On Mott's formula for the ac-conductivity in the Anderson model

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Abstract

We study the ac-conductivity in linear response theory in the general framework of ergodic magnetic Schrödinger operators. For the Anderson model, if the Fermi energy lies in the localization regime, we prove that the ac-conductivity is bounded from above by $C\nu^2(\log \frac{1}{\nu})^{d+2}$ at small frequencies ν. This is to be compared to Mott's formula, which predicts the leading term to be $C\nu^2(\log \frac{1}{\nu})^{d+1}$.

1. Introduction

The occurrence of localized electronic states in disordered systems was first noted by Anderson in 1958 [An], who argued that for a simple Schrödinger operator in a disordered medium, "at sufficiently low densities transport does not take place; the exact wave functions are localized in a small region of space." This phenomenon was then studied by Mott, who wrote in 1968 [Mo1]: "The idea that one can have a continuous range of energy values, in which all the wave functions are localized, is surprising and does not seem to have gained universal acceptance." This led Mott to examine Anderson's result in terms of the Kubo–Greenwood formula for $\sigma_{E_F}(\nu)$, the electrical alternating current (ac) conductivity at Fermi energy E_F and zero temperature, with ν being the frequency. Mott used its value at $\nu = 0$ to reformulate localization: If a range of values of the Fermi energy E_F exists in which $\sigma_{E_F}(0) = 0$, the states with these energies are said to be localized; if $\sigma_{E_F}(0) \neq 0$, the states are nonlocalized.

Mott then argued that the direct current (dc) conductivity $\sigma_{E_F}(0)$ indeed vanishes in the localized regime. In the context of Anderson's model, he studied the behavior of $\operatorname{Re}\sigma_{E_F}(\nu)$ as $\nu \to 0$ at Fermi energies E_F in the localization region (note $\operatorname{Im}\sigma_{E_F}(0) = 0$). The result was the well-known *Mott's formula* for the ac-conductivity at zero temperature [Mo1], [Mo2], which we state as in

[MoD, Eq. (2.25)] and [LGP, Eq. (4.25)]:

$$(1.1) \qquad \operatorname{Re} \sigma_{E_F}(\nu) \sim n(E_F)^2 \, \tilde{\ell}_{E_F}^{d+2} \, \nu^2 \left(\log \tfrac{1}{\nu}\right)^{d+1} \quad \text{as} \quad \nu \downarrow 0,$$

where d is the space dimension, $n(E_F)$ is the density of states at energy E_F, and $\tilde{\ell}_{E_F}$ is a localization length at energy E_F.

Mott's calculation was based on a fundamental assumption: the leading mechanism for the ac-conductivity in localized systems is the resonant tunneling between pairs of localized states near the Fermi energy E_F, the transition from a state of energy $E \in \,]E_F - \nu, E_F]$ to another state with resonant energy $E + \nu$, the energy for the transition being provided by the electrical field. Mott also argued that the two resonating states must be located at a spatial distance of $\sim \log \tfrac{1}{\nu}$. Kirsch, Lenoble and Pastur [KLP] have recently provided a careful heuristic derivation of Mott's formula along these lines, incorporating also ideas of Lifshitz [L].

In this article we give the first mathematically rigorous treatment of Mott's formula. The general nature of Mott's arguments leads to the belief in physics that Mott's formula (1.1) describes the generic behavior of the low-frequency conductivity in the localized regime, irrespective of model details. Thus we study it in the most popular model for electronic properties in disordered systems, the Anderson tight-binding model [An] (see (2.1)), where we prove a result of the form

$$(1.2) \qquad \operatorname{Re} \overline{\sigma}_{E_F}(\nu) \leqslant c \, \tilde{\ell}_{E_F}^{d+2} \, \nu^2 \left(\log \tfrac{1}{\nu}\right)^{d+2} \quad \text{for small } \nu > 0.$$

The precise result is stated in Theorem 2.3; formally

$$(1.3) \qquad \operatorname{Re} \overline{\sigma}_{E_F}(\nu) = \frac{1}{\nu} \int_0^\nu \mathrm{d}\nu' \, \operatorname{Re} \sigma_{E_F}(\nu'),$$

so that $\operatorname{Re} \overline{\sigma}_{E_F}(\nu) \approx \operatorname{Re} \sigma_{E_F}(\nu)$ for small $\nu > 0$. The discrepancy in the exponents of $\log \tfrac{1}{\nu}$ in (1.2) and (1.1), namely $d+2$ instead of $d+1$, is discussed in Remarks 2.5 and 4.10.

We believe that a result similar to Theorem 2.3 holds for the continuous Anderson Hamiltonian, which is a random Schrödinger operator on the continuum with an alloy-type potential. All steps in our proof of Theorem 2.3 can be redone for such a continuum model, except the finite volume estimate of Lemma 4.9. The missing ingredient is Minami's estimate [M], which we recall in (4.47). It is not yet available for that continuum model. In fact, proving a continuum analogue of Minami's estimate would not only yield Theorem 2.3 for the continuous Anderson Hamiltonian, but it would also establish, in the localization region, simplicity of eigenvalues as in [KlM] and Poisson statistics for eigenvalue spacing as in [M].

To get to Mott's formula, we conduct what seems to be the first careful mathematical analysis of the ac-conductivity in linear response theory, and introduce a new concept, the conductivity measure. This is done in the general

framework of ergodic magnetic Schrödinger operators, in both the discrete and continuum settings. We give a controlled derivation in linear response theory of a Kubo formula for the ac-conductivity along the lines of the derivation for the dc-conductivity given in [BoGKS]. This Kubo formula (see Corollary 3.5) is written in terms of $\Sigma_{E_F}(\mathrm{d}\nu)$, the conductivity measure at Fermi energy E_F (see Definition 3.3 and Theorem 3.4). If $\Sigma_{E_F}(\mathrm{d}\nu)$ was known to be an absolutely continuous measure, $\operatorname{Re}\sigma_{E_F}(\nu)$ would then be well-defined as its density. The conductivity measure $\Sigma_{E_F}(\mathrm{d}\nu)$ is thus an analogous concept to the density of states measure $\mathcal{N}(\mathrm{d}E)$, whose formal density is the density of states $n(E)$. The conductivity measure has also an expression in terms of the velocity-velocity correlation measure (see Proposition 3.10).

The first mathematical proof of localization [GoMP] appeared almost twenty years after Anderson's seminal paper [An]. This first mathematical treatment of Mott's formula is appearing about thirty seven years after its formulation [Mo1]. It relies on some highly nontrivial research on random Schrödinger operators conducted during the last thirty years, using a good amount of what is known about the Anderson model and localization. The first ingredient is linear response theory for ergodic Schrödinger operators with Fermi energies in the localized region [BoGKS], from which we obtain an expression for the conductivity measure. To estimate the low frequency ac-conductivity, we restrict the relevant quantities to finite volume and estimate the error. The key ingredients here are the Helffer–Sjöstrand formula for smooth functions of self-adjoint operators [HS] and the exponential estimates given by the fractional moment method in the localized region [AM], [A], [ASFH]. The error committed in the passage from spectral projections to smooth functions is controlled by Wegner's estimate for the density of states [W]. The finite volume expression is then controlled by Minami's estimate [M], a crucial ingredient. Combining all these estimates, and choosing the size of the finite volume to optimize the final estimate, we get (1.2).

This paper is organized as follows. In Section 2 we introduce the Anderson model, define the region of complete localization, give a brief outline of how electrical conductivities are defined and calculated in linear response theory, and state our main result (Theorem 2.3). In Section 3, we give a detailed account of how electrical conductivities are defined and calculated in linear response theory, within the noninteracting particle approximation. This is done in the general framework of ergodic magnetic Schrödinger operators; we treat simultaneously the discrete and continuum settings. We introduce and study the conductivity measure (Definition 3.3), and derive a Kubo formula (Corollary 3.5). In Section 4 we give the proof of Theorem 2.3, reformulated as Theorem 4.1.

In this article $|B|$ denotes either Lebesgue measure if B is a Borel subset of $\mathbb{R}^n$, or the counting measure if $B \subset \mathbb{Z}^n$ $(n = 1, 2, \dots)$. We always use χ_B to

denote the characteristic function of the set B. By $C_{a,b,\dots}$, etc., we will always denote some finite constant depending only on $a, b, \dots$.

2. The Anderson model and the main result

The Anderson tight binding model is described by the random Schrödinger operator H, a measurable map $\omega \mapsto H_\omega$ from a probability space $(\Omega, \mathbb{P})$ (with expectation $\mathbb{E}$) to bounded self-adjoint operators on $\ell^2(\mathbb{Z}^d)$, given by

$$(2.1) \qquad\qquad H_\omega := -\Delta + V_\omega.$$

Here Δ is the centered discrete Laplacian,

$$(2.2) \qquad (\Delta\varphi)(x) := - \sum_{y \in \mathbb{Z}^d;\, |x-y|=1} \varphi(y) \quad \text{for} \quad \varphi \in \ell^2(\mathbb{Z}^d),$$

and the random potential V consists of independent identically distributed random variables $\{V(x); x \in \mathbb{Z}^d\}$ on $(\Omega, \mathbb{P})$, such that the common single site probability distribution μ has a bounded density ρ with compact support.

The *Anderson Hamiltonian H* given by (2.1) is $\mathbb{Z}^d$-ergodic, and hence its spectrum, as well as its spectral components in the Lebesgue decomposition, are given by nonrandom sets $\mathbb{P}$-almost surely [KM], [CL], [PF].

There is a wealth of localization results for the Anderson model in arbitrary dimension, based either on the multiscale analysis [FS], [FMSS], [Sp], [DK], or on the fractional moment method [AM], [A], [ASFH]. The spectral region of applicability of both methods turns out to be the same, and in fact it can be characterized by many equivalent conditions [GK1], [GK2]. For this reason we call it the *region of complete localization* as in [GK2]; the most convenient definition for our purposes is by the conclusions of the fractional moment method.

Definition 2.1. The *region of complete localization* Ξ^{CL} for the Anderson Hamiltonian H is the set of energies $E \in \mathbb{R}$ for which there are an open interval $I_E \ni E$ and an exponent $s = s_E \in {]0, 1[}$ such that

$$(2.3) \qquad \sup_{E' \in I_E} \sup_{\eta \neq 0} \mathbb{E}\big\{|\langle \delta_x, R(E' + \mathrm{i}\eta)\delta_y\rangle|^s\big\} \leqslant K\, \mathrm{e}^{-\frac{1}{\ell}|x-y|} \quad \text{for all } x, y \in \mathbb{Z}^d,$$

where $K = K_E$ and $\ell = \ell_E > 0$ are constants, and $R(z) := (H - z)^{-1}$ is the resolvent of H.

Remark 2.2.   (i)   The constant ℓ_E admits the interpretation of a localization length at energies near E.

(ii)   The fractional moment condition (2.3) is known to hold under various circumstances, for example, large disorder or extreme energies [AM], [A],

[ASFH]. Condition (2.3) implies spectral localization with exponentially decaying eigenfunctions [AM], dynamical localization [A], [ASFH], exponential decay of the Fermi projection [AG], and absence of level repulsion [M].

(iii) The single site potential density ρ is assumed to be bounded with compact support, so condition (2.3) holds with any exponent $s \in \left]0, \frac{1}{4}\right[$ and appropriate constants $K(s)$ and $\ell(s) > 0$ at all energies where a multiscale analysis can be performed [ASFH]. Since the converse is also true, that is, given (2.3) one can perform a multiscale analysis as in [DK] at the energy E, the energy region Ξ^{CL} given in Definition 2.1 is the same region of complete localization defined in [GK2].

We briefly outline how electrical conductivities are defined and calculated in linear response theory following the approach adopted in [BoGKS]; a detailed account in the general framework of ergodic magnetic Schrödinger operators, in both the discrete and continuum settings, is given in Section 3.

