Annals of Mathematics, **166** (2007), 317–345

Stability conditions on triangulated categories

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1. Introduction

This paper introduces the notion of a stability condition on a triangulated category. The motivation comes from the study of Dirichlet branes in string theory, and especially from M.R. Douglas's work on Π-stability. From a mathematical point of view, the most interesting feature of the definition is that the set of stability conditions $\operatorname{Stab}(\mathcal{D})$ on a fixed category $\mathcal{D}$ has a natural topology, thus defining a new invariant of triangulated categories. In a separate article [6] I give a detailed description of this space of stability conditions in the case that $\mathcal{D}$ is the bounded derived category of coherent sheaves on a K3 surface. The present paper though is almost pure homological algebra. After setting up the necessary definitions I prove a deformation result which shows that the space $\operatorname{Stab}(\mathcal{D})$ with its natural topology is a manifold, possibly infinite-dimensional.

1.1. Before going any further let me describe a simple example of a stability condition on a triangulated category. Let X be a nonsingular projective curve and let $\mathcal{D}(X)$ denote its bounded derived category of coherent sheaves. Recall [11] that any nonzero coherent sheaf E on X has a unique Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$$

whose factors E_j/E_{j-1} are semistable sheaves with descending slope $\mu = \deg / \operatorname{rank}$. Torsion sheaves should be thought of as having slope $+\infty$ and come first in the filtration. On the other hand, given an object $E \in \mathcal{D}(X)$, the truncations $\sigma_{\leqslant j}(E)$ associated to the standard t-structure on $\mathcal{D}(X)$ fit into triangles

$$\begin{tikzcd}
\cdots \arrow[r] & \sigma_{\leqslant j-1}(E) \arrow[rr] & & \sigma_{\leqslant j}(E) \arrow[rr] \arrow[dl] & & \sigma_{\leqslant j+1}(E) \arrow[r] \arrow[dl] & \cdots \\
& & A_j \arrow[ul, dashed] & & A_{j+1} \arrow[ul, dashed] & &
\end{tikzcd}$$

which allow one to break up E into its shifted cohomology sheaves $A_j = H^j(E)[-j]$. Combining these two ideas, one can concatenate the Harder-Narasimhan filtrations of the cohomology sheaves $H^j(E)$ to obtain a kind of filtration of any nonzero object $E \in \mathcal{D}(X)$ by shifts of semistable sheaves.

Let $K(X)$ denote the Grothendieck group of $\mathcal{D}(X)$. Define a group homomorphism $Z\colon K(X) \to \mathbb{C}$ by the formula

$$Z(E) = -\deg(E) + i\,\mathrm{rank}(E).$$

For each nonzero sheaf E on X, there is a unique branch $\phi(E)$ of $(1/\pi)\arg Z(E)$ lying in the interval $(0, 1]$. If one defines

$$\phi\big(E[k]\big) = \phi(E) + k,$$

for each integer k, then the filtration described above is by objects of descending phase ϕ, and in fact is unique with this property. Thus each nonzero object of $\mathcal{D}(X)$ has a kind of generalised Harder-Narasimhan filtration. Note that not all objects of $\mathcal{D}(X)$ have a well-defined phase, indeed many objects of $\mathcal{D}(X)$ define the zero class in $K(X)$. Nonetheless, the phase function is well-defined on the generating subcategory $\mathcal{P} \subset \mathcal{D}(X)$ consisting of shifts of semistable sheaves.

1.2. The definition of a stability condition on a triangulated category is obtained by abstracting these generalised Harder-Narasimhan filtrations of nonzero objects of $\mathcal{D}(X)$ together with the map Z as follows. Throughout the paper the Grothendieck group of a triangulated category $\mathcal{D}$ is denoted $K(\mathcal{D})$.

Definition 1.1. A stability condition $(Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ called the *central charge*, and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(a) if $E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,

(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,

(d) for each nonzero object $E \in \mathcal{D}$ there are a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd},$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j.

I shall always assume that the category $\mathcal{D}$ is essentially small, that is, that $\mathcal{D}$ is equivalent to a category in which the class of objects is a set. One can then consider the set of all stability conditions on $\mathcal{D}$. In fact it is convenient to restrict attention to stability conditions satisfying a certain technical condition called local-finiteness (Definition 5.7). I show how to put a natural topology on the set $\operatorname{Stab}(\mathcal{D})$ of such stability conditions, and prove the following theorem.

THEOREM 1.2. *Let $\mathcal{D}$ be a triangulated category. For each connected component $\Sigma \subset \operatorname{Stab}(\mathcal{D})$ there are a linear subspace $V(\Sigma) \subset \operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$, with a well-defined linear topology, and a local homeomorphism $\mathcal{Z}\colon \Sigma \to V(\Sigma)$ which maps a stability condition $(Z, \mathcal{P})$ to its central charge Z.*

It follows immediately from this theorem that each component $\Sigma \subset \operatorname{Stab}(\mathcal{D})$ is a manifold, locally modelled on the topological vector space $V(\Sigma)$.

