z} \to B^2\mathcal{Z}$ classifying the extensions, and we claim that in fact the bottom triangle in the diagram

$$\begin{tikzcd}
& B\mathcal{N}/\mathcal{Z} \arrow[dl] \arrow[dr] & \\
BX/\mathcal{Z} \arrow[rr] \arrow[dr] & & BX'/\mathcal{Z} \arrow[dl] \\
& B^2\mathcal{Z} &
\end{tikzcd}$$

commutes up to homotopy. By construction the outer square commutes up to homotopy (since both composites agree with the classifying map $B\mathcal{N}/\mathcal{Z} \to B^2\mathcal{Z}$ since j and j' are fixed). Since the top triangle also commutes up to homotopy, an application of the transfer [56, Thm. 9.13], using that $B^2\mathcal{Z}$ is a product of Eilenberg-Mac Lane spaces and that $\chi((X/\mathcal{Z})/(\mathcal{N}/\mathcal{Z})) = 1$, shows that the bottom triangle commutes up to homotopy as well. Since we have constructed a map $BX/\mathcal{Z} \to BX'/\mathcal{Z}$ over $B^2\mathcal{Z}$ we get an induced homotopy equivalence $BX \to BX'$. (Note that this construction does not *a priori* give this map as a map under $B\mathcal{N}$.)

We now want to get the second statement about automorphism groups. Consider the homotopy commutative diagram

$$\begin{tikzcd} B\mathcal{N} \arrow[r, "f'"] \arrow[d] & B\mathcal{N}/\mathcal{Z} \arrow[d] \\ BX \arrow[r, "f"] & BX/\mathcal{Z} \end{tikzcd}$$

where we can suppose that the two horizontal maps f' and f are fibrations.

We first claim that we can replace $B\operatorname{Aut}(f)$ with $B\operatorname{Aut}(BX)$ and $B\operatorname{Aut}(f')$ with $B\operatorname{Aut}(B\mathcal{N})$. As in the case of the component group (see the proof of Lemma 6.6) we just have to justify that in the appropriate fibration sequences we have equivalences $\mathcal{B} \to \operatorname{map}(\mathcal{F}, \mathcal{B})_0$ and $\pi_0(\operatorname{Aut}(f)) \to \pi_0(\operatorname{Aut}(\mathcal{E}))$. The map $BX/\mathcal{Z} \to \operatorname{map}(B\mathcal{Z}, BX/\mathcal{Z})_0$ is a homotopy equivalence since the trivial map is central [57, Prop. 10.1]. That $B\mathcal{N}/\mathcal{Z} \to \operatorname{map}(B\mathcal{Z}, B\mathcal{N}/\mathcal{Z})_0$ is an equivalence follows by a similar (but easier) argument.

By Lemma 4.1 a self-equivalence of BX induces a unique self-equivalence of $B\mathcal{N}$, and hence a canonical self-equivalence of $B\mathcal{Z}$. Now, by the description of $X/\mathcal{Z}$ as a Borel construction (given in [56, Pf. of Prop. 8.3]) we get a canonical self-equivalence of $BX/\mathcal{Z}$. This self-equivalence is furthermore unique, in the sense that given a diagram

$$\begin{tikzcd} BX \arrow[r, "g"] \arrow[d] & BX \arrow[d] \\ BX/\mathcal{Z} \arrow[r, "g'"] & BX/\mathcal{Z} \end{tikzcd}$$

the homotopy type of g' is uniquely given by that of g. To see this note that by Lemma 4.1 the diagram restricts to a unique diagram

$$\begin{tikzcd} B\mathcal{N} \arrow[r, "\tilde{g}"] \arrow[d] & B\mathcal{N} \arrow[d] \\ B\mathcal{N}/\mathcal{Z} \arrow[r, "\tilde{g}'"] & B\mathcal{N}/\mathcal{Z}. \end{tikzcd}$$

By looking at discrete approximations we see that the homotopy class of $\tilde{g}'$ is determined by $\tilde{g}$. Since by assumption the homotopy class of g' is determined by $\tilde{g}'$, we conclude that a self-equivalence of BX induces a unique self-equivalence of $BX/\mathcal{Z}$, and so $\pi_0(\mathrm{Aut}(f)) \cong \pi_0(\mathrm{Aut}(BX))$. The last part of the argument furthermore shows that also $\pi_0(\mathrm{Aut}(f')) \cong \pi_0(\mathrm{Aut}(B\mathcal{N}))$.

We hence have the following diagram where the rows are fibration sequences

$$\begin{tikzcd}
\mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f)} \arrow[r] \arrow[d] & B\,\mathrm{Aut}(BX) \arrow[r] \arrow[d] & B\,\mathrm{Aut}(BX/\mathcal{Z}) \arrow[d] \\
\mathrm{map}(B\mathcal{N}/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f')} \arrow[r] & B\,\mathrm{Aut}(B\mathcal{N}) \arrow[r] & B\,\mathrm{Aut}(B\mathcal{N}/\mathcal{Z}).
\end{tikzcd}$$

Examining when the middle vertical arrow is a homotopy equivalence reduces to finding out when the restriction map $\mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f)} \to \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f')}$ is a homotopy equivalence, which we now analyze.

Note that since $B\mathcal{Z}$ is a product of Eilenberg-Mac Lane spaces we have a fibration sequence

$$B^2\mathcal{Z} \to B\,\mathrm{Aut}(B\mathcal{Z}) \to B\,\mathrm{Aut}(\breve{\mathcal{Z}})$$

where $\breve{\mathcal{Z}}$ is the discrete approximation to $\mathcal{Z}$ and $\mathrm{Aut}(\breve{\mathcal{Z}})$ is the discrete group of automorphisms. Since our extensions are central this gives a diagram of fibration sequences

$$\begin{tikzcd}
\mathrm{map}(BX/\mathcal{Z}, B^2\mathcal{Z})_{C(f)} \arrow[r] \arrow[d] & \mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f)} \arrow[r] \arrow[d] & \mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(\breve{\mathcal{Z}}))_0 \arrow[d] \\
\mathrm{map}(B\mathcal{N}/\mathcal{Z}, B^2\mathcal{Z})_{C(f')} \arrow[r] & \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_{C(f')} \arrow[r] & \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B\,\mathrm{Aut}(\breve{\mathcal{Z}}))_0.
\end{tikzcd}$$

Again, in this diagram the map between the base spaces is obviously an equivalence, so we are reduced to studying

$$\mathrm{map}(BX/\mathcal{Z}, B^2\mathcal{Z})_{C(f)} \to \mathrm{map}(B\mathcal{N}/\mathcal{Z}, B^2\mathcal{Z})_{C(f')}. \tag{6.2}$$

Since $B^2\mathcal{Z}$ is a product of Eilenberg-Mac Lane spaces a transfer argument (cf. [56, Thm. 9.13]) shows that this gives an embedding as a retract. Since we assume $\Phi : \pi_0(\mathrm{Aut}(BX/\mathcal{Z})) \cong \pi_0(B\,\mathrm{Aut}(B\mathcal{N}/\mathcal{Z}))$ we furthermore get that this is an isomorphism on π_0 by the definition of $C(f)$ and $C(f')$. Let (W, L') denote the Weyl group of $X/\mathcal{Z}$. Write $B\mathcal{Z} \simeq B^2A \times BA'$, where A is a finite sum of copies of $\mathbf{Z}_p$ and A' is finite (cf. [57, Thm. 1.1]). On π_1 the map (6.2) identifies with

$$H^1(BX/\mathcal{Z}; A') \oplus H^2(BX/\mathcal{Z}; A) \to H^1(B\mathcal{N}/\mathcal{Z}; A') \oplus H^2(B\mathcal{N}/\mathcal{Z}; A).$$

The group $H^1(B\mathcal{N}/\mathcal{Z}; A')$ is zero since $\pi_1(B\mathcal{N}/\mathcal{Z}) = W$ is generated by elements of order prime to p, since p is assumed to be odd.

Furthermore, $H^2(B\mathcal{N}/\mathcal{Z};A)$ is related via the Serre spectral sequence to the groups

$$H^2(BW;H^0(B^2L';A)),\ H^1(BW;H^1(B^2L';A)),\ \text{and}\ H^0(BW;H^2(B^2L';A)).$$

The first of these groups is zero since W is generated by elements of order prime to p by the assumption that p is odd. The second is obviously zero, and the last group is zero since $H^0(W;\mathrm{Hom}(L',\mathbf{Z}_p)) = \mathrm{Hom}((L')_W,\mathbf{Z}_p) = 0$ because $(L')_W$ is finite.

Hence we get an isomorphism on π_1, since we already know that the map is injective. On π_2 and π_3 the map identifies with

$$H^0(BX/\mathcal{Z};A')\oplus H^1(BX/\mathcal{Z};A)\to H^0(B\mathcal{N}/\mathcal{Z};A')\oplus H^1(B\mathcal{N}/\mathcal{Z};A)$$

and $H^0(BX/\mathcal{Z};A)\to H^0(B\mathcal{N}/\mathcal{Z};A)$ respectively, and these maps are obviously isomorphisms. Hence $\mathrm{map}(BX/\mathcal{Z},B^2\mathcal{Z})_{C(f)}\to \mathrm{map}(B\mathcal{N}/\mathcal{Z},B^2\mathcal{Z})_{C(f')}$ is a homotopy equivalence, which via the fibration sequences above implies that $B\operatorname{Aut}(BX)\to B\operatorname{Aut}(B\mathcal{N})$ is a homotopy equivalence as wanted. $\square$

Remark 6.9. Consider $BX = B(\mathrm{SU}(3)\times S^1)\hat{_2}$. This has center $\mathcal{Z} = (S^1)\hat{_2}$ and $X/\mathcal{Z} = \mathrm{SU}(3)\hat{_2}$. By direct calculation (or appeal to [83, Cor. 3.5]) we have $B\operatorname{Aut}(BX/\mathcal{Z}) \xrightarrow{\simeq} B\operatorname{Aut}(B\mathcal{N}/\mathcal{Z})$. However $\Phi : \pi_0(\operatorname{Aut}(BX)) \to \pi_0(\operatorname{Aut}(B\mathcal{N}))$ is not surjective by Proposition 5.2, since $\mathrm{Hom}(W_{\mathrm{SU}(3)},\mathbf{Z}/2^\infty) = \mathbf{Z}/2$. This shows that the assumption that p is odd is necessary in the last part of the above lemma.

Remark 6.10. Suppose that X is a connected p-compact group. Fibration sequences with base space $B^2\pi_1(X)$ and fiber $B(X\langle 1\rangle)$ are in one-to-one correspondence with the set of maps $[B^2\pi_1(X), B\operatorname{Aut}(B(X\langle 1\rangle))]$. Likewise self-equivalences of BX can be expressed in terms of self-equivalences of $B(X\langle 1\rangle)$ and $\pi_1(X)$, analogously to the lemmas above. Hence if we *a priori* knew that Theorem 1.7 held true, i.e., if we could read off $\pi_1(X)$ from $\mathcal{N}_X$ then the above methods would reduce the proof of the main theorems to the simply connected case, which could be used advantageously in the proofs. (See also Remark 10.3.)

Remark 6.11. The assumption in Lemma 6.8(1) that $\Phi : \pi_0(\operatorname{Aut}(BX/\mathcal{Z})) \to \pi_0(\operatorname{Aut}(B\mathcal{N}/\mathcal{Z}))$ is surjective has the following origin. We have a canonical restriction map $H^2(BX/\mathcal{Z};\breve{\mathcal{Z}})\to H^2(B\mathcal{N}/\mathcal{Z};\breve{\mathcal{Z}})$, which is injective by a transfer argument. Two extension classes in $H^2(BX/\mathcal{Z};\breve{\mathcal{Z}})$ give rise to isomorphic total spaces if the extension classes are conjugate via the actions of $\operatorname{Aut}(BX/\mathcal{Z})$ and $\operatorname{Aut}(\breve{\mathcal{Z}})$ on $H^2(BX/\mathcal{Z};\breve{\mathcal{Z}})$. The total spaces have isomorphic maximal torus normalizers if the extension classes have images in $H^2(B\mathcal{N}/Z;\breve{\mathcal{Z}})$ which are conjugate under the actions of $\operatorname{Aut}(B\mathcal{N}/\mathcal{Z})$ and $\operatorname{Aut}(\breve{\mathcal{Z}})$, which could *a priori* be a weaker notion.

7. An integral version of a theorem of Nakajima and realization of p-compact groups

The goal of this section is to prove an integral version of an algebraic result of Nakajima (Theorem 7.1) and use this to prove Theorem 7.3 which, as part of our inductive proof of Theorem 1.1, will allow us to construct the center-free p-compact groups corresponding to $\mathbf{Z}_p$-reflection groups (W, L) such that $\mathbf{Z}_p[L]^W$ is a polynomial algebra. This will provide the existence part of Theorem 1.1. We feel that this way of showing existence, is perhaps more straightforward than previous approaches; compare for instance [110]. (We refer to the introduction for the history behind this result.)

THEOREM 7.1. *Let p be an odd prime and let (W, L) be a finite $\mathbf{Z}_p$-reflection group. For a subspace V of $L \otimes \mathbf{F}_p$ let W_V denote the pointwise stabilizer of V in W. Then the following conditions are equivalent*:

(1) $\mathbf{Z}_p[L]^W$ *is a polynomial algebra.*

(2) $\mathbf{Z}_p[L]^{W_V}$ *is a polynomial algebra for all nontrivial subspaces $V \subseteq L \otimes \mathbf{F}_p$.*

(3) (W_V, L) *is a $\mathbf{Z}_p$-reflection group for all nontrivial subspaces $V \subseteq L \otimes \mathbf{F}_p$.*

Remark 7.2. An analog of the implication (1) $\Rightarrow$ (2) where the ring $\mathbf{Z}_p$ is replaced by a field was proved by Nakajima [102, Lem. 1.4] (in the case of finite fields see also [61, Thm. 1.4] and [104, Cor. 10.6.1]). For fields of positive characteristic the implication (3) $\Rightarrow$ (1) does not hold; see [86, Ex. 2.2] for more information about this case. Our proof unfortunately involves the classification of finite $\mathbf{Z}_p$-reflection groups and some case-by-case checking. (See the discussion following the proof of Theorem 1.8 for related information.)

Proof of Theorem 7.1. To start, note that the implication (2) $\Rightarrow$ (3) follows from the fact that if $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra then $\mathbf{Q}_p[L \otimes \mathbf{Q}]^{W_V}$ is as well, so (W_V, L) is a $\mathbf{Z}_p$-reflection group by the Shephard-Todd-Chevalley theorem ([10, Thm. 7.2.1] or [127, Thm. 7.4.1]).

To go further we want to see that the theorem is well behaved under products, i.e., that if $(W, L) = (W', L') \times (W'', L'')$, then the theorem holds for (W, L) if it holds for (W', L') and (W'', L''). This follows from the fact that the stabilizer in $W' \times W''$ of an arbitrary subgroup of $(L' \otimes \mathbf{F}_p) \oplus (L'' \otimes \mathbf{F}_p)$ equals the stabilizer of the smallest product subgroup containing it, combined with the fact that the tensor product of two algebras is a polynomial algebra if and only if each of the factors is. Hence to prove the remaining implications it follows from Theorem 11.1 that it suffices to consider separately the cases where (W, L) comes from a compact connected Lie group and the cases where (W, L) is one of the exotic $\mathbf{Z}_p$-reflection groups.

Assume first that $(W, L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for a compact connected Lie group G. If $\mathbf{Z}_p[L]^W$ is a polynomial algebra then by Theorem 12.2 (which involves case-by-case considerations and p odd) $BX = BG\hat{_p}$ satisfies $H^*(BX;\mathbf{Z}_p) \cong H^*(B^2L;\mathbf{Z}_p)^W$. We can identify $V \subseteq L \otimes \mathbf{F}_p$ with a toral elementary abelian p-subgroup in X and by [61, Thm. 1.3] $H^*(B\mathcal{C}_X(V);\mathbf{Z}_p)$ is again a polynomial algebra concentrated in even degrees. In particular $\mathcal{C}_X(V)$ is connected and by [57, Thm. 7.6(1)] $W_{\mathcal{C}_X(V)} = W_V$. Hence, by Theorem 12.1, $H^*(B\mathcal{C}_X(V);\mathbf{Z}_p) \cong H^*(B^2L;\mathbf{Z}_p)^{W_V}$, so $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra. This shows that (1) $\Rightarrow$ (2) when (W,L) comes from a compact connected Lie group. To prove (3) $\Rightarrow$ (1) in the Lie group case suppose that (W,L) is a finite $\mathbf{Z}_p$-reflection group corresponding to a p-compact group $X = G\hat{_p}$ such that (W_V,L) is a $\mathbf{Z}_p$-reflection group for all nontrivial $V \subseteq L\otimes\mathbf{F}_p$. Since p is odd it follows by [57, Thm. 7.6] that $\mathcal{C}_X(V)$ is connected for all nontrivial $V \subseteq L\otimes\mathbf{F}_p$. Hence, since X is assumed to come from a compact connected Lie group [11, Thm. B] (or [132, Thm. 2.28]) implies that $H^*(BX;\mathbf{Z}_p)$ does not have torsion and hence $H^*(BX;\mathbf{Z}_p) \cong H^*(B^2L;\mathbf{Z}_p)^W$ (cf. Theorem 12.1). So $\mathbf{Z}_p[L]^W$ is a polynomial algebra as wanted.

Next we assume that (W,L) is one of the exotic $\mathbf{Z}_p$-reflection groups. By Theorem 12.2, $\mathbf{Z}_p[L]^W$ is a polynomial algebra, so we only need to prove that $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra for any nontrivial $V \subseteq L \otimes \mathbf{F}_p$. Furthermore, by Theorem 12.2(2), $\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra. Nakajima's result [102, Lem. 1.4] shows that $\mathbf{F}_p[L\otimes\mathbf{F}_p]^{W_V}$ is a polynomial algebra as well. Thus we are done if $p \nmid |W_V|$ by Lemma 12.6, which in particular covers the cases where $p \nmid |W|$.

If (W,L) belongs to family number 2 on the Clark-Ewing list, then since p is odd, it is easily seen from the form of the representing matrices (see Section 11 for a concrete description) that reduction mod p gives a bijection between reflections in (W,L) and $(W,L\otimes\mathbf{F}_p)$. As $\mathbf{F}_p[L\otimes\mathbf{F}_p]^{W_V}$ is a polynomial algebra it follows by the Shephard-Todd-Chevalley theorem [10, Thm. 7.2.1] that $W_V \subseteq \mathrm{GL}(L\otimes\mathbf{F}_p)$ is a reflection group. Thus (W_V,L) is a $\mathbf{Z}_p$-reflection group. Since the representing matrices are monomial, it follows by [102, Thm. 2.4] that $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra.

By Theorem 11.1 only four cases remain, namely the Zabrodsky-Aguadé cases $(W_{12}, p=3)$, $(W_{29}, p=5)$, $(W_{31}, p=5)$ and $(W_{34}, p=7)$. For each of these a direct computation, for instance easily done with the aid of a computer, shows that if S is a Sylow p-subgroup of W, then $U = (L\otimes\mathbf{F}_p)^S$ is 1-dimensional and (W_U,L) is isomorphic to $(\Sigma_p, L_{\mathrm{SU}(p)}\otimes\mathbf{Z}_p)$. (A more *ad hoc* construction of this reflection subgroup can also be found in Aguadé [4].) Hence we see that if $V\subseteq L\otimes\mathbf{F}_p$ is nontrivial then either $p\nmid|W_V|$ or V is W-conjugate to U. But in these cases we already know that $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra. □

THEOREM 7.3 (Inductive Polynomial Realization Theorem). *Let p be an odd prime and let (W,L) be a finite $\mathbf{Z}_p$-reflection group with the property that $\mathbf{Z}_p[L]^W$ is a polynomial algebra over $\mathbf{Z}_p$. Note that for any nontrivial elemen-*

tary abelian p-subgroup V of $\breve{T} = L \otimes \mathbf{Z}/p^\infty$, the subgroup W_V of W fixing V pointwise is again a finite $\mathbf{Z}_p$-reflection group by Theorem 7.1.

Assume that for all such V there exists a p-compact group $F(V)$ with discrete approximation to its maximal torus normalizer given by $\breve{T} \rtimes W_V$ such that $F(V)$ is determined by $\mathcal{N}_{F(V)}$, $\Phi : \mathrm{Aut}(BF(V)) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N}_{F(V)})$, and $H^*(BF(V); \mathbf{Z}_p) \xrightarrow{\cong} H^*(B^2L; \mathbf{Z}_p)^{W_V}$. Then there exists a connected p-compact group X with discrete approximation to its maximal torus normalizer given by $\breve{T} \rtimes W$ satisfying the same properties as listed for $F(V)$.

Proof. First note that by Theorem 7.1 (W_V, L) is again a $\mathbf{Z}_p$-reflection group and $\mathbf{Z}_p[L]^{W_V}$ is a polynomial algebra, so the assumptions make sense. Set $\breve{\mathcal{N}} = \breve{T} \rtimes W$. We want to construct a candidate 'centralizer decomposition' diagram. Let $\mathbf{A}$ be the category with objects the nontrivial elementary abelian p-subgroups V of $\breve{T}$ and morphisms the homomorphisms between them induced by inclusions of subgroups and conjugation by elements in W. We now define a functor F from $\mathbf{A}^{\mathrm{op}}$ to p-compact groups and conjugation classes of morphisms. On objects we send V to $F(V)$. By assumption $j_V : C_{\breve{\mathcal{N}}}(V) \to F(V)$ is a discrete approximation to the maximal torus normalizer in $F(V)$. Now let $\varphi : V \to V'$ be a morphism in $\mathbf{A}$, induced by conjugation by an element $x \in W$ and consider the diagram

$$\begin{tikzcd} V' \arrow[r] & C_{\breve{\mathcal{N}}}(V') \arrow[r, "c_{x^{-1}}"] \arrow[d, "j_{V'}"] & C_{\breve{\mathcal{N}}}(V) \arrow[d, "j_V"] \\ & F(V') & F(V). \end{tikzcd}$$

Taking the centralizer of the composite map $x^{-1} : V' \to F(V)$ we get a space $\mathcal{C}_{F(V)}(x^{-1}) = \Omega\,\mathrm{map}(BV', BF(V))_{Bx^{-1}}$, which has discrete approximation to its maximal torus normalizer equal to $C_{\breve{\mathcal{N}}}(V')$. By assumption we get a unique (up to conjugacy) isomorphism $F(V') \to \mathcal{C}_{F(V)}(x^{-1})$ under $C_{\breve{\mathcal{N}}}(V')$. By composing with the evaluation $\mathcal{C}_{F(V)}(x^{-1}) \to F(V)$, we get a morphism $F(\varphi) : F(V') \to F(V)$. We need to check that this gives us a well-defined functor from $\mathbf{A}^{\mathrm{op}}$ to the homotopy category of spaces, i.e., that for $V \xrightarrow{\varphi} V' \xrightarrow{\psi} V''$, $F(\psi\varphi)$ is conjugate to $F(\varphi)F(\psi)$. To see this suppose that ψ is induced by conjugation by $y \in W$ and consider the following diagram with obvious maps:

$$\begin{tikzcd} F(V) & \mathcal{C}_{F(V)}(x^{-1}) \arrow[l, "\mathrm{ev}"'] & F(V') \arrow[l, "\cong"'] & \mathcal{C}_{F(V')}(y^{-1}) \arrow[l, "\mathrm{ev}"'] \arrow[dl, "\cong"] & F(V'') \arrow[l, "\cong"'] \arrow[ddll, "\cong"] \\ & & \mathcal{C}_{\mathcal{C}_{F(V)}(x^{-1})}(\widetilde{x^{-1}}y^{-1}) \arrow[ul, "\mathrm{ev}"] \arrow[d, "\cong"] & & \\ & & \mathcal{C}_{F(V)}(x^{-1}y^{-1}). \arrow[uull, "\mathrm{ev}"] & & \end{tikzcd}$$

(Here $\widetilde{(\cdot)}$ denotes the adjoint map which is explained in Construction 2.1.) Note that the bottom composite from $F(V'')$ to $F(V)$ is $F(\psi\varphi)$ and the top composite is $F(\varphi)F(\psi)$. The top triangle is commutative, since the lower isomorphism in that triangle is just the map obtained by taking centralizers of the upper one. The rightmost square is homotopy commutative, since the corresponding square of isomorphisms between centralizers in $\breve{\mathcal{N}}$ is commutative, by our assumption that maps are detected here. Finally, the leftmost square is homotopy commutative, by definition of the adjoint construction.

We hence get a well-defined functor $BF : \mathbf{A}^{\mathrm{op}} \to Ho(\mathrm{Spaces})$, where $Ho(\mathrm{Spaces})$ denotes the homotopy category of spaces, on objects given by $V \mapsto BF(V)$. By construction the functor obtained when taking cohomology of this diagram, can be identified with the canonical functor which on objects is given by $V \mapsto H^*(B\breve{T}; \mathbf{Z}_p)^{W_V}$.

We want to lift this to a diagram in the category of spaces. The obstruction theory for doing this is described in [49, Thm. 1.1], when we note that by [57, Lem. 11.15] our diagram is a so-called centric diagram so the assumptions of that theorem are satisfied.

By looking at their cohomology we see that all the spaces $F(V)$ are connected and hence by [57, Thm. 7.5] have center given by $\breve{T}^{W_V}$, since p is odd. In particular (see e.g. Lemma 9.2) the homotopy groups of $\mathcal{Z}F(V)$ are given by $\pi_0(\mathcal{Z}F(V)) = H^1(W_V; L)$ and $\pi_1(\mathcal{Z}F(V)) = L^{W_V}$. By [55, §8] (for details see Section 9) $\lim^*_{V\in\mathbf{A}} \pi_*(F(-)) = 0$, so by [49, Thm. 1.1] there exists a (unique) lift of our functor BF to a functor $\widetilde{BF}$ landing in Spaces. Set $BX = (\mathrm{hocolim}_{\mathbf{A}} \widetilde{BF})\hat{_p}$.

The spectral sequence for calculating the cohomology of a homotopy colimit [22, XII.4.5] has E_2-term given by $E_2^{i,j} = \lim^i_{V\in\mathbf{A}} H^j(B\breve{T}; \mathbf{Z}_p)^{W_V}$. But again by [55, §8] these groups vanish for $i > 0$ and for $i = 0$ give $\lim^0_{\mathbf{A}} H^*(B\breve{T}; \mathbf{Z}_p)^{W_V} \cong H^*(B\breve{T}; \mathbf{Z}_p)^W$. Hence the spectral sequence collapses onto the vertical axis, and we get $H^*(BX; \mathbf{Z}_p) \cong H^*(B\breve{T}; \mathbf{Z}_p)^W$.

Since $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra, $H^*(X; \mathbf{Z}_p)$ will be an exterior algebra on odd degree generators (cf. Theorem 12.1), so X is indeed a connected p-compact group. The fact that X is determined by $\mathcal{N}$ and satisfies $\Phi : \mathrm{Aut}(BX) \xrightarrow{\cong} \mathrm{Aut}(B\mathcal{N})$, also follows easily from the above—the details are given in the proof of Theorem 2.2. □

Remark 7.4. Note that Theorem 7.3 in itself does not quite give a standalone proof of the realization and uniqueness of all center-free p-compact groups with Weyl group satisfying that $\mathbf{Z}_p[L]^W$ is a polynomial algebra, since (W_V, L) is not center-free which prevents the obvious induction from working. Compare to the proof of Theorem 1.1; the main problem is that unitary groups will occur in most decompositions, but their adjoint forms do not have cohomology rings which are polynomial algebras.

8. Nontoral elementary abelian p-subgroups of simple center-free Lie groups

In this section we determine, for an odd prime p, all conjugacy classes of nontoral elementary abelian p-subgroups E, of any simple center-free compact Lie group G, as well as their centralizers $C_G(E)$ and *Weyl groups* $W(E) = N_G(E)/C_G(E)$. (Recall that a subgroup of G is called *toral* if it is contained in a torus in G and *nontoral* otherwise.)

Our strategy is as follows. Since p is odd, the groups G we need to consider are the projective unitary groups $\mathrm{PU}(n)$ and the exceptional groups. The groups $\mathrm{PU}(n)$ are easy to deal with and we only expand slightly on the work of Griess [70]. For the exceptional groups the maximal nontoral elementary abelian p-subgroups are also determined by Griess [70]. We first find these subgroups explicitly and then get lower bounds for their Weyl groups by producing explicit elements in their normalizers. From this we are able to identify the nonmaximal nontoral elementary abelian p-subgroups and get lower bounds for their Weyl groups. Finally we get exact results on the Weyl groups by computing centralizers.

In accordance with the standard literature we will in this section state and prove all theorems in the context of linear algebraic groups over the complex numbers $\mathbf{C}$—we state in Proposition 8.4 why this is equivalent to considering compact Lie groups. (The results for $G(\mathbf{C})$ can furthermore be translated into results for $G(F)$ for any algebraically closed field F of characteristic prime to p; see [71, Thm. 1.22] and [67].)

This section is divided into five subsections. The first recalls some results from the theory of linear algebraic groups and discusses the relationship with compact Lie groups. In the second subsection we determine the elementary abelian p-subgroups of the projective unitary groups and the final subsections deal with the elementary abelian 3-subgroups of the groups of type E_6, E_7 and E_8 respectively. (The remaining nontrivial cases $E_8(\mathbf{C}), p = 5$ and $F_4(\mathbf{C}), p = 3$ are treated completely in [70, Lem. 10.3 and Thm. 7.4].)

For some of our computations for the groups $3E_6(\mathbf{C})$ and $E_8(\mathbf{C})$ we have used the computer algebra system MAGMA [16], although this reliance on computers could if needed be replaced by some rather tedious hand calculations.

Notation 8.1. We now collect some notation which will be used multiple times throughout the computations in this section. We use standard names for the linear algebraic groups considered, e.g. $3E_6(\mathbf{C})$ denotes the simply connected group of type E_6 over $\mathbf{C}$ and $E_6(\mathbf{C})$ denotes its adjoint version. We let $\mathbf{T}_n$ denote an n-dimensional torus, i.e., $\mathbf{T}_n = (\mathbf{C}^\times)^n$.

To describe centralizers we follow standard notation for extensions of groups, cf. the $\mathbb{ATLAS}$ [38, p. xx]. Thus $A : B$ denotes a semidirect product, $A \cdot B$ denotes a nonsplit extension and $A \circ_C B$ denotes a central product.

Whenever E is a concrete elementary abelian p-group of rank n we will always fix an ordered basis of E, so that $\mathrm{GL}(E)$ identifies with $\mathrm{GL}_n(\mathbf{F}_p)$. We make the standing convention that all matrices acts on columns.

We identify a permutation σ in the symmetric group Σ_n with its permutation matrix $A = [a_{ij}]$ given by $a_{ij} = \delta_{i,\sigma(j)}$ where δ is the Kronecker delta.

If K is a field, we let $M_n(K)$ denote the set of $n \times n$-matrices over K. For $a_1, \dots, a_n \in K$ we let $\mathrm{diag}(a_1, \dots, a_n) \in M_n(K)$ denote the diagonal matrix with the a_i's in the diagonal. For $1 \le i, j \le n$, $e_{ij} \in M_n(K)$ denotes the matrix whose only nonzero entry is 1 in position (i, j). Given matrices $A_1 \in M_{n_1}(K)$, $\dots$, $A_m \in M_{n_m}(K)$ we let $A_1 \oplus \dots \oplus A_m$ denote the $n \times n$-block matrix with the A_i's in the diagonal, $n = n_1 + \dots + n_m$. We also need the 'blowup' homomorphism $\Delta_{n,m} : M_n(K) \longrightarrow M_{mn}(K)$ defined by replacing each entry a_{ij} by $a_{ij}I_m$, where $I_m \in M_m(K)$ is the identity matrix.

As $p = 3$ for all the exceptional groups we consider, we use some special notation. An arbitrary element of $\mathbf{F}_3$ is denoted by $*$, and ε denotes an element of the multiplicative group $\mathbf{F}_3^\times$. We let $\omega = e^{2\pi i/3}$ and $\eta = e^{2\pi i/9}$ and define elements $\beta, \gamma, \tau_1, \tau_2 \in \mathrm{SL}_3(\mathbf{C})$ by $\beta = \mathrm{diag}(1, \omega, \omega^2)$,

$$\gamma = (1,2,3) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \tau_1 = \frac{e^{-\pi i/18}}{\sqrt{3}} \begin{bmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{bmatrix}$$

and $\tau_2 = \mathrm{diag}(\eta, \eta^{-2}, \eta)$. Note that $\beta^{\tau_1} = \beta\gamma$, $\gamma^{\tau_1} = \gamma$, $\beta^{\tau_2} = \beta$ and $\gamma^{\tau_2} = \beta\gamma$.

To distinguish subgroups we use *class distributions*. As an example the group $3E_6(\mathbf{C})$ contains seven conjugacy classes of elements of order 3 labeled **3A**, **3B**, **3B′**, **3C**, **3D**, **3E** and **3E′**, cf. 8.7. The fact that the subgroup $E^4_{3E_6}$ from Theorem 8.8 below has class distribution $\mathbf{3C^{78}3E^{1}3E'^{1}}$ means that it (apart from the identity) contains 78 elements from the conjugacy class **3C** and one element from each of the conjugacy classes **3E** and **3E′**.

8.1. *Recollection of some results on linear algebraic groups.* Recall that a (not necessarily connected) linear algebraic group G is called *reductive* if its unipotent radical, i.e., the largest normal connected unipotent subgroup of G, is trivial.

THEOREM 8.2. *Let G be a linear algebraic group over an algebraically closed field K.*

(1) *If A is a subgroup of G and S is some subset of A, then A is toral in G if and only if A is toral in $C_G(S)$.*

(2) *If H is a maximal torus of G, then two subsets of H are conjugate in G if and only if they are conjugate in $N_G(H)$. If A is a toral subgroup of G, then $W(A) = N_G(A)/C_G(A)$ is isomorphic to a subquotient of the Weyl group $W = N_G(H)/H$ of G.*

(3) *Assume that G is a connected reductive group such that the commutator subgroup G' is simply connected. Then the centralizer of any semisimple element in G is connected. In particular, if A is an abelian subgroup of G consisting of semisimple elements generated by at most two elements, then A is toral.*

(4) *If G is reductive and σ is a semisimple automorphism of G, then the fixed point subgroup G^σ is reductive and contains a regular element of G.*

(5) *Assume that G is a connected reductive group, let $Z \subseteq G$ be a central subgroup, and let $\pi : G \to G/Z$ be the quotient homomorphism. If A is a subgroup of G, then A is toral in G if and only if $\pi(A)$ is toral in G/Z.*

(6) *Assume* $\operatorname{char} K = 0$ *and let $\mathfrak{g}$ be the Lie algebra of G. If $S \subseteq G$ is a finite subset of G, then the Lie algebra of $C_G(S)$ is given by*

$$\mathfrak{c}_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid \operatorname{Ad}(s)(x) = x \textit{ for all } s \in S\}.$$

In particular, if $S \subseteq G$ is a finite subgroup, then

$$\dim C_G(S) = \frac{1}{|S|} \sum_{s \in S} \operatorname{tr}_{\mathfrak{g}} \operatorname{Ad}(s).$$

Proof. (1): Obviously, if A is toral in $C_G(S)$ then A is toral in G. Conversely, if A is toral in G, then $A \subseteq H$ for a torus H in G. Since $S \subseteq A$ we get $H \subseteq C_G(S)$ and thus A is toral in $C_G(S)$.

(2): The first part follows by the Frattini argument: Assume that $A, A^g \subseteq H$ are conjugate subsets of H. Then H and $H^{g^{-1}}$ are maximal tori of $C_G(A)$ and thus conjugate in $C_G(A)$ (cf. [76, Cor. 21.3.A]). Thus we may write $H = H^{g^{-1}c}$ for some $c \in C_G(A)$ and we conclude that $n = g^{-1}c \in N_G(H)$. Then $A^{n^{-1}} = A^{c^{-1}g} = A^g$, which proves the first part. The second part follows similarly; cf. [90, Prop. 1.1(i)].

(3): The first part which is due to Steinberg is proved in [28, Thm. 3.5.6]. The second part follows from the first; cf. [130, II.5.1].

(4): We can assume G to be connected. In case G is semisimple and simply connected the first claim is proved in [131, Thm. 8.1] and the general case reduces to this one. Indeed we can find a finite cover $\tilde{G}$ of G which is a direct product of a semisimple simply connected group and a torus, and σ lifts to a semisimple automorphism of $\tilde{G}$ by [131, 9.16]. For the second claim see [142, Thms. 2 and 3] or [130, Pf. of Thm. II.5.16] in case G is semisimple; the general case clearly reduces to this one.