Consider a system at zero temperature, modeled by the Anderson Hamiltonian H. At the reference time $t = -\infty$, the system is in equilibrium in the state given by the (random) Fermi projection $P_{E_F} := \chi_{]-\infty, E_F]}(H)$, where we assume that $E_F \in \Xi^{\mathrm{CL}}$; that is, the Fermi energy lies in the region of complete localization. A spatially homogeneous, time-dependent electric field $\mathbf{E}(t)$ is then introduced adiabatically: Starting at time $t = -\infty$, we switch on the electric field $\mathbf{E}_\eta(t) := \mathrm{e}^{\eta t}\mathbf{E}(t)$ with $\eta > 0$, and then let $\eta \to 0$. On account of isotropy we assume without restriction that the electric field is pointing in the x_1-direction: $\mathbf{E}(t) = \mathcal{E}(t)\widehat{x}_1$, where $\mathcal{E}(t)$ is the (real-valued) amplitude of the electric field, and $\widehat{x}_1$ is the unit vector in the x_1-direction. We assume that

$$(2.4) \qquad \mathcal{E}(t) = \int_{\mathbb{R}} \mathrm{d}\nu\, \mathrm{e}^{\mathrm{i}\nu t}\widehat{\mathcal{E}}(\nu), \quad \text{where } \widehat{\mathcal{E}} \in C_c(\mathbb{R}) \text{ and } \widehat{\mathcal{E}}(\nu) = \overline{\widehat{\mathcal{E}}(-\nu)}.$$

For each $\eta > 0$ this results in a time-dependent random Hamiltonian $H(\eta, t)$, written in an appropriately chosen gauge. The system is then described at time t by the density matrix $\varrho(\eta, t)$, given as the solution to the Liouville equation

$$(2.5) \qquad \begin{cases} \mathrm{i}\partial_t \varrho(\eta, t) = [H(\eta, t), \varrho(\eta, t)] \\ \lim_{t \to -\infty} \varrho(\eta, t) = P_{E_F} \end{cases}.$$

The adiabatic electric field generates a time-dependent electric current, which, thanks to reflection invariance in the other directions, is also oriented along the x_1-axis, and has amplitude

$$(2.6) \qquad J_\eta(t; E_F, \mathcal{E}) = -\mathcal{T}\left(\varrho(\eta, t)\dot{X}_1(t)\right),$$

where $\mathcal{T}$ stands for the trace per unit volume and $\dot{X}_1(t)$ is the first component of the velocity operator at time t in the Schrödinger picture (the time dependence coming from the particular gauge of the Hamiltonian). In Section 3 we

calculate the *linear response current*

$$(2.7) \qquad J_{\eta,\mathrm{lin}}(t; E_F, \mathcal{E}) := \frac{\mathrm{d}}{\mathrm{d}\alpha} \, J_\eta(t; E_F, \alpha\mathcal{E})\big|_{\alpha=0}.$$

The resulting Kubo formula may be written as

$$(2.8) \qquad J_{\eta,\mathrm{lin}}(t; E_F, \mathcal{E}) = \mathrm{e}^{\eta t} \int_{\mathbb{R}} \mathrm{d}\nu \, \mathrm{e}^{\mathrm{i}\nu t} \, \sigma_{E_F}(\eta, \nu) \, \widehat{\mathcal{E}}(\nu),$$

with the (regularized) *conductivity* $\sigma_{E_F}(\eta, \nu)$ given by

$$(2.9) \qquad \sigma_{E_F}(\eta, \nu) := -\tfrac{\mathrm{i}}{\pi} \int_{\mathbb{R}} \Sigma_{E_F}(\mathrm{d}\lambda) \, (\lambda + \nu - \mathrm{i}\eta)^{-1},$$

where Σ_{E_F} is a finite, positive, even Borel measure on $\mathbb{R}$, the *conductivity measure* at Fermi Energy E_F—see Definition 3.3 and Theorem 3.4.

It is customary to decompose $\sigma_{E_F}(\eta, \nu)$ into its real and imaginary parts:

$$(2.10) \qquad \sigma_{E_F}^{\mathrm{in}}(\eta, \nu) := \mathrm{Re}\, \sigma_{E_F}(\eta, \nu) \quad \text{and} \quad \sigma_{E_F}^{\mathrm{out}}(\eta, \nu) := \mathrm{Im}\, \sigma_{E_F}(\eta, \nu),$$

the *in phase* or *active* conductivity $\sigma_{E_F}^{\mathrm{in}}(\eta, \nu)$ being an even function of ν, and the *out of phase* or *passive* conductivity $\sigma_{E_F}^{\mathrm{out}}(\eta, \nu)$ an odd function of ν. This induces a decomposition $J_{\eta,\mathrm{lin}} = J_{\eta,\mathrm{lin}}^{\mathrm{in}} + J_{\eta,\mathrm{lin}}^{\mathrm{out}}$ of the linear response current into an *in phase* or *active* contribution

$$(2.11) \qquad J_{\eta,\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E}) := \mathrm{e}^{\eta t} \int_{\mathbb{R}} \mathrm{d}\nu \, \mathrm{e}^{\mathrm{i}\nu t} \, \sigma_{E_F}^{\mathrm{in}}(\eta, \nu) \, \widehat{\mathcal{E}}(\nu),$$

and an *out of phase* or *passive* contribution

$$(2.12) \qquad J_{\eta,\mathrm{lin}}^{\mathrm{out}}(t; E_F, \mathcal{E}) := \mathrm{i}\,\mathrm{e}^{\eta t} \int_{\mathbb{R}} \mathrm{d}\nu \, \mathrm{e}^{\mathrm{i}\nu t} \, \sigma_{E_F}^{\mathrm{out}}(\eta, \nu) \, \widehat{\mathcal{E}}(\nu).$$

The adiabatic limit $\eta \downarrow 0$ is then performed, yielding

$$(2.13) \qquad J_{\mathrm{lin}}(t; E_F, \mathcal{E}) = J_{\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E}) + J_{\mathrm{lin}}^{\mathrm{out}}(t; E_F, \mathcal{E}).$$

In particular we obtain the following expression for the linear response in phase current (see Corollary 3.5):

$$(2.14) \qquad J_{\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E}) := \lim_{\eta \downarrow 0} J_{\eta,\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E}) = \int_{\mathbb{R}} \Sigma_{E_F}(\mathrm{d}\nu) \, \mathrm{e}^{\mathrm{i}\nu t} \, \widehat{\mathcal{E}}(\nu).$$

The terminology comes from the fact that if the time dependence of the electric field is given by a pure sine (cosine), then $J_{\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E})$ also varies like a sine (cosine) as a function of time, and hence is in phase with the field, while $J_{\mathrm{lin}}^{\mathrm{out}}(t; E_F, \mathcal{E})$ behaves like a cosine (sine), and hence is out of phase. Thus the work done by the electric field on the current $J_{\mathrm{lin}}(t; E_F, \mathcal{E})$ relates only to $J_{\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E})$ when averaged over a period of oscillation. The passive part $J_{\mathrm{lin}}^{\mathrm{out}}(t; E_F, \mathcal{E})$ does not contribute to the work.

It turns out that the in phase conductivity

$$(2.15) \qquad \sigma_{E_F}^{\mathrm{in}}(\nu) = \operatorname{Re}\sigma_{E_F}(\nu) := \lim_{\eta\downarrow 0}\sigma_{E_F}^{\mathrm{in}}(\eta,\nu),$$

appearing in Mott's formula (1.1), and more generally in physics (e.g., [LGP, KLP]), may not be a well defined function. It is the conductivity measure Σ_{E_F} that is a well defined mathematical quantity. If the measure Σ_{E_F} happens to be absolutely continuous, then the two are related by $\sigma_{E_F}^{\mathrm{in}}(\nu) := \frac{\Sigma_{E_F}(\mathrm{d}\nu)}{\mathrm{d}\nu}$, and (2.14) can be recast in the form

$$(2.16) \qquad J_{\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E}) = \int_{\mathbb{R}} \mathrm{d}\nu\, \mathrm{e}^{\mathrm{i}\nu t}\, \sigma_{E_F}^{\mathrm{in}}(\nu)\, \widehat{\mathcal{E}}(\nu).$$

Since the in phase conductivity $\sigma_{E_F}^{\mathrm{in}}(\nu)$ may not be well defined as a function, we state our result in terms of the *average in phase conductivity*, an even function (Σ_{E_F} is an even measure) defined by

$$(2.17) \qquad \overline{\sigma}_{E_F}^{\mathrm{in}}(\nu) := \tfrac{1}{\nu}\Sigma_{E_F}([0,\nu]) \quad \text{for } \nu > 0.$$

Our main result is given in the following theorem, proved in Section 4.

THEOREM 2.3. *Let H be the Anderson Hamiltonian and consider a Fermi energy in its region of complete localization: $E_F \in \Xi^{\mathrm{CL}}$. Then*

$$(2.18) \qquad \limsup_{\nu\downarrow 0} \frac{\overline{\sigma}_{E_F}^{\mathrm{in}}(\nu)}{\nu^2 \left(\log\frac{1}{\nu}\right)^{d+2}} \leqslant C^{d+2}\pi^3\,\|\rho\|_\infty^2\,\ell_{E_F}^{d+2},$$

where ℓ_{E_F} is as given in (2.3), ρ is the density of the single site potential, and the constant C is independent of all parameters.

Remark 2.4. The estimate (2.18) is the first mathematically rigorous version of Mott's formula (1.1). The proof in Section 4 estimates the constant: $C \leqslant 205$; tweaking the proof would improve this numerical estimate to $C \leqslant 36$. The length ℓ_{E_F}, which controls the decay of the s-th fractional moment of the Green's function in (2.3), is the effective localization length that enters our proof and, as such, is analogous to $\tilde{\ell}_{E_F}$ in (1.1). The appearance of the term $\|\rho\|_\infty^2$ in (2.18) is also compatible with (1.1) in view of Wegner's estimate [W]: $n(E) \leqslant \|\rho\|_\infty$ for a.e. energy $E \in \mathbb{R}$.

Remark 2.5. A comparison of the estimate (2.18) with the expression in Mott's formula (1.1) would note the difference in the power of $\log\frac{1}{\nu}$, namely $d+2$ instead of $d+1$. This comes from a finite volume estimate (see Lemma 4.9) based on a result of Minami [M], which tells us that we only need to consider pairs of resonating localized states with energies E and $E + \nu$ in a volume of diameter $\sim \log\frac{1}{\nu}$, which gives a factor of $(\log\frac{1}{\nu})^d$. On the other hand, Mott's argument [Mo1], [Mo2], [MoD], [KLP] assumes that these localized states must be at a distance $\sim \log\frac{1}{\nu}$ from each other, which only gives a surface area

factor of $(\log \frac{1}{\nu})^{d-1}$. We have not seen any convincing argument for Mott's assumption. (See Remark 4.10 for a more precise analysis based on the proof of Theorem 2.3.)

Remark 2.6. A zero-frequency (or dc) conductivity at zero temperature may also be calculated by using a constant (in time) electric field. This dc-conductivity is known to exist and to be equal to zero for the Anderson model in the region of complete localization [N, Th. 1.1], [BoGKS, Cor. 5.12].

3. Linear response theory and the conductivity measure

In this section we study the ac-conductivity in linear response theory and introduce the conductivity measure. We work in the general framework of ergodic magnetic Schrödinger operators, following the approach in [BoGKS]. (See [BES], [SB] for an approach incorporating dissipation.) We treat simultaneously the discrete and continuum settings. But we will concentrate on the zero temperature case for simplicity, the general case being not very different.

3.1. *Ergodic magnetic Schrödinger operators.* We consider an ergodic magnetic Schrödinger operator H on the Hilbert space $\mathcal{H}$, where $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$ in the continuum setting and $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ in the discrete setting. In either case $\mathcal{H}_c$ denotes the subspace of functions with compact support. The ergodic operator H is a measurable map from the probability space $(\Omega, \mathbb{P})$ to the self-adjoint operators on $\mathcal{H}$. The probability space $(\Omega, \mathbb{P})$ is equipped with an ergodic group $\{\tau_a;\ a \in \mathbb{Z}^d\}$ of measure preserving transformations. The crucial property of the ergodic system is that it satisfies a covariance relation: there exists a unitary projective representation $U(a)$ of $\mathbb{Z}^d$ on $\mathcal{H}$, such that for all $a, b \in \mathbb{Z}^d$ and $\mathbb{P}$-a.e. $\omega \in \Omega$ we have

$$(3.1) \qquad U(a)H_\omega U(a)^* = H_{\tau_a(\omega)},$$

$$(3.2) \qquad U(a)\chi_b U(a)^* = \chi_{b+a},$$

$$(3.3) \qquad U(a)\delta_b = \delta_{b+a} \quad \text{if} \quad \mathcal{H} = \ell^2(\mathbb{Z}^d),$$

where χ_a denotes the multiplication operator by the characteristic function of a unit cube centered at a, also denoted by χ_a. In the discrete setting the operator χ_a is just the orthogonal projection onto the one-dimensional subspace spanned by δ_a; in particular, (3.2) and (3.3) are equivalent in the discrete setting.

We assume the ergodic magnetic Schrödinger operator to be of the form

$$(3.4) \qquad H_\omega = \begin{cases} H(\mathbf{A}_\omega, V_\omega) := (-\mathrm{i}\,\nabla - \mathbf{A}_\omega)^2 + V_\omega & \text{if} \quad \mathcal{H} = \mathrm{L}^2(\mathbb{R}^d) \\ H(\vartheta_\omega, V_\omega) := -\Delta(\vartheta_\omega) + V_\omega & \text{if} \quad \mathcal{H} = \ell^2(\mathbb{Z}^d) \end{cases}.$$

The precise requirements in the continuum are described in [BoGKS, §4]. Briefly, the random magnetic potential $\mathbf{A}$ and the random electric potential

V belong to a very wide class of potentials which ensures that $H(\mathbf{A}_\omega, V_\omega)$ is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ and uniformly bounded from below for $\mathbb{P}$-a.e. ω, and hence there is $\gamma \geqslant 0$ such that

$$(3.5) \qquad H_\omega + \gamma \geqslant 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

In the discrete setting ϑ is a lattice random magnetic potential and we require the random electric potential V to be $\mathbb{P}$-almost surely bounded from below. Thus, if we let $\mathcal{B}(\mathbb{Z}^d) := \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d; |x - y| = 1\}$, the set of oriented bonds in $\mathbb{Z}^d$, we have $\vartheta_\omega \colon \mathcal{B}(\mathbb{Z}^d) \to \mathbb{R}$, with $\vartheta_\omega(x, y) = -\vartheta_\omega(y, x)$ a measurable function of ω, and

$$(3.6) \qquad \big(\Delta(\vartheta_\omega)\varphi\big)(x) := - \sum_{y \in \mathbb{Z}^d;\, |x-y|=1} \mathrm{e}^{-\mathrm{i}\vartheta_\omega(x,y)}\varphi(y).$$

The operator $\Delta(\vartheta_\omega)$ is bounded (uniformly in ω), $H(\vartheta_\omega, V_\omega)$ is essentially self-adjoint on $\mathcal{H}_c$, and (3.5) holds for some $\gamma \geqslant 0$. *The Anderson Hamiltonian given in* (2.1) *satisfies these assumptions with* $\vartheta_\omega = 0$.

