Annals of Mathematics, **167** (2008), 1–52

On the homology of algebras of Whitney functions over subanalytic sets

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Abstract

In this article we study several homology theories of the algebra $\mathcal{E}^\infty(X)$ of Whitney functions over a subanalytic set $X \subset \mathbb{R}^n$ with a view towards noncommutative geometry. Using a localization method going back to Teleman we prove a Hochschild-Kostant-Rosenberg type theorem for $\mathcal{E}^\infty(X)$, when X is a regular subset of $\mathbb{R}^n$ having regularly situated diagonals. This includes the case of subanalytic X. We also compute the Hochschild cohomology of $\mathcal{E}^\infty(X)$ for a regular set with regularly situated diagonals and derive the cyclic and periodic cyclic theories. It is shown that the periodic cyclic homology coincides with the de Rham cohomology, thus generalizing a result of Feigin-Tsygan. Motivated by the algebraic de Rham theory of Grothendieck we finally prove that for subanalytic sets the de Rham cohomology of $\mathcal{E}^\infty(X)$ coincides with the singular cohomology. For the proof of this result we introduce the notion of a bimeromorphic subanalytic triangulation and show that every bounded subanalytic set admits such a triangulation.

Contents

Introduction

Methods originating from noncommutative differential geometry have proved to be very successful not only for the study of noncommutative algebras, but also have given new insight to the geometric analysis of smooth manifolds, which are the typical objects of commutative differential geometry. As three particular examples for this we mention the following results:

1. The isomorphism between the de Rham homology of a smooth manifold and the periodic cyclic cohomology of its algebra of smooth functions (Connes [9], [10]),

2. The local index formula in noncommutative geometry by Connes-Moscovici [11],

3. The algebraic index theorem of Nest-Tsygan [40].

It is a common feature of these examples that the underlying space has to be smooth, so that the natural question arises, whether noncommutative methods can also be effectively applied to the study of singular spaces. This is exactly the question we want to address in this work.

In noncommutative geometry, one obtains essential mathematical information about a certain (topological) space from "its" algebra of functions. In the special case, when the underlying space is smooth, i.e. either a smooth complex variety or a smooth manifold, one can recover topological and geometric properties from the algebra of regular, analytic or smooth functions. In particular, as a consequence of the classical Hochschild-Kostant-Rosenberg theorem [28] and Connes' topological version [9], [10], the complex resp. singular cohomology of a smooth space can be obtained as the (periodic) cyclic cohomology of the algebra of global sections of the natural structure sheaf. However, in the presence of singularities, the situation is more complicated. For example, if X is an analytic variety with singularities, the singular cohomology coincides, in general, neither with the de Rham cohomology of the algebra of analytic functions (see Herrera [24] for a specific counterexample) nor with the (periodic) cyclic homology (this can be concluded from the last theorem of Burghelea-Vigué-Poirrier [8]). One can even prove that the vanishing of higher degree Hochschild homology groups of the algebra of regular resp. analytic functions is a criterion for smoothness (see Rodicio [45] or Avramov-Vigué-Poirrier [1]). Computational and structural problems related to singularities appear also, when one tries to compute the Hochschild or cyclic homology of function algebras over a stratified space. For work in this direction see Brasselet-Legrand [5] or Brasselet-Legrand-Teleman [6], [7], where the relation to intersection cohomology [5], [7] and the case of piecewise differentiable functions [7] have been examined.

In this work we propose to consider Whitney functions over singular spaces under a noncommutative point of view. We hope to convince the reader that this is a reasonable approach by showing among other things that the periodic cyclic homology of the algebra $\mathcal{E}^\infty(X)$ of Whitney functions on a subanalytic set $X \subset \mathbb{R}^n$, the de Rham cohomology of $\mathcal{E}^\infty(X)$ (which we call the Whitney-de Rham cohomology of X) and the singular cohomology of X naturally coincide. Besides the de Rham cohomology and the periodic cyclic homology of algebras of Whitney functions we also study their Hochschild homology and cohomology. In fact, we compute these homology theories at first by application of a variant of the localization method of Teleman [48] and then derive the (periodic) cyclic homology from the Hochschild homology.

We have been motivated to study algebras of Whitney functions in a noncommutative setting by two reasons. First, the theory of jets and Whitney functions has become an indispensable tool in real analytic geometry and the differential analysis of spaces with singularities [2], [3], [37], [50], [52]. Second, we have been inspired by the algebraic de Rham theory of Grothendieck [21] (see also [23], [25]) and by the work of Feigin-Tsygan [15] on the (periodic) cyclic homology of the formal completion of the coordinate ring of an affine algebraic variety.

Recall that the formal completion of the coordinate ring of an affine complex algebraic variety $X \subset \mathbb{C}^n$ is the I-adic completion of the coordinate ring of $\mathbb{C}^n$ with respect to the vanishing ideal of X in $\mathbb{C}^n$. Thus, the formally completed coordinate ring of X can be interpreted as the algebraic analogue of the algebra of Whitney functions on X. Now, Grothendieck [21] has proved that the de Rham cohomology of the formal completion coincides with the complex cohomology of the variety, and Feigin-Tsygan [15] have shown that the periodic cyclic cohomology of the formal completion coincides with the algebraic de Rham cohomology, if the affine variety is locally a complete intersection. By the analogy between algebras of formal completions and algebras of Whitney functions it was natural to conjecture that these two results should also hold for Whitney functions over appropriate singular spaces. Theorems 6.4 and 7.1 confirm this conjecture in the case of a subanalytic space.

Our article is set up as follows. In the first section we have collected some basic material from the theory of jets and Whitney functions. Later on in this work we also explain necessary results from Hochschild resp. cyclic homology theory. We have tried to be fairly explicit in the presentation of the preliminaries, so that a noncommutative geometer will find himself going easily through the singularity theory used in this article and vice versa. At the end of Section 1 we also present a short discussion about the dependence of the algebra $\mathcal{E}^\infty(X)$ on the embedding of X in some Euclidean space and how to construct a natural category of ringed spaces $(X, \mathcal{E}^\infty)$.

Since the localization method used in this article provides a general approach to the computation of the Hochschild (co)homology of quite a large class of function algebras over singular spaces, we introduce this method in a separate section, namely Section 2. In Section 3 we treat Peetre-like theorems for local operators on spaces of Whitney functions and on spaces of G-invariant functions. These results will later be used for the computation of the Hochschild cohomology of Whitney functions, but may be of interest on their own.

Section 4 is dedicated to the computation of the Hochschild homology of $\mathcal{E}^\infty(X)$. Using localization methods we first prove that it is given by the homology of the so-called diagonal complex. This complex is naturally isomorphic to the tensor product of $\mathcal{E}^\infty(X)$ with the Hochschild chain complex of the algebra of formal power series. The homology of the latter complex can be computed via a Koszul-resolution, so we obtain the Hochschild homology of $\mathcal{E}^\infty(X)$. In the next section we consider the cohomological case. Interestingly, the Hochschild cohomology of $\mathcal{E}^\infty(X)$ is more difficult to compute, as several other tools besides localization methods are involved, as for example a generalized Peetre's theorem and operations on the Hochschild cochain complex. In Section 6 we derive the cyclic and periodic cyclic homology from the Hochschild homology by standard arguments of noncommutative geometry.

In Section 7 we prove that the Whitney-de Rham cohomology over a subanalytic set coincides with the singular cohomology of the underlying topological space. The claim follows essentially from a Poincaré lemma for Whitney functions over subanalytic sets. This Poincaré lemma is proved with the help of a so-called bimeromorphic subanalytic triangulation of the underlying subanalytic set. The existence of such a triangulation is shown in the last section.

With respect to the above list of (some of) the achievements of noncommutative geometry in geometric analysis we have thus shown that the first result can be carried over to a wide class of singular spaces with the structure sheaf given by Whitney functions. It would be interesting and tempting to examine whether the other two results also have singular analogues involving Whitney functions.

Acknowledgment. The authors gratefully acknowledge financial support by the European Research Training Network *Geometric Analysis on Singular Spaces*. Moreover, the authors thank André Legrand, Michael Puschnigg and Nicolae Teleman for helpful discussions on cyclic homology in the singular setting.

1. Preliminaries on Whitney functions

1.1. *Jets.* The variables $x, x_0, x_1, \ldots, y$ and so on will always stand for elements of some $\mathbb{R}^n$; the coordinates are denoted by $x_{\underline{i}}, x_{0\,\underline{i}}, \ldots, y_{\underline{i}}$, respectively, where $i = 1, \ldots, n$. By $\alpha = (\alpha_{\underline{1}}, \cdots, \alpha_{\underline{n}})$ and β we will always denote multi-indices lying in $\mathbb{N}^n$. Moreover, we write $|\alpha| = \alpha_{\underline{1}} + \ldots + \alpha_{\underline{n}}$, $\alpha! = \alpha_{\underline{1}}! \cdot \ldots \cdot \alpha_{\underline{n}}!$ and $x^\alpha = x_{\underline{1}}^{\alpha_{\underline{1}}} \cdot \ldots \cdot x_{\underline{n}}^{\alpha_{\underline{n}}}$. By $|x|$ we denote the euclidian norm of x, and by $d(x, y)$ the euclidian distance between two points.

In this article X will always mean a locally closed subset of some $\mathbb{R}^n$ and, if not stated differently, $U \subset \mathbb{R}^n$ an open subset such that $X \subset U$ is relatively closed. By a *jet* of *order* m on X (with $m \in \mathbb{N} \cup \{\infty\}$) we understand a family $F = (F^\alpha)_{|\alpha| \leq m}$ of continuous functions on X. The space of jets of order m on X will be denoted by $\mathsf{J}^m(X)$. We write $F(x) = F^0(x)$ for the evaluation of a jet at some point $x \in X$, and $F_{|x}$ for the restricted family $(F^\alpha(x))_{|\alpha| \leq m}$. More generally, if $Y \subset X$ is locally closed, the restriction of continuous functions gives rise to a natural map $\mathsf{J}^m(X) \to \mathsf{J}^m(Y)$, $(F^\alpha)_{|\alpha| \leq m} \mapsto (F^\alpha_{|Y})_{|\alpha| \leq m}$. Given $|\alpha| \leq m$, we denote by $D^\alpha : \mathsf{J}^m(X) \to \mathsf{J}^{m-|\alpha|}(X)$ the linear map, which associates to every $(F^\beta)_{|\beta| \leq m}$ the jet $(F^{\beta+\alpha})_{|\beta| \leq m-|\alpha|}$. If $\alpha = (0, \ldots, 1, \ldots, 0)$ with 1 at the i-th spot, we denote D^α by $D_{\underline{i}}$.

For every natural number $r \leq m$ and every $K \subset X$ compact, $|F|^K_r = \sup_{\substack{x \in K \\ |\alpha| \leq r}} |F^\alpha(x)|$ is a seminorm on $\mathsf{J}^m(X)$. Sometimes, in particular if K consists only of one point, we write only $|\cdot|_r$ instead of $|\cdot|^K_r$. The topology defined by the seminorms $|\cdot|^K_r$ gives $\mathsf{J}^m(X)$ the structure of a Fréchet space. Moreover, D^α and the restriction maps are continuous with respect to these topologies.

The space $\mathsf{J}^m(X)$ carries a natural algebra structure where the product FG of two jets has components $(FG)^\alpha = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} F^\beta \, G^{\alpha-\beta}$. One checks easily that $\mathsf{J}^m(X)$ with this product becomes a unital Fréchet algebra.

For $U \subset \mathbb{R}^n$ open we denote by $\mathcal{C}^m(U)$ the space of $\mathcal{C}^m$-functions on U. Then $\mathcal{C}^m(U)$ is a Fréchet space with topology defined by the seminorms

$$|f|^K_r = \sup_{\substack{x \in K \\ |\alpha| \leq r}} |\partial^\alpha_x f(x)| \,,$$

where K runs through the compact subsets of U and r through all natural numbers $\leq m$. Note that for $X \subset U$ closed there is a continuous linear map $\mathsf{J}^m_X : \mathcal{C}^m(U) \to \mathsf{J}^m(X)$ which associates to every $\mathcal{C}^m$-function f the jet $\mathsf{J}^m_X(f) = \left(\partial^\alpha_x f_{\,|X}\right)_{|\alpha| \leq m}$. Jets of this kind are sometimes called *integrable jets.*

1.2. *Whitney functions.* Given $y \in X$ and $F \in \mathsf{J}^m(X)$, the *Taylor polynomial* (of *order* m) of F is defined as the polynomial

$$T^m_y F(x) = \sum_{|\alpha| \leq m} \frac{F^\alpha(y)}{\alpha!}(x-y)^\alpha, \quad x \in U.$$

Moreover, one sets $R^m_y F = F - \mathsf{J}^m(T^m_y F)$. Then, if $m \in \mathbb{N}$, a *Whitney function* of *class* $\mathcal{C}^m$ on X is an element $F \in \mathsf{J}^m(X)$ such that for all $|\alpha| \leq m$

$$(R^m_y F)(x) = o(|y-x|^{m-|\alpha|}) \quad \text{for } |x-y| \to 0,\ x, y \in X.$$

The space of all Whitney functions of class $\mathcal{C}^m$ on X will be denoted by $\mathcal{E}^m(X)$. It is a Fréchet space with topology defined by the seminorms

$$\|F\|_m^K = |F|_m^K + \sup_{\substack{x,y\in K\\ x\neq y\\ |\alpha|\leq m}} \frac{|(R_y^m F)^\alpha(x)|}{|y-x|^{m-|\alpha|}},$$

where K runs through the compact subsets of X. The projective limit $\varprojlim_r \mathcal{E}^r(X)$ will be denoted by $\mathcal{E}^\infty(X)$; its elements are called *Whitney functions* of *class* $\mathcal{C}^\infty$ on X. By construction, $\mathcal{E}^\infty(X)$ can be identified with the subspace of all $F \in \mathsf{J}^\infty(X)$ such that $\mathsf{J}^r F \in \mathcal{E}^r(X)$ for every natural number r. Moreover, the Fréchet topology of $\mathcal{E}^\infty(X)$ then is given by the seminorms $\|\cdot\|_r^K$ with $K \subset X$ compact and $r \in \mathbb{N}$. It is not very difficult to check that for $U \subset \mathbb{R}^n$ open, $\mathcal{E}^m(U)$ coincides with $\mathcal{C}^m(U)$ (even for $m = \infty$).

Each one of the spaces $\mathcal{E}^m(X)$ inherits from $\mathsf{J}^m(X)$ the associative product; thus $\mathcal{E}^m(X)$ becomes a subalgebra of $\mathsf{J}^m(X)$ and a Fréchet algebra. It is straightforward that the spaces $\mathcal{E}^m(V)$ with V running through the open subsets of X form the sectional spaces of a sheaf $\mathcal{E}_X^m$ of Fréchet algebras on X and that this sheaf is fine. We will denote by $\mathcal{E}_{X,x}^m$ the stalk of this sheaf at some point $x \in X$ and by $[F]_x \in \mathcal{E}_{X,x}^m$ the germ (at x) of a Whitney function $F \in \mathcal{E}^m(V)$ defined on a neighborhood V of x.

For more details on the theory of jets and Whitney functions the reader is referred to the monographs of Malgrange [37] and Tougeron [50], where he or she will also find explicit proofs.

1.3. *Regular sets.* For an arbitrary compact subset $K \subset \mathbb{R}^n$ the seminorms $|\cdot|_m^K$ and $\|\cdot\|_m^K$ are in general not equivalent. The notion of regularity essentially singles out those sets for which $\|\cdot\|_m^K$ can be majorized by a seminorm of the form $C\,|\cdot|_{m'}^K$ with $C > 0$, $m' \geq m$. Following [50, Def. 3.10], a compact set K is defined to be *p-regular*, if it is connected by rectifiable arcs and if the geodesic distance δ satisfies $\delta(x,y) \leq C\,|x-y|^{1/p}$ for all $x, y \in K$ and some $C > 0$ depending only on K. Then, if K is 1-regular, the seminorms $|\cdot|_m^K$ and $\|\cdot\|_m^K$ have to be equivalent and $\mathcal{E}^m(K)$ is a closed subspace of $\mathsf{J}^m(K)$. More generally, if K is p-regular for some positive integer p, there exists a constant $C_m > 0$ such that $\|F\|_m^K \leq C_m\,|F|_{pm}^K$ for all $F \in \mathcal{E}^{pm}(K)$ (see [50]).

Generalizing the notion of regularity to not necessarily compact locally closed subsets one calls a closed subset $X \subset U$ *regular*, if for every point $x \in X$ there exist a positive integer p and a p-regular compact neighborhood $K \subset X$. For X regular, the Fréchet space $\mathcal{E}^\infty(X)$ is a closed subspace of $\mathsf{J}^\infty(X)$ which means in other words that the topology given by the seminorms $|\cdot|_r^K$ is equivalent to the original topology defined by the seminorms $\|\cdot\|_r^K$.

1.4. *Whitney's extension theorem.* Let $Y \subset X$ be closed and denote by $\mathcal{J}^m(Y;X)$ the ideal of all Whitney functions $F \in \mathcal{E}^m(X)$ which are *flat* of *order*

m on Y, which means those which satisfy $F_{|Y} = 0$. The Whitney extension theorem (Whitney [52], see also [37, Thm. 3.2, Thm. 4.1] and [50, Thm. 2.2, Thm. 3.1]) then says that for every $m \in \mathbb{N} \cup \{\infty\}$ the sequence

$$0 \longrightarrow \mathcal{J}^m(Y;X) \longrightarrow \mathcal{E}^m(X) \longrightarrow \mathcal{E}^m(Y) \longrightarrow 0 \tag{1.1}$$

is exact, where the third arrow is given by restriction. In particular this means that $\mathcal{E}^m(Y)$ coincides with the space of integrable m-jets on Y. For finite m and compact X such that Y lies in the interior of X there exists a linear splitting of the above sequence or in other words an extension map $W : \mathcal{E}^m(Y) \to \mathcal{E}^m(X)$ which is continuous in the sense that $|W(F)|^X_m \leq C \, \|F\|^Y_m$ for all $F \in \mathcal{E}^m(Y)$. If in addition X is 1-regular this means that the sequence (1.1) is split exact. These complements on the continuity of W are due to Glaeser [18]. Note that for $m = \infty$ a continuous linear extension map does in general not exist.

Under the assumption that X is 1-regular, m finite and Y in the interior of X, the subspace of all Whitney functions of class $\mathcal{C}^\infty$ on X which vanish in a neighborhood of Y is dense in $\mathcal{J}^m(Y;X)$ (with respect to the topology of $\mathcal{E}^m(X)$).

Assume to be given two relatively closed subsets $X \subset U$ and $Y \subset V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^N$ are open. Further let $g : U \to V$ be a smooth map such that $g(X) \subset Y$. Then, by Whitney's extension theorem, there exists for every $F \in \mathcal{E}^\infty(Y)$ a uniquely determined Whitney function $g^*(F) \in \mathcal{E}^\infty(X)$ such that for every $f \in \mathcal{C}^\infty(V)$ with $\mathsf{J}^\infty_Y(f) = F$ the function $f \circ g \in \mathcal{C}^\infty(U)$ satisfies $\mathsf{J}^\infty_X(f \circ g) = g^*(F)$. The Whitney function $g^*(F)$ will be called the *pull-back* of F by g.

1.5. *Regularly situated sets.* Two closed subsets X, Y of an open subset $U \subset \mathbb{R}^n$ are called *regularly situated* [50, Chap. IV, Def. 4.4], if either $X \cap Y = \emptyset$ or if for every point $x_0 \in X \cap Y$ there exists a neighborhood $W \subset U$ of x_0 and a pair of constants $C > 0$ and $\lambda \geq 0$ such that

$$d(x, Y) \geq C \, d(x, X \cap Y)^\lambda \quad \text{for all } x \in W \cap X.$$

It is a well-known result by Łojasiewicz [33] that X, Y are regularly situated if and only if the sequence

$$0 \longrightarrow \mathcal{E}^\infty(X \cup Y) \xrightarrow{\delta} \mathcal{E}^\infty(X) \oplus \mathcal{E}^\infty(Y) \xrightarrow{\pi} \mathcal{E}^\infty(X \cap Y) \longrightarrow 0$$

is exact, where the maps δ and π are given by $\delta(F) = (F_{|X}, F_{|Y})$ and $\pi(F, G) = F_{|X \cap Y} - G_{|X \cap Y}$.

1.6. *Multipliers.* If $Y \subset U$ is closed we denote by $\mathcal{M}^\infty(Y;U)$ the set of all $f \in \mathcal{C}^\infty(U \setminus Y)$ which satisfy the following condition: For every compact $K \subset U$ and every $\alpha \in \mathbb{N}^n$ there exist constants $C > 0$ and $\lambda > 0$ such that

$$|\partial^\alpha_x f(x)| \leq \frac{C}{(d(x, Y))^\lambda} \quad \text{for all } x \in K \setminus Y.$$

The space $\mathcal{M}^\infty(Y;U)$ is an algebra of *multipliers* for $\mathcal{J}^\infty(Y;U)$ which means that for every $f \in \mathcal{J}^\infty(Y;U)$ and $g \in \mathcal{M}^\infty(Y;U)$ the product gf on $U \setminus Y$ has a unique extension to an element of $\mathcal{J}^\infty(Y;U)$. More generally, if X and Y are closed subsets of U, then we denote by $\mathcal{M}^\infty(Y;X)$ the injective limit $\varinjlim_{W} \mathrm{J}^\infty_{X\setminus Y}\mathcal{M}^\infty(Y;W)$, where W runs through all open sets of U which satisfy $X \cup Y \subset W$. In case X and Y are regularly situated, then $\mathcal{M}^\infty(Y;X)$ is an algebra of multipliers for $\mathcal{J}^\infty(X \cap Y;X)$ (see [37, IV.1]).

1.7. *Subanalytic sets.* A set $X \subset \mathbb{R}^n$ is called *subanalytic* [26, Def. 3.1], if for every point $x \in X$ there exist an open neighborhood U of x in $\mathbb{R}^n$, a finite system of real analytic maps $f_{ij} : U_{ij} \to U$ $(i = 1, \ldots, p,\ j = 1, 2)$ defined on open subsets $U_{ij} \subset \mathbb{R}^{n_{ij}}$ and a family of closed analytic subsets $A_{ij} \subset U_{ij}$ such that every restriction $f_{ij|A_{ij}} : A_{ij} \to U$ is proper and

$$X \cap U = \bigcup_{i=1}^{p} f_{i1}(A_{i1}) \setminus f_{i2}(A_{i2}).$$

The set of all subanalytic sets is closed under the operations of finite intersection, finite union and complement. Moreover, the image of a subanalytic set under a proper analytic map is subanalytic. From these properties one can derive that for every subanalytic $X \subset \mathbb{R}^n$ the interior $\overset{\circ}{X}$, the closure $\overline{X}$ and the frontier $\operatorname{fr} X = \overline{X} \setminus X$ are subanalytic as well. For details and proofs see Hironaka [26] or Bierstone-Milman [4].

Note that every subanalytic set $X \subset \mathbb{R}^n$ is regular [31, Cor. 2], and that any two relatively closed subanalytic sets $X, Y \subset U$ are regularly situated [4, Cor. 6.7].

1.8. *Łojasiewicz's inequality.* Under the assumption that X and Y are closed in $U \subset \mathbb{R}^n$, one usually says (cf. [50, §V.4]) that a function $f : X \setminus Y \to \mathbb{R}^N$ satisfies *Łojasiewicz's inequality* or is *Łojasiewicz* with respect to Y, if for every compact $K \subset X$ there exist two constants $C > 0$ and $\lambda \geq 0$ such that

$$|f(x)| \geq C\, d(x,Y)^\lambda \quad \text{for all } x \in K \setminus Y.$$

More generally, we say that f is *Łojasiewicz* with respect to the pair (Y, Z), where $Z \subset \mathbb{R}^N$ is a closed subset, if for every K as above there exist $C > 0$ and $\lambda \geq 0$ such that

$$d(f(x), Z) \geq C\, d(x,Y)^\lambda \quad \text{for all } x \in K \setminus Y.$$

In case $g_1, g_2 : X \to \mathbb{R}$ are two subanalytic functions with compact graphs such that $g_1^{-1}(0) \subset g_2^{-1}(0)$, there exist $C > 0$ and $\lambda > 0$ such that g_1 and g_2 satisfy the following relation, also called the *Łojasiewicz inequality*:

$$|g_1(x)| \geq C\, |g_2(x)|^\lambda \quad \text{for all } x \in X. \tag{1.2}$$

For a proof of this property see [4, Thm. 6.4].