1.3. Suppose now that $\mathcal{D}$ is linear over a field k. This means that the morphisms of $\mathcal{D}$ have the structure of a vector space over k, with respect to which the composition law is bilinear. Suppose further that $\mathcal{D}$ is of finite type, that is that for every pair of objects E and F of $\mathcal{D}$ the vector space $\bigoplus_i \operatorname{Hom}_{\mathcal{D}}(E, F[i])$ is finite-dimensional. In this situation one can define a bilinear form on $K(\mathcal{D})$, known as the Euler form, via the formula

$$\chi(E, F) = \sum_i (-1)^i \dim_k \operatorname{Hom}_{\mathcal{D}}(E, F[i]),$$

and a free abelian group $\mathcal{N}(\mathcal{D}) = K(\mathcal{D})/K(\mathcal{D})^{\perp}$ called the *numerical Grothendieck group* of $\mathcal{D}$. If this group $\mathcal{N}(\mathcal{D})$ has finite rank the category $\mathcal{D}$ is said to be *numerically finite*.

Suppose then that $\mathcal{D}$ is of finite type over a field, and define $\operatorname{Stab}_{\mathcal{N}}(\mathcal{D})$ to be the subspace of $\operatorname{Stab}(\mathcal{D})$ consisting of *numerical* stability conditions, that is, those for which the central charge $Z\colon K(\mathcal{D}) \to \mathbb{C}$ factors through the quotient group $\mathcal{N}(\mathcal{D})$. The following result is an immediate consequence of Theorem 1.2.

COROLLARY 1.3. *Suppose $\mathcal{D}$ is numerically finite. For each connected component $\Sigma \subset \operatorname{Stab}_{\mathcal{N}}(\mathcal{D})$ there are a subspace $V(\Sigma) \subset \operatorname{Hom}_{\mathbb{Z}}(\mathcal{N}(\mathcal{D}), \mathbb{C})$ and a local homeomorphism $\mathcal{Z}\colon \Sigma \to V(\Sigma)$ which maps a stability condition to its central charge Z. In particular Σ is a finite-dimensional complex manifold.*

There are two large classes of examples of numerically finite triangulated categories. Firstly, if A is a finite-dimensional algebra over a field, then the bounded derived category $\mathcal{D}(A)$ of finite-dimensional left A-modules is numerically finite. The corresponding space of numerical stability conditions will be denoted $\operatorname{Stab}(A)$. Secondly, if X is a smooth projective variety over $\mathbb{C}$ then the Riemann-Roch theorem shows that the bounded derived category $\mathcal{D}(X)$ of coherent sheaves on X is numerically finite. In this case the space of numerical stability conditions will be denoted $\mathrm{Stab}(X)$.

Obviously one would like to be able to compute these spaces of stability conditions in some interesting examples. The only case considered in this paper involves X as an elliptic curve. Here the answer is rather straightforward: $\mathrm{Stab}(X)$ is connected, and there is a local homeomorphism

$$\mathcal{Z}\colon\ \mathrm{Stab}(X) \to \mathbb{C}^2.$$

The image of this map is $\mathrm{GL}^+(2,\mathbb{R})$, the group of rank two matrices with positive determinant, considered as an open subset of $\mathbb{C}^2$ in the obvious way, and $\mathrm{Stab}(X)$ is the universal cover of this space. Perhaps of more interest is the quotient of $\mathrm{Stab}(X)$ by the group of autoequivalences of $\mathcal{D}(X)$. One has

$$\mathrm{Stab}(X)\,/\mathrm{Aut}\,\mathcal{D}(X) \cong \mathrm{GL}^+(2,\mathbb{R})\,/\mathrm{SL}(2,\mathbb{Z}),$$

which is a $\mathbb{C}^*$-bundle over the modular curve.

1.4. The motivation for the definition of a stability condition given above came from the work of Douglas on Π-stability for Dirichlet branes. It therefore seems appropriate to include here a short summary of some of Douglas' ideas. However the author is hardly an expert in this area, and this section will inevitably contain various inaccuracies and over-simplifications. The reader would be well-advised to consult the original papers of Douglas [7], [8], [9] and Aspinwall-Douglas [1]. Of course, those with no interest in string theory can happily skip to the next section.

String theorists believe that the supersymmetric nonlinear sigma model allows them to associate a $(2,2)$ superconformal field theory (SCFT) to a set of data consisting of a compact, complex manifold X with trivial canonical bundle, a Kähler class $\omega \in H^2(X,\mathbb{R})$ and a class $B \in H^2(X,\mathbb{R}/\mathbb{Z})$ induced by a closed 2-form on X known as the B-field. Assume for simplicity that X is a simply-connected threefold. The set of possible choices of these data up to equivalence then defines an open subset $\mathcal{U}_X$ of the moduli space $\mathcal{M}$ of SCFTs. This moduli space $\mathcal{M}$ has two foliations, which when restricted to $\mathcal{U}_X$ correspond to those obtained by holding constant either the complex structure of X or the complexified Kähler class $B + i\omega$.

It is worth bearing in mind that the open subset $\mathcal{U}_X \subset \mathcal{M}$ described above is just a neighbourhood of a particular 'large volume limit' of $\mathcal{M}$; a given component of $\mathcal{M}$ may contain points corresponding to sigma models on topologically distinct manifolds X and also points that do not correspond to sigma models at all. One of the long-term goals of the present work is to try to gain a clearer mathematical understanding of this moduli space $\mathcal{M}$.

The next step is to consider branes. These are boundary conditions in the SCFT and naturally form the objects of a category, with the space of morphisms between a pair of branes being the spectrum of open strings with boundaries on them. One of the most striking claims of recent work in string theory is that the SCFT corresponding to a nonlinear sigma model admits a 'topological twisting' in which the corresponding category of branes is equivalent to $\mathcal{D}(X)$, the bounded derived category of coherent sheaves on X. In particular this category does not depend on the so-called stringy Kähler moduli space of X, that is, the leaf $\mathcal{M}_K(X) \subset \mathcal{M}$ corresponding to a fixed complex structure on X.