(5): By [76, Cor. 21.3.C] we know that if H is a maximal torus of G, then $\pi(H)$ is a maximal torus of G/Z, and all maximal tori of G/Z are of this form. Thus if A is toral in G, then $\pi(A)$ is toral in G/Z. Conversely, if H' is a maximal torus of G/Z containing $\pi(A)$, then by the above we have $H' = \pi(H)$

for some maximal torus H of G. Thus we get $A \subseteq \langle H, Z\rangle$. However since G is connected and reductive, we get $Z \subseteq H$ by [76, Cor. 26.2.A(b)]. Thus $A \subseteq H$ and we are done.

(6): In case S consists of a single element, the first part follows from [76, Thm. 13.4(a)] (note that the connectivity assumption in [76, Thm. 13.4] is only used in [76, Thm. 13.4(b)]). The general case follows from this by applying [76, Thm. 12.5] to the centralizers $C_G(s)$, $s \in S$.

Now assume that $S \subseteq G$ is a finite subgroup, and let χ denote the character of the adjoint representation of G restricted to S. Then the dimension of

$$\mathfrak{c}_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid \mathrm{Ad}(s)(x) = x \text{ for all } s \in S\}$$

equals the multiplicity of the trivial character in χ. By the orthogonality relations this is given by

$$(\chi \mid 1) = \frac{1}{|S|} \sum_{s \in S} \chi(s),$$

and we are done. □

We also need the following result whose proof is extracted from [122].

Theorem 8.3. *Let G be a reductive linear algebraic group, H a maximal torus of G and let $N = N_G(H)$. Let $U \subseteq N$ be a subgroup consisting of semisimple elements such that $U/(U \cap H)$ is cyclic. Let S be the identity component of H^U (the subgroup of H fixed by U), and assume that S is a maximal torus of $C_G(U)$. Then $C_N(U) = N_{C_G(U)}(S)$ and in particular $C_N(U)$ is a maximal torus normalizer in $C_G(U)$.*

Proof. As any element of $C_N(U)$ normalizes H^U and hence also its identity component S, the inclusion $C_N(U) \subseteq N_{C_G(U)}(S)$ is clear. Suppose conversely that $x \in N_{C_G(U)}(S)$. Let $C = U \cap H$. From [15, 2.15(d)] it follows that G^C is reductive. By assumption the cyclic group U/C acts by semisimple automorphisms on G^C. It now follows from Theorem 8.2(4) that $G^U = (G^C)^{U/C}$ is reductive and that every maximal torus of G^U is contained in a unique maximal torus of G^C. Since $C \subseteq H$, we see that H is the maximal torus of G^C containing S. As H^x is also a maximal torus of G^C and $H^x \supseteq S^x = S$ we conclude that $H^x = H$. Thus $x \in C_N(U)$ proving the result. □

We now explain the relationship between reductive complex linear algebraic groups and compact Lie groups. If G is a complex linear algebraic group then the underlying variety of G is an affine complex variety. By endowing this variety with the usual Euclidean topology instead of the Zariski topology we may view G as a complex Lie group since the group operations are given by polynomial maps.

PROPOSITION 8.4. *Let G be a complex linear algebraic group.*

(1) *Viewed as a Lie group, G contains a maximal compact subgroup which is unique up to conjugacy, and for any such subgroup K there is a diffeomorphism $G \cong K \times \mathbf{R}^s$ for some s.*

(2) *Let K be a maximal compact subgroup of G, and let $S, S' \subseteq K$ be two subsets. If $S' = S^g$ for some $g \in G$, then there exists $k \in K$ such that $x^k = x^g$ for all $x \in S$.*

(3) *Assume that G is reductive. If S is a finite subgroup of G, then $C_G(S)$ is also reductive. If K is a maximal compact subgroup of G containing S, then $C_K(S)$ is a maximal compact subgroup of $C_G(S)$.*

(4) *If G is reductive and K is a maximal compact subgroup of G, then there is a diffeomorphism $Z(G) \cong Z(K) \times \mathbf{R}^s$ for some s.*

Proof. Note first that the identity component G_1 of G seen as a Lie group coincides with the identity component of G seen as a linear algebraic group [115, Ch. 3, §3, no. 1]. Thus G/G_1 is finite by [76, Prop. 7.3(a)]. The first claim is now part of the Cartan-Chevalley-Iwasawa-Malcev-Mostow theorem [74, Ch. XV, Thm. 3.1] and the second claim also follows from this; cf. [14, Ch. V, §24.7, Prop. 2].

In case G is reductive it is possible to give a more explicit form of the decomposition above. By [76, Thm. 8.6] we may assume that G is a closed subgroup of $\mathrm{GL}(V)$ for some complex vector space V. From [115, Thm. 5.2.8] it follows that G has a compact real form K and we may thus choose a non-degenerate Hermitian inner product on V which is invariant under K (e.g. by [115, Thm. 3.4.2]). Let $\mathrm{U}(V)$ denote the set of operators in $\mathrm{GL}(V)$ which are unitary with respect to the chosen inner product. Using [115, Problems 5.2.3 and 5.2.4] we see that $G \subseteq \mathrm{GL}(V)$ is self-adjoint and that $K = G \cap \mathrm{U}(V)$. The last part now follows by combining [115, Cor. 2 of Thm. 5.2.2] with [115, Cor. 2 of Thm. 5.2.1].

If S is a subgroup of K, then S is self-adjoint since K consists of unitary operators. In particular $C_G(S)$ is also a self-adjoint subgroup of $\mathrm{GL}(V)$, and so by [115, Cor. 3 of Thm. 5.2.1], $C_K(S) = C_G(S) \cap \mathrm{U}(V)$ is a maximal compact subgroup of $C_G(S)$.

It only remains to prove that $C_G(S)$ is reductive for a finite subgroup S of G. However by [115, Problem 6.11] and [115, Ch. 4, §1, no. 2] we see that a complex linear algebraic group is reductive if and only if its Lie algebra is reductive. Thus it suffices to prove that the Lie algebra of $C_G(S)$ is reductive. However by Theorem 8.2(6) this Lie algebra equals

$$\mathfrak{c}_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid \mathrm{Ad}(s)(x) = x \text{ for all } s \in S\},$$

where $\mathfrak{g}$ denotes the Lie algebra of G. The claim now follows from [33, Ch. V, §2, no. 2, Prop. 8]. ☐

8.2. *The projective unitary groups.* The purpose of this short subsection is to describe the nontoral elementary abelian subgroups of $\mathrm{PGL}_n(\mathbf{C})$, which by Proposition 8.4 is equivalent to finding them for its maximal compact subgroup $\mathrm{PU}(n)$, as well as to give information about centralizers and Weyl groups. The subgroups are easily determined and are described in [70, Thm. 3.1]—we here just add some extra information about centralizers and Weyl groups which we need in our proof of Theorem 1.1.

We first introduce a useful subgroup. If p^r divides n write $n = p^r k$ and consider the extra special group p_+^{1+2r} embedded in $\mathrm{GL}_n(\mathbf{C})$ by taking k copies of one of the $p-1$ faithful irreducible p^r-dimensional representations. (They all have the same image; see [80, Satz V.16.14].) Note that this embedding maps the center of p_+^{1+2r} to the elements of order p in the center of $\mathrm{GL}_n(\mathbf{C})$. Let Γ_r denote the subgroup of $\mathrm{GL}_n(\mathbf{C})$ generated by the image of p_+^{1+2r} and the center of $\mathrm{GL}_n(\mathbf{C})$. Note that as an abstract group Γ_r fits into an extension sequence

$$1 \to \mathbf{C}^\times \to \Gamma_r \to \bar{\Gamma}_r \to 1,$$

where $\mathbf{C}^\times$ identifies with the center of $\mathrm{GL}_n(\mathbf{C})$ and $\bar{\Gamma}_r \cong (\mathbf{Z}/p)^{2r}$ identifies with the image of Γ_r in $\mathrm{PGL}_n(\mathbf{C})$. (The matrices for Γ_r are written explicitly for $k = 1$ in [114, p. 56–57] where it is denoted $\Gamma^U_{p^r}$.)

THEOREM 8.5. *Suppose E is a nontoral elementary abelian p-subgroup of $\mathrm{PGL}_n(\mathbf{C})$ for an arbitrary prime p. Then, up to conjugacy, E can be written as $E = \bar{\Gamma}_r \times \bar{A}$, for some $r \geq 1$ with $n = p^r k$ and some abelian subgroup A of $C_{\mathrm{GL}_n(\mathbf{C})}(\Gamma_r) \cong \mathrm{GL}_k(\mathbf{C})$.*

For a given r, the conjugacy classes of such subgroups E are in one-to-one correspondence with the conjugacy classes of toral elementary abelian p-subgroups $\bar{A}$ of $\mathrm{PGL}_k(\mathbf{C}) \cong C_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r)_1$ (allowing the trivial subgroup), and the centralizer of E is given by $C_{\mathrm{PGL}_n(\mathbf{C})}(E) \cong \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbf{C})}(\bar{A})$.

The Weyl group equals

$$W_{\mathrm{PGL}_n(\mathbf{C})}(E) = \begin{bmatrix} \mathrm{Sp}(\bar{\Gamma}_r) & 0 \\ * & W_{\mathrm{PGL}_k(\mathbf{C})}(\bar{A}) \end{bmatrix}.$$

Here $\mathrm{Sp}(\bar{\Gamma}_r)$ is the symplectic group relative to the symplectic product coming from the commutator product $[\cdot,\cdot] : \bar{\Gamma}_r \times \bar{\Gamma}_r \to \mathbf{Z}/p \subseteq \mathbf{C}^\times$ and the symbol $*$ denotes a $\operatorname{rank} \bar{A} \times 2r$ matrix with arbitrary entries.

An element $\alpha \in \mathrm{Sp}(\bar{\Gamma}_r) \subseteq W_{\mathrm{PGL}_n(\mathbf{C})}(E)$ acts up to conjugacy as $\alpha \times 1$ on $C_{\mathrm{PGL}_n(\mathbf{C})}(E) \cong \bar{\Gamma}_r \times C_{\mathrm{PGL}_k(\mathbf{C})}(\bar{A})$.

Sketch of proof. The existence of the decomposition $E = \bar{\Gamma}_r \times \bar{A}$ follows from Griess [70, Thm. 3.1] and the statements about uniqueness follow by representation theory of the extra special p-groups (cf. [69, Ch. 5.5] or [80, Satz V.16.14]). Since the image of p_+^{1+2r} is the sum of k identical irreducible representations we have $C_{\mathrm{GL}_n(\mathbf{C})}(\Gamma_r) \cong \mathrm{GL}_k(\mathbf{C})$ by Schur's lemma (see also [114, Prop. 4]). From this the centralizer in $\mathrm{PGL}_n(\mathbf{C})$ can easily be worked out.

In the case where $\bar{A}$ is trivial the statement about Weyl groups is given in [114, Thm. 6] (and just uses elementary character theory). The general case follows similarly, again using character theory.

For the statement about the Weyl group action, first note that

$$\mathrm{Out}(\bar{\Gamma}_r \times \mathrm{PGL}_k(\mathbf{C})) \cong \mathrm{Aut}(\bar{\Gamma}_r) \times \mathrm{Out}(\mathrm{PGL}_k(\mathbf{C})).$$

An element $\alpha \in \mathrm{Sp}(\bar{\Gamma}_r) = W_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r)$ acts as an inner automorphism on $\mathrm{PGL}_k(\mathbf{C})$ since this is true for the action on $C_{\mathrm{GL}_n(\mathbf{C})}(\Gamma_r) \cong \mathrm{GL}_k(\mathbf{C})$ by character theory. Hence we can choose a representative $g \in N_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r)$ of α which acts as $\alpha \times 1$ on $C_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r) \cong \bar{\Gamma}_r \times \mathrm{PGL}_k(\mathbf{C})$. Hence g is also a representative of $\alpha \in \mathrm{Sp}(\bar{\Gamma}_r) \subseteq W_{\mathrm{PGL}_n(\mathbf{C})}(\bar{\Gamma}_r \times \bar{A})$. The claim now follows. □

8.3. *The groups $E_6(\mathbf{C})$ and $3E_6(\mathbf{C})$, $p = 3$.* In this subsection we consider the elementary abelian 3-subgroups of the groups of type E_6 over $\mathbf{C}$. The group $3E_6(\mathbf{C})$ has two nonisomorphic faithful irreducible 27-dimensional representations. These have highest weight λ_1 and λ_6 respectively and are dual to each other. An explicit construction of $3E_6(\mathbf{C})$ based on one of these representations was originally given by Freudenthal [65]. This construction is described in more detail in [37, §2] from which we take most of our notation. In particular we let $\mathbf{K}$ be the 27-dimensional complex vector space consisting of triples $m = (m_1, m_2, m_3)$ of complex 3×3-matrices m_i, $1 \le i \le 3$, where addition and scalar multiplication are defined coordinatewise. We define a cubic form $\langle\cdot\rangle$ on $\mathbf{K}$ by

$$\langle m\rangle = \det(m_1) + \det(m_2) + \det(m_3) - \mathrm{tr}(m_1 m_2 m_3).$$

Then $3E_6(\mathbf{C})$ is the subgroup of $\mathrm{GL}(\mathbf{K})$ preserving the form $\langle\cdot\rangle$. Moreover the stabilizer in $3E_6(\mathbf{C})$ of the element $(I_3, 0, 0) \in \mathbf{K}$ is the group $F_4(\mathbf{C})$. For $g_1, g_2, g_3 \in \mathrm{SL}_3(\mathbf{C})$ we have the element s_{g_1,g_2,g_3} of $3E_6(\mathbf{C})$ given by

$$s_{g_1,g_2,g_3}(m_1, m_2, m_3) = \left(g_1 m_1 g_2^{-1}, g_2 m_2 g_3^{-1}, g_3 m_3 g_1^{-1}\right)$$

for $m = (m_1, m_2, m_3) \in \mathbf{K}$. This gives a representation of $\mathrm{SL}_3(\mathbf{C})^3$ which has kernel C_3 generated by $(\omega I_3, \omega I_3, \omega I_3)$, and we thus get an embedding of $\mathrm{SL}_3(\mathbf{C})^3/C_3$ in $3E_6(\mathbf{C})$. We will denote the element s_{g_1,g_2,g_3} by $[g_1, g_2, g_3]$.

We let $\{e^i_{j,k}\}$, $1 \le i, j, k \le 3$ be the natural basis of $\mathbf{K}$ consisting of the elements $e^i_{j,k}$ whose entries are all 0 except for the (j,k)-entry of the ith matrix which equals 1. The elements of $3E_6(\mathbf{C})$ which act diagonally with respect to this basis of $\mathbf{K}$ form a maximal torus H in $3E_6(\mathbf{C})$. Let $m_i^{j,k}$ denote the

(j,k)-entry of the matrix m_i. We then have H-invariant subgroups

$$u_{\alpha_1}(t) = [I_3, I_3 + te_{1,3}, I_3], \qquad u_{-\alpha_1}(t) = [I_3, I_3 + te_{3,1}, I_3],$$

$$u_{\alpha_2}(t) = [I_3 + te_{2,1}, I_3, I_3], \qquad u_{-\alpha_2}(t) = [I_3 + te_{1,2}, I_3, I_3],$$

$$u_{\alpha_3}(t) = [I_3, I_3 + te_{2,1}, I_3], \qquad u_{-\alpha_3}(t) = [I_3, I_3 + te_{1,2}, I_3],$$

$$u_{\alpha_4}(t) : (m_i)_{i=1,2,3} \mapsto \left(m_i + t \cdot \begin{bmatrix} 0 & -m_{i+2}^{2,3} & 0 \\ 0 & 0 & 0 \\ 0 & m_{i+2}^{2,1} & 0 \end{bmatrix} \right)_{i=1,2,3},$$

$$u_{-\alpha_4}(t) : (m_i)_{i=1,2,3} \mapsto \left(m_i + t \cdot \begin{bmatrix} 0 & 0 & 0 \\ m_{i+1}^{3,2} & 0 & -m_{i+1}^{1,2} \\ 0 & 0 & 0 \end{bmatrix} \right)_{i=1,2,3},$$

$$u_{\alpha_5}(t) = [I_3, I_3, I_3 + te_{2,1}], \qquad u_{-\alpha_5}(t) = [I_3, I_3, I_3 + te_{1,2}],$$

$$u_{\alpha_6}(t) = [I_3, I_3, I_3 + te_{1,3}], \qquad u_{-\alpha_6}(t) = [I_3, I_3, I_3 + te_{3,1}].$$

Here, in the description of $u_{\pm\alpha_4}(t)$, the m_i's should be counted cyclicly mod 3, e.g. $m_{i+2} = m_1$ for $i = 2$.

The associated roots α_i, $1 \le i \le 6$, of these root subgroups form a simple system in the root system $\Phi(E_6)$ of $3E_6(\mathbf{C})$ (our numbering agrees with [18, Planche V]). For this simple system, the highest weight of $\mathbf{K}$ is λ_1. Furthermore the root subgroups $u_{\pm\alpha_i}$, $1 \le i \le 6$, have been chosen so that they satisfy the conditions in [129, Prop. 8.1.1(i) and Lem. 8.1.4(i)]; i.e., they form part of a *realization* ([129, p. 133]) of $\Phi(E_6)$ in $3E_6(\mathbf{C})$. For $\alpha = \pm\alpha_i$, $1 \le i \le 6$, and $t \in \mathbf{C}^\times$, we define the elements

$$n_\alpha(t) = u_\alpha(t) u_{-\alpha}(-1/t) u_\alpha(t), \qquad h_\alpha(t) = n_\alpha(t) n_\alpha(1)^{-1}.$$

Then the maximal torus consists of the elements

$$h(t_1, t_2, t_3, t_4, t_5, t_6) = \prod_{i=1}^{6} h_{\alpha_i}(t_i)$$

and the normalizer $N(H)$ of the maximal torus is generated by H and the elements $n_i = n_{\alpha_i}(1)$, $1 \le i \le 6$. It should be noted that this notation differs from that used in [37]. More precisely, the element $h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ in [37, p. 109] equals $h(\delta, \alpha^{-1}, \gamma^{-1}, \beta, \varepsilon^{-1}, \zeta)$ in our notation, and the elements n_1, n_2, n_3, n_4, n_5 and n_6 in [37, p. 109] equal $n_1 h_{\alpha_1}(-1) h_{\alpha_3}(-1)$, $n_2 h(-1,1,1,-1,1,-1)$, $n_3 h_{\alpha_1}(-1)$, n_4, $n_5 h_{\alpha_6}(-1)$ and $n_6 h_{\alpha_5}(-1) h_{\alpha_6}(-1)$ respectively in our notation.

From the description of the root system of type E_6 in [18, Planche V] we see that the center Z of $3E_6(\mathbf{C})$ is cyclic of order 3 and is generated by the element $z = \left[I_3, \omega^2 I_3, \omega I_3\right]$. We also consider the element $a = [\omega I_3, I_3, I_3]$. A straightforward computation shows that the roots of the centralizer $C_{3E_6(\mathbf{C})}(a)$ are

$$\{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_5, \pm\alpha_6, \pm\widetilde{\alpha}, \pm(\alpha_1+\alpha_3), \pm(\alpha_5+\alpha_6), \pm(\alpha_2-\widetilde{\alpha})\},$$

where $\widetilde{\alpha}$ is the longest root. The Dynkin diagram for this centralizer is the same as the extended Dynkin diagram for E_6 with the node α_4 removed. In particular it has type $A_2A_2A_2$ and a simple system of roots is given by $\{\alpha_3, \alpha_1, \alpha_5, \alpha_6, \alpha_2, -\widetilde{\alpha}\}$. Since $3E_6(\mathbf{C})$ is simply connected, Theorem 8.2(3) implies that the centralizer $C_{3E_6(\mathbf{C})}(a)$ is connected, and thus it is generated by the maximal torus H and the root subgroups $u_{\pm\alpha}(t)$ where α runs through the simple roots $\{\alpha_3, \alpha_1, \alpha_5, \alpha_6, \alpha_2, -\widetilde{\alpha}\}$. Now note that $u_{\widetilde{\alpha}}(t) = [I_3 + te_{3,1}, I_3, I_3]$ and $u_{-\widetilde{\alpha}}(t) = [I_3 + te_{1,3}, I_3, I_3]$ are root subgroups with associated roots $\widetilde{\alpha}$ and $-\widetilde{\alpha}$ respectively. Since these along with H and the root subgroups $u_{\pm\alpha_1}$, $u_{\pm\alpha_2}$, $u_{\pm\alpha_3}$, $u_{\pm\alpha_5}$ and $u_{\pm\alpha_6}$ generate the subgroup $\mathrm{SL}_3(\mathbf{C})^3/C_3$ of $3E_6(\mathbf{C})$ from above, we conclude that $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$.

To describe the conjugacy classes of elementary abelian 3-subgroups we introduce the following elements in $\mathrm{SL}_3(\mathbf{C})^3/C_3 \subseteq 3E_6(\mathbf{C})$:

$$x_1 = [I_3, \beta, \beta], \qquad x_2 = [\beta, \beta, \beta], \qquad y_1 = \left[I_3, \gamma, \gamma^2\right], \qquad y_2 = [\gamma, \gamma, \gamma].$$

We also need the following elements in $N(H)$:

$$\begin{aligned} s_1 &= n_1n_3n_4n_2n_5n_4n_3n_1n_6n_5n_4n_2n_3n_4n_5n_6, \\ s_2 &= n_1n_2n_3n_1n_4n_2n_3n_1n_4n_3n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_2n_3n_1n_4 \\ &\quad \cdot n_3n_5n_4n_2n_6n_5n_4n_3n_1. \end{aligned}$$

The actions of these elements are as follows:

$$s_1(m_1, m_2, m_3) = (m_3, m_1, m_2), \qquad s_2(m_1, m_2, m_3) = (m_3^{\mathtt{T}}, m_2^{\mathtt{T}}, m_1^{\mathtt{T}}),$$

where $m_i^{\mathtt{T}}$ denotes the transpose of m_i. Thus these elements act by conjugation on the subgroup $\mathrm{SL}_3(\mathbf{C})^3/C_3$ as follows:

$$[g_1, g_2, g_3]^{s_1} = [g_2, g_3, g_1], \qquad [g_1, g_2, g_3]^{s_2} = \left[\left(g_1^{-1}\right)^{\mathtt{T}}, \left(g_3^{-1}\right)^{\mathtt{T}}, \left(g_2^{-1}\right)^{\mathtt{T}}\right].$$

Lemma 8.6. *We have*

$$\begin{aligned} &z = h(\omega, 1, \omega^2, 1, \omega, \omega^2), \quad a = h(\omega, 1, \omega^2, 1, \omega^2, \omega), \quad x_1 = h(\omega, 1, \omega, 1, \omega, \omega), \\ &x_2 = h(1, \omega^2, \omega^2, 1, \omega^2, 1), \quad y_1 = n_1n_3n_5n_6 \cdot h_{\alpha_5}(-1), \\ &y_2 = n_1n_2n_3n_4n_3n_1n_5n_4n_2n_3n_4n_5n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_6n_5 \cdot h_{\alpha_2}(-1). \end{aligned}$$

Moreover conjugation by the element

$$n_1n_4n_2n_3n_1n_4n_5n_4n_6n_5n_4n_2n_3n_1n_4 \cdot h_{\alpha_2}(-1)h_{\alpha_4}(-1)$$

acts as follows:

$$a \mapsto x_2, \quad x_2 \mapsto a, \quad y_1 \mapsto s_1, \quad y_2 \mapsto y_2^2, \quad x_2x_1^{-1} \mapsto h_{\alpha_4}(\omega) = [\tau_2, \tau_2, \tau_2].$$

Proof. Both parts of the lemma may be checked by direct computation. The second part also follows from the first by using the following relations in $N(H)$: The element n_i has image s_{α_i} in W ([129, Lem. 8.1.4(i)]), we have $n_i^2 = h_{\alpha_i}(-1)$ ([129, Lem. 8.1.4(ii)]) and

$$n_i n_j n_i \ldots = n_j n_i n_j \ldots$$

for $1 \leq i, j \leq 6$, where the number of factors on both sides equals the order of $s_{\alpha_i} s_{\alpha_j}$ in W ([129, Prop. 9.3.2]). $\square$

Notation 8.7. For our calculations, we need some information on the conjugacy classes of elements of order 3 in $3E_6(\mathbf{C})$. These are given in [37, Table 2]: There are seven such conjugacy classes, which we label **3A**, **3B**, $\mathbf{3B'}$, **3C**, **3D**, **3E** and $\mathbf{3E'}$, where $\mathbf{3B'}$ and $\mathbf{3E'}$ denote the inverses of the classes **3B** and **3E**. This notation is almost identical to the notation in [37, Table 2], but differs from [70, Table VI]. We will need the following, which comes quickly from [37, Table 2] using the action of W on H: We have $z \in \mathbf{3E}$, $a, x_2, y_2 \in \mathbf{3C}$, $x_1, y_1 \in \mathbf{3D}$ and $x_2 x_1^{-1} \in \mathbf{3A}$. Multiplication by z acts as follows on the conjugacy classes:

$$\mathbf{3A} \mapsto \mathbf{3B},\ \mathbf{3B} \mapsto \mathbf{3B'},\ \mathbf{3B'} \mapsto \mathbf{3A},\ \mathbf{3C} \mapsto \mathbf{3C},\ \mathbf{3D} \mapsto \mathbf{3D},\ \mathbf{3E} \mapsto \mathbf{3E'},\ \mathbf{3E'} \mapsto \mathbf{1},$$

where **1** denotes the conjugacy class consisting of the identity element.

THEOREM 8.8. *The conjugacy classes of nontoral elementary abelian 3-subgroups of $3E_6(\mathbf{C})$ are given by the following table*:

rank	*name*	*ordered basis*	$3E_6(\mathbf{C})$*-class distribution*	$C_{3E_6(\mathbf{C})}(E)$
3	$E^3_{3E_6}$	$\{a, x_2, y_2\}$	$\mathbf{3C^{26}}$	$E^4_{3E_6}$
4	$E^4_{3E_6}$	$\{z, a, x_2, y_2\}$	$\mathbf{3C^{78}3E^{1}3E'^{1}}$	$E^4_{3E_6}$

Their Weyl groups with respect to the given ordered bases are:

$$W(E^3_{3E_6}) = \mathrm{SL}_3(\mathbf{F}_3), \ \ W(E^4_{3E_6}) = \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right].$$

Proof. *Nontoral subgroups.* By [70, Thm. 11.13], there are two conjugacy classes of nontoral elementary abelian 3-subgroups in $3E_6(\mathbf{C})$, one nonmaximal of rank 3 and one maximal of rank 4. We may concretely realize these as follows. Consider the subgroups

$$E^3_{3E_6} = \langle a, x_2, y_2 \rangle \text{ and } E^4_{3E_6} = \langle z, a, x_2, y_2 \rangle\,,$$

which are readily seen to be elementary abelian 3-subgroups of rank 3 and 4 respectively. Both subgroups are contained in $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$, and since $\beta, \gamma \in \mathrm{SL}_3(\mathbf{C})$ do not commute, we see that the preimages of $E^3_{3E_6}$ and

$E^4_{3E_6}$ under the projection $\mathrm{SL}_3(\mathbf{C})^3 \to \mathrm{SL}_3(\mathbf{C})^3/C_3$ are non-abelian. Thus by Theorem 8.2(5) $E^3_{3E_6}$ and $E^4_{3E_6}$ are nontoral in $\mathrm{SL}_3(\mathbf{C})^3/C_3 = C_{3E_6(\mathbf{C})}(a)$ and hence also nontoral in $3E_6(\mathbf{C})$ by Theorem 8.2(1). Hence these two subgroups represent the two conjugacy classes of nontoral elementary abelian 3-subgroups in $3E_6(\mathbf{C})$.

Lower bounds for Weyl groups. By [70, Thm. 7.4] there is a unique nontoral elementary abelian 3-subgroup E of $F_4(\mathbf{C})$ of rank 3 whose Weyl group in $F_4(\mathbf{C})$ equals $\mathrm{SL}_3(\mathbf{F}_3)$. Since we have an inclusion $F_4(\mathbf{C}) \subseteq 3E_6(\mathbf{C})$ this subgroup may also be considered as a subgroup of $3E_6(\mathbf{C})$ and its Weyl group in $3E_6(\mathbf{C})$ must contain $\mathrm{SL}_3(\mathbf{F}_3)$. In particular it has order divisible by 13 and since $13 \nmid |W(E_6)|$, we conclude by Theorem 8.2(2) that E is nontoral in $3E_6(\mathbf{C})$ as well. Thus by the above, E must be conjugate to $E^3_{3E_6}$, and hence $W(E^3_{3E_6})$ contains $\mathrm{SL}_3(\mathbf{F}_3)$. From this we immediately see that $W(E^4_{3E_6})$ contains the group $1 \times \mathrm{SL}_3(\mathbf{F}_3)$.

Note that the element $\left[I_3, \beta, \beta^2\right]$ commutes with z, a and x_2 and conjugates y_2 to $y_2 z$. Thus it normalizes $E^4_{3E_6}$ and produces the element $I_4 + e_{1,4}$ in $W(E^4_{3E_6})$. As a result we see that $W(E^4_{3E_6})$ contains the group

$$\left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right].$$

Class distributions. Since $a \in \mathbf{3C}$ by 8.7 and $W(E^3_{3E_6})$ contains $\mathrm{SL}_3(\mathbf{F}_3)$ which acts transitively on $E^3_{3E_6} - \{1\}$, the class distribution of $E^3_{3E_6}$ follows immediately. Using this and the information given in 8.7 about multiplication by z, the class distribution of $E^4_{3E_6}$ follows.

Centralizers. Since $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$ we directly get

$$C_{3E_6(\mathbf{C})}(a, x_2) = C_{\mathrm{SL}_3(\mathbf{C})^3/C_3}(x_2) = \langle y_2, (\mathbf{T}_2 \times \mathbf{T}_2 \times \mathbf{T}_2)/C_3 \rangle,$$
$$C_{3E_6(\mathbf{C})}(a, x_2, y_2) = \langle x_2, y_2, (\langle \omega I_3 \rangle \times \langle \omega I_3 \rangle \times \langle \omega I_3 \rangle)/C_3 \rangle = E^4_{3E_6},$$

proving that $C_{3E_6(\mathbf{C})}(E^3_{3E_6}) = C_{3E_6(\mathbf{C})}(E^4_{3E_6}) = E^4_{3E_6}$.

Exact Weyl groups. From the lower bounds above and the fact that z is central we get $\mathrm{SL}_3(\mathbf{F}_3) \subseteq W(E^3_{3E_6}) \subseteq \mathrm{GL}_3(\mathbf{F}_3)$ and

$$\left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right] \subseteq W(E^4_{3E_6}) \subseteq \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{GL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right].$$

As $C_{3E_6(\mathbf{C})}(a, x_2) = \langle y_2, (\mathbf{T}_2 \times \mathbf{T}_2 \times \mathbf{T}_2)/C_3 \rangle$, we see that no element in $C_{3E_6(\mathbf{C})}(a, x_2)$ conjugates y_2 to y_2^{-1}. Hence $\mathrm{diag}(1,1,2) \notin W(E^3_{3E_6})$ and

$\mathrm{diag}(1,1,1,2) \notin W(E^4_{3E_6})$ which shows that the Weyl groups are the ones given in the theorem. $\square$

We now turn to the group $E_6(\mathbf{C})$. As above, let Z be the center of $3E_6(\mathbf{C})$ and let $\pi : 3E_6(\mathbf{C}) \to E_6(\mathbf{C}) = 3E_6(\mathbf{C})/Z$ denote the quotient homomorphism. For $g \in 3E_6(\mathbf{C})$ we write $\overline{g}$ instead of $\pi(g)$ and similarly we let $\overline{S} = \pi(S)$ for a subset $S \subseteq 3E_6(\mathbf{C})$.

LEMMA 8.9. *Let E be a rank two nontoral elementary abelian 3-subgroup of $E_6(\mathbf{C})$. Then the Weyl group $W(E)$ is a subgroup of $\mathrm{SL}_2(\mathbf{F}_3)$.*

Proof. Let $\{\overline{g_1}, \overline{g_2}\}$ be an ordered basis of E. By Theorem 8.2 parts (5) and (3) the subgroup $\langle g_1, g_2\rangle \subseteq 3E_6(\mathbf{C})$ is non-abelian. Thus setting $z' = [g_1, g_2] \in Z$ we have $z' \neq 1$. Assume that $\sigma \in W(E)$ is represented by the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$, i.e., we have $\sigma(\overline{g_1}) = (\overline{g_1})^{a_{11}}(\overline{g_2})^{a_{21}}$ and $\sigma(\overline{g_2}) = (\overline{g_1})^{a_{12}}(\overline{g_2})^{a_{22}}$. Since σ is given by a conjugation in $E_6(\mathbf{C})$, it lifts to a conjugation in $3E_6(\mathbf{C})$. Now the relation $[g_1, g_2] = z' \in Z$ shows that $(z')^{a_{11}\cdot a_{22} - a_{12}\cdot a_{21}} = z'$, so $\sigma \in \mathrm{SL}_2(\mathbf{F}_3)$ since $z' \neq 1$. $\square$

THEOREM 8.10. *The conjugacy classes of nontoral elementary abelian 3-subgroups of $E_6(\mathbf{C})$ are given by the following table*:

rank	*name*	*ordered basis*	$3E_6(\mathbf{C})$*-class distribution*	$C_{E_6(\mathbf{C})}(E)$	$Z(C_{E_6(\mathbf{C})}(E))$
2	$E^{2a}_{E_6}$	$\{\overline{y_1}, \overline{x_2}\}$	$\mathbf{3C^{18}3D^{6}3E^{1}3E'^{1}}$	$E^{2a}_{E_6} \times \mathrm{PSL}_3(\mathbf{C})$	$E^{2a}_{E_6}$
2	$E^{2b}_{E_6}$	$\{\overline{y_1}, \overline{x_1}\}$	$\mathbf{3D^{24}3E^{1}3E'^{1}}$	$E^{2b}_{E_6} \times G_2(\mathbf{C})$	$E^{2b}_{E_6}$
3	$E^{3a}_{E_6}$	$\{\overline{a}, \overline{y_1}, \overline{x_2}\}$	$\mathbf{3C^{60}3D^{18}3E^{1}3E'^{1}}$	$E^{3a}_{E_6} \circ_{\langle\overline{a}\rangle} (\mathbf{T}_2 : \langle\overline{y_2}\rangle)$	$E^{3a}_{E_6}$
3	$E^{3b}_{E_6}$	$\{\overline{a}, \overline{x_2}, \overline{y_2}\}$	$\mathbf{3C^{78}3E^{1}3E'^{1}}$	$E^{3b}_{E_6} \cdot (C_3)^3$	$E^{3b}_{E_6}$
3	$E^{3c}_{E_6}$	$\{\overline{a}, \overline{y_1}, \overline{x_1}\}$	$\mathbf{3C^{6}3D^{72}3E^{1}3E'^{1}}$	$E^{3c}_{E_6} \circ_{\langle\overline{a}\rangle} \mathrm{SL}_3(\mathbf{C})$	$E^{3c}_{E_6}$
3	$E^{3d}_{E_6}$	$\{\overline{x_2x_1^{-1}}, \overline{y_1}, \overline{x_1}\}$	$\mathbf{3A^{2}3B^{2}3B'^{2}3C^{48}3D^{24}3E^{1}3E'^{1}}$	$E^{3d}_{E_6} \circ_{\langle\overline{x_2x_1^{-1}}\rangle} \mathrm{GL}_2(\mathbf{C})$	$E^{3d}_{E_6} \circ_{\langle\overline{x_2x_1^{-1}}\rangle} \mathbf{T}_1$
4	$E^{4a}_{E_6}$	$\{\overline{a}, \overline{y_2}, \overline{y_1}, \overline{x_2}\}$	$\mathbf{3C^{186}3D^{54}3E^{1}3E'^{1}}$	$E^{4a}_{E_6}$	$E^{4a}_{E_6}$
4	$E^{4b}_{E_6}$	$\{\overline{a}, \overline{x_2x_1^{-1}}, \overline{y_1}, \overline{x_1}\}$	$\mathbf{3A^{6}3B^{6}3B'^{6}3C^{150}3D^{72}3E^{1}3E'^{1}}$	$E^{4b}_{E_6} \circ_{\langle\overline{a}, \overline{x_2x_1^{-1}}\rangle} \mathbf{T}_2$	$E^{4b}_{E_6} \circ_{\langle\overline{a}, \overline{x_2x_1^{-1}}\rangle} \mathbf{T}_2$

In particular ${}_3Z(C_{E_6(\mathbf{C})}(E)) = E$ (where ${}_pA = \ker(A \xrightarrow{\cdot p} A)$ for an abelian group A) for all nontoral elementary abelian 3-subgroups E of $E_6(\mathbf{C})$. (In the table the $3E_6(\mathbf{C})$-class distribution of $E \subseteq E_6(\mathbf{C})$ denotes the class distribution of $\pi^{-1}(E) \subseteq 3E_6(\mathbf{C})$.) The Weyl groups of these subgroups with respect to the given ordered bases are as follows:

$$W(E^{2a}_{E_6}) = \begin{bmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_1 \end{bmatrix}, \quad W(E^{2b}_{E_6}) = \mathrm{SL}_2(\mathbf{F}_3), \quad W(E^{3a}_{E_6}) = \begin{bmatrix} \varepsilon_1 & * & * \\ 0 & \varepsilon_2 & * \\ 0 & 0 & \varepsilon_2 \end{bmatrix},$$

$$W(E^{3b}_{E_6}) = \mathrm{SL}_3(\mathbf{F}_3), \quad W(E^{3c}_{E_6}) = \left[\begin{array}{c|cc} \varepsilon & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbf{F}_3)$}} \\ 0 & & \end{array}\right],$$

$$W(E_{E_6}^{3d}) = \mathrm{GL}_1(\mathbf{F}_3) \times \mathrm{SL}_2(\mathbf{F}_3), \quad W(E_{E_6}^{4a}) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbf{F}_3)} & * & * \\ & & * & * \\ \hline 0 & 0 & \det & * \\ 0 & 0 & 0 & \det \end{array}\right],$$

$$W(E_{E_6}^{4b}) = \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\mathrm{SL}_2(\mathbf{F}_3)} \\ 0 & 0 & & \end{array}\right],$$

where det *denotes the determinant of the matrix from* $\mathrm{GL}_2(\mathbf{F}_3)$ *in the description of* $W(E_{E_6}^{4a})$.