1.9. *Topological tensor products and nuclearity.* Recall that on the tensor product $V \otimes W$ of two locally convex real vector spaces V and W one can consider many different locally convex topologies arising from the topologies on V and W (see Grothendieck [20] or Trèves [51, Part. III]). For our purposes, the most natural topology is the π-*topology*, i.e. the finest locally convex topology on $V \otimes W$ for which the natural mapping $\otimes : V \times W \to V \otimes W$ is continuous. With this topology, $V \otimes W$ is denoted by $V \otimes_\pi W$ and its completion by $V \hat{\otimes} W$. In fact, the π-topology is the strongest topology compatible with $\otimes$ in the sense of Grothendieck [20, I. §3, n° 3]. The weakest topology compatible with $\otimes$ is usually called the ε-*topology*; in general it is different from the π-topology. A locally convex space V is called *nuclear*, if all the compatible topologies on $V \otimes W$ agree for every locally convex spaces W.

1.10. Proposition. *The algebra $\mathcal{E}^\infty(X)$ of Whitney functions over a locally closed subset $X \subset \mathbb{R}^n$ is nuclear. Moreover, if $X' \subset \mathbb{R}^{n'}$ is a further locally closed subset, then $\mathcal{E}^\infty(X) \hat{\otimes} \mathcal{E}^\infty(X') \cong \mathcal{E}^\infty(X \times X')$.*

Proof. For open $U \subset \mathbb{R}^n$ the Fréchet space $\mathcal{C}^\infty(U)$ is nuclear [20, II. §2, n° 3], [51, Chap. 51]. Choose U such that X is closed in U. Recall that every Hausdorff quotient of a nuclear space is again nuclear [51, Prop. 50.1]. Moreover, by Whitney's extension theorem, $\mathcal{E}^\infty(X)$ is the quotient of $\mathcal{C}^\infty(U)$ by the closed ideal $\mathcal{J}^\infty(X;U)$; hence one concludes that $\mathcal{E}^\infty(X)$ is nuclear.

Now choose an open set $U' \subset \mathbb{R}^{n'}$ such that X' is closed in U'. Then we have the following commutative diagram of continuous linear maps:

$$\begin{array}{ccc}
\mathcal{C}^\infty(U) \otimes_\pi \mathcal{C}^\infty(U') & \longrightarrow & \mathcal{E}^\infty(X) \otimes_\pi \mathcal{E}^\infty(X') \\
\big\downarrow & & \big\downarrow \\
\mathcal{C}^\infty(U \times U') & \longrightarrow & \mathcal{E}^\infty(X \times X').
\end{array}$$

Clearly, the horizontal arrows are surjective and the vertical arrows injective. Since the completion of $\mathcal{C}^\infty(U) \otimes_\pi \mathcal{C}^\infty(U')$ coincides with $\mathcal{C}^\infty(U \times U')$, the completion of $\mathcal{E}^\infty(X) \otimes_\pi \mathcal{E}^\infty(X')$ coincides with $\mathcal{E}^\infty(X \times X')$. This proves the claim. $\square$

1.11. *Remark.* Note that for finite m and nonfinite but compact X the space $\mathcal{E}^m(X)$ is not nuclear, since a normed space is nuclear if and only if it is finite dimensional [51, Cor. 2 to Prop. 50.2].

1.12. *The category of Whitney ringed spaces.* Given a subanalytic (or more generally a stratified) set X, the algebra $\mathcal{E}^\infty(X)$ of Whitney functions on X depends on the embedding $X \hookrightarrow \mathbb{R}^n$. This phenomenon already appears in the algebraic de Rham theory of Grothendieck, where the formal completion $\hat{\mathcal{O}}$ of the algebra of regular functions on a complex algebraic variety X depends on the choice of an embedding of X in some affine $\mathbb{C}^n$. The dependence of the ringed space $(X, \mathcal{E}^\infty)$ resp. $(X, \hat{\mathcal{O}})$ on such embeddings appears to be unnatural, since the structure sheaf should be an intrinsic property of X. Following ideas of Grothendieck [22] on crystalline cohomology we will now briefly sketch an approach showing how to remedy this situation and how to give Whitney functions a more intrinsic interpretation. The essential ansatz hereby consists of regarding the category of all local (smooth or analytic) embeddings of the underlying subanalytic set X in some Euclidean space $\mathbb{R}^n$ instead of just a global one. Note that the following considerations will not be needed in the sequel and that they are of a more fundamental nature.

Now assume X to be a stratified space. By a *smooth chart* on X we understand a homeomorphism $\iota : U \to \tilde{U} \subset \mathbb{R}^n$ from an open subset of X onto a locally closed subset $\tilde{U}$ in some Euclidean space such that for every stratum $S \subset X$ the restriction $\iota_{|U\cap S}$ is a diffeomorphism onto a smooth submanifold of $\mathbb{R}^n$. Such a smooth chart will often be denoted by (ι, U) or $(\iota, U, \mathbb{R}^n)$. Given smooth charts $(\iota, U, \mathbb{R}^n)$ and $(\kappa, V, \mathbb{R}^m)$ such that $U \subset V$ and $n \geq m$, a *morphism* $(\iota, U) \to (\kappa, V)$ is a (vector valued) Whitney function $H : \iota(U) \to \mathbb{R}^n$ such that the following holds true:

(i) H is diffeomorphic which means that H can be extended to a diffeomorphism from an open neighborhood of $\iota(U)$ to an open subset of $\mathbb{R}^n$,

(ii) $H \circ \iota = i^n_m \circ \kappa_{|U}$, where $i^n_m : \mathbb{R}^m \to \mathbb{R}^n$ is the canonical injection $(x_{\underline{1}}, \cdots, x_{\underline{m}}) \mapsto (x_{\underline{1}}, \cdots, x_{\underline{m}}, 0 \cdots, 0)$.

For convenience, we sometimes denote such a morphism as a pair $(H, \mathbb{R}^n)$. In case $(G, \mathbb{R}^m) : (\kappa, V) \to (\lambda, W)$ is a second morphism, the composition $(G, \mathbb{R}^m) \circ (H, \mathbb{R}^n)$ is defined as the morphism $((G \times \mathrm{id}_{\mathbb{R}^{n-m}}) \circ H, \mathbb{R}^n)$. It is immediate to check that the smooth charts on X thus form a small category with pullbacks.

Two charts (κ_1, V_1) and (κ_2, V_2) are called *compatible*, if for every $x \in V_1 \cap V_2$ there exists an open neighborhood $U \subset V_1 \cap V_2$ and a chart (ι, U) such that there are morphisms $(\iota, U) \to (\kappa_i, V_i)$ for $i = 1, 2$. If $U \subset X$ is an open subspace, a *covering* of U is a family (ι_i, U_i) of smooth charts such that $U = \bigcup_i U_i$. A covering for X will be called an *atlas*. If an atlas is maximal with respect to inclusion we call it a *smooth structure* for X. This notion has been introduced in [44, §1.3]. Clearly, algebraic varieties, semialgebraic sets and subanalytic sets carry natural smooth structures inherited from their canonical embedding in some $\mathbb{R}^n$. In [44] it has been shown also that orbit spaces of proper Lie group actions and symplectically reduced spaces carry a natural smooth structure.

Given such a smooth structure $\mathcal{A}$ on X we now construct a Grothendieck topology on X (or better on $\mathcal{A}$), and then the sheaf of Whitney functions. Observe first that $\mathcal{A}$ is a small category with pullbacks. By a *covering* of a smooth chart $(\iota, U) \in \mathcal{A}$ we mean a family $\big(H_i : (\iota_i, U_i) \to (\iota, U)\big)$ of morphisms in $\mathcal{A}$ such that $U = \bigcup_i U_i$. It is immediate to check that assigning to every $(\iota, U) \in \mathcal{A}$ the set $\mathrm{Cov}(\iota, U)$ of all its coverings gives rise to a (basis of a) Grothendieck topology on $\mathcal{A}$ (see [36] for details on Grothendieck topologies). To every $(\iota, U) \in \mathcal{A}$ we now associate the algebra $\mathcal{E}^\infty(\iota, U) := \mathcal{E}^\infty(\iota(U))$ of Whitney functions over $\iota(U) \subset \mathbb{R}^n$. Moreover, every morphism $H : (\iota, U) \to (\kappa, V)$ gives rise to a *generalized restriction map*

$$H^* : \mathcal{E}^\infty(\kappa, V) \to \mathcal{E}^\infty(\iota, U), \quad F \mapsto F \circ H^{-1} \circ i^n_m.$$

It is immediate to check that $\mathcal{E}^\infty$ thus becomes a separated presheaf on the site $(\mathcal{A}, \mathrm{Cov})$. Let $\hat{\mathcal{E}}^\infty$ be the associated sheaf. Then $(X, \hat{\mathcal{E}}^\infty)$ is a ringed space in a generalized sense; we call it a *Whitney ringed space* and the structure sheaf $\hat{\mathcal{E}}^\infty$ the *sheaf of Whitney functions* on X. This sheaf depends only on the smooth structure on X and not on a particular embedding of X in some $\mathbb{R}^n$. So the sheaf of Whitney functions $\hat{\mathcal{E}}^\infty$ is intrinsically defined, and the main results of this article can be interpreted as propositions about the local homological properties of $\hat{\mathcal{E}}^\infty$ (resp. $\mathcal{E}^\infty$) in case X is subanalytic. Finally let us mention that one can also define morphisms of Whitney ringed spaces. These are just morphisms of ringed spaces which in local charts are given by vector-valued Whitney functions. Thus the Whitney ringed spaces form a category, which we expect to be quite useful in singular analysis and geometry.

2. Localization techniques

In this section we introduce a localization method for the computation of the Hochschild homology of a fine commutative algebra. This method works also for the computation of (co)homology groups with values in a module and generalizes the approach of Teleman [48] and Brasselet-Legrand-Teleman [7].

2.1. Let $X \subset \mathbb{R}^n$ be a locally closed subset and d the euclidian metric. Let $\mathcal{A}$ be a sheaf of commutative unital $\mathbb{R}$-algebras on X and denote by $A = \mathcal{A}(X)$ its space of global sections. We assume that $\mathcal{A}$ is an $\mathcal{E}^\infty_X$-module sheaf, which implies in particular that $\mathcal{A}$ is a fine sheaf. Additionally, we assume that the sectional spaces of $\mathcal{A}$ carry the structure of a Fréchet algebra, that all the restriction maps are continuous and that for every open $U \subset X$ the action of $\mathcal{E}^\infty(U)$ on $\mathcal{A}(U)$ is continuous. This implies in particular that A is a commutative Fréchet algebra. The premises on $\mathcal{A}$ are satisfied for example in the case when $\mathcal{A}$ is the sheaf of Whitney functions or the sheaf of smooth functions on X.

From $\mathcal{A}$ one constructs for every $k \in \mathbb{N}^*$ the exterior tensor product sheaf $\mathcal{A}^{\hat{\boxtimes} k}$ over X^k. Its space of sections over a product of the form $U_1 \times \ldots \times U_k$ with $U_i \subset X$ open is given by the completed π-tensor product $\mathcal{A}(U_1) \hat{\otimes} \ldots \hat{\otimes} \mathcal{A}(U_k)$. Using the fact that $\mathcal{A}$ is a topological $\mathcal{E}^\infty_X$-module sheaf and that $\mathcal{E}^\infty(X)$ is fine one checks immediately that the presheaf defined by these conditions is in fact a sheaf, hence $\mathcal{A}^{\hat{\boxtimes} k}$ is well-defined. Throughout this article we will often make silent use of the sheaf $\mathcal{A}^{\hat{\boxtimes} k}$ by writing an element of the topological tensor product $A^{\hat{\otimes} k}$ as a section $c(x_0, \ldots, x_{k-1})$, where $c \in \mathcal{A}^{\hat{\boxtimes} k}(X^k)$ and $x_0, \ldots, x_{k-1} \in X$.

Next we will introduce a few objects often used in the sequel. First choose a smooth function $\varrho : \mathbb{R} \to [0,1]$ with $\operatorname{supp} \varrho = (-\infty, \frac{3}{4}]$ and $\varrho(s) = 1$ for $s \leq \frac{1}{2}$. For every $t > 0$ denote by ϱ_t the rescaled function $\varrho_t(s) = \varrho(\frac{s}{t})$, $s \in \mathbb{R}$. By $\Delta_k : \mathbb{R}^n \to \mathbb{R}^{kn}$ or briefly Δ we denote the diagonal map $x \mapsto (x, \cdots, x)$ and by $d_k : \mathbb{R}^{kn} \to \mathbb{R}$ the following distance to the diagonal:

$$d_k(x_0, x_1, \cdots, x_{k-1}) = \sqrt{d^2(x_0, x_1) + d^2(x_1, x_2) + \cdots + d^2(x_{k-1}, x_0)}.$$

Finally, let $U_{k,t} = \{(x_0, \cdots, x_{k-1}) \in X^k \mid d_k^2(x_0, \cdots, x_{k-1}) < t\}$ be the so-called *t-neighborhood* of the diagonal $\Delta_k(X)$.

In the following we want to show how the computation of the Hochschild homology of A can be essentially reduced to the computation of the local Hochschild homology groups of $\mathcal{A}$. Since we consider the topological version of Hochschild homology theory, we will use in the definition of the Hochschild (co)chain complex the completed π-tensor product $\hat{\otimes}$ and the functor Hom_A of continuous A-linear maps between A-Fréchet modules.

2.2. Now assume to be given an $\mathcal{A}$-module sheaf $\mathcal{M}$ of symmetric Fréchet modules and denote by $M = \mathcal{M}(X)$ the Fréchet space of global sections. Denote by $C_\bullet(A, M)$ the Hochschild chain complex with components $M \hat{\otimes} A^{\hat{\otimes} k}$ and by $C^\bullet(A, M)$ the Hochschild cochain complex, where $C^k(A, M)$ is given by $\operatorname{Hom}_A(C_k(A, A), M)$. Denote by $b_k : C_k(A, M) \to C_{k-1}(A, M)$ the Hochschild boundary and by $b^k : C^k(A, M) \to C^{k+1}(A, M)$ the Hochschild coboundary. This means that $b_k = \sum_{i=0}^{k} (-1)^i \, (b_{k,i})_*$ and $b^k = \sum_{i=0}^{k+1} (-1)^i \, b_{k+1,i}^*$, where the $b_{k,i} : C_k \to C_{k-1}$ with $C_k := C_k(A, A)$ are the face maps which act on an element $c \in C_k$ as follows:

$$b_{k,i} c(x_0, \ldots, x_{k-1}) = \begin{cases} c(x_0, x_0, \ldots, x_{k-1}), & \text{if } i = 0, \\ c(x_0, \ldots, x_i, x_i, \ldots, x_{k-1}), & \text{if } 1 \leq i < k, \\ c(x_0, \ldots, x_{k-1}, x_0), & \text{if } i = k. \end{cases}$$

Hereby, $x_0, \ldots, x_{k-1}$ are elements of X, and the fact has been used that C_k can be identified with the space of global sections of the sheaf $\mathcal{A}^{\hat{\boxtimes}(k+1)}$. The Hochschild homology of A with values in M now is the homology $H_\bullet(A, M)$ of the complex $(C_\bullet(A, M), b_\bullet)$. Likewise, the Hochschild cohomology $H^\bullet(A, M)$ is given by the cohomology of the cochain complex $(C^\bullet(A, M), b^\bullet)$. As usual we will denote the homology space $H_\bullet(A, A)$ briefly by $HH_\bullet(A)$.

2.3. *Remark.* In general, the particular choice of the topological tensor product used in the definition of the Hochschild homology of a topological algebra is crucial for the theory to work well (see Taylor [47] for general information on this topic and Brasselet-Legrand-Teleman [6] for a particular example of a topological algebra, where the ε-tensor product has to be used in the definition of the topological Hochschild complex). But since the Fréchet space $\mathcal{E}^\infty(X)$ is nuclear, this question does not arise in the main application we are interested in, namely the definition and computation of the Hochschild homology of $\mathcal{E}^\infty(X)$.

2.4. As $C_k(A,M)$ is the space of global sections of a sheaf, the notion of support of a chain $c \in C_k(A,M)$ makes sense: $\operatorname{supp} c = \overline{\{x \in X^{k+1} \mid [c]_x \neq 0\}}$. To define the support of a cochain note first that C_k inherits from A the structure of a commutative algebra and secondly that C_k acts on $C^k(A,M)$ by $cf(c') = f(c\,c')$, where $c, c' \in C_k$ and $f \in C^k(A,M)$. The support of $f \in C^k(A,M)$ then is given by the complement of all $x \in X^{k+1}$ for which there exists an open neighborhood U such that $cf = 0$ for all $c \in C_k$ with $\operatorname{supp} c \subset U$.

The following two observations are fundamental for localization à la Teleman.

1. Localization on the first factor: For $a \in A$ the chain $a_k = a \otimes 1 \otimes \ldots \otimes 1 \in A^{\hat{\otimes}(k+1)}$ acts in a natural way on $C_k(A,M)$ and $C^k(A,M)$. Since A is commutative and M a symmetric A-module, the resulting endomorphisms give rise to chain maps $a_\bullet : C_\bullet(A,M) \to C_\bullet(A,M)$ and $a^\bullet : C^\bullet(A,M) \to C^\bullet(A,M)$ such that $\operatorname{supp} a_\bullet c \subset (\operatorname{supp} a \times X^k) \cap \operatorname{supp} c$ and $\operatorname{supp} a^\bullet f \subset (\operatorname{supp} a \times X^k) \cap \operatorname{supp} f$.
2. Localization around the diagonal: For any $t > 0$ and $k \in \mathbb{N}$ let $\Psi_{k,t} : A^{\hat{\otimes}(k+1)} \to A^{\hat{\otimes}(k+1)}$ be defined by

$$\Psi_{k,t}(x_0, \cdots, x_k) = \prod_{j=0}^{k} \varrho_t\big(d^2(x_j, x_{j+1})\big), \quad \text{where } x_{k+1} := x_0. \tag{2.1}$$

   Then the action by $\Psi_{k,t}$ gives rise to chain maps $\Psi_{\bullet,t} : C_\bullet(A,M) \to C_\bullet(A,M)$ and $\Psi_t^\bullet : C^\bullet(A,M) \to C^\bullet(A,M)$ such that $\operatorname{supp}(\Psi_{k,t} c) \subset U_{k+1,t}$ and $\operatorname{supp}(\Psi_t^k f) \subset U_{k+1,t}$.

We now will construct a homotopy operator between the identity and $\Psi_{\bullet,t}$ resp. $\Psi_t^\bullet$. To this end define A-module maps $\eta_{k,i,t} : C_k \to C_{k+1}$ for every integer $k \geq -1$ and $i = 1, \cdots, k+2$ by

$$\tag{2.2} \begin{aligned} &\eta_{k,i,t}(c)(x_0, \cdots, x_{k+1}) \\ &= \begin{cases} \Psi_{k+1,i,t}(x_0, \cdots, x_{k+1})\, c(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}) & \text{for } i < k+1, \\ \Psi_{k+1,k+1,t}(x_0, \cdots, x_{k+1})\, c(x_0, \cdots, x_k) & \text{for } i = k+1, \\ 0 & \text{for } i = k+2 \end{cases} \end{aligned}$$

where $c \in C_k$, $x_0, \cdots, x_{k+1} \in X$ and, since $x_{k+2} := x_0$, the functions $\Psi_{k+1,i,t}$, $i = 1, \cdots, k+2$ are given by $\Psi_{k+1,i,t}(x_0, \cdots, x_{k+1}) = \prod_{j=0}^{i-1} \varrho_t(d^2(x_j, x_{j+1}))$. For $i = 2, \cdots, k$ one then computes

$$
\begin{aligned}
\big((b_{k+1}\eta_{k,i,t} + \eta_{k-1,i,t}b_k)c\big)(x_0, \cdots, x_k) &= \Psi_{k,i-1,t}(-1)^{i-1}c(x_0, \cdots, x_k) \\
&+\Psi_{k,i-1,t}\sum_{j=0}^{i-2}(-1)^j\, c(x_0, \cdots, x_j, x_j, \cdots, x_{i-2}, x_i, \cdots, x_k) \\
&+(-1)^i\Psi_{k,i,t}c(x_0, \cdots, x_k) \\
&+\Psi_{k,i,t}\sum_{j=0}^{i-1}(-1)^j\, c(x_0, \cdots, x_j, x_j, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k).
\end{aligned} \tag{2.3}
$$

For the two remaining cases $i = 1$ and $i = k+1$,

$$
\begin{aligned}
&\big((b_{k+1}\eta_{k,1,t} + \eta_{k-1,1,t}b_k)c\big)(x_0, \cdots, x_k) \\
&\quad = c(x_0, \cdots, x_k) - \Psi_{k,1,t}c(x_0, \cdots, x_k) + \Psi_{k,1,t}c(x_0, x_0, x_2, \cdots, x_k),
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
&\big((b_{k+1}\eta_{k,k+1,t} + \eta_{k-1,k+1,t}b_kc)\big)(x_0, \cdots, x_k) \\
&\quad = \Psi_{k,k,t}(-1)^k c(x_0, \cdots, x_k) \\
&\qquad +\Psi_{k,k,t}\sum_{j=0}^{k-1}(-1)^j\, c(x_0, \cdots, x_j, x_j, \cdots, x_{k-1}) \\
&\qquad +(-1)^{k+1}\Psi_{k,t}\, c(x_0, \cdots, x_k).
\end{aligned} \tag{2.5}
$$

Note that by definition every $\eta_{k,i,t}$ is a morphism of A-modules, which means that one can apply the functors $M\hat{\otimes}-$ and $\operatorname{Hom}_A(-, M)$ to these morphisms. By the computations above we thus obtain our first result.

2.5. Proposition. *The map*

$$
\begin{aligned}
H_{k,t} &= \sum_{i=1}^{k+1}(-1)^{i+1}\,(\eta_{k,i,t})_* : C_k(A, M) \to C_{k+1}(A, M) \quad \textit{resp.} \\
H_t^k &= \sum_{i=1}^{k}(-1)^{i+1}\,\eta^*_{k-1,i,t} : C^k(A, M) \to C^{k-1}(A, M)
\end{aligned}
$$

gives rise to a homotopy between the identity and the localization morphism $\Psi_{\bullet,t}$ *resp.* $\Psi_t^\bullet$. *More precisely,*

$$
\big(b_{k+1}H_{k,t} + H_{k-1,t}b_k\big)c = c - \Psi_{k,t}\, c \quad \textit{for all } c \in C_k(A, M) \textit{ and} \tag{2.6}
$$

$$
\big(b^{k-1}H_t^k + H_t^{k+1}b^k\big)f = f - \Psi_{k,t}\, f \quad \textit{for all } f \in C^k(A, M). \tag{2.7}
$$

2.6. *Remark.* The localization morphisms given in Teleman, which form the analogue of the morphisms $\eta_{k,i,t}$ defined above, are not A-linear, hence can be used only for localization of the complex $C_\bullet(A, A)$ but not for the localization of Hochschild cohomology or of Hochschild homology with values in an arbitrary module M.

Following Teleman [48] we denote by $C_k^t(A,M) \subset C_k(A,M)$ resp. $C_t^k(A,M) \subset C^k(A,M)$ the space of Hochschild (co)chains with support disjoint from $U_{k+1,t}$ and by $C_k^0(A,M)$ resp. $C_0^k(A,M)$ the inductive limit $\bigcup_{t>0} C_k^t(A,M)$ resp. $\bigcup_{t>0} C_t^k(A,M)$. Finally denote by $\mathcal{H}_\bullet$ the sheaf associated to the presheaf with sectional spaces $H_\bullet(\mathcal{A}(V),\mathcal{M}(V))$, where V runs through the open subsets of X. The proposition then implies the following results.