The (random) velocity operator in the x_j-direction is $\dot{X}_j := \mathrm{i}\,[H, X_j]$, where X_j denotes the operator of multiplication by the j-th coordinate x_j. In the continuum $\dot{X}_{\omega,j}$ is the closure of the operator $2(-\mathrm{i}\partial_{x_j} - \mathbf{A}_{\omega,j})$ defined on $\mathcal{C}_c^\infty(\mathbb{R}^d)$, and there is $C_\gamma < \infty$ such that [BoGKS, Prop. 2.3]

$$(3.7) \qquad \big\| \dot{X}_{\omega,j}\, (H_\omega + \gamma)^{-\frac{1}{2}} \big\| \leqslant C_\gamma \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

In the lattice $\dot{X}_{\omega,j}$ there is a bounded operator (uniformly in ω), given by

$$(3.8) \qquad \begin{aligned} \dot{X}_{\omega,j} &= D_j(\vartheta_\omega) + \big(D_j(\vartheta_\omega)\big)^*, \\ \big(D_j(\vartheta_\omega)\varphi\big)(x) &:= \mathrm{e}^{-\mathrm{i}\vartheta_\omega(x,x+\widehat{x}_j)}\varphi(x + \widehat{x}_j) - \varphi(x). \end{aligned}$$

3.2. *The mathematical framework for linear response theory.* The derivation of the Kubo formula will require normed spaces of measurable covariant operators, which we now briefly describe. We refer to [BoGKS, §3] for background, details, and justifications.

By $\mathcal{K}_{mc}$ we denote the vector space of measurable covariant operators $A \colon \Omega \to \mathrm{Lin}\big(\mathcal{H}_c, \mathcal{H}\big)$, identifying measurable covariant operators that agree $\mathbb{P}$-a.e.; all properties stated are assumed to hold for $\mathbb{P}$-a.e. $\omega \in \Omega$. Here $\mathrm{Lin}\big(\mathcal{H}_c, \mathcal{H}\big)$ is the vector space of linear operators from $\mathcal{H}_c$ to $\mathcal{H}$. Recall that A is measurable if the functions $\omega \to \langle \phi, A_\omega \phi \rangle$ are measurable for all $\phi \in \mathcal{H}_c$, A is covariant if

$$(3.9) \qquad U(x)A_\omega U(x)^* = A_{\tau_x(\omega)} \quad \text{for all} \quad x \in \mathbb{Z}^d,$$

and A is locally bounded if $\|A_\omega \chi_x\| < \infty$ and $\|\chi_x A_\omega\| < \infty$ for all $x \in \mathbb{Z}^d$. The subspace of locally bounded operators is denoted by $\mathcal{K}_{mc,lb}$. If $A \in \mathcal{K}_{mc,lb}$, we have $\mathcal{D}(A_\omega^*) \supset \mathcal{H}_c$, and hence we may set $A_\omega^\ddagger := A_\omega^*\big|_{\mathcal{H}_c}$. Note that $(\mathcal{J}A)_\omega := A_\omega^\ddagger$ defines a conjugation in $\mathcal{K}_{mc,lb}$.

We introduce norms on $\mathcal{K}_{mc,lb}$ given by

$$\|A\|_\infty := \|\,\|A_\omega\|\,\|_{\mathrm{L}^\infty(\Omega,\mathbb{P})}$$

(3.10)
$$\|A\|_p^p := \mathbb{E}\big\{\mathrm{tr}\{\chi_0|\overline{A_\omega}|^p\chi_0\}\big\}, \qquad p = 1,2,$$

$$= \mathbb{E}\big\{\langle\delta_0, |\overline{A_\omega}|^p\delta_0\rangle\big\} \qquad \text{if } \mathcal{H} = \ell^2(\mathbb{Z}^d),$$

and consider the normed spaces

(3.11) $$\mathcal{K}_p := \{A \in \mathcal{K}_{mc,lb}; \|A\|_p < \infty\}, \quad p = 1,2,\infty.$$

It turns out that $\mathcal{K}_\infty$ is a Banach space and $\mathcal{K}_2$ is a Hilbert space with inner product

(3.12)
$$\langle\!\langle A, B\rangle\!\rangle := \mathbb{E}\big\{\mathrm{tr}\{\chi_0 A_\omega^* B_\omega \chi_0\}\big\}$$

$$= \mathbb{E}\big\{\langle A_\omega\delta_0, B_\omega\delta_0\rangle\big\} \qquad \text{if} \quad \mathcal{H} = \ell^2(\mathbb{Z}^d).$$

Since $\mathcal{K}_1$ is not complete, we introduce its (abstract) completion $\overline{\mathcal{K}_1}$. The conjugation $\mathcal{J}$ is an isometry on each $\mathcal{K}_p$, $p = 1,2,\infty$. Moreover, $\mathcal{K}_p^{(0)} := \mathcal{K}_p \cap \mathcal{K}_\infty$ is dense in $\mathcal{K}_p$ for $p = 1,2$.

Note that in the discrete setting we have

(3.13) $$\|A\|_1 \leqslant \|A\|_2 \leqslant \|A\|_\infty \quad \text{and hence} \quad \mathcal{K}_\infty \subset \mathcal{K}_2 \subset \mathcal{K}_1;$$

in particular, $\mathcal{K}_\infty = \mathcal{K}_p^{(0)}$ is dense in $\mathcal{K}_p$, $p = 1,2$. Moreover, in this case $\Delta(\vartheta)$ and $\dot{X}_j$ are in $\mathcal{K}_\infty$.

Given $A \in \mathcal{K}_\infty$, we identify A_ω with its closure $\overline{A_\omega}$, a bounded operator in $\mathcal{H}$. We may then introduce a product in $\mathcal{K}_\infty$ by pointwise operator multiplication, and $\mathcal{K}_\infty$ becomes a C^*-algebra. ($\mathcal{K}_\infty$ is actually a von Neumann algebra [BoGKS, Subsection 3.5].) This C^*-algebra acts by left and right multiplication in $\mathcal{K}_p$, $p = 1,2$. Given $A \in \mathcal{K}_p$, $B \in \mathcal{K}_\infty$, left multiplication $B \odot_L A$ is simply defined by $(B \odot_L A)_\omega = B_\omega A_\omega$. Right multiplication is more subtle; we set $(A \odot_R B)_\omega = A_\omega^{\ddagger*} B_\omega$ (see [BoGKS, Lemma 3.4] for a justification), and note that $(A \odot_R B)^\ddagger = B^* \odot_L A^\ddagger$. Moreover, left and right multiplication commute:

(3.14) $$B \odot_L A \odot_R C := B \odot_L (A \odot_R C) = (B \odot_L A) \odot_R C$$

for $A \in \mathcal{K}_p$, $B, C \in \mathcal{K}_\infty$. (We refer to [BoGKS, §3] for an extensive set of rules and properties which facilitate calculations in these spaces of measurable covariant operators.)

Given $A \in \mathcal{K}_p$, $p = 1,2$, we define

(3.15) $$\mathcal{U}_L^{(0)}(t)A := \mathrm{e}^{-\mathrm{i}tH} \odot_L A,$$

(3.16) $$\mathcal{U}_R^{(0)}(t)A := A \odot_R \mathrm{e}^{-\mathrm{i}tH}, \qquad \text{i.e., } \mathcal{U}_R^{(0)}(t) = \mathcal{J}\mathcal{U}_L^{(0)}(-t)\mathcal{J},$$

(3.17) $$\mathcal{U}^{(0)}(t)A := \mathrm{e}^{-\mathrm{i}tH} \odot_L A \odot_R \mathrm{e}^{\mathrm{i}tH}, \quad \text{i.e., } \mathcal{U}^{(0)}(t) = \mathcal{U}_L^{(0)}(t)\mathcal{U}_R^{(0)}(-t).$$

Then $\mathcal{U}^{(0)}(t), \mathcal{U}_L^{(0)}(t), \mathcal{U}_R^{(0)}(t)$ are strongly continuous, one-parameter groups of operators on $\mathcal{K}_p$ for $p = 1,2$, which are unitary on $\mathcal{K}_2$ and isometric on $\mathcal{K}_1$,

and hence extend to isometries on $\overline{\mathcal{K}_1}$. (See [BoGKS, Cor. 4.12] for $\mathcal{U}^{(0)}(t)$; the same argument works for $\mathcal{U}_L^{(0)}(t)$ and $\mathcal{U}_R^{(0)}(t)$.) These one-parameter groups of operators commute with each other, and hence can be simultaneously diagonalized by the spectral theorem. Using Stone's theorem, we define commuting self-adjoint operators $\mathcal{L}, \mathcal{H}_L, \mathcal{H}_R$ on $\mathcal{K}_2$ by

$$(3.18) \qquad \mathrm{e}^{-\mathrm{i}t\mathcal{L}} := \mathcal{U}^{(0)}(t), \quad \mathrm{e}^{-\mathrm{i}t\mathcal{H}_L} := \mathcal{U}_L^{(0)}(t), \quad \mathrm{e}^{-\mathrm{i}t\mathcal{H}_R} := \mathcal{U}_R^{(0)}(t).$$

The operator $\mathcal{L}$ is the *Liouvillian*; now we have

$$(3.19) \qquad \mathcal{L} = \overline{\mathcal{H}_L - \mathcal{H}_R} \quad \text{and} \quad \mathcal{H}_R = \mathcal{J}\mathcal{H}_L\mathcal{J}.$$

If the ergodic magnetic Schrödinger operator H is bounded, e.g., the Anderson Hamiltonian in (2.1), then $H \in \mathcal{K}_\infty$, and $\mathcal{L}, \mathcal{H}_L, \mathcal{H}_R$ are bounded commuting self-adjoint operators on $\mathcal{K}_2$, with

$$(3.20) \qquad \mathcal{H}_L A = H \odot_L A, \quad \mathcal{H}_R A = A \odot_R H, \quad \text{and} \quad \mathcal{L} = \mathcal{H}_L - \mathcal{H}_R.$$

The *trace per unit volume* is given by

$$(3.21) \qquad \begin{aligned} \mathcal{T}(A) &:= \mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 A_\omega \chi_0\right\}\right\} \quad \text{for} \quad A \in \mathcal{K}_1, \\ &= \mathbb{E}\left\{\langle \delta_0, A_\omega \delta_0 \rangle\right\} \qquad \text{if} \quad \mathcal{H} = \ell^2(\mathbb{Z}^d), \end{aligned}$$

a well defined linear functional on $\mathcal{K}_1$ with $|\mathcal{T}(A)| \leqslant \|A\|_1$, and hence can be extended to $\overline{\mathcal{K}_1}$. Note that $\mathcal{T}$ is indeed the trace per unit volume:

$$(3.22) \qquad \mathcal{T}(A) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \mathrm{tr}\left\{\chi_{\Lambda_L} A_\omega \chi_{\Lambda_L}\right\} \quad \text{for } \mathbb{P}\text{-a.e. } \omega,$$

where Λ_L denotes the cube of side L centered at 0 (see [BoGKS, Prop. 3.20]).

3.3. *The linear response current.* We consider a quantum system at zero temperature, modeled by an ergodic magnetic Schrödinger operator H as in (3.4). We fix a Fermi energy E_F and the x_1-direction, and make the following assumption on the (random) Fermi projection $P_{E_F} := \chi_{]-\infty, E_F]}(H)$.

ASSUMPTION 3.1.

$$(3.23) \qquad Y_{E_F} := \mathrm{i}\left[X_1, P_{E_F}\right] \in \mathcal{K}_2.$$

Under Assumption 3.1 we have $Y_{E_F} = Y_{E_F}^{\ddagger}$ and $Y_{E_F} \in \mathcal{D}(\mathcal{L})$ by [BoGKS, Lemma 5.4(iii) and Cor. 4.12]. Moreover, we also have $Y_{E_F} \in \mathcal{K}_1$ (see [BoGKS, Rem. 5.2]). (Condition (3.23) is the main assumption in [BoGKS]; it was originally identified in [BES].)

If H is the Anderson Hamiltonian we always have (3.23) *if the Fermi energy lies in the region of complete localization, i.e.,* $E_F \in \Xi^{\mathrm{CL}}$ [AG, Th. 2], [GK2, Th. 3]. (*In fact, in this case* $[X_j, P_{E_F}] \in \mathcal{K}_2$ *for all* $j = 1, 2, \ldots, d$.)