Proof. Maximal nontoral subgroups. By [70, Thm. 11.14], there are two conjugacy classes of maximal nontoral elementary abelian 3-subgroups in $E_6(\mathbf{C})$, both of rank 4. We may concretely realize these as follows. Consider the subgroups

$$E_a = \langle z, a, y_1, y_2, x_2 \rangle \text{ and } E_b = \left\langle z, a, x_2 x_1^{-1}, y_1, x_1 \right\rangle$$

of $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$. Since the commutator subgroup of both of these equals Z, we see that $E_{E_6}^{4a} = \pi(E_a)$ and $E_{E_6}^{4b} = \pi(E_b)$ are elementary abelian 3-subgroups of rank 4 in $E_6(\mathbf{C})$. It follows from Theorem 8.2(5) that both $E_{E_6}^{4a}$ and $E_{E_6}^{4b}$ are nontoral in $E_6(\mathbf{C})$. We will see below that their class distributions are the ones given in the table. From this it follows that they are not conjugate and thus represent the two conjugacy classes of maximal elementary abelian 3-subgroups in $E_6(\mathbf{C})$.

Lower bounds for Weyl groups of maximal nontoral subgroups. We now find lower bounds for the Weyl groups of the maximal nontoral elementary abelian 3-subgroups by conjugating with elements from the centralizer $C_{3E_6(\mathbf{C})}(a) = \mathrm{SL}_3(\mathbf{C})^3/C_3$ and the normalizer $N(H)$ of the maximal torus.

The elements $\overline{[\beta^2, I_3, I_3]}$, $\overline{[I_3, \tau_1, \tau_1^2]}$, $\overline{s_1}$ and $\overline{s_2}$ normalize $E_{E_6}^{4a}$ and conjugation by these elements induces the automorphisms on $E_{E_6}^{4a}$ given by the matrices $I_4 + e_{1,2}$, $I_4 + e_{3,4}$, $I_4 + e_{2,3}$ and $\mathrm{diag}(2, 1, 2, 2)$. Moreover, by Lemma 8.6 we may conjugate the ordered basis of $E_{E_6}^{4a}$ into the ordered basis $\{\overline{x_2}, \overline{y_2^2}, \overline{s_1}, \overline{a}\}$. Noting that the element $\overline{[\tau_1, \tau_1, \tau_1]}$ commutes with $\overline{y_2}$, $\overline{s_1}$ and $\overline{a}$ and conjugates $\overline{x_2}$ into $\overline{x_2 y_2}$, we see that $W(E_{E_6}^{4a})$ contains the element $I_4 + e_{2,1}$. The above matrices are easily seen to generate the group

$$W'(E_{E_6}^{4a}) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbf{F}_3)} & * & * \\ & & * & * \\ \hline 0 & 0 & \det & * \\ 0 & 0 & 0 & \det \end{array}\right]$$

and thus $W_{E_6}^{4a}$ contains this group.

Now consider $E_{E_6}^{4b}$ and let $\sigma = -(2,3) \in \mathrm{SL}_3(\mathbf{C})$. We then see that the elements $\overline{[I_3, \tau_1, \tau_1^2]}$, $\overline{[I_3, \tau_2\beta, \tau_2^2]}$, $\overline{[\sigma, I_3, I_3]}$, $\overline{[\gamma, I_3, I_3]}$, $\overline{[I_3, \beta^2, I_3]}$ and $\overline{s_2}$ normalize $E_{E_6}^{4b}$, and conjugation by these elements induces the automorphisms on $E_{E_6}^{4b}$ given by the matrices $I_4 + e_{3,4}$, $I_4 + e_{4,3}$, $\mathrm{diag}(1,2,1,1)$, $I_4 + e_{1,2}$, $I_4 + e_{1,3}$ and $-I_4$. These matrices generate the group

$$W'(E_{E_6}^{4b}) = \left[\begin{array}{cc|cc} \varepsilon_1 & * & * & * \\ 0 & \varepsilon_2 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{SL}_2(\mathbf{F}_3)$}} \\ 0 & 0 & & \end{array}\right]$$

and thus $W_{E_6}^{4b}$ contains this group.

Orbit computation. Any elementary abelian 3-subgroup of rank one is toral since $E_6(\mathbf{C})$ is connected. Since $E_{E_6}^{4a}$ and $E_{E_6}^{4b}$ are representatives of the maximal nontoral elementary abelian 3-subgroups, we may find the conjugacy classes of nontoral elementary abelian 3-subgroups of ranks 2 and 3 by studying subgroups of these.

Under the action of $W'(E_{E_6}^{4a})$, the set of rank 2 subgroups of $E_{E_6}^{4a}$ has orbit representatives

$$E_{E_6}^{2a} = \langle \overline{y_1}, \overline{x_2} \rangle, \langle \overline{a}, \overline{x_2} \rangle, \langle \overline{a}, \overline{y_1} \rangle \text{ and } \langle \overline{a}, \overline{y_2} \rangle,$$

and under the action of $W'(E_{E_6}^{4b})$, the set of rank 2 subgroups of $E_{E_6}^{4b}$ has orbit representatives

$$E_{E_6}^{2a} = \langle \overline{y_1}, \overline{x_2} \rangle, E_{E_6}^{2b} = \langle \overline{y_1}, \overline{x_1} \rangle, \langle \overline{a}, \overline{x_2} \rangle, \langle \overline{a}, \overline{y_1} \rangle, \left\langle \overline{a}, \overline{x_2 x_1^{-1}} \right\rangle \text{ and } \left\langle \overline{x_2 x_1^{-1}}, \overline{x_1} \right\rangle.$$

Similarly we find that under the action of $W'(E_{E_6}^{4a})$, the set of rank 3 subgroups of $E_{E_6}^{4a}$ has orbit representatives

$$E_{E_6}^{3a} = \langle \overline{a}, \overline{y_1}, \overline{x_2} \rangle, E_{E_6}^{3b} = \langle \overline{a}, \overline{y_2}, \overline{x_2} \rangle \text{ and } \langle \overline{a}, \overline{y_1}, \overline{y_2} \rangle,$$

and that under the action of $W'(E_{E_6}^{4b})$, the set of rank 3 subgroups of $E_{E_6}^{4b}$ has orbit representatives

$$E_{E_6}^{3a} = \langle \overline{a}, \overline{y_1}, \overline{x_2} \rangle, E_{E_6}^{3c} = \langle \overline{a}, \overline{y_1}, \overline{x_1} \rangle, E_{E_6}^{3d} = \left\langle \overline{x_2 x_1^{-1}}, \overline{y_1}, \overline{x_1} \right\rangle \text{ and } \left\langle \overline{a}, \overline{x_2 x_1^{-1}}, \overline{x_1} \right\rangle.$$

Other nontoral subgroups. The subgroups $\langle \overline{a}, \overline{x_2} \rangle$, $\left\langle \overline{a}, \overline{x_2 x_1^{-1}} \right\rangle$, $\left\langle \overline{x_2 x_1^{-1}}, \overline{x_1} \right\rangle$ and $\left\langle \overline{a}, \overline{x_2 x_1^{-1}}, \overline{x_1} \right\rangle$ are visibly toral. Since the elements β and γ are conjugate in $\mathrm{SL}_3(\mathbf{C})$, the subgroup $\langle \overline{a}, \overline{y_1}, \overline{y_2} \rangle$ is conjugate to the subgroup $\left\langle \overline{a}, \overline{[I_3, \beta, \beta^2]}, \overline{x_2} \right\rangle$ which is obviously toral. Thus the subgroups $\langle \overline{a}, \overline{y_1}, \overline{y_2} \rangle$, $\langle \overline{a}, \overline{y_1} \rangle$ and $\langle \overline{a}, \overline{y_2} \rangle$ are also toral. Using the fact that $[y_1, x_1] = [y_1, x_2] = z$ we see from Theorem 8.2(5) that both $E_{E_6}^{2a}$ and $E_{E_6}^{2b}$ are nontoral in $E_6(\mathbf{C})$. Since the subgroups $E_{E_6}^{3a}$, $E_{E_6}^{3c}$ and $E_{E_6}^{3d}$ all contain either $E_{E_6}^{2a}$ or $E_{E_6}^{2b}$ they are also nontoral. Using Theorem 8.2(5) we see that the subgroup $E_{E_6}^{3b}$ is nontoral in $E_6(\mathbf{C})$, since $\pi^{-1}(E_{E_6}^{3b}) = E_{3E_6}^{4}$ is nontoral in $3E_6(\mathbf{C})$ by Theorem 8.8.

Class distributions. Using 8.7 and the actions of the groups $W'(E_{E_6}^{4a})$ and $W'(E_{E_6}^{4b})$ we easily verify the class distributions in the table. As an example consider the subgroup $E_{E_6}^{4b}$. From the action of $W'(E_{E_6}^{4b})$ we see that $\underline{E_{E_6}^{4b}} - \{1\}$ contains two elements conjugate to $\overline{a}$, six elements conjugate to $\overline{x_2x_1^{-1}}$, 24 elements conjugate to $\overline{x_1}$ and 48 elements conjugate to $\overline{x_2}$. Thus by 8.7, the set $\pi^{-1}(E_{E_6}^{4b} - \{1\})$ contains six elements from each of the classes **3A**, **3B** and **3B′**, $3 \cdot (2 + 48) = 150$ elements from the class **3C** and $3 \cdot 24 = 72$ elements from the class **3D**. Including the elements z and z^2 from the classes **3E** and **3E′** respectively, we get the class distribution of $\pi^{-1}(E_{E_6}^{4b}) - \{1\}$ given in the table. Similar computations give the remaining entries in the table. Since these distributions are different we see that the subgroups in the table are not conjugate and thus they provide a set of representatives for the conjugacy classes of nontoral elementary abelian 3-subgroups of $E_6(\mathbf{C})$.

Lower bounds for other Weyl groups. We now show that the other matrix groups in the theorem are all lower bounds for the remaining Weyl groups. To do this consider one of the nonmaximal subgroups E from the table. We then have $E \subseteq E_{E_6}^{4a}$ or $E \subseteq E_{E_6}^{4b}$, and we get a lower bound on $W(E)$ by considering the action on E of the subgroup of $W'(E_{E_6}^{4a})$ or $W'(E_{E_6}^{4b})$ stabilizing E. As an example we see that $E_{E_6}^{2a} \subseteq E_{E_6}^{4a}$ and that the stabilizer of $E_{E_6}^{2a}$ inside $W'(E_{E_6}^{4a})$ is

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbf{F}_3)$}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \det & x \\ 0 & 0 & 0 & \det \end{array}\right]$$

where det is the determinant of the matrix from $\mathrm{GL}_2(\mathbf{F}_3)$. The action of such a matrix on $E_{E_6}^{2a}$ is given by

$$\overline{y_1} \mapsto (\overline{y_1})^{\det}, \qquad \overline{x_2} \mapsto (\overline{y_1})^{x}(\overline{x_2})^{\det}.$$

Thus $W(E_{E_6}^{2a})$ contains the group

$$W'(E_{E_6}^{2a}) = \begin{bmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_1 \end{bmatrix}$$

as claimed. Similar computations show that for the subgroups $E = E_{E_6}^{2b}$, $E_{E_6}^{3a}$, $E_{E_6}^{3c}$ and $E_{E_6}^{3d}$, the group $W'(E)$ occurring in the theorem is a lower bound for the Weyl group $W(E)$.

For the subgroup $E_{E_6}^{3b} = \langle \overline{a}, \overline{x_2}, \overline{y_2} \rangle$ we know the structure of $W(\pi^{-1}(E_{E_6}^{3b})) = W(E_{3E_6}^{4})$ by Theorem 8.8. From this we immediately get $W(E_{E_6}^{3b}) = \mathrm{SL}_3(\mathbf{F}_3)$.

Exact Weyl groups. We now prove that the lower bounds on the Weyl groups established above are in fact equalities. By Lemma 8.9 the Weyl groups $W(E^{2a}_{E_6})$ and $W(E^{2b}_{E_6})$ are subgroups of $\mathrm{SL}_2(\mathbf{F}_3)$. From this we see that $W(E^{2b}_{E_6}) = \mathrm{SL}_2(\mathbf{F}_3)$ and that $W(E^{2a}_{E_6})$ is equal to either $W'(E^{2a}_{E_6})$ or $\mathrm{SL}_2(\mathbf{F}_3)$, since these are the only subgroups of $\mathrm{SL}_2(\mathbf{F}_3)$ containing $W'(E^{2a}_{E_6})$. We have $E^{2a}_{E_6} = \langle \overline{y_1}, \overline{x_2}\rangle$, and by 8.7 the elements $\overline{y_1}$ and $\overline{x_2}$ are not conjugate in $E_6(\mathbf{C})$. In particular $W(E^{2a}_{E_6})$ cannot act transitively on the nontrivial elements of $E^{2a}_{E_6}$, and we conclude that $W(E^{2a}_{E_6}) = W'(E^{2a}_{E_6})$ is the group from above.

For each of the remaining nontoral subgroups we now show that a strictly larger Weyl group would contradict the Weyl group results already established. The subgroups $E = E^{3a}_{E_6}$, $E^{3d}_{E_6}$, and $E^{4b}_{E_6}$ all contain $E^{2a}_{E_6}$. A direct computation shows that any proper overgroup of $W'(E)$ in $\mathrm{GL}(E)$ contains an element which normalizes the subgroup $E^{2a}_{E_6}$ and induces an automorphism which does not lie in $W(E^{2a}_{E_6})$. Hence $W(E) = W'(E)$. If $E = E^{3c}_{E_6}$ a similar argument, using the subgroup $E^{2b}_{E_6}$, again shows that $W(E) = W'(E)$. Consider finally $E = E^{4a}_{E_6}$. Each proper overgroup of $W'(E)$ contains an element which normalizes one of the subgroups $E^{2a}_{E_6}$ or $E^{3b}_{E_6}$ and induces an automorphism on it not contained in its Weyl group. Hence $W(E) = W'(E)$. This concludes the proof that the Weyl groups listed in the theorem are the correct ones.

Centralizers. Let $\Theta : \mathrm{SL}_3(\mathbf{C}) \longrightarrow \mathrm{SL}_3(\mathbf{C})^3/C_3 \subseteq 3E_6(\mathbf{C})$ denote the homomorphism given by $\Theta(g) = [g, g, g]$ for $g \in \mathrm{SL}_3(\mathbf{C})$. By Lemma 8.6 the subgroup $E^{2a}_{E_6} = \langle \overline{x_2}, \overline{y_1}\rangle$ is conjugate to the subgroup $\langle \overline{a}, \overline{s_1}\rangle$. Since $a^{s_1} = az^2$ we obtain $C_{E_6(\mathbf{C})}(\overline{a}) = \overline{\langle s_1, \mathrm{SL}_3(\mathbf{C})^3/C_3\rangle}$, and hence

$$\begin{aligned} C_{E_6(\mathbf{C})}(\overline{a}, \overline{s_1}) &= \overline{\langle a, s_1, z, \Theta(\mathrm{SL}_3(\mathbf{C}))\rangle} \\ &= \overline{\langle a, s_1, z\rangle \times \Theta(\mathrm{SL}_3(\mathbf{C}))} = \langle \overline{a}, \overline{s_1}\rangle \times \mathrm{PSL}_3(\mathbf{C}), \end{aligned}$$

proving the claims for $E^{2a}_{E_6}$. By a slight abuse of notation, we let $\overline{g}$ denote the image of $g \in \mathrm{SL}_3(\mathbf{C})$ in the quotient $\mathrm{PSL}_3(\mathbf{C})$. From Lemma 8.6 we then see that the elements $\overline{a}$, $\overline{y_2}$ and $\overline{x_2 x_1^{-1}}$ in $C_{E_6(\mathbf{C})}(E^{2a}_{E_6})$ correspond to the elements $\overline{\beta}$, $\overline{\gamma^2}$ and $\overline{\tau_2}$ in the $\mathrm{PSL}_3(\mathbf{C})$-component of $C_{E_6(\mathbf{C})}(E^{2a}_{E_6})$. Thus we immediately get

$$\begin{aligned} C_{E_6(\mathbf{C})}(E^{3a}_{E_6}) &= E^{2a}_{E_6} \times C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}), & C_{E_6(\mathbf{C})}(E^{3d}_{E_6}) &= E^{2a}_{E_6} \times C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\tau_2}), \\ C_{E_6(\mathbf{C})}(E^{4a}_{E_6}) &= E^{2a}_{E_6} \times C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}, \overline{\gamma^2}), & C_{E_6(\mathbf{C})}(E^{4b}_{E_6}) &= E^{2a}_{E_6} \times C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}, \overline{\tau_2}). \end{aligned}$$

Note that $C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}) = \overline{\mathbf{T}_2 : \langle \gamma\rangle}$, giving $C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}, \overline{\gamma^2}) = \langle \overline{\beta}, \overline{\gamma}\rangle$ and $C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\beta}, \overline{\tau_2}) = \overline{\mathbf{T}_2}$. From this the results on $E^{3a}_{E_6}$, $E^{4a}_{E_6}$ and $E^{4b}_{E_6}$ follow directly. Note also that $C_{\mathrm{PSL}_3(\mathbf{C})}(\overline{\tau_2}) \cong \mathrm{GL}_2(\mathbf{C})$ from which we deduce the claims about $E^{3d}_{E_6}$.

Now consider the subgroup $E^{3b}_{E_6}$. Since $C_{E_6(\mathbf{C})}(\overline{a}) = \overline{\langle s_1, \mathrm{SL}_3(\mathbf{C})^3/C_3\rangle}$ we get

$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{x_2}) = \overline{\langle s_1, y_1, y_2, (\mathbf{T}_2 \times \mathbf{T}_2 \times \mathbf{T}_2)/C_3\rangle},$$
$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{x_2}, \overline{y_2}) = \overline{\langle s_1, y_1, y_2, [I_3, \beta, \beta^2], x_2, (\langle \omega I_3\rangle \times \langle \omega I_3\rangle \times \langle \omega I_3\rangle)/C_3\rangle},$$

and thus $C_{E_6(\mathbf{C})}(E^{3b}_{E_6}) = \left\langle E^{3b}_{E_6}, \overline{s_1}, \overline{y_1}, \overline{[I_3, \beta, \beta^2]}\right\rangle$. It is now easy to check that $C_{E_6(\mathbf{C})}(E^{3b}_{E_6})$ has the structure $E^{3b}_{E_6} \cdot (C_3)^3$ and that $Z(C_{E_6(\mathbf{C})}(E^{3b}_{E_6})) = E^{3b}_{E_6}$. For the subgroup $E^{3c}_{E_6}$ we obtain

$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{x_1}) = \overline{\langle y_1, (\mathrm{SL}_3(\mathbf{C}) \times \mathbf{T}_2 \times \mathbf{T}_2)/C_3\rangle},$$
$$C_{E_6(\mathbf{C})}(\overline{a}, \overline{x_1}, \overline{y_1}) = \overline{\langle y_1, x_1, (\mathrm{SL}_3(\mathbf{C}) \times \langle \omega I_3\rangle \times \langle \omega I_3\rangle)/C_3\rangle}.$$

Thus $C_{E_6(\mathbf{C})}(E^{3c}_{E_6})$ equals the central product $E^{3c}_{E_6} \circ_{\langle \overline{a}\rangle} \mathrm{SL}_3(\mathbf{C})$ and we obtain the claims about $E^{3c}_{E_6}$.

Finally consider the subgroup $E^{2b}_{E_6} = \langle \overline{y_1}, \overline{x_1}\rangle$. If $g \in \pi^{-1}(C_{E_6(\mathbf{C})}(E^{2b}_{E_6}))$ then $[g, y_1], [g, x_2] \in Z$, and since $[y_1, x_2] = z$ it follows that $g \in \pi^{-1}(E^{2b}_{E_6}) \circ_Z C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6}))$. Thus we have $C_{E_6(\mathbf{C})}(E^{2b}_{E_6}) = E^{2b}_{E_6} \times \overline{C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6}))}$.

A direct computation shows that $C_{3E_6(\mathbf{C})}(x_1)$ has type $T_2 D_4$ and a system of simple roots of the centralizer is given by $\{\alpha_1 + \alpha_3 + \alpha_4, \alpha_2, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5\}$. From this we see that the 2-dimensional torus $\mathbf{T}_2$ consists of the elements $h(\alpha, 1, \gamma, 1, \alpha, \gamma)$ where $\alpha, \gamma \in \mathbf{C}^\times$. Moreover we see that $C_{3E_6(\mathbf{C})}(x_1) = \mathbf{T}_2 \circ_C \mathrm{Spin}(8, \mathbf{C})$, where $C = Z(\mathrm{Spin}(8, \mathbf{C})) = C_2 \times C_2$ consists of the elements $h(\alpha, 1, \gamma, 1, \alpha, \gamma)$, $\alpha, \gamma = \pm 1$.

Let σ denote the automorphism of $C_{3E_6(\mathbf{C})}(x_1)$ given by conjugation with y_1. A direct check shows that the map from C to C given by $x \mapsto x^{-1}x^\sigma$ is surjective. It then follows that

$$C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6})) = (\mathbf{T}_2 \circ_C \mathrm{Spin}(8, \mathbf{C}))^\sigma = \mathbf{T}_2^\sigma \circ_{C^\sigma} \mathrm{Spin}(8, \mathbf{C})^\sigma.$$

We have $\mathbf{T}_2^\sigma = \langle z\rangle$, so $C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6})) = \langle z\rangle \times \mathrm{Spin}(8, \mathbf{C})^\sigma$. Using the class distribution of $\pi^{-1}(E^{2b}_{E_6})$ found above together with [37, Table 2] and Theorem 8.2(6) we find

$$\dim C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6})) = \frac{1}{3^3} \cdot \left(3 \cdot 78 + 24 \cdot (30 + 24\omega + 24\omega^2)\right) = 14.$$

Thus $\mathrm{Spin}(8, \mathbf{C})^\sigma$ has dimension 14 and since $Z(\mathrm{Spin}(8, \mathbf{C}))^\sigma = 1$ we also see that $\mathrm{Spin}(8, \mathbf{C})^\sigma$ has rank less than 4. From this it follows that the identity component of $\mathrm{Spin}(8, \mathbf{C})^\sigma$ must have type G_2. By [131, Thm. 8.1], $\mathrm{Spin}(8, \mathbf{C})^\sigma$ is connected, so $\mathrm{Spin}(8, \mathbf{C})^\sigma = G_2(\mathbf{C})$ and hence $C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6})) = \langle z\rangle \times G_2(\mathbf{C})$. Combining this with the computation from above we conclude $C_{E_6(\mathbf{C})}(E^{2b}_{E_6}) = E^{2b}_{E_6} \times G_2(\mathbf{C})$. □

8.4. *The group $E_8(\mathbf{C})$, $p = 3$.* In this section we consider the elementary abelian 3-subgroups of the group $E_8(\mathbf{C})$. By using [19, Table 2, p. 214] we see that the smallest faithful representation of $E_8(\mathbf{C})$ is the adjoint representation, i.e., the representation given by the action of $E_8(\mathbf{C})$ on its Lie algebra $\mathfrak{e}_8$, which has dimension 248. For our computations, we explicitly construct this representation on a computer by following the recipe in [27, Ch. 4]. As explained in [27, Ch. 4] there is some ambiguity in choosing a Chevalley basis of $\mathfrak{e}_8$; we return to this problem below.

Letting $\Phi(E_8)$ denote the root system of type E_8 (we use the notation of [18, Planche VII]), we have in particular a maximal torus H generated by the elements $h_{\alpha_i}(t)$, $1 \le i \le 8$, $t \in \mathbf{C}^\times$ ([27, p. 92, p. 97]) and root subgroups $u_\alpha(t)$, $\alpha \in \Phi(E_8)$, $t \in \mathbf{C}$. The normalizer $N(H)$ of the maximal torus, is generated by H and the elements $n_i = n_{\alpha_i}(1)$, $1 \le i \le 8$ ([27, p. 93, p. 101]). We let

$$h(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8) = \prod_{i=1}^{8} h_{\alpha_i}(t_i).$$

Note that by [27, p. 100 and Lem. 6.4.4] the root subgroups u_α form a *realization* ([129, p. 133]) of $\Phi(E_8)$ in $E_8(\mathbf{C})$. In particular we have the following relations: The element n_i has image s_{α_i} in $W = W(E_8)$ ([129, Lem. 8.1.4(i)]), $n_i^2 = h_{\alpha_i}(-1)$ ([129, Lem. 8.1.4(ii)]) and

$$n_i n_j n_i \ldots = n_j n_i n_j \ldots$$

for $1 \le i, j \le 8$, where the number of factors on both sides equals the order of $s_{\alpha_i} s_{\alpha_j}$ in W ([129, Prop. 9.3.2]).

Now let $\overline{a} = h_{\alpha_1}(\omega) h_{\alpha_2}(\omega) h_{\alpha_3}(\omega^2) \in E_8(\mathbf{C})$. Direct computation shows that for any root $\alpha \in \Phi(E_8)$ we have $\alpha(\overline{a}) = \omega^{2(\lambda_2,\alpha)}$. From this we see that the Dynkin diagram of the centralizer $C_{E_8(\mathbf{C})}(\overline{a})$ is the same as the extended Dynkin diagram of E_8 with the node α_2 removed. Thus it has type A_8 and a simple system of roots is given by

$$\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, -\widetilde{\alpha}\},$$

where $\widetilde{\alpha}$ is the longest root. As in [18, Planche I] we identify $\Phi(A_8)$ with the set of elements in $\mathbf{R}^9$ of the form $e_i - e_j$ with $i \neq j$ and $1 \le i, j \le 9$, where e_i denotes the ith canonical basis vector in $\mathbf{R}^9$. We now consider $\mathrm{SL}_9(\mathbf{C})$, which is the simply connected group of type A_8 over $\mathbf{C}$. Given a root $\alpha' = e_i - e_j \in \Phi(A_8)$ we let $u'_{\alpha'}(t) = I_9 + t e_{i,j}$ for $t \in \mathbf{C}$. With respect to the maximal torus consisting of diagonal matrices, this is a root subgroup of $\mathrm{SL}_9(\mathbf{C})$ corresponding to the root α'. The roots $\alpha'_i = e_i - e_{i+1}$, $1 \le i \le 8$, form a simple system in $\Phi(A_8)$. It now follows that we can choose the Chevalley basis of $\mathfrak{e}_8$ in such a way that

$$\begin{aligned}
&u'_{\pm\alpha'_1}(t) \mapsto u_{\pm\alpha_1}(t), && u'_{\pm\alpha'_2}(t) \mapsto u_{\pm\alpha_3}(t), && u'_{\pm\alpha'_3}(t) \mapsto u_{\pm\alpha_4}(t),\\
&u'_{\pm\alpha'_4}(t) \mapsto u_{\pm\alpha_5}(t), && u'_{\pm\alpha'_5}(t) \mapsto u_{\pm\alpha_6}(t), && u'_{\pm\alpha'_6}(t) \mapsto u_{\pm\alpha_7}(t),\\
&u'_{\pm\alpha'_7}(t) \mapsto u_{\pm\alpha_8}(t), && u'_{\pm\alpha'_8}(t) \mapsto u_{\mp\widetilde{\alpha}}(t)
\end{aligned}$$

defines a homomorphism $\mathrm{SL}_9(\mathbf{C}) \to E_8(\mathbf{C})$ onto the centralizer $C_{E_8(\mathbf{C})}(\overline{a})$, and we fix a certain such choice. It is easy to check that this homomorphism has kernel $C_3 = \langle \omega I_9 \rangle$ and thus we may make the identification $C_{E_8(\mathbf{C})}(\overline{a}) = \mathrm{SL}_9(\mathbf{C})/C_3$. For any $g \in \mathrm{SL}_9(\mathbf{C})$ we denote by $\overline{g}$ its image in $\mathrm{SL}_9(\mathbf{C})/C_3 = C_{E_8(\mathbf{C})}(\overline{a}) \subseteq E_8(\mathbf{C})$. In particular we see that $a = \eta I_9$ corresponds to the element $\overline{a}$ from above. Define the following elements in $\mathrm{SL}_9(\mathbf{C})$:

$$x_1 = \operatorname{diag}(1,\omega,\omega^2,1,\omega,\omega^2,1,\omega,\omega^2), \qquad x_2 = \operatorname{diag}(1,1,1,\omega,\omega,\omega,\omega^2,\omega^2,\omega^2),$$
$$x_3 = \operatorname{diag}(1,1,1,1,1,1,\omega,\omega,\omega), \qquad y_1 = (1,2,3)(4,5,6)(7,8,9),$$
$$y_2 = (1,4,7)(2,5,8)(3,6,9).$$

From the explicit homomorphism above we easily find

$$\overline{a} = h_{\alpha_1}(\omega)h_{\alpha_2}(\omega)h_{\alpha_3}(\omega^2), \qquad \overline{x_1} = h_{\alpha_1}(\omega)h_{\alpha_5}(\omega)h_{\alpha_8}(\omega),$$
$$\overline{x_2} = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega), \qquad \overline{x_3} = h_{\alpha_1}(\omega^2)h_{\alpha_3}(\omega)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega).$$

With our particular choice of Chevalley basis a direct computation in $E_8(\mathbf{C})$ shows that

$$\begin{aligned} n_{-\widetilde{\alpha}} = {} & n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8 \\ & \cdot n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8. \end{aligned}$$

(A different choice of Chevalley basis may effect this expression by an order two element in H. If a Chevalley basis is chosen such that the above formula holds then all further formulas will be independent of the choice.)

From this and the explicit homomorphism above we find, either by direct computation or by using the relations in $N(H)$, that

$$\begin{aligned} \overline{y_1} = {} & n_1n_3n_5n_6n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3n_1n_7n_6n_5 \\ & \cdot n_4n_2n_3n_4n_5n_6n_7n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5n_4n_3 \\ & \cdot n_1n_7n_6n_5n_4n_2n_3n_4n_5n_6n_7n_8 \cdot h_{\alpha_1}(-1)h_{\alpha_2}(-1)h_{\alpha_7}(-1), \end{aligned}$$

$$\begin{aligned} \overline{y_2} = {} & n_2n_3n_1n_4n_2n_3n_4n_5n_4n_2n_3n_4n_6n_5n_4n_2n_3n_1n_4n_7n_6n_5n_4 \\ & \cdot n_2n_3n_1n_4n_3n_5n_6n_7n_8n_7n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_2n_6n_5 \\ & \cdot n_4n_3n_1n_7n_6n_5n_4n_2n_3n_4n_5n_8n_7n_6 \cdot h_{\alpha_2}(-1)h_{\alpha_5}(-1). \end{aligned}$$

Notation 8.11. To distinguish subgroups of $E_8(\mathbf{C})$, we need some information on the conjugacy classes of elements of order 3. These are given in [70, Table VI] (which is taken from [36, Table 4]): There are four such conjugacy classes, which we label **3A**, **3B**, **3C** and **3D**. Moreover these classes may be distinguished by their traces on $\mathfrak{e}_8$. Since the trace of the element $h \in H$ is given by $8 + \sum_{\alpha\in\Phi(E_8)} \alpha(h)$ we get $\overline{a} \in$ **3A**, $\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{y_1}, \overline{y_2} \in$ **3B** and $\overline{x_3 a^{-1}} \in$ **3D**.

Notation 8.12. If K is a field and n is a natural number, we define *the group of symplectic similitudes* as $\mathrm{GSp}_{2n}(K) = \{X \in \mathrm{GL}_{2n}(K) | X^tBX = cB, c \in K^\times\}$, where

$$B = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{n \text{ times}}.$$

We define the homomorphism $\chi : \mathrm{GSp}_{2n}(K) \to K^\times$ by $\chi(X) = c$, where $X^tBX = cB$. The kernel of χ is *the symplectic group* $\mathrm{Sp}_{2n}(K)$. (The notation CSp is also used in the literature; cf. e.g. [91].)

THEOREM 8.13. *The conjugacy classes of nontoral elementary abelian 3-subgroups of $E_8(\mathbf{C})$ are given by the following table*:

rank	*name*	*ordered basis*	$E_8(\mathbf{C})$*-class dist.*	$C_{E_8(\mathbf{C})}(E)$	$Z(C_{E_8(\mathbf{C})}(E))$
3	$E^{3a}_{E_8}$	$\{\overline{x_1}, \overline{y_1}, \overline{a}\}$	$\mathbf{3A^{18}3B^{8}}$	$E^{3a}_{E_8} \times \mathrm{PSL}_3(\mathbf{C})$	$E^{3a}_{E_8}$
3	$E^{3b}_{E_8}$	$\{\overline{x_1}, \overline{y_1}, \overline{x_3}\}$	$\mathbf{3B^{26}}$	$E^{3b}_{E_8} \times G_2(\mathbf{C})$	$E^{3b}_{E_8}$
4	$E^{4a}_{E_8}$	$\{\overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3a^{-1}}\}$	$\mathbf{3A^{52}3B^{26}3D^{2}}$	$E^{4a}_{E_8} \circ_{\langle \overline{x_3a^{-1}} \rangle} \mathrm{GL}_2(\mathbf{C})$	$E^{4a}_{E_8} \circ_{\langle \overline{x_3a^{-1}} \rangle} \mathbf{T}_1$
4	$E^{4b}_{E_8}$	$\{\overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a}\}$	$\mathbf{3A^{54}3B^{26}}$	$E^{4b}_{E_8} \circ_{\langle \overline{x_2} \rangle} (\mathbf{T}_2 : \langle \overline{y_2} \rangle)$	$E^{4b}_{E_8}$
4	$E^{4c}_{E_8}$	$\{\overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3}\}$	$\mathbf{3B^{80}}$	$E^{4c}_{E_8} \circ_{\langle \overline{x_2} \rangle} \mathrm{SL}_3(\mathbf{C})$	$E^{4c}_{E_8}$
5	$E^{5a}_{E_8}$	$\{\overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3a^{-1}}\}$	$\mathbf{3A^{156}3B^{80}3D^{6}}$	$E^{5a}_{E_8} \circ_{\langle \overline{x_2}, \overline{x_3a^{-1}} \rangle} \mathbf{T}_2$	$E^{5a}_{E_8} \circ_{\langle \overline{x_2}, \overline{x_3a^{-1}} \rangle} \mathbf{T}_2$
5	$E^{5b}_{E_8}$	$\{\overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2}, \overline{a}\}$	$\mathbf{3A^{162}3B^{80}}$	$E^{5b}_{E_8}$	$E^{5b}_{E_8}$

In particular ${}_3Z(C_{E_8(\mathbf{C})}(E)) = E$ *for all nontoral elementary abelian* 3*-subgroups* E *of* $E_8(\mathbf{C})$. *The Weyl groups of these subgroups with respect to the given ordered bases are as follows*:

$$W(E^{3a}_{E_8}) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbf{F}_3)} & * \\ & & * \\ \hline 0 & 0 & \det \end{array}\right],$$

$$W(E^{3b}_{E_8}) = \mathrm{SL}_3(\mathbf{F}_3), \quad W(E^{4a}_{E_8}) = \mathrm{SL}_3(\mathbf{F}_3) \times \mathrm{GL}_1(\mathbf{F}_3),$$

$$W(E^{4b}_{E_8}) = \left[\begin{array}{c|cc|c} \varepsilon & * & * & * \\ \hline 0 & \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbf{F}_3)} & * \\ 0 & & & * \\ \hline 0 & 0 & 0 & \det \end{array}\right], \quad W(E^{4c}_{E_8}) = \left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right],$$

$$W(E^{5a}_{E_8}) = \left[\begin{array}{c|ccc|c} \varepsilon_1 & * & * & * & * \\ \hline 0 & & & & 0 \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & & 0 \\ 0 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & \varepsilon_2 \end{array}\right], \quad W(E^{5b}_{E_8}) = \left[\begin{array}{cccc|c} & & & & * \\ & \mathrm{GSp}_4(\mathbf{F}_3) & & & * \\ & & & & * \\ & & & & * \\ \hline 0 & 0 & 0 & 0 & \chi \end{array}\right],$$

where det *is the determinant of the matrix from* $\mathrm{GL}_2(\mathbf{F}_3)$ *in the description of* $W(E^{3a}_{E_8})$ *and* $W(E^{4b}_{E_8})$. *In the description of* $W(E^{5b}_{E_8})$, χ *denotes the value of the homomorphism* $\chi : \mathrm{GSp}_4(\mathbf{F}_3) \to \mathbf{F}_3^\times$ *defined in* 8.12 *evaluated on the matrix from* $\mathrm{GSp}_4(\mathbf{F}_3)$.