2.7. Corollary. *The complexes $C_\bullet^0(A,M)$ and $C_0^\bullet(A,M)$ are acyclic.*

2.8. Corollary. *The Hochschild homology of A coincides with the global sections of $\mathcal{H}_\bullet$ which means that $H_\bullet(A,M) = \mathcal{H}_\bullet(X)$.*

3. Peetre-like theorems

In this section we will show that a continuous local operator acting on Whitney functions of class $\mathcal{C}^\infty$ and with values in $\mathcal{E}^m$, $m \in \mathbb{N}$, is locally given by a differential operator. Thus we obtain a generalization of Peetre's theorem [42] which says that every local operator acting on the algebra of smooth functions on $\mathbb{R}^n$ has to be a differential operator, locally.

3.1. Recall that a k-linear operator $D : \mathcal{E}^m(X) \times \ldots \times \mathcal{E}^m(X) \to \mathcal{E}^r(X)$ (with $m, r \in \mathbb{N} \cup \{\infty\}$) is said to be *local*, if for all $F_1, \ldots, F_k \in \mathcal{E}^m(X)$ and every $x \in X$ the value $D(F_1,\ldots,F_k)_{|x} \in \mathcal{E}^r(\{x\})$ depends only on the germs $[F_1]_x, \ldots, [F_k]_x$. In other words this means that D can be regarded as a morphism of sheaves $\Delta_{|X}^*(\mathcal{E}_X^m \otimes \ldots \otimes \mathcal{E}_X^m) \to \mathcal{E}_X^r$.

The following result forms the basic tool for our proof of a Peetre-like theorem for Whitney functions.

3.2. Proposition. *Let E be a Banach space with norm $\|\cdot\|$ and $W \xrightarrow{q} V \to 0$ an exact sequence of Fréchet spaces and continuous linear maps such that the topology of W is given by a countable family of norms $\|\cdot\|_l$, $l \in \mathbb{N}$. Then for every continuous k-linear operator $f : V \times \ldots \times V \to E$ there exists a constant $C > 0$ and a natural number r such that*

$$\|f(v_1,\ldots,v_k)\| \leq C \; \left| v_1 \right|_r \cdot \ldots \cdot \left| v_k \right|_r \quad \textit{for all } v_1,\ldots,v_k \in V,$$

where $|\cdot|_r$ is the seminorm $|v|_r = \inf_{w\in q^{-1}(v)} \sup_{l\leq r} \|w\|_l$.

Proof. Let us first consider the case, where $W = V$ and q is the identity map. Assume that in this situation the claim does not hold. Then one can find sequences $(v_{ij})_{j\in\mathbb{N}} \subset V$ for $i = 1,\ldots,k$ such that

$$\|f(v_{1j},\ldots,v_{kj})\| > j \; \left| v_{1j} \right|_j \cdot \ldots \cdot \left| v_{kj} \right|_j \quad \text{for all } j \in \mathbb{N}.$$

Let $w_{ij} = \frac{1}{\sqrt[k]{j}\,|v_{ij}|_j} v_{ij}$. Then $\lim_{j\to\infty}(w_{1j},\ldots,w_{kj}) = 0$, but $\|f(w_{1j},\ldots,w_{kj})\| \geq 1$ for all $j \in \mathbb{N}$, which is a contradiction to the continuity of f. Hence the claim must be true for $W = V$ and $q = \mathrm{id}$.

Let us now consider the general case of an exact sequence $W \xrightarrow{q} V \to 0$, where the topology of W is given by a countable family of norms. Define $F : W \times \ldots \times W \to E$ by $F(w_1,\ldots,w_k) = f(q(w_1),\ldots,q(w_k))$, $w_i \in W$. By the result proven so far one concludes that there exist a $C > 0$ and a natural r such that

$$\|F(w_1,\ldots,w_k)\| \leq C \sup_{l\leq r} \|w_1\|_l \cdot \ldots \cdot \sup_{l\leq r} \|w_k\|_l \quad \text{for all } w_1,\ldots,w_k \in V.$$

But this entails

$$\begin{aligned}\|f(v_1,\ldots,v_k)\| &= \inf_{w_1\in q^{-1}(v_1)} \cdot \ldots \cdot \inf_{w_k\in q^{-1}(v_k)} \|F(w_1,\ldots,w_k)\| \\ &\leq C\ |v_1|_r \cdot \ldots \cdot |v_k|_r\,;\end{aligned}$$

hence the claim follows. □

3.3. *Peetre's theorem for Whitney functions. Let X be a regular locally closed subset of $\mathbb{R}^n$, $m \in \mathbb{N}$ and $D : \mathcal{E}^\infty(X)\times\cdots\times\mathcal{E}^\infty(X) \to \mathcal{E}^m(X)$ a k-linear continuous and local operator. Then for every compact $K \subset X$ there exists a natural number r such that for all Whitney functions $F_1, G_1, \ldots, F_k, G_k \in \mathcal{E}^\infty(X)$ and every point $x \in K$ the relation $\mathsf{J}^r F_i(x) = \mathsf{J}^r G_i(x)$ for $i = 1,\cdots,k$ implies $D(F_1,\cdots,F_k)_{|x} = D(G_1,\cdots,G_k)_{|x}$.*

Proof. By a straightforward partition of unity argument one can reduce the claim to the case of compact X. So let us assume that X is compact and p-regular for some positive integer p. Then $\mathcal{E}^m(X)$ is a Banach space with norm $\|\cdot\|^X_m$, and $\mathcal{E}^\infty(X)$ is Fréchet with topology defined by the seminorms $|\cdot|^X_l$, $l \in \mathbb{N}$. Choose a compact cube Q such that X lies in the interior of Q. Then the sequence $\mathcal{E}^\infty(Q) \to \mathcal{E}^\infty(X) \to 0$ is exact by Whitney's extension theorem and the topology of $\mathcal{E}^\infty(Q)$ is generated by the norms $|\cdot|^Q_l$, $l \in \mathbb{N}$. Since the sequence $\mathcal{E}^l(Q) \to \mathcal{E}^l(X) \to 0$ is exact and the topology of $\mathcal{E}^l(Q)$ is defined by the norm $|\cdot|^Q_l$, Proposition 3.2 yields the fact that the operator D extends to a continuous k-linear map $D : \mathcal{E}^r(X)\times\cdots\times\mathcal{E}^r(X) \to \mathcal{E}^m(X)$, if r is chosen sufficiently large. Now assume that $F_i, G_i \in \mathcal{E}^\infty(X)$ are Whitney functions with $\mathsf{J}^{pr}F_i(x) = \mathsf{J}^{pr}G_i(x)$ for $i = 1,\cdots,k$. According to 1.4 we can then choose sequences $(d_{ij})_{j\in\mathbb{N}} \subset \mathcal{E}^\infty(X)$ for $i = 1,\ldots,k$ such that d_{ij} vanishes in a neighborhood of x and such that $|F_i - G_i - d_{ij}|^X_{pr} < 2^{-j}$. But then $G_i + d_{ij}$ converges to F_i in $\mathcal{E}^r(X)$; hence by continuity

$$\lim_{j\to\infty} D(G_1 + d_{1j},\ldots,G_k + d_{kj})_{|x} = D(F_1,\ldots,F_k)_{|x}.$$

On the other hand we have $D(G_1 + d_{1j},\ldots,G_k + d_{kj})_{|x} = D(G_1,\ldots,G_k)_{|x}$ for all j by the locality of D. Hence the claim follows. □

3.4. *Remark.* In case $m = \infty$, a continuous and local operator $D : \mathcal{E}^\infty(X) \to \mathcal{E}^m(X)$ need not be a differential operator, as the following example shows. Let X be the $x_{\underline{1}}$-axis of $\mathbb{R}^2$ and let D be the operator $D = \sum_{k\in\mathbb{N}} \delta_k D^k_{\underline{2}}$, where $\delta_k = \mathsf{J}^\infty_X x^k_{\underline{2}}$. Then D is continuous and local, but DF depends over every compact set on infinitely many jets of the argument F.

The following theorem will not be needed in the rest of this work but appears to be of independent interest. Since the proof goes along the same lines as the one for Peetre's theorem for Whitney functions, we leave it to the reader.

3.5. *Peetre's theorem for G-invariant functions. Let G be a compact Lie group acting by diffeomorphisms on a smooth manifold M and let $E, E_1, \cdots, E_k$ be smooth vector bundles over M with an equivariant G-action. Let $D : \Gamma^\infty(E_1)^G \times \cdots \times \Gamma^\infty(E_k)^G \to \Gamma^\infty(E)^G$ be a k-linear continuous and local operator. Then for every compact set $K \subset M$ there exists a natural r such that for all sections $s_1, t_1, \ldots, s_k, t_k \in \Gamma^\infty(E_i)^G$ and every point $x \in K$ the relation $\mathsf{J}^r s_i(x) = \mathsf{J}^r t_i(x)$ for $i = 1, \cdots, k$ implies $D(s_1, \cdots, s_k)(x) = D(t_1, \cdots, t_k)(x)$.*

4. Hochschild homology of Whitney functions

4.1. Our next goal is to apply the localization techniques established in Section 2 to the computation of the Hochschild homology of the algebra $\mathcal{E}^\infty(X)$ of Whitney functions on X. Note that this algebra is the space of global sections of the sheaf $\mathcal{E}^\infty_X$; hence the premises of Section 2 are satisfied. Throughout this section we will assume that X is a regular subset of $\mathbb{R}^n$ and that X has *regularly situated diagonals*. By the latter we mean that X^k and $\Delta_k(\mathbb{R}^n) \cap U^k$ are regularly situated subsets of U^k for every $k \in \mathbb{N}^*$, where $U \subset \mathbb{R}^n$ open is chosen such that $X \subset U$ is closed. Denote by $C_\bullet$ the complex $C_\bullet(\mathcal{E}^\infty(X), \mathcal{E}^\infty(X))$. By Proposition 1.10 we then have $C_k = \mathcal{E}^\infty(X^{k+1})$. Now let $J_\bullet \subset C_\bullet$ be the subspace of chains infinitely flat on the diagonal which means that $J_k = \mathcal{J}^\infty(\Delta_{k+1}(X); X^{k+1})$. Obviously, every face map $b_{k,i}$ maps J_k to J_{k-1}, hence $J_\bullet$ is a subcomplex of $C_\bullet$.

4.2. Proposition. *Assume that $\mathcal{M}$ is a finitely generated projective $\mathcal{E}^\infty_X$-module sheaf of symmetric Fréchet modules, M the $\mathcal{E}^\infty(X)$-module $\mathcal{M}(X)$ and $m \in \mathbb{N} \cup \{\infty\}$. Then the complexes*

$$J_\bullet \hat{\otimes}_{\mathcal{E}^\infty(X)} M \text{ and } \operatorname{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet, M \hat{\otimes}_{\mathcal{E}^\infty(X)} \mathcal{E}^m(X))$$

are acyclic.

Before we can prove the proposition we have to set up a few preliminaries. First let us denote by $e_{k,i} : C_k \to C_{k+1}$ for $i = 1, \ldots, k+1$ the extension morphism such that

$$(e_{k,i}c)(x_0,\ldots,x_{k+1}) = c(x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_{k+1}).$$

Clearly, $e_{k,i}$ is continuous and satisfies $e_{k,i}(J_k)\subset J_{k+1}$. Secondly recall the definition of the functions $\Psi_{k,t}$ and $\Psi_{k,i,t}$ in 2.4. The following two lemmas now hold true.

4.3. Lemma. *Let $\varphi_{k,t}\in\mathcal{C}^\infty(\mathbb{R}^{(k+1)n})$ be one of the functions $\Psi_{k,t}$ or $\Psi_{k,i,\varepsilon t}\,e_{k-1,i}(\partial_t\Psi_{k-1,t})$, where $\varepsilon>0$, $t>0$ and $i=1,\ldots,k$. Then for every compact set $K\subset\mathbb{R}^{(k+1)n}$, $T>0$ and $\alpha\in\mathbb{N}^{(k+1)n}$ there exist a constant $C>0$ and an $m\in\mathbb{N}$ such that*

(4.1)
$$|D^\alpha\varphi_{k,t}(x)|\le C\,\frac{t}{(d(x,\Delta_{k+1}(\mathbb{R}^n)))^m}\quad\textit{for all } x\in K\setminus\Delta_{k+1}(\mathbb{R}^n)\textit{ and } t\in(0,T].$$

Proof. If $\varphi_{k,t}=\Psi_{k,t}$ and $\alpha=0$ the estimate (4.1) is obvious since $\Psi_{k,t}(x)$ is bounded as a function of x and t. Now assume $|\alpha|\ge1$ and compute

$$(D^\alpha\Psi_{k,t})(x)=\sum_{\substack{l_0,\ldots,l_k\in\mathbb{N}\\ 1\le\sum l_j\le|\alpha|}}\prod_{j=0}^{k}\frac{1}{|t|^{l_j}}\,\varrho^{(l_j)}\left(\frac{d^2(x_j,x_{j+1})}{t}\right)d_{l_j,\alpha}(x_j,x_{j+1}),\tag{4.2}$$

where $x=(x_0,\ldots,x_k)$, $x_{k+1}:=x_0$ and the functions $d_{l_j,\alpha}$ are polynomials in the derivatives of the euclidian distance, and so in particular are bounded on compact sets. Now, by definition of the function ϱ_t we have $\varrho_t'(s)=0$ for $0<s\le\frac{t}{2}$ and $\varrho_t(s)=0$ for $s>t$; hence,

$$(D^\alpha\Psi_{k,t})(x)=0\quad\text{for all } x\in U_{k+1,\frac{t}{2}}\text{ and all } x\in\mathbb{R}^{(k+1)n}\setminus U_{k+1,(k+1)t}.\tag{4.3}$$

On the other hand, there exists by Equation (4.2) a constant $C'>0$ such that for all $t\in(0,T]$ and $x\in\left(K\cap\overline{U}_{k+1,(k+1)t}\right)\setminus U_{k+1,\frac{t}{2}}$

$$|(D^\alpha\Psi_{k,t})(x)|\le C'\,\frac{1}{t^{|\alpha|}}<(k+1)^{|\alpha|+1}C'\,\frac{t}{(d_{k+1}(x_0,\ldots,x_k))^{2|\alpha|+2}}.\tag{4.4}$$

But the estimates (4.3) and (4.4) imply that (4.1) holds true for appropriate C and m, hence the claim follows for $\Psi_{k,t}$. By a similar argument one shows the claim for the functions $\Psi_{k,i,\varepsilon t}\,e_{k-1,i}(\partial_t\Psi_{k-1,t})$. ☐

4.4. Lemma. *Each one of the mappings*

$$\mu_k: J_k\times[0,1]\to J_k,\quad (c,t)\mapsto\begin{cases}\Psi_{k,t}c & \textit{if } t>0,\\ 0, & \textit{if } t=0,\end{cases}\tag{4.5}$$

$$\mu_{k,i}: J_k\times[0,1]\to J_k,\quad (c,t)\mapsto\begin{cases}\Psi_{k,i,\varepsilon t}e_{k-1,i}(\partial_t\Psi_{k-1,t})c, & \textit{if } t>0,\\ 0, & \textit{if } t=0,\end{cases}\tag{4.6}$$

is continuous.

Proof. Since X^{k+1} and $\Delta(U) := \Delta_{k+1}(\mathbb{R}^n) \cap U^{k+1}$ are regularly situated there exists a smooth function $\tilde{c} \in \mathcal{J}^\infty(\Delta(U); U^{k+1})$ whose image in $\mathcal{E}^\infty(X^{k+1})$ equals c. By Taylor's expansion one then concludes that for every compact set $K \subset U^{k+1}$, $\alpha \in \mathbb{N}^{(k+1)n}$ and $N \in \mathbb{N}$ there exists a second compact set $L \subset U^{k+1}$ and a constant $C_{\alpha,N}$ such that

$$|D^\alpha \tilde{c}(x)| \leq C_{\alpha,N} \left(d(x, \Delta(U))\right)^N |\tilde{c}|^L_{N+|\alpha|} \quad \text{for all } x \in K. \tag{4.7}$$

By Leibniz rule and Lemma 4.3 the continuity of $\mu_{k,i}$ follows immediately. Analogously, one shows the continuity of μ_k. □

Proof of Proposition 4.2. By the assumptions on $\mathcal{M}$ it suffices to show that the complexes $J_\bullet$ and $\mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet, \mathcal{E}^m(X))$ are acyclic. To prove the claim in the homology case we will construct a (continuous) homotopy $K_k : J_k \to J_{k+1}$ such that

$$(b_{k+1}K_k + K_{k-1}b_k)c = \Psi_{k,1}c \quad \text{for all } c \in J_k. \tag{4.8}$$

By Proposition 2.5 the complex $J_\bullet$ then has to be acyclic. Using the homotopy $H_{\bullet,t}$ of Proposition 2.5 we first define $K_{k,t} : C_k \to C_{k+1}$ by

$$K_{k,t}c = \int_t^1 H_{k,\frac{s}{2(k+1)}}(\partial_s \Psi_{k,s}\, c)\, ds, \quad c \in C_k.$$

Since $\Psi_{\bullet,s}$ is a chain map, we obtain by Equation (2.6)

$$\begin{aligned}
(4.9)\ & (b_{k+1}K_{k,t} + K_{k-1,t}b_k)c \\
&= \int_t^1 b_{k+1}H_{k,\frac{s}{2(k+1)}}(\partial_s \Psi_{k,s}\, c) + H_{k-1,\frac{s}{2(k+1)}} b_k\, (\partial_s \Psi_{k,s}\, c) ds \\
&= \int_t^1 \partial_s \Psi_{k,s} c - \Psi_{k,\frac{s}{2(k+1)}}\, \partial_s \Psi_{k,s} c\, ds = \int_t^1 \partial_s \Psi_{k,s} c\, ds = \Psi_{k,1}c - \Psi_{k,t}c.
\end{aligned}$$

Hereby we have used the relation $\Psi_{k,\frac{s}{2(k+1)}}\, \partial_s \Psi_{k,s} = 0$ which follows from the fact that $\partial_s \Psi_{k,s}(x)$ vanishes on $U_{k+1,\frac{s}{2}}$ and that $\mathrm{supp}\, \Psi_{k,\frac{s}{2(k+1)}} \subset U_{k+1,\frac{s}{2}}$. Let us now assume that $c \in J_k$. Since

$$\begin{aligned}
K_{k,t}c &= \sum_{i=1}^{k+1} (-1)^{i+1} \int_t^1 \Psi_{k+1,i,\frac{s}{2(k+1)}}\, e_{k,i}(\partial_s \Psi_{k,s}) e_{k,i}(c)\, ds \\
&= \sum_{i=1}^{k+1} (-1)^{i+1} \int_t^1 \mu_{k+1,i}(e_{k,i}(c), s)\, ds
\end{aligned}$$

and $e_{k+1,i}(c) \in J_{k+1}$ one concludes by Lemma 4.4 that the map $K_k : J_k \to J_{k+1}$, $c \mapsto \lim_{t \searrow 0} K_{k,t}c$ is well-defined and continuous. So we can pass to the limit $t \to 0$ in Equation (4.9) and obtain (4.8), because $\lim_{t \searrow 0} \Psi_{k,t}c = 0$ by Lemma 4.4.

Since every K_k is continuous and $\mathcal{E}^\infty(X)$-linear, the map

$$K^k : \mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^m(X)) \to \mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_{k-1}, \mathcal{E}^m(X)), \quad f \mapsto f \circ K_{k-1}$$

gives rise to a homotopy such that

$$(b^{k-1}K^k + K^{k+1}b^k)f = \Psi_{k,1}f \quad \text{for all } f \in \mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^m(X)). \tag{4.10}$$

Hence the complex $\mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet, \mathcal{E}^m(X))$ is acyclic as well. □

Consider now the following short exact sequence of complexes:

$$0 \longrightarrow J_\bullet \longrightarrow C_\bullet \longrightarrow E_\bullet \longrightarrow 0, \tag{4.11}$$

where $E_k = C_k/J_k$. As a consequence of the proposition the homology of the complexes $C_\bullet$ and $E_\bullet$ have to coincide. Following Teleman [48] we call $E_\bullet$ the *diagonal complex*. By Whitney's extension theorem its k-th component is given by $E_k = \mathcal{E}^\infty(\Delta_{k+1}(X))$. Since M is a finitely generated projective $\mathcal{E}^\infty(X)$-module, the tensor product of M with the above sequence remains exact. We thus obtain the following result.

4.5. Corollary. *The Hochschild homology $H_\bullet(\mathcal{E}^\infty(X), M)$ is naturally isomorphic to the homology of the tensor product of the diagonal complex and M, i.e. to the homology of the complex $E_\bullet \hat{\otimes} M$.*

The following proposition can be interpreted as a kind of Borel lemma with parameters.

4.6. Proposition. *There is a canonical topologically linear isomorphism of $\mathcal{E}^\infty(X)$-modules*

$$j^\infty_\Delta : \mathcal{E}^\infty(\Delta_{k+1}(X)) \to \mathcal{E}^\infty(X)\hat{\otimes}_\pi \mathcal{F}^\infty_{kn},$$
$$F \mapsto \sum_{\alpha_1,\ldots,\alpha_k \in \mathbb{N}^n} F_{\alpha_1,\ldots,\alpha_k}\, y_1^{\alpha_1} \cdot \ldots \cdot y_k^{\alpha_k}, \quad F_{\alpha_1,\ldots,\alpha_k} = \Delta^*_{k+1}(D^{\alpha_1}_{y_1} \ldots D^{\alpha_k}_{y_k} F),$$

where $\mathcal{F}^\infty_n$ denotes the formal power series algebra in n (real) indeterminates and, for every $i = 1, \ldots, k$, the symbols $y_i = (y_{i\underline{1}}, \ldots, y_{i\underline{n}})$ denote indeterminates.