In the distant past, taken to be $t = -\infty$, the system is in equilibrium in the state given by this Fermi projection P_{E_F}. A spatially homogeneous,

time-dependent electric field $\mathbf{E}(t)$ is then introduced adiabatically: Starting at time $t = -\infty$, we switch on the electric field $\mathbf{E}_\eta(t) := \mathrm{e}^{\eta t}\mathbf{E}(t)$ with $\eta > 0$, and then let $\eta \to 0$. *We here assume that the electric field is pointing in the x_1-direction: $\mathbf{E}(t) = \mathcal{E}(t)\widehat{x}_1$, where the amplitude $\mathcal{E}(t)$ is a continuous function such that $\int_{-\infty}^{t} \mathrm{d}s\, \mathrm{e}^{\eta s}|\mathcal{E}(s)| < \infty$ for all $t \in \mathbb{R}$ and $\eta > 0$. Note that the relevant results in [BoGKS], although stated for constant electric fields $\mathbf{E}$, are valid under this assumption.* We set $\mathcal{E}_\eta(t) := \mathrm{e}^{\eta t}\mathcal{E}(t)$, and

$$(3.24) \qquad\qquad F_\eta(t) := \int_{-\infty}^{t} \mathrm{d}s\, \mathcal{E}_\eta(s).$$

For each fixed $\eta > 0$ the dynamics are now generated by a time-dependent ergodic Hamiltonian. Following [BoGKS, Subsection 2.2], we resist the impulse to take $H_\omega + \mathcal{E}_\eta(t)X_1$ as the Hamiltonian, and instead consider the physically equivalent (but bounded below) Hamiltonian

$$(3.25) \qquad\qquad H_\omega(\eta, t) := G(\eta, t)H_\omega G(\eta, t)^*,$$

where $G(\eta, t) := \mathrm{e}^{\mathrm{i}F_\eta(t)X_1}$ is a time-dependent gauge transformation. We get

$$(3.26) \qquad \begin{aligned} H_\omega(\eta, t) &= H(\mathbf{A}_\omega + F_\eta(t)\widehat{x}_1, V_\omega) &&\text{if}\quad \mathcal{H} = \mathrm{L}^2(\mathbb{R}^d), \\ H_\omega(\eta, t) &= H(\vartheta_\omega + F_\eta(t)\gamma_1, V_\omega) &&\text{if}\quad \mathcal{H} = \ell^2(\mathbb{Z}^d), \end{aligned}$$

where $\gamma_1(x, y) := y_1 - x_1$ for $(x, y) \in \mathcal{B}(\mathbb{Z}^d)$.

Remark 3.2. If H_ω is the Anderson Hamiltonian given in (2.1), there is no difficulty in defining $\widetilde{H}_\omega(\eta, t) := H_\omega + \mathcal{E}_\eta(t)X_1$ as an (unbounded) self-adjoint operator. Moreover, in this case $H_\omega(\eta, t)$ is actually a bounded operator. It follows that if $\widetilde{\psi}(t)$ is a strong solution of the Schrödinger equation $\mathrm{i}\partial_t\widetilde{\psi}(t) = \widetilde{H}_\omega(\eta, t)\widetilde{\psi}(t)$, then $\psi(t) = G(\eta, t)\widetilde{\psi}(t)$ is a strong solution of $\mathrm{i}\partial_t\psi(t) = H_\omega(\eta, t)\psi(t)$. A similar statement holds in the opposite direction for *weak* solutions. (See the discussion in [BoGKS, Subsection 2.2].) At the formal level, one can easily see that the linear response current given in (2.7) is independent of the choice of gauge.

The system was described at time $t = -\infty$ by the Fermi projection P_{E_F}. It is then described at time t by the density matrix $\varrho(\eta, t)$, the unique solution to the Liouville equation (2.5) in both spaces $\mathcal{K}_2$ and $\overline{\mathcal{K}_1}$. (See [BoGKS, Th. 5.3] for a precise statement.)

The adiabatic electric field generates a time-dependent electric current. Its amplitude in the x_1-direction is given by (2.6), where $\dot{X}_1(t) := G(\eta, t)\dot{X}_1 G(\eta, t)^*$ is the first component of the velocity operator at time t in the Schrödinger picture. The *linear response current* is then defined as in (2.7), its existence is proven in [BoGKS, Th. 5.9] with

$$(3.27) \qquad J_{\eta,\mathrm{lin}}(t; E_F, \mathcal{E}) = \mathcal{T}\left\{ \int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{E}(r)\dot{X}_1\, \mathcal{U}^{(0)}(t - r)Y_{E_F} \right\}.$$

Since the integral in (3.27) is a Bochner integral in the Banach space $\overline{\mathcal{K}_1}$, where $\mathcal{T}$ is a bounded linear functional, they can be interchanged, and hence, using [BoGKS, Eq. (5.88)], we obtain

$$(3.28) \qquad J_{\eta,\mathrm{lin}}(t; E_F, \mathcal{E}) = -\int_{-\infty}^{t} \mathrm{d}r\ \mathrm{e}^{\eta r} \mathcal{E}(r) \langle\!\langle Y_{E_F}, \mathrm{e}^{-\mathrm{i}(t-r)\mathcal{L}} \mathcal{L} \mathcal{P}_{E_F} Y_{E_F} \rangle\!\rangle.$$

Here $\mathcal{P}_{E_F}$ is the bounded self-adjoint operator on $\mathcal{K}_2$ given by

$$(3.29) \qquad \begin{aligned} \mathcal{P}_{E_F} &:= \chi_{]-\infty, E_F]}(\mathcal{H}_L) - \chi_{]-\infty, E_F]}(\mathcal{H}_R); \quad \text{that is,} \\ \mathcal{P}_{E_F} A\ &= P_{E_F} \odot_L A - A \odot_R P_{E_F} \quad \text{for}\ A \in \mathcal{K}_2. \end{aligned}$$

Note that $\mathcal{P}_{E_F}$ commutes with $\mathcal{L}, \mathcal{H}_L, \mathcal{H}_R$; in particular $\mathcal{P}_{E_F} Y_{E_F} \in \mathcal{D}(\mathcal{L})$. Moreover, we have $\mathcal{P}_{E_F}^2 Y_{E_F} = Y_{E_F}$ [BoGKS, Lemma 5.13].

3.4. *The conductivity measure and a Kubo formula for the ac-conductivity.* Suppose now that the amplitude $\mathcal{E}(t)$ of the electric field satisfies assumption (2.4). We can then rewrite (3.28), first using the Fubini–Tonelli theorem, and then proceeding as in [BoGKS, Eq. (5.89)], as

(3.30)

$$\begin{aligned} J_{\eta,\mathrm{lin}}(t; E_F, \mathcal{E}) &= -\int_{\mathbb{R}} \mathrm{d}\nu\ \widehat{\mathcal{E}}(\nu) \int_{-\infty}^{t} \mathrm{d}r\ \mathrm{e}^{(\eta + \mathrm{i}\nu)r} \langle\!\langle Y_{E_F}, \mathrm{e}^{-\mathrm{i}(t-r)\mathcal{L}} \mathcal{L} \mathcal{P}_{E_F} Y_{E_F} \rangle\!\rangle \\ &= -\mathrm{i}\,\mathrm{e}^{\eta t} \int_{\mathbb{R}} \mathrm{d}\nu\ \mathrm{e}^{\mathrm{i}\nu t} \widehat{\mathcal{E}}(\nu) \langle\!\langle Y_{E_F}, (\mathcal{L} + \nu - \mathrm{i}\,\eta)^{-1} (-\mathcal{L}\mathcal{P}_{E_F})\, Y_{E_F} \rangle\!\rangle. \end{aligned}$$

This leads us to the following definition, which is justified in the subsequent theorem.

Definition 3.3. The *conductivity measure* (x_1-x_1 *component*) at Fermi energy E_F is defined as

$$(3.31) \qquad \Sigma_{E_F}(B) := \pi \langle\!\langle Y_{E_F}, \chi_B(\mathcal{L}) (-\mathcal{L}\mathcal{P}_{E_F}) Y_{E_F} \rangle\!\rangle \quad \text{for a Borel set } B \subset \mathbb{R}.$$

THEOREM 3.4. *Let E_F be a Fermi energy satisfying Assumption* 3.1. *Then Σ_{E_F} is a finite, positive, even, Borel measure on $\mathbb{R}$. Moreover, for an electric field with amplitude $\mathcal{E}(t)$ satisfying assumption* (2.4),

$$(3.32) \qquad J_{\eta,\mathrm{lin}}(t; E_F, \mathcal{E}) = \mathrm{e}^{\eta t} \int_{\mathbb{R}} \mathrm{d}\nu\ \mathrm{e}^{\mathrm{i}\nu t}\, \sigma_{E_F}(\eta, \nu)\, \widehat{\mathcal{E}}(\nu)$$

with

$$(3.33) \qquad \sigma_{E_F}(\eta, \nu) := -\tfrac{\mathrm{i}}{\pi} \int_{\mathbb{R}} \Sigma_{E_F}(\mathrm{d}\lambda) (\lambda + \nu - \mathrm{i}\,\eta)^{-1}.$$

Proof. Recall that $\mathcal{H}_L$ and $\mathcal{H}_R$ are commuting self-adjoint operators on $\mathcal{K}_2$, and hence can be simultaneously diagonalized by the spectral theorem. Thus it follows from (3.19) and (3.29) that

$$(3.34) \qquad\qquad\qquad -\mathcal{L}\mathcal{P}_{E_F} \geqslant 0.$$

Since $Y_{E_F} \in \mathcal{D}(\mathcal{L})$ and $\mathcal{P}_{E_F}$ is bounded, we conclude that Σ_{E_F} is a finite positive Borel measure. To show that it is even, note that $\mathcal{J}\mathcal{L}\mathcal{J} = -\mathcal{L}$, $\mathcal{J}\mathcal{P}_{E_F}\mathcal{J} = -\mathcal{P}_{E_F}$, and $\mathcal{J}\chi_B(\mathcal{L})\mathcal{L}\mathcal{P}_{E_F}\mathcal{J} = \chi_B(-\mathcal{L})\mathcal{L}\mathcal{P}_{E_F} = \chi_{-B}(\mathcal{L})\mathcal{L}\mathcal{P}_{E_F}$. Since $\mathcal{J}Y_{E_F} = Y_{E_F}$, we get $\Sigma_{E_F}(B) = \Sigma_{E_F}(-B)$.

Since (3.33) may be rewritten as

$$(3.35) \qquad \sigma_{E_F}(\eta, \nu) = -\mathrm{i} \left\langle\!\!\left\langle Y_{E_F}, (\mathcal{L} + \nu - \mathrm{i}\,\eta)^{-1} \left(-\mathcal{L}\mathcal{P}_{E_F}\right) Y_{E_F} \right\rangle\!\!\right\rangle,$$

the equality (3.32) follows from (3.30).   □

COROLLARY 3.5. *Let E_F be a Fermi energy satisfying Assumption 3.1, and let $\mathcal{E}(t)$ be the amplitude of an electric field satisfying assumption (2.4). Then the adiabatic limit $\eta \downarrow 0$ of the linear response in phase current given in (2.11) exists:*

$$(3.36) \qquad J_{\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E}) := \lim_{\eta \downarrow 0} J_{\eta,\mathrm{lin}}^{\mathrm{in}}(t; E_F, \mathcal{E}) = \int_{\mathbb{R}} \Sigma_{E_F}(\mathrm{d}\nu)\, \mathrm{e}^{\mathrm{i}\nu t}\, \widehat{\mathcal{E}}(\nu).$$

If in addition $\mathcal{E}(t)$ is uniformly Hölder continuous, then the adiabatic limit $\eta \downarrow 0$ of the linear response out of phase current also exists:

$$
(3.37)
\begin{aligned}
J_{\mathrm{lin}}^{\mathrm{out}}(t; E_F, \mathcal{E}) :&= \lim_{\eta \downarrow 0} J_{\eta,\mathrm{lin}}^{\mathrm{out}}(t; E_F, \mathcal{E}) \\
&= \tfrac{1}{\pi \mathrm{i}} \int_{\mathbb{R}} \Sigma_{E_F}(\mathrm{d}\lambda)\ \mathrm{PV}\!\!\int_{\mathbb{R}} \mathrm{d}\nu\, \frac{\mathrm{e}^{\mathrm{i}\nu t}\, \widehat{\mathcal{E}}(\nu)}{\nu - \lambda},
\end{aligned}
$$

where the integral over ν in (3.37) is to be understood in the principal-value sense.

Proof. This corollary is an immediate consequence of (3.32), (3.33), and well known properties of the Cauchy (Borel, Stieltjes) transform of finite Borel measures. The limit in (3.36) follows from [StW, Th. 2.3]. We can establish the limit in (3.37) by using Fubini's theorem and the existence (with bounds) of the principal value integral for uniformly Hölder continuous functions (see [Gr, Rem. 4.1.2]).   □

Remark 3.6. The out of phase (or passive) conductivity does not appear to be the subject of extensive study; but see [LGP].

3.5. *Correlation measures.* For each $A \in \mathcal{K}_2$ we define a finite Borel measure Υ_A on $\mathbb{R}^2$ by

$$(3.38) \qquad \Upsilon_A(C) := \left\langle\!\!\left\langle A, \chi_C(\mathcal{H}_L, \mathcal{H}_R) A \right\rangle\!\!\right\rangle \quad \text{for a Borel set } C \subset \mathbb{R}^2.$$

Note that it follows from (3.19) that

$$(3.39) \qquad \Upsilon_A(B_1 \times B_2) = \Upsilon_{A^\ddagger}(B_2 \times B_1) \quad \text{for all Borel sets } B_1, B_2 \subset \mathbb{R}.$$

The correlation measure we obtain by taking $A = Y_{E_F}$ plays an important role in our analysis.