Remark 8.14. Our information on the subgroups $E^{5a}_{E_8}$ and $E^{3b}_{E_8}$ corrects [70, Table II and Lem. 11.5] and [70, 13.2] respectively.

Proof of Theorem 8.13. *Maximal nontoral subgroups.* By [70, Lems. 11.7 and 11.9], any maximal nontoral elementary abelian 3-subgroup of $E_8(\mathbf{C})$ contains an element of type **3A**. We may thus find representatives inside $C_{E_8(\mathbf{C})}(\overline{a}) = \mathrm{SL}_9(\mathbf{C})/C_3$. From [70, Cor. 11.10], it follows that there are two conjugacy classes of maximal nontoral elementary abelian 3-subgroups both of rank 5. Moreover, by [70, Lem. 11.5], their preimages in $\mathrm{SL}_9(\mathbf{C})$ may be chosen to have the form $(3^{1+2}_+ \circ_{C_3} C_9) \times C_3 \times C_3$ and $3^{1+4}_+ \circ_{C_3} C_9$. Using the representation theory of extra special p-groups (cf. [69, Ch. 5.5] or [80, Satz V.16.14]) we find that they are represented by $E^{5a}_{E_8} = \left\langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3 a^{-1}} \right\rangle$ and $E^{5b}_{E_8} = \langle \overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2}, \overline{a} \rangle$.

Lower bounds for Weyl groups of maximal nontoral subgroups. We can find lower bounds for the Weyl groups of $E^{5a}_{E_8}$ and $E^{5b}_{E_8}$ by conjugating with elements in the centralizer $C_{E_8(\mathbf{C})}(\overline{a}) = \mathrm{SL}_9(\mathbf{C})/C_3$ and the normalizer $N(H)$ of the maximal torus. Note that

$$a = \eta I_3 \oplus \eta I_3 \oplus \eta I_3, \qquad x_1 = \beta \oplus \beta \oplus \beta, \qquad x_2 = I_3 \oplus \omega I_3 \oplus \omega^2 I_3,$$
$$x_3 = I_3 \oplus I_3 \oplus \omega I_3, \qquad y_1 = \gamma \oplus \gamma \oplus \gamma$$

and $(A \oplus B \oplus C)^{y_2} = B \oplus C \oplus A$. Conjugation by $\tau_1 \oplus \tau_1 \oplus \tau_1$, $\tau_2 \oplus \tau_2 \oplus \tau_2$ and $I_3 \oplus \beta^2 \oplus \beta$ gives

(8.1)
$$\tau_1 \oplus \tau_1 \oplus \tau_1 : a \mapsto a, \quad x_1 \mapsto x_1 y_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2.$$
(8.2)
$$\tau_2 \oplus \tau_2 \oplus \tau_2 : a \mapsto a, \quad x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto x_1 y_1, \quad y_2 \mapsto y_2.$$
(8.3)
$$I_3 \oplus \beta^2 \oplus \beta : a \mapsto a, \quad x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_1 \mapsto x_2 y_1, \quad y_2 \mapsto x_1 y_2.$$

Now consider the subgroup $E^{5a}_{E_8}$. From (8.1)–(8.3) we see that the elements $\overline{\tau_1 \oplus \tau_1 \oplus \tau_1}$, $\overline{\tau_2 \oplus \tau_2 \oplus \tau_2}$ and $\overline{I_3 \oplus \beta^2 \oplus \beta}$ normalize $E^{5a}_{E_8}$ and that conjugation by these elements induces the automorphisms on $E^{5a}_{E_8}$ given by the matrices $I_5 + e_{3,2}$, $I_5 + e_{2,3}$ and $I_5 + e_{1,3}$.

Letting $\sigma = -(1,4)(2,5)(3,6) \in \mathrm{SL}_9(\mathbf{C})$ we have $(A \oplus B \oplus C)^{\sigma} = B \oplus A \oplus C$. Using this and the above we obtain that $\overline{\sigma}$, $\overline{y_2}$ and $\overline{I_3 \oplus I_3 \oplus \beta^2}$ normalize $E^{5a}_{E_8}$ and that conjugation by these elements induces the automorphisms on $E^{5a}_{E_8}$ given by the matrices $\mathrm{diag}(2,1,1,1,1)$, $I_5 + e_{1,4} + e_{1,5}$ and $I_5 + e_{4,3}$. By using the relations in $N(H)$ given above or by direct computation, it may be checked that conjugation by the element

$$n_1 n_2 n_4 n_2 n_3 n_5 n_4 n_2 n_3 n_1 n_4 n_3 n_5 n_4 n_6 n_5 n_4 n_2 n_3 n_4 n_7 n_6 n_5 n_4 n_8 n_7 n_6$$
$$\cdot\, n_5 n_4 n_2 n_3 n_1 n_4 n_3 n_5 n_4 n_2 n_6 n_5 n_4 n_7 \cdot h(1,1,-1,-1,-1,1,-1,-1)$$

induces the automorphism on $E^{5a}_{E_8}$ represented by the matrix $\mathrm{diag}(1,1,1,1,2)+e_{2,4}$. It is easy to see that the above matrices generate the group

$$W'(E^{5a}_{E_8}) = \left[\begin{array}{c|ccc|c} \varepsilon_1 & * & * & * & * \\ \hline 0 & & & & 0 \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & & 0 \\ 0 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & \varepsilon_2 \end{array}\right]$$

and thus $W(E^{5a}_{E_8})$ contains this group.

Next consider the subgroup $E^{5b}_{E_8}$. From (8.1)–(8.3) we see that the elements $\overline{\tau_1 \oplus \tau_1 \oplus \tau_1}$, $\overline{\tau_2 \oplus \tau_2 \oplus \tau_2}$ and $\overline{I_3 \oplus \beta^2 \oplus \beta}$ normalize $E^{5b}_{E_8}$ and that conjugation by these elements induces the automorphisms of $E^{5b}_{E_8}$ given by the matrices $I_5 + e_{2,1}$, $I_5 + e_{1,2}$ and $I_5 + e_{1,4} + e_{3,2}$. Now note that $a = \Delta_{3,3}(\eta I_3)$, $x_2 = \Delta_{3,3}(\beta)$ and $y_2 = \Delta_{3,3}(\gamma)$. Since $\Delta_{3,3}(A)$ commutes with $B \oplus B \oplus B$ for any $A, B \in M_3(\mathbf{C})$, the elements $\overline{\Delta_{3,3}(\tau_1)}$ and $\overline{\Delta_{3,3}(\tau_2)}$ normalize $E^{5b}_{E_8}$. The automorphisms induced on $E^{5b}_{E_8}$ by conjugation with these elements have the matrices $I_5 + e_{4,3}$ and $I_5 + e_{3,4}$. By using the relations in $N(H)$ given above or by direct computation, we get that conjugation by the element

$$\begin{gathered} n_2 n_8 n_7 n_6 n_5 n_4 n_2 n_3 n_1 n_4 n_3 n_5 n_4 n_2 n_6 n_5 n_4 n_3 n_1 n_7 n_6 \\ \cdot\, n_5 n_4 n_2 n_3 n_4 n_5 n_6 n_7 n_8 \cdot h(1,-1,-1,-1,-1,1,1,1) \end{gathered}$$

induces the automorphism on $E^{5b}_{E_8}$ represented by the matrix $\mathrm{diag}(1,2,1,2,2)+e_{3,5}$. It now follows that $W(E^{5b}_{E_8})$ contains the group

$$W'(E^{5b}_{E_8}) = \left[\begin{array}{cccc|c} & & & & * \\ & \mathrm{GSp}_4(\mathbf{F}_3) & & & * \\ & & & & * \\ & & & & * \\ \hline 0 & 0 & 0 & 0 & \chi \end{array}\right].$$

Lower bounds for other Weyl groups. We now show that the other Weyl groups in the theorem are all lower bounds. To do this consider one of the nonmaximal subgroups E from the table. We then have $E \subseteq E^{5a}_{E_8}$, and we get a lower bound on $W(E)$ by considering the action on E of the subgroup of $W'(E^{5a}_{E_8})$ stabilizing E. As an example we find that the stabilizer of $E^{3a}_{E_8}$ inside $W'(E^{5a}_{E_8})$ is

$$\begin{bmatrix} \varepsilon_1 & 0 & 0 & x & x \\ 0 & a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{21} & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & \det & 0 \\ 0 & 0 & 0 & 0 & \det \end{bmatrix}$$

where $\det = a_{11} \cdot a_{22} - a_{12} \cdot a_{21} \neq 0$. The action of such a matrix on $E^{3a}_{E_8}$ is given by

$$\overline{x_1} \mapsto (\overline{x_1})^{a_{11}} (\overline{y_1})^{a_{21}}, \qquad \overline{y_1} \mapsto (\overline{x_1})^{a_{12}} (\overline{y_1})^{a_{22}}, \qquad \overline{a} \mapsto (\overline{x_1})^{a_{13}} (\overline{y_1})^{a_{23}} (\overline{a})^{\det}.$$

Thus $W(E^{3a}_{E_8})$ contains the group

$$W'(E^{3a}_{E_8}) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbf{F}_3)$}} & * \\ & & * \\ \hline 0 & 0 & \det \end{array}\right]$$

as claimed. Similar computations show that for the remaining subgroups $E = E^{3b}_{E_8}$, $E^{4a}_{E_8}$, $E^{4b}_{E_8}$ and $E^{4c}_{E_8}$, the group $W'(E)$ occurring in the theorem is a lower bound for the Weyl group $W(E)$.

Orbit computation. Note first that all elementary abelian 3-subgroups of rank at most two are toral by Theorem 8.2(3). By using the lower bounds on the Weyl groups of $E^{5a}_{E_8}$ and $E^{5b}_{E_8}$ established above, we may find a set of representatives for the conjugacy classes of subgroups of $E^{5a}_{E_8}$ and $E^{5b}_{E_8}$ of ranks three and four.

Under the action of $W'(E^{5a}_{E_8})$, the set of rank 3 subgroups of $E^{5a}_{E_8}$ has orbit representatives

$$E^{3a}_{E_8} = \langle \overline{x_1}, \overline{y_1}, \overline{a} \rangle, E^{3b}_{E_8} = \langle \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle, \langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle,$$
$$\langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_2}, \overline{x_3} \rangle,$$

and under the action of $W'(E^{5b}_{E_8})$, the set of rank 3 subgroups of $E^{5b}_{E_8}$ has orbit representatives

$$E^{3a}_{E_8} = \langle \overline{x_1}, \overline{y_1}, \overline{a} \rangle, \langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle \text{ and } \langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle.$$

Similarly we find that under the action of $W'(E^{5a}_{E_8})$, the set of rank 4 subgroups of $E^{5a}_{E_8}$ has orbit representatives

$$E^{4a}_{E_8} = \left\langle \overline{x_1}, \overline{y_1}, \overline{x_3}, \overline{x_3 a^{-1}} \right\rangle, E^{4b}_{E_8} = \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a} \rangle,$$
$$E^{4c}_{E_8} = \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle,$$

and that under the action of $W'(E^{5b}_{E_8})$, the set of rank 4 subgroups of $E^{5b}_{E_8}$ has orbit representatives

$$E^{4b}_{E_8} = \langle \overline{x_2}, \overline{x_1}, \overline{y_1}, \overline{a} \rangle \text{ and } E_0 = \langle \overline{x_1}, \overline{x_2}, \overline{y_1}, \overline{y_2} \rangle.$$

Class distributions. Recall that by 8.11, $\overline{a}$ is in the conjugacy class **3A**, $\overline{x_1}$ and $\overline{x_2}$ are in the class **3B** and $\overline{x_3 a^{-1}}$ belongs to the class **3D**. Using the actions of $W'(E^{5a}_{E_8})$ and $W'(E^{5b}_{E_8})$ it is then straightforward to verify the class distributions given in the table. As an example consider the subgroup $E^{5a}_{E_8}$. Under the action of $W'(E^{5a}_{E_8})$ it contains 156 elements conjugate to $\overline{a}$, 78 elements conjugate to $\overline{x_1}$, two elements conjugate to $\overline{x_2}$ and six elements conjugate to $\overline{x_3 a^{-1}}$, which gives the class distribution in the table. Similar computations give the results for the remaining subgroups.

We also see that the subgroup $E_0 = \langle \overline{x_1}, \overline{x_2}, \overline{y_1}, \overline{y_2} \rangle$ has class distribution $\mathbf{3B^{80}}$ and from the class distribution of $E^{5b}_{E_8}$ we get $E_0 = (E^{5b}_{E_8} \cap \mathbf{3B}) \cup \{1\}$. It then follows from [70, Lem. 11.5] that E_0 is toral.

Other nontoral subgroups. We see directly that the subgroups

$$\langle \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_2} \rangle, \langle \overline{a}, \overline{x_1}, \overline{x_3} \rangle \text{ and } \langle \overline{a}, \overline{x_2}, \overline{x_3} \rangle$$

are toral. Since the subgroup $\langle \overline{x_1}, \overline{x_2}, \overline{y_1} \rangle$ is a subgroup of E_0 it is also toral. Alternatively, from the action of $W'(E^{5a}_{E_8})$ we see that it is conjugate to the subgroup $\langle \overline{x_1}, \overline{x_2}, \overline{x_3} \rangle$, which is visibly toral. Thus any nontoral elementary abelian 3-subgroup of $E_8(\mathbf{C})$ is conjugate to a subgroup in the table. Moreover, since their class distributions differ, none of these subgroups are conjugate.

To see that the subgroups in the table are nontoral we may proceed as follows. The subgroup $E^{3a}_{E_8}$ contains the element $\overline{a}$; so by Theorem 8.2(1) it is toral in $E_8(\mathbf{C})$ if and only if it is toral in $C_{E_8(\mathbf{C})}(\overline{a}) = \mathrm{SL}_9(\mathbf{C})/C_3$. However this is not the case by Theorem 8.2(5), since its preimage in $\mathrm{SL}_9(\mathbf{C})$ is non-abelian. The subgroups $E^{4a}_{E_8}$ and $E^{4b}_{E_8}$ are thus also nontoral since they contain $E^{3a}_{E_8}$. We saw above that the Weyl group of $E^{3b}_{E_8}$ contains $\mathrm{SL}_3(\mathbf{F}_3)$, which has order divisible by 13. Since $13 \nmid |W(E_8)|$ it follows from Theorem 8.2(2) that $E^{3b}_{E_8}$ is nontoral. Since $E^{4c}_{E_8}$ contains $E^{3b}_{E_8}$ it is also nontoral.

Centralizers. The subgroups $E = E^{3a}_{E_8}$, $E^{4a}_{E_8}$, $E^{4b}_{E_8}$, $E^{5a}_{E_8}$ and $E^{5b}_{E_8}$ are easy to deal with since they all contain $\overline{a}$, and hence we have $C_{E_8(\mathbf{C})}(E) = C_{\mathrm{SL}_9(\mathbf{C})/C_3}(E)$ for these. It is however notationally convenient first to change the representatives as follows. Define $x_4 = \tau_2^{-1} \oplus \tau_2^{-1} \oplus \tau_2^{-1} \in \mathrm{SL}_9(\mathbf{C})$, and note that conjugation by $(2,7,3,4)(5,8,9,6) \in \mathrm{SL}_9(\mathbf{C})$ acts as follows:

$$a \mapsto a, \qquad x_1 \mapsto x_2, \qquad x_2 \mapsto x_1^2, \qquad x_3 a^{-1} \mapsto x_4, \qquad y_1 \mapsto y_2, \qquad y_2 \mapsto y_1^2.$$

In particular we see that $E^{3a}_{E_8}$ is conjugate to $\langle \overline{x_2}, \overline{y_2}, \overline{a} \rangle$. Moreover we have

$$C_{E_8(\mathbf{C})}(\overline{a}, \overline{x_2}) = \overline{\langle y_2, \{A \oplus B \oplus C \mid \det ABC = 1\} \rangle}.$$

From this we directly get

$$\begin{aligned} C_{E_8(\mathbf{C})}(\overline{a}, \overline{x_2}, \overline{y_2}) &= \overline{\left\langle x_2, y_2, \{A \oplus A \oplus A \mid (\det A)^3 = 1\} \right\rangle} \\ &= \overline{\langle x_2, y_2, a, \{A \oplus A \oplus A \mid \det A = 1\} \rangle} \\ &\cong \langle \overline{x_2}, \overline{y_2}, \overline{a} \rangle \times \mathrm{PSL}_3(\mathbf{C}). \end{aligned}$$

Thus $C_{E_8(\mathbf{C})}(E^{3a}_{E_8}) = E^{3a}_{E_8} \times \mathrm{PSL}_3(\mathbf{C})$ and $Z(C_{E_8(\mathbf{C})}(E^{3a}_{E_8})) = E^{3a}_{E_8}$. From the above we see that the elements $\overline{x_2}$, $\overline{x_3 a^{-1}}$ and $\overline{y_2}$ in $C_{E_8(\mathbf{C})}(E^{3a}_{E_8})$ correspond to the elements $\overline{\beta^2}$, $\overline{\tau_2^{-1}}$ and $\overline{\gamma^2}$ in the $\mathrm{PSL}_3(\mathbf{C})$-component of $C_{E_8(\mathbf{C})}(E^{3a}_{E_8})$. From this we easily compute the structure of $C_{E_8(\mathbf{C})}(E)$ for the subgroups E which contain $E^{3a}_{E_8}$; cf. the proof of Theorem 8.10.

For the computation of the centralizers of $E^{3b}_{E_8}$ and $E^{4c}_{E_8}$ we consider the element $g = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2) \in E_8(\mathbf{C})$. By using [36, Tables 4 and 6] we see that g belongs to the conjugacy class **3B** and that the centralizer $C_{E_8(\mathbf{C})}(g)$ has type E_6A_2. The precise structure of this centralizer may be found as follows. Since $E_8(\mathbf{C})$ is simply connected, Theorem 8.2(3) implies that $C_{E_8(\mathbf{C})}(g)$ is connected. Setting

$$\alpha' = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$$

we see that $\{\alpha_5, \alpha_8, \alpha_6, \alpha_7, \alpha', \alpha_2\} \cup \{\alpha_1, \alpha_3\}$ is a system of simple roots of $C_{E_8(\mathbf{C})}(g)$. (The simple systems of the components of type E_6 and A_2 have been ordered so that the numbering is consistent with [18, Planches I and V].) From this we get an explicit homomorphism $3E_6(\mathbf{C}) \times \mathrm{SL}_3(\mathbf{C}) \to E_8(\mathbf{C})$ onto the centralizer $C_{E_8(\mathbf{C})}(g)$. The kernel is given by $\langle (z, \omega^2 I_3)\rangle$, where $z \in 3E_6(\mathbf{C})$ denotes the central element defined in Section 8.3. Thus $C_{E_8(\mathbf{C})}(g) = 3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$, and we denote elements in this central product by $A \cdot B$ where $A \in 3E_6(\mathbf{C})$ and $B \in \mathrm{SL}_3(\mathbf{C})$. In particular we have $g = z \cdot I_3 = 1 \cdot \omega I_3$.

Now consider the subgroup $E = \langle z \cdot I_3, x_1 \cdot \beta, y_1 \cdot \gamma\rangle$ which is seen to be an elementary abelian 3-subgroup of rank 3 (here the elements $x_1, y_1 \in 3E_6(\mathbf{C})$ from Section 8.3 should not be confused with the elements $x_1, y_1 \in \mathrm{SL}_9(\mathbf{C})$ from above). We have

$$\begin{aligned} C_{E_8(\mathbf{C})}(z \cdot I_3, x_1 \cdot \beta) &= C_{3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})}(x_1 \cdot \beta) \\ &= \left\langle y_1 \cdot \gamma, C_{3E_6(\mathbf{C})}(x_1) \circ_{C_3} C_{\mathrm{SL}_3(\mathbf{C})}(\beta) \right\rangle. \end{aligned}$$

We note that $y_1 \cdot \gamma$ is not conjugate to its inverse in $C_{E_8(\mathbf{C})}(z \cdot I_3, x_1 \cdot \beta)$ since no element in $C_{\mathrm{SL}_3(\mathbf{C})}(\beta)$ conjugates γ into γ^{-1} times a power of ωI_3. Thus $\mathrm{diag}(1,1,-1) \notin W(E)$ and $W(E) \neq \mathrm{GL}_3(\mathbf{F}_3)$. From the above we also get

$$\begin{aligned} C_{E_8(\mathbf{C})}(E) &= \left\langle y_1 \cdot \gamma, x_1 \cdot \beta, C_{3E_6(\mathbf{C})}(x_1, y_1) \circ_{C_3} C_{\mathrm{SL}_3(\mathbf{C})}(\beta, \gamma)\right\rangle \\ &= \left\langle y_1 \cdot \gamma, x_1 \cdot \beta, C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6})) \circ_{C_3} Z(\mathrm{SL}_3(\mathbf{C}))\right\rangle \\ &= \langle y_1 \cdot \gamma, x_1 \cdot \beta, (\langle z\rangle \times G_2(\mathbf{C})) \circ_{C_3} Z(\mathrm{SL}_3(\mathbf{C}))\rangle \\ &= E \times G_2(\mathbf{C}), \end{aligned}$$

using the computation of $C_{3E_6(\mathbf{C})}(\pi^{-1}(E^{2b}_{E_6}))$ from the last part of the proof of Theorem 8.10. Since the preimage of E in $3E_6(\mathbf{C}) \times \mathrm{SL}_3(\mathbf{C})$ is non-abelian it follows from Theorem 8.2(5) that E is nontoral in $3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$. Now Theorem 8.2(1) shows that E is nontoral in $E_8(\mathbf{C})$ (alternatively observe that $C_{E_8(\mathbf{C})}(E)$ has rank less than 8). From what we have already proved we then see that E is conjugate to either $E^{3a}_{E_8}$ or $E^{3b}_{E_8}$ in $E_8(\mathbf{C})$. Since we have already calculated $C_{E_8(\mathbf{C})}(E^{3a}_{E_8})$ we conclude that E must be conjugate to $E^{3b}_{E_8}$ (alternatively one can also compute the class distribution of E directly). In particular we have $C_{E_8(\mathbf{C})}(E^{3b}_{E_8}) = E^{3b}_{E_8} \times G_2(\mathbf{C})$ and $W(E^{3b}_{E_8}) \neq \mathrm{GL}_3(\mathbf{F}_3)$.

Using the inclusion $3E_6(\mathbf{C}) \subseteq 3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C}) \subseteq E_8(\mathbf{C})$ we may also consider the subgroup $E^4_{3E_6} \subseteq 3E_6(\mathbf{C})$ from Theorem 8.8 as a subgroup of $E_8(\mathbf{C})$. Since $E^4_{3E_6}$ is nontoral in $3E_6(\mathbf{C})$, it is also nontoral in $3E_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$, and hence also in $E_8(\mathbf{C})$ by Theorem 8.2(1). Thus $E^4_{3E_6}$ must be conjugate in $E_8(\mathbf{C})$ to one of the subgroups $E^{4a}_{E_8}$, $E^{4b}_{E_8}$ or $E^{4c}_{E_8}$. Comparing the class distributions we can rule out $E^{4a}_{E_8}$ and $E^{4b}_{E_8}$, and so $E^4_{3E_6}$ is conjugate to $E^{4c}_{E_8}$. From Theorem 8.8 we have $C_{3E_6(\mathbf{C})}(E^4_{3E_6}) = E^4_{3E_6}$. Hence $C_{E_8(\mathbf{C})}(E^4_{3E_6}) = E^4_{3E_6} \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$ and thus $C_{E_8(\mathbf{C})}(E^{4c}_{E_8}) = E^{4c}_{E_8} \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$. We determine the precise structure of the central product below after the computation of $W(E^{4c}_{E_8})$.

Exact Weyl groups. Recall from above that $E^{3a}_{E_8}$ is conjugate to $\langle \overline{x_2}, \overline{y_2}, \overline{a} \rangle$. If $W(E^{3a}_{E_8})$ was strictly larger than the group $W'(E^{3a}_{E_8})$ from above, $W(E^{3a}_{E_8})$ would have to contain one of the groups

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathrm{GL}_2(\mathbf{F}_3)$}} & * \\ & & * \\ \hline 0 & 0 & \varepsilon \end{array}\right] \quad \text{or} \quad \mathrm{SL}_3(\mathbf{F}_3)$$

since these are the minimal overgroups of $W'(E^{3a}_{E_8})$ inside $\mathrm{GL}_3(\mathbf{F}_3)$. Thus $W(E^{3a}_{E_8})$ would have to contain one of the matrices $\mathrm{diag}(1,2,1)$ or $I_3 + e_{3,2}$. This would imply the existence of an element in $C_{E_8(\mathbf{C})}(\overline{x_2}, \overline{a})$ which conjugates $\overline{y_2}$ into either $\overline{y_2}^2$ or $\overline{y_2 a}$. However we saw above that

$$C_{E_8(\mathbf{C})}(\overline{x_2}, \overline{a}) = \overline{\langle y_2, \{A \oplus B \oplus C \mid \det ABC = 1\} \rangle},$$

and from this it follows that no such element exists. Thus $W(E^{3a}_{E_8}) = W'(E^{3a}_{E_8})$ as claimed. For the subgroup $E^{3b}_{E_8}$ we have $\mathrm{SL}_3(\mathbf{F}_3) \subseteq W(E^{3b}_{E_8}) \neq \mathrm{GL}_3(\mathbf{F}_3)$ and hence $W(E^{3b}_{E_8}) = \mathrm{SL}_3(\mathbf{F}_3)$.

As in the proof of Theorem 8.10 we show that the remaining Weyl groups equal the lower bounds already established, by looking at what a strictly larger Weyl group would imply for the subgroups $E^{3a}_{E_8}$ and $E^{3b}_{E_8}$. For $E = E^{4a}_{E_8}$, $E^{4b}_{E_8}$, $E^{5a}_{E_8}$, and $E^{5b}_{E_8}$, any proper overgroup of $W(E)$ contains an element which normalizes $E^{3a}_{E_8}$ but induces an automorphism on it not contained in its Weyl group. For $E^{4c}_{E_8}$ the result follows by considering the subgroup $E^{3b}_{E_8}$.

It remains only to determine the precise structure of the central product $C_{E_8(\mathbf{C})}(E^{4c}_{E_8}) = E^{4c}_{E_8} \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$. From the structure of $W(E^{4c}_{E_8})$ we see that the subgroup $\langle \overline{x_2} \rangle$ is invariant under the action of $W(E^{4c}_{E_8})$. Thus a conjugation which sends $E^{4c}_{E_8}$ to $E^4_{3E_6}$ must send $\langle \overline{x_2} \rangle$ to a $W(E^4_{3E_6})$-invariant subgroup of $E^4_{3E_6}$ of rank one. From the structure of $W(E^4_{3E_6})$ we see that there is only one such subgroup, namely $\langle z \cdot I_3 \rangle = \langle 1 \cdot \omega I_3 \rangle$. As this is exactly the center of the $\mathrm{SL}_3(\mathbf{C})$-component of $C_{E_8(\mathbf{C})}(E^4_{3E_6}) = E^4_{3E_6} \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$, we see that $C_{E_8(\mathbf{C})}(E^{4c}_{E_8}) = E^{4c}_{E_8} \circ_{\langle \overline{x_2}\rangle} \mathrm{SL}_3(\mathbf{C})$. □

8.5. *The group* $2E_7(\mathbf{C})$, $p=3$. In this section we consider the elementary abelian 3-subgroups of $2E_7(\mathbf{C})$. We let H be a maximal torus of $2E_7(\mathbf{C})$, $\Phi(E_7)$ be the root system relative to H, and choose a *realization* ([129, p. 133]) $(u_\alpha)_{\alpha\in\Phi(E_7)}$ of $\Phi(E_7)$ in $2E_7(\mathbf{C})$. By [129, Lem. 8.1.4(iv)] we may suppose that the root subgroups $(u'_\alpha)_{\alpha\in\Phi(E_6)}$ in $3E_6(\mathbf{C}) \subseteq 2E_7(\mathbf{C})$ coming from the choice of root subgroups for $3E_6(\mathbf{C})$ from Section 8.3 satisfy $u_\alpha = u'_\alpha$ for $\alpha \in \Phi(E_6)$. For $\alpha = \alpha_i$, $1 \le i \le 7$, and $t \in \mathbf{C}^\times$ we define the elements

$$n_\alpha(t) = u_\alpha(t)u_{-\alpha}(-1/t)u_\alpha(t), \qquad h_\alpha(t) = n_\alpha(t)n_\alpha(1)^{-1}.$$

Then the maximal torus consists of the elements $\prod_{i=1}^7 h_{\alpha_i}(t_i)$ and the normalizer $N(H)$ of the maximal torus is generated by H and the elements $n_i = n_{\alpha_i}(1)$, $1 \le i \le 7$. As in Section 8.3 we define the following elements in $3E_6(\mathbf{C}) \subseteq 2E_7(\mathbf{C})$:

$$\begin{aligned} z &= h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega)h_{\alpha_6}(\omega^2), \qquad a = h_{\alpha_1}(\omega)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2)h_{\alpha_6}(\omega),\\ x_2 &= h_{\alpha_2}(\omega^2)h_{\alpha_3}(\omega^2)h_{\alpha_5}(\omega^2),\\ y_2 &= n_1n_2n_3n_4n_3n_1n_5n_4n_2n_3n_4n_5n_6n_5n_4n_2n_3n_1n_4n_3n_5n_4n_6n_5 \cdot h_{\alpha_2}(-1). \end{aligned}$$

Notation 8.15. The conjugacy classes of elements of order 3 in $2E_7(\mathbf{C})$ are given in [70, Table VI] and [36, Table 6] from which we take our notation. There are five such conjugacy classes, which we label **3A**, **3B**, **3C**, **3D** and **3E**. (We take this opportunity to note the following corrections to these references: In [70, Table VI] the eigenvalue multiplicities for the class **3C** in $2E_7(\mathbf{C})$ should be 43, 45, 45; for the class **3E** they should be 67, 33, 33; the centralizer type of the class **3E** should be D_6T_1; in [36, Table 6] the centralizer types for the classes **3A** and **3E** should be A_6T_1 and D_6T_1 respectively.) By direct computation we easily obtain the inclusions

$$\mathbf{3C}[3E_6] \subseteq \mathbf{3C}[2E_7], \qquad \mathbf{3E}[3E_6] \subseteq \mathbf{3B}[2E_7], \qquad \mathbf{3E}'[3E_6] \subseteq \mathbf{3B}[2E_7],$$

corresponding to the inclusion $3E_6(\mathbf{C}) \subseteq 2E_7(\mathbf{C})$.

Theorem 8.16. *The conjugacy classes of nontoral elementary abelian 3-subgroups of* $2E_7(\mathbf{C})$ *are given by the following table*:

rank	*name*	*ordered basis*	$2E_7(\mathbf{C})$*-class dist.*	$C_{2E_7(\mathbf{C})}(E)$	$Z(C_{2E_7(\mathbf{C})}(E))$
3	$E^3_{2E_7}$	$\{a, x_2, y_2\}$	$\mathbf{3C^{26}}$	$E^3_{2E_7} \times \mathrm{SL}_2(\mathbf{C})$	$E^3_{2E_7} \times Z(2E_7(\mathbf{C}))$
4	$E^4_{2E_7}$	$\{z, a, x_2, y_2\}$	$\mathbf{3B^2 3C^{78}}$	$E^4_{2E_7} \circ_{\langle z\rangle} \mathbf{T}_1$	$E^4_{2E_7} \circ_{\langle z\rangle} \mathbf{T}_1$

In particular ${}_3Z(C_{2E_7(\mathbf{C})}(E)) = E$ *for all nontoral elementary abelian 3-subgroups* E *of* $2E_7(\mathbf{C})$. *The Weyl groups of these subgroups with respect to the* *given ordered bases are as follows*:

$$W(E^3_{2E_7}) = \mathrm{SL}_3(\mathbf{F}_3),\ \ W(E^4_{2E_7}) = \left[\begin{array}{c|ccc} \varepsilon & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right].$$

Remark 8.17. Our information on the rank 3 subgroup $E^3_{2E_7}$ corrects [70, Table II and Thm. 11.16].

Proof of Theorem 8.16. *Nontoral subgroups.* From the way the realization $(u_\alpha)_{\alpha\in\Phi(E_7)}$ is chosen above, it follows from Theorem 8.8 that $E^3_{2E_7}$ and $E^4_{2E_7}$ are elementary abelian 3-subgroups of $2E_7(\mathbf{C})$ and that

$$W(E^3_{2E_7}) \supseteq \mathrm{SL}_3(\mathbf{F}_3),\ \ W(E^4_{2E_7}) \supseteq \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{SL}_3(\mathbf{F}_3) & \\ 0 & & & \end{array}\right].$$

In particular both $W(E^3_{2E_7})$ and $W(E^4_{2E_7})$ have orders divisible by 13 and since $13 \nmid |W(E_7)|$, we conclude by Theorem 8.2(2) that $E^3_{2E_7}$ and $E^4_{2E_7}$ are nontoral in $2E_7(\mathbf{C})$. By [70, Thm. 11.16] there are precisely two conjugacy classes of nontoral elementary abelian 3-subgroups in $2E_7(\mathbf{C})$, and thus $E^3_{2E_7}$ and $E^4_{2E_7}$ represent these two conjugacy classes.

Class distributions. The class distributions follows directly from the class distributions of the subgroups $E^3_{3E_6}$ and $E^4_{3E_6}$ given in Theorem 8.8 and the information in 8.15 about the behavior of conjugacy classes in $3E_6(\mathbf{C})$ under the inclusion $3E_6(\mathbf{C}) \subseteq 2E_7(\mathbf{C})$.

Weyl groups. Using our realization $(u_\alpha)_{\alpha\in\Phi(E_7)}$ we define a canonical map $\phi : W \to N(H)$ as follows ([129, 9.3.3]): If $w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_r}}$ is a reduced expression for $w \in W$ we let $\phi(w) = n_{i_1} \dots n_{i_r}$ (by [129, Props. 8.3.3 and 9.3.2] this does not depend on the reduced expression for w). Note that the element $\phi(w)$ is a representative of $w \in W$ in $N(H)$. Now let $w_0 \in W$ be the longest element in W, and let $n_0 = \phi(w_0)$. From [18, Planche VI] it follows that w_0 equals the scalar transformation -1, and so conjugation by n_0 acts as inversion on H. Now let $w \in W$ and define w' by $ww' = w_0$. Since w_0 is central in our case, we have $(ww')\,w^{-1} = w^{-1}\,(ww') = w'$ so we conclude that $w'w = ww' = w_0$. Now let ℓ be the length function on W. By [79, p. 16] we have $\ell(w) + \ell(w') = \ell(w_0)$. In general the map ϕ is not a homomorphism, but we do have $\phi(w_1w_2) = \phi(w_1)\phi(w_2)$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ by [129, Ex. 9.3.4(1)]. From this it follows that $\phi(w)\phi(w') = \phi(w')\phi(w) = \phi(w_0) = n_0$, and we conclude that n_0 commutes with $\phi(w)$ for all $w \in W$.

Now consider the element

$$w = s_1 s_2 s_3 s_4 s_3 s_1 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_5 s_4 s_2 s_3 s_1 s_4 s_3 s_5 s_4 s_6 s_5.$$

Since the length of an element is given by the number of positive roots it sends to negative roots ([79, Cor. 1.7]), the above product is a reduced expression for w. Thus we have $y_2 = \phi(w) h_{\alpha_2}(-1)$. From the above we then conclude that conjugation by n_0 acts as follows:

$$z \mapsto z^2, \qquad a \mapsto a^2, \qquad x_2 \mapsto x_2^2, \qquad y_2 \mapsto y_2.$$

Thus n_0 normalizes $E^4_{2E_7}$ and gives the element $\mathrm{diag}(2,2,2,1)$ in $W(E^4_{2E_7})$. Combined with the above we conclude that

$$W(E^4_{2E_7}) \supseteq \left[\begin{array}{c|c} \varepsilon & * \;\; * \;\; * \\ \hline 0 & \\ 0 & \mathrm{SL}_3(\mathbf{F}_3) \\ 0 & \end{array}\right].$$

From the inclusion $\Phi(E_7) \subseteq \Phi(E_8)$ we get the inclusion $2E_7(\mathbf{C}) \subseteq E_8(\mathbf{C})$, and so we may consider $E^3_{2E_7}$ and $E^4_{2E_7}$ as subgroups of $E_8(\mathbf{C})$ as well. Since the orders of their Weyl groups in $2E_7(\mathbf{C})$ are divisible by 13 and $13 \nmid |W(E_8)|$, we see from Theorem 8.2(2) that $E^3_{2E_7}$ and $E^4_{2E_7}$ remain nontoral in $E_8(\mathbf{C})$. Using Theorem 8.13 and the class distributions from above we conclude that $E^3_{2E_7}$ and $E^4_{2E_7}$ are conjugate to $E^{3b}_{E_8}$ and $E^{4c}_{E_8}$ respectively in $E_8(\mathbf{C})$. Theorem 8.13 now shows that the lower bounds found above are indeed the Weyl groups of $E^3_{2E_7}$ and $E^4_{2E_7}$ in $2E_7(\mathbf{C})$.