Proof. Clearly, the map j^∞_Δ is continuous and injective. By an immediate computation one checks that j^∞_Δ is a morphism of $\mathcal{E}^\infty(X)$-modules. So it remains to prove surjectivity; since $\mathcal{E}^\infty(X) \otimes_\pi \mathcal{F}^\infty_{kn}$ is a Fréchet space the claim then follows by the open mapping theorem. To prove surjectivity we use an argument similar to the one used in the proof of Borel's lemma. For simplicity we assume that X is compact; the general case can be deduced from that by a partition of unity argument. Given a series $\sum F_{\alpha_1,\ldots,\alpha_k}\, y_1^{\alpha_1} \cdot \ldots \cdot y_k^{\alpha_k}$ let us define a Whitney function $F \in \mathcal{E}^\infty(\Delta_{k+1}(X))$ by

$$F_{|(x_0,x_1,\ldots,x_k)} = \sum_{\alpha_1,\ldots,\alpha_k \in \mathbb{N}^n} \frac{F_{\alpha_1,\ldots,\alpha_k|x_0}}{\alpha_1! \cdot \ldots \cdot \alpha_k!}\, \mu\big(A_{\alpha_1,\ldots,\alpha_k} d_{k+1}(x_0,\ldots,x_k)\big)\, (x_1-x_0)^{\alpha_1} \cdot \ldots \cdot (x_k-x_0)^{\alpha_k},$$

where μ is a C^∞-function whose value is 1 in a neighborhood of 0 and whose support is contained in $[-1,1]$, $d_{k+1}(x_0,\ldots,x_k)$ is the distance to the diagonal previously defined and

$$A_{\alpha_1,\ldots,\alpha_k} = \sup\Big(1, \sup_{\beta_1\le\alpha_1,\ldots,\beta_k\le\alpha_k, m\le|\alpha_1|+\ldots+|\alpha_k|} |F_{\beta_1,\ldots,\beta_k}|^X_m\Big).$$

The function $\mu\big(A_\alpha d_{k+1}(x_0,\ldots,x_k)\big)$ is C^∞, because $\mu(t)=1$ near $t=0$. It is straightforward to check that the above series converges to an element $F\in\mathcal{E}^\infty(\Delta_{k+1}(X))$ which satisfies $j^\infty_\Delta(F)=\sum F_{\alpha_1,\ldots,\alpha_k}\, y_1^{\alpha_1}\cdot\ldots\cdot y_k^{\alpha_k}$. □

4.7. Before we formulate a Hochschild-Kostant-Rosenberg type theorem for Whitney functions let us briefly explain what we mean by the space of Whitney differential forms. Recall that the space of Kähler differentials of $\mathcal{E}^\infty(X)$ is the (up to isomorphism uniquely defined) $\mathcal{E}^\infty(X)$-module $\Omega^1_{\mathcal{E}^\infty}(X)$ with a derivation $d:\mathcal{E}^\infty(X)\to\Omega^1_{\mathcal{E}^\infty}(X)$ which is universal with respect to derivations $\delta:\mathcal{E}^\infty(X)\to M$, where M is an $\mathcal{E}^\infty(X)$-module (see Matsumura [38, Ch. 10]). Given an open $U\subset\mathbb{R}^n$ and an X closed in U, the spaces of smooth differential 1-forms over U and $\Omega^1_{\mathcal{E}^\infty}(X)$ are related by the following second exact sequence for Kähler differentials [38, Thm. 58]:

$$\mathcal{J}^\infty(X;U)/\big(\mathcal{J}^\infty(X;U)\big)^2 \to \mathcal{E}^\infty(X)\otimes_{\mathcal{C}^\infty(U)}\Omega^1(U)\to\Omega^1_{\mathcal{E}^\infty}(X)\to 0.$$

Since $\mathcal{J}^\infty(X;U)=\big(\mathcal{J}^\infty(X;U)\big)^2$ this means that there is a canonical isomorphism

$$\Omega^\bullet_{\mathcal{E}^\infty}(X)\cong\mathcal{E}^\infty(X)\otimes_{\mathcal{C}^\infty(U)}\Omega^\bullet(U). \tag{4.12}$$

Hereby, $\Omega^\bullet_{\mathcal{E}^\infty}(X)$ is the exterior power $\Lambda^\bullet\Omega^1_{\mathcal{E}^\infty}(X)$ called the space of *Whitney differential forms* over X. The differential $d:\mathcal{E}^\infty(X)\to\Omega^1_{\mathcal{E}^\infty}(X)$ extends naturally to $\Omega^\bullet_{\mathcal{E}^\infty}(X)$ and gives rise to the *Whitney-de Rham complex*:

$$0\longrightarrow\mathcal{E}^\infty(X)\xrightarrow{d}\Omega^1_{\mathcal{E}^\infty}(X)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^k_{\mathcal{E}^\infty}(X)\xrightarrow{d}\cdots.$$

The cohomology $H^\bullet_{\mathrm{WdR}}(X)$ of this complex will be called the *Whitney-de Rham cohomology* of X and will be computed for subanalytic X later in this work. Clearly, the spaces $\Omega^k_{\mathcal{E}^\infty}(V)$, where V runs through the open subsets of X, are the sectional spaces of a fine sheaf over X which we denote by $\Omega^k_{\mathcal{E}^\infty_X}$. We thus obtain a sheaf complex and, taking global sections, again the Whitney-de Rham complex.

4.8. Theorem. *Let $X\subset\mathbb{R}^n$ be a regular subset with regularly situated diagonals, and $m\in\mathbb{N}\cup\{\infty\}$. Assume that $\mathcal{M}$ is a finitely generated projective $\mathcal{E}^\infty_X$-module sheaf of symmetric Fréchet modules and denote by M the $\mathcal{E}^\infty(X)$-module $\mathcal{M}(X)$. Then the Hochschild homology of $\mathcal{E}^\infty(X)$ with values in M coincides with the local Hochschild homology $H_\bullet(E_\bullet,M)$ and is given by*

$$H_\bullet(\mathcal{E}^\infty(X),M)=\Omega^\bullet_{\mathcal{E}^\infty}(X)\hat\otimes_{\mathcal{E}^\infty(X)}M\cong M\otimes\Lambda^\bullet(T^*_0\mathbb{R}^n). \tag{4.13}$$

4.9. *Remark.* Since a subanalytic set $X \subset \mathbb{R}^n$ is always regular and possesses regularly situated diagonals (the diagonal is obviously subanalytic and two subanalytic sets are always regularly situated), the statement of the theorem holds in particular for subanalytic sets.

Proof. Since the sheaf $\mathcal{M}$ is finitely generated projective we can reduce the claim to the case $\mathcal{M} = \mathcal{E}^\infty_X$. We will present two ways to prove the result in this case; both of them show that

$$HH_k(\mathcal{E}^\infty(X)) \cong \mathcal{E}^\infty(X) \otimes_{\mathcal{C}^\infty(U)} \Omega^k(U).$$

The first proof follows Teleman's procedure in [49] (see also [7]). The homology of the diagonal complex $E_\bullet$ coincides with the homology of the nondegenerated complex $E^T_\bullet$, i.e. the complex generated by nondegenerated monomials (non lacunary in the terminology of [49]). The nondegenerated complex $E^T_\bullet$ is itself identified with the direct product of its components $E^r_\bullet$ where $E^r_\bullet$ is the subcomplex of $E^T_\bullet$ generated by all monomials of (total) degree r. Proposition 4.6 shows that the elements of $E^T_\bullet$ can be interpreted as infinite jets vanishing at the origin, regarding the variables $y_1, \ldots, y_k$, and with coefficients in $\mathcal{E}^\infty(X)$. An argument similar to Teleman's spectral sequence computation [49], but here with coefficients in $\mathcal{E}^\infty(X)$, proves that the homology of $E^r_\bullet$ is $\mathcal{E}^\infty(X) \otimes \Lambda^r(T^*_0(\mathbb{R}^n))$ and we have the desired result.

The second way to prove the result is to consider the isomorphism j^∞_Δ of Proposition 4.6 and carry the boundary map b_k from E_k to $\mathcal{E}^\infty(X) \otimes_\pi \mathcal{F}^\infty_{kn}$ such that $b_k(j^\infty_\Delta F) = j^\infty_\Delta(b_k F)$ for all $F \in \mathcal{E}^\infty(\Delta_{k+1}(X))$. Writing an element $\sigma \in \mathcal{E}^\infty(X) \otimes_\pi \mathcal{F}^\infty_{kn}$ as a section $\sigma(x_0, y_1, \ldots, y_k)$ of the module sheaf $\mathcal{E}^\infty_X \otimes \mathcal{F}^\infty_{kn}$ one now computes

$$\begin{aligned} b_k\sigma(x_0, y_1, \ldots, y_{k-1}) = {} & \sigma(x_0, 0, y_1, \ldots, y_{k-1}) \\ & + \sum_{i=1}^{k-1} (-1)^i \sigma(x_0, y_1, \ldots, y_i, y_i, \ldots, y_{k-1}) \\ & + (-1)^k \sigma(x_0, y_1, \ldots, y_{k-1}, 0). \end{aligned}$$

This shows that the homology of the complex $(\mathcal{E}^\infty_X \otimes \mathcal{F}^\infty_{kn}, b)$ is nothing else but the Hochschild homology $H_\bullet(\mathcal{F}^\infty_n, \mathcal{E}^\infty(X))$, where $\mathcal{E}^\infty(X)$ is given the $\mathcal{F}^\infty_n$-module structure such that $y_{\underline{i}} F = 0$ for each of the indeterminates $y_{\underline{1}}, \ldots, y_{\underline{n}}$ and for every $F \in \mathcal{E}^\infty(X)$. Now, since Hochschild homology can be interpreted as a derived functor homology (see [32, Prop. 1.1.13] in the algebraic and [44, §6.3] in the topological case), we can use the Koszul resolution for the computation of $H_\bullet(\mathcal{F}^\infty_n, \mathcal{E}^\infty(X))$; this yields the following topologically projective resolution:

$$K_\bullet : 0 \longleftarrow \mathcal{F}^\infty_n \longleftarrow \mathcal{F}^\infty_n \otimes \Lambda^1(T^*_0(\mathbb{R}^n)) \xleftarrow{i_Y} \cdots \xleftarrow{i_Y} \mathcal{F}^\infty_n \otimes \Lambda^k(T^*_0(\mathbb{R}^n)) \xleftarrow{i_Y} \cdots,$$

where i_Y denotes the insertion of the radial (formal) vector field $Y = y_{\underline{1}}\partial_{y_{\underline{1}}} + \ldots + y_{\underline{n}}\partial_{y_{\underline{n}}}$ in an alternating form. Hence

$$H_\bullet(\mathcal{F}_n^\infty, \mathcal{E}^\infty(X)) = H_\bullet(K_\bullet \otimes_{\mathcal{F}_n^\infty} \mathcal{E}^\infty(X)) = \mathcal{E}^\infty(X) \otimes \Lambda^\bullet(T_0^*(\mathbb{R}^n)).$$

The result then is a Hochschild-Kostant-Rosenberg type theorem for Whitney functions. □

In the spirit of the last part of the preceding proof we finally show in this section that there exists a Koszul resolution for Whitney functions in case the set $X \subset \mathbb{R}^n$ has the *extension property* which means that for an open subset $U \subset \mathbb{R}^n$ in which X is closed there exists a continuous linear splitting $\mathcal{E}^\infty(X) \to \mathcal{C}^\infty(U)$ of the canonical map $\mathcal{C}^\infty(U) \to \mathcal{E}^\infty(X)$ (cf. [2], where it is in particular shown that a subanalytic subset $X \subset \mathbb{R}^n$ has the extension property if and only if it has a dense interior).

4.10. Proposition. *Let $X \subset \mathbb{R}^n$ be a locally closed and regular subset. Then the complex of topological $\mathcal{E}^\infty(X)$-bimodules*

$$0 \longleftarrow \mathcal{E}^\infty(X) \longleftarrow \mathcal{E}^\infty(X \times X) \xleftarrow{i_Y} \cdots \xleftarrow{i_Y} \mathcal{E}^\infty(X \times X) \otimes \Lambda^k(T_0^*(\mathbb{R}^n)) \xleftarrow{i_Y} \cdots,$$

where i_Y denotes the insertion of the radial vector field

$$Y(x,y) = (x-y)_{\underline{1}}\partial_{y_{\underline{1}}} + \ldots + (x-y)_{\underline{n}}\partial_{y_{\underline{n}}},$$

is exact, hence gives rise to a resolution $R_\bullet(\mathcal{E}^\infty(X))$ of $\mathcal{E}^\infty(X)$ by topologically projective $\mathcal{E}^\infty(X \times X)$-modules $R_k(\mathcal{E}^\infty(X)) = \mathcal{E}^\infty(X \times X) \otimes \Lambda^k(T_0^*(\mathbb{R}^n))$. In case $X \subset \mathbb{R}^n$ satisfies the extension property, then the above exact sequence even has a contracting homotopy by continuous linear maps which in other words means that in this case $R_\bullet(\mathcal{E}^\infty(X))$ is a topological projective resolution of $\mathcal{E}^\infty(X)$.

Proof. Let $U \subset \mathbb{R}^n$ be an open subset such that $X \subset U$ is relatively closed. By [10] one knows that

$$0 \longleftarrow \mathcal{C}^\infty(U) \longleftarrow \mathcal{C}^\infty(U \times U) \xleftarrow{i_Y} \cdots \xleftarrow{i_Y} \mathcal{C}^\infty(U \times U) \otimes \Lambda^k(T_0^*(\mathbb{R}^n)) \xleftarrow{i_Y} \cdots$$

is a topological projective resolution of $\mathcal{C}^\infty(U)$ as $\mathcal{C}^\infty(U)\hat{\otimes}\mathcal{C}^\infty(U)$-module. Since $\mathcal{E}^\infty(X) = \mathcal{C}^\infty(U)/\mathcal{J}^\infty(X;U)$ and

$$\mathcal{E}^\infty(X \times X) = \mathcal{C}^\infty(U \times U)/\mathcal{J}^\infty(X \times X; U \times U),$$

the complex $R_\bullet(\mathcal{E}^\infty(X))$ has to be acyclic, if one can show exactness for the complex

$$\begin{aligned} 0 \longleftarrow \mathcal{J}^\infty(X;U) \longleftarrow & \mathcal{J}^\infty(X \times X; U \times U) \\ & \xleftarrow{i_Y} \cdots \xleftarrow{i_Y} \mathcal{J}^\infty(X \times X; U \times U) \otimes \Lambda^k(T_0^*(\mathbb{R}^n)) \xleftarrow{i_Y} \cdots. \end{aligned}$$

We first prove that $\mathcal{J}^\infty(X;U) \longleftarrow \mathcal{J}^\infty(X \times X; U \times U)$ is surjective. Let $f \in \mathcal{J}^\infty(X;U)$. Since $\mathcal{J}^\infty(X;U)^2 = \mathcal{J}^\infty(X;U)$ there exist $f_1, f_2 \in \mathcal{J}^\infty(X;U)$

with $f = f_1 f_2$. Put $F(x,y) = f_1(x) f_2(y)$ for $x, y \in U$. Then one has $F \in \mathcal{J}^\infty(X \times X; U \times U)$, and f is the image of F under the map $\mathcal{J}^\infty(X \times X; U \times U) \to \mathcal{J}^\infty(X; U)$. Next, we show for $k > 0$ exactness at $\mathcal{J}^\infty(X \times X; U \times U) \otimes \Lambda^k(T_0^*(\mathbb{R}^n))$. Assume that

$$F = \sum_{1 \leq i_1 < \cdots < i_k < n} F_{i_1, \cdots, i_k} dx_{i_1} \wedge \cdots \wedge x_{i_k} \in \mathcal{J}^\infty(X \times X; U \times U) \otimes \Lambda^k(T_0^*(\mathbb{R}^n))$$

with $i_Y(F) = 0$. By [50, V. Lem. 2.4] there exist

$$G, \widetilde{F}_{i_1, \cdots, i_k} \in \mathcal{J}^\infty(X \times X; U \times U)$$

such that

$$G(x,y) > 0 \text{ for } (x,y) \notin X \times X$$

and

$$F = G\,\widetilde{F} \text{ for } \widetilde{F} := \sum_{1 \leq i_1 < \cdots < i_k < n} \widetilde{F}_{i_1, \cdots, i_k} dx_{i_1} \wedge \cdots \wedge x_{i_k}.$$

But then one has $i_Y \widetilde{F} = 0$; hence by the exactness of $R_\bullet(\mathcal{C}^\infty(U))$ there now exists a function $H \in \mathcal{J}^\infty(X \times X; U \times U) \otimes \Lambda^{k+1}(T_0^*(\mathbb{R}^n))$ with $i_Y H = \widetilde{F}$. Hence $i_Y(GH) = G\,(i_Y H) = G\widetilde{F} = F$, which shows exactness at $k > 0$. Likewise, one proves exactness at $k = 0$. Since, obviously, each of the spaces $\mathcal{E}^\infty(X \times X) \otimes \Lambda^k(T_0^*(\mathbb{R}^n))$ is topologically projective over $\mathcal{E}^\infty(X \times X)$, the first claim now is proven.

For each $k \in \mathbb{N}$, denote by E_k (resp. F_k) the image of the map $R_k(\mathcal{E}^\infty(X)) \to R_{k-1}(\mathcal{E}^\infty(X))$ (resp. the quotient space $R_{k+1}(\mathcal{E}^\infty(X))/\ker(i_Y)$). Then for $R_\bullet(\mathcal{E}^\infty(X))$ to be a topologically projective resolution of $\mathcal{E}^\infty(X)$ it is necessary and sufficient that for each $k \in \mathbb{N}$ the short exact sequence

$$0 \longrightarrow F_k \longrightarrow \mathcal{E}^\infty(X \times X) \otimes \Lambda^k T^* \mathbb{R}^n \longrightarrow E_k \longrightarrow 0, \tag{4.14}$$

of (nuclear) Fréchet spaces splits topologically (cf. [47, §1]). We prove that this sequence splits in case $X \subset \mathbb{R}^n$ has the extension property. For simplicity, we also assume that X is compact, since by an appropriate localization argument as above one can reduce the claim to the compact case. Hereby, we will use a splitting theorem for short exact sequences of nuclear Fréchet-spaces by Vogt (cf. [39, §30]). More precisely, we will show that under the assumptions made, F_k has property (Ω) and E_k has property (DN), which will imply the claim (see again [39, §30] for the necessary functional analytic notation). Since property (Ω) passes to (complete) quotient spaces by [39, Lem. 29.11], and since $\mathcal{C}^\infty(U \times U)$ has property (Ω) (see [39, Cor. 31.13]), one concludes that F_k has property (Ω). Since X is compact and has the extension property, there exists a continuous splitting $\mathcal{E}^\infty(X) \to \mathcal{S}$ of the canonical restriction map $\mathcal{S} \to \mathcal{E}^\infty(X)$, where $\mathcal{S}$ denotes the space of rapidly decreasing smooth functions on $\mathbb{R}^n$. Since $\mathcal{S}$ has property (DN) ([39, Thm. 31.5]), and property (DN) passes to closed subspaces ([39, Lem. 29.2]), E_k satisfies property (DN), too. Hence (4.14) splits topologically. This finishes the proof. □

5. Hochschild cohomology of Whitney functions

5.1. After having determined the Hochschild homology of $\mathcal{E}^\infty(X)$ we now consider its Hochschild cohomology. In particular we want to compute the cohomology of the Hochschild cochain complexes $C^\bullet(\mathcal{E}^\infty(X),(\mathcal{E}^\infty(X))')$ and $C^\bullet(\mathcal{E}^\infty(X),\mathcal{E}^\infty(X))$, where $(\mathcal{E}^\infty(X))'$ denotes the strong dual of $\mathcal{E}^\infty(X)$. Note that $(\mathcal{E}^\infty(X))'$ is nuclear by [51, Prop. 50.6].

By [47, Prop. 4.5] and Prop. 4.10, the cohomology $H^\bullet(\mathcal{E}^\infty(X))$ of the cochain complex $C^\bullet(\mathcal{E}^\infty(X),(\mathcal{E}^\infty(X))')$ can be computed as the cohomology of the cochain complex $\mathrm{Hom}_{\mathcal{E}^\infty(X\times X)}(R_\bullet(\mathcal{E}^\infty(X),(\mathcal{E}^\infty(X))')$. One then obtains immediately:

5.2. Theorem. *For every regular $X\subset\mathbb{R}^n$ the Hochschild cohomology $H^k(\mathcal{E}^\infty(X))$ coincides with $(\mathcal{E}^\infty(X)\otimes\Lambda^kT_0^*\mathbb{R}^n)'$, the space of de Rham k-currents on $\mathbb{R}^n$ with support X.*

The computation of the cohomology $H^\bullet(\mathcal{E}^\infty(X),\mathcal{E}^\infty(X))$ is harder. In the following we briefly denote the cochain complex $C^\bullet(\mathcal{E}^\infty(X),\mathcal{E}^\infty(X))$ by $C^\bullet$. As in the previous section we assume from now on that $X\subset\mathbb{R}^n$ is a regular locally closed subset and that X has regularly situated diagonals. We then apply the functor $\mathrm{Hom}_{\mathcal{E}^\infty(X)}(-,\mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)}M)$ to (4.11) and obtain the following sequence

$$\begin{aligned}(5.1)\qquad 0&\longrightarrow \mathrm{Hom}_{\mathcal{E}^\infty(X)}(E_\bullet,\mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)}M)\\ &\longrightarrow \mathrm{Hom}_{\mathcal{E}^\infty(X)}(C_\bullet,\mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)}M)\\ &\longrightarrow \mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet,\mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)}M)\longrightarrow 0.\end{aligned}$$

Since the Hom-functor is left exact, this sequence is exact at the first two (nontrivial) spots. For $m=\infty$ it is not necessarily exact at the third spot, but we have the following.

5.3. Proposition. *For $m\in\mathbb{N}\cup\{\infty\}$ denote by $Q_m^\bullet$ the quotient complex making the following sequence exact*:

$$\begin{aligned}(5.2)\qquad 0&\longrightarrow \mathrm{Hom}_{\mathcal{E}^\infty(X)}(E_\bullet,\mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)}M)\\ &\longrightarrow \mathrm{Hom}_{\mathcal{E}^\infty(X)}(C_\bullet,\mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)}M)\longrightarrow Q_m^\bullet\longrightarrow 0.\end{aligned}$$

For finite m, $Q_m^\bullet$ then is exact and coincides with

$$\mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet,\mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)}M).$$

In case X has the extension property, $Q_\infty^\bullet$ is exact as well.

Proof. Note first that one can reduce the claim to compact X by a localization argument involving an appropriate partition of unity. Moreover, we can reduce the claim to the case where $M=\mathcal{E}^\infty(X)$, since M is finitely generated projective. So we assume without loss of generality that X is compact and that $M = \mathcal{E}^\infty(X)$.

We now consider the case $m \in \mathbb{N}$. Under this assumption it suffices by Proposition 4.2 to show that the sequence (5.1) is exact at the third spot. So we have to check that for every k the natural map

$$\mathrm{Hom}_{\mathcal{E}^\infty(X)}(C_k, \mathcal{E}^m(X)) \to \mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^m(X)) \tag{5.3}$$

is surjective. Choose a compact cube Q such that X lies in the interior of Q. Then $\mathcal{J}^\infty(\Delta_{k+1}(X); Q^{k+1})$ is a Fréchet space, the topology of which is defined by the norms $|\cdot|_r^Q$, $r \in \mathbb{N}$. Moreover, the Fréchet topology of J_k is the quotient topology with respect to the canonical projection $\mathcal{J}^\infty(\Delta_{k+1}(X); Q^{k+1}) \to J_k$. Hence, given $f \in \mathrm{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^m(X))$ there exists by Proposition 3.2 a natural number $r \geq m$ such that f extends to a continuous $\mathcal{E}^r(X)$-linear map $f_r : \mathcal{J}^r(\Delta_{k+1}(X); X^{k+1}) \to \mathcal{E}^r(X^{k+1})$.

Using the notion introduced in Proposition 4.6 let us now define a map $j_\Delta^r : \mathcal{E}^\infty(\Delta_{k+1}(X)) \to \mathcal{E}^\infty(X^{k+1})$ by

$$j_\Delta^r(F)_{|(x_0,x_1,\ldots,x_k)} = \sum_{|\alpha_1|+\ldots+|\alpha_k| \leq r} \frac{F_{\alpha_1,\ldots,\alpha_k|x_0}}{\alpha_1! \cdot \ldots \cdot \alpha_k!} (x_1 - x_0)^{\alpha_1} \cdot \ldots \cdot (x_k - x_0)^{\alpha_k}.$$

Like j_Δ^∞ the map j_Δ^r is continuous, linear and a morphism of $\mathcal{E}^\infty(X)$-modules. Moreover, using Taylor's formula, one checks easily that

$$F - j_\Delta^r(\Delta_{k+1}^*(F)) \in \mathcal{J}^r(\Delta_{k+1}(X); X^{k+1}) \quad \text{for all } F \in \mathcal{E}^\infty(X^{k+1}). \tag{5.4}$$

Since $j_\Delta^r(\Delta_{k+1}^*(G)) = 0$ for $G \in \mathcal{J}^r(\Delta_{k+1}(X); X^{k+1})$, the map $\widetilde{f} : \mathcal{E}^\infty(X^{k+1}) \to \mathcal{E}^m(X)$, defined by $\widetilde{f}(F) = f_r(F - j_\Delta^r(\Delta_{k+1}^*(F)))$, lies in $\mathrm{Hom}_{\mathcal{E}^\infty(X)}(C_k, \mathcal{E}^m(X))$ and satisfies $\widetilde{f}(G) = f(G)$ for all $G \in \mathcal{J}^r(\Delta_{k+1}(X); X^{k+1})$. This proves the claim for $m \in \mathbb{N}$.