PROPOSITION 3.7. *Let E_F be a Fermi energy satisfying Assumption 3.1 and set $\Psi_{E_F} := \Upsilon_{Y_{E_F}}$. Then*

$$(3.40) \qquad \Sigma_{E_F}(B) = \pi \int_{\mathbb{R}^2} \Psi_{E_F}(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2) \, |\lambda_1 - \lambda_2| \, \chi_B(\lambda_1 - \lambda_2)$$

for all Borel sets $B \subset \mathbb{R}$. Moreover, the measure Ψ_{E_F} is supported by the set $\mathbb{S}_{E_F}$; i.e., $\Psi_{E_F}(\mathbb{R}^2 \setminus \mathbb{S}_{E_F}) = 0$, where

$$(3.41) \qquad \mathbb{S}_{E_F} := \{]-\infty, E_F] \times]E_F, \infty[\} \cup \{]E_F, \infty[\times]-\infty, E_F] \} \subset \mathbb{R}^2.$$

Proof. If we set

$$(3.42) \qquad Q_{E_F}(\lambda_1, \lambda_2) := \chi_{\mathbb{S}_{E_F}}(\lambda_1, \lambda_2) = \left| \chi_{]-\infty, E_F]}(\lambda_1) - \chi_{]-\infty, E_F]}(\lambda_2) \right|,$$

then, from (3.29),

$$(3.43) \qquad \mathcal{Q}_{E_F} = \mathcal{P}_{E_F}^2, \quad \text{where} \quad \mathcal{Q}_{E_F} := Q_{E_F}(\mathcal{H}_L, \mathcal{H}_R).$$

Thus $\mathcal{Q}_{E_F} Y_{E_F} = Y_{E_F}$, and the measure Ψ_{E_F} is supported by the set $\mathbb{S}_{E_F}$. Hence

$$(3.44) \qquad \Sigma_{E_F}(B) = \pi \langle\!\langle Y_{E_F}, \chi_B(\mathcal{L}) | \mathcal{L} | Y_{E_F} \rangle\!\rangle \quad \text{for all Borel sets } B \subset \mathbb{R},$$

and (3.40) follows. $\qquad \square$

3.6. *The velocity-velocity correlation measure.* The velocity-velocity correlation measure Φ is formally given by $\Phi = \Upsilon_{\dot{X}_1}$, but note that $\dot{X}_1 \notin \mathcal{K}_2$ in the continuum setting.

Definition 3.8. The *velocity-velocity correlation measure (x_1-x_1 component)* is the positive σ-finite Borel measure on $\mathbb{R}^2$ defined on bounded Borel sets $C \subset \mathbb{R}^2$ by

$$(3.45) \qquad \Phi(C) := \langle\!\langle \dot{X}_{1,\alpha}, (\mathcal{H}_L + \gamma)^{2\alpha} \chi_C(\mathcal{H}_L, \mathcal{H}_R) (\mathcal{H}_R + \gamma)^{2\alpha} \dot{X}_{1,\alpha} \rangle\!\rangle$$

$$(3.46) \qquad \qquad = \Upsilon_{\dot{X}_1}(C) \quad \text{if} \quad \mathcal{H} = \ell^2(\mathbb{Z}^d),$$

where

$$(3.47) \qquad \begin{aligned} \dot{X}_{1,\alpha} &:= \left\{ (H + \gamma)^{-\alpha} \dot{X}_1 (H + \gamma)^{-\frac{1}{2}} \right\} \odot_L (H + \gamma)^{-[[\frac{d}{4}]]} \in \mathcal{K}_2, \\ \alpha &:= \tfrac{1}{2} + [[\tfrac{d}{4}]] \quad \text{with } [[\tfrac{d}{4}]] \text{ the smallest integer bigger than } \tfrac{d}{4}. \end{aligned}$$

Note that (3.47) is justified since we have $\dot{X}_1 (H + \gamma)^{-\frac{1}{2}} \in \mathcal{K}_\infty$ by (3.7) and $(H + \gamma)^{-[[\frac{d}{4}]]} \in \mathcal{K}_2$ by [BoGKS, Prop. 4.2(i)]; note that $\dot{X}_{1,\alpha}^{\ddagger} = \dot{X}_{1,\alpha}$. In the discrete setting, $\dot{X}_1 \in \mathcal{K}_2$ and hence $\Phi = \Upsilon_{\dot{X}_1}$, a finite measure.

The following lemma relates the measure Ψ_{E_F} of Proposition 3.7 to the measure Φ.

LEMMA 3.9. *The correlation measure Ψ_{E_F} is absolutely continuous with respect to the velocity-velocity correlation measure Φ, with*

$$(3.48) \qquad \frac{\mathrm{d}\Psi_{E_F}}{\mathrm{d}\Phi}(\lambda_1, \lambda_2) = \frac{Q_{E_F}(\lambda_1, \lambda_2)}{(\lambda_1 - \lambda_2)^2}.$$

Proof. The key observation is that (use [BoGKS, Lemma 5.4(iii) and Cor. 4.12])

$$(3.49) \qquad \begin{aligned} (\mathcal{H}_L + \gamma)^{-\alpha} (\mathcal{H}_R + \gamma)^{-\alpha} \mathcal{L}Y_{E_F} &= -\mathcal{P}_{E_F}\dot{X}_{1,\alpha}, \\ \mathcal{L}Y_{E_F} &= -\mathcal{P}_{E_F}\dot{X}_1 \quad \text{if} \quad \mathcal{H} = \ell^2(\mathbb{Z}^d). \end{aligned}$$

Now, for all Borel sets $C \subset \mathbb{R}^2$,

$$
\begin{aligned}
(3.50) \qquad \int_C \Psi_{E_F}(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\,(\lambda_1 - \lambda_2)^2 &= \langle\!\langle \mathcal{L}Y_{E_F}, \chi_C(\mathcal{H}_L, \mathcal{H}_R)\mathcal{L}Y_{E_F}\rangle\!\rangle \\
&= \langle\!\langle \mathcal{P}_{E_F}\dot{X}_{1,\alpha}, (\mathcal{H}_L + \gamma)^{2\alpha}\chi_C(\mathcal{H}_L, \mathcal{H}_R)\,(\mathcal{H}_R + \gamma)^{2\alpha}\mathcal{P}_{E_F}\dot{X}_{1,\alpha}\rangle\!\rangle \\
&= \langle\!\langle \dot{X}_{1,\alpha}, \mathcal{P}^2_{E_F}(\mathcal{H}_L + \gamma)^{2\alpha}\chi_C(\mathcal{H}_L, \mathcal{H}_R)\,(\mathcal{H}_R + \gamma)^{2\alpha}\dot{X}_{1,\alpha}\rangle\!\rangle \\
&= \langle\!\langle \dot{X}_{1,\alpha}, (\mathcal{H}_L + \gamma)^{2\alpha}\chi_{C\cap\mathbb{S}_{E_F}}(\mathcal{H}_L, \mathcal{H}_R)\,(\mathcal{H}_R + \gamma)^{2\alpha}\dot{X}_{1,\alpha}\rangle\!\rangle \\
&= \int_C \Phi(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\,Q_{E_F}(\lambda_1, \lambda_2).
\end{aligned}
$$

Since Ψ_{E_F} is supported on $\mathbb{S}_{E_F}$, the lemma follows. $\qquad\square$

We can now write the conductivity measure in terms of the velocity-velocity correlation measure.

PROPOSITION 3.10. *Let E_F be a Fermi energy satisfying Assumption 3.1. Then*

$$(3.51) \qquad \Sigma_{E_F}(B) = \pi \int_{\mathbb{S}_{E_F}} \Phi(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\,|\lambda_1 - \lambda_2|^{-1}\chi_B(\lambda_1 - \lambda_2)$$

for all Borel sets $B \subset \mathbb{R}$.

Proof. The representation (3.51) is an immediate consequence of (3.40) and (3.48). $\qquad\square$

Remark 3.11. If we assume, as is customary in physics, that the conductivity measure Σ_{E_F} is absolutely continuous, its density being the in phase conductivity $\sigma^{\mathrm{in}}_{E_F}(\nu)$, and that in addition the velocity-velocity correlation measure Φ is absolutely continuous with a continuous density $\phi(\lambda_1, \lambda_2)$, then (3.51) yields the well-known formula (cf. [P], [KLP])

$$(3.52) \qquad \sigma^{\mathrm{in}}_{E_F}(\nu) = \frac{\pi}{\nu} \int_{E_F - \nu}^{E_F} \mathrm{d}E\,\phi(E + \nu, E).$$

The existence of the densities $\sigma_{E_F}^{\text{in}}(\nu)$ and $\phi(\lambda_1, \lambda_2)$ is currently an open question, and hence (3.52) is only known as a formal expression. In contrast, the integrated version (3.51) is mathematically well established. (See also [BH] for some recent work on the velocity-velocity correlation function.)

3.7. *Bounds on the average in phase conductivity.* The average in phase conductivity $\overline{\sigma}_{E_F}^{\text{in}}(\nu)$ defined in (2.17) can be bounded from above and below by the correlation measure Ψ_{E_F}. Note that since Σ_{E_F} is an even measure it suffices to consider frequencies $\nu > 0$.
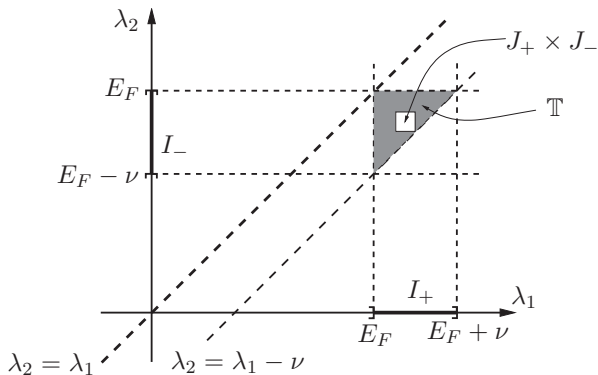

Figure 1: $J_+ \times J_- \subset \mathbb{T} \subset I_+ \times I_-$.

PROPOSITION 3.12. *Let E_F be a Fermi energy satisfying Assumption 3.1. Given $\nu > 0$, define the pairs of disjoint energy intervals*

(3.53)
$$I_- := \,]E_F - \nu, E_F] \quad and \quad I_+ := \,]E_F, E_F + \nu],$$
$$J_- := \,]E_F - \tfrac{\nu}{2}, E_F - \tfrac{\nu}{4}] \quad and \quad J_+ := \,]E_F + \tfrac{\nu}{4}, E_F + \tfrac{\nu}{2}].$$

Then

(3.54)
$$\tfrac{\pi}{2}\,\Psi_{E_F}(J_+ \times J_-) \leqslant \overline{\sigma}_{E_F}^{\text{in}}(\nu) \leqslant \pi\,\Psi_{E_F}(I_+ \times I_-).$$

Proof. It follows immediately from the representation (3.40) that

(3.55)
$$\overline{\sigma}_{E_F}^{\text{in}}(\nu) \leqslant \pi \int_{\mathbb{S}_{E_F}} \Psi_{E_F}(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\,\chi_{[0,\nu]}(\lambda_1 - \lambda_2) = \pi\,\Psi_{E_F}(\mathbb{T}),$$

where

(3.56)
$$\mathbb{T} := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_2 \leqslant E_F < \lambda_1 \text{ and } \lambda_1 - \lambda_2 \leqslant \nu\}$$

is the triangle in Figure 1. Since $\mathbb{T} \subset I_+ \times I_-$, as can be seen there, the upper bound in (3.54) follows from (3.55).

Similarly, we have $J_+ \times J_- \subset \mathbb{T}$ (see Figure 1) and the lower bound in (3.54). $\qquad\square$

4. The proof of Theorem 2.3

In this section we let H be the Anderson Hamiltonian and fix a Fermi energy $E_F \in \Xi^{\mathrm{CL}}$. Thus (2.3) holds, and hence, using the exponential decay of the Fermi projection given in [AG, Th. 2] and $\|P_{E_F}\| \leqslant 1$, we have

$$(4.1) \qquad \mathbb{E}\left\{|\langle \delta_x, P_{E_F}\delta_y\rangle|^p\right\} \leqslant C\mathrm{e}^{-c|x-y|} \qquad \text{for all } p \in [1, \infty[\text{ and } x, y \in \mathbb{Z}^d,$$

where C and $c > 0$ are constants depending on E_F and ρ. In particular, Assumption 3.1 is satisfied, and we can use the results of Section 3. In view of Proposition 3.12, Theorem 2.3 is an immediate consequence of the following result.

THEOREM 4.1. *Let H be the Anderson Hamiltonian and consider a Fermi energy in its region of complete localization: $E_F \in \Xi^{\mathrm{CL}}$. Consider the finite Borel measure Ψ_{E_F} on $\mathbb{R}^2$ of Proposition 3.7, and, given $\nu > 0$, let I_- and I_+ be the disjoint energy intervals given in (3.53). Then*

$$(4.2) \qquad \limsup_{\nu \downarrow 0} \frac{\Psi_{E_F}(I_+ \times I_-)}{\nu^2 \left(\log\frac{1}{\nu}\right)^{d+2}} \leqslant 205^{d+2}\pi^2 \|\rho\|_\infty^2 \ell_{E_F}^{d+2},$$

where ℓ_{E_F} is as in (2.3) and ρ is the density of the single site potential.