Centralizers. For the computation of the centralizer of $E^3_{2E_7}$ we consider the element $g = h_{\alpha_1}(\omega) h_{\alpha_3}(\omega^2) \in 2E_7(\mathbf{C})$. By using [36, Table 6] we see that g belongs to the conjugacy class **3C** and that the centralizer $C_{2E_7(\mathbf{C})}(g)$ has type $A_5 A_2$. The precise structure of this centralizer may be found as follows. Since $2E_7(\mathbf{C})$ is simply connected, Theorem 8.2(3) implies that $C_{2E_7(\mathbf{C})}(g)$ is connected. Setting

$$\alpha' = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$$

we see that $\{\alpha_5, \alpha_6, \alpha_7, \alpha', \alpha_2\} \cup \{\alpha_1, \alpha_3\}$ is a system of simple roots of $C_{2E_7(\mathbf{C})}(g)$. (The simple systems of the components of type A_5 and A_2 have been ordered so that the numbering is consistent with [18, Planche I].) From this we get an explicit homomorphism $\mathrm{SL}_6(\mathbf{C}) \times \mathrm{SL}_3(\mathbf{C}) \to 2E_7(\mathbf{C})$ onto the centralizer $C_{2E_7(\mathbf{C})}(g)$. The kernel is given by $\langle(\omega I_6, \omega^2 I_3)\rangle$. Thus $C_{2E_7(\mathbf{C})}(g) = \mathrm{SL}_6(\mathbf{C}) \circ_{C_3} \mathrm{SL}_3(\mathbf{C})$, and we denote elements in this central product by $A \cdot B$ where $A \in \mathrm{SL}_6(\mathbf{C})$ and $B \in \mathrm{SL}_3(\mathbf{C})$. In particular we have $g = \omega I_6 \cdot I_3 = I_6 \cdot \omega I_3$.

Now consider the subgroup $E = \langle \omega I_6 \cdot I_3, (\beta \oplus \beta) \cdot \beta, (\gamma \oplus \gamma) \cdot \gamma^2 \rangle$ which is seen to be an elementary abelian 3-subgroup of rank 3. We have

$$\begin{aligned}C_{2E_7(\mathbf{C})}(\omega I_6\cdot I_3,(\beta\oplus\beta)\cdot\beta) &= C_{\mathrm{SL}_6(\mathbf{C})\circ_{C_3}\mathrm{SL}_3(\mathbf{C})}((\beta\oplus\beta)\cdot\beta)\\ &= \big\langle(\gamma\oplus\gamma)\cdot\gamma^2, C_{\mathrm{SL}_6(\mathbf{C})}(\beta\oplus\beta)\circ_{C_3} C_{\mathrm{SL}_3(\mathbf{C})}(\beta)\big\rangle\,.\end{aligned}$$

From this we get

$$\begin{aligned}C_{2E_7(\mathbf{C})}(E) &= \big\langle(\gamma\oplus\gamma)\cdot\gamma^2,(\beta\oplus\beta)\cdot\beta, C_{\mathrm{SL}_6(\mathbf{C})}(\beta\oplus\beta,\gamma\oplus\gamma)\circ_{C_3} C_{\mathrm{SL}_3(\mathbf{C})}(\beta,\gamma)\big\rangle\\ &= \big\langle(\gamma\oplus\gamma)\cdot\gamma^2,(\beta\oplus\beta)\cdot\beta, C_{\mathrm{SL}_6(\mathbf{C})}(\beta\oplus\beta,\gamma\oplus\gamma)\circ_{C_3} Z(\mathrm{SL}_3(\mathbf{C}))\big\rangle\,.\end{aligned}$$

Here $C_{\mathrm{SL}_6(\mathbf{C})}(\beta\oplus\beta,\gamma\oplus\gamma)=\Delta_{2,3}(\{A\in\mathrm{GL}_2(\mathbf{C})|(\det A)^3=1\})$ is generated by $\Delta_{2,3}(\omega I_2)=\omega I_6$ and $\Delta_{2,3}(\mathrm{SL}_2(\mathbf{C}))$. From this we get

$$C_{2E_7(\mathbf{C})}(E)=\langle E,\Delta_{2,3}(\mathrm{SL}_2(\mathbf{C}))\rangle\cong E\times\mathrm{SL}_2(\mathbf{C}).$$

Since the preimage of E in $\mathrm{SL}_6(\mathbf{C})\times\mathrm{SL}_3(\mathbf{C})$ is non-abelian it follows from Theorem 8.2(5) that E is nontoral in $\mathrm{SL}_6(\mathbf{C})\circ_{C_3}\mathrm{SL}_3(\mathbf{C})$. Now Theorem 8.2(1) shows that E is nontoral in $2E_7(\mathbf{C})$ (alternatively one could also just observe that $C_{2E_7(\mathbf{C})}(E)$ has rank less than 7). It follows that E is conjugate to $E^3_{2E_7}$ in $2E_7(\mathbf{C})$, and so $C_{2E_7(\mathbf{C})}(E^3_{2E_7})=E^3_{2E_7}\times\mathrm{SL}_2(\mathbf{C})$. Hence $Z(C_{2E_7(\mathbf{C})}(E^3_{2E_7}))=E^3_{2E_7}\times Z(2E_7(\mathbf{C}))$ since the center of $2E_7(\mathbf{C})$ has order 2.

To compute the centralizer of $E^4_{2E_7}$ we note that $C_{2E_7(\mathbf{C})}(z)$ has centralizer type E_6T_1, and that the E_6-component corresponds to the subgroup $3E_6(\mathbf{C})\subseteq 2E_7(\mathbf{C})$. A computation shows that the T_1-component is given by $\mathbf{T}_1=\{h(t^2,t^3,t^4,t^6,t^5,t^4,t^3)|t\in\mathbf{C}^\times\}$, and thus we get $C_{2E_7(\mathbf{C})}(z)=3E_6(\mathbf{C})\circ_{\langle z\rangle}\mathbf{T}_1$. Theorem 8.8 now shows that $C_{2E_7(\mathbf{C})}(E^4_{2E_7})=C_{3E_6(\mathbf{C})}(E^4_{2E_7})\circ_{\langle z\rangle}\mathbf{T}_1=E^4_{2E_7}\circ_{\langle z\rangle}\mathbf{T}_1$. □

9. Calculation of the obstruction groups

In this section we show that the existence and uniqueness obstructions to lifting our diagram in the homotopy category to a diagram in the category of spaces identically vanish. More precisely, we will show the following theorem.

Theorem 9.1 (Obstruction Vanishing Theorem). *Suppose that X is any of the following p-compact groups $(F_4)\hat{_3}$, $(E_6)\hat{_3}$, $(E_7)\hat{_3}$, $(E_8)\hat{_3}$, $(E_8)\hat{_5}$ or $\mathrm{PU}(n)\hat{_p}$ (any p), or suppose that p is odd and X is connected with $H^*(BX;\mathbf{Z}_p)$ a polynomial algebra. Then*

$$\lim_{\mathbf{A}(X)}{}^{i}\pi_j(B\mathcal{Z}(\mathcal{C}_X(-)))=0,\ \textit{for all } i,j.$$

(See Theorem 12.2 for an explanation of why exactly these p-compact groups need attention.) Note that for the purpose of Theorem 1.4 we only need to calculate the above groups for $j=1,2$ and $i=j$ or $i=j+1$.

We prove the theorem by filtering the functor $F_j=\pi_j(B\mathcal{Z}(\mathcal{C}_X(-)))$, and showing that all filtration quotients vanish (with a small twist for $\mathrm{PU}(2)\hat{_2}$).

First we show that the quotient functor of F_j concentrated on the toral elementary abelian p-subgroups has vanishing limits, using a Mackey functor argument which first appeared in [55]. This takes care of the case where $H^*(BX;\mathbf{Z}_p)$ is a polynomial algebra since in this case all subgroups are toral by Lemma 10.8. For the exceptional compact connected Lie groups we then continue and filter the nontoral part of the functor such that the filtration quotients are concentrated on only one nontoral subgroup, and use a formula of Oliver [113] to show that the higher limits of these subquotient functors all vanish. For $\mathrm{PU}(n)$ we use a variant of this technique by suitably grouping the nontoral subgroups and using a combination of Oliver's formula and the Mackey functor argument we used for the toral part. We divide the proof up in three subsections corresponding to the toral part, the nontoral part for the exceptional groups, and the nontoral part for the projective unitary groups.

The notation St_G denotes the Steinberg module over $\mathbf{Z}_p$ of a finite group of Lie type G of characteristic p, defined as the top homology group with $\mathbf{Z}_p$ coefficients of the Tits building of G (see e.g., [78]). In the special case $G = \mathrm{GL}(E)$ we also write $\mathrm{St}(E)$ for the Steinberg module.

9.1. *The toral part.* Define a quotient functor F_j^{tor} of F_j by setting $F_j^{\mathrm{tor}}(V) = F_j(V)$ if V is toral and $F_j^{\mathrm{tor}}(V) = 0$ if V is nontoral. Let $\mathbf{A}^{\mathrm{tor}}(X)$ denote the full subcategory of $\mathbf{A}(X)$ consisting of toral subgroups. From the chain complex defining higher limits (see e.g., [68, App. II, 3.3]), it follows that

$$\lim_{\mathbf{A}(X)}{}^{*} F_j^{\mathrm{tor}} \cong \lim_{\mathbf{A}^{\mathrm{tor}}(X)}{}^{*} F_j^{\mathrm{tor}}.$$

In order to use a Mackey functor argument on the right-hand side we need to see that the functor F_j^{tor} is indeed a Mackey functor. This is accomplished by the following lemma, whose assumption are always satisfied for p odd by [57, Thm. 7.5].

LEMMA 9.2. *Fix a connected p-compact group X and let $\breve{T}$ be the discrete approximation to a maximal torus T in X. Let $V \subseteq \breve{T}$ be a nontrivial elementary abelian p-subgroup.*

If $\breve{T}^{W_{\mathcal{C}_X(V)_1}}$ is a discrete approximation to $\mathcal{Z}(\mathcal{C}_X(V)_1)$ then $\breve{T}^{W_{\mathcal{C}_X(V)}}$ is a discrete approximation to $\mathcal{Z}(\mathcal{C}_X(V))$. In particular in this case $\pi_1(B\mathcal{Z}(\mathcal{C}_X(V)))$ $= H^1(W_{\mathcal{C}_X(V)}; L_X)$ and $\pi_2(B\mathcal{Z}(\mathcal{C}_X(V))) = (L_X)^{W_{\mathcal{C}_X(V)}}$, where $L_X = \pi_1(T)$.

Remark 9.3. For a connected p-compact group X and p odd, the fixed point set $\breve{T}^{W_X}$ always equals a discrete approximation to the center of X by [57, Thm. 7.5]. If X is the $\mathbf{F}_p$-completion of a compact connected Lie group then this is likewise the case for $p = 2$ unless X contains a direct factor isomorphic to $\mathrm{SO}(2n+1)\hat{_2}$, by [92, Thm. 1.6].

Proof of Lemma 9.2. Set $Y = \mathcal{C}_X(V)$ and $\pi = \pi_0(Y)$ for short. Since V is toral, $\breve{T}$ is in a canonical way a discrete approximation to a maximal torus in Y.

First observe that the center of Y has discrete approximation in $\breve{T}$. Indeed, otherwise there would by [57, Thm. 6.4] exist a central homomorphism $f : \mathbf{Z}/p^n \to \breve{\mathcal{N}}_{p,Y}$ with image not in $\breve{T}$, which would produce a homomorphism $f' : \mathbf{Z}/p^n \to \breve{\mathcal{N}}_{p,X}$ commuting with $\breve{T}$ but not in $\breve{T}$, which contradicts the fact that T is self-centralizing in X by [56, Prop. 9.1], since X is connected.

Suppose that $\breve{T}^{W_{\mathcal{C}_X(V)_1}}$ is a discrete approximation to $\mathcal{Z}(Y_1)$ and set $C = \breve{T}^{W_{\mathcal{C}_X(V)}}$. We want to show that C is central in Y. Let $f : BC \to BY_1$ be the natural inclusion. We have an obvious diagram with horizontal maps fibrations

$$\begin{tikzcd}
\mathrm{map}(BC, BY_1)_{\{f\}} \arrow[r] \arrow[d] & \mathrm{map}(BC, BY)_f \arrow[r] \arrow[d] & \mathrm{map}(BC, B\pi)_0 \arrow[d, equal] \\
BY_1 \arrow[r] & BY \arrow[r] & B\pi
\end{tikzcd}$$

where $\{f\}$ denotes the set of homotopy classes of maps $BC \to BY_1$ generated by f under the π-action on BY_1. If we can show that $\{f\}$ consists of just f then it follows from the five-lemma that the middle vertical map is a homotopy equivalence, since our assumption implies that C is central in Y_1.

To see that the action is trivial consider the following diagram:

$$\begin{tikzcd}
 & B\mathcal{N}_{Y_1} \arrow[r, "\tilde{g}"] \arrow[d] & B\mathcal{N}_{Y_1} \arrow[d] \\
BC \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & BY_1 \arrow[r, "g"] & BY_1
\end{tikzcd}$$

where $\tilde{f}$ is the natural inclusion of BC in $B\mathcal{N}_{Y_1}$, g is an element in $\mathrm{Aut}(BY_1)$ induced by an element in π and $\tilde{g}$ is the corresponding self-map of $B\mathcal{N}_{Y_1}$ defined via Lemma 4.1. However by the definition of C, the composite $\tilde{g}\tilde{f}$ is homotopic to $\tilde{f}$ for all g induced by an element in π, so f is homotopic to gf as well. Hence we have shown that C is central in Y and since the center of Y has discrete approximation in $\breve{T}$ it is obviously the largest subgroup with this property. So C is a discrete approximation to $\mathcal{Z}Y$ as wanted.

The last statement about the homotopy groups now follows easily from the long exact sequence in group cohomology. $\square$

Remark 9.4. The above lemma should be compared to Lemma 6.6 and Remark 6.7 which have slightly different assumptions and conclusions.

We are now ready for the proposition, essentially contained in [55, §8], which will take care of the toral part. See [57, Def. 7.3] and Remark 9.6 for the definition of the singular set $\sigma(s)$ of a reflection s, and note that the assumption is always satisfied for p odd by [57, Thm. 7.5] and the definition of $\sigma(s)$.

Proposition 9.5. *Let X be a connected p-compact group, and assume that for each nontrivial toral elementary abelian p-subgroup $V \in \mathbf{A}(X)$ the fixed point set $\breve{T}^{W_{\mathcal{C}_X(V)_1}}$ is a discrete approximation to $\mathcal{Z}(\mathcal{C}_X(V)_1)$. Then*

$$\lim_{\mathbf{A}(X)}{}^i F_j^{\mathrm{tor}} = \begin{cases} H^{2-j}(W_X; L_X) & \text{if } i = 0 \text{ and } j = 1,\, 2, \\ 0 & \text{otherwise} \end{cases}$$

where $H^{2-j}(W_X; L_X) \cong \pi_j(B\mathcal{Z}(X))$ if $\breve{T}^W$ is a discrete approximation to $\mathcal{Z}(X)$. In particular if for all reflections $s \in W_X$ the singular set $\sigma(s)$ equals the fixed point set $\breve{T}^{\langle s \rangle}$ then the assumption above is satisfied and

$$\lim_{\mathbf{A}(X)}{}^i F_j^{\mathrm{tor}} = \begin{cases} \pi_j(B\mathcal{Z}(X)) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first part of the proof consists of a translation of [55, §8] into the current notation. By [58, Prop. 3.4] all morphisms in $\mathbf{A}(X)$ between toral subgroups $V \to X$ and $V' \to X$ are induced by inclusions and action by elements of W_X. Hence we can identify $\mathbf{A}^{\mathrm{tor}}(X)$, up to equivalence of categories with a category which has objects the nontrivial subgroups of ${}_p\breve{T} \cong (\mathbf{Z}/p)^r$ (where r is the rank of T) and morphisms the homomorphisms between subgroups induced by inclusions and action by W_X. Also, by [57, Thm. 7.6(1)], $W_{\mathcal{C}_X(V)}$ consists of the elements in W_X which pointwise fix V. Hence Lemma 9.2 shows that the functor F_2^{tor} on $\mathbf{A}^{\mathrm{tor}}(X)$ is isomorphic to the functor $\alpha^0_{\Gamma,M}$ on $\mathbf{A}_\Gamma$ from [55, §8], where $\Gamma = W_X$ and $M = L_X$. Likewise F_1^{tor} is isomorphic to $\alpha^1_{\Gamma,M}$. (Note that there is the difference in formulation from [55, §8] where M is a $\mathbf{F}_p\Gamma$-module rather than an $\mathbf{Z}_p\Gamma$-module, but the proof is identical.) Therefore [55, §8] (which is a Mackey functor argument, which can also be deduced from [54] or [81]) implies the first part of the lemma about obstruction groups.

To see the last part about the singular set recall that for an abelian subgroup $A \subseteq \breve{T}$,

$$\bigcap_{\substack{\text{reflections } s \in W_X \\ \text{such that } A \subseteq \sigma(s)}} \sigma(s)$$

is a discrete approximation to $\mathcal{Z}(\mathcal{C}_X(A)_1)$ by [57, Thms. 7.5 and 7.6]. Hence if $\sigma(s) = \breve{T}^{\langle s \rangle}$ for all reflections $s \in W_X$, then the assumption of the first part is obviously satisfied. $\square$

Remark 9.6. Let G be a compact connected Lie group with maximal torus T, and let α be a root of G relative to T with corresponding reflection s_α. In this case the singular set $\sigma(s_\alpha)$ is just the discrete approximation of the kernel U_α of α on T. To see this note that by [20, §4, no. 5] the reflection s_α lifts to an element n_α (denoted by $\nu(\theta)$ in [20]) which satisfies $n_\alpha^2 = \exp(\alpha^\vee/2)$; the statement now follows; cf. [92, Pf. of Prop. 3.1(ii)].

Explicit calculations [92, Prop. 3.1(ii)] (see also [62], [83, Prop. 3.2(vi)], and [116, §4]) show that for a compact connected Lie group G, $\sigma(s)$ in fact always equals $\breve{T}^{\langle s\rangle}$ except when G contains a direct factor isomorphic to $\mathrm{SO}(2n+1)$, $p=2$ and s is a reflection corresponding to a short root. Combining this with Proposition 9.5, now gives the following calculation of the toral part of the obstruction groups, whose full strength at $p=2$ we will however not use here.

COROLLARY 9.7. *Let G be a compact connected Lie group and p a prime. Set $X = G\hat{_p}$ and assume that G contains no direct factor isomorphic to $\mathrm{SO}(2n+1)$ if $p=2$. Then*

$$\lim_{\mathbf{A}(X)}{}^{i} F_j^{\mathrm{tor}} = \begin{cases} \pi_j(B\mathcal{Z}(X)) & \textit{if } i=0,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof of Theorem 9.1 *when $H^*(BX;\mathbf{Z}_p)$ is polynomial, p odd.* If $H^*(BX;\mathbf{Z}_p)$ is a polynomial algebra concentrated in even degrees then all elementary abelian p-subgroups are toral by Lemma 10.8, so $F=F^{\mathrm{tor}}$. Since p is odd the assumption of Proposition 9.5 holds and Theorem 9.1 follows. □

9.2. *The nontoral part for the exceptional groups.* In this subsection we prove Theorem 9.1 when X is the $\mathbf{F}_p$-completion of one of the exceptional groups and p is odd. Let F_j^E denote the subquotient functor of F_j concentrated on a nontoral elementary abelian p-subgroup E. By Oliver's formula [113, Prop. 4]

$$\lim_{\mathbf{A}(X)}{}^{i} F_j^{E} = \begin{cases} \mathrm{Hom}_W(\mathrm{St}(E),F_j(E)) & \text{if } i=\mathrm{rk}\,E-1,\\ 0 & \text{otherwise.}\end{cases}$$

We now embark on proving some lemmas which will be used to show that these obstructions groups identically vanish. (For Theorem 1.4 we actually only need this when E has rank at most four.)

Since $\mathcal{Z}(\mathcal{C}_X(E))$ is the $\mathbf{F}_p$-completion of an abelian compact Lie group (cf. [57, Thm. 1.1]), $F_j=0$ unless $j=1,2$. The following lemma reduces the problem of showing that the obstruction groups vanish to showing that $\mathrm{Hom}_{W(E)}(\mathrm{St}(E), {}_p\breve{\mathcal{Z}}(\mathcal{C}_X(E)))=0$, where ${}_p\breve{\mathcal{Z}}(\mathcal{C}_X(E))$ is the finite group of elements of order p in the discrete approximation $\breve{\mathcal{Z}}(\mathcal{C}_X(E))$.

LEMMA 9.8. *Let A be an abelian compact Lie group, and let ${}_pA$ and A^p denote the kernel and the image of the pth power map on A (with multiplicative notation). Let P be a finitely generated projective $\mathbf{Z}_pW$-module for a finite group W, and assume that A has a module action of W. Then* $\mathrm{Hom}_W(P,{}_pA)=0$ *if and only if* $\mathrm{Hom}_W(P,\pi_1(A)\otimes\mathbf{Z}_p)=\mathrm{Hom}_W(P,\pi_0(A)\otimes\mathbf{Z}_p)=0$.

Proof. The long exact sequence of homotopy groups associated to the exact sequence of groups $1 \to A^p \to A \to A/A^p \to 1$ shows that the inclusion $A^p \hookrightarrow A$ induces an isomorphism $\pi_1(A^p) \xrightarrow{\cong} \pi_1(A)$ and an injection $\pi_0(A^p) \hookrightarrow \pi_0(A)$.

Hence the exact sequence $1 \to {}_pA \to A \xrightarrow{p} A^p \to 1$ produces the following diagram, where the row, as well as the sequence going through $\pi_i(A)$ instead of $\pi_i(A^p)$, is exact:

$$\begin{tikzcd}
\pi_1(A) \arrow[r, "p"] \arrow[dr, "p"'] & \pi_1(A^p) \arrow[r] \arrow[d, "\cong"] & \pi_0({}_pA) \arrow[r] & \pi_0(A) \arrow[r, "p"] \arrow[dr, "p"'] & \pi_0(A^p) \arrow[d, hook] \\
& \pi_1(A) & & & \pi_0(A).
\end{tikzcd}$$

Apply the exact functor $\mathrm{Hom}_W(P, - \otimes \mathbf{Z}_p)$ to this diagram. The lemma now follows from Nakayama's lemma, since $\pi_0(A)$ is finite and $\pi_1(A)$ is finitely generated. $\square$

The following elementary observation is so useful that it is worth stating explicitly.

LEMMA 9.9. *Suppose that p is odd and that W is a subgroup of* $\mathrm{GL}(E)$, *with $-1 \in W$. Then* $\mathrm{Hom}_W(\mathrm{St}(E), E) = 0$.

Proof. Set $Z = \langle -1 \rangle$. Since Z acts trivially on $\mathrm{St}(E)$ we have

$$\mathrm{Hom}_W(\mathrm{St}(E), E) \subseteq \mathrm{Hom}_Z(\mathrm{St}(E), E) = \mathrm{Hom}_Z(\mathrm{St}(E), E^Z) = 0. \qquad \square$$

We also need the following lemma, which follows from a theorem of S. D. Smith [128].

LEMMA 9.10. *Let G be a finite group of Lie type of characteristic p, and let P be a parabolic subgroup of G with corresponding unipotent radical U and Levi subgroup $L \cong P/U$. Suppose that W is a subgroup with $U \subseteq W \subseteq P$, and let M be an $\mathbf{F}_pW$-module.*

(1) *If U acts trivially on M, then* $\mathrm{Hom}_W(\mathrm{St}_G, M) = \mathrm{Hom}_{W/U}(\mathrm{St}_L, M)$.

(2) *If $\mathrm{St}_L \otimes \mathbf{F}_p$ is irreducible as an $\mathbf{F}_pW/U$-module and if M has a finite filtration as an $\mathbf{F}_pW$-module, with filtration quotients of $\mathbf{F}_p$-dimension strictly less than* $\mathrm{rank}_{\mathbf{Z}_p} \mathrm{St}_L$ *then* $\mathrm{Hom}_W(\mathrm{St}_G, M) = 0$.

Proof. Since U acts trivially on M,

$$\mathrm{Hom}_W(\mathrm{St}_G, M) = \mathrm{Hom}_{W/U}((\mathrm{St}_G)_U, M)$$

where $(-)_U$ denotes coinvariants. But since the Steinberg module is self-dual, as is clear from its definition as a homology module, $(\mathrm{St}_G)_U \cong (\mathrm{St}_G)^U$. Now

Smith's theorem [128] says that $(\mathrm{St}_G)^U \cong \mathrm{St}_L$, which proves the first part of the lemma.

For the second part, we can assume that the filtration quotients are simple $\mathbf{F}_pW$-modules. Since $U \subseteq O_p(W)$, U acts trivially on any irreducible $\mathbf{F}_pW$-module, by elementary representation theory. Hence the second part follows from the first together with a dimension consideration. □

The above lemma is often used in conjunction with the following obvious observation.

LEMMA 9.11. *Let E be a nontoral elementary abelian p-subgroup of a compact Lie group G. Then the $\mathbf{F}_p$-dimension of ${}_pZC_G(E)$ is at most equal to the maximal dimension of a nontoral elementary abelian p-subgroup of G, and E is a $W(E)$-submodule of ${}_pZC_G(E)$.* □

The last lemma we shall need is a concrete calculation. First we need the following.

LEMMA 9.12. *Let Γ be a subgroup of $\mathrm{GL}_n(\mathbf{F}_p)$ and let M be a $\mathbf{Z}_{(p)}\Gamma$-module. Then*

$$\mathrm{Hom}_{\mathbf{Z}_{(p)}\Gamma}(\mathrm{St}_{\mathrm{GL}_n(\mathbf{F}_p)}, M) \cong \sum_{I\subseteq\{1,\dots,n-1\}} (-1)^{|I|} \bigoplus_{g\in\Gamma\backslash \mathrm{GL}_n(\mathbf{F}_p)/P_I} M^{\Gamma\cap gP_Ig^{-1}},$$

where P_I is the parabolic subgroup of $\mathrm{GL}_n(\mathbf{F}_p)$ corresponding to the subset I.

Proof. This follows easily from the fact that $\mathrm{St}_{\mathrm{GL}_n(\mathbf{F}_p)}$ is isomorphic to $\sum_{I\subseteq\{1,\dots,n-1\}}(-1)^{|I|}1_{P_I}^{\mathrm{GL}_n(\mathbf{F}_p)}$ as $\mathbf{Z}_{(p)}\,\mathrm{GL}_n(\mathbf{F}_p)$-modules [87, Cor. 1.2] combined with Frobenius reciprocity and the double coset formula (see e.g. [9, Prop. 3.3.1(ii) and Cor. 3.3.5(iv)]). □

LEMMA 9.13. *Let $E = (\mathbf{F}_3)^4$ and let $W = \mathrm{SL}_3(\mathbf{F}_3) \times 1 \subseteq \mathrm{GL}(E)$. Then $\mathrm{Hom}_W(\mathrm{St}(E), E) = 0$.*

Proof. This is most easily checked by implementing the formula from Lemma 9.12 on a computer, e.g., using MAGMA [16]. However in this case the calculation is sufficiently small to be redone by hand with some effort. Alternatively one can use some *ad hoc* Lie theoretic arguments. (We are grateful to A. Kleschev and H. H. Andersen for sketching a couple of such arguments to us—however, since these arguments are rather involved compared to the size of the calculation at hand we will not provide them here.) □

Before we start going through the exceptional groups, we need to introduce a bit of notation. For an $\mathbf{F}_p$-vector space $E = \langle e_1, \dots, e_n\rangle$, we let $E_{ij\dots}$ denote the subspace generated by $e_i, e_j, \dots$. Likewise we let $P_{ij\dots}$ (resp. $U_{ij\dots}$) denote the parabolic subgroup (resp. its unipotent radical) of $\mathrm{GL}(E)$ corresponding to the simple roots $\alpha_i, \alpha_j, \ldots$ in the standard notation. For example in $\mathrm{GL}_3(\mathbf{F}_p)$, U_2 is the subgroup

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof of Theorem 9.1 *when* $X = G\hat{_p}$, $(G,p) = (E_8,5)$, $(F_4,3)$, $(E_6,3)$, $(E_7,3)$, *or* $(E_8,3)$. By Lemma 9.8 it is enough to see that

$$\mathrm{Hom}_{W(E)}(\mathrm{St}(E), {}_pZC_G(E)) = 0$$

for all nontoral elementary abelian p-subgroups E of G. We proceed case-by-case.

$(E_8,5)$ and $(F_4,3)$: By [70, Lem. 10.3 and Thm. 7.4] G has, up to conjugacy, exactly one nontoral elementary abelian p-subgroup E, which has rank 3, Weyl group $W = \mathrm{SL}(E)$, and (since E is necessarily maximal) $E = {}_pZC_G(E)$. Since $\mathrm{St}(E)$ is an irreducible $\mathrm{SL}(E)$-module of dimension p^3 we have $\mathrm{Hom}_W(\mathrm{St}(E), E) = 0$.

$(E_7,3)$: By Theorem 8.16 E_7 has, up to conjugacy, two nontoral elementary abelian 3-subgroups $E^3_{2E_7}$ and $E^4_{2E_7}$ of rank 3 and 4 respectively. Since $W(E^3_{2E_7}) = \mathrm{SL}_3(\mathbf{F}_3)$ a dimension consideration as above gives

$$\mathrm{Hom}_{W(E^3_{2E_7})}(\mathrm{St}(E^3_{2E_7}), {}_3ZC_G(E^3_{2E_7})) = 0.$$

For $E^4_{2E_7}$ (whose Weyl group is listed in Theorem 8.16) we use Lemma 9.10(2), taking $U = U_{23}$, which immediately gives that also

$$\mathrm{Hom}_{W(E^4_{2E_7})}(\mathrm{St}(E^4_{2E_7}), {}_3ZC_G(E^4_{2E_7})) = 0.$$

$(E_6,3)$: By Theorem 8.10 E_6 has, up to conjugacy, eight nontoral elementary abelian 3-subgroups all of rank at most 4. For $E = E^{2b}_{E_6}$, $E^{3b}_{E_6}$, $E^{3c}_{E_6}$ and $E^{4a}_{E_6}$ we have $\mathrm{Hom}_W(\mathrm{St}(E), {}_3ZC_G(E)) = 0$ by Lemma 9.10(2) taking $U = 1$, 1, U_2, and U_1 respectively (note that we do not need to know ${}_3ZC_G(E)$ exactly since the rough bound from Lemma 9.11 is sufficient.) For $E = E^{2a}_{E_6}$, $E^{3a}_{E_6}$, $E^{3d}_{E_6}$, and $E^{4b}_{E_6}$ we have $-1 \in W(E)$ and $E = {}_3ZC_G(E)$ (a fact we did not need above) by Theorem 8.10, so Lemma 9.9 shows that $\mathrm{Hom}_W(\mathrm{St}(E), {}_3ZC_G(E)) = 0$.

$(E_8,3)$: By Theorem 8.13 E_8 has seven conjugacy classes of nontoral elementary abelian 3-subgroups. If $E = E^{3a}_{E_8}$, $E^{3b}_{E_8}$, $E^{4b}_{E_8}$, or $E^{4c}_{E_8}$ then

$$\mathrm{Hom}_W(\mathrm{St}(E), {}_3ZC_G(E)) = 0$$

by Lemma 9.10(2) when $U = U_1$, 1, U_2, and U_{23} respectively (note again that we do not need to know ${}_3ZC_G(E)$ exactly). Now consider $E = E^{4a}_{E_8}$. By Theorem 8.13 we have ${}_3ZC_G(E) = E$, and by Lemma 9.13 $\mathrm{Hom}_{W(E)}(\mathrm{St}(E), E) = 0$. Next, suppose that $E = E^{5a}_{E_8}$. Then E has the W-invariant subspace E_1 and $U = U_{234}$ acts trivially on E_1 and E/E_1. By Lemma 9.10(1) we get $\mathrm{Hom}_W(\mathrm{St}(E), E_1) = \mathrm{Hom}_{W/U}(\mathrm{St}(E_1) \otimes \mathrm{St}(E_{2345}), E_1)$. Since W/U contains $\mathrm{GL}_1(\mathbf{F}_3) \times 1$ and $\mathrm{Hom}_{\mathrm{GL}_1(\mathbf{F}_3)}(\mathrm{St}(E_1), E_1) = 0$ by Lemma 9.9 we get $\mathrm{Hom}_W(\mathrm{St}(E), E_1) = 0$. Lemma 9.10(1) also shows that $\mathrm{Hom}_W(\mathrm{St}(E), E/E_1) = \mathrm{Hom}_{W/U}(\mathrm{St}(E_1) \otimes \mathrm{St}(E_{2345}), E/E_1)$, and since W/U contains $1 \times \mathrm{SL}_3(\mathbf{F}_3) \times 1$ we get $\mathrm{Hom}_W(\mathrm{St}(E), E/E_1) = 0$ by Lemma 9.13. Thus $\mathrm{Hom}_W(\mathrm{St}(E), E) = 0$ as desired. Finally, let $E = E^{5b}_{E_8}$. The subspace E_{1234} is W-invariant and $U = U_{123}$ acts trivially on E_{1234} and E/E_{1234}. By Lemma 9.10(1) we have $\mathrm{Hom}_W(\mathrm{St}(E), M) = \mathrm{Hom}_{W/U}(\mathrm{St}(E_{1234}) \otimes \mathrm{St}(E_5), M)$ for $M = E_{1234}$ and $M = E/E_{1234}$ and since W/U contains $\mathrm{Sp}_4(\mathbf{F}_3) \times 1$ it suffices to prove that $\mathrm{Hom}_{\mathrm{Sp}_4(\mathbf{F}_3)}(\mathrm{St}(E_{1234}), E_{1234}) = 0$ and $\mathrm{Hom}_{\mathrm{Sp}_4(\mathbf{F}_3)}(\mathrm{St}(E_{1234}), \mathbf{F}_3) = 0$. Since $-1 \in \mathrm{Sp}_4(\mathbf{F}_3)$, the first claim follows from Lemma 9.9. The second claim follows from [5] or from a direct computer calculation based on Lemma 9.12. Thus $\mathrm{Hom}_W(\mathrm{St}(E), E) = 0$ in this case as well. This exhausts the list. $\square$

9.3. *The nontoral part for the projective unitary groups.* We now embark on proving Theorem 9.1 for $X = \mathrm{PU}(n)\hat{_p}$. We will throughout this subsection use the notation for elementary abelian p-subgroups of X introduced in Section 8.2. We first treat the toral case directly (see also Corollary 9.7).

LEMMA 9.14. *Let $X = \mathrm{PU}(n)\hat{_p}$. Then*

$$\lim_{\mathbf{A}(X)}{}^i F_j^{\mathrm{tor}} = \begin{cases} \mathbf{Z}/2 & \text{if } n = p = 2,\ i = 0 \text{ and } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $n \neq 2$ then it is immediate to check that $\breve{T}^{\langle s \rangle}$ is p-divisible for an arbitrary reflection $s \in W_X$, and so $\sigma(s) = \breve{T}^{\langle s \rangle}$ by the definition of $\sigma(s)$. Hence if $n \neq 2$ or p odd the lemma follows by Proposition 9.5.

Now suppose that $X = \mathrm{PU}(2)\hat{_2}$. Since for the nontrivial elementary abelian 2-subgroup $V \subseteq \breve{T}$ we have $W_X(V)_1 = 1$ and $\mathcal{C}_X(V)_1 \cong T$, the first part of Proposition 9.5 still applies to finish the proof also in this case. $\square$

We next record the following general lemma, which is obvious from the Künneth formula.