The proof of the remaining claim will be postponed until the end of this section; the reader will notice that no circular argument will result. □

Propositions 5.3 and 4.2 now result in the following.

5.4. Corollary. *If m is finite or X has the extension property, the Hochschild cohomology $H^\bullet(\mathcal{E}^\infty(X), \mathcal{E}^m(X) \hat{\otimes}_{\mathcal{E}^\infty(X)} M)$ is naturally isomorphic to the corresponding local Hochschild cohomology, i.e. the cohomology of the cochain complex* $\mathrm{Hom}_{\mathcal{E}^\infty(X)}(E_\bullet; \mathcal{E}^m(X) \hat{\otimes}_{\mathcal{E}^\infty(X)} M)$.

5.5. Before we come to the computation of the cohomology of

$$\mathrm{Hom}_{\mathcal{E}^\infty(X)}(E_\bullet; \mathcal{E}^m(X) \hat{\otimes}_{\mathcal{E}^\infty(X)} M)$$

we will introduce two operations on the Hochschild cochain complex, namely the cup product and the Gerstenhaber bracket. The latter was originally defined in [16] and has been used in the deformation theory of algebras [13], [17],

[29]. For two cochains $f_1 \in C^{k_1}$ and $f_2 \in C^{k_2}$ one defines $f_1 \circ f_2 \in C^{k_1+k_2-1}$ by $f_1 \circ f_2 = 0$, if $k_1 = 0$, and otherwise by

$$\begin{aligned} & f_1 \circ f_2(F_0, \ldots, F_{k_1+k_2-1}) \\ = & \sum_{j=1}^{k_1} (-1)^{(j-1)(k_2-1)} f_1(F_0, \ldots, F_{j-1}, f_2(1, F_j, \ldots, F_{j+k_2-1}), F_{k_2+j}, \ldots, F_{k_1+k_2-1}), \end{aligned}$$

where $F_0, \ldots, F_{k_1+k_2-1} \in \mathcal{E}^\infty(X)$. The *Gerstenhaber bracket* of f_1 and f_2 then is defined by

$$[f_1, f_2] = f_1 \circ f_2 - (-1)^{(k_1-1)(k_2-1)} f_2 \circ f_1.$$

Moreover, the *cup product* of f_1 and f_2 is given by

$$f_1 \smile f_2(F_0, \ldots, F_{k_1+k_2}) = f_1(F_0, \ldots, F_{k_1})\, f_2(1, F_{k_1+1}, \ldots, F_{k_1+k_2}).$$

It is well-known that the complex $C^{\bullet-1}$ together with the Gerstenhaber bracket becomes a graded Lie algebra and that $C^\bullet$ is a graded algebra with respect to the cup product. Note that the cup product $f_1 \smile f_2$ is also well-defined for $f_1, f_2 \in C^\bullet(\mathcal{E}^\infty(X), \mathcal{E}^m(X))$ and that $f \circ G$ even makes sense, if $f \in C^\bullet(\mathcal{E}^\infty(X), \mathcal{E}^m(X))$ and $G \in C^0(\mathcal{E}^\infty(X), \mathcal{E}^m(X)) = \mathcal{E}^\infty(X)$.

Next recall that the inclusion of the normalized cochain complex $\overline{C}^\bullet \hookrightarrow C^\bullet$ is a quasi-isomorphism. Thereby, $\overline{C}^k$ consists of all *normalized* cochains, that means of all $f \in C^k$ such that $f(F_0, \ldots, F_k) = 0$, whenever one of the Whitney functions F_i, $i > 0$ is constant. Likewise, the inclusion of the normalized cochain complex $\overline{\mathrm{Hom}}_{\mathcal{E}^\infty(X)}(E_\bullet; \mathcal{E}^m(X) \hat{\otimes}_{\mathcal{E}^\infty(X)} M)$ is a quasi isomorphism.

5.6. Let us proceed to the computation of the cohomology of the cochain complex $E^{m,\bullet} := \mathrm{Hom}_{\mathcal{E}^\infty(X)}(E_\bullet; \mathcal{E}^m(X))$ respectively of its normalization $\overline{E}^{m,\bullet}$. We denote elements of $E^{m,\bullet}$ as $D, D_1, \ldots$, since every $D \in E^{m,k}$ can be regarded as a local and continuous k-linear operator $\mathcal{E}^\infty(X) \times \ldots \times \mathcal{E}^\infty(X) \to \mathcal{E}^m(X)$ and, at least for finite m, such a D is locally given by a differential cochain according to Peetre's Theorem 3.3 for Whitney functions. Recall that by a *differential cochain* of *degree* k and *order* $\leq d \in \mathbb{N}$ (and *class* $\mathcal{C}^m$) one understands an element $D \in E^{m,k}$ such that

$$D(F_0, \ldots, F_k) = \sum_{\substack{\alpha_1, \ldots, \alpha_k \in \mathbb{N}^n \\ |\alpha_1| + \ldots + |\alpha_k| \leq d}} d_{\alpha_1, \ldots, \alpha_k}\, F_0\, (D^{\alpha_1} F_1) \ldots (D^{\alpha_k} F_k),$$

where the coefficients $d_{\alpha_1, \ldots, \alpha_k}$ are elements of $\mathcal{E}^m(X)$. A differential cochain is called *homogeneous* of *order* d, if it is a linear combination of *monomial cochains* of *order* d, i.e. of cochains of the form $d_{\alpha_1, \ldots, \alpha_k} D^{\alpha_1} \smile \ldots \smile D^{\alpha_k}$ with $d = |\alpha_1| + \ldots + |\alpha_k|$.

In a first step we will now determine $H^\bullet(\overline{E}^{m,\bullet}_{\mathrm{diff}})$ and then show in a second step that the cohomology of $\overline{E}^{m,\bullet}_{\mathrm{diff}}$ coincides with $H^\bullet(\overline{E}^{m,\bullet})$. For m finite the second step follows trivially from the localization results of Section 2, but for $m = \infty$ we need some more arguments to prove that.

Let us denote by $\mathcal{X}^\infty$ the sheaf of smooth vector fields on $\mathbb{R}^n$ and let $V_1, \dots, V_k$ be elements of $\mathcal{X}_{\mathcal{E}^m}(X) := \mathcal{E}^m(X) \otimes_{\mathcal{C}^\infty(U)} \mathcal{X}^\infty(U)$. Such elements will be called *Whitney vector fields* of *class* $\mathcal{C}^m$ on X. Clearly, a Whitney vector field V (of class $\mathcal{C}^m$) on X defines, for every $F \in \mathcal{E}^\infty(X)$, a Whitney function $VF \in \mathcal{E}^m(X)$ by $VF = \sum_{j=1}^n v_j D_{\underline{j}} F$, where the v_j are the coefficient Whitney functions of V with respect to the standard basis of $\mathbb{R}^n$. Hence the skew symmetric product $V_1 \wedge \ldots \wedge V_k$, which we regard as an element of $\Lambda^k \mathcal{X}_{\mathcal{E}^m}(X) := \mathcal{E}^m(X) \otimes_{\mathcal{C}^\infty(U)} \Lambda^k \mathcal{X}^\infty(U)$, defines a Hochschild cocycle with values in $\mathcal{E}^m(X)$ by

$$(F_0, \dots, F_k) \mapsto \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) F_0 \, (V_{\sigma(1)} F_1) \cdot \ldots \cdot (V_{\sigma(k)} F_k).$$

In the following we will show that the inclusion $\Lambda^\bullet \mathcal{X}_{\mathcal{E}^m}(X) \hookrightarrow \overline{E}^{m,\bullet}_{\mathrm{diff}}$ is a quasi isomorphism by constructing an appropriate homotopy. As the essential tool we will use the homotopy operator introduced by deWilde-Lecomte in [14]. The principal idea there is to decrease the order of a differential Hochschild cocycle while staying in the same cohomology class until one arrives at a skew symmetric differential Hochschild cocycle of order 1 in each nontrivial argument, or in other words, at a linear combination of skew symmetric products of Whitney vector fields. Note that by a nontrivial argument of a cochain $(F_0, F_1, \dots, F_k) \mapsto D(F_0, F_1, \dots, F_k)$ we understand one of the arguments $F_1, \dots, F_k$, since D is $\mathcal{E}^\infty(X)$-linear in F_0.

Following deWilde-Lecomte [14] we first define two maps on $\overline{E}^{m,k}_{\mathrm{diff}}$, where $k \geq 1$. Put

$$Q^k D(F_0, \dots, F_{k-1}) = \sum_{l=1}^n \sum_{0<i<j<k} (-1)^i D(F_0, \dots, F_{i-1}, x_{\underline{l}}, \dots, D_{\underline{l}} F_j, \dots, F_{k-1})$$

and

$$P^k D = \sum_{l=1}^n [x_{\underline{l}}, D] \smile D_{\underline{l}} = (-1)^k \sum_{l=1}^n (D \circ x_{\underline{l}}) \smile D_{\underline{l}}.$$

The proof of Proposition 4.1 in [14] can now be literally transfered to the case of Whitney functions, so we obtain

5.7. Proposition. *Assume that $D \in \overline{E}^{m,k}_{\mathrm{diff}}$ with $k > 0$ is a differential cochain homogeneous of order d. Then*

$$(Q^{k+1} b^k + b^{k-1} Q^k) D = -(d + P^k) D.$$

Next let us define for every $l \in \mathbb{N}$ a homogeneous map $P^\bullet_l : \overline{E}^{m,\bullet}_{\mathrm{diff}} \to \overline{E}^{m,\bullet}_{\mathrm{diff}}$ of degree 0 as follows:

$$P_l^k D = \begin{cases} D, & \text{if } l = 0, \\ \sum\limits_{j_1,\ldots,j_l=1}^{n} (\operatorname{ad} x_{\underline{j_1}} \cdots \operatorname{ad} x_{\underline{j_l}}(D)) \smile D_{\underline{j_1}} \smile \ldots \smile D_{\underline{j_l}}, & \text{if } 1 \leq l \leq k, \\ 0, & \text{if } l > k. \end{cases}$$

Hereby, $\operatorname{ad} G$ is the adjoint action $E^{m,k} \ni D \mapsto [G, D] = (-1)^k D \circ G \in E^{m,k-1}$ of some element $G \in \mathcal{E}^\infty(X)$. Since we have

$$\operatorname{ad} G_1 \operatorname{ad} G_2 = -\operatorname{ad} G_2 \operatorname{ad} G_1 \quad \text{for all } G_1, G_2 \in \mathcal{E}^\infty(X),$$

the cochain $P_k^k D$ is skew symmetric in the nontrivial arguments, hence a linear combination of skew symmetric products of Whitney vector fields.

5.8. Proposition. *The operators P_l^k satisfy the recursive relations $P_{l+1}^k = P^k P_l^k + l P_l^k$. Moreover, $\frac{(-1)^k}{k!} P_k^k : \overline{E}_{\mathrm{diff}}^{m,k} \to \overline{E}_{\mathrm{diff}}^{m,k}$ is a projection onto the space of normalized differential cochains which are homogeneous of order k and skew symmetric in the nontrivial arguments. P_k^k vanishes on every monomial cochain which is of order > 1 in some argument or which is symmetric with respect to at least two of its nontrivial arguments. Finally, $P_l^\bullet$ is a chain map.*

Proof. Repeating the proof of [14, Prop. 4.2] immediately gives the claim. □

Using the maps P^k and Q^k, deWilde-Lecomte define iteratively operators $Q_l^k : \overline{E}^{m,k} \to \overline{E}^{m,k-1}$, $0 \leq l \leq k$, by

$$Q_0^k D = D, \quad Q_1^k D = -\frac{1}{d} Q^k D, \quad Q_{l+1}^k D = -\frac{1}{d-l}((P^{k-1} + d) Q_l^k + Q^k) D,$$

where $D \in \overline{E}^{m,k}$ is homogeneous of order d. Note that $d \geq k$, since D is normalized.

The operators Q_k^k will turn out to comprise a homotopy between the identity and the antisymmetrization. Let us show this by induction as in [14, §4]. By Proposition 5.7 the formula

$$D - \lambda_{k,l,d} P_l^k D = (b^{k-1} Q_l^k + Q_l^{k+1} b^k) D \tag{5.5}$$

holds true for $l = 1$ and $\lambda_{k,1,d} = -\frac{1}{d}$. Assume that it is true for some l with $1 \leq l < k$ and apply P^k to both sides. By Proposition 5.8 and the definition of the Q_j^k one concludes that it holds for $l+1$ with $\lambda_{k,l+1,d} = -\frac{1}{d-l} \lambda_{k,l}$. Hence the formula is true for $l = k$ and $\lambda_{k,k,d} = (-1)^k \frac{(d-k)!}{d!}$. Note that $P_k^k D = 0$, if $d > k$, and that $\lambda_{k,k,k} = (-1)^k \frac{1}{k!}$, so we finally obtain

5.9. Proposition. *Let $A^\bullet : \overline{E}_{\mathrm{diff}}^{m,\bullet} \to \Lambda^\bullet \mathcal{X}_{\mathcal{E}^m}(X)$ with $A^k = \frac{(-1)^k}{k!} P_k^k$ be the skew symmetrization operator. Then*

$$(b^{k-1} Q_k^k - Q_k^{k+1} b^k) = D - A^k D \quad \text{for all } D \in \overline{E}_{\mathrm{diff}}^{m,k}. \tag{5.6}$$

Thus, the inclusion $\Lambda^\bullet \mathcal{X}_{\mathcal{E}^m}(X) \hookrightarrow \overline{E}_{\mathrm{diff}}^{m,\bullet}$ is a quasi isomorphism.

Since the subcomplex $\Lambda^\bullet \mathcal{X}_{\mathcal{E}^m}(X)$ has coboundary 0, the proposition gives the cohomology of the complex $\overline{E}^{m,\bullet}_{\mathrm{diff}}$. Let us show that it coincides with the cohomology of $\overline{E}^{m,\bullet}$. By Peetre's Theorem 3.3 for Whitney functions one concludes that every element $D \in E^{m,k}$ has a representation of the form

$$D = \sum_{\alpha_1,\ldots,\alpha_k \in \mathbb{N}^n} d_{\alpha_1,\ldots,\alpha_k}\, D^{\alpha_1} \smile \ldots \smile D^{\alpha_k},$$

where the $d_{\alpha_1,\ldots,\alpha_k}$ are uniquely determined elements of $\mathcal{E}^m(X)$ and where the differential operators D_j, $j \in \mathbb{N}$, with

$$D_j = \sum_{\substack{\alpha_1,\ldots,\alpha_k \in \mathbb{N}^n,\\ |\alpha_1|+\ldots+|\alpha_k| \leq j}} d_{\alpha_1,\ldots,\alpha_k}\, D^{\alpha_1} \smile \ldots \smile D^{\alpha_k}$$

converge to the operator D in such a way that for every natural $r \leq m$ and every compact $K \subset X$ there exists a number $j_{r,K}$ such that $\|D_i F - D_j F\|^K_r = 0$ for all $i, j \geq j_{r,K}$. Thus, the sequence of differential operators D_j converges uniformly on its domain to D. If now $D \in \overline{E}^{m,k}$, the construction of the operators P^k_l and Q^k_l shows that the operator sequences $(P^k_l D_j)_{j\in\mathbb{N}}$ and $(Q^k_l D_j)_{j\in\mathbb{N}}$ converge uniformly to $P^k_l D$, respectively to an operator $Q^k_l D \in \overline{E}^{m,k-1}$. But this entails that Equation (5.6) holds for all $D \in \overline{E}^{m,k}$, so that the inclusion $\Lambda^\bullet \mathcal{X}_{\mathcal{E}^m}(X) \hookrightarrow \overline{E}^{m,\bullet}$ is a quasi isomorphism as well. This proves the first part of the following main result of this section.

5.10. Theorem. *Let $X \subset \mathbb{R}^n$ be a regular subset with regularly situated diagonals, and $m \in \mathbb{N} \cup \{\infty\}$. Assume that $\mathcal{M}$ is a finitely generated projective $\mathcal{E}^\infty_X$-module sheaf of symmetric Fréchet modules and that M is the $\mathcal{E}^\infty(X)$-module $\mathcal{M}(X)$. Then the local Hochschild cohomology of $\mathcal{E}^\infty(X)$ with values in $\mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)} M$ is given by*

(5.7)
$$H^\bullet(E_\bullet, \mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)} M) = \Lambda^\bullet \mathcal{X}_{\mathcal{E}^m}(X)\hat{\otimes}_{\mathcal{E}^\infty(X)} M \cong \mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)} M \otimes \Lambda^\bullet \mathbb{R}^n.$$

If m is finite or X has the extension property, then the local Hochschild cohomology $H^\bullet(E_\bullet, \mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)} M)$ coincides naturally with the Hochschild cohomology $H^\bullet(\mathcal{E}^\infty(X), \mathcal{E}^m(X)\hat{\otimes}_{\mathcal{E}^\infty(X)} M)$.

Proof. The second claim follows immediately from Corollary 5.4, if m is finite. Assume now that X has the extension property, and assume for simplicity that $M = \mathcal{E}^\infty(X)$. Using the resolution from Proposition 4.10 one infers then that $H^\bullet(\mathcal{E}^\infty(X), \mathcal{E}^\infty(X))$ is naturally isomorphic to $\mathcal{E}^\infty(X) \otimes \Lambda^\bullet \mathbb{R}^n$, hence coincides naturally with $H^\bullet(E_\bullet, \mathcal{E}^\infty(X))$. This also implies that the cochain complex Q_∞ in Proposition 5.3 has to be exact, if X has the extension property. □

6. Cyclic homology of Whitney functions

6.1. Following the presentation by Loday [32, Chap. 2] let us recall the classical operators defining cyclic homology: the usual cyclic group action on the module $(\mathcal{E}^\infty(X))^{\hat{\otimes} k+1}$ is denoted by t, the classical norm operator by $N = 1 + t + \cdots + t^k$ and the extra degeneracy operator by s. More precisely:

$$\begin{aligned} &t(F_0 \otimes F_1 \otimes \ldots \otimes F_k) = (-1)^k F_k \otimes F_0 \otimes \ldots \otimes F_{k-1} \quad \text{and} \\ &s(F_0 \otimes F_1 \otimes \ldots \otimes F_k) = 1 \otimes F_0 \otimes F_1 \otimes \ldots \otimes F_k \quad \text{for all } F_0, \ldots, F_k \in \mathcal{E}^\infty(X). \end{aligned}$$

Moreover, there is a canonical map

$$\pi_k : C_k = (\mathcal{E}^\infty(X))^{\hat{\otimes} k+1} \to \Omega^k_{\mathcal{E}^\infty}(X), \quad F_0 \otimes F_1 \otimes \ldots \otimes F_k \mapsto F_0 \, dF_1 \wedge \ldots \wedge F_k,$$

which, as a consequence of Theorem 4.8 and under the assumptions made there, induces an isomorphism $HH_k(\mathcal{E}^\infty(X)) \to \Omega^k_{\mathcal{E}^\infty}(X)$, still denoted by π_k.

On the one hand, the Connes boundary map $B = (1-t)sN : C_k \to C_{k+1}$ induces a boundary map $B : E_k \to E_{k+1}$. This map gives rise to a map $B_* : HH_k(\mathcal{E}^\infty(X)) \to HH_{k+1}(\mathcal{E}^\infty(X))$ and, by [32, Prop. 2.3.4], there is a commutative diagram:

$$\begin{tikzcd} HH_k(\mathcal{E}^\infty(X)) \arrow[r, "B_*"] \arrow[d, "\pi_k"', "\cong"] & HH_{k+1}(\mathcal{E}^\infty(X)) \arrow[d, "\pi_{k+1}"', "\cong"] \\ \Omega^k_{\mathcal{E}^\infty}(X) \arrow[r, "(k+1)d"] & \Omega^{k+1}_{\mathcal{E}^\infty}(X), \end{tikzcd} \tag{6.1}$$

where $d : \Omega^k_{\mathcal{E}^\infty}(X) \to \Omega^{k+1}_{\mathcal{E}^\infty}(X)$ is the differential of the Whitney-de Rham complex. The factor $(k+1)$ appears in the same way as in [32].

On the other hand, we have two mixed complexes (see [32, 2.5.13] for the definition of a mixed complex). The first one, $(C_\bullet, b, B)$, defines the (topological) bicomplex $\mathcal{B}(\mathcal{E}^\infty(X))$ (cf. [32, 2.1.7])

$$\begin{tikzcd} \arrow[d] & \arrow[d] & \arrow[d] \\ (\mathcal{E}^\infty(X))^{\hat{\otimes} 3} \arrow[d, "b"'] & (\mathcal{E}^\infty(X))^{\hat{\otimes} 2} \arrow[l, "B"'] \arrow[d, "b"] & \mathcal{E}^\infty(X) \arrow[l, "B"'] \\ (\mathcal{E}^\infty(X))^{\hat{\otimes} 2} \arrow[d, "b"'] & \mathcal{E}^\infty(X) \arrow[l, "B"'] & \\ \mathcal{E}^\infty(X), & & \end{tikzcd}$$

the second one $(\Omega^\bullet_{\mathcal{E}^\infty}(X), 0, d)$ determines the bicomplex

$$\begin{array}{ccccc}
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
\Omega^2_{\mathcal{E}^\infty}(X) & \xleftarrow{d} & \Omega^1_{\mathcal{E}^\infty}(X) & \xleftarrow{d} & \Omega^0_{\mathcal{E}^\infty}(X) \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\Omega^1_{\mathcal{E}^\infty}(X) & \xleftarrow{d} & \Omega^0_{\mathcal{E}^\infty}(X) & & \\
\downarrow{\scriptstyle 0} & & & & \\
\Omega^0_{\mathcal{E}^\infty}(X). & & & &
\end{array}$$

Note that all spaces involved in these bicomplexes are Fréchet spaces and all maps are continuous. The (topological) cyclic homology of $\mathcal{E}^\infty(X)$ is defined as the homology of the total complex of $\mathcal{B}(\mathcal{E}^\infty(X))$, in signs $HC_\bullet(\mathcal{E}^\infty(X)) := H_\bullet(\mathrm{Tot}_\bullet\, \mathcal{B}(\mathcal{E}^\infty(X)))$. Now, the map $(1/k!)\pi_k$ gives rise to a map of mixed complexes $(C_\bullet, b, B) \to (\Omega^\bullet_{\mathcal{E}^\infty}(X), 0, d)$, hence by [32, Prop. 2.3.7] to a morphism

$$HC_k(\mathcal{E}^\infty(X)) \to \Omega^k_{\mathcal{E}^\infty}(X)/d\Omega^{k-1}_{\mathcal{E}^\infty}(X) \oplus H^{k-2}_{\mathrm{WdR}}(X) \oplus H^{k-4}_{\mathrm{WdR}}(X) \oplus \cdots .$$

Since the maps $\pi_\bullet$ in the diagram (6.1) are isomorphisms, this morphism has to be an isomorphism. In the homological setting we thus obtain an equivalent of Connes' result [10, III.2α] for Whitney functions.