Theorem 4.1 will be proved by a reduction to finite volume, a cube of side L, where the relevant quantity will be controlled by Minami's estimate. Optimizing the final estimate will lead to a choice of $L \sim \log\frac{1}{\nu}$, which is responsible for the factor of $\left(\log\frac{1}{\nu}\right)^{d+2}$ in (4.2). By improving some of the estimates in the proof (at the price of making them more cumbersome), the numerical constant 205 in (4.2) may be reduced to 36.

4.1. *Some properties of the measure Ψ_{E_F}.* We briefly recall some facts about the Anderson Hamiltonian. If $I \subset \Xi^{\mathrm{CL}}$ is a compact interval, then for all Borel functions f with $|f| \leqslant 1$ we have ([A], [AG])

$$(4.3) \qquad \mathbb{E}\left\{|\langle \delta_x, f(H)\chi_I(H)\delta_y\rangle|\right\} \leqslant C_I\, \mathrm{e}^{-c_I|x-y|} \qquad \text{for all } x, y \in \mathbb{Z}^d,$$

for suitable constants C_I and $c_I > 0$, and hence

$$(4.4) \qquad\qquad [X_1, f(H)\chi_I(H)] \in \mathcal{K}_2.$$

We also recall Wegner's estimate [W], which yields

$$(4.5) \qquad |\mathbb{E}\left\{\langle \delta_x, \chi_B(H)\delta_y\rangle\right\}| \leqslant \mathbb{E}\left\{\langle \delta_0, \chi_B(H)\delta_0\rangle\right\} \leqslant \|\rho\|_\infty |B|$$

for all Borel sets $B \subset \mathbb{R}$ and $x, y \in \mathbb{Z}^d$.

We begin by proving a preliminary bound on $\Psi_{E_F}(I_+ \times I_-)$, a consequence of Wegner's estimate.

LEMMA 4.2. *Given $\beta \in \,]0,1[$, there exists a constant W_β such that*

$$(4.6) \qquad \Psi_{E_F}(B_+ \times B_-) \leqslant W_\beta \left(\min\{|B_+|, |B_-|\}\right)^\beta$$

for all Borel sets $B_\pm \subset \mathbb{R}$.

Proof. Since

$$(4.7) \qquad \Psi_{E_F}(B_+ \times B_-) \leqslant \min\left\{\Psi_{E_F}(B_+ \times \mathbb{R}), \Psi_{E_F}(\mathbb{R} \times B_-)\right\},$$

$$(4.8) \qquad \Psi_{E_F}(B_+ \times B_-) = \Psi_{E_F}(B_- \times B_+),$$

and, for all Borel sets $B \subset \mathbb{R}$,

$$(4.9) \qquad \Psi_{E_F}(B \times \mathbb{R}) = \langle\!\langle Y_{E_F}, \chi_B(\mathcal{H}_L) Y_{E_F}\rangle\!\rangle,$$

it suffices to show that for $\beta \in \,]0,1[$ there exists a constant W_β such that

$$(4.10) \qquad \langle\!\langle Y_{E_F}, \chi_B(\mathcal{H}_L) Y_{E_F}\rangle\!\rangle \leqslant W_\beta |B|^\beta \quad \text{for all Borel sets } B \subset \mathbb{R}.$$

Using $X_1 \delta_0 = 0$, we obtain

$$\langle\!\langle Y_{E_F}, \chi_B(\mathcal{H}_L) Y_{E_F}\rangle\!\rangle = \mathbb{E}\left\{\langle X_1 P_{E_F} \delta_0, \chi_B(H) X_1 P_{E_F} \delta_0\rangle\right\}$$

$$(4.11) \qquad \leqslant \sum_{x,y \in \mathbb{Z}^d} |x_1||y_1|\, \mathbb{E}\left\{|\langle \delta_0, P_{E_F} \delta_x\rangle|\,|\langle \delta_x, \chi_B(H)\delta_y\rangle|\,|\langle \delta_y, P_{E_F} \delta_0\rangle|\right\}$$

$$\leqslant W_\beta |B|^\beta,$$

where we used Hölder's inequality plus the estimates (4.1) and (4.5). $\qquad \square$

Remark 4.3. In the case of the Anderson Hamiltonian, the self-adjoint operators $\mathcal{H}_L$ and $\mathcal{H}_R$ on the Hilbert space $\mathcal{K}_2$ have absolutely continuous spectrum. The proof is a variation of the argument in Lemma 4.2. Recalling that in the discrete setting $\mathcal{K}_\infty$ is a dense subset of $\mathcal{K}_2$, to show that $\mathcal{H}_L$ has absolutely continuous spectrum it suffices to prove that for each $A \in \mathcal{K}_\infty$ the measure $\Upsilon_A^{(L)}$ on $\mathbb{R}$, given by $\Upsilon_A^{(L)}(B) := \Upsilon_A(B \times \mathbb{R})$ (see (3.38)) is absolutely continuous. Since $\chi_B(H) \in \mathcal{K}_\infty \subset \mathcal{K}_2$, we have, similarly to (4.11), that

$$\Upsilon_A^{(L)}(B) = \langle\!\langle A, \chi_B(\mathcal{H}_L) A\rangle\!\rangle = \|\chi_B(H) \odot_L A\|_2^2 = \|A^\ddagger \odot_R \chi_B(H)\|_2^2$$

$$(4.12) \qquad \leqslant \|A\|_\infty^2 \|\chi_B(H)\|_2^2 = \|A\|_\infty^2\, \mathbb{E}\left\{\langle \delta_0, \chi_B(H)\delta_0\rangle\right\}$$

$$\leqslant \|\rho\|_\infty \|A\|_\infty^2 |B|.$$

Unfortunately, knowing that $\mathcal{H}_L$, and hence also $\mathcal{H}_R$, has absolutely continuous spectrum does not imply that the Liouvillian $\mathcal{L} = \mathcal{H}_L - \mathcal{H}_R$ has no nonzero eigenvalues. (Note that 0 is always an eigenvalue for $\mathcal{L}$.)

The next lemma rewrites $\Psi_{E_F}(I_+ \times I_-)$ in ordinary $\ell^2(\mathbb{Z}^d)$-language. Recall that $f(H) \in \mathcal{K}_2 \cap \mathcal{K}_\infty$ and $[X_1, f(H)] \in \mathcal{K}_2$ if either $f \in \mathcal{S}(\mathbb{R})$, or f is a bounded Borel function with $f\chi_I = f$ for some bounded interval $I \subset \Xi^{\mathrm{CL}}$, or $f = \chi_{]-\infty, E]}$ with $E \in \Xi^{\mathrm{CL}}$ [BoGKS, Prop. 4.2].

LEMMA 4.4. *Let $F_\pm := f_\pm(H)$, where $f_\pm \geqslant 0$ are bounded Borel measurable functions on $\mathbb{R}$. Suppose*

$$F_- P_{E_F} = F_-, \quad F_+ P_{E_F} = 0, \quad and \quad F_\pm, [X_1, F_\pm] \in \mathcal{K}_2. \tag{4.13}$$

Then

$$\int_{\mathbb{R}^2} \Psi_{E_F}(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\, f_+^2(\lambda_1) f_-^2(\lambda_2) = \mathbb{E}\big\{\langle \delta_0, F_- X_1 F_+^2 X_1 F_- \delta_0 \rangle\big\}. \tag{4.14}$$

Proof. It follows from (3.38) that

$$\int_{\mathbb{R}^2} \Psi_{E_F}(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\, f_+^2(\lambda_1) f_-^2(\lambda_2) = \|F_+ \odot_L Y_{E_F} \odot_R F_-\|_2^2. \tag{4.15}$$

In view of (3.23) and (4.13), it follows from [BoGKS, Eq. (4.8)] that

$$-\mathrm{i}Y_{E_F} \odot_R F_- = [X_1, F_- P_{E_F}] - P_{E_F} \odot_L [X_1, F_-] = [X_1, F_-] - P_{E_F} \odot_L [X_1, F_-], \tag{4.16}$$

and hence

$$F_+ \odot_L Y_{E_F} \odot_R F_- = \mathrm{i}\, F_+ \odot_L [X_1, F_-]. \tag{4.17}$$

Thus, from (4.15),

$$\int_{\mathbb{R}^2} \Psi_{E_F}(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\, f_+^2(\lambda_1) f_-^2(\lambda_2) = \mathbb{E}\big\{\|F_+ X_1 F_- \delta_0\|_2^2\big\}, \tag{4.18}$$

which implies (4.14). □

Lemma 4.4 has the following corollary, which will be used to justify the replacement of spectral projections by smooth functions of H.

LEMMA 4.5. *Let $B_\pm$ be bounded Borel subsets of the region of complete localization Ξ^{CL} with $B_- \subset\,]-\infty, E_F]$ and $B_+ \cap\,]-\infty, E_F] = \varnothing$, so that*

$$P_- P_{E_F} = P_- \quad and \quad P_+ P_{E_F} = 0, \quad where \quad P_\pm := \chi_{B_\pm}(H), \tag{4.19}$$

and let $f_\pm$ and $F_\pm$ be as in Lemma 4.4 obeying $\chi_{B_\pm} \leqslant f_\pm \leqslant 1$. Then

$$\Psi_{E_F}(B_+ \times B_-) = \mathbb{E}\big\{\langle \delta_0, P_- X_1 P_+ X_1 P_- \delta_0 \rangle\big\} \tag{4.20}$$
$$\leqslant \mathbb{E}\big\{\langle \delta_0, F_- X_1 F_+ X_1 F_- \delta_0 \rangle\big\}. \tag{4.21}$$

Proof. The equality (4.20) follows from Lemma 4.4 with $f_\pm = \chi_{B_\pm}$. To prove the bound (4.21), note that we also have $\chi_{B_\pm} \leqslant f_\pm^2 \leqslant f_\pm \leqslant 1$, and hence, since

$$\Psi_{E_F}(B_+ \times B_-) \leqslant \int_{\mathbb{R}^2} \Psi_{E_F}(\mathrm{d}\lambda_1 \mathrm{d}\lambda_2)\, f_+^2(\lambda_1) f_-^2(\lambda_2), \tag{4.22}$$

(4.21) follows from (4.14) since $F_+^2 \leqslant F_+$. □

4.2. *Passage to finite volume.* Restricting the Anderson Hamiltonian to finite volume leads to a natural minimal distance between its eigenvalues, as shown in [KlM, Lemma 2] using Minami's estimate [M]. It is this natural distance that allows control over an eigenvalue correlation like (4.20).

The finite volumes will be cubes Λ_L with $L \geqslant 3$. Here Λ_L is the largest cube in $\mathbb{Z}^d$, centered at the origin and oriented along the coordinate axes, with $|\Lambda_L| \leqslant L^d$. We denote by H_L the (random) *finite-volume restriction* of the Anderson Hamiltonian H to $\ell^2(\Lambda_L)$ *with periodic boundary condition.* We will think of $\ell^2(\Lambda_L)$ as being naturally embedded into $\ell^2(\mathbb{Z}^d)$, with all operators defined on $\ell^2(\Lambda_L)$ acting on $\ell^2(\mathbb{Z}^d)$ via their trivial extension. In addition, it will be convenient to consider another extension of H_L to $\ell^2(\mathbb{Z}^d)$, namely

$$(4.23) \qquad \widehat{H}_L := H_L + \chi_{\Lambda_L^c} H \chi_{\Lambda_L^c},$$

where by S^c we denote the complement of the set S. We set $\partial S := \{x \in S : \text{there exists } y \in S^c \text{ with } |x - y| = 1\}$, the boundary of a subset S in $\mathbb{Z}^d$. Moreover, when convenient we use the notation $A(x, y) := \langle \delta_x, A \delta_y \rangle$ for the matrix elements of a bounded operator A on $\ell^2(\mathbb{Z}^d)$.

To prove (4.2), we rewrite $\Psi_{E_F}(I_+ \times I_-)$ as in (4.20), estimate the corresponding finite-volume quantity, and calculate the error committed in going from infinite to finite volume. To do so, we would like to express the spectral projections in (4.20) in terms of resolvents, where we can control the error by the resolvent identity. This can be done by means of the Helffer–Sjöstrand formula for *smooth* functions f of self-adjoint operators [HS], [HuS]. More precisely, it requires finiteness in one of the norms

$$(4.24) \qquad \{\{f\}\}_m := \sum_{r=0}^{m} \int_{\mathbb{R}} \mathrm{d}u \, |f^{(r)}(u)| \, (1 + |u|^2)^{\frac{r-1}{2}}, \quad m = 1, 2, \ldots .$$

If $\{\{f\}\}_m < \infty$ with $m \geqslant 2$, then for any self-adjoint operator K we have

$$(4.25) \qquad f(K) = \int_{\mathbb{R}^2} \mathrm{d}\tilde{f}(z) \, (K - z)^{-1},$$

where the integral converges absolutely in operator norm. Here $z = x + \mathrm{i}y$, $\tilde{f}(z)$ is an *almost analytic extension* of f to the complex plane, $\mathrm{d}\tilde{f}(z) := \frac{1}{2\pi} \partial_{\bar{z}} \tilde{f}(z) \, \mathrm{d}x \, \mathrm{d}y$, with $\partial_{\bar{z}} = \partial_x + \mathrm{i}\partial_y$, and $|\mathrm{d}\tilde{f}(z)| := (2\pi)^{-1} |\partial_{\bar{z}} \tilde{f}(z)| \, \mathrm{d}x \, \mathrm{d}y$. Moreover, for all $p \geqslant 0$ we have

$$(4.26) \qquad \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}(z)| \, \frac{1}{|\mathrm{Im}\, z|^p} \leqslant c_p \, \{\{f\}\}_m < \infty \quad \text{for} \quad m \geqslant p + 1$$

with a constant c_p (see [HuS, App. B] for details).