Douglas starts from this point of view and proceeds to argue that at each point in $\mathcal{M}_K(X)$ there is a subcategory $\mathcal{P} \subset \mathcal{D}(X)$ whose objects are the physical or BPS branes for the corresponding SCFT. He also gives a precise criterion 'Π-stability' for describing how this subcategory $\mathcal{P}$ changes along continuous paths in $\mathcal{M}_K(X)$. An important point to note is that whilst the category of BPS branes is well-defined at any point in $\mathcal{M}_K(X)$, the embedding $\mathcal{P} \subset \mathcal{D}(X)$ is not, so that monodromy around loops in the Kähler moduli space leads to different subcategories $\mathcal{P} \subset \mathcal{D}(X)$, related to each other by autoequivalences of $\mathcal{D}(X)$.

The definition of a stability condition given above was an attempt to abstract the properties of the subcategories $\mathcal{P} \subset \mathcal{D}(X)$. Thus the points of the Kähler moduli space $\mathcal{M}_K(X)$ should be thought of as defining points in the quotient $\operatorname{Stab}(X)/\operatorname{Aut}\mathcal{D}(X)$, and the category $\mathcal{P} = \bigcup_\phi \mathcal{P}(\phi)$ should be thought of as the category of BPS branes at the corresponding point of $\mathcal{M}_K(X)$.

There is also a mirror side to this story. According to the predictions of mirror symmetry there is an involution σ of the moduli space $\mathcal{M}$ which identifies some part of the open subset $\mathcal{U}_X$ defined above with part of the corresponding set $\mathcal{U}_{\check{X}}$ associated to a mirror manifold $\check{X}$. This identification exchanges the two foliations, so that the Kähler moduli space of X becomes identified with the moduli of complex structures on $\check{X}$ and vice versa.

Kontsevich's homological mirror conjecture [13] predicts that the derived category $\mathcal{D}(X)$ is equivalent to the derived Fukaya category $\mathcal{D}\operatorname{Fuk}(\check{X})$. Roughly speaking, this equivalence is expected to take the subcategory $\mathcal{P}(\phi) \subset \mathcal{D}(X)$ at a particular point of $\mathcal{M}_K(X)$ to the subcategory of $\mathcal{D}\operatorname{Fuk}(\check{X})$ consisting of special Lagrangians of phase ϕ with respect to the corresponding complex structure on $\check{X}$. For more on this side of the picture see for example [18], [19].

Notation. The term generalised metric will be used to mean a distance function $d\colon X \times X \to [0,\infty]$ on a set X satisfying all the usual metric space axioms except that it need not be finite. Any such function defines a topology on X in the usual way and induces a metric space structure on each connected component of X.

The reader is referred to [10], [12], [20] for background on triangulated categories. I always assume that my categories are essentially small. I write

[1] for the shift (or translation) functor of a triangulated category and draw my triangles as follows

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, dashed] & \end{tikzcd}$$

where the dotted arrow means a morphism $C \to A[1]$. Sometimes I just write

$$A \longrightarrow B \longrightarrow C.$$

The Grothendieck group of a triangulated category $\mathcal{D}$ is denoted $K(\mathcal{D})$. Similarly, the Grothendieck group of an abelian category $\mathcal{A}$ is denoted $K(\mathcal{A})$.

A full subcategory $\mathcal{A}$ of a triangulated category $\mathcal{D}$ will be called extension-closed if whenever $A \to B \to C$ is a triangle in $\mathcal{D}$ as above, with $A \in \mathcal{A}$ and $C \in \mathcal{A}$, then $B \in \mathcal{A}$ also. The extension-closed subcategory of $\mathcal{D}$ generated by a full subcategory $\mathcal{S} \subset \mathcal{D}$ is the smallest extension-closed full subcategory of $\mathcal{D}$ containing $\mathcal{S}$.

Acknowledgements. My main debt is to Michael Douglas whose papers on Π-stability provided the key idea for this paper. I'm also indebted to Dmitry Arinkin and Vladimir Drinfeld who pointed out a simpler way to prove Theorem 7.1. Finally I'd like to thank Alexei Bondal, Mark Gross, Alastair King, Antony Maciocia, So Okada, Aidan Schofield and Richard Thomas for their comments and corrections.

2. Stability functions and Harder-Narasimhan filtrations

The definition of a stable vector bundle on a curve has two fundamental ingredients, namely the partial ordering $E \subset F$ arising from the notion of a subbundle, and the numerical ordering coming from the slope function $\mu(E)$. Both of these ingredients were generalised by A.N. Rudakov [16] to give an abstract notion of a stability condition on an abelian category. For the purposes of this paper, it will not be necessary to adopt the full generality of Rudakov's approach, which allowed for arbitrary orderings on abelian categories. In fact one need only consider orderings induced by certain phase functions, as follows.