LEMMA 9.15. *Suppose $\mathbf{D}_1$ and $\mathbf{D}_2$ are two categories with only finitely many morphisms. Let $C\mathbf{D}_i$ be 'the cone on $\mathbf{D}_i$', i.e., the category constructed from $\mathbf{D}_i$ by adding an initial object e to $\mathbf{D}_i$, and let $\mathbf{D}_1 \star \mathbf{D}_2 = C\mathbf{D}_1 \times C\mathbf{D}_2 - \{(e,e)\}$, 'the join of $\mathbf{D}_1$ and $\mathbf{D}_2$' (see [119, §1]). If $F_i : C\mathbf{D}_i \to \mathbf{Z}_p$-mod, $i = 1, 2$ are functors then*

$$C^*(C\mathbf{D}_1 \times C\mathbf{D}_2, \mathbf{D}_1 \star \mathbf{D}_2; F_1 \otimes F_2) \cong C^*(C\mathbf{D}_1, \mathbf{D}_1; F_1) \otimes C^*(C\mathbf{D}_2, \mathbf{D}_2; F_2).$$

In particular if one of the chain complexes has torsion-free homology or if everything is defined over $\mathbf{F}_p$ *then*

$$H^*(C\mathbf{D}_1 \times C\mathbf{D}_2, \mathbf{D}_1 \star \mathbf{D}_2; F_1 \otimes F_2) \cong H^*(C\mathbf{D}_1, \mathbf{D}_1; F_1) \otimes H^*(C\mathbf{D}_2, \mathbf{D}_2; F_2).$$

□

The following result gives that certain filtration quotients have (almost) vanishing cohomology.

THEOREM 9.16. *Set* $X = \mathrm{PU}(n)\hat{_p}$ *and let* $r > 0$ *with* $p^r \,|\, n$. *Let* $F_j^r : \mathbf{A}(X) \to \mathbf{Z}_p\text{-}\mathrm{mod}$ *denote the functor on objects given by* $F_j^r(E) = \pi_j(B\mathcal{Z}\mathcal{C}_X(E))$ *if* E *is of the form* $\bar\Gamma_r \times \bar A$ (*in the notation of Section* 8.2) *and zero otherwise. Then*

$$\lim_{\mathbf{A}(X)}{}^i F_j^r = \begin{cases} \mathbf{Z}/2 & \text{if } n = p = 2 \text{ and } r = i = j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Define a functor $\tilde F_j^r$ by $\tilde F_j^r(E) = \pi_j(BZC_{\mathrm{PU}(k)}(\bar A)\hat{_p})$, with $k = n/p^r$, if $E = \bar\Gamma_r \times \bar A$ for a fixed r and zero otherwise. This is a subfunctor of F_j^r via the identification $\mathrm{PU}(k) \cong C_{\mathrm{PU}(n)}(\bar\Gamma_r)_1$. Set $\tilde{\tilde F}{}_j^r = F_j^r / \tilde F_j^r$ and observe that this is the trivial functor unless $j = 1$ where it is given by $\tilde{\tilde F}{}_j^r(E) = \bar\Gamma_r$ if E is of the form $\bar\Gamma_r \times \bar A$ and zero otherwise. Consider the category

$$\mathbf{D} = \mathbf{A}^e(X)_{\subseteq \bar\Gamma_r} \times \mathbf{A}^e(\mathrm{PU}(k)\hat{_p}) - \{(e,e)\},$$

where the superscript e means that we do not exclude the trivial subgroup. We have a natural inclusion of categories $\iota : \mathbf{D} \to \mathbf{A}(X)$ on objects given by $(V, \bar A) \mapsto V \times \bar A$.

Step 1. We claim that this map induces an isomorphism

$$\lim_{\mathbf{A}(X)}{}^* F_j^r \to \lim_{\mathbf{D}}{}^* F_j^r.$$

By filtering the functor and using Nakayama's lemma it is enough to show this for $\tilde F_j^r \otimes \mathbf{F}_p$ and $\tilde{\tilde F}{}_j^r \otimes \mathbf{F}_p$. We can furthermore replace these functors by the subquotient functors which are only concentrated on one subgroup $\bar\Gamma_r \times \bar A$.

Consider first such a subquotient of $\tilde F_j^r \otimes \mathbf{F}_p$. In this case the formula of Oliver [113, Prop. 4] together with Lemma 9.15 shows that the higher limits on both sides are only nonzero in a single degree, where the map identifies with the restriction map

$$\begin{aligned} &\mathrm{Hom}_{W_X(\bar\Gamma_r \times \bar A)}(\mathrm{St}(\bar\Gamma_r \times \bar A), \pi_j(BZC_{\mathrm{PU}(k)}(\bar A)) \otimes \mathbf{F}_p) \\ &\qquad \longrightarrow \mathrm{Hom}_{\mathrm{Sp}(\bar\Gamma_r) \times W_{\mathrm{PU}(k)}(\bar A)}(\mathrm{St}(\bar\Gamma_r) \otimes \mathrm{St}(\bar A), \pi_j(BZC_{\mathrm{PU}(k)}(\bar A)) \otimes \mathbf{F}_p). \end{aligned}$$

Let U be the subgroup of elements in $W_{\mathrm{PU}(n)}(\bar\Gamma_r \times \bar A)$ which act as the identity on $\bar A$ and $(\bar\Gamma_r \times \bar A)/\bar A$. Then U acts trivially on $\pi_j(BZC_{\mathrm{PU}(k)}(\bar A)) \otimes \mathbf{F}_p$.

Furthermore, by the theorem of Smith [128] (and self-duality of the Steinberg module) $\mathrm{St}(\bar\Gamma_r \times \bar A)_U \cong \mathrm{St}(\bar\Gamma_r \times \bar A)^U \cong \mathrm{St}(\bar\Gamma_r) \otimes \mathrm{St}(\bar A)$, where $(-)_U$ and $(-)^U$ denote coinvariants and invariants respectively. Hence this map is an isomorphism. The case of a subquotient of $\tilde{\tilde F}{}^r_j \otimes \mathbf{F}_p$ is completely analogous. This shows the isomorphism.

Step 2. We now proceed to calculate the higher limits over $\mathbf{D}$, which we do by calculating the limits of $\tilde F^r_j$ and $\tilde{\tilde F}{}^r_j$. We first consider $\tilde{\tilde F}{}^r_j$. We have already remarked that only $\tilde{\tilde F}{}^r_1 \neq 0$. Furthermore if $k \neq 1$ then

$$H^*(C\mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(k)\hat{}_p), \mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(k)\hat{}_p); \mathbf{Z}_p) = 0$$

by [55, §8] so $\lim^*_{\mathbf{D}} \tilde{\tilde F}{}^r_1 = 0$ by Lemma 9.15. If $k = 1$ we get $\lim^i_{\mathbf{D}} \tilde{\tilde F}{}^r_1 \cong \mathrm{Hom}_{\mathrm{Sp}(\bar\Gamma_r)}(\mathrm{St}(\bar\Gamma_r), \bar\Gamma_r)$ if $i = 2r-1$ and zero otherwise, by [113, Prop. 4].

Now consider $\tilde F^r_j$. By Lemma 9.15

$$\lim_{\mathbf{D}}{}^i \tilde F^r_j = \mathrm{Hom}_{\mathrm{Sp}(\bar\Gamma_r)}(\mathrm{St}(\bar\Gamma_r), \mathbf{Z}_p) \otimes \lim_{\mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(k)\hat{}_p)}{}^{i-2r} \pi_j(B\mathcal{Z}\mathcal{C}_{\mathrm{PU}(k)\hat{}_p}(\bar A)).$$

By Lemma 9.14 $\lim^{i-2r}_{\mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(k)\hat{}_p)} \pi_j(B\mathcal{Z}\mathcal{C}_{\mathrm{PU}(k)\hat{}_p}(\bar A)) = 0$ unless $p = k = 2$, $j = 1$ and $i - 2r = 1$ where we get $\lim^1_{\mathbf{A}^{\mathrm{tor}}(\mathrm{PU}(2)\hat{}_2)} \pi_j(B\mathcal{Z}\mathcal{C}_{\mathrm{PU}(2)\hat{}_2}(\bar A)) = \mathbf{Z}/2$. By an argument of H. H. Andersen and C. Stroppel [5] we have

$$\mathrm{Hom}_{\mathrm{Sp}(\bar\Gamma_r)}(\mathrm{St}(\bar\Gamma_r), \mathbf{Z}_p) = 0$$

for all r and p. To sum up we get

$$\lim_{\mathbf{A}(X)}{}^i F^r_j \cong \lim_{\mathbf{D}}{}^i \tilde{\tilde F}{}^r_j \cong \mathrm{Hom}_{\mathrm{Sp}(\bar\Gamma_r)}(\mathrm{St}(\bar\Gamma_r), \bar\Gamma_r)$$

if $j = k = 1$ and $i = 2r - 1$ and zero otherwise. However the same argument of H. H. Andersen and C. Stroppel [5] shows that

$$\mathrm{Hom}_{\mathrm{Sp}(\bar\Gamma_r)}(\mathrm{St}(\bar\Gamma_r), \bar\Gamma_r) = 0$$

unless $r = 1$ and $p = 2$ where it equals $\mathbf{Z}/2$. (Note that this is obvious if p is odd by Lemma 9.9.) This shows the wanted formula. □

Remark 9.17. Note that slightly nontrivial statements from [5] are only used above for $p = 2$ and furthermore become trivial when $r = 1$, where $\mathrm{Sp}(\bar\Gamma_1) \cong \mathrm{SL}(\bar\Gamma_1)$, and this is in fact the only case which involves obstruction groups in the range needed for the proof of (the $p = 2$ version of) Theorem 1.4.

Remark 9.18. It is in fact possible to give a short proof of Smith's theorem in the special case used above, from the geometric definition of the Steinberg module $\mathrm{St}(E)$ via flags.

Proof of Theorem 9.1 *for* $X = \mathrm{PU}(n)\hat{_p}$. Theorem 9.16 and Lemma 9.14 directly show the conclusion unless $n = 2$ and $p = 2$, so assume this is the case. From the definition it follows that $\lim^0_{\mathbf{A}(X)} F_1 = 0$. Now, consider the exact sequence of functors $0 \to F_1^1 \to F_1 \to F_1^{\mathrm{tor}} \to 0$. The long exact sequence of higher limits starts out as

$$0 \to 0 \to 0 \to \lim_{\mathbf{A}(X)}{}^0 F_1^{\mathrm{tor}} \to \lim_{\mathbf{A}(X)}{}^1 F_1^1 \to \lim_{\mathbf{A}(X)}{}^1 F_1 \to 0 \to \cdots .$$

So, since $\lim^0_{\mathbf{A}(X)} F_1^{\mathrm{tor}} \cong \mathbf{Z}/2 \cong \lim^1_{\mathbf{A}(X)} F_1^1$, we get $\lim^i_{\mathbf{A}(X)} F_1 = 0$ for $i > 0$ as well. This concludes the proof of the last case of Theorem 9.1. □

10. Consequences of the main theorem

In this section we prove the theorems listed in the introduction which are consequences of the main theorem.

Proof of Theorem 1.2. The theorem follows directly from Theorem 1.1 together with the classification of finite $\mathbf{Z}_p$-reflection groups (Theorem 11.1), since by the proof of Theorem 1.1 (and Theorem 12.1) all exotic p-compact groups have torsion-free $\mathbf{Z}_p$-cohomology. □

Proof of Theorem 1.5. By [100, Thm. 1.4] X is isomorphic to a p-compact group of the form $(X' \times T'')/A$, where X' is a simply connected p-compact group, T'' is a p-compact torus, and A is a finite central subgroup of the product. Hence we have $X/T \simeq X'/T'$, where T and T' are maximal tori of X and X' respectively. So we can without restriction assume that X is simply connected.

For compact connected Lie groups the statement of this theorem is the celebrated result of Bott [17, Thm. A]. Hence by Theorem 1.2 it is enough to prove the theorem when X is an exotic p-compact group. In that case $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra with generators in even degrees, and the number of generators equals the rank of X (by the proof of Theorem 1.4). The same is true over $\mathbf{F}_p$, and since $H^*(BT; \mathbf{F}_p)$ is finitely generated over $H^*(BX; \mathbf{F}_p)$ by [56, Prop. 9.11], $H^*(BX; \mathbf{F}_p) \to H^*(BT; \mathbf{F}_p)$ is injective by a Krull dimension consideration. But since they are both polynomial algebras it follows by e.g., [60, §11] that $H^*(BT; \mathbf{F}_p)$ is in fact free over $H^*(BX; \mathbf{F}_p)$. Hence the Eilenberg-Moore spectral sequence of the fibration $X/T \to BT \to BX$ collapses and

$$H^*(X/T; \mathbf{F}_p) \cong \mathbf{F}_p \otimes_{H^*(BX;\mathbf{F}_p)} H^*(BT; \mathbf{F}_p).$$

In particular $H^*(X/T; \mathbf{F}_p)$ is concentrated in even degrees so the rank equals the Euler characteristic $\chi(X/T)$ which again equals $|W_X|$ by [56, Prop. 9.5]. By the long exact sequence in cohomology and Nakayama's lemma, $H^*(X/T; \mathbf{Z}_p)$ is a free $\mathbf{Z}_p$-module of rank $|W_X|$ as wanted. □

Remark 10.1. Let $H^*_{\mathbf{Q}_p}(\cdot) = H^*(\cdot; \mathbf{Z}_p) \otimes \mathbf{Q}$. For any connected p-compact group X the natural map $X/T \to BT$ induces an isomorphism

$$H^*_{\mathbf{Q}_p}(X/T) \cong \mathbf{Q}_p \otimes_{H^*_{\mathbf{Q}_p}(BX)} H^*_{\mathbf{Q}_p}(BT)$$

since the Eilenberg-Moore spectral sequence of the fibration $X/T \to BT \to BX$ collapses by [56, Prop. 9.7] and [60, §11]. It then follows from [32] that the natural W_X-action on $H^*(X/T; \mathbf{Z}_p) \otimes \mathbf{Q} = H^*_{\mathbf{Q}_p}(X/T)$ is isomorphic to the regular representation of W_X when the grading is ignored. Just as for compact connected Lie groups this is not true over $\mathbf{Z}_p$ when $p \,|\, |W_X|$ (cf. e.g. [127, p. 221]).

Proof of Theorem 1.6. By Theorem 1.2 it is enough to prove the statement in the case where X is the $\mathbf{F}_p$-completion of a compact connected Lie group and the case where X is an exotic p-compact group separately. The case where X is the $\mathbf{F}_p$-completion of a compact connected Lie group of course follows directly from the classical Peter-Weyl theorem (cf. e.g. [23, Thm. III.4.1]), and so we can concentrate on the case where X is exotic. If p does not divide the order of the Weyl group the statement is also obvious: The inclusion $\breve{T} \to \mathrm{U}(r)$ induces a map $\breve{T} \rtimes W \to \mathrm{U}(r|W|)$ whose $\mathbf{F}_p$-completion is a monomorphism. The remaining cases have been shown to have faithful representations by Castellana: If (W, L) is in family $2a$ then this is carried out in [30, Thm. E] and if (W, L) is one of the pairs $(G_{12}, p = 3)$, $(G_{29}, p = 5)$, $(G_{31}, p = 5)$, or $(G_{34}, p = 7)$ this is carried out in [29]. $\square$

We now turn to Theorem 1.7 which in fact follows easily from the classification. But let us first state the part which one can see by elementary means. (See also [100, Cor. 5.6] and [60, Lem. 9.3].) For a space Y, define $H_n^{\mathbf{Z}_p}(Y) = \lim_k H_n(Y; \mathbf{Z}/p^k)$.

Proposition 10.2 (cf. [58, Lem. 6.11] and [60, Lem. 9.3]). *Let X be a connected p-compact group. Then the natural composite map*

$$(L_X)_W \cong H_0(W; H_2^{\mathbf{Z}_p}(BT)) \to H_2^{\mathbf{Z}_p}(BX) \cong \pi_1(X)$$

induced by the inclusion $T \to X$ is surjective with finite kernel. In particular if $(L_X)_W$ is torsion-free then it is an isomorphism.

Proof. By [100, Thm. 1.4] X is isomorphic to a p-compact group of the form $(X' \times T'')/A$, where X' is a simply connected p-compact group, T'' is a p-compact torus, and A is a finite central subgroup of the product. Since the center of a connected p-compact group is contained in a maximal torus by [57, Thm. 7.5] we can assume that A is a subgroup of $T' \times T''$, where T' is a maximal torus for X', and hence $(T' \times T'')/A$ is a maximal torus for X.

Therefore we get the following diagram of fibration sequences:

$$\begin{tikzcd} BA \arrow[r] \arrow[d, equal] & BT' \times BT'' \arrow[r] \arrow[d] & B((T' \times T'')/A) \arrow[d] \\ BA \arrow[r] & BX' \times BT'' \arrow[r] & BX. \end{tikzcd}$$

The long exact sequence of homotopy groups and the five-lemma now show that $\pi_2(B((T \times T'')/A)) \to \pi_2(BX)$ is surjective which is the first statement in the proposition. To see that the kernel is finite note that by [56, Thm. 9.7(iii)] $H^2_{\mathbf{Q}_p}(BX) \to H^2_{\mathbf{Q}_p}(BT)^W$ is an isomorphism, which by dualizing to homology shows the claim.

That we get an isomorphism when $(L_X)_W$ is torsion-free is obvious from the general statement. □

Remark 10.3. One easily shows that the image of the differential d_3 : $H_3(W; \mathbf{Z}_p) \to H_0(W; H_2^{\mathbf{Z}_p}(BT))$ in the Serre spectral sequence for the fibration $BT \to B\mathcal{N}_X \to BW$ is always in the kernel of the surjective map of Proposition 10.2. By standard group cohomology (cf. [31]) the image of this differential identifies with the image of the map given by capping with the k-invariant $\gamma \in H^3(W; H_2^{\mathbf{Z}_p}(BT))$ of the extension. If one knew that the double coset formula held for p-compact groups (more precisely that $H^*(B\mathcal{N}; \mathbf{Z}_p) \xrightarrow{\mathrm{tr}^*} H^*(BX; \mathbf{Z}_p) \xrightarrow{\mathrm{res}} H^*(BT; \mathbf{Z}_p)$ is the restriction map, cf. [64, Ex. VI.4]) then it would easily follow that this image is in fact equal to the kernel of the map in Proposition 10.2, which would give a conceptual proof of the formula for the fundamental group. Note that by a result of Tits [135] (see also [63], [103], [6]) the extension class γ is always of order 2 for compact connected Lie groups. The next proposition gives the complete answer in the Lie group case.

PROPOSITION 10.4. *Let G be a compact connected Lie group. Then the map $\pi_1(T)_W \to \pi_1(G)$ is surjective with kernel $(\mathbf{Z}/2)^s$, where s is the number of direct factors of G isomorphic to a symplectic group* $\mathrm{Sp}(n)$, $n \geq 1$.

Proof. That the map is surjective follows as in the p-compact case, so we just have to identify the kernel. By [92, Thm. 1.6], for any compact connected Lie group G, $T^W = Z(G) \oplus (\mathbf{Z}/2)^s$, where s is the number of direct factors of G isomorphic to a special orthogonal group $\mathrm{SO}(2n+1)$, $n \geq 1$.

Consider the dual group $G^\vee$ of G obtained from the dual root diagram (see [20, §4, no. 8]). Then $G^\vee$ has fundamental group isomorphic to $\widehat{Z(G)}$, where the hat denotes the Poincaré dual group (see [20, §4, no. 9]). Likewise $\widehat{(L_{G^\vee})_W}$ is canonically isomorphic to T^W. Since duality is an involution on the set of compact connected Lie groups which sends direct factors to direct factors and $\mathrm{SO}(2n+1)$ to $\mathrm{Sp}(n)$ the claim about the fundamental group follows directly from the dual result about the center. □

Proof of Theorem 1.7. By Theorem 11.1 $(L_X)_{W_X} = 0$ for all exotic p-compact groups X, so Proposition 10.2 shows the formula in this case. By Theorem 1.2 we are hence reduced to showing the formula for X of the form $G\hat{_p}$ for some compact connected Lie group G. In this case the formula is well known and easy. Namely it follows from Remark 10.3 that the kernel of $(L_X)_{W_X} \to \pi_1(X)$ is an elementary abelian 2-group. Alternatively the same conclusion follows from the formula for the fundamental group of a compact connected Lie group (see [20, §4, no. 6, Prop. 11] or [1, Thm. 5.47], where in the notation of [1], $(1 - \varphi_r)\gamma_r = 2\gamma_r$). □

We now start to prove Theorems 1.8 and 1.9.

LEMMA 10.5. *Suppose X and X' are two connected p-compact groups with the same maximal torus normalizer $\mathcal{N}$. Then all elementary abelian p-subgroups of X are toral if and only if all elementary abelian p-subgroups of X' are toral.*

Furthermore, if for all toral elementary abelian p-subgroups $V \to X$ the centralizer $\mathcal{C}_X(V)$ is connected then all elementary abelian p-subgroups in X are toral.

Proof. Suppose that X has a nontoral elementary abelian p-subgroup $V \to X$. We can assume that it is minimal, in the sense that any elementary abelian p-subgroup of smaller rank is toral. Write $V = V' \oplus V''$, where V' has rank one. We can assume that $V \to X$ factors through $\mathcal{N}$ (indeed through $\mathcal{N}_p$ by [57, Prop. 2.14]) and that the restriction to V'' factors through T (by the minimality of V).

We want to show that the resulting map $V \to \mathcal{N} \to X'$ is also nontoral, by proving that the adjoint $V' \to \mathcal{C}_{X'}(V'')$ does not factor through the identity component $\mathcal{C}_{X'}(V'')_1$, which would be the case if $V \to X'$ was toral. In detail, proceed as follows: Let $\mathcal{N}''$ denote the maximal torus normalizer in $\mathcal{C}_X(V'')_1$, which by [57, Thm. 7.6(2)] can be described in terms of V'' and $\mathcal{N}$. The adjoint map $V' \to \mathcal{C}_{\mathcal{N}}(V'')$ cannot factor through $\mathcal{N}''$ since otherwise $V' \to \mathcal{C}_X(V'')$ would factor through $\mathcal{C}_X(V'')_1$ and hence be toral in $\mathcal{C}_X(V'')$ by [56, Prop. 5.6], contradicting that V is assumed to be nontoral. Note that $\breve{\mathcal{N}}''$ is normal in $C_{\breve{\mathcal{N}}}(V'')$ and $C_{\breve{\mathcal{N}}}(V'')/\breve{\mathcal{N}}'' \cong \pi_0(\mathcal{C}_X(V'')) \cong \pi_0(\mathcal{C}_{X'}(V''))$ (see [57, Rem. 2.11]). Hence $V' \to \pi_0(\mathcal{C}_{X'}(V''))$ is nontrivial, so $V \to \mathcal{N} \to X'$ is nontoral in X' as desired. The last part of the lemma is clear from the proof of the first part. □

Remark 10.6. Despite the above lemma it is not *a priori* clear how to determine whether a p-compact group X has the property that all elementary abelian p-subgroups are toral just by looking at $\mathcal{N}_X$ (but see [132, Thm. 2.28] for the Lie group case). However, by a case-by-case analysis (Theorem 1.8), this is the case if and only if all toral elementary abelian p-subgroups have connected centralizers.

Remark 10.7. Note that by Lannes' theory [88, Thm. 0.4] the property that every elementary abelian p-subgroup of X is toral is equivalent to $H^*(BX;\mathbf{F}_p) \to H^*(BT;\mathbf{F}_p)^{W_X}$ being an F-isomorphism. (See also Theorem 12.1 and Remark 12.3.)

We state the following well known lemma for easy reference.

LEMMA 10.8. *Suppose that X is a connected p-compact group such that $H^*(BX;\mathbf{Z}_p)$ is a polynomial algebra with generators concentrated in even degrees. Then all elementary abelian p-subgroups of X are toral.*

Proof. Let $\nu : E \to X$ be an elementary abelian p-subgroup. Then $H^*(B\mathcal{C}_X(\nu);\mathbf{F}_p)$ is a polynomial algebra with generators concentrated in even degrees by [61, Thm. 1.3] (note that Lannes' T-functor preserves objects concentrated in even degrees by [88, Prop. 2.1.3]), so in particular $\mathcal{C}_X(\nu)$ is connected. Lemma 10.5 now shows that all elementary abelian p-subgroups of X are toral. (Alternatively one can use Remark 12.3.) $\square$

Proof of Theorem 1.8. First note that the implications (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) follow from Theorem 12.1. The implication (3) $\Rightarrow$ (4) follows easily from Theorem 7.1. Namely, for all toral elementary abelian p-subgroups $V \to X$, Theorem 7.1 implies that $W_{\mathcal{C}_X(V)}$ is a reflection group, so by [57, Thm. 7.6] $\mathcal{C}_X(V)$ is connected, using the assumption that p is odd. But this implies that all elementary abelian p-subgroups are toral by Lemma 10.5.

We now prove the implication (4) $\Rightarrow$ (1). First note that by Theorem 11.1 and [58, Thm. 1.4] we can write $X \cong X' \times X''$ where X' has Weyl group $(W_G, L_G \otimes \mathbf{Z}_p)$, for some compact connected Lie group G, and $(W_{X''}, L_{X''})$ is a product of exotic finite $\mathbf{Z}_p$-reflection groups. Furthermore, since the normalizer of a connected p-compact group is split for p odd by [6, Thm. 1.2] we have $\mathcal{N}_{G\hat{_p}} \cong \mathcal{N}_{X'}$. Since by Lemma 10.5 the property of having all elementary abelian p-subgroups toral is a property which only depends on $\mathcal{N}$ we conclude that G has this property as well. But this implies that G has torsion-free $\mathbf{Z}_p$-cohomology by [11, Thm. B] (see also [132, Thm. 2.28]). In the exotic case, we know by the proof of Theorem 1.4 that we can find a p-compact group $\tilde{X}''$ which has the same maximal torus normalizer as X'' and which has torsion-free $\mathbf{Z}_p$-cohomology. Hence we have found a p-compact group $G\hat{_p} \times \tilde{X}''$ which has the same maximal torus normalizer as X and has torsion-free $\mathbf{Z}_p$-cohomology. Since by Theorem 1.4 a p-compact group is determined by its maximal torus normalizer we conclude that X in fact has torsion-free $\mathbf{Z}_p$-cohomology. (Alternatively, one can appeal to Remark 10.11 which shows that the property of having torsion-free $\mathbf{Z}_p$-cohomology only depends on $\mathcal{N}$.)

Finally we prove the implication (2) $\Rightarrow$ (1), where we seem to need the full strength of Theorem 1.1. Note that by Theorem 1.2 we can write $X \cong$

$G_p^{\wedge} \times X'$ where G is a compact connected Lie group and X' has torsion-free $\mathbf{Z}_p$-cohomology. Likewise if $BG_p^{\wedge}$ has torsion-free $\mathbf{Z}_p$-cohomology then $G_p^{\wedge}$ has torsion-free $\mathbf{Z}_p$-cohomology by [12, p. 93]. □

Remark 10.9. We make some remarks about Theorem 1.8. Since the implication (1) $\Rightarrow$ (4) follows from Lemma 10.8, we see that the implications (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (2), and (1) $\Rightarrow$ (4) follow by general arguments. In the case of compact connected Lie groups the implication (4) $\Rightarrow$ (3) has a general proof, by combining [132, Thm. 2.28] with [43], and likewise (2) $\Rightarrow$ (1) has a general proof by [12, p. 93]. We do not know non-case-by-case proofs of these implications for p-compact groups. (The implication (2) $\Rightarrow$ (1) is stated in [101, Thm. 4.2] but the proof is incorrect.) The remaining implications do not seem to have general proofs even for compact connected Lie groups. See also [132, §4], Remark 10.11 and Theorem 12.1.

Proof of Theorem 1.9. By adjointness (cf. Construction 2.1) it is obvious that (2) $\Rightarrow$ (3) since a rank one elementary abelian p-subgroup of a connected p-compact group is toral by [56, Prop. 5.6]. We prove the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) by reducing them to theorems for Lie groups via the classification of finite $\mathbf{Z}_p$-reflection groups. For the proof that (1) implies (2) or (3) we also have to rely on Theorem 1.7.

By Theorem 11.1 and [58, Thm. 1.4] we can write $X \cong X' \times X''$ where X' has Weyl group $(W_G, L_G \otimes \mathbf{Z}_p)$, for some compact connected Lie group G, and the Weyl group (W'', L'') of X'' is a product of exotic $\mathbf{Z}_p$-reflection groups. Note that each of the properties (1), (2) and (3) holds for X if and only if it holds for X' and X''.

By Theorem 11.1 and Proposition 10.2 we have $\pi_1(X'') = 0$ so X'' satisfies (1). By Theorems 12.2 and 7.1 combined with [57, Thm. 7.6], (2) also holds for X''. So we can without restriction assume that $X \cong X'$. Furthermore we have $\mathcal{N}_{G_p^{\wedge}} \cong \mathcal{N}_X$, by [6, Thm. 1.2] since p is odd. By [57, Thm. 7.6], (2) is a property which only depends on the maximal torus normalizer. By Lemma 10.5 and its proof, this is also the case for (3). Since (2) $\Leftrightarrow$ (3) for compact connected Lie groups by [132, Thm. 2.27] we see that (2) and (3) are equivalent for p-compact groups as well. If X satisfies (2), then so does $G_p^{\wedge}$ by the above and hence $\pi_1(G_p^{\wedge})$ is torsion-free by [132, Thm. 2.27]. Combining Propositions 10.4 and 10.2 then shows that $\pi_1(X)$ is torsion-free which proves (2) $\Rightarrow$ (1). Finally to see (1) $\Rightarrow$ (2), observe that by Theorem 1.7, $\pi_1(X) = \pi_1(G_p^{\wedge})$, which reduces the statement to the Lie group case, where the statement again follows from [132, Thm. 2.27]. □

Remark 10.10. It is not hard to see that the conjectural classification for $p = 2$ implies that Theorem 1.9 and the implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Rightarrow$ (3) in Theorem 1.8 also hold true for $p = 2$. However, in Theorem 1.8, (3) is not equivalent to the other conditions when $p = 2$. To see this observe that

$\mathbf{Z}_2[L_{\mathrm{SO}(2n+1)} \otimes \mathbf{Z}_2]^{W_{\mathrm{SO}(2n+1)}}$ is a polynomial algebra (because this holds for $\mathrm{Sp}(n)$, which has the same Weyl group) even though $\mathrm{SO}(2n+1)$ has 2-torsion.

Remark 10.11. Notbohm states his classification of connected p-compact groups with $\mathbf{Z}_p[L]^W$ a polynomial algebra in the setup of spaces BX with polynomial cohomology (cf. [108], [110]). This means that his uniqueness statement is *a priori* only uniqueness among p-compact groups with torsion-free $\mathbf{Z}_p$-cohomology (cf. Theorem 12.1). We will here briefly sketch a direct but case-by-case way (following a line of argument given in a special case in [101, Pf. of Thm. 5.3]) to show that for a p-compact group the property of having torsion-free $\mathbf{Z}_p$-cohomology depends only on (W, L), which allows us to remove the extra assumption.

Assume that X is a connected p-compact group, p odd, such that $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra. We want to show that $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra as well. By Theorem 12.2(1), $(L_X)_{W_X}$ is torsion-free, and so $\pi_1(X) = (L_X)_{W_X}$ by Proposition 10.2. By the Serre and Eilenberg-Moore spectral sequences $H^*(BX; \mathbf{Z}_p)$ is a polynomial algebra if and only if $H^*(B(X\langle 1\rangle); \mathbf{Z}_p)$ is a polynomial algebra. Furthermore by construction $L_{X\langle 1\rangle} = SL_X$ and by Theorem 12.2(1) $\mathbf{Z}_p[L_{X\langle 1\rangle}]^{W_X}$ is also a polynomial algebra, so we can without loss of generality assume that X is simply connected. By [58, Thm. 1.4 and Rem. 1.6] we can furthermore assume that X is a simple p-compact group.

By [6, Thm. 1.2] $\breve{\mathcal{N}}_X = \breve{T}_X \rtimes W_X$. Using Theorems 11.1 and 12.2 we first show that the cohomology of $\breve{\mathcal{N}}_X$ is detected by elementary abelian p-subgroups. More precisely we show that in each case there is a compact connected Lie group H such that $\breve{\mathcal{N}}_X$ contains a subgroup isomorphic to $\breve{\mathcal{N}}_{H\hat{_p}}$ with index prime to p having the required property. When $p \nmid |W|$ we take H to be a torus, and if (W_X, L_X) is in family 1 or family $2a$ we take $H = \mathrm{SU}(n)$ and $H = \mathrm{U}(n)$ respectively. If (W_X, L_X) is one of the exotic $\mathbf{Z}_p$-reflection groups $(G_{12}, p = 3)$, $(G_{29}, p = 5)$, $(G_{31}, p = 5)$, or $(G_{34}, p = 7)$ we take $H = \mathrm{SU}(p)$; cf. the proof of Theorem 7.1. The only remaining cases are the ones where $(W_X, L_X) = (W_G, L_G \otimes \mathbf{Z}_p)$ for one of the following pairs (G, p): $(G_2, p = 3)$, $(3E_6, p = 5)$, $(2E_7, p = 5)$, $(2E_7, p = 7)$, and $(E_8, p = 7)$. In these cases we can by [82, Prop. 6.11] take $H = \mathrm{SU}(3)$, $\mathrm{SU}(2) \times_{C_2} \mathrm{SU}(6)$, $\mathrm{SU}(8)/C_2$, $\mathrm{SU}(8)/C_2$ and $\mathrm{SU}(9)/C_3$ respectively. Since both $\breve{\mathcal{N}}_{\mathrm{U}(n)\hat{_p}}$ and $\breve{\mathcal{N}}_{\mathrm{SU}(n)\hat{_p}}$ have cohomology which is detected by elementary abelian p-subgroups by [117, Prop. 3.4] (for $\mathcal{N}_{\mathrm{U}(n)\hat{_p}}$; $\mathcal{N}_{\mathrm{SU}(n)\hat{_p}}$ follows from this, cf. [105, Lem. 12.6]) we see that in all cases the cohomology of $\breve{\mathcal{N}}_{H\hat{_p}}$ is detected by elementary abelian p-subgroups. Hence, by a transfer argument, the mod p cohomology of BX is detected by elementary abelian p-subgroups.

Next, we want to show that all elementary abelian p-subgroups of X factor through a maximal torus. By Lemma 10.5 we just have to show that we can find *some* p-compact group X' with the same maximal torus normalizer which has this property. If (W_X, L_X) is of Lie type this follows by combining

Theorem 12.2(2) with Borel's theorem [11, Thm. B]. If (W_X, L_X) is exotic this is also true since we know (by Theorem 7.3 or Notbohm's work [110]) that there exists a p-compact group with Weyl group (W_X, L_X) and classifying space having polynomial $\mathbf{Z}_p$-cohomology algebra.

The fact that all elementary abelian p-subgroups of X are toral combined with the fact that the cohomology is detected by elementary abelian p-subgroups implies that the mod p cohomology of BX is concentrated in even degrees. Hence $H^*(BX; \mathbf{Z}_p)$ is torsion-free as wanted.

Proof of Theorem 1.10. Let X be a connected finite loop space with maximal torus $i : T \to X$. Note that $(X/T)\hat{_p} \simeq X\hat{_p}/T\hat{_p}$ by the fiber lemma [22, II.5.1], and consequently, by the definition of the Euler characteristic, $\chi(X/T) = \chi(X\hat{_p}/T\hat{_p})$. Hence $T\hat{_p} \to X\hat{_p}$ will be a maximal torus for the p-compact group $X\hat{_p}$, for all primes p.

For our connected finite loop space X, define $W_X(T)$ to be the set of conjugacy classes of self-equivalences φ of T such that i and $i\varphi$ are conjugate. We obviously have an injective homomorphism $W_X \to W_{X\hat{_p}}$ for all primes p and we now want to see that this map is surjective as well, so that we can naturally identify $(W_X, \pi_1(T) \otimes \mathbf{Z}_p)$ with $(W_{X\hat{_p}}, L_{X\hat{_p}})$.

First note that by [56, Pf. of Thm. 9.7] we can view $W_{X\hat{_p}}$ as the Galois group of the extension of polynomial algebras $H^*_{\mathbf{Q}_p}(BX) \to H^*_{\mathbf{Q}_p}(BT)$. But, since BX has finitely many cells in each dimension and since BX is nilpotent, we can identify

$$\begin{tikzcd} H^*(BX;\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p \arrow[r] \arrow[d, "\cong"] & H^*(BT;\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p \arrow[d, "\cong"] \\ H^*_{\mathbf{Q}_p}(BX) \arrow[r] & H^*_{\mathbf{Q}_p}(BT) \end{tikzcd}$$

so the extensions $H^*(BX; \mathbf{Q}) \to H^*(BT; \mathbf{Q})$ and $H^*_{\mathbf{Q}_p}(BX) \to H^*_{\mathbf{Q}_p}(BT)$ have canonically isomorphic Galois groups. Hence any element in $W_{X\hat{_p}}$ lifts to a canonical element in the Galois group of the extension $H^*(BX; \mathbf{Q}) \to H^*(BT; \mathbf{Q})$. However, since $BX_{\mathbf{Q}}$ and $BT_{\mathbf{Q}}$ are products of Eilenberg-Mac Lane spaces (cf. e.g. [124, Ch. V, §4, Prop. 6]), this Galois group identifies with the self-equivalences $BT_{\mathbf{Q}} \to BT_{\mathbf{Q}}$ over $Bi_{\mathbf{Q}} : BT_{\mathbf{Q}} \to BX_{\mathbf{Q}}$, where as usual $Bi_{\mathbf{Q}}$ has been replaced by an equivalent fibration. Hence any element in $W_{X\hat{_p}}$ gives rise to a compatible family of self-equivalences of $BT_{\mathbf{Q}}$ and $BT\hat{_l}$, for all primes l. Hence, by the arithmetic square [22, VI.8.1], we get a self-equivalence of BT over BX, i.e., an element in W_X. The constructed element is a lift of the element in $W_{X\hat{_p}}$ we started with, and so the map $W_X \to W_{X\hat{_p}}$ is surjective as well.