Thus we will pick appropriate smooth functions $f_\pm$ and estimate the error between the quantity in (4.21) and the corresponding finite volume quantity. The error will then be controlled by the following lemma.

LEMMA 4.6. *Let $I \subset \Xi^{\mathrm{CL}}$ be a compact interval, so that (2.3) holds for all $E \in I$ with the same ℓ and s. Then there exists a constant C such that for all C^4-functions $f_\pm$ with $\mathrm{supp}\, f_\pm \subset I$ and $|f_\pm| \leqslant 1$,*

$$(4.27) \quad \left| \mathbb{E}\left\{ \langle \delta_0, F_- X_1 F_+ X_1 F_- \delta_0 \rangle - \langle \delta_0, F_{-,L} X_1 F_{+,L} X_1 F_{-,L} \delta_0 \rangle \right\} \right|$$
$$\leqslant C \big(1 + \{\!\{ f_- \}\!\}_3 \big)^{\frac{2}{3}} \big(\{\!\{ f_+ \}\!\}_4 + \{\!\{ f_- \}\!\}_4 \big)^{\frac{1}{3}} L^{\frac{4}{3} d} \, \mathrm{e}^{-\frac{1}{12\ell} L}$$

for all $L \geqslant 3$, where $F_\pm := f_\pm(H)$ and $F_{\pm,L} := f_\pm(H_L)$.

Proof. Since $f_\pm = f_\pm \chi_I$ and $I \subset \Xi^{\mathrm{CL}}$ with I a compact interval, and $|f_\pm| \leqslant 1$, it follows from (4.3) that

$$(4.28) \quad \mathbb{E}\left\{ |\langle \delta_x, F_\pm \delta_y \rangle|^p \right\} \leqslant C_I \mathrm{e}^{-c_I |x-y|} \qquad \text{for all } p \in [1, \infty[\text{ and } x, y \in \mathbb{Z}^d,$$

where the constants C_I and $c_I > 0$ are independent of $f_\pm$. The corresponding estimates for $F_{\pm,L}$ and $F_\pm - \widehat{F}_{\pm,L}$, the two main technical estimates needed for the proof of Lemma 4.6, are isolated in the following sublemma.

SUBLEMMA 4.7. *Let the interval I be as in Lemma 4.6. Then there exist constants C_1, C_2 such that for all all C^4-functions f with $\mathrm{supp}\, f \subset I$, $L \geqslant 3$, and all $x, y \in \mathbb{Z}^d$,*

$$(4.29) \quad \mathbb{E}\left\{ |\langle \delta_x, (F - \widehat{F}_L)\delta_y \rangle| \right\} \leqslant C_1 \{\!\{ f \}\!\}_4 \, L^{2d-2} \, \mathrm{e}^{-\frac{1}{2\ell}\{\mathrm{dist}(x, \partial\Lambda_L) + \mathrm{dist}(y, \partial\Lambda_L)\}}$$

and

$$(4.30) \quad \mathbb{E}\left\{ |\langle \delta_0, \widehat{F}_L \delta_x \rangle| \right\} \leqslant C_2 \{\!\{ f \}\!\}_3 \, L^{d-1} \, \mathrm{e}^{-\frac{1}{\ell}|x|} \, \chi_{\Lambda_L}(x),$$

where $F := f(H)$ and $\widehat{F}_L := f(\widehat{H}_L)$.

Proof. Let $R(z) := (H - z)^{-1}$ and $\widehat{R}_L(z) := (\widehat{H}_L - z)^{-1}$ be the resolvents for H and $\widehat{H}_L$. It follows from the resolvent identity that

$$(4.31) \quad \widehat{R}_L(z) = R(z) + R(z)\Gamma_L \widehat{R}_L(z)$$

$$(4.32) \quad = R(z) + R(z)\Gamma_L R(z) - R(z)\Gamma_L \widehat{R}_L(z)\Gamma_L R(z),$$

where $\Gamma_L := H - \widehat{H}_L$. Note that either $\Gamma_L(x,y) = 0$ or $|\Gamma_L(x,y)| = 1$, and if $(x,y) \in \mathcal{E}_L := \{(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d : \Gamma_L(x,y) \neq 0\}$ we must have either $x \in \partial\Lambda_L$ or $y \in \partial\Lambda_L$ (or both, because we use periodic boundary conditions), and moreover $|\mathcal{E}_L| \leqslant 8d^2 L^{d-1}$.

To prove (4.29), we first apply the Helffer–Sjöstrand formula (4.25) to both F and $\widehat{F}_L$, use (4.32) and the crude estimate $\|\widehat{R}_L(z)\| \leqslant |\mathrm{Im}\, z|^{-1}$ to get

$$\mathbb{E}\left\{ |\langle \delta_x, (F - \widehat{F}_L)\delta_y \rangle| \right\}$$

$$(4.33) \quad \leqslant |\mathcal{E}_L| \sup_{(u,v) \in \mathcal{E}_L} \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}(z)| \, \mathbb{E}\left\{ |R(z; x, u)| \, |R(z; v, y)| \right\}$$

$$+ |\mathcal{E}_L|^2 \sup_{\substack{(u,v) \in \mathcal{E}_L \\ (w',w) \in \mathcal{E}_L}} \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}(z)| \, |\mathrm{Im}\, z|^{-1} \mathbb{E}\left\{ |R(z; x, u)| \, |R(z; w, y)| \right\}.$$

We now exploit the crude bound $\|R(z)\| \leqslant |\operatorname{Im} z|^{-1}$ and the Cauchy–Schwarz inequality to obtain fractional moments. This allows the use of (2.3) for $\operatorname{Re} z \in \operatorname{supp} f \subset I \subset \Xi^{\mathrm{CL}}$, obtaining,

$$
\mathbb{E}\{|R(z;x,u)|\,|R(z;v,y)|\}
$$

(4.34)
$$
\leqslant |\operatorname{Im} z|^{s-2}\, \mathbb{E}\{|R(z;x,u)|^s\}^{\frac{1}{2}} \mathbb{E}\{|R(z;v,y)|^s\}^{\frac{1}{2}}
$$

$$
\leqslant K|\operatorname{Im} z|^{s-2}\, \mathrm{e}^{-\frac{1}{2\ell}(|x-u|+|v-y|)}
$$

for all $x, u, v, y \in \mathbb{Z}^d$. Plugging the bound (4.34) into (4.33), and using (4.26) and properties of the set $\mathcal{E}_L$, we get the estimate (4.29).

The estimate (4.30) is proved along the same lines. We may assume $x \in \Lambda_L$, since otherwise the left-hand side is clearly zero. Proceeding as above, we get

$$
\mathbb{E}\{|\langle \delta_0, \widehat{F}\delta_x \rangle|\} \leqslant \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}(z)|\, \mathbb{E}\{|\widehat{R}_L(z;0,x)|\}
$$

(4.35)
$$
\leqslant \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}(z)|\, \mathbb{E}\{|R(z;0,x)|\}
$$

$$
+ |\mathcal{E}_L| \sup_{(u,v)\in\mathcal{E}_L} \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}(z)|\,|\operatorname{Im} z|^{-1}\mathbb{E}\{|R(z;0,u)|\}
$$

and

(4.36) $\quad \mathbb{E}\{|R(z;0,x)|\} \leqslant |\operatorname{Im} z|^{s-1}\, \mathbb{E}\{|R(z;0,x)|^s\} \leqslant K|\operatorname{Im} z|^{s-1}\, \mathrm{e}^{-\frac{1}{\ell}|x|}.$

The estimate (4.30) now follows. $\qquad\square$

We may now finish the proof of Lemma 4.6 using the fact that

(4.37) $\quad \langle \delta_0, F_{-,L} X_1 F_{+,L} X_1 F_{-,L} \delta_0 \rangle = \langle \delta_0, \widehat{F}_{-,L} X_1 \widehat{F}_{+,L} X_1 \widehat{F}_{-,L} \delta_0 \rangle,$

since $\chi_{\Lambda_L} F_{\pm,L} \chi_{\Lambda_L} = \chi_{\Lambda_L} \widehat{F}_{\pm,L} \chi_{\Lambda_L}$ and the operators $F_{\pm,L}$ and $\widehat{F}_{\pm,L}$ commute with χ_{Λ_L}. Thus

$$
\left| \mathbb{E}\left\{ \langle \delta_0, F_- X_1 F_+ X_1 F_- \delta_0 \rangle - \langle \delta_0, F_{-,L} X_1 F_{+,L} X_1 F_{-,L} \delta_0 \rangle \right\} \right|
$$

(4.38)
$$
\leqslant \left| \mathbb{E}\left\{ \langle \delta_0, (F_- - \widehat{F}_{-,L}) X_1 F_+ X_1 F_- \delta_0 \rangle \right\} \right|
$$

(4.39)
$$
+ \left| \mathbb{E}\left\{ \langle \delta_0, \widehat{F}_{-,L} X_1 F_+ X_1 (F_- - \widehat{F}_{-,L}) \delta_0 \rangle \right\} \right|
$$

(4.40)
$$
+ \left| \mathbb{E}\left\{ \langle \delta_0, \widehat{F}_{-,L} X_1 (F_+ - \widehat{F}_{+,L}) X_1 \widehat{F}_{-,L} \delta_0 \rangle \right\} \right|.
$$

Each term in the above inequality can be estimated by Hölder's inequality:

$$
|\mathbb{E}\left\{ \langle \delta_0, A_1 X_1 A_2 X_1 A_3 \delta_0 \rangle \right\}|
$$

$$
\leqslant \sum_{x,y\in\mathbb{Z}^d} |x_1|\,|y_1|\, \mathbb{E}\left\{ |A_1(0,x)|\,|A_2(x,y)|\,|A_3(y,0)| \right\}
$$

(4.41)
$$
\leqslant \sum_{x,y\in\mathbb{Z}^d} |x_1|\,|y_1|\, \mathbb{E}\left\{ |A_1(0,x)|^3 \right\}^{\frac{1}{3}} \mathbb{E}\left\{ |A_2(x,y)|^3 \right\}^{\frac{1}{3}} \mathbb{E}\left\{ |A_3(y,0)|^3 \right\}^{\frac{1}{3}},
$$

where A_j, $j = 1, 2, 3$, may be either $F_\pm$, $\widehat{F}_{-,L}$, or $F_\pm - \widehat{F}_{\pm,L}$. We estimate $\mathbb{E}\{|F_\pm(x, y)|^3\}$ by (4.28) and $\mathbb{E}\{|\widehat{F}_{-,L}(0, x)|^3\}$ by (4.30). If follows from (4.29) that

$$\mathbb{E}\{|(F_- - \widehat{F}_{-,L})(0, x)|^3\} \leqslant 4\,\mathbb{E}\{|(F_- - \widehat{F}_{-,L})(0, x)|\}$$

$$(4.42) \qquad\qquad \leqslant 4C_1\,\{\{f_-\}\}_4\,L^{2d-2}\,\mathrm{e}^{-\frac{1}{2\ell}(\mathrm{dist}(0,\partial\Lambda_L)+\mathrm{dist}(x,\partial\Lambda_L))}$$

$$(4.43) \qquad\qquad \leqslant 4C_1\,\{\{f_-\}\}_4\,L^{2d-2}\,\mathrm{e}^{-\frac{1}{2\ell}\frac{L-3}{2}},$$

since $|(F_- - \widehat{F}_{-,L})(0, x)| \leqslant 2$ and $\mathrm{dist}(0, \partial\Lambda_L) \geqslant \frac{L-3}{2}$. Thus we get, with some constant C,

$$(4.44) \qquad (4.38) + (4.39) \leqslant C\big(1 + \{\{f_-\}\}_3^{\frac{1}{3}}\big)\,\{\{f_-\}\}_4^{\frac{1}{3}}\,L^{d-1}\,\mathrm{e}^{-\frac{1}{6\ell}\frac{L-3}{2}}.$$

To estimate (4.40), we control $\mathbb{E}\{|(F_+ - \widehat{F}_{+,L})(x, y)|^3\}$ from (4.29) as in (4.42). We get, with constant C',

$$(4.40) \leqslant C'L^{\frac{4}{3}(d-1)}\,\{\{f_-\}\}_3^{\frac{2}{3}}\,\{\{f_+\}\}_4^{\frac{1}{3}}$$

$$(4.45) \qquad\qquad \times \sum_{x,y\in\Lambda_L} |x_1|\,|y_1|\,\mathrm{e}^{-\frac{1}{3\ell}(|x|+|y|)}\,\mathrm{e}^{-\frac{1}{6\ell}(\mathrm{dist}(x,\partial\Lambda_L)+\mathrm{dist}(y,\partial\Lambda_L))}$$

$$\leqslant C'L^{\frac{4}{3}(d-1)}\,\{\{f_-\}\}_3^{\frac{2}{3}}\,\{\{f_+\}\}_4^{\frac{1}{3}}\,\mathrm{e}^{-\frac{1}{6\ell}\frac{L-3}{2}},$$

since for $x \in \Lambda_L$ we have

$$(4.46) \qquad\qquad |x| + \mathrm{dist}(x, \partial\Lambda_L) \geqslant \mathrm{dist}(0, \partial\Lambda_L) \geqslant \frac{L-3}{2}.$$

The desired estimate (4.27) now follows from (4.38)–(4.40), (4.44), and (4.45), with a suitable constant C. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.3. *The finite volume estimate.* For the finite volume Anderson Hamiltonian H_L we have available a beautiful estimate due to Minami [M], which may be stated as

$$(4.47) \qquad \mathbb{E}\{\{\mathrm{tr}\,\chi_I(H_L)\}^2 - \mathrm{tr}\,\chi_I(H_L)\} \leqslant \pi^2\|\rho\|_\infty^2 |I|^2 |\Lambda_L|^2$$

for all intervals $I \subset \mathbb{R}$ and length scales $L \geqslant 1$. (See [KlM, App. A] for an outline of the argument.) Although Minami wrote his original proof for Dirichlet boundary condition, the result is valid for the usual boundary conditions, and in particular for periodic boundary conditions.