Definition 2.1. A *stability function* on an abelian category $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$ the complex number $Z(E)$ lies in the strict upper half-plane

$$H = \{r \exp(i\pi\phi) : r > 0 \text{ and } 0 < \phi \leqslant 1\} \subset \mathbb{C}.$$

Given a stability function $Z\colon K(\mathcal{A}) \to \mathbb{C}$, the *phase* of an object $0 \neq E \in \mathcal{A}$ is defined to be

$$\phi(E) = (1/\pi) \arg Z(E) \in (0, 1].$$

The function ϕ allows one to order the nonzero objects of the category $\mathcal{A}$ and thus leads to a notion of stability for objects of $\mathcal{A}$. Of course one could equally well define this ordering using the function $-\operatorname{Im} Z(E)/\operatorname{Re} Z(E)$ taking values in $(-\infty, +\infty]$, but in what follows it will be important to use the phase function ϕ instead.

Definition 2.2. Let $Z\colon K(\mathcal{A}) \to \mathbb{C}$ be a stability function on an abelian category $\mathcal{A}$. An object $0 \neq E \in \mathcal{A}$ is said to be *semistable* (with respect to Z) if every subobject $0 \neq A \subset E$ satisfies $\phi(A) \leqslant \phi(E)$.

Of course one could equivalently define a semistable object $0 \neq E \in \mathcal{A}$ to be one for which $\phi(E) \leqslant \phi(B)$ for every nonzero quotient $E \twoheadrightarrow B$. The importance of semistable objects in this paper is that they provide a way to filter objects of $\mathcal{A}$. This is the so-called Harder-Narasimhan property, which was first proved for bundles on curves in [11].

Definition 2.3. Let $Z\colon K(\mathcal{A}) \to \mathbb{C}$ be a stability function on an abelian category $\mathcal{A}$. A *Harder-Narasimhan filtration* of an object $0 \neq E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are semistable objects of $\mathcal{A}$ with

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

The stability function Z is said to have the *Harder-Narasimhan property* if every nonzero object of $\mathcal{A}$ has a Harder-Narasimhan filtration.

Note that if $f\colon E \to F$ is a nonzero map between semistable objects then by considering $\operatorname{im} f \cong \operatorname{coim} f$ in the usual way, one sees that $\phi(E) \leqslant \phi(F)$. It follows easily from this that Harder-Narasimhan filtrations (when they exist) are unique. The following slight strengthening of a result of Rudakov [16] shows that the existence of Harder-Narasimhan filtrations is actually a rather weak assumption.

Proposition 2.4. *Suppose $\mathcal{A}$ is an abelian category and $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a stability function satisfying the chain conditions*

(a) *there are no infinite sequences of subobjects in $\mathcal{A}$*

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

with $\phi(E_{j+1}) > \phi(E_j)$ for all j,

(b) *there are no infinite sequences of quotients in* $\mathcal{A}$

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots$$

with $\phi(E_j) > \phi(E_{j+1})$ *for all* j.

Then Z *has the Harder-Narasimhan property.*

Proof. First note that if $E \in \mathcal{A}$ is nonzero then either E is semistable or there is a subobject $0 \neq E' \subset E$ with $\phi(E') > \phi(E)$. Repeating the argument and using the first chain condition we see that every nonzero object of $\mathcal{A}$ has a semistable subobject $A \subset E$ with $\phi(A) \geqslant \phi(E)$. A similar argument using the second chain condition gives the dual statement: every nonzero object of $\mathcal{A}$ has a semistable quotient $E \twoheadrightarrow B$ with $\phi(E) \geqslant \phi(B)$.

A maximally destabilising quotient (mdq) of an object $0 \neq E \in \mathcal{A}$ is defined to be a nonzero quotient $E \twoheadrightarrow B$ such that any nonzero quotient $E \twoheadrightarrow B'$ satisfies $\phi(B') \geqslant \phi(B)$, with equality holding only if $E \twoheadrightarrow B'$ factors via $E \twoheadrightarrow B$. By what was said above it is enough to check this condition under the additional assumption that B' is semistable. Note also that if $E \twoheadrightarrow B$ is an mdq then B must be semistable with $\phi(E) \geqslant \phi(B)$. The first step in the proof of the proposition is to show that mdqs always exist.

Take a nonzero object $E \in \mathcal{A}$. Clearly if E is semistable then the identity map $E \to E$ is an mdq. Otherwise, as above, there is a short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow E' \longrightarrow 0$$

with A semistable and $\phi(A) > \phi(E) > \phi(E')$. I claim that if $E' \twoheadrightarrow B$ is an mdq for E' then the induced quotient $E \twoheadrightarrow B$ is an mdq for E. Indeed, if $E \twoheadrightarrow B'$ is a quotient with B' semistable and $\phi(B') \leqslant \phi(B)$ then $\phi(B') < \phi(A)$ so that there is no map $A \to B'$ and the quotient $E \twoheadrightarrow B'$ factors via E', which proves the claim. Thus I can replace E by E' and repeat the argument. By the second chain condition, this process must eventually terminate. It follows that every nonzero object of $\mathcal{A}$ has an mdq.