Likewise, the argument above showed that W_X is the Galois group of the extension $H^*(BX;\mathbf{Q}) \to H^*(BT;\mathbf{Q})$, so we have an isomorphism

$$H^*(BX;\mathbf{Q}) \xrightarrow{\cong} H^*(BT;\mathbf{Q})^{W_X}.$$

Since $H^*(BX;\mathbf{Q})$ is a polynomial algebra, $(W_X, \pi_1(T))$ is a $\mathbf{Z}$-reflection group by the Shephard-Todd-Chevalley theorem (see [10, Thm. 7.2.1]). Hence, by Theorem 11.1, $(W_X, \pi_1(T))$ is the Weyl group of some compact connected Lie group G.

For each p we have an extension class $\gamma_p \in H^3(W_X; \pi_1(T) \otimes \mathbf{Z}_p)$ corresponding to the fibration sequence $BT\hat{_p} \to B\mathcal{N}_{X\hat{_p}} \to BW_X$. Since $H^3(W_X;\pi_1(T))$ is a finite abelian group, and hence given as a sum of its p-primary parts, these extension classes identify with a unique extension class $\gamma \in H^3(W_X;\pi_1(T))$. We define the loop space $\mathcal{N}_X$ to be the loop space of the total space in the fibration sequence $BT \to B\mathcal{N}_X \to BW_X$ with the canonical action of W_X on BT and extension class γ. Since the fiber-wise $\mathbf{F}_p$-completion of $B\mathcal{N}_X$ with respect to this defining fibration identifies with $B\mathcal{N}_{X\hat{_p}}$, the arithmetic square produces a canonical morphism $\mathcal{N}_X \to X$. ($\mathcal{N}_X$ is, quite naturally, called the maximal torus normalizer of the finite loop space X [101, Def. 1.3].)

By [6] (see also [103], [63]) the extension classes defining $T \to \mathcal{N}_X \to W_X$ and $T \to N_G(T) \to W_G(T)$ are both 2-torsion. Let $\widetilde{BN}$ denote the fiber-wise $\mathbf{Z}[\frac{1}{2}]$-localization of the total space of the fibration $BT \to BN_G(T) \to BW_G(T)$ or equivalently the corresponding fibration with $B\mathcal{N}_X$. We hence have embeddings

$$\begin{tikzcd} & \widetilde{BN} \arrow[dl] \arrow[dr] & \\ BX[\frac{1}{2}] & & BG[\frac{1}{2}]. \end{tikzcd}$$

By the arithmetic square [22, VI.8.1], the following square is a pullback

$$\begin{tikzcd} BX[\frac{1}{2}] \arrow[r] \arrow[d] & \prod_{p\neq 2} BX\hat{_p} \arrow[d] \\ BX_{\mathbf{Q}} \arrow[r] & (\prod_{p\neq 2} BX\hat{_p})_{\mathbf{Q}} \end{tikzcd}$$

and similarly for BG. By Theorem 1.4 we can construct unique maps between $\mathbf{F}_p$-completions under $\widetilde{BN}$, and we obviously also have a unique map between the rationalizations. By construction (as maps under $\widetilde{BN}$) these maps agree on the rationalization of the product of the $\mathbf{F}_p$-completions, so by the arithmetic square we get an induced map $BX[\frac{1}{2}] \to BG[\frac{1}{2}]$ which by construction is an $\mathbf{F}_p$-equivalence for all primes p. Since both spaces are one-connected this implies that the map is a homotopy equivalence. $\square$

11. Appendix: The classification of finite $\mathbf{Z}_p$-reflection groups

The purpose of this appendix is to give a short proof of the classification of finite $\mathbf{Z}_p$-reflection groups (Theorem 11.1), simplifying work of Notbohm [107], [109], and as a by-product extending his results to all primes. We likewise explain how this classification relates to the classification of finite $\mathbf{Z}$-reflection groups and the classification of compact Lie groups (Theorem 11.5). We start by recalling some definitions. Let R be an integral domain with field of fractions K. An *R-reflection group* is a pair (W, L) where L is a finitely generated free R-module, and W is a subgroup of $\mathrm{Aut}(L)$ generated by elements α such that $1-\alpha$ has rank one viewed as a matrix over K. Two finite R-reflection groups (W, L) and (W', L') are called *isomorphic*, if we can find an R-linear isomorphism $\varphi : L \to L'$ such that the group $\varphi W \varphi^{-1}$ equals W'. A finite R-reflection group (W, L) is said to be *irreducible* if the corresponding representation of W on $L \otimes_R K$ is irreducible. If R has characteristic zero we define the *character field* of an R-reflection group (W, L) as the field extension of $\mathbf{Q}$ generated by the values of the character of the representation $W \hookrightarrow \mathrm{Aut}(L)$. For $R = \mathbf{Z}_p$ or $\mathbf{Q}_p$ we define an *exotic* R-reflection group to be a finite irreducible R-reflection group with character field strictly containing $\mathbf{Q}$.

The classification of finite $\mathbf{Z}_p$-reflection groups is based on the work of Clark-Ewing [34] and Dwyer-Miller-Wilkerson [52], which is again based on the classification of finite $\mathbf{C}$-reflection groups by Shephard-Todd [126] (see also [35]). The result of Clark-Ewing and Dwyer-Miller-Wilkerson is that there is a bijection between finite $\mathbf{Q}_p$-reflection groups and finite $\mathbf{C}$-reflection groups whose character field embeds in $\mathbf{Q}_p$ (for details see [52, Prop. 5.4, Prop. 5.5 and Pf. of Thm. 1.5]). The classification of finite complex reflection groups by Shephard-Todd [126] is as follows: Up to isomorphism, the irreducible finite complex reflection groups fall into three infinite families and 34 sporadic cases. We follow the notation of Shephard-Todd and label the three infinite families as 1, 2 and 3 and the sporadic cases as G_i, $4 \le i \le 37$. Moreover any finite complex reflection group can be written as a direct product of irreducible finite complex reflection groups; cf. [62, Rem. 2.3] (in fact this holds over any field of characteristic 0).

It is convenient to split family 2 further depending on the character field. The associated complex reflection group is the group $G(m, r, n)$ (where m, r and n are integers with $m, n \ge 2$, $r \ge 1$, $r \mid m$ and $(m, r, n) \ne (2, 2, 2)$) from [126, p. 277] which consists of monomial $n \times n$-matrices such that the nonzero entries are mth roots of unity and the product of the nonzero entries is an (m/r)th root of unity. Thus $G(m, r, n)$ is the semidirect product of its subgroup $A(m, r, n)$ of diagonal matrices with the subgroup of permutation matrices. Let $\zeta_m = e^{2\pi i/m}$. For $n \ge 3$ or $n = 2$ and $r \ne m$ the character field of $G(m, r, n)$ equals $\mathbf{Q}(\zeta_m)$, and for $n = 2$ and $r = m$ it equals $\mathbf{Q}(\zeta_m + \zeta_m^{-1})$

(see [34, p. 432–433]). Following [34] these are denoted as family $2a$ and family $2b$ respectively.

A complete list of the irreducible finite complex reflection groups, their character fields and the primes for which these embed in $\mathbf{Q}_p$ can be found in [85, p. 165], [10, Table 7.1], or [6, Table 1].

If (W, V) is a finite $\mathbf{Q}_p$-reflection group, then by [40, Prop. 23.16] we can find a (nonunique) finitely generated $\mathbf{Z}_pW$-submodule $L \subseteq V$ with $L \otimes \mathbf{Q} = V$. Thus any finite $\mathbf{Q}_p$-reflection group may be obtained from a finite $\mathbf{Z}_p$-reflection group by extension of scalars, but in general there are several nonisomorphic $\mathbf{Z}_p$-reflection groups which give rise to the same $\mathbf{Q}_p$-reflection group. The following result extends [109, Thm. 1.5 and Prop. 1.6] to all primes. (See also the addendum Theorem 11.5 for an elaboration.)

Theorem 11.1 (The classification of finite $\mathbf{Z}_p$-reflection groups). *Let (W, L) be a finite $\mathbf{Z}_p$-reflection group. Then there exists a decomposition*

$$(W, L) = (W_1 \times W_2, L_1 \oplus L_2)$$

where $(W_1, L_1) \cong (W_G, L_G \otimes \mathbf{Z}_p)$, for some (nonunique) compact connected Lie group G with Weyl group W_G and integral lattice L_G, and (W_2, L_2) is a (up to permutation unique) direct product of exotic $\mathbf{Z}_p$-reflection groups.

The canonical map $(W, L) \mapsto (W, L \otimes \mathbf{Q})$ gives a one-to-one correspondence between exotic $\mathbf{Z}_p$-reflection groups up to isomorphism and exotic $\mathbf{Q}_p$-reflection groups up to isomorphism.

If (W, L) is an exotic $\mathbf{Z}_p$-reflection group, then $L \otimes \mathbf{F}_p$ is an irreducible $\mathbf{F}_pW$-module, and in particular $(L \otimes \mathbf{Z}/p^\infty)^W = 0$ and $H_0(W; L) = 0$.

Remark 11.2. For odd primes p the last two statements say that any exotic $\mathbf{Z}_p$-reflection group is respectively *center-free* and *simply connected*; cf. [107]. Note also that [109, Thm. 1.5] imposes the unnecessarily strong condition that the invariant ring $\mathbf{Z}_p[L]^W$ is a polynomial algebra, but this condition is not actually used in [109].

Before the proof of Theorem 11.1 we need two lemmas. First recall the following elementary fact about elements of finite order in $\mathrm{GL}_n(\mathbf{Z}_p)$.

Lemma 11.3. *Let $G \subseteq \mathrm{GL}_n(\mathbf{Z}_p)$ be a finite subgroup. Then the* mod p *reduction $G \hookrightarrow \mathrm{GL}_n(\mathbf{Z}_p) \to \mathrm{GL}_n(\mathbf{F}_p)$ is injective if p is odd. For $p = 2$ the kernel of the composition is an elementary abelian 2-subgroup of rank at most n^2. In particular the kernel is contained in $O_2(G)$, the largest normal 2-subgroup of G.*

Proof. It is easy to see directly that any nontrivial finite order element in $\mathrm{GL}_n(\mathbf{Z}_p)$ has nontrivial reduction mod p if p is odd (cf. [127, Pf. of Lem. 10.7.1]). For $p = 2$ the same argument shows that this is true if we reduce mod 4. The result now follows. □

Lemma 11.4. *For any exotic $\mathbf{Q}_p$-reflection group (W, V) there exists a finitely generated $\mathbf{Z}_pW$-submodule $L \subseteq V$ with $L \otimes \mathbf{Q} = V$, such that $L \otimes \mathbf{F}_p$ is an irreducible $\mathbf{F}_pW$-module.*

Proof. Assume first that $p \nmid |W|$. By [40, Prop. 23.16] we can find a finitely generated $\mathbf{Z}_pW$-submodule $L \subseteq V$ with $L \otimes \mathbf{Q} = V$. It follows from [39, 75.6 and 76.15] that $L \otimes \mathbf{F}_p$ is automatically an irreducible $\mathbf{F}_pW$-module.

Assume now that W has order divisible by p. From the Clark-Ewing list we see that the only exotic $\mathbf{Q}_p$-reflection groups satisfying this condition are the groups $G(m, r, n)$ from family $2a$ and the groups G_{12} for $p = 3$, G_{24} for $p = 2$, G_{29} and G_{31} for $p = 5$ and G_{34} for $p = 7$.

In case $W = G(m, r, n)$ from family $2a$ we get the extra conditions $m \geq 3$, $p \equiv 1 \pmod{m}$ and $p \leq n$. Note in particular that $n \geq 3$. The description above directly gives a representation with entries in $\mathbf{Z}_p$ since the multiplicative group of $\mathbf{Z}_p$ contains the $(p-1)$th roots of unity. Let $L = (\mathbf{Z}_p)^n$ be the natural $\mathbf{Z}_pW$-module, i.e., the set of columns with entries in $\mathbf{Z}_p$. Assume that $0 \neq M \subseteq L \otimes \mathbf{F}_p$ is an $\mathbf{F}_pW$-submodule of $L \otimes \mathbf{F}_p$. Choose $x \in M$ with $x \neq 0$ and let $\theta \in \mathbf{F}_p$ be a primitive mth root of unity. Since W contains the permutation matrices and the diagonal matrix $\operatorname{diag}(\theta, \theta^{-1}, 1, \ldots, 1)$ we see that M contains an element of the form $x' = (x_1, x_2, 0, \ldots, 0)$ with $x_1 \neq 0$. Since $n \geq 3$, W also contains the diagonal matrix $\operatorname{diag}(\theta, 1, \theta^{-1}, 1, \ldots, 1)$ and hence M contains $((1-\theta)x_1, 0, \ldots, 0)$. As $\theta \neq 1$ and W contains all permutation matrices we conclude that $M = L \otimes \mathbf{F}_p$, proving the claim for the groups from family $2a$.

Next consider $W = G_{12}$ at $p = 3$. Since W is isomorphic to $\mathrm{GL}_2(\mathbf{F}_3)$, Lemma 11.3 shows that for any finitely generated $\mathbf{Z}_3W$-submodule $L \subseteq (\mathbf{Q}_3)^2$ of rank 2, we may identify $L \otimes \mathbf{F}_3$ with the natural $\mathbf{F}_3\,\mathrm{GL}_2(\mathbf{F}_3)$-module. In particular $L \otimes \mathbf{F}_3$ is an irreducible $\mathbf{F}_3W$-module.

For $W = G_{24}$ at $p = 2$ we have $W \cong \mathbf{Z}/2 \times \mathrm{GL}_3(\mathbf{F}_2)$. Hence Lemma 11.3 shows that for any finitely generated $\mathbf{Z}_2W$-submodule $L \subseteq (\mathbf{Q}_2)^3$ of rank 3, we may identify $L \otimes \mathbf{F}_2$ with the $\mathbf{F}_2(\mathbf{Z}/2 \times \mathrm{GL}_3(\mathbf{F}_2))$-module where $\mathbf{Z}/2$ acts trivially and $\mathrm{GL}_3(\mathbf{F}_2)$ acts naturally. In particular $L \otimes \mathbf{F}_2$ is an irreducible $\mathbf{F}_2W$-module.

Next consider the groups G_{29} and G_{31} at $p = 5$. Since G_{29} is contained in G_{31} it suffices to show the result for $W = G_{29}$. The representation in [126, p. 298] is defined over $\mathbf{Z}[\frac{1}{2}, i]$ and hence we get a representation over $\mathbf{Z}_5$ by mapping i to a primitive 4th root of unity in $\mathbf{Z}_5$. Let $L = (\mathbf{Z}_5)^4$ be the natural $\mathbf{Z}_5W$-module. There are 40 reflections in G_{29}: The 24 reflections in the hyperplanes of the form $x_j - i^\alpha x_k = 0$, $j \neq k$, and the 16 reflections in the hyperplanes of the form $\sum_{j=1}^4 i^{\alpha_j} x_j = 0$ with $\sum_{j=1}^4 \alpha_j \equiv 0 \pmod 4$. In particular G_{29} contains the reflections in the hyperplanes $x_j - x_k = 0$ and thus G_{29} contains all permutation matrices. The product of the reflections in the hyperplanes $x_1 - ix_2 = 0$ and $x_1 - x_2 = 0$ equals the diagonal matrix

$\mathrm{diag}(i, -i, 1, 1)$ and thus this element is also contained in G_{29}. Now the same argument used in the case of the groups from family $2a$ shows that $L \otimes \mathbf{F}_5$ is an irreducible $\mathbf{F}_5 W$-module.

The argument for the group $W = G_{34}$ at $p = 7$ is similar. The representation given in [126, p. 298] is defined over $\mathbf{Z}[\frac{1}{3}, \omega]$, $\omega = \zeta_3$ and hence we get a representation over $\mathbf{Z}_7$ by mapping ω to a primitive 3rd root of unity in $\mathbf{Z}_7$. Let $L = (\mathbf{Z}_7)^6$ be the natural $\mathbf{Z}_7 W$-module. There are 126 reflections in G_{34}: The 45 reflections in the hyperplanes of the form $x_j - \omega^\alpha x_k = 0$, $j \neq k$, and the 81 reflections in the hyperplanes of the form $\sum_{j=1}^{6} \omega^{\alpha_j} x_j = 0$ with $\sum_{j=1}^{6} \alpha_j \equiv 0 \pmod 3$. In particular G_{34} contains all permutation matrices. The product of the reflections in the hyperplanes $x_1 - \omega x_2 = 0$ and $x_1 - x_2 = 0$ equals the diagonal matrix $\mathrm{diag}(\omega, \omega^2, 1, 1, 1, 1)$ and thus this element is also contained in G_{34}. As above we then see that $L \otimes \mathbf{F}_7$ is an irreducible $\mathbf{F}_7 W$-module. $\square$

Proof of Theorem 11.1. Assume first that (W, V) is an exotic $\mathbf{Q}_p$-reflection group. Lemma 11.4 shows that there exists a finitely generated $\mathbf{Z}_p W$-submodule $L \subseteq V$ with $L \otimes \mathbf{Q} = V$, such that $L \otimes \mathbf{F}_p$ is an irreducible $\mathbf{F}_p W$-module. It then follows from [125, 15.2, Ex. 3] that L is unique up to a homothety (i.e. up to scaling by a unit in $\mathbf{Q}_p$). This gives the bijection between exotic $\mathbf{Z}_p$-reflection groups and exotic $\mathbf{Q}_p$-reflection groups.

Since $L \otimes \mathbf{F}_p$ is an irreducible $\mathbf{F}_p W$-module we also conclude that $(L \otimes \mathbf{F}_p)^W = 0$ and $H_0(W; L \otimes \mathbf{F}_p) = 0$. Hence we get $(L \otimes \mathbf{Z}/p^\infty)^W = 0$ as claimed. We also see that multiplication by p is surjective on $H_0(W; L)$ and from this we obtain $H_0(W; L) = 0$ by Nakayama's lemma. This proves the part of the theorem pertaining to exotic $\mathbf{Z}_p$-reflection groups.

Now consider a finite $\mathbf{Z}_p$-reflection group (W, L) such that there is a direct sum decomposition $L \otimes \mathbf{Q} = V_1 \oplus V_2$ as $\mathbf{Q}_p W$-modules. Let W_1 (resp. W_2) be the subgroup of W which fixes V_2 (resp. V_1) pointwise. It it easy to see (cf. [58, Lem. 6.3]) that (W_i, V_i) is a $\mathbf{Q}_p$-reflection group and that we get the decomposition $(W, L \otimes \mathbf{Q}) = (W_1 \times W_2, V_1 \oplus V_2)$.

We now claim that if (W_2, V_2) is an exotic $\mathbf{Q}_p$-reflection group, then we have the decomposition $(W, L) = (W_1 \times W_2, L_1 \oplus L_2)$ with $L_i = L \cap V_i$. Let $\alpha : L_1 \oplus L_2 \longrightarrow L$ be the addition map. As in [58, Pf. of Thm. 1.5] it suffices to prove that $\alpha \otimes \mathbf{Z}/p^\infty : (L_1 \otimes \mathbf{Z}/p^\infty) \oplus (L_2 \otimes \mathbf{Z}/p^\infty) \longrightarrow L \otimes \mathbf{Z}/p^\infty$ is injective. Assume that (x_1, x_2) is in the kernel of $\alpha \otimes \mathbf{Z}/p^\infty$, $x_i \in L_i \otimes \mathbf{Z}/p^\infty$. Thus $x_1 + x_2 = 0$. If $s \in W_2$ is a reflection we have $s \cdot x_1 = x_1$ by definition, and hence s also fixes $x_2 = -x_1$. Since W_2 is generated by reflections we get $x_2 \in (L_2 \otimes \mathbf{Z}/p^\infty)^W$ and hence $x_2 = 0$ by the results already proved for exotic $\mathbf{Z}_p$-reflection groups. Hence $x_1 = 0$ as well, and thus $\alpha \otimes \mathbf{Z}/p^\infty$ is injective proving the claim.

Since any finite $\mathbf{Q}_p$-reflection group may be decomposed into a (up to permutation unique) product of finite irreducible ones, we see by using the claim repeatedly that any finite $\mathbf{Z}_p$-reflection group (W, L) may be decomposed as a product $(W, L) \cong (W_1 \times W_2, L_1 \oplus L_2)$ where (W_1, L_1) is a $\mathbf{Z}_p$-reflection group with character field equal to $\mathbf{Q}$ and (W_2, L_2) is as in the theorem.

To finish the proof we thus need to show that for any finite $\mathbf{Z}_p$-reflection group (W, L) with character field equal to $\mathbf{Q}$ we may find a compact connected Lie group G such that (W, L) is isomorphic to $(W_G, L_G \otimes \mathbf{Z}_p)$. We start by reducing the problem to finite $\mathbf{Z}$-reflection groups. The representation $W \to \mathrm{GL}(L \otimes \mathbf{Q})$ is a reflection representation and hence has Schur index 1 by [34, Cor. p. 429]. Thus this representation is equivalent to a representation defined over $\mathbf{Q}$. Hence [40, Cor. 30.10] applied to $R = \mathbf{Z}_{(p)}$ shows that there exists a (unique) finitely generated $\mathbf{Z}_{(p)}W$-submodule $L' \subseteq L$ with $L' \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_p = L$. Now [40, Cor. 23.14] applied to $R = \mathbf{Z}$ shows that L' contains a (nonunique) finitely generated $\mathbf{Z}W$-submodule $L'' \subseteq L'$ with $L' = L'' \otimes \mathbf{Z}_{(p)}$. We conclude in particular that $(W, L) \cong (W, L'' \otimes \mathbf{Z}_p)$.

We finish the proof by showing that there exists a (nonunique) compact connected Lie group G whose Weyl group (W_G, L_G) is isomorphic to (W, L''). For each reflection $s \in W$ the group $\{x \in L'' \mid s(x) = -x\}$ is an infinite cyclic group with two generators which we label $\pm\alpha_s$. Let $\Phi = \{\pm\alpha_s \mid s \text{ is a reflection in } W\}$ and $L''_0 = (L'')^W$. It then follows (cf. [116, p. 85]) that (L'', L''_0, Φ) is a reduced root diagram whose associated $\mathbf{Z}$-reflection group equals (W, L'') (see [20, §4, no. 8] for definitions). From the classification of compact connected Lie groups ([20, §4, no. 9, Prop. 16]) it then follows that there exists a compact connected Lie group G whose root diagram equals (L'', L''_0, Φ). In particular $(W_G, L_G) \cong (W, L'')$ and we are done. $\square$

We now analyze concretely when two compact connected Lie groups give rise to the same p-compact group. For a compact connected Lie group G, let $G\langle 1\rangle$ denote the universal cover of G. Furthermore let H be the direct product of the identity component of the center, $Z(G)_1$, with the universal cover of the derived group of G. We have a canonical covering homomorphism $\varphi : H \to G$ with finite kernel (cf. [20, §1, no. 4, Prop. 4]). If p is a prime number, we let $\mathrm{Cov}^{p'}(G)$ denote the covering of G corresponding to the subgroup of $\pi_1(G)$ given as the preimage of the Sylow p-subgroup of $\pi_1(G)/\varphi(\pi_1(H))$, and let K denote the kernel of $H \to \mathrm{Cov}^{p'}(G)$. Write $H = R_1 \times \cdots \times R_n \times S \times T'$ where each R_i is a special unitary group, S is a simply connected compact Lie group which contains no direct factors isomorphic to a special unitary group, and T' is a torus. Suppose that K_1 and K_2 are finite central p-subgroups of H. We say that K_1 and K_2 are *p-equivalent* subgroups of H if there exist integers $k_1, \ldots, k_n, k$ prime to p such that the homomorphism

$$\Psi = \psi^{k_1} \times \cdots \times \psi^{k_n} \times 1 \times \alpha_k : T_{R_1} \times \cdots T_{R_n} \times T_S \times T' \to T_{R_1} \times \cdots T_{R_n} \times T_S \times T'$$

induces an isomorphism from K_1 onto K_2, where T_{R_i} is a maximal torus of R_i, ψ^l is the lth power map, and $\alpha_k : T' \to T'$ is a homomorphism which with respect to some splitting $T' = S^1 \times \cdots \times S^1$ has the form $1 \times \ldots \times 1 \times \psi^k$. More generally we say that H/K and H'/K' are *p-equivalent* if there exists an isomorphism between H and H' such that the image of K in H' is p-equivalent to K'. This terminology is justified by the following theorem.

THEOREM 11.5 (Addendum to Theorems 1.1 and 11.1). *Let G and G' be two compact connected Lie groups and p a prime number. Then*

(1) *(W_G, L_G) and $(W_{G'}, L_{G'})$ are isomorphic if and only if G is isomorphic to G' up to the substitution of direct factors isomorphic to* $\mathrm{Sp}(n)$ *with direct factors isomorphic to* $\mathrm{SO}(2n+1)$.

(2) *$(W_G, L_G \otimes \mathbf{Z}_2)$ and $(W_{G'}, L_{G'} \otimes \mathbf{Z}_2)$ are isomorphic if and only if* $\mathrm{Cov}^{2'}(G)$ *and* $\mathrm{Cov}^{2'}(G')$ *are 2-equivalent up to the substitution of direct factors isomorphic to* $\mathrm{Sp}(n)$ *with direct factors isomorphic to* $\mathrm{SO}(2n+1)$. *Moreover the following conditions are equivalent*:

 (a) *$(W_G, L_G \otimes \mathbf{Z}_2, L_{G\langle 1\rangle} \otimes \mathbf{Z}_2)$ and $(W_{G'}, L_{G'} \otimes \mathbf{Z}_2, L_{G'\langle 1\rangle} \otimes \mathbf{Z}_2)$ are isomorphic.*

 (b) $\mathrm{Cov}^{2'}(G)$ *is 2-equivalent to* $\mathrm{Cov}^{2'}(G')$.

 (c) $(BG)\hat{_2} \simeq (BG')\hat{_2}$.

(3) *For p odd the following conditions are equivalent*:

 (a) *$(W_G, L_G \otimes \mathbf{Z}_p)$ and $(W_{G'}, L_{G'} \otimes \mathbf{Z}_p)$ are isomorphic.*

 (b) $\mathrm{Cov}^{p'}(G)$ *and* $\mathrm{Cov}^{p'}(G')$ *are p-equivalent up to the substitution of direct factors isomorphic to* $\mathrm{Sp}(n)$ *with direct factors isomorphic to* $\mathrm{Spin}(2n+1)$.

 (c) $(BG)\hat{_p} \simeq (BG')\hat{_p}$.

Note that two simple compact Lie groups G and G' of the form $\mathrm{Cov}^{p'}(\cdot)$ are p-equivalent if and only if they are isomorphic. The next two examples show how this fails in general.

Example 11.6. Let $\zeta = e^{2\pi i/p}$ and $G = \mathrm{SU}(p) \times \mathrm{SU}(p)$, and let Δ_1 be the central subgroup of G generated by $(\zeta I, \zeta I)$ and Δ_2 the central subgroup generated by $(\zeta I, \zeta^2 I)$. Then G/Δ_1 and G/Δ_2 are nonisomorphic as Lie groups if $p \geq 5$, but they are p-equivalent; similar examples can be constructed for $p = 2, 3$.

Example 11.7. Let $p \geq 5$, $\zeta = e^{2\pi i/p}$, and $G = \mathrm{SU}(p) \times \mathrm{SU}(p) \times \mathrm{SU}(p) \times S^1 \times S^1$. Consider the subgroups

$$\Delta_1 = \langle(\zeta I, I, \zeta I, \zeta, 1), (I, \zeta I, \zeta I, 1, \zeta)\rangle\,,\ \Delta_2 = \langle(\zeta I, I, \zeta I, \zeta, 1), (I, \zeta I, \zeta I, 1, \zeta^2)\rangle.$$

The quotients G/Δ_1 and G/Δ_2 are again p-equivalent but not isomorphic. One can check that in this example α_k cannot be chosen to be the identity. Similar examples can be constructed for $p = 2, 3$.

Before proving Theorem 11.5, we need the following lemma.

Lemma 11.8. *Let G be a simple simply connected compact Lie group not isomorphic to a special unitary group. Suppose that $\varphi \in N_{\mathrm{GL}(L_G \otimes \mathbf{Z}_p)}(W_G)$. Then there exists $\sigma \in \mathrm{Aut}(G)$ such that φ and σ induces the same automorphism of $\breve{\mathcal{Z}}(G)$.*

Proof. If $G = \mathrm{Spin}(2n+1)$, $\mathrm{Sp}(n)$, $2E_7$, E_8, F_4 or G_2, the claim is obvious, since there are no nontrivial automorphisms of the center. For $G = \mathrm{Spin}(2n)$, $n \geq 4$, it follows from Theorem 13.1 that $N_{\mathrm{GL}(L_G \otimes \mathbf{Z}_p)}(W_G)$ is generated by the scalars $\mathbf{Z}_p^\times$, W and the automorphisms of the Dynkin diagram. Thus the only potential problem occurs for $p = 2$ and n odd where $\breve{\mathcal{Z}}(G) \cong \mathbf{Z}/4$. However it follows by [18, Planche IV(XI)] that the nontrivial automorphism of the center is induced by the nontrivial graph automorphism. Finally, by [18, Planche V(XI)], the same argument works in the case $G = 3E_6$, $p = 3$ where $\breve{\mathcal{Z}}(G) \cong \mathbf{Z}/3$. □

Proof of Theorem 11.5. By [116, §4] or [83, Prop. 3.2(vi)] we can recover the root datum of a compact connected Lie group from its integral lattice up to substitution of direct factors isomorphic to $\mathrm{Sp}(n)$ with direct factors isomorphic to $\mathrm{SO}(2n+1)$. Part (1) now follows.

Suppose that G and G' are p-equivalent. Then by assumption the associated covering groups H and H' are isomorphic, in such a way that the image of K in H' differs from K' by an endomorphism of $T_{H'}$ of the type Ψ. The map $\breve{T}_H \to \breve{T}_{H'}$ induced by the composite of the isomorphism with the endomorphism Ψ is hence an isomorphism sending K to K'. Hence we get an induced isomorphism $(W_G, L_G \otimes \mathbf{Z}_p) \to (W_{G'}, L_{G'} \otimes \mathbf{Z}_p)$.

Conversely, suppose that $G = H/K$ and $G' = H'/K'$ do not contain any direct factors isomorphic to $\mathrm{Sp}(n)$, and that $(W_G, L_G \otimes \mathbf{Z}_p)$ is isomorphic to $(W_{G'}, L_{G'} \otimes \mathbf{Z}_p)$. By Proposition 10.4 the fundamental group of G equals the coinvariants $(L_G)_W$ and hence $L_H = (L_G)^W \oplus SL_G$. This shows that $(W, L_H \otimes \mathbf{Z}_p)$ can be reconstructed from $(W, L_G \otimes \mathbf{Z}_p)$. By the classification of simply connected compact Lie groups we can for $p = 2$ reconstruct H, up to isomorphism, from $(W, L_H \otimes \mathbf{Z}_p)$. For p odd the only ambiguity arises from direct factors isomorphic to $\mathrm{Sp}(n)$ or $\mathrm{Spin}(2n+1)$, but in this case by the assumption on G, H cannot contain any direct factors isomorphic to $\mathrm{Sp}(n)$, and we conclude that in all cases we can reconstruct H, up to isomorphism, from $(W, L_G \otimes \mathbf{Z}_p)$. Hence we can without loss of generality assume that $H = H'$. Note that K is the cokernel of the inclusion $L_H \otimes \mathbf{Z}_p \to L_G \otimes \mathbf{Z}_p$, so we can also recover the inclusion $K \subseteq L_H \otimes \mathbf{Z}/p^\infty$. In other words an isomorphism between $(W, L_G \otimes \mathbf{Z}_p)$ and $(W, L_{G'} \otimes \mathbf{Z}_p)$ induces an automorphism of $\breve{T}_H$ taking K to K'. We have to see that if there exists such an automorphism then there exists an automorphism of H, followed by an endomorphism of the type Ψ, which also takes K to K', since this will show that H/K and H/K' are p-equivalent.

Write $H = U \times T'$, where $U = R_1 \times \cdots \times R_n \times S$, in the notation introduced before the theorem. By Proposition 5.4 any automorphism of $(W_H, L_H \otimes \mathbf{Z}_p)$ induces an automorphism φ of $\breve{T}_U \times \breve{T}'$ which is of the form $\varphi = \varphi_1 \times \varphi_2$. *A priori* φ_2 is an element in $\mathrm{Aut}(\breve{T}') \cong \mathrm{GL}(L_{T'} \otimes \mathbf{Z}_p)$, but since K_1 and K_2 are finite we can without loss of generality replace it by a matrix with integer coefficients and determinant prime to p. By the 'elementary divisor theorem' in linear algebra over a Euclidean domain (cf. e.g., [8, Thm. 12.4.3]) we can find a basis for $L_{T'}$ in which φ_2 is given by AD where A is a product of elementary matrices over $\mathbf{Z}$ and D is a diagonal matrix with determinant prime to p. Since we are only interested in the effect over $\mathbf{Z}/p^s$ for some fixed large s and D consists of units modulo p^s we can furthermore change φ_2 and A such that D can be assumed to be of the form $\mathrm{diag}(1, \ldots, 1, k)$, with k prime to p, since if x is a unit in $\mathbf{Z}/p^s$ then the matrix $\mathrm{diag}(x, x^{-1})$ can be written as a product of elementary matrices by a straightforward calculation (done in [41, 40.25]). This shows that we can put φ_2 on the correct form. For the map φ_1 this follows directly from Proposition 5.4 together with Lemma 11.8 and Theorem 13.1. Hence we have seen that for any φ we can find a map of the stated form which takes K to K'.

The above analysis directly shows the first claim in (2) as well as (3a) $\Leftrightarrow$ (3b). From the first claim in (2), (2a) $\Leftrightarrow$ (2b) follows, since $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+1)$ have different $\mathbf{Z}_2$-reflection data $(W_G, L_G \otimes \mathbf{Z}_2, L_{G\langle 1\rangle} \otimes \mathbf{Z}_2)$. The implication (2b) $\Rightarrow$ (2c) follows from the existence of unstable Adams operations on $\mathrm{SU}(n)$, first constructed by Sullivan [134, p. 142], realizing Ψ on the level of $\mathbf{F}_p$-completed classifying spaces (or for an overkill, use Theorem 1.1 directly). The implication (3b) $\Rightarrow$ (3c) also follows from this together with the fact that $B\,\mathrm{SO}(2n+1)\hat{_p}$ is homotopy equivalent to $B\,\mathrm{Sp}(n)\hat{_p}$ for $p > 2$, as originally proved by Friedlander [66, Thm. 2.1] (or again as a very special case of Theorem 1.1). The remaining implications (2c) $\Rightarrow$ (2a) and (3c) $\Rightarrow$ (3a) follow directly from the fundamental properties of the Weyl group of a p-compact group. □

12. Appendix: Invariant rings of finite $\mathbf{Z}_p$-reflection groups, p odd (following Notbohm)

The purpose of this appendix is to recall Notbohm's determination [109] of finite $\mathbf{Z}_p$-reflection groups (W, L), p odd, such that the invariant ring $\mathbf{Z}_p[L]^W$ is a polynomial algebra.

Before stating it let us however for easy reference recall the following 'classical' characterizations of a 'p-torsion-free' p-compact group, which has a proof by general arguments which we will sketch below.

THEOREM 12.1. *Let X be a connected p-compact group with maximal torus T and Weyl group W_X. The following statements are equivalent*:

(1) *$H^*(X;\mathbf{Z}_p)$ is torsion-free.*

(2) *$H^*(X;\mathbf{Z}_p)$ is an exterior algebra over $\mathbf{Z}_p$ with generators in odd degrees (or equivalently with $\mathbf{F}_p$ instead of $\mathbf{Z}_p$).*

(3) *$H^*(BX;\mathbf{Z}_p)$ is a polynomial algebra over $\mathbf{Z}_p$ with generators in even degree (or equivalently with $\mathbf{F}_p$ instead of $\mathbf{Z}_p$).*

(4) *$H^*(BX;\mathbf{Z}_p)$ is a polynomial algebra and $H^*(BX;\mathbf{Z}_p) \xrightarrow{\cong} H^*(BT;\mathbf{Z}_p)^{W_X}$.*

We now give Notbohm's classification, Theorem 12.2 below. The first part (which is a reduction to the simply connected case) is [109, Thm. 1.3] and the second (which is a case-by-case argument in the simply connected case) is a slight extension of [109, Thm. 1.4]. For the benefit of the reader we give a streamlined proof of the second part. Recall that for a finite $\mathbf{Z}_p$-reflection group (W, L) we define SL to be the submodule of L generated by elements of the form $(1-w)x$ with $w \in W$ and $x \in L$. We call (W, L) *simply connected* if $L \cong SL'$ for some $\mathbf{Z}_pW$-lattice L'. (Note that for p odd this is equivalent to $SL = L$ since $S^2L' = SL'$; cf. the discussion of $\mathbf{Z}_p$-reflection data in the introduction.)