Remark 4.8. The dependence on $L \sim |\Lambda_L|^{\frac{1}{d}}$ on the right-hand side of (4.47) is optimal; it cannot be improved. Ergodicity implies that

$$(4.48) \qquad \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\,\mathrm{tr}\,\chi_B(H_L) = \mathbb{E}\{\langle\delta_0, \chi_B(H)\delta_0\rangle\} = \mathcal{N}(B) \quad \mathbb{P}\text{-a.s.,}$$

where $\mathcal{N}(B)$ is the density of states measure. If I and $I_\pm$ are intervals of nonzero lengths contained in the spectrum of H, we must have $\mathcal{N}(I), \mathcal{N}(I_\pm)$

> 0, and hence

(4.49) $\lim\limits_{L\to\infty} \frac{1}{|\Lambda_L|^2} \, \mathbb{E}\big\{\{\operatorname{tr}\chi_I(H_L)\}^2 - \operatorname{tr}\chi_I(H_L)\big\} = \mathcal{N}(I)^2 > 0,$

(4.50) $\lim\limits_{L\to\infty} \frac{1}{|\Lambda_L|^2} \, \mathbb{E}\big\{\{\operatorname{tr}\chi_{I_+}(H_L)\}\{\operatorname{tr}\chi_{I_-}(H_L)\}\big\} = \mathcal{N}(I_+)\mathcal{N}(I_-) > 0.$

LEMMA 4.9. *Let $J_\pm \subset \mathbb{R}$ be intervals such that $J_- \cap J_+ = \varnothing$, and consider an interval $J \supset J_- \cup J_+$. Then, with Borel functions $f_\pm$ on $\mathbb{R}$ with $0 \leqslant f_\pm \leqslant \chi_{J_\pm}$,*

(4.51) $\mathbb{E}\big\{\langle \delta_0, F_{-,L} X_1 F_{+,L} X_1 F_{-,L} \delta_0 \rangle\big\} \leqslant \frac{\pi^2}{4} \|\rho\|_\infty^2 |J|^2 L^{d+2}$

for all $L \geqslant 3$, where $F_{\pm,L} = f_\pm(H_L)$.

Proof. With periodic boundary condition finite volume expectations are invariant with respect to translations (in the torus). This, combined with $F_{-,L}F_{+,L} = 0$, gives

(4.52)
$$\begin{aligned} \mathbb{E}\big\{\langle \delta_0, &F_{-,L} X_1 F_{+,L} X_1 F_{-,L} \delta_0 \rangle\big\} \\ &= \tfrac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \mathbb{E}\big\{\langle \delta_x, F_{-,L} X_1 F_{+,L} X_1 F_{-,L} \delta_x \rangle\big\} \\ &= \tfrac{1}{|\Lambda_L|} \, \mathbb{E}\big\{\operatorname{tr}\{F_{-,L} X_1 F_{+,L} X_1 F_{-,L}\}\big\}, \end{aligned}$$

where the trace is taken in $\ell^2(\Lambda_L)$. Since $\|X_{1,L}\| \leqslant \frac{L}{2}$, where $X_{1,L} = X_1 \chi_{\Lambda_L}$ is the restriction of X_1 to $\ell^2(\Lambda_L)$, $0 \leqslant F_{\pm,L} \leqslant Q_{\pm,L} := \chi_{J_\pm}(H_L)$, and $Q_{+,L} + Q_{-,L} \leqslant Q_L := \chi_J(H_L)$, we have

(4.53) $\operatorname{tr}\big\{F_{-,L} X_1 F_{+,L} X_1 F_{-,L}\big\} \leqslant \|X_{1,L} F_{+,L} X_{1,L}\| \big(\operatorname{tr} F_{-,L}^2\big)$

(4.54) $\leqslant \tfrac{L^2}{4} \|F_{+,L}\| \big(\operatorname{tr} F_{-,L}^2\big) \leqslant \tfrac{L^2}{4} \big(\operatorname{tr} F_{+,L}\big)\big(\operatorname{tr} F_{-,L}^2\big)$

(4.55) $\leqslant \tfrac{L^2}{4} \big(\operatorname{tr} Q_{+,L}\big)\big(\operatorname{tr} Q_{-,L}\big) \leqslant \tfrac{L^2}{4}\big\{\big(\operatorname{tr} Q_L\big)^2 - \operatorname{tr} Q_L\big\}.$

Combining (4.52) and (4.53) – (4.55), and using Minami's estimate (4.47), we get

(4.56) $\mathbb{E}\big\{\langle \delta_0, F_{-,L} X_1 F_{+,L} X_1 F_{-,L} \delta_0 \rangle\big\} \leqslant \frac{\pi^2}{4} \|\rho\|_\infty^2 |J|^2 |\Lambda_L| L^2,$

which yields (4.51). $\square$

4.4. *The proof of Theorem* 4.1. We now have all the ingredients to prove Theorem 4.1. Since $E_F \in \Xi^{\mathrm{CL}}$, there is $\nu_0 \in \,]0,1[$ such that $I_0 := [E_F - \nu_0, E_F + \nu_0] \subset \Xi^{\mathrm{CL}}$, and (2.3) holds for all $E \in I_0$ with the same exponent $s = s_{E_F}$ and localization length $\ell = \ell_{E_F}$. Given $\nu \in \,]0,\nu_0]$, we define compact intervals

(4.57)
$$\begin{aligned} I &:= [E_F - \nu, E_F + \nu], \\ I_- &:= \,]E_F - \nu, E_F] \quad \text{and} \quad J_- := \,]E_F - \nu + \nu^4, E_F - \nu^4], \\ I_+ &:= \,]E_F, E_F + \nu] \quad \text{and} \quad J_+ := \,]E_F + \nu^4, E_F + \nu - \nu^4]. \end{aligned}$$

Note that $J_\pm \subset I_\pm \subset I \subset \Xi^{\mathrm{CL}}$ and $I_- \cap I_+ = \varnothing$. Moreover, we have $I_\pm \backslash J_\pm = J_{\pm,1} \cup J_{\pm,2}$, where $J_{\pm,j}$, $j = 1, 2$, are intervals of length $|J_{\pm,j}| = \nu^4$. Thus

$$\begin{aligned}
(4.58) \qquad \Psi_{E_F}(I_+ \times I_-) &= \Psi_{E_F}(J_+ \times J_-) + \Psi_{E_F}(I_+ \times J_{-,1}) + \Psi_{E_F}(I_+ \times J_{-,2}) \\
&\quad + \Psi_{E_F}(J_{+,1} \times J_-) + \Psi_{E_F}(J_{+,2} \times J_-) \\
&\leqslant \Psi_{E_F}(J_+ \times J_-) + 4 W_{\frac{1}{2}} \nu^2,
\end{aligned}$$

where the four terms containing $J_{\pm,1}$ and $J_{\pm,2}$ were estimated by Lemma 4.2 (with $\beta = 1/2$).

To estimate $\Psi_{E_F}(J_+ \times J_-)$, we exploit the existence of C^4-functions $f_\pm$ such that $\chi_{J_\pm} \leqslant f_\pm \leqslant \chi_{I_\pm}$ and $|f_\pm^{(k)}| \leqslant 2\nu^{-4k}\chi_{I_\pm \backslash J_\pm}$, $k = 1, 2, 3, 4$. Note that

$$(4.59) \qquad \{\{f_\pm\}\}_3 \leqslant \{\{f_\pm\}\}_4 \leqslant C\nu^{-16}\nu = C\nu^{-15},$$

where the constant C is independent of $\nu \in {]0, \nu_0]}$ and $f_\pm$. Using first (4.21) in Lemma 4.5 (with $B_\pm = J_\pm$ and $F_\pm = f_\pm(H)$) to replace the spectral projections by smooth functions of H, followed by Lemma 4.6 to achieve the passage to finite volume, we get

$$\begin{aligned}
(4.60) \qquad \Psi_{E_F}(J_+ \times J_-) &\leqslant \mathbb{E}\{\langle \delta_0, F_- X_1 F_+ X_1 F_- \delta_0 \rangle\} \\
&\leqslant \mathbb{E}\{\langle \delta_0, F_{-,L} X_1 F_{+,L} X_1 F_{-,L} \delta_0 \rangle\} + C'\nu^{-15} L^{\frac{4}{3}d} \, \mathrm{e}^{-\frac{1}{12\ell}L}
\end{aligned}$$

for all $L \geqslant 3$, where $F_{\pm,L} = f_\pm(H_L)$.

Combining (4.58) and (4.60), and using Lemma 4.9 to estimate the finite volume quantity, we get

$$(4.61) \qquad \Psi_{E_F}(I_+ \times I_-) \leqslant \pi^2 \|\rho\|_\infty^2 \nu^2 L^{d+2} + C'\nu^{-15} L^{\frac{4}{3}d} \, \mathrm{e}^{-\frac{1}{12\ell}L} + 4W_{\frac{1}{2}}\nu^2.$$

If we now choose

$$(4.62) \qquad L = (17 \cdot 12 + 1)\ell \log \tfrac{1}{\nu} = 205\ell \log \tfrac{1}{\nu},$$

then there exists $\nu_0' \in {]0, \nu_0]}$, such that for all $\nu \in {]0, \nu_0']}$,

$$(4.63) \qquad \Psi_{E_F}(I_+ \times I_-) \leqslant 205^{d+2}\pi^2 \|\rho\|_\infty^2 \ell^{d+2}\nu^2 \left(\log \tfrac{1}{\nu}\right)^{d+2} + C''\nu^2,$$

from which (4.2) follows.

Theorem 4.1 is proved, yielding Theorem 2.3.

Remark 4.10. As discussed in Remark 2.5, in our estimate for Mott's formula, namely (2.18) (or, equivalently, (4.2)), the exponent of $\log \tfrac{1}{\nu}$ is $d + 2$, instead of $d + 1$ as in (1.1). This comes from (4.56), where we get a factor of L^{d+2}. As seen in Remark 4.8, the power of L we acquire in the passage from (4.55) to (4.56) cannot be improved. The factor of L^2 obtained going from (4.53) to (4.54) must also be correct because of (4.62), since we need L^{d+2} in (4.56) to get ℓ^{d+2} in (4.63). To obtain a factor of $(\log \tfrac{1}{\nu})^{d+1}$ as in (1.1), we would need to improve the estimate in (4.53), (4.54) to gain an extra factor of $(\log \tfrac{1}{\nu})^{-1}$.

Remark 4.11. Starting from the lower bound given in Proposition 3.12, and proceeding as in the derivation of (4.60), we obtain the lower bound

$$(4.64) \qquad \overline{\sigma}_{E_F}^{\mathrm{in}}(\nu) \geqslant \tfrac{\pi}{2}\mathbb{E}\big\{\langle \delta_0, G_{-,L}X_1 G_{+,L}X_1 G_{-,L}\delta_0\rangle\big\} + C''\nu^{-15}L^{4d}\, \mathrm{e}^{-\frac{1}{12\ell}L},$$

where $G_{\pm,L} := g_\pm(H_L)$ and the functions $g_\pm$ satisfy $\chi_{B_\pm} \leqslant g_\pm \leqslant \chi_{J_\pm}$ with $J_\pm$ as in (3.53), $B_- := \,]E_F - \tfrac{\nu}{2} + \nu^4, E_F - \tfrac{\nu}{4} - \nu^4]$, and $B_+ := \,]E_F + \tfrac{\nu}{4} + \nu^4, E_F + \tfrac{\nu}{2} - \nu^4]$. Moreover, the functions $g_\pm$ are supposed to satisfy the hypotheses of Lemma 4.6 with respect to the intervals $B_\pm$, the estimate (4.59), and $f_\pm = \sqrt{g_\pm}$ satisfy the hypotheses of Lemma 4.5 with respect to the intervals $B_\pm$. Unfortunately, we are not able to obtain a useful lower bound for $\overline{\sigma}_{E_F}^{\mathrm{in}}(\nu)$ from (4.64) because we do not have a lower bound for the finite volume term; Minami's estimate gives only an upper bound.

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