Take a nonzero object $E \in \mathcal{A}$. If E is semistable then $0 \subset E$ is a Harder-Narasimhan filtration of E. Otherwise there is a short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow B \longrightarrow 0$$

with $E \twoheadrightarrow B$ an mdq and $\phi(E') > \phi(E)$. Suppose $E' \twoheadrightarrow B'$ is an mdq. Consider the following diagram of short exact sequences:

$$
(\dagger) \qquad \begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0. \\
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

It follows from the definition of B that $\phi(Q) > \phi(B)$ and hence $\phi(B') > \phi(B)$. Replacing E by E' and repeating the process, one obtains a sequence of subobjects of E

$$\cdots \subset E^i \subset E^{i-1} \subset \cdots \subset E^1 \subset E^0 = E$$

such that $\phi(E^i) > \phi(E^{i-1})$ and with semistable factors $F^i = E^i/E^{i-1}$ of ascending phase. This sequence must terminate by the first chain condition, and renumbering gives a Harder-Narasimhan filtration of E. □

3. t-structures and slicings

The notion of a t-structure was introduced by A. Beilinson, J. Bernstein and P. Deligne [3]. t-structures are the tool which allows one to see the different abelian categories embedded in a given triangulated category. A slightly different way to think about t-structures is that they provide a way to break up objects of a triangulated category into pieces (cohomology objects) indexed by the integers. The aim of this section is to introduce the notion of a slicing, which allows one to break up objects of the category into finer pieces indexed by the real numbers. I start by recalling the definition of a t-structure.

Definition 3.1. A t-*structure* on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{F} \subset \mathcal{D}$, satisfying $\mathcal{F}[1] \subset \mathcal{F}$, such that if one defines

$$\mathcal{F}^{\perp} = \{G \in \mathcal{D} : \mathrm{Hom}_{\mathcal{D}}(F, G) = 0 \text{ for all } F \in \mathcal{F}\},$$

then for every object $E \in \mathcal{D}$ there is a triangle $F \to E \to G$ in $\mathcal{D}$ with $F \in \mathcal{F}$ and $G \in \mathcal{F}^{\perp}$.

The motivating example is the *standard* t-*structure* on the bounded derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$, obtained by taking $\mathcal{F}$ to consist of all those objects of $\mathcal{D}(\mathcal{A})$ whose cohomology objects $H^i(E) \in \mathcal{A}$ are zero for all $i > 0$.

The *heart* of a t-structure $\mathcal{F} \subset \mathcal{D}$ is the full subcategory

$$\mathcal{A} = \mathcal{F} \cap \mathcal{F}^{\perp}[1] \subset \mathcal{D}.$$

It was proved in [3] that $\mathcal{A}$ is an abelian category, with the short exact sequences in $\mathcal{A}$ being precisely the triangles in $\mathcal{D}$ all of whose vertices are objects of $\mathcal{A}$.

A t-structure $\mathcal{F} \subset \mathcal{D}$ is said to be *bounded* if

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{F}[i] \cap \mathcal{F}^{\perp}[j].$$

A bounded t-structure $\mathcal{F} \subset \mathcal{D}$ is determined by its heart $\mathcal{A} \subset \mathcal{D}$. In fact $\mathcal{F}$ is the extension-closed subcategory generated by the subcategories $\mathcal{A}[j]$ for integers $j \geqslant 0$. The following easy result gives another characterisation of bounded t-structures. The proof is a good exercise in manipulating the definitions.

LEMMA 3.2. *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. Then $\mathcal{A}$ is the heart of a bounded* t*-structure $\mathcal{F} \subset \mathcal{D}$ if and only if the following two conditions hold*:

(a) *if $k_1 > k_2$ are integers then* $\mathrm{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) = 0$ *for all A, B of $\mathcal{A}$*,

(b) *for every nonzero object $E \in \mathcal{D}$ there are a finite sequence of integers*

$$k_1 > k_2 > \cdots > k_n$$

and a collection of triangles

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \mathcal{A}[k_j]$ for all j.

Taking Lemma 3.2 as a guide, one can now replace the integers k_j with real numbers ϕ_j to give the notion of a slicing. This is the key ingredient in the definition of a stability condition on a triangulated category. Some explicit examples will be given in Section 5.

Definition 3.3. A *slicing* $\mathcal{P}$ of a triangulated category $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying the following axioms:

(a) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

(b) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,

(c) for each nonzero object $E \in \mathcal{D}$ there are a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j.

Let $\mathcal{P}$ be a slicing of a triangulated category $\mathcal{D}$. It is an easy exercise to check that the decompositions of axiom (c) are uniquely defined up to isomorphism. Given a nonzero object $0 \neq E \in \mathcal{D}$ define real numbers $\phi^+_{\mathcal{P}}(E) = \phi_1$ and $\phi^-_{\mathcal{P}}(E) = \phi_n$. One has an inequality $\phi^-_{\mathcal{P}}(E) \leqslant \phi^+_{\mathcal{P}}(E)$ with equality holding precisely when $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$. When the slicing $\mathcal{P}$ is clear from the context I often drop it from the notation and write $\phi^{\pm}(E)$.

For any interval $I \subset \mathbb{R}$, define $\mathcal{P}(I)$ to be the extension-closed subcategory of $\mathcal{D}$ generated by the subcategories $\mathcal{P}(\phi)$ for $\phi \in I$. Thus, for example, the full subcategory $\mathcal{P}((a, b))$ consists of the zero objects of $\mathcal{D}$ together with those objects $0 \neq E \in \mathcal{D}$ which satisfy $a < \phi^-(E) \leqslant \phi^+(E) < b$.

Lemma 3.4. *Let $\mathcal{P}$ be a slicing of a triangulated category $\mathcal{D}$ and let $I \subset \mathbb{R}$ be an interval of length at most one. Suppose*

$$\begin{tikzcd}
A \arrow[rr] & & E \arrow[dl] \\
& B \arrow[ul, dashed] &
\end{tikzcd}$$

is a triangle in $\mathcal{D}$, all of whose vertices are nonzero objects of $\mathcal{P}(I)$. Then there are inequalities $\phi^+(A) \leqslant \phi^+(E)$ and $\phi^-(E) \leqslant \phi^-(B)$.