Theorem 12.2 (Finite $\mathbf{Z}_p$-reflection groups with polynomial invariants, p odd). *Let p be an odd prime and (W, L) a finite $\mathbf{Z}_p$-reflection group. Then the following statements hold*:

(1) *$\mathbf{Z}_p[L]^W$ is a polynomial algebra if and only if $\mathbf{Z}_p[SL]^W$ is a polynomial algebra and the group of coinvariants L_W is torsion-free.*

(2) *Suppose (W, L) is irreducible and simply connected. The following conditions are equivalent*:

  (a) *$\mathbf{Z}_p[L]^W$ is a polynomial algebra.*

  (b) *$\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra.*

  (c) *(W, L) is not isomorphic to $(W_G, L_G \otimes \mathbf{Z}_p)$ for the following pairs (G, p): $(F_4, 3)$, $(3E_6, 3)$, $(2E_7, 3)$, $(E_8, 3)$ and $(E_8, 5)$.*

In particular, if X is an exotic p-compact group then $\mathbf{Z}_p[L_X]^{W_X}$ is a polynomial algebra and if $(W, L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for a compact connected Lie group G then $\mathbf{Z}_p[L_G \otimes \mathbf{Z}_p]^{W_G}$ is a polynomial algebra if and only if $H^*(G;\mathbf{Z}_p)$ is torsion-free.

Sketch of proof of Theorem 12.1. The equivalence of (1), (2), and (3) is proved by old H-space and loop space arguments which we first very briefly sketch. By a Bockstein spectral sequence argument (cf. e.g., [85, §11-2]) $H^*(X;\mathbf{Z}_p)$ is torsion-free if and only if $H^*(X;\mathbf{Z}_p)$ is an exterior algebra on odd dimensional generators so (1) is equivalent to (2). This is again equivalent to $H^*(BX;\mathbf{Z}_p)$ being a polynomial algebra on even dimensional generators (using the Eilenberg-Moore and the cobar spectral sequence; see e.g., [85, §7-4]); thus, (2) is equivalent to (3).

That (4) implies (3) is obvious. The fact that (1)–(3) also imply (4) requires more machinery and is probably first found in [52, Thm. 2.11]—we quickly sketch an argument. We want to show that the map $r : H^*(BX;\mathbf{Z}_p) \to H^*(BT;\mathbf{Z}_p)^W$ is an isomorphism. By [56, Thm. 9.7(iii)]

$$H^*(BX;\mathbf{Z}_p)\otimes \mathbf{Q} \xrightarrow{\cong} H^*(BT;\mathbf{Z}_p)^W \otimes \mathbf{Q}. \tag{12.1}$$

This implies by comparison of Krull dimensions that the number of polynomial generators equals the rank of T. Since $H^*(BT;\mathbf{F}_p)$ is finitely generated over $H^*(BX;\mathbf{F}_p)$ by [56, Prop. 9.11] it follows by comparing Krull dimensions again that $H^*(BX;\mathbf{F}_p) \to H^*(BT;\mathbf{F}_p)$ is injective. Hence $H^*(BX;\mathbf{Z}_p) \to H^*(BT;\mathbf{Z}_p)$ has to be injective by Nakayama's lemma. Likewise r has to be surjective: By (12.1) the cokernel of r has to be p-torsion. Since the reduction mod p of r is still injective (as seen above) the cokernel of r has to be p-torsion-free as well (since $\mathrm{Tor}(\mathrm{coker}(r),\mathbf{F}_p) = 0$). □

Remark 12.3. If p is odd then $\mathbf{F}_p$-coefficients can also be used in Theorem 12.1(4) by a Galois theory argument using Lemma 11.3. For $p = 2$, this is not true as can be seen by taking $X = \mathrm{SU}(2)\hat{_2}$. See [53] for a version for $p = 2$.

Remark 12.4. If (W,L) is a finite $\mathbf{Z}_p$-reflection group then $\mathbf{Z}_p[L]^W$ is a polynomial algebra if and only if $\mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ is a polynomial algebra and the canonical monomorphism $\mathbf{Z}_p[L]^W \otimes \mathbf{F}_p \longrightarrow \mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ is an isomorphism, as shown in [109, Lem. 2.3]. Note that this can be reformulated as saying that $\mathbf{Z}_p[L]^W$ is a polynomial algebra if and only if $\mathbf{F}_p[L\otimes\mathbf{F}_p]^W$ is a polynomial algebra with generators in the same degrees as the generators of $\mathbf{Q}_p[L\otimes\mathbf{Q}]^W$, since $\dim_{\mathbf{Q}_p}(\mathbf{Q}_p[L\otimes\mathbf{Q}]^W)_n = \dim_{\mathbf{F}_p}(\mathbf{Z}_p[L]^W\otimes\mathbf{F}_p)_n \le \dim_{\mathbf{F}_p}(\mathbf{F}_p[L\otimes\mathbf{F}_p]^W)_n$ for any n.

Remark 12.5. The $\mathbf{Z}_3$-reflection group $(W,L) = (W_{\mathrm{PU}(3)}, L_{\mathrm{PU}(3)}\otimes\mathbf{Z}_3)$ does not have invariant ring a polynomial ring (e.g., since $L_W \cong \mathbf{Z}/3$ is not torsion-free). However a short calculation shows that $\mathbf{F}_3[L\otimes\mathbf{F}_3]^W$ *is* a polynomial ring with generators in degrees 1 and 6 (as opposed to the degrees over $\mathbf{Q}_3$ which are 2 and 3). (See also [52, Rem. 5.3].) It turns out that this example is essentially the only one since it can be proved that if (W,L) is a finite $\mathbf{Z}_p$-reflection group, p odd, such that $\mathbf{F}_p[L]^W$ is a polynomial algebra, then $\mathbf{Z}_p[L]^W$ is also a polynomial algebra unless $p = 3$ and (W,L) contains $(W_{\mathrm{PU}(3)}, L_{\mathrm{PU}(3)}\otimes\mathbf{Z}_3)$ as a direct factor. We omit the proof which is an extension of the technique used in the examples in Section 7 in a preprint version of [61], which can at the time of writing be found on Wilkerson's homepage.

LEMMA 12.6. *Assume that L is a finitely generated free $\mathbf{Z}_p$-module and that W is a finite subgroup of* $\mathrm{GL}(L)$. *If $p \nmid |W|$ and $\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra, then $\mathbf{Z}_p[L]^W$ is also a polynomial algebra.*

Proof. By assumption we have the averaging homomorphisms $\mathbf{Z}_p[L] \longrightarrow \mathbf{Z}_p[L]^W$ and $\mathbf{F}_p[L \otimes \mathbf{F}_p] \longrightarrow \mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ given by $f \mapsto \frac{1}{|W|}\sum_{w \in W} w \cdot f$. These are obviously surjective and hence the commutative diagram

$$\begin{tikzcd}
\mathbf{Z}_p[L] \arrow[r] \arrow[d] & \mathbf{Z}_p[L]^W \arrow[d] \\
\mathbf{F}_p[L \otimes \mathbf{F}_p] \arrow[r] & \mathbf{F}_p[L \otimes \mathbf{F}_p]^W
\end{tikzcd}$$

shows that the reduction homomorphism $\mathbf{Z}_p[L]^W \to \mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is surjective. The result now follows easily from Nakayama's lemma (cf. [109, Lem. 2.3]). $\square$

Proof of Theorem 12.2. Part (1) is contained in [109, Thm. 1.3]. To prove part (2) note that by Notbohm [107] (see also [109, Thm. 1.2(iii)] and Theorem 11.1), there is a unique finite irreducible simply connected $\mathbf{Z}_p$-reflection group for each group on the Clark-Ewing list. We now go through the list, verifying the result in each case.

If $p \nmid |W|$ the invariant ring $\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra by the Shephard-Todd-Chevalley theorem ([10, Thm. 7.2.1] or [127, Thm. 7.4.1]), and thus Lemma 12.6 shows that $\mathbf{Z}_p[L]^W$ is a polynomial algebra.

Next, assume that (W, L) is an exotic $\mathbf{Z}_p$-reflection group. If (W, L) belongs to family 2, the representing matrices with respect to the standard basis are monomial and so $\mathbf{Z}_p[L]^W$ is a polynomial algebra by [102, Thm. 2.4].

An inspection of the Clark-Ewing list now shows that only four exotic cases remain, namely $(G_{12}, p = 3)$, $(G_{29}, p = 5)$, $(G_{31}, p = 5)$ and $(G_{34}, p = 7)$. In the first case we have $G_{12} \cong \mathrm{GL}_2(\mathbf{F}_3)$ and Lemma 11.3 shows that the action on $L \otimes \mathbf{F}_3 = (\mathbf{F}_3)^2$ is the canonical one. The invariant ring $\mathbf{F}_3[L \otimes \mathbf{F}_3]^{\mathrm{GL}_2(\mathbf{F}_3)}$ was computed by Dickson [44]. In the remaining three cases the mod p invariant ring was calculated by Xu [144], [145] using a computer; see also Kemper-Malle [86, Prop. 6.1]. The conclusion of these computations is that in all four cases the invariant ring $\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra with generators in the same degrees as the generators of $\mathbf{Q}_p[L \otimes \mathbf{Q}]^W$. By Remark 12.4 we then see that $\mathbf{Z}_p[L]^W$ is a polynomial algebra in these cases.

The only remaining cases are the finite simply connected $\mathbf{Z}_p$-reflection groups which are not exotic. Since p is odd and $\pi_1(G)$ and $(L_G)_{W_G}$ only differ by an elementary abelian 2-group (cf. the proof of Theorem 1.7 and Proposition 10.4), we may assume that $(W, L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for some simply connected compact Lie group G. In this case Demazure [43] shows that if p is not a torsion prime for the root system associated to G, then the invariant rings $\mathbf{Z}_p[L_G \otimes \mathbf{Z}_p]^{W_G}$ and $\mathbf{F}_p[L_G \otimes \mathbf{F}_p]^{W_G}$ are polynomial algebras.

By the calculation of torsion primes for the simple root systems, [43, §7], the excluded pairs (G,p) in the last part of the theorem are exactly the cases where the root system of G has p-torsion. In these cases Kemper-Malle [86, Prop. 6.1 and Pf. of Thm. 8.5] shows that $\mathbf{F}_p[L_G \otimes \mathbf{F}_p]^{W_G}$ is not a polynomial algebra. Hence in these cases $\mathbf{Z}_p[L_G \otimes \mathbf{Z}_p]^{W_G}$ is not a polynomial algebra by [109, Lem. 2.3(i)]. This proves the second claim.

Finally, let G be a compact connected Lie group with Weyl group $W = W_G$ and integral lattice $L = L_G$. We now prove that $\mathbf{Z}_p[L \otimes \mathbf{Z}_p]^W$ is a polynomial algebra if and only if $H^*(G;\mathbf{Z}_p)$ is torsion-free. (See also [110, Prop. 1.11].) One direction follows from Theorem 12.1, so assume now that $\mathbf{Z}_p[L \otimes \mathbf{Z}_p]^W$ is a polynomial algebra. From Theorem 12.2(1) we see that $\mathbf{Z}_p[S(L \otimes \mathbf{Z}_p)]^W$ is a polynomial algebra and that $(L \otimes \mathbf{Z}_p)_W$ is torsion-free. Since p is odd, we have $(L \otimes \mathbf{Z}_p)_W = \pi_1(G) \otimes \mathbf{Z}_p$ and $S(L \otimes \mathbf{Z}_p) = L_{G\langle 1\rangle} \otimes \mathbf{Z}_p$; cf. the proofs of Theorem 1.7 and Proposition 10.4. From the above we conclude that $H^*(G\langle 1\rangle;\mathbf{Z}_p)$ is torsion-free. Since $\pi_1(G)$ has no p-torsion, it now follows easily from the Serre spectral sequence that $H^*(G;\mathbf{Z}_p)$ is torsion-free. $\square$

Remark 12.7. Let p be an odd prime and (W,L) a finite $\mathbf{Z}_p$-reflection group. We claim that the following conditions are equivalent:

(1) $\mathbf{Z}_p[L]^W$ is a polynomial algebra.

(2) $\mathbf{F}_p[L \otimes \mathbf{F}_p]^W$ is a polynomial algebra and L_W is torsion-free.

(3) $\mathbf{F}_p[SL \otimes \mathbf{F}_p]^W$ is a polynomial algebra and L_W is torsion-free.

Indeed we have (1) $\Leftrightarrow$ (3) by Theorem 12.2 since (W,SL) can be decomposed as a direct product of finite irreducible simply connected $\mathbf{Z}_p$-reflection groups by [107, Thm. 1.4]. The implication (1) $\Rightarrow$ (2) follows from [109, Thm. 1.3 and Lem. 2.3]. Finally (2) $\Rightarrow$ (3) follows from [102, Prop. 4.1] as L_W torsion-free implies that $SL \otimes \mathbf{F}_p \to L \otimes \mathbf{F}_p$ is injective.

13. Appendix: Outer automorphisms of finite $\mathbf{Z}_p$-reflection groups

Theorem 1.1 states that the outer automorphism group of a connected p-compact group X, p odd, equals $N_{\mathrm{GL}(L_X)}(W_X)/W_X$, which makes it useful to have a complete case-by-case calculation of this group. The purpose of this appendix is to provide such a calculation based on results of Broué-Malle-Michel [26, Prop. 3.13] over the complex numbers. Calculations in the case where W is one of the exotic groups from family $2a$ were given in [108, §6] (where the nonstandard notation $G(q,r;n)$ for $G(q,q/r,n)$ is used).

Theorem 11.1 and Proposition 5.4 reduce the calculation of $N_{\mathrm{GL}(L)}(W)/W$ to the case where (W,L) is exotic or $(W,L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for some compact connected Lie group G. In the second case we can write $G = H/K$ where H

is a direct product of a torus and a simply connected compact Lie group and K is a finite central subgroup of H, and it is easy to use coverings to compute $N_{\mathrm{GL}(L_G\otimes\mathbf{Z}_p)}(W_G)/W_G$ from $N_{\mathrm{GL}(L_H\otimes\mathbf{Z}_p)}(W_H)/W_H$. By Proposition 5.4 this again reduces to the case where H is simple.

We can hence restrict to the case where (W,L) is exotic or $(W,L) = (W_G, L_G\otimes\mathbf{Z}_p)$ for a simple simply connected compact Lie group G. For the statement of our result in these cases (which will take place in the theorem below as well as in the following elaborations), we fix the realizations $G(m,r,n)$ of the groups from family 2 as described in Section 11. Moreover we also fix the realizations of the complex reflection groups G_i, $4\le i\le 37$, to be the ones described in [126]. Let μ_n denote the group of nth roots of unity. If G is a simply connected compact Lie group, its integral lattice L_G equals the coroot lattice. Hence the automorphism group Γ of the Dynkin diagram of G can be considered as a subgroup of $N_{\mathrm{GL}(L_G)}(W_G)$; cf. [77, §12.2]. For $G=\mathrm{Spin}(5)$, F_4 or G_2, there is an automorphism φ_l of $L_G\otimes\mathbf{Z}[1/\sqrt{l}]$ of order 2 (here $l=2$ for Spin(5) and F_4 and $l=3$ for G_2); see [26, p. 182–183] or [27, p. 217] for details.

Theorem 13.1 (Outer automorphisms of finite $\mathbf{Z}_p$-reflection groups). *Let (W,L) be a finite irreducible simply connected $\mathbf{Z}_p$-reflection group, i.e., (W,L) is exotic or of the form $(W_G, L_G\otimes\mathbf{Z}_p)$ for a simple simply connected compact Lie group G. Let (W,V) be the associated complex reflection group. Then $N_{\mathrm{GL}(V)}(W) = \langle W, \mathbf{C}^\times\rangle$ and hence $N_{\mathrm{GL}(L)}(W)/W = \mathbf{Z}_p^\times/Z(W)$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W = 1$ except in the following cases*:

(1) *$W=G(m,r,n)$ is exotic and belongs to family* 2, *$(m,r,n)\neq(4,2,2)$, $(3,3,3)$: $N_{\mathrm{GL}(V)}(W) = \langle G(m,1,n),\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W = C_{\gcd(r,n)}$; cf.* 13.4.

(2) *$W=G(4,2,2)$: $N_{\mathrm{GL}(V)}(W)=\langle G_8,\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W = \Sigma_3$; cf.* 13.5.

(3) *$W=G(3,3,3)$: $N_{\mathrm{GL}(V)}(W)=\langle G_{26},\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W = A_4$; cf.* 13.6.

(4) *$W=G_5$: $N_{\mathrm{GL}(V)}(W)=\langle G_{14},\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W = C_2$; cf.* 13.7.

(5) *$W=G_7$: $N_{\mathrm{GL}(V)}(W)=\langle G_{10},\mathbf{C}^\times\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W = C_2$; cf.* 13.8.

(6) *$(W,L)=(W_G,L_G\otimes\mathbf{Z}_p)$ for $G=\mathrm{Spin}(4n)$, $n\ge 2$: $N_{\mathrm{GL}(V)}(W) = \langle W,\mathbf{C}^\times,\Gamma\rangle$ and $N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W\cong\Gamma$; cf.* 13.9.

(7) $(W,L) = (W_G, L_G \otimes \mathbf{Z}_p)$ *for* $G = \mathrm{Spin}(5)$, F_4 *or* G_2: $N_{\mathrm{GL}(V)}(W) = \langle W, \mathbf{C}^\times, \varphi_l \rangle$. *Moreover*

$$N_{\mathrm{GL}(L)}(W)/\mathbf{Z}_p^\times W = \begin{cases} 1 & \text{for } p = l, \\ C_2 & \text{for } p \neq l \end{cases}$$

cf. 13.10.

Lemma 13.2. *Let $K \subseteq K'$ be fields of characteristic zero, and $W \subseteq \mathrm{GL}_n(K)$ an irreducible reflection group. Then*

$$N_{\mathrm{GL}_n(K')}(W) = \left\langle N_{\mathrm{GL}_n(K)}(W), {K'}^\times \right\rangle.$$

Proof. The inclusion '$\supseteq$' is clear, so suppose $g \in N_{\mathrm{GL}_n(K')}(W)$. Consider the system of equations $Xw = gwg^{-1}X$, $w \in W$ where X is an $n \times n$-matrix. Over K' this has the solution $X = g$. By [62, Lem. 2.10], the representation $W \to \mathrm{GL}_n(K')$ is irreducible, so the solution space is the 1-dimensional space spanned by g. Since the coefficients lie in K, the solution space over K is 1-dimensional as well, so we can write $g = \lambda g_1$ with $\lambda \in K'$ and $g_1 \in M_n(K)$. As $g \neq 0$ we get $\lambda \neq 0$ and $g_1 \in N_{\mathrm{GL}_n(K)}(W)$. ☐

We can now start the proof of Theorem 13.1. The results on $N_{\mathrm{GL}(V)}(W)$ follow directly from [26, Prop. 3.13] except when W belongs to family 2 or $W = G_{28}$. The structure of $N_{\mathrm{GL}(V)}(W)$ in the cases (1), (2) and (3) also follows from [26, Prop. 3.13] since $\langle G(4,1,2), G_6, \mathbf{C}^\times \rangle = \langle G_8, \mathbf{C}^\times \rangle$ and $G(3,1,3) \subseteq G_{26}$.

Now assume that W does not belong to family 2 and $W \neq G_5, G_7, G_{28}$. Let n denote the rank of W and K the field extension of $\mathbf{Q}$ generated by the entries of the matrices representing W. Our assumption ensures that $N_{\mathrm{GL}(V)}(W) = \langle W, \mathbf{C}^\times \rangle$. Since W is a reflection group it has Schur index 1 and we can assume that K equals the character field of W; cf. [34, Cor. p. 429]. Then $N_{\mathrm{GL}_n(K)}(W) = \langle W, K^\times \rangle$ and Lemma 13.2 now shows that $N_{\mathrm{GL}_n(\mathbf{Q}_p)}(W) = \left\langle W, \mathbf{Q}_p^\times \right\rangle$. Hence we get $N_{\mathrm{GL}(L)}(W) = \left\langle W, \mathbf{Z}_p^\times \right\rangle$ and since W is irreducible we have $W \cap \mathbf{Z}_p^\times = Z(W)$; cf. [62, Lem. 2.9].

This proves Theorem 13.1 in case W does not belong to family 2 and $W \neq G_5, G_7, G_{28}$. In the cases (1), (2), (3), (4) and (5) we only need to find the structure of $N_{\mathrm{GL}(L)}(W)$. This is done in Elaborations 13.4, 13.5, 13.6, 13.7 and 13.8 below.

This leaves the cases where $(W,L) = (W_G, L_G \otimes \mathbf{Z}_p)$ for a simple simply connected compact Lie group G such that W_G belongs to family 2 and $W_G \neq G_{28}$, i.e., $G = \mathrm{Spin}(2n+1)$ for $n \geq 2$, $\mathrm{Sp}(n)$ for $n \geq 3$, $\mathrm{Spin}(2n)$ for $n \geq 4$, G_2 and F_4. In the first two cases W_G equals $G(2,1,n)$ and hence $N_{\mathrm{GL}(V)}(W_G) = \langle W_G, \mathbf{C}^\times \rangle$ when $n \geq 3$ by [26, Prop. 3.13]. As above this proves Theorem 13.1 in these cases. The case $G = \mathrm{Spin}(2n)$, $n \geq 4$ is dealt with in Elaboration 13.9, and the cases $G = \mathrm{Spin}(5)$, G_2 and F_4 are handled in Elaboration 13.10.

To treat the dihedral group $G(m,m,2)$ from family 2 we need the following auxiliary result.

LEMMA 13.3. *Let $m \geq 3$ and $p \equiv \pm 1 \pmod m$ so that $\zeta_m + \zeta_m^{-1} \in \mathbf{Z}_p$. Then $2 + \zeta_m + \zeta_m^{-1}$ is a unit in $\mathbf{Z}_p$.*

Proof. It suffices to prove that the norm $N_{\mathbf{Q}(\zeta_m+\zeta_m^{-1})/\mathbf{Q}}(2+\zeta_m+\zeta_m^{-1})$ is not divisible by p. Since its square equals the norm $N_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}(2+\zeta_m+\zeta_m^{-1})$ it is enough to see that this norm is not divisible by p. In $\mathbf{Q}(\zeta_m)$ we have $2+\zeta_m+\zeta_m^{-1} = (1+\zeta_m)^2/\zeta_m$ and since ζ_m is a unit it is enough to see that $N_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}(1+\zeta_m)$ is not divisible by p. By definition

$$N_{\mathbf{Q}(\zeta_m)/\mathbf{Q}}(1+\zeta_m) = \prod_{\substack{0 \leq k \leq m \\ \gcd(k,m)=1}} (1+\zeta_m^k) = (-1)^{\phi(m)} \prod_{\substack{0 \leq k \leq m \\ \gcd(k,m)=1}} (-1-\zeta_m^k) = \Phi_m(-1).$$

The claim now follows from [138, Lem. 2.9]. □

Elaboration 13.4 (Family 2, generic case). Let $W = G(m,r,n)$ from family 2 and let p be a prime number such that W is an exotic $\mathbf{Z}_p$-reflection group. Thus if $n \geq 3$ or $n = 2$ and $r < m$ we have $m \geq 3$ and $p \equiv 1 \pmod m$, and for $n = 2$ and $m = r$ we have $m \geq 5, m \neq 6$ and $p \equiv \pm 1 \pmod m$. Assume moreover that $(m,r,n) \neq (4,2,2), (3,3,3)$ (these two cases are dealt with in Elaborations 13.5 and 13.6 below).

Assume first that $p \equiv 1 \pmod m$. The realizations of the groups $G(m,r,n)$ and $G(m,1,n)$ from above are both defined over the ring $\mathbf{Z}[\zeta_m]$ which embeds in $\mathbf{Z}_p$. Lemma 13.2 shows that $N_{\mathrm{GL}_n(\mathbf{Z}_p)}(W) = \langle G(m,1,n), \mathbf{Z}_p^\times \rangle$ whence the natural homomorphism $(A(m,1,n)/A(m,r,n)) \times \mathbf{Z}_p^\times \longrightarrow N_{\mathrm{GL}_n(\mathbf{Z}_p)}(W)/W$ is surjective. The kernel is the cyclic group generated by the element $([\zeta_m I_n], \zeta_m^{-1})$ (here $[\zeta_m I_n] \in A(m,1,n)/A(m,r,n)$ denotes the coset of $\zeta_m I_n$) and thus

$$N_{\mathrm{GL}_n(\mathbf{Z}_p)}(W)/W = (A(m,1,n)/A(m,r,n)) \circ_{C_m} \mathbf{Z}_p^\times.$$

Note that $A(m,1,n)/A(m,r,n)$ is cyclic of order r generated by the element $x = [\mathrm{diag}(1,\dots,1,\zeta_m)]$ and that $[\zeta_m I_n] = x^n$.

If $p \not\equiv 1 \pmod m$, then $W = G(m,m,2)$ is the dihedral group of order $2m$ with $m \geq 5, m \neq 6$ and $p \equiv -1 \pmod m$. Conjugation of the realization of $G(m,m,2)$ from above with the element

$$g = \begin{bmatrix} 1 & -\zeta_m^{-1} \\ 1 & -\zeta_m \end{bmatrix}$$

gives a realization $G(m,m,2)^g$ defined over the character field $\mathbf{Q}(\zeta_m + \zeta_m^{-1})$. Note that if m is odd, then

$$N_{\mathrm{GL}_2(\mathbf{C})}(G(m,m,2)) = \langle G(m,1,2), \mathbf{C}^\times \rangle = \langle G(m,m,2), \mathbf{C}^\times \rangle$$

and hence $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g)/G(m,m,2)^g = \mathbf{Z}_p^\times$. Thus, we may assume that m is even. Since $G(m,1,2)$ is generated by $G(m,m,2)$ and $\mathrm{diag}(1,\zeta_m)$ we find

$$N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g) = \left\langle G(m,m,2)^g, \begin{bmatrix} 1 & 1 \\ -1 & 1+\zeta_m+\zeta_m^{-1} \end{bmatrix}, \mathbf{Q}_p^\times \right\rangle \cap \mathrm{GL}_2(\mathbf{Z}_p)$$

using Lemma 13.2. From Lemma 13.3 we see that the above matrix is invertible over $\mathbf{Z}_p$ and hence

$$N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g) = \left\langle G(m,m,2)^g, \begin{bmatrix} 1 & 1 \\ -1 & 1+\zeta_m+\zeta_m^{-1} \end{bmatrix}, \mathbf{Z}_p^\times \right\rangle.$$

Thus the homomorphism $\mathbf{Z}\times(\mathbf{Z}_p^\times/\mu_2) \longrightarrow N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g)/G(m,m,2)^g$ which maps $(k,[\lambda])$ to the coset of $\lambda \begin{bmatrix} 1 & 1 \\ -1 & 1+\zeta_m+\zeta_m^{-1} \end{bmatrix}^k$ is surjective. The kernel is easily seen to be the infinite cyclic group generated by the element $(-2,[1+\zeta_m+\zeta_m^{-1}])$ and thus we get $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g)/G(m,m,2)^g \cong \mathbf{Z}\circ_{\mathbf{Z}}(\mathbf{Z}_p^\times/\mu_2)$. It is easily checked that $[2+\zeta_m+\zeta_m^{-1}]$ has a square root in $\mathbf{Z}_p^\times/\mu_2$ if and only if either $m \equiv 0 \pmod 4$ or $m \equiv 2 \pmod 4$ and $p \equiv -1 \pmod{2m}$. In this case we have $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(m,m,2)^g)/G(m,m,2)^g \cong C_2 \times (\mathbf{Z}_p^\times/\mu_2)$.

Elaboration 13.5 ($G(4,2,2)$). The realization of the group $G(4,2,2)$ from above and the realization of the group G_8 from [126, Table II] are both defined over their common character field $\mathbf{Q}(i)$. Thus the relevant primes p are the ones satisfying $p \equiv 1 \pmod 4$. More precisely the representations are defined over $\mathbf{Z}[\frac{1}{2}, i]$ and as this ring embeds in $\mathbf{Z}_p$ for all p as above, we get $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(4,2,2)) = \langle G_8, \mathbf{Z}_p^\times \rangle$ by Lemma 13.2. It is easily checked that $G_8 = \langle G(4,2,2), H\rangle$, where H is the group of order 24 generated by the elements

$$\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}, \qquad \frac{1+i}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

Since $G(4,2,2) \cap \langle H, \mathbf{Z}_p^\times \rangle = Z(H) = \mu_4$ we conclude that

$$N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G(4,2,2))/G(4,2,2) \cong (H/Z(H)) \times (\mathbf{Z}_p^\times/\mu_4) \cong \Sigma_3 \times (\mathbf{Z}_p^\times/\mu_4).$$

Elaboration 13.6 ($G(3,3,3)$). The realization of the group $G(3,3,3)$ from above and the realization of the group G_{26} from [126, p. 297] are both defined over their common character field $\mathbf{Q}(\omega)$ where $\omega = e^{2\pi i/3}$. Thus the relevant primes p are the ones satisfying $p \equiv 1 \pmod 3$. More precisely the representations are defined over $\mathbf{Z}[\frac{1}{3}, \omega]$ and as this ring embeds in $\mathbf{Z}_p$ for all p as above, we see that $N_{\mathrm{GL}_3(\mathbf{Z}_p)}(G(3,3,3)) = \langle G_{26}, \mathbf{Z}_p^\times \rangle$ using Lemma 13.2. It is easily checked that G_{26} is the semidirect product of $G(3,3,3)$ with the group

$H \cong \mathrm{SL}_2(\mathbf{F}_3)$ generated by the elements

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \qquad R_2 = \frac{1}{\sqrt{-3}} \begin{bmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{bmatrix}.$$

The center of H is generated by the element

$$z = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and $G(3,3,3) \cap \langle H, \mathbf{Z}_p^\times \rangle = \langle -z, \mu_3 \rangle$. Thus

$$N_{\mathrm{GL}_3(\mathbf{Z}_p)}(G(3,3,3))/G(3,3,3) \cong H \circ_{C_2} (\mathbf{Z}_p^\times/\mu_3) \cong \mathrm{SL}_2(\mathbf{F}_3) \circ_{C_2} (\mathbf{Z}_p^\times/\mu_3)$$

where the central product is given by identifying $z \in H$ with $[-1] \in \mathbf{Z}_p^\times/\mu_3$.

Elaboration 13.7 (G_5). The realization of the group G_5 from [126, Table I] is defined over the field $\mathbf{Q}(\zeta_{12})$, but the character field is $\mathbf{Q}(\omega)$ and thus the relevant primes p are the ones satisfying $p \equiv 1 \pmod 3$. Conjugation by the matrix

$$g = \begin{bmatrix} 2 & \sqrt{3}-1 \\ (\sqrt{3}-1)(1-i) & i-1 \end{bmatrix}$$

gives a realization defined over $\mathbf{Z}[\frac{1}{3}, \omega]$ which embeds in $\mathbf{Z}_p$ for all p as above. Its easily checked that G_{14} is generated by G_5 and the reflection

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix}.$$

By Lemma 13.2 we then get:

$$N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_5^g) = \left\langle G_5^g, \begin{bmatrix} 0 & 1 \\ -2\omega & 0 \end{bmatrix}, \mathbf{Z}_p^\times \right\rangle$$

and thus the homomorphism $\mathbf{Z} \times (\mathbf{Z}_p^\times/\mu_6) \longrightarrow N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_5^g)/G_5^g$ which maps $(k, [\lambda])$ to the coset of $\lambda \begin{bmatrix} 0 & 1 \\ -2\omega & 0 \end{bmatrix}^k$ is surjective. The kernel is easily seen to be the infinite cyclic group generated by the element $(-2, [2])$ and we get $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_5^g)/G_5^g \cong \mathbf{Z} \circ_{\mathbf{Z}} (\mathbf{Z}_p^\times/\mu_6)$. It is easy to check that the element [2] has a square root in $\mathbf{Z}_p^\times/\mu_6$ if and only if $p \equiv 1, 7, 19 \pmod{24}$ (that is unless $p \equiv 13 \pmod{24}$). In this case we get the simpler description $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_5^g)/G_5^g \cong C_2 \times (\mathbf{Z}_p^\times/\mu_6)$.

Elaboration 13.8 (G_7). The realizations of the groups G_7 and G_{10} given in [126, Tables I and II] are both defined over their common character field $\mathbf{Q}(\zeta_{12})$. Thus the relevant primes p are the ones satisfying $p \equiv 1 \pmod{12}$.

More precisely the representations are defined over $\mathbf{Z}[\frac{1}{2},\zeta_{12}]$ and as this ring embeds in $\mathbf{Z}_p$ for all p as above, we get $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_7) = \left\langle G_{10}, \mathbf{Z}_p^\times \right\rangle$ by Lemma 13.2. It is easily checked that $G_{10} = \langle G_7, C_4\rangle$, where C_4 is the cyclic group generated by $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$. Since $G_7 \cap (C_4 \times \mathbf{Z}_p^\times) = C_2 \times \mu_{12}$ we conclude that $N_{\mathrm{GL}_2(\mathbf{Z}_p)}(G_7)/G_7 \cong C_2 \times (\mathbf{Z}_p^\times/\mu_{12})$.

Elaboration 13.9 (Spin($2n$), $n \geq 4$). The group $G = \mathrm{Spin}(2n)$, $n \geq 4$ has Weyl group $G(2,2,n)$ and by [26, Prop. 3.13] $N_{\mathrm{GL}(V)}(W)$ equals $\langle G(2,1,n), \mathbf{C}^\times\rangle$ for $n \geq 5$. For n odd we have $\langle G(2,1,n), \mathbf{C}^\times\rangle = \langle W_G, \mathbf{C}^\times\rangle$, and as above this proves Theorem 13.1 in these cases.

Now assume that $n \geq 4$ is even. The automorphism of the Dynkin diagram which exchanges α_{n-1} and α_n equals $\mathrm{diag}(1, \dots, 1, -1)$ and hence $N_{\mathrm{GL}(V)}(W)$ equals $\langle W_G, \mathbf{C}^\times, \Gamma\rangle$ for $n \geq 6$. For $n = 4$ this also holds since

$$N_{\mathrm{GL}(V)}(W) = \left\langle G(2,1,n), W(F_4), \mathbf{C}^\times \right\rangle = \left\langle W(F_4), \mathbf{C}^\times \right\rangle = \left\langle W_G, \mathbf{C}^\times, \Gamma \right\rangle$$

by [26, Prop. 3.13] and a direct computation. Since $\Gamma \subseteq \mathrm{GL}(L_G)$, Lemma 13.2 shows that $N_{\mathrm{GL}(L_G\otimes\mathbf{Z}_p)}(W_G) = \left\langle W_G, \mathbf{Z}_p^\times, \Gamma \right\rangle$ in all cases and hence

$$N_{\mathrm{GL}(L_G\otimes\mathbf{Z}_p)}(W_G)/\mathbf{Z}_p^\times W_G \cong \Gamma$$

since $\Gamma \cap \mathbf{Z}_p^\times W_G = 1$.

Elaboration 13.10 (Spin(5), F_4 and G_2). For $G = F_4$ the first claim follows directly from [26, Prop. 3.13]. For $G = \mathrm{Spin}(5)$, $W(G)$ is conjugate to $G(4,4,2)$ in $\mathrm{GL}_2(\mathbf{C})$ and [26, Prop. 3.13] shows that

$$N_{\mathrm{GL}_2(\mathbf{C})}(G(4,4,2)) = \left\langle G(4,1,2), \mathbf{C}^\times \right\rangle.$$

Similarly, for $G = G_2$, $W(G)$ is conjugate to $G(6,6,2)$ in $\mathrm{GL}_2(\mathbf{C})$ and [26, Prop. 3.13] shows $N_{\mathrm{GL}_2(\mathbf{C})}(G(6,6,2)) = \langle G(6,1,2), \mathbf{C}^\times\rangle$. From this it is easy to check the first claim in these cases. Thus $N_{\mathrm{GL}(V)}(W) = \left\langle W, \mathbf{C}^\times, \sqrt{l}\varphi_l \right\rangle$ in all cases. By construction $\sqrt{l}\varphi_l$ stabilizes L_G, so Lemma 13.2 shows that $N_{\mathrm{GL}(L\otimes\mathbf{Z}_p)}(W) = \left\langle W, \mathbf{Z}_p^\times, \sqrt{l}\varphi_l \right\rangle$. Since $\varphi_l^2 = 1$ and $\sqrt{l}\varphi_l$ has determinant a power of l the remaining claims follow.

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