6.2. Theorem. *For every regular subset $X \subset \mathbb{R}^n$ having regularly situated diagonals the cyclic homology $HC_\bullet(\mathcal{E}^\infty(X))$ coincides with*

$$\Omega^\bullet_{\mathcal{E}^\infty}(X)/d\Omega^{\bullet-1}_{\mathcal{E}^\infty}(X) \oplus H^{\bullet-2}_{\mathrm{WdR}}(X) \oplus H^{\bullet-4}_{\mathrm{WdR}}(X) \oplus \cdots .$$

Arguing as in [32, 2.5.13] one obtains as a corollary a Connes' periodicity exact sequence:

$$\begin{aligned}\cdots \longrightarrow HH_k(\mathcal{E}^\infty(X)) \xrightarrow{I} HC_k(\mathcal{E}^\infty(X)) &\xrightarrow{S} HC_{k-2}(\mathcal{E}^\infty(X)) \\ &\xrightarrow{B} HH_{k-1}(\mathcal{E}^\infty(X)) \xrightarrow{I} \cdots .\end{aligned}$$

6.3. We finally determine the periodic cyclic homology of $\mathcal{E}^\infty(X)$. It is given by the homology $H_\bullet(\mathrm{Tot}^\Pi(CC^{\,\mathrm{per}}))$ of the (product) total complex of the periodic bicomplex $\mathcal{B}(\mathcal{E}^\infty(X))^{\mathrm{per}}$ below and will be denoted by $HP_\bullet(\mathcal{E}^\infty(X))$ (cf. [32, 5.1.7]).

$$
\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \arrow[d] \\
\cdots \arrow[d, "b"'] & (\mathcal{E}^\infty(X))^{\hat\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & (\mathcal{E}^\infty(X))^{\hat\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & \mathcal{E}^\infty(X) \arrow[l, "B"'] \\
\cdots \arrow[d, "b"'] & (\mathcal{E}^\infty(X))^{\hat\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & \mathcal{E}^\infty(X) \arrow[l, "B"'] & \\
\cdots \arrow[d, "b"'] & \mathcal{E}^\infty(X) \arrow[l, "B"'] & & \\
{} & & &
\end{tikzcd}
$$

From the exact sequence [32, 5.1.9],

$$0 \longrightarrow \varprojlim_r{}^1 HC_{k+2r+1} \longrightarrow HP_k(\mathcal{E}^\infty(X)) \longrightarrow \varprojlim_r HC_{k+2r} \longrightarrow 0$$

and the fact that the periodicity map $S : HC_k(\mathcal{E}^\infty(X)) \to HC_{k-2}(\mathcal{E}^\infty(X))$ is surjective one can conclude by [32, 5.1.10] that $HP_k(\mathcal{E}^\infty(X)) = \varprojlim_r HC_{k+2r}$. This proves the last result of this section.

6.4. Theorem. *For every regular set $X \subset \mathbb{R}^n$ having regularly situated diagonals the periodic cyclic homology of $\mathcal{E}^\infty(X)$ is given by $HP_0 = H^{\mathrm{ev}}_{\mathrm{WdR}}(X)$ and $HP_1 = H^{\mathrm{odd}}_{\mathrm{WdR}}(X)$, where $H^{\mathrm{ev/odd}}_{\mathrm{WdR}}(X)$ denotes the Whitney-de Rham cohomology in even, resp. odd, degree.*

6.5. *Remark.* The reader might ask whether it is possible to use the excision result in periodic cyclic homology due to Cuntz-Quillen [12] for the computation, since according to Whitney's extension theorem there is an exact sequence

$$0 \longrightarrow \mathcal{J}^\infty(X;U) \longrightarrow \mathcal{C}^\infty(U) \longrightarrow \mathcal{E}^\infty(X) \longrightarrow 0, \tag{6.2}$$

and the periodic cyclic homology of $\mathcal{C}^\infty(U)$ is well-known. But unfortunately one cannot apply excision to compute $HP_\bullet(\mathcal{E}^\infty(X))$, since, in general, the sequence (6.2) does not possess a continuous splitting.

7. Whitney-de Rham cohomology of subanalytic spaces

In this section we will compute the cohomology of the Whitney-de Rham complex over a subanalytic set by proving the following.

7.1. Theorem. *For every subanalytic $X \subset \mathbb{R}^n$ the sequence*

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{E}^\infty_X \xrightarrow{d} \Omega^1_{\mathcal{E}^\infty_X} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k_{\mathcal{E}^\infty_X} \xrightarrow{d} \cdots \tag{7.1}$$

comprises a fine resolution of the sheaf $\mathbb{R}_X$ of locally constant real-valued functions on X.

Before we prepare the proof of the theorem let us remark that a subanalytic X is a locally path-connected and locally contractible locally compact topological Hausdorff space (this can be concluded for example from the fact that a subanalytic set is regular [31, Cor. 2] and possesses a Whitney stratification [26, Thm. 4.8]). Hence, if $\mathcal{S}_X^k$ denotes the sheaf associated to the presheaf of singular k-cochains on X, the complex of sheaves

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{S}_X^0 \longrightarrow \mathcal{S}_X^1 \longrightarrow \cdots \longrightarrow \mathcal{S}_X^k \longrightarrow \cdots ,$$

is a soft resolution of $\mathbb{R}_X$ (cf. Godement [19, Ex. 3.9.1]). By the theorem it thus follows that the Whitney-de Rham cohomology coincides with the cohomology of the complex of global sections of $\mathcal{S}_X^\bullet$, i.e. the singular cohomology of X (with values in $\mathbb{R}$). So we obtain:

7.2. Corollary. *The Whitney-de Rham cohomology*

$$H^\bullet_{\mathrm{WdR}}(X) = H^\bullet(\Omega^\bullet_{\mathcal{E}^\infty}(X))$$

coincides with the singular cohomology $H^\bullet_{\mathrm{sing}}(X;\mathbb{R})$.

The nontrivial part in the proof of the theorem is to show that the sequence (7.1) is exact or in other words that Poincaré's lemma holds true for Whitney functions. The essential tool for proving Poincaré's lemma for Whitney functions will be a so-called bimeromorphic subanalytic triangulation of X together with a particular system of tubular neighborhoods for the strata defined by the triangulation. From 7.3 to 7.8 we set up the material needed for the proof of the theorem. The proof will then be given in 7.9. Let us finally mention that a de Rham theorem for Whitney functions over a stratified space with a so-called curvature moderate control datum has already been proved in [44, §5.4]. Moreover, it has been claimed in [44, Rem. 5.4.6] that the assumptions of this de Rham theorem are satisfied by subanalytic sets, but an explicit proof of this claim could not have been given. Thus a complete proof of the statement in Corollary 7.2 appears here for the first time.

7.3 . Recall that by a *finite* (resp. *locally finite*) *subanalytic triangulation* of a closed subanalytic set X one understands a pair $T = (h, \mathcal{K})$, where $\mathcal{K}$ is a finite (resp. locally finite) simplicial complex in some $\mathbb{R}^n$ and $h : |\mathcal{K}| \to X$ is a subanalytic homeomorphism such that for every simplex $\Delta \in \mathcal{K}$ the following holds true:

(TRG1) The image $\widetilde{\Delta} := h(\Delta)$ is a *subanalytic leaf* that means a subanalytic, connected and locally closed smooth real-analytic submanifold of $\mathbb{R}^n$.

(TRG2) The homeomorphism h induces a real-analytic isomorphism $h_\Delta : \Delta \to \widetilde{\Delta}$.

Note that we always assume a simplex to be open, if it is not stated otherwise.

If $X_1, \ldots, X_k \subset X$ are subanalytic subsets, one calls the triangulation T *compatible* with the X_j, if every one of the sets X_j is a union of simplices $h(\Delta)$,

$\Delta \in \mathcal{K}$. The following result is well-known (see Łojasiewicz [35] and Hironaka [27]).

7.4. THEOREM. *For every family $X_1, \dots, X_l \subset \mathbb{R}^n$ of bounded subanalytic sets there exists a compact parallelotope $Q \subset \mathbb{R}^n$ containing the X_j in its interior and a finite subanalytic triangulation $(h : Q \to Q, \mathcal{K})$ which is compatible with the X_j.*

A subanalytic triangulation has the following further property which follows immediately from the lemma below.

(TRG3) The triangulation map h is *bi-Hölder*, i.e. h and its inverse are Hölder continuous.

Recall that a map $f : X \to \mathbb{R}^N$ is said to be *Hölder continuous*, if there exist $C > 0$ and $\lambda > 0$ such that

$$|f(x) - f(y)| \leq C\,|x - y|^\lambda \quad \text{for all } x, y \in X.$$

In case $Y \subset X$ is a closed subset, this estimate implies that for f Hölder, $Z = f(Y)$ and $K \subset X$ compact we have

$$d(f(x), Z) \leq C\, d(x, Y)^\lambda \quad \text{for all } x \in K \setminus Y. \tag{7.2}$$

Let us assume more generally that $Z \subset \mathbb{R}^N$ is an arbitrary closed subset. Then, if one can find for every compact $K \subset X$ constants $C > 0$ and $\lambda > 0$ such that (7.2) holds true, we say that f is *Hölder* with respect to the pair (Y, Z).

7.5. LEMMA. *Every subanalytic function $f : X \to \mathbb{R}^N$ with compact graph is Hölder continuous. Moreover, if f is continuous and $Y \subset X$ is a closed subanalytic subset such that $f(Y) \cap f(X \setminus Y) = \emptyset$, then f is Łojasiewicz with respect to the pair $(Y, f(Y))$.*

Proof. Let $g_1(x, y) = |x - y|$ and $g_2(x, y) = |f(x) - f(y)|$ for $x, y \in X$. Since these functions are subanalytic and $g_1^{-1}(0) \subset g_2^{-1}(0)$, Łojasiewicz's inequality (1.2) immediately yields the first claim.

To prove the second claim we observe that the function $X \ni x \mapsto d(f(x), f(Y)) \in \mathbb{R}$ is subanalytic and, by assumption, vanishes only on Y. Łojasiewicz's inequality then also yields the second claim. □

7.6. PROPOSITION. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^N$ be open, $X, Y \subset U$ be closed and regularly situated and let $Z \subset V$ be closed. Furthermore, assume that $F \in \mathcal{M}^\infty(Y; X)$ is a (vector-valued) Whitney function and $f : X \setminus Y \to V$ a continuous map such that $f = F^0$. Then the following hold true*:

(1) *If f is Łojasiewicz with respect to the pair (Y, Z'), where $Z' = \overline{Z} \cup (\mathbb{R}^N \setminus V)$, $f(K \setminus Y) \subset V$ is relatively compact for every compact $K \subset X$ and if f satisfies $f(X \setminus Y) \subset V \setminus Z$, then $F^* \mathcal{M}^\infty(\overline{Z}; \overline{V}) \subset \mathcal{M}^\infty(Y; X)$.*

(2) *If f is Hölder with respect to the pair* (Y,Z), *then* $F^*\mathcal{J}^\infty(Z;V)\subset\mathcal{J}^\infty(X\cap Y;X)$.

We call a Whitney function $F\in\mathcal{M}^\infty(Y;X)$ *meromorphic from* (X,Y) *to* (V,Z), if $f=F^0$ satisfies all the conditions stated in (1) and (2). By abuse of language, we sometimes call a continuous function $f:X\backslash Y\to V$ meromorphic, if there is a meromorphic F with $f=F^0$ and if it is clear by the context which F is meant. A Whitney function F is said to be *bimeromorphic from* (X,Y) *to* (I,Z), where I is the set $\overline{f(X\setminus Y)}\cap V$, if f is a homeomorphism onto its image and if there exists a Whitney function $G\in\mathcal{M}^\infty(I;Z)$ which is meromorphic from (I,Z) to (U,Y) and which satisfies $f^{-1}=G^0$.

Proof. We will only show (1), since the proof of (2) is similar. As the claim is essentially a local statement, we can assume without loss of generality that X is compact. Let us choose an open neighborhood W of $X\setminus Y$ and an element $\widetilde{f}\in\mathcal{M}^\infty(Y;W)$ such that $F=\mathrm{J}^\infty_{X\setminus Y}\widetilde{f}$. We will show first that the restriction of $\widetilde{f}$ to an appropriate neighborhood of $X\setminus Y$ is Łojasiewicz with respect to (Y,Z'). To this end we need several estimates which will be proved in the following. Since f is Łojasiewicz with respect to (Y,Z'), there exist $C_{\mathrm{L}},\lambda_{\mathrm{L}}>0$ such that

$$d(f(x),Z')\geq C_{\mathrm{L}}\,d(x,Y)^{\lambda_{\mathrm{L}}}\quad\text{for all }x\in X\setminus Y.\tag{7.3}$$

By assumption, X and Y are regularly situated. Hence there exist C_r, $\lambda_r>0$ such that

$$d(x,Y)\geq C_r\,d(x,X\cap Y)^{\lambda_r}\quad\text{for all }x\in X.\tag{7.4}$$

Moreover, one can find a compact neighborhood $K\subset W$ of $X\setminus Y$ such that $K\cap Y=X\cap Y$ and such that $W\setminus\overset{\circ}{K}$ and X are regularly situated. Finally, the fact that X and Y are regularly situated, implies that there are $C_K>0$ and $\lambda_K>0$ such that

$$d(x,X)+d(x,Y)\geq C_K\,d(x,X\cap Y)^{\lambda_K}\quad\text{for all }x\in K.\tag{7.5}$$

As $\widetilde{f}\in\mathcal{M}^\infty(Y;W)$, there exist $C>0$ and $\lambda>0$ such that

$$\|D\widetilde{f}(x)\|\leq C\,\frac{1}{d(x,X\cap Y)^{\lambda}}\quad\text{for all }x\in K\setminus Y.\tag{7.6}$$

By possibly shrinking K we can now achieve that for all $x\in K\setminus Y$

$$\begin{aligned}d(x,X)<\min\Big\{&\frac{1}{4}\frac{C_{\mathrm{L}}\,C_r^{\lambda_{\mathrm{L}}}}{C}\Big(\frac{3}{4}\Big)^{\lambda+\lambda_{\mathrm{L}}\,\lambda_r}d(x,X\cap Y)^{\lambda+\lambda_{\mathrm{L}}\,\lambda_r},\\&\frac{1}{4}d(x,X\cap Y),\frac{1}{4}\,C_K\,d(x,X\cap Y)^{\lambda_K}\Big\}.\end{aligned}\tag{7.7}$$

For every $x \in \mathbb{R}^n$ let x_{d} be an element of X such that $d(x, X) = d(x, x_{\mathrm{d}})$. By (7.7) we then have for every $s \in [0, 1]$

$$
\begin{aligned}
d(x + s(x_{\mathrm{d}} - x), X \cap Y) &\geq d(x, X \cap Y) - d(x, x + s(x_{\mathrm{d}} - x)) \\
&\geq d(x, X \cap Y) - d(x, x_{\mathrm{d}}) \geq \frac{3}{4} d(x, X \cap Y)
\end{aligned}
\tag{7.8}
$$

for all $x \in K \setminus Y$. One then concludes for $x \in K \setminus Y$

$$
\begin{aligned}
d(\widetilde{f}(x), Z') &\geq d(f(x_{\mathrm{d}}), Z') - d(\widetilde{f}(x), f(x_{\mathrm{d}})) \\
&\geq d(f(x_{\mathrm{d}}), Z') - d(x, x_{\mathrm{d}}) \sup_{s \in [0,1]} \| D\widetilde{f}(x + s(x_{\mathrm{d}} - x)) \| \\
&\geq C_{\mathrm{L}}\, d(x_{\mathrm{d}}, Y)^{\lambda_{\mathrm{L}}} - C\, d(x, x_{\mathrm{d}}) \frac{1}{d(x + s(x_{\mathrm{d}} - x), X \cap Y)^{\lambda}} \\
&\geq C_{\mathrm{L}}\, d(x_{\mathrm{d}}, Y)^{\lambda_{\mathrm{L}}} - C \left(\frac{4}{3}\right)^{\lambda} d(x, x_{\mathrm{d}}) \frac{1}{d(x, X \cap Y)^{\lambda}} \\
&\geq C_{\mathrm{L}}\, d(x_{\mathrm{d}}, Y)^{\lambda_{\mathrm{L}}} - \frac{1}{4} C_{\mathrm{L}} C_r^{\lambda_{\mathrm{L}}} \left(\frac{3}{4}\right)^{\lambda_{\mathrm{L}} \lambda_r} d(x, X \cap Y)^{\lambda_{\mathrm{L}} \lambda_r} \\
&\geq \frac{3}{4} C_{\mathrm{L}}\, d(x_{\mathrm{d}}, Y)^{\lambda_{\mathrm{L}}}.
\end{aligned}
\tag{7.9}
$$

Here, we have used Taylor's formula for the second inequality, (7.3) and (7.6) for the third, and (7.8) for the fourth inequality. The fifth inequality follows from (7.7), the last one is a consequence of (7.8) and (7.4). Now, (7.7) and (7.5) imply that

$$
d(x, X) \leq \frac{1}{4} C_K\, d(x, X \cap Y)^{\lambda_K} \leq \frac{1}{4} (d(x, X) + d(x, Y)) \quad \text{for all } x \in K.
\tag{7.10}
$$

Hence,

$$
d(x_{\mathrm{d}}, Y) \geq d(x, Y) - d(x_{\mathrm{d}}, x) \geq \frac{2}{3} d(x, Y) \quad \text{for all } x \in K \setminus Y.
\tag{7.11}
$$

After redefining C_{L} and λ_{L} we thus obtain from (7.9)

$$
d(\widetilde{f}(x), Z') \geq C_{\mathrm{L}}\, d(x, Y)^{\lambda_{\mathrm{L}}} \quad \text{for all } x \in K \setminus Y;
\tag{7.12}
$$

hence $\widetilde{f}$ is Łojasiewicz with respect to (Y, Z').

In the second part we will show that for every $g \in \mathcal{M}^\infty(\overline{Z}; \overline{V})$ the map $\psi\,(g \circ \widetilde{f})$ lies in $\mathcal{M}^\infty(Y; W)$, where $\psi \in \mathcal{M}^\infty(X \cap Y; W)$ is a function which is identical to 1 on a neighborhood of $X \setminus Y$ and vanishes on $W \setminus K$. Obviously, this then proves the claim. The existence of a function ψ with the stated properties is provided by the fact that X and $W \setminus \overset{\circ}{K}$ are regularly situated and by Chapter IV, Lemma 4.5 in [50]. Let us now estimate the growth of the derivatives of $g \circ \widetilde{f}$ near $X \cap Y$. By the above consideration on $\widetilde{f}$, the composition $g \circ \widetilde{f}$ is well-defined over $\overset{\circ}{K}$ and smooth. Moreover, by the assumption on f and since $\widetilde{f} \in \mathcal{M}^\infty(Y;W)$, one can shrink K in such a way that the set $\widetilde{f}(K \setminus Y)$ is relatively compact in V and such that X and $W \setminus \overset{\circ}{K}$ are still regularly situated. Given $r \in \mathbb{N}$ there thus exist $C_1, C_2 > 0$ and $\lambda_1, \lambda_2 > 0$ such that

$$|\partial_y^\beta g(y)| \leq \frac{C_1}{d(y,Z')^{\lambda_1}} \quad \text{for all } y \in f(K \setminus Y) \text{ and } |\beta| \leq r,\ \beta \in \mathbb{N}^N, \tag{7.13}$$

$$|\partial_x^\gamma \widetilde{f}(x)| \leq \frac{C_2}{d(x,Y)^{\lambda_2}} \quad \text{for all } x \in K \setminus Y \text{ and } |\gamma| \leq r,\ \gamma \in \mathbb{N}^n. \tag{7.14}$$

By virtue of the chain and Leibniz rule these estimates together with (7.12) entail that there are constants $C' > 0$ and $C > 0$ such that for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq r$ and all $x \in K \setminus Y$

$$\begin{aligned} |\partial_x^\alpha (g \circ \widetilde{f})\,(x)| &\leq C' \sup_{|\beta|,|\gamma| \leq r} |\partial_y^\beta g(\widetilde{f}(x))| \left(1 + |\partial_x^\gamma \widetilde{f}(x)|\right)^r \\ &\leq C \, \frac{1}{d(x,Y)^{\lambda_1 \lambda_L}} \left(1 + \frac{1}{d(x,Y)^{\lambda_2}}\right). \end{aligned} \tag{7.15}$$

Hence $\psi\,(g \circ \widetilde{f}) \in \mathcal{M}^\infty(Y;W)$ and the proof of (1) is finished. □

7.7. By a *tubular neighborhood* of a subanalytic leaf $\Gamma \subset \mathbb{R}^n$ we will understand a triple $(E, \varepsilon, \varphi)$ such that the following hold:

(TUB1) $\pi_E : E \to \Gamma$ is a smooth real-analytic subbundle of $T_{|\Gamma}\mathbb{R}^n$ which is complementary to the tangent bundle $T\Gamma$.

(TUB2) $\varepsilon : \Gamma \to \mathbb{R}_{>0}$ is a continuous map.

(TUB3) φ is a real-analytic open embedding from $E_{2\varepsilon}$ into $\mathbb{R}^n$ such that $\varphi(x,0) = x$ for all $x \in \Gamma$. Hereby, E_δ denotes for every positive function $\delta : \Gamma \to \mathbb{R}_{>0}$ the open set $\{(x,v) \in E \mid |v| < 2\delta(x)\}$.

Given a tubular neighborhood for Γ we denote by U_δ for every positive $\delta : \Gamma \to \mathbb{R}_{>0}$ the open set $\varphi(E_\delta \cap E_{2\varepsilon})$ and by $\pi_{E,\Gamma}$ or briefly π_Γ the *projection* of the tubular neighborhood that means the map $U_{2\varepsilon} \ni x \mapsto \varphi(\pi_E(\varphi^{-1}(x)) \in \Gamma$.

By shrinking the map ε one can achieve that a tubular neighborhood $(E, \varepsilon, \varphi)$ has the following two further properties:

(TUB4a) The map φ extends to a homeomorphism $\overline{\varphi} : \overline{E_{2\varepsilon}} \to \overline{U_{2\varepsilon}}$, and $\overline{E_\varepsilon} \setminus (\operatorname{fr}\Gamma \times \{0\}) \subset E_{2\varepsilon}$.

(TUB4b) $\overline{U_\varepsilon} \setminus \operatorname{fr}\Gamma \subset U_{2\varepsilon}$.

A tubular neighborhood is called subanalytic, if (TUB4a), (TUB4b) and the following two axioms hold true:

(TUB5a) The bundle E and the maps ε and φ are subanalytic.

(TUB5b) The projection π_Γ is subanalytic.

Finally, we call a tubular neighborhood *bimeromorphic*, if it satisfies (TUB4a), (TUB4b) and the last two axioms:

(TUB6a) φ, or more precisely the map $U_{2\varepsilon} \times \mathbb{R}^n \ni (x,v) \mapsto \varphi\big(\pi_\Gamma(x), v - T_{\pi_\Gamma(x)}\pi_\Gamma(v)\big)$, restricts to a Whitney function which is bimeromorphic from $(\overline{E_\varepsilon}, \operatorname{fr}\Gamma \times \{0\})$ to $(\overline{U_\varepsilon}, \operatorname{fr}\Gamma)$.

(TUB6b) The projection π_Γ is meromorphic from $(\overline{U_\varepsilon}, \operatorname{fr}\Gamma)$ to $(\mathbb{R}^n, \operatorname{fr}\Gamma)$.

It is straightforward to check that every b-axiom is a consequence of the corresponding a-axiom (under the assumption that (TUB1) to (TUB3) hold true).

The simplices $\Delta \in \mathcal{K}$ of a simplicial complex $\mathcal{K}$ have natural subanalytic tubular neighborhoods $(E_\Delta, \varepsilon_\Delta, \pi_\Delta)$, where E_Δ is the bundle normal to $T\Delta$ with respect to the euclidian metric and π_Δ is the orthogonal projection onto Δ. Clearly, one can choose the ε_Δ in such a way that for two different simplices Δ, Δ' of the same dimension $U_{\varepsilon_\Delta} \cap U_{\varepsilon_{\Delta'}} = \emptyset$. Moreover, (TUB4b) entails $\varepsilon_\Delta(x) < d(x, \operatorname{fr}\Delta)$ for all $x \in \Delta$ sufficiently close to $\operatorname{fr}\Delta$ or in other words sufficiently close to the faces of Δ.

7.8. By a *bimeromorphic triangulation* of a closed subanalytic $X \subset \mathbb{R}^n$ we now understand a subanalytic triangulation $(h, \mathcal{K})$ of X together with a system of bimeromorphic subanalytic tubular neighborhoods $(E_{\widetilde{\Delta}}, \varepsilon_{\widetilde{\Delta}}, \pi_{\widetilde{\Delta}})$ for the leaves $\widetilde{\Delta} = h(\Delta)$ such that the conditions (BMT1) to (BMT4) below are fulfilled for every simplex $\Delta \in \mathcal{K}$:

(BMT1) The tubular neighborhoods satisfy $U_{\varepsilon_{\widetilde{\Delta}}} \cap U_{\varepsilon_{\widetilde{\Delta}'}} = \emptyset$ for every $\widetilde{\Delta}'$ of the same dimension as $\widetilde{\Delta}$ but disjoint from $\widetilde{\Delta}$.