Proof. It is enough to prove the first inequality since the second then follows in the same way. One can also assume that $I = [t, t+1]$ for some $t \in \mathbb{R}$. By definition, if $\phi = \phi^+(A)$ there is an object $A^+ \in \mathcal{P}(\phi)$ with a nonzero morphism $f\colon A^+ \to A$. Suppose for a contradiction that $\phi > \phi^+(E)$. Then there are no nonzero morphisms $A^+ \to E$ and so f factors via $B[-1]$. But $B[-1] \in \mathcal{P}(\leqslant t)$ and so this implies that $\phi \leqslant t$. Since $\phi^+(E) \geqslant t$ this gives the required contradiction. $\square$

Let $\mathcal{P}$ be a slicing of a triangulated category $\mathcal{D}$. For any $\phi \in \mathbb{R}$ one has pairs of orthogonal subcategories $(\mathcal{P}(>\phi), \mathcal{P}(\leqslant \phi))$ and $(\mathcal{P}(\geqslant \phi), \mathcal{P}(<\phi))$.

Note that the subcategories $\mathcal{P}(>\phi)$ and $\mathcal{P}(\geqslant\phi)$ are closed under left shifts and thus define t-structures[1] on $\mathcal{D}$. So for each $\phi \in \mathbb{R}$ there are t-structures $\mathcal{P}(> \phi) \subset \mathcal{P}(\geqslant \phi)$ on $\mathcal{D}$, indexed by the real numbers, which are compatible in the sense that

$$\phi \geqslant \psi \implies \mathcal{P}(> \phi) \subset \mathcal{P}(> \psi) \text{ and } \mathcal{P}(\geqslant \phi) \subset \mathcal{P}(\geqslant \psi).$$

Of course one could axiomatise these compatible t-structures to give a slightly weaker notion than that of a slicing. Note that the heart of the t-structure $\mathcal{P}(> \phi)$ is the subcategory $\mathcal{P}((\phi, \phi + 1]) \subset \mathcal{D}$, and similarly, the t-structure $\mathcal{P}(\geqslant \phi)$ has heart $\mathcal{P}([\phi, \phi + 1))$. As a matter of convention, the heart of the slicing $\mathcal{P}$ is defined to be the abelian subcategory $\mathcal{P}((0, 1]) \subset \mathcal{D}$.

4. Quasi-abelian categories

Let $\mathcal{P}$ be a slicing of a triangulated category $\mathcal{D}$. It was observed in the last section that for any real number ϕ the full subcategories $\mathcal{P}((\phi, \phi+1])$ and $\mathcal{P}([\phi, \phi + 1))$ of $\mathcal{D}$ are the hearts of t-structures on $\mathcal{D}$ and hence are abelian. Suppose instead that $I \subset \mathbb{R}$ is an interval of length < 1 and consider the corresponding full subcategory $\mathcal{A} = \mathcal{P}(I) \subset \mathcal{D}$. In general this category $\mathcal{A}$ will not be abelian, but it does have a natural exact structure [15], obtained by defining a short exact sequence in $\mathcal{A}$ to be a triangle in $\mathcal{D}$ all of whose vertices are objects of $\mathcal{A}$. In fact this exact structure is intrinsic to $\mathcal{A}$ and can be derived from the fact that $\mathcal{A}$ is a so-called *quasi-abelian category*. Although this notion is not strictly necessary for the proof of Theorem 1.2, it seems worthwhile to summarise the basic definitions concerning quasi-abelian categories, since they undoubtedly provide the right context for discussing these subcategories $\mathcal{P}(I) \subset \mathcal{D}$. At a first reading it might be a good idea to skip this section, since it is really only used in Section 7. The main reference for quasi-abelian categories is J.-P. Schneiders' paper [17]; see also [5, Appendix B].

Suppose then that $\mathcal{A}$ is an additive category with kernels and cokernels. Note that any such category has pushouts and pullbacks. Given a morphism $f \colon E \to F$ in $\mathcal{A}$, the image of f is the kernel of the canonical map $F \to \operatorname{coker} f$, and the coimage of f is the cokernel of the canonical map $\ker f \to E$. There is a canonical map $\operatorname{coim} f \to \operatorname{im} f$, and f is called *strict* if this map is an isomorphism. An abelian category is by definition an additive category with kernels and cokernels in which all morphisms are strict. The following definition gives a weaker notion.

[1] There is an unavoidable clash of notation here: in the standard notation for t-structures $\operatorname{Hom}_{\mathcal{D}}(E, F)$ vanishes providing $E \in \mathcal{D}^{\leqslant k}$ and $F \in \mathcal{D}^{>k}$, but in the notation for stability $\operatorname{Hom}_A(E, F)$ vanishes for E and F semistable providing E has slope $> k$ and F has slope $\leqslant k$.

Definition 4.1. A *quasi-abelian category* is an additive category $\mathcal{A}$ with kernels and cokernels such that every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism.

A strict short exact sequence in a quasi-abelian category $\mathcal{A}$ is a diagram

$$(*) \qquad 0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

in which i is the kernel of j and j is the cokernel of i. It follows that i is a strict monomorphism and j is a strict epimorphism. Conversely, if $i\colon A \to B$ is a strict monomorphism, the cokernel of i is a strict epimorphism $j\colon B \to C$ whose kernel is i. Similarly, a strict epimorphism $j\colon B \to C$ has a kernel i fitting into a strict short exact sequence as above. The class of strict monomorphisms (respectively epimorphisms) is closed under composition, and if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

is a commutative diagram, then h a strict monomorphism implies that f is a strict monomorphism, and similarly, h a strict epimorphism implies that g is a strict epimorphism. These facts are enough to show that a quasi-abelian category together with its class of strict short exact sequences is an exact category [15]. The Grothendieck group of $\mathcal{A}$ is defined to be the abelian group $K(\mathcal{A})$ generated by the objects of $\mathcal{A}$, with a relation $[B] = [A] + [C]$ for each strict short exact sequence $(*)$.

The following characterization of quasi-abelian categories was proved by Schneiders [17, Lemma 1.2.34].

LEMMA 4.2. *An additive category $\mathcal{A}$ is quasi-abelian if and only if there are abelian categories $\mathcal{A}^\sharp$ and $\mathcal{A}^\flat$ and fully faithful embeddings $\mathcal{A} \subset \mathcal{A}^\sharp$ and $\mathcal{A} \subset \mathcal{A}^\flat$ such that*

(a) *if $A \to E$ is a monomorphism in $\mathcal{A}^\sharp$ with $E \in \mathcal{A}$ then $A \in \mathcal{A}$ also*,

(b) *if $E \to B$ is an epimorphism in $\mathcal{A}^\flat$ with $E \in \mathcal{A}$ then $B \in \mathcal{A}$ also.*

If these conditions hold, *the strict short exact sequences in $\mathcal{A}$ are precisely those sequences $(*)$ which are exact in both $\mathcal{A}^\sharp$ and $\mathcal{A}^\flat$.*

A good example to bear in mind is the category $\mathcal{A}$ of torsion-free sheaves on a smooth projective variety. I leave it to the reader to check that this category is quasi-abelian. A monomorphism in $\mathcal{A}$ is just an injective morphism of sheaves. An epimorphism is a morphism of sheaves whose cokernel is torsion.

The kernel of a morphism of torsion-free sheaves in $\mathcal{A}$ is just the usual sheaf-theoretic kernel, but the cokernel in $\mathcal{A}$ is the usual cokernel modded out by its torsion subsheaf. All epimorphisms are strict, whereas a monomorphism is strict precisely if its cokernel as a map of sheaves is torsion-free.

LEMMA 4.3. *Let $\mathcal{P}$ be a slicing of a triangulated category $\mathcal{D}$. For any interval $I \subset \mathbb{R}$ of length < 1, the full subcategory $\mathcal{P}(I) \subset \mathcal{D}$ is quasi-abelian. The strict short exact sequences in $\mathcal{A}$ are in one-to-one correspondence with triangles in $\mathcal{D}$ all of whose vertices are objects of $\mathcal{A}$.*

Proof. Assume for definiteness that $I = (a, b)$ with $0 < b - a < 1$. The other cases are equally easy. The result then follows by application of Lemma 4.2 to the embeddings $\mathcal{P}((a, b)) \subset \mathcal{P}((a, a + 1])$ and $\mathcal{P}((a, b)) \subset \mathcal{P}(([b - 1, b))$ and by use of Lemma 3.4. $\square$

In what follows I shall abuse notation in a number of ways. Suppose A, B and E are objects of a quasi-abelian category $\mathcal{A}$. Then I shall write $A \subset E$ to mean that there is a strict monomorphism $i\colon A \to E$. I shall also call A a strict subobject of E and write E/A for the cokernel of i. Similarly, I write $E \twoheadrightarrow B$ to mean that there is a strict epimorphism $E \to B$ in $\mathcal{A}$ and refer to B as a strict quotient of E.

As in the case of an abelian category, the partial order $\subset$ allows one to say what it means for a quasi-abelian category $\mathcal{A}$ to be artinian or noetherian. Thus, for example, $\mathcal{A}$ is artinian if any infinite chain

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

of strict subobjects in $\mathcal{A}$ stabilises. If $\mathcal{A}$ is artinian and noetherian then it is said to be of finite length. For example, the category $\mathcal{A}$ of torsion-free sheaves described above is of finite length, because the rank function is additive on $\mathcal{A}$ and every nonzero object of $\mathcal{A}$ has positive rank.

Using the notion of a strict subobject in a quasi-abelian category, one can give a definition of semistability in a quasi-abelian category, depending on a choice of stability function $Z\colon K(\mathcal{A}) \to \mathbb{C}$. Of course there is no reason to expect the resulting notion to have good properties. Nonetheless, the proof of Theorem 1.2 will hinge on showing that in certain cases this notion of stability in a quasi-abelian category does in fact behave nearly as well as in the abelian case.

It will be convenient to extend the definition in order to include possibly skewed stability functions as follows.

Definition 4.4. A *skewed stability function* on a quasi-abelian category $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that there is a strict half-plane

$$H_\alpha = \{r \exp(i\pi\phi) : r > 0 \text{ and } \alpha < \phi \leqslant \alpha + 1\} \subset \mathbb{C},$$

defined by some $\alpha \in \mathbb{R}$, such that $Z(E) \in H_\alpha$ for all objects $0 \neq E \in \mathcal{A}$.