(BMT2) For a sufficiently small neighborhood U of $\overline{U_{\varepsilon_\Delta}}$ the map $h_\Delta \circ \pi_\Delta : \overline{U_{\varepsilon_\Delta}} \setminus \operatorname{fr}\Delta \to \widetilde{\Delta}$ can be extended to a continuous map $h_{\pi_\Delta} : U \to \mathbb{R}^n$ which lies in the multiplier algebra $\mathcal{M}^\infty(\operatorname{fr}\Delta; U)$ and is meromorphic from $(U, \operatorname{fr}\Delta)$ to $(\mathbb{R}^n, \operatorname{fr}\widetilde{\Delta})$.

(BMT3) For a sufficiently small neighborhood $\widetilde{U}$ of $\overline{U_{\varepsilon_{\widetilde{\Delta}}}}$ the map $h^{-1}_{\widetilde{\Delta}} \circ \pi_{\widetilde{\Delta}} : \overline{U_{\varepsilon_{\widetilde{\Delta}}}} \setminus \operatorname{fr}\widetilde{\Delta} \to \Delta$ can be extended to a continuous map $h^-_{\pi_{\widetilde{\Delta}}} : \widetilde{U} \to \mathbb{R}^n$ which lies in the multiplier algebra $\mathcal{M}^\infty(\operatorname{fr}\widetilde{\Delta}; \widetilde{U})$ and is meromorphic from $(\widetilde{U}, \operatorname{fr}\widetilde{\Delta})$ to $(\mathbb{R}^n, \operatorname{fr}\Delta)$.

(BMT4) For a sufficiently small neighborhood $\widetilde{U}$ of $\overline{U_{\varepsilon_{\widetilde{\Delta}}}}$ the map

$$(\overline{U_{\varepsilon_{\widetilde{\Delta}}}} \setminus \operatorname{fr}\widetilde{\Delta}) \times [0,1] \to \mathbb{R}^n, \quad (x,t) \mapsto \varphi_{\widetilde{\Delta}}(t\,\varphi^{-1}_{\widetilde{\Delta}}(x))$$

can be extended to a continuous homotopy $H_{\widetilde{\Delta}} : \widetilde{U} \times [0,1] \to \mathbb{R}^n$ which lies in the multiplier algebra $\mathcal{M}^\infty(\operatorname{fr}\widetilde{\Delta} \times [0,1]; \widetilde{U} \times [0,1])$, satisfies $H_{\widetilde{\Delta},1} = \mathrm{id}$ and is meromorphic from $(\widetilde{U} \times [0,1], \operatorname{fr}\widetilde{\Delta} \times [0,1])$ to $(\mathbb{R}^n, \widetilde{\operatorname{fr}\Delta})$. Hereby, $H_{\widetilde{\Delta},t}$ denotes the map $H_{\widetilde{\Delta}}(\cdot, t)$ for every $t \in [0,1]$.

The second and third axiom imply that for every $\Delta \in \mathcal{K}$ the restriction $h_{|\Delta}$ (or more precisely the Whitney function $\mathsf{J}^\infty_\Delta(h \circ \pi_\Delta)$)is bimeromorphic from $(\overline{\Delta}, \operatorname{fr}\Delta)$ to $(\overline{\widetilde{\Delta}}, \operatorname{fr}\widetilde{\Delta})$.

7.9. *Proof of Theorem* 7.1. As already mentioned we only have to show that for every $x_0 \in X$ there exists a basis of contractible neighborhoods $V \subset X$ of x_0 such that for all $\omega \in \Omega^k_{\mathcal{E}^\infty}(V)$ with $d\omega = 0$ there exists a form $\eta \in \Omega^{k-1}_{\mathcal{E}^\infty}(V)$ satisfying $d\eta = \omega$.

We can assume that x_0 is not an element of the interior $\overset{\circ}{X}$, because otherwise the classical Poincaré lemma could be applied. Since the claim is essentially a local statement, we can even assume furthermore without loss of generality that X is a compact and connected subanalytic set. Now choose a bimeromorphic subanalytic triangulation $(h : Q \to Q, \mathcal{K})$ compatible with X and the one-point set $\{x_0\}$; in the following section we will show that this is possible indeed. Clearly, we can choose the triangulation in such a way that 0 is a simplex of $\mathcal{K}$ and $h(0) = x_0$. As a further tool for our construction we need a particular integral operator $K_M : \Omega^{k+1}(M) \times [0,1] \to \Omega^k(M)$, where M is an arbitrary smooth manifold. This operator is defined by

$$K_M\omega = \int_0^1 \iota_s^*(\partial_s \lrcorner\, \omega)ds, \quad \text{for all } \omega \in \Omega^{k+1}(M),$$

where s denotes the last coordinate of an element of $M \times [0,1]$, $\partial_s\lrcorner$ means the insertion of the vector field ∂_s in a differential form at the first position and $\iota_s : M \to M \times [0,1]$ is the map $y \mapsto (y,s)$. By Cartan's magic formula

$$d\,K_M + K_M\,d = \iota_1^* - \iota_0^*. \tag{7.16}$$

Now let $B \subset \mathbb{R}^n$ be an open ball around the origin such that B does not contain any other 0-simplex of $\mathcal{K}$ besides the origin and let $\mathcal{K}_j$ for $j = 0, \ldots, n$ be the set of all j-simplices of $\mathcal{K}$ which meet B and $h^{-1}(X)$. Let $H : B \times [0,1] \to B$ be the radial homotopy $(x,t) \mapsto tx$. Then, $H((\Delta \cap B) \times [0,1]) \subset (\Delta \cap B) \cup \{0\}$ for all Δ meeting B. Next let $\widetilde{B} = h(B)$, $V = \widetilde{B} \cap X$ and let $\omega \in \Omega^k_{\mathcal{E}^\infty}(V)$ be closed. Choose a smooth differential form $\widetilde{\omega} \in \Omega^k(\widetilde{B})$ which induces ω over V in the sense of Whitney's extension theorem. We will construct inductively smooth differential forms $\widetilde{\eta}_0, \ldots, \widetilde{\eta}_n \in \Omega^{k-1}(\widetilde{B})$ such that

(7.17)
$$\widetilde{\omega} - d(\widetilde{\eta}_0 + \ldots + \widetilde{\eta}_j) \in \mathcal{J}^\infty(K_0 \cup \ldots \cup K_j; \widetilde{B})\,\Omega^k(\widetilde{B}) \quad \text{for } j = 1, \ldots, n,$$

where $K_j = \bigcup_{\Delta \in \mathcal{K}_j} h(\Delta)$. Clearly, this proves the claim, since the element $\eta \in \Omega^{k-1}_{\mathcal{E}^\infty}(V)$ induced by $\widetilde{\eta}_0 + \ldots + \widetilde{\eta}_n$ satisfies $\omega = d\eta$.

For the construction of the $\widetilde{\eta}_j$ we mention first that $\pi_\Delta^{-1}(\Delta \cap B) \subset B$ and $\pi_{\widetilde{\Delta}}^{-1}(\widetilde{\Delta} \cap \widetilde{B}) \subset \widetilde{B}$ after possibly passing to smaller ε_Δ and $\varepsilon_{\widetilde{\Delta}}$. Secondly, we simplify notation by writing, respectively, $\Delta, \widetilde{\Delta}, \partial\Delta, U_{\varepsilon_\Delta}, \ldots$ instead of $\Delta \cap B, \widetilde{\Delta} \cap \widetilde{B}, \partial\Delta \cap B, U_{\varepsilon_\Delta} \cap B, \ldots$. Let us now come to the construction of $\widetilde{\eta}_0$. Since $\widetilde{B}$ is contractible, there exists a smooth homotopy $G : \widetilde{B} \times [0,1] \to \widetilde{B}$ such that $G(x_0, t) = x_0$, $G(x, 0) = x_0$ and $G(x, 1) = x$ for all $x \in \widetilde{B}$ and $t \in [0,1]$. Let $\widetilde{\eta}_0 = K_{\widetilde{B}} G^* \widetilde{\omega}$. By Equation (7.16) we then have $d\widetilde{\eta}_0 + K_{\widetilde{B}} G^* d\widetilde{\omega} = \widetilde{\omega}$. Using the assumption on G and $d\omega = 0$ one concludes that $K_{\widetilde{B}} G^* d\widetilde{\omega} \in \mathcal{J}^\infty(\{x_0\}, \widetilde{B})$; hence $\widetilde{\eta}_0$ has the desired property. Now assume that we have constructed $\widetilde{\eta}_0, \ldots, \widetilde{\eta}_j$ for $0 \leq j < n$. If $\mathcal{K}_{j+1} = \emptyset$ we are done, since we can then put $\widetilde{\eta}_{j+1} = \ldots = \widetilde{\eta}_n = 0$. So assume $\mathcal{K}_{j+1} \neq \emptyset$. Let $\widetilde{\omega}' = \widetilde{\omega} - d(\widetilde{\eta}_0 + \ldots + \widetilde{\eta}_j)$. The following constructions can be performed separately for every $\Delta \in \mathcal{K}_{j+1}$ and so we assume for simplicity that there is only one simplex $\Delta \in \mathcal{K}_{j+1}$. We proceed in three steps.

1. *Step.* Consider the homotopy $H_{\widetilde{\Delta}} : \widetilde{U} \times [0,1] \to \mathbb{R}^n$ of (BMT4). After possibly changing $H_{\widetilde{\Delta}}$ outside a sufficiently small neighborhood of $\overline{U_{\varepsilon_{\widetilde{\Delta}}}}$ and extending the homotopy appropriately we see that $H_{\widetilde{\Delta}}$ is a homotopy which is defined on $\widetilde{B} \times [0,1]$, has values in $\widetilde{B}$ and has the properties stated in (BMT4). Let $\widetilde{\eta}' = K_{\widetilde{B}} H^*_{\widetilde{\Delta}} \widetilde{\omega}'$. Since $\widetilde{\omega}' \in \mathcal{J}^\infty(\operatorname{fr}\widetilde{\Delta}; \widetilde{B})\,\Omega^k(\widetilde{B})$ and $H_{\widetilde{\Delta}}(x,t) = x$ for all x in the closure of $\widetilde{\Delta}$ and all $t \in [0,1]$, (BMT4) and Proposition 7.6 entail that $\widetilde{\eta}'$ is well-defined and lies in $\mathcal{J}^\infty(\operatorname{fr}\widetilde{\Delta}; \widetilde{B})\,\Omega^{k-1}(\widetilde{B})$. Moreover, we have $d\widetilde{\omega}' \in \mathcal{J}^\infty(X; \widetilde{B}) \otimes \Omega^k(\widetilde{B})$. By virtue of (7.16) and Proposition 7.6 one concludes that

$$\widetilde{\omega}' - H^*_{\widetilde{\Delta},0}\widetilde{\omega}' - d\widetilde{\eta}' = K_{\widetilde{B}} H^*_{\widetilde{\Delta}} \widetilde{d}\omega' \in \mathcal{J}^\infty(\overline{\widetilde{\Delta}}; \widetilde{B})\,\Omega^{k-1}(\widetilde{B}), \tag{7.18}$$

where $H_{\widetilde{\Delta},0} = H_{\widetilde{\Delta}}(\cdot, 0)$. Note that the restriction of $H_{\widetilde{\Delta},0}$ to $\overline{U_{\varepsilon_{\widetilde{\Delta}}}}$ coincides with $\pi_{\widetilde{\Delta}}$.

2. *Step.* Next consider the map h_{π_Δ} of (BMT2). Clearly, $h_{\pi_\Delta}(\overline{U_{\varepsilon_\Delta}} \cap B) \subset \widetilde{B}$, so after possibly redefining h_{π_Δ} outside a sufficiently small neighborhood of $\overline{U_{\varepsilon_\Delta}}$ and appropriate extension we can assume that h_{π_Δ} is defined on B and has image in $\widetilde{B}$, while the properties of (BMT2) remain valid. We have $h_{\pi_\Delta}(\operatorname{fr}\Delta) \subset \operatorname{fr}\widetilde{\Delta}$ and $\widetilde{\omega}' \in \mathcal{J}^\infty(\operatorname{fr}\widetilde{\Delta}; \widetilde{B})\,\Omega^k(\widetilde{B})$, hence by Proposition 7.6 the pull-back $h^*_{\pi_\Delta}\widetilde{\omega}'$ has to be in $\mathcal{J}^\infty(\operatorname{fr}\Delta; B)\,\Omega^k(B)$. Moreover, $dh^*_{\pi_\Delta}\widetilde{\omega}' \in \mathcal{J}^\infty(\Delta; B)\,\Omega^{k+1}(B)$. Let $\widetilde{\nu} = K_B H^* h^*_{\pi_\Delta}\widetilde{\omega}'$. Observe that $\widetilde{\nu} \in \mathcal{J}^\infty(\operatorname{fr}\Delta; B)\,\Omega^{k-1}(B)$ and $K_B H^* dh^*_{\pi_\Delta}\widetilde{\omega}' \in \mathcal{J}^\infty(\Delta; B)\,\Omega^k(B)$. Hence by (7.16)

$$h^*_{\pi_\Delta}\widetilde{\omega}' - d\widetilde{\nu} \in \mathcal{J}^\infty(\Delta; B)\,\Omega^k(B). \tag{7.19}$$

3. *Step.* Analogously to the second step let us now consider the map $h^-_{\pi_{\widetilde{\Delta}}}$ of (BMT3). We can assume after appropriate alteration that $h^-_{\pi_{\widetilde{\Delta}}}$ is defined on $\widetilde{B}$, has image in B and still has the properties stated in (BMT3). Let $\widetilde{\eta}'' = (h^-_{\pi_{\widetilde{\Delta}}})^*\widetilde{\nu}$. Similarly, as in the second step one checks that $\widetilde{\eta}'' \in \mathcal{J}^\infty(\operatorname{fr}\widetilde{\Delta}; B)\,\Omega^{k-1}(\widetilde{B})$. Since $(h^-_{\pi_{\widetilde{\Delta}}})^* h^*_{\pi_\Delta}\widetilde{\omega}' = \pi^*_{\widetilde{\Delta}}\widetilde{\omega}' = H^*_{\widetilde{\Delta},0}\widetilde{\omega}'$ over $\overline{U_{\varepsilon_{\widetilde{\Delta}}}}$, Equations (7.18) and (7.19) yield

$$\widetilde{\omega}' - d\widetilde{\eta}'' - d\widetilde{\eta}' \in \mathcal{J}^\infty(\overline{\widetilde{\Delta}}; \widetilde{B})\,\Omega^{k-1}(\widetilde{B}). \tag{7.20}$$

So, if we now put $\widetilde{\eta}_{j+1} := \widetilde{\eta}' + \widetilde{\eta}''$, the induction step is finished and the theorem is proven. $\square$

8. Bimeromorphic triangulations

In the following we will show that for every subanalytic $X \subset \mathbb{R}^n$ there exists a compatible bimeromorphic triangulation. But before we come to the details of the existence proof we have to introduce many preliminaries.

8.1. Throughout this section Γ will always denote a subanalytic leaf of $\mathbb{R}^n$ and $(E, \varepsilon, \varphi)$ a tubular neighborhood of Γ in $\mathbb{R}^n$. Moreover, for every $k \leq n$ we will denote by $\pi^n_k : \mathbb{R}^n \to \mathbb{R}^k$ the canonical projection $(x_{\underline{1}}, \dots, x_{\underline{n}}) \mapsto (x_{\underline{1}}, \dots, x_{\underline{k}})$. In the particular case $k = n-1$ we will briefly write π instead of $\pi^n_{(n-1)}$.

Let us assume that the tubular neighborhood $(E, \varepsilon, \varphi)$ is bimeromorphic and subanalytic. Moreover, let $N \subset \Gamma \times \Gamma$ be a subanalytic neighborhood of the diagonal $\Delta_2(\Gamma)$ such that $d(x,y) < \frac{1}{2} \min\{d(x, \operatorname{fr}\Gamma), d(y, \operatorname{fr}\Gamma)\}$ and $(x,y) \in N$ for all $(y,x) \in N$. Finally, let $p : N \to \mathbb{R}^n$ be the projection $(y,x) \mapsto x$. By a *bimeromorphic subanalytic parallel transport* for $(E, \varepsilon, \varphi)$ (*defined* over N) we then understand a smooth map $// : p^*E \to E$ having the following properties:

(PT1) the restriction $//_{yx} := //_{|\{y\} \times E_x} : \{y\} \times E_x \to E$ is a linear isomorphism onto E_y for $(y,x) \in N$ and satisfies $//_{xy} \circ //_{yx} = \mathrm{id}_{E_x}$,

(PT2) $//_{|p^* E_{2\varepsilon}} : p^* E_{2\varepsilon} \to E \subset \mathbb{R}^{2n}$ is subanalytic,

(PT3) $//_{|p^* \overline{E_\varepsilon}}$ is bimeromorphic from $(p^* \overline{E_\varepsilon}, \Delta_2(\operatorname{fr}\Gamma) \times \{0\})$ to $(\mathbb{R}^{2n}, \operatorname{fr}\Gamma \times \{0\})$.

8.2. Lemma. *Let $\Gamma \subset \mathbb{R}^n$ be a bounded subanalytic leaf of dimension $k < n$ such that the restriction $\pi^n_{k\,|\Gamma} : \Gamma \to \mathbb{R}^k$ is an analytic isomorphism onto an open subanalytic set $G \subset \mathbb{R}^k$. Then the map $\varrho : G \to \Gamma \subset \mathbb{R}^n$ inverse to $\pi^n_{k\,|\Gamma}$ is meromorphic from $(\overline{G}, \operatorname{fr} G)$ to $(\mathbb{R}^n, \operatorname{fr}\Gamma)$.*

Clearly, $\pi^n_{k\,|\Gamma}$ is meromorphic from $(\overline{\Gamma}, \operatorname{fr}\Gamma)$ to $(\mathbb{R}^k, \operatorname{fr} G)$, and so the lemma shows that $\pi^n_{k\,|\Gamma}$ is in fact bimeromorphic.

Proof. We first show that $\varrho \in \mathcal{M}^\infty(\operatorname{fr} G; \overline{G})$. Let $P : \Gamma \to \operatorname{End}(\mathbb{R}^n)$ be the smooth map which associates to every $x \in \Gamma$ the orthogonal projection P_x onto the tangent space $T_x\Gamma$. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and set for $x \in \Gamma$ and $1 \leq i \leq k$, $f_i(x) = P_x e_i$. Then the maps $f_i : \Gamma \to \mathbb{R}^n$ are subanalytic; this follows for example by the proof of (II.1.5), p. 99 in Shiota [46]. Hence, by the assumptions on Γ and Łojasiewicz's inequality, the following estimate holds for appropriate $C > 0$ and $\lambda \geq 1$:

$$C\, d(x, \operatorname{fr}\Gamma)^\lambda < \|f_i(x)\| \leq 1 \quad \text{for all } x \in \Gamma. \tag{8.1}$$

Next let $A : \Gamma \to \mathbb{R}^{k \times k}$ be the matrix valued function such that

$$A_{ij}(x) = \langle e_i, f_j(x) \rangle \quad \text{for } 1 \leq i, j \leq k \text{ and } x \in \Gamma.$$

Then, A is subanalytic and, since $\pi^n_{k\,|\Gamma}$ is an analytic isomorphism, fulfills $0 < \det A(x) \leq 1$. Let A^{-1} be the matrix-valued function such that $A^{-1}(x)A(x) = A(x)A^{-1}(x) = 1$. By Cramer's rule, the estimate (8.1) and Łojasiewicz's inequality for $\det A(x)$ entail that there are constants $C > 0$ and $\lambda > 0$ such that

$$0 < \|A^{-1}(x)\| \leq C \, \frac{1}{d(x, \operatorname{fr}\Gamma)^\lambda} \quad \text{for all } x \in \Gamma. \tag{8.2}$$

With the help of the equality $\pi^n_k f_j(x) = \sum_{l=1}^{k} A_{lj} e_l$ one then computes for $x \in \Gamma$ and $y = \pi(x)$

$$\begin{aligned} T_y\varrho(e_i) &= \sum_{j=1}^{k} A^{-1}_{ji}(x)\, T_y\varrho(\pi^n_k f_i(x)) = \sum_{j=1}^{k} A^{-1}_{ji}(x)\, T_x(\varrho \circ \pi^n_k)(f_i(x)) \\ &= \sum_{j=1}^{k} A^{-1}_{ji}(x)\, f_i(x). \end{aligned}$$

By (8.2) and since $\pi^n_{k\,|\overline{\Gamma}}$ is Hölder according to Lemma 7.5, one can find constants $C_1 > 0$ and $\lambda_1 > 0$ such that

$$\|T_y\varrho\| \leq C \, \frac{1}{d(\varrho(y), \operatorname{fr}\Gamma)^\lambda} \leq C_1 \, \frac{1}{d(y, \operatorname{fr} G)^{\lambda_1}} \quad \text{for all } y \in G. \tag{8.3}$$

Let us now consider the bounded subanalytic leaf $\Gamma_1 := T\Gamma \cap (\Gamma \times B) \subset \mathbb{R}^{2n}$, where $B \subset \mathbb{R}^n$ is the open unit ball. The restricted projection $\pi^{2n}_{2k\,|\Gamma_1} : \Gamma_1 \to \mathbb{R}^{2k}$ then is an analytic isomorphism onto its image G_1, and G_1 is open and subanalytic in $\mathbb{R}^{2k}$. Note also, that $\operatorname{fr} G_1$ and $\operatorname{fr} G \times \{0\}$ are both subanalytic subsets of $\mathbb{R}^{2k}$, hence regularly situated. From this and the arguments above one concludes that there are $C_2 > 0$ and $\lambda_2 > 0$ such that

$$|\partial^\alpha_y \varrho(y)| \leq C_2 \, \frac{1}{d(y, \operatorname{fr} G)^{\lambda_2}} \quad \text{for all } y \in G \text{ and } \alpha \in \mathbb{R}^{2k} \text{ with } |\alpha| \leq 2.$$

Iteratively one thus obtains for every $r \in \mathbb{N}$ constants $C_r > 0$ and $\lambda_r > 0$ such that

$$|\partial^\alpha_y \varrho(y)| \leq C_r \, \frac{1}{d(y, \operatorname{fr} G)^{\lambda_r}} \quad \text{for all } y \in G \text{ and } \alpha \in \mathbb{R}^{2k} \text{ with } |\alpha| \leq r. \tag{8.4}$$

This proves that $\varrho \in \mathcal{M}^\infty(\operatorname{fr} G; \overline{G})$.

By definition, ϱ satisfies $\varrho(G) \subset \Gamma \subset \mathbb{R}^n \setminus \operatorname{fr}\Gamma$. Moreover, since $\pi^n_{k\,|\overline{\Gamma}}$ is subanalytic with compact graph, the map ϱ has to be Łojasiewicz and Hölder with respect to the pair $(\operatorname{fr} G, \operatorname{fr}\Gamma)$. Hence ϱ is meromorphic from $(\overline{G}, \operatorname{fr} G)$ to $(\mathbb{R}^n, \operatorname{fr}\Gamma)$. □

The following result is a straightforward consequence of the lemma.