Clearly one can always reduce to the unskewed case $\alpha = 0$ but in fact it will not always be convenient to do so. Given a skewed stability function $Z\colon K(\mathcal{A}) \to \mathbb{C}$, define the phase of an object $0 \neq E \in \mathcal{A}$ to be

$$\phi(E) = (1/\pi) \arg Z(E) \in (\alpha, \alpha + 1].$$

An object $0 \neq E \in \mathcal{A}$ is then defined to be semistable if for every strict subobject $0 \neq A \subset E$ one has $\phi(A) \leqslant \phi(E)$. An equivalent condition is that $\phi(E) \leqslant \phi(B)$ for every nonzero strict quotient $E \twoheadrightarrow B$.

A Harder-Narasimhan filtration of an object $0 \neq E \in \mathcal{A}$ is a finite chain of strict subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are semistable objects of $\mathcal{A}$ with

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

Recall that when $\mathcal{A}$ is abelian, Harder-Narasimhan filtrations are unique, essentially because if $f\colon E \to F$ is a nonzero map between semistable objects then $\phi(E) \leqslant \phi(F)$. But the proof of this fact depends on the assumption that all morphsims are strict, so there is no reason to expect the corresponding result to hold in the quasi-abelian context.

5. Stability conditions

This section introduces the idea of a stability condition on a triangulated category, which combines the notions of slicing and stability function. The mathematical justification for this combination seems to be that, as Theorem 1.2 shows, it leads to nice deformation properties.

Definition 5.1. A *stability condition* $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ and a slicing $\mathcal{P}$ of $\mathcal{D}$ such that if $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$.

The linear map $Z\colon K(\mathcal{D}) \to \mathbb{C}$ will be referred to as the *central charge* of the stability condition. The following lemma shows that each category $\mathcal{P}(\phi)$ is abelian. The nonzero objects of $\mathcal{P}(\phi)$ are said to be *semistable* in σ of phase ϕ, and the simple objects of $\mathcal{P}(\phi)$ are said to be *stable*.

LEMMA 5.2. *If $\sigma = (Z, \mathcal{P})$ is a stability condition on a triangulated category $\mathcal{D}$ then each subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ is abelian.*

Proof. The category $\mathcal{P}(\phi)$ is a full additive subcategory of the abelian category $\mathcal{A} = \mathcal{P}((\phi - 1, \phi])$. It will therefore be enough to show that if $f\colon E \to F$

is a morphism in $\mathcal{P}(\phi)$ then the kernel and cokernel of f, considered as a morphism of $\mathcal{A}$, actually lie in $\mathcal{P}(\phi)$. But if

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$ and E is an object of $\mathcal{P}(\phi)$ then Lemma 3.4 implies that $B \in \mathcal{P}(\phi)$ and drawing a picture one sees that $A \in \mathcal{P}(\phi)$ also. $\square$

Let $\sigma = (Z, \mathcal{P})$ be a stability condition on a triangulated category $\mathcal{D}$. Recall that the decomposition of an object $0 \neq E \in \mathcal{D}$ given in the definition of a slicing is unique; the objects A_j will be called the *semistable factors* of E with respect to σ. I shall write $\phi^{\pm}_{\sigma}(E)$ for $\phi^{\pm}_{\mathcal{P}}(E)$; thus $\phi^{+}_{\sigma}(E) \geqslant \phi^{-}_{\sigma}(E)$ with equality precisely if E is semistable in σ. The *mass* of E is defined to be the positive real number $m_{\sigma}(E) = \sum_i |Z(A_i)|$. By the triangle inequality one has $m_{\sigma}(E) \geqslant |Z(E)|$. When the stability condition σ is clear from the context I often drop it from the notation and write $\phi^{\pm}(E)$ and $m(E)$.

The following result shows the relationship between t-structures and stability conditions.

Proposition 5.3. *To give a stability condition on a triangulated category $\mathcal{D}$ is equivalent to giving a bounded* t*-structure on $\mathcal{D}$ and a stability function on its heart with the Harder-Narasimhan property.*

Proof. Note first that if $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ then $K(\mathcal{A})$ can be identified with $K(\mathcal{D})$. If $\sigma = (Z, \mathcal{P})$ is a stability condition on $\mathcal{D}$, the t-structure $\mathcal{P}(> 0)$ is bounded with heart $\mathcal{A} = \mathcal{P}((0, 1])$. The central charge Z defines a stability function on $\mathcal{A}$ and it is easy to check that the corresponding semistable objects are precisely the nonzero objects of the categories $\mathcal{P}(\phi)$ for $0 < \phi \leqslant 1$. The decompositions of objects of $\mathcal{A}$ given by Definition 3.3(c) are Harder-Narasimhan filtrations.

For the converse, suppose $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ and $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a stability function on $\mathcal{A}$ with the Harder-Narasimhan property. Define a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ as follows. For each $\phi \in (0, 1]$ let $\mathcal{P}(\phi)$ be the full additive subcategory of $\mathcal{D}$ consisting of semistable objects of $\mathcal{A}$ with phase ϕ, together with the zero objects of $\mathcal{D}$. The first condition of Definition 3.3 then determines $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$ and condition (b) is easily verified. For any nonzero $E \in \mathcal{D}$ a filtration as in Definition 3.3(c) can be obtained by combining the decompositions of Lemma 3.2 with the Harder-Narasimhan filtrations of nonzero objects of $\mathcal{A}$. $\square$

I shall now give some examples of stability conditions.

Example 5.4. Let $\mathcal{A}$ be the category of coherent sheaves on a nonsingular projective curve X over an algebraically closed field k of characteristic zero, and