8.3. Corollary. *There exists a bimeromorphic subanalytic tubular neighborhood* (E,ε,φ) *of* Γ *in* $\mathbb{R}^n$ *and a bimeromorphic subanalytic parallel transport* $//$ *for* (E,ε,φ). *More precisely, the triple* (E,ε,φ) *with*

$$\begin{aligned} E &= \{(x,v) \in \Gamma \times \mathbb{R}^n \mid \pi^n_k(v) = 0\}, \\ \varepsilon(x) &= \frac{1}{4} d(x, \operatorname{fr}\Gamma) \quad \textit{for all } x \in \Gamma, \textit{ and} \\ \varphi(x,v) &= x + v \quad \textit{for all } (x,v) \in E \end{aligned} \tag{8.5}$$

has these properties and satisfies

$$\pi_\Gamma(x) = \varrho(\pi^n_k(x)) \quad \textit{and} \quad \varphi^{-1}(x) = (\pi_\Gamma(x), x - \pi_\Gamma(x)) \quad \textit{for all } x \in U_{2\varepsilon}.$$

The parallel transport $//$ *is defined over*

$$N = \left\{(x,y) \in \Gamma \times \Gamma \mid d(x,y) < \frac{1}{2} \min\{d(x,\operatorname{fr}\Gamma), d(y, \operatorname{fr}\Gamma)\}\right\}$$

and is given by $//_{yx} v = v$ *for all* $(y,x) \in N$ *and all* $v \in \mathbb{R}^n$ *with* $\pi^n_k(v) = 0$.

8.4. Lemma. *Let* $\Gamma \subset \mathbb{R}^n$ *be bounded and* (E,ε,φ) *a bimeromorphic subanalytic tubular neighborhood of* Γ. *Then, after possibly passing to a smaller subanalytic* ε *there exists a continuous homotopy* $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ *with the following properties*:

(1) $H(x,t) = \varphi(t\varphi^{-1}(x))$ *for all* $x \in U_\varepsilon$ *and* $t \in [0,1]$,

(2) $H(x,1) = x$ *for all* $x \in U$, *and*

(3) H *is meromorphic from* $(\mathbb{R}^n \times [0,1], \operatorname{fr}\Gamma \times [0,1])$ *to* $(\mathbb{R}^n, \operatorname{fr}\Gamma)$.

Proof. Consider the subanalytic map $\overline{E_\varepsilon} \ni (x,v) \mapsto d(\varphi(x,v),x)$. Since $d(\varphi(x,0),x) = 0$ for all $x \in \overline{\Gamma}$, one can find by Łojasiewicz's inequality a constant $C > 0$ and a rational number $\lambda > 0$ such that

$$d(\varphi(x,v),x) \leq C\,|v|^\lambda \quad \text{for all } (x,v) \in \overline{E_\varepsilon}. \tag{8.6}$$

Let $\varepsilon' : \Gamma \to \mathbb{R}_{>0}$ be the subanalytic function $x \mapsto \min\left\{\frac{1}{2}\varepsilon(x), \frac{1}{8}(\frac{1}{C}\, d(x,\operatorname{fr}\Gamma))^{1/\lambda}\right\}$. Then

$$d(\varphi(x,v),x) \leq \frac{1}{4}\, d(x,\operatorname{fr}\Gamma) \quad \text{for all } (x,v) \in \overline{E_{2\varepsilon'}}. \tag{8.7}$$

This entails that for every $t \in [0,1]$

$$d(\varphi(x,tv),\operatorname{fr}\Gamma) \geq d(x,\operatorname{fr}\Gamma) - d(\varphi(x,tv),x) \geq \frac{3}{4}\, d(x,\operatorname{fr}\Gamma) \tag{8.8}$$

and

$$d(\varphi(x,v),\varphi(x,tv)) \leq \frac{1}{2}\, d(x,\operatorname{fr}\Gamma). \tag{8.9}$$

Consequently

$$(8.10)\quad s\varphi(x,v)+(1-s)\varphi(x,tv)\notin \operatorname{fr}\Gamma \quad \text{for all } (x,v)\in \overline{E_{2\varepsilon'}} \text{ and } s,t\in[0,1].$$

Now, the sets $\overline{U_{\varepsilon'}}$ and $\mathbb{R}^n \setminus U_{2\varepsilon'}$ are subanalytic, hence regularly situated. Thus, by Chap. IV, Lemma 4.5 of [50] there exists a function $\psi \in \mathcal{M}^\infty(\operatorname{fr}\Gamma;\mathbb{R}^n)$ with values in $[0,1]$ such that $\psi = 1$ over $\overline{U_{\varepsilon'}} \setminus \operatorname{fr}\Gamma$ and $\psi = 0$ over $\mathbb{R}^n \setminus \overline{U_{2\varepsilon'}}$. With the help of the function ψ we define:

$$H(x,t)=\begin{cases} x & \text{for } x\in \operatorname{fr}\Gamma,\\ \psi(x)\,\varphi(t\varphi^{-1}(x))+(1-\psi(x))\,x & \text{for } x\in\mathbb{R}^n\setminus \operatorname{fr}\Gamma. \end{cases}$$

By construction we then have $H \in \mathcal{M}^\infty(\operatorname{fr}\Gamma \times [0,1], \mathbb{R}^n \times [0,1])$, $H(\operatorname{fr}\Gamma \times [0,1]) \subset \operatorname{fr}\Gamma$ and $H(x,1) = x$ for all $x \in \mathbb{R}^n$. Moreover, (8.10) entails that $H(\mathbb{R}^n \times [0,1] \setminus \operatorname{fr} G) \subset \mathbb{R}^n \setminus \operatorname{fr}\Gamma$. Finally, using the fact that φ is subanalytic, one concludes that H is a meromorphic map from $(\mathbb{R}^n \times [0,1], \operatorname{fr}\Gamma \times [0,1])$ to $(\mathbb{R}^n, \operatorname{fr}\Gamma)$. Hence H satisfies the claim. □

8.5. Let us consider the following situation which will appear in the construction of a bimeromorphic triangulation. Let $\Delta \subset \mathbb{R}^n$ be an open affine simplex of dimension n and $h : \overline{\Delta} \to \mathbb{R}^N$ a subanalytic homeomorphism onto its image with the following properties:

(1) $h_{|\Delta} : \Delta \to \widetilde{\Delta} = h(\Delta)$ is an analytic isomorphism onto a subanalytic leaf of $\mathbb{R}^N$,

(2) h is bimeromorphic from $(\overline{\Delta}, \operatorname{fr}\Delta)$ to $(\overline{\widetilde{\Delta}}, \operatorname{fr}\widetilde{\Delta})$.

Additionally we assume that $\widetilde{\Delta}$ has a bimeromorphic subanalytic tubular neighborhood $(E_{\widetilde{\Delta}}, \varepsilon_{\widetilde{\Delta}}, \varphi_{\widetilde{\Delta}})$ in $\mathbb{R}^N$ with a bimeromorphic subanalytic parallel transport $//_{\widetilde{\Delta}}$. Now let $\Lambda \subset \mathbb{R}^n$ be an affine simplex of dimension $< n$ with total space lying in Δ. Clearly, Λ has a natural bimeromorphic subanalytic tubular neighborhood $(E_\Lambda, \varepsilon_\Lambda, \varphi_\Lambda)$ in $\mathbb{R}^n$ given by orthogonal projection. After possibly shrinking ε_Λ we can assume that $U_{2\varepsilon_\Lambda}$ lies in Δ as well. The image of Λ under the map h is a subanalytic leaf Γ in $\mathbb{R}^N$. Moreover, Γ has a tubular neighborhood $(E, \varepsilon, \varphi)$ induced by $(E_{\widetilde{\Delta}}, \varepsilon_{\widetilde{\Delta}}, \varphi_{\widetilde{\Delta}})$, $(E_\Lambda, \varepsilon_\Lambda, \varphi_\Lambda)$ and h. Its components are given as follows:

(1) $E = E_{\widetilde{\Delta}|\Gamma} \oplus h_* E_\Lambda$,

(2) $\varepsilon : \Gamma \to \mathbb{R}_{>0}$ is a subanalytic map such that for all $(x,v,w) \in E_{2\varepsilon}$

$$\begin{aligned} &|T_x h^{-1}(w)| < \varepsilon_\Lambda\big(h^{-1}(x)\big),\\ &(y,x)\in N_{\widetilde{\Delta}} \quad \text{with } y = h(\varphi_\Lambda(T_x h^{-1}(w))),\\ &|//_{\widetilde{\Delta},yx} v| < \varepsilon_{\widetilde{\Delta}}(y), \end{aligned}$$

(3) $\varphi(x,v,w) = \varphi_{\widetilde{\Delta}}(y, //_{\widetilde{\Delta},yx} v)$ for all $(x,v,w) \in E_{2\varepsilon}$ and $y = h(\varphi_\Lambda(T_x h^{-1}(w)))$.

Finally, if $N = N_{\widetilde{\Delta}} \cap (\Gamma \times \Gamma)$, then

$$\begin{aligned} & //_{yx}(x,v,w) \\ & \quad = \big(y, //_{\widetilde{\Delta},yx} v, T_{h^{-1}(y)} h(T_x h^{-1}(w))\big) \text{ with } (y,x) \in N \text{ and } (x,v,w) \in E_x, \end{aligned}$$

defines a parallel transport for $(E, \varepsilon, \varphi)$ over N. By a tedious but straightforward argument involving Proposition 7.6 and well-known properties of subanalytic maps one derives the following result.

8.6. LEMMA. *The tubular neighborhood $(E, \varepsilon, \varphi)$ of Γ induced by*

$$(E_{\widetilde{\Delta}}, \varepsilon_{\widetilde{\Delta}}, \varphi_{\widetilde{\Delta}}), (E_\Lambda, \varepsilon_\Lambda, \varphi_\Lambda)$$

and h is bimeromorphic and subanalytic. Moreover, the parallel transport $//$ is bimeromorphic subanalytic.

8.7. For the construction of a bimeromorphic triangulation we repeat essentially the argument of Łojasiewicz [35, §2] (cf. also Hironaka [27]), which shows that a bounded subanalytic set in $\mathbb{R}^n$ has a subanalytic triangulation, and we add the necessary details which prove that the triangulation can be chosen to be bimeromorphic. One of the key ingredients in the proof is the following subanalytic version of a lemma due to Koopman-Brown [30] (see also [34] and [27]).

8.8. LEMMA. *Let $X \subset \mathbb{R}^n$ be a bounded and nowhere dense subanalytic set. Then there exists an open and dense subset $O \subset \mathbb{P}^{n-1}$ such that for every direction $\xi \in O$ and every $v \in \mathbb{R}^n$ the intersection $X \cap (v + \xi)$ has finitely many points.*

As in [35, §2] we say that a subset $X \subset \mathbb{R}^n$ has property (f), if every one of the fibers $\pi^{-1}(y)$ with $y \in \mathbb{R}^{n-1}$ has only finitely many elements. The following two results correspond to Lemma 2 and Lemma 3 in [35, §2].

8.9. LEMMA. *Let $X \subset \mathbb{R}^n$ be a compact subanalytic set having property* (f). *Then there exists a closed subanalytic set $Y \supset X$ with property* (f) *such that the projection $\pi_{|Y} : Y \to \mathbb{R}^{n-1}$ is open.*

8.10. LEMMA. *Let $X \subset \mathbb{R}^n$ be a bounded subanalytic set having property* (f). *Then there exists a partition of X into finitely many subanalytic leaves Γ such that every image $\pi(\Gamma)$ is a subanalytic leaf of $\mathbb{R}^{n-1}$ and such that the restriction $\pi_{|\Gamma} : \Gamma \to \pi(\Gamma)$ is an analytic isomorphism.*

8.11. *Remark.* Concerning the notation of projections restricted to a leaf Γ we depart somewhat from the presentation in [35]. In our work, the symbol π_Γ always denotes the projection of a tubular neighborhood of Γ, and not as in [35], the restriction of the projection $\pi = \pi^n_{(n-1)}$ to Γ.

8.12. THEOREM. *For every family $X_1, \ldots, X_l \subset \mathbb{R}^n$ of bounded subanalytic sets there exists a compact parallelotope $Q \subset \mathbb{R}^n$ containing the X_j in its interior and a finite bimeromorphic subanalytic triangulation $T = (h : Q \to Q, \mathcal{K})$ which is compatible with the X_j.*

Proof. The claim is proved by induction on the dimension n. For $n = 1$ it is trivial, since every bounded subanalytic set in $\mathbb{R}$ is the union of finitely many points and bounded open intervals. Assume that the claim holds for all $\mathbb{R}^k$ with $k < n$ and that for finitely many bounded subanalytic sets in $\mathbb{R}^k$ one can always find a compatible bimeromorphic subanalytic triangulation of a large enough parallelotope such that the tubular neighborhoods of the simplices of the triangulation all have a bimeromorphic subanalytic parallel transport. It will be shown that then one can also find a bimeromorphic subanalytic triangulation with this property for $X_1, \ldots, X_l \subset \mathbb{R}^n$. As in [35, §2] one argues that it suffices to assume that the X_j are pairwise disjoint and nowhere dense in $\mathbb{R}^n$. By Lemma 8.8 the compact subanalytic set $\bigcup_j \overline{X_j}$ has property (f) after an appropriate coordinate transformation. Moreover, by Lemma 8.9 there exists a closed subanalytic $F_0 \supset \bigcup_j \overline{X_j}$ such that F_0 has property (f) and such that the restricted projection $\pi_{|F_0} : F_0 \to \mathbb{R}^{n-1}$ is open. Following Łojasiewicz [35] let us put $u = (x_{\underline{1}}, \ldots, x_{\underline{n-1}})$ and choose $C > R > 0$ in such a way that the truncated cone $D = \{x \in \mathbb{R}^n \mid |u| < C - |x_{\underline{n}}|, |x_{\underline{n}}| < R\}$ contains $\bigcup_j \overline{X_j}$. Let

$$\Gamma_1 = \{x \in \mathbb{R}^n \mid x_{\underline{n}} = R\}, \quad \Gamma_2 = -Y_1 \quad \Gamma_3 = \{x \in \mathbb{R}^n \mid |u| > C,\ x_{\underline{n}} = 0\} \quad \text{and}$$
$$Y = \{x \in \mathbb{R}^n \mid |u| = C - |x_{\underline{n}}|,\ |x_{\underline{n}}| < R\}.$$

Then $F = (F_0 \cap D) \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup Y$ is a closed subanalytic set, has property (f) and the projection $\pi_{|F} : F \to \mathbb{R}^{n-1}$ is open. By Lemmas 8.8 and 8.10 one can now find (after an appropriate orthogonal transformation of the coordinates $x_{\underline{1}}, \ldots, x_{\underline{n-1}}$) a partition of each one of the sets $X_1, \ldots, X_l$, Y and $(F_0 \cap D) \setminus \bigcup_j X_j$ into finitely many subanalytic leaves Γ such that every one of the projections $\pi^n_{k|\Gamma}$ with $k = \dim \Gamma$ is an analytic isomorphism onto an open subanalytic set of $\mathbb{R}^k$. Let $\Gamma_4, \Gamma_5, \ldots, \Gamma_r$ be all these leaves. Thus one obtains a finite partition $F = \bigcup_{i=1}^r \Gamma_i$ which is compatible with the X_j. Note that by Corollary 8.3 every leaf Γ_i has a bimeromorphic subanalytic tubular neighborhood in $\mathbb{R}^n$ together with a bimeromorphic subanalytic parallel transport.

According to the induction hypothesis one can choose a sufficiently large parallelotope $\widetilde{Q} \subset \mathbb{R}^{n-1}$ containing the ball of radius $2C$ and a bimeromorphic subanalytic triangulation $\widetilde{T} = (\widetilde{h} : \widetilde{Q} \to \widetilde{Q}, \widetilde{\mathcal{K}})$ which is compatible with the sets $\pi(\Gamma_4), \pi(\Gamma_5), \ldots, \pi(\Gamma_r)$. Obviously, the triangulation $\widetilde{T}$ then is compatible with all the leaves $\pi(\Gamma_i)$, $i = 1, \ldots, r$. The Γ_i are graphs of analytic functions (denoted by the same symbol); hence for every simplex Λ of $\widetilde{T}$ the non-empty restrictions $\Gamma_{i|\Lambda}$ are pairwise disjoint and give rise to a finite and strictly increasing sequence of analytic functions

$$-R = \Psi_{\Lambda,i} < \ldots < \Psi_{\Lambda,r_\Lambda} = R \quad \text{over } \Lambda.$$

Each of the functions $\Psi_{\Lambda,i}$ is subanalytic, and the Whitney function $\mathsf{J}^\infty_\Lambda(\Psi_{\Lambda,i} \circ \pi_\Lambda)$ lies in $\mathcal{M}^\infty(\overline{\Lambda}, \operatorname{fr}\Lambda)$ by Corollary 8.3, the assumption on $\widetilde{T}$ and Proposition 7.6. Since F is closed, the closure $\overline{\Psi_{\Lambda,i}}$ is the graph of a subanalytic function. Moreover, given a face Λ' of Λ, the restriction of $\overline{\Psi_{\Lambda,i}}$ to Λ' equals one of the functions $\Psi_{\Lambda',j}$. Vice versa, every $\Psi_{\Lambda',j}$ is the restriction of some $\overline{\Psi_{\Lambda,i}}$, since $\pi_{|F}$ is open. After replacing $\widetilde{\mathcal{K}}$ by its barycentric subdivision, one has $\overline{\Psi_{\Lambda,i}}(v) < \overline{\Psi_{\Lambda,i+1}}(v)$ for at least one vertex v of Λ. Thus the sets

$$\Psi_{\Lambda,i} \quad \text{and} \quad \{x \in \mathbb{R}^n \mid u \in \Lambda,\ \Psi_{\Lambda,i}(u) < x_{\underline{n}} < \Psi_{\Lambda,i+1}(u)\}$$

form a subanalytic stratification $\mathcal{N}$ of the parallelotope $Q = \widetilde{Q} \times [-R, R]$, and this stratification is compatible with the X_j (see [44, Chap. 1] for details on stratification theory).

Łojasiewicz [34] then uses the subanalytic homeomorphism $\widetilde{h}^{-1} \times e : Q \to Q$, where e is the identity on $[-R, R]$, to map $\mathcal{N}$ to a stratification $\mathcal{M}$ of Q into the following subanalytic leaves:

$$\Phi_{\Delta,i} = \Psi_{\widetilde{h}(\Delta),i} \circ \widetilde{h}_{|\Delta} \quad \text{and} \quad \{x \in Q \mid u \in \Delta,\ \Phi_{\Delta,i}(u) < x_{\underline{n}} < \Phi_{\Delta,i+1}(u)\},$$

where $\Delta \in \widetilde{K}$. The stratification $\mathcal{M}$ inherits the following properties from $\mathcal{N}$:

(1) for every face Δ' of Δ the map $\Phi_{\Delta',j}$ is the restriction of some $\overline{\Phi_{\Delta,i}}$ to Δ',

(2) $\overline{\Phi_{\Delta,i}}(v) < \overline{\Phi_{\Delta,i+1}}(v)$ for at least one vertex v of Δ,

(3) $\mathsf{J}^\infty_\Delta(\Phi_{\Delta,i} \circ \pi_\Delta) \in \mathcal{M}^\infty(\overline{\Delta}, \operatorname{fr}\Delta)$.

By the assumption on $\widetilde{T}$ it is clear that the restriction of $\widetilde{h} \times e$ to a leaf Γ of $\mathcal{M}$ is an analytic isomorphism onto a leaf $\widetilde{\Gamma}$ of $\mathcal{N}$. Moreover, by (3) and also since for every Δ the Whitney function $\mathsf{J}^\infty_\Delta(\widetilde{h} \circ \pi_{|\Delta})$ is bimeromorphic from $(\overline{\Delta}, \operatorname{fr}\Delta)$ to $\big(\overline{\widetilde{h}(\Delta)}, \operatorname{fr}\widetilde{h}(\Delta)\big)$, one concludes that $(\widetilde{h} \times e) \circ \pi_{|\Gamma}$ is bimeromorphic from $(\overline{\Gamma}, \operatorname{fr}\Gamma)$ to $(\overline{\widetilde{\Gamma}}, \operatorname{fr}\widetilde{\Gamma})$. Hereby, for $\Gamma = \Phi_{\Delta,i}$, the projection π_Γ is given by $\pi_\Gamma(u, x_{\underline{n}}) = \Phi_{\Delta,i}(\pi_\Delta(u))$ with $(u, x_{\underline{n}})$ in a sufficiently small subanalytic neighborhood of Γ.

Next denote by $\Xi_{\Delta,i} : \Delta \to \mathbb{R}$ the subanalytic map which coincides with the restriction of the affine function having value $\overline{\Phi_{\Delta,i}}(v)$ at a vertex $v \in \overline{\Delta}$. Then (2) shows

(4) $\Xi_{\Delta,i} < \Xi_{\Delta,i+1}$ over Δ.

Hence the subanalytic leaves

$$\Xi_{\Delta,i} \quad \text{and} \quad \{x \in Q \mid u \in \Delta,\ \Xi_{\Delta,i}(u) < x_{\underline{n}} < \Xi_{\Delta,i+1}(u)\}$$

form a stratification of Q and a finite cellular complex $\mathcal{L}$ in $\mathbb{R}^n$. By virtue of properties (1), (2) and (4) above Łojasiewicz [34] obtains a subanalytic homeomorphism

$$h^* : Q \to Q, \quad (u,t) \mapsto (u, a(u,t)),$$

where for every $u \in \Delta$ the map $[-R,R] \ni t \mapsto a(u,t) \in [-R,R]$ is defined over each interval $[\Xi_{\Delta,i}(u), \Xi_{\Delta,i+1}(u)]$ as the affine function such that $a(\Xi_{\Delta,i}(u)) = \Phi_{\Delta,i}(u)$. According to [34, §2] the image of $\mathcal{L}$ under h^* coincides with the stratification $\mathcal{M}$ and the restriction of h^* to a cell Γ of $\mathcal{L}$ is an analytic isomorphism onto its image $\widetilde{\Gamma}$. Additionally, h^* is bimeromorphic from $(\overline{\Gamma}, \operatorname{fr}\Gamma)$ to $(\overline{\widetilde{\Gamma}}, \operatorname{fr}\widetilde{\Gamma})$ by construction and (3). If now $\mathcal{K}$ denotes the barycentric subdivision of $\mathcal{L}$ and h the composition $(\widetilde{h} \times e) \circ h^*$, then $T = (h, \mathcal{K})$ is a subanalytic triangulation of Q which is compatible with the X_i.

It remains to prove that the simplices of T have bimeromorphic tubular neighborhoods with bimeromorphic parallel transport and that T satisfies the axioms (BMT1) to (BMT4). To this end check first (using Proposition 7.6 and the corresponding property of $\widetilde{h} \times e$ and h^*) that for every $\Delta \in \mathcal{K}$ the Whitney function $\mathsf{J}^\infty_{\overline{\Delta}}(h \circ \pi_{|\Delta})$ is bimeromorphic from $(\overline{\Delta}, \operatorname{fr}\Delta)$ to $(\overline{\widetilde{\Delta}}, \operatorname{fr}\widetilde{\Delta})$, where $\widetilde{\Delta} = h(\Delta)$. Then recall that the leaves Γ_i (even Γ_1, Γ_2 and Γ_3 after passing to the intersection with a sufficiently large open ball) satisfy the hypothesis of Lemma 8.2. Hence Corollary 8.3 and Lemma 8.6 entail the fact that every simplex of T possesses a bimeromorphic subanalytic tubular neighborhood together with a bimeromorphic subanalytic parallel transport. By passing to possibly smaller functions $\varepsilon_{\widetilde{\Delta}}$ one can achieve that (BMT1) is fulfilled. (BMT4) is an immediate consequence of Lemma 8.4. The remaining two axioms are consequences of the following facts:

- $\mathsf{J}^\infty_{\overline{\Delta}}(h \circ \pi_{|\Delta})$ is bimeromorphic,
- the projections π_Δ and $\pi_{\widetilde{\Delta}}$ of the tubular neighborhoods of Δ resp. $\widetilde{\Delta}$ are meromorphic,
- $\mathbb{R}^n \setminus U_{\varepsilon_\Delta}$ and $\overline{\Delta}$ are regularly situated,
- $\mathbb{R}^n \setminus U_{\varepsilon_{\widetilde{\Delta}}}$ and $\overline{\widetilde{\Delta}}$ are regularly situated.

The details of the argument leading to (BMT2) and (BMT3) are similar to the proof of Lemma 8.4, and so we leave them to the reader. This finishes the proof of the bimeromorphic triangulation theorem. $\square$

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(Received March 17, 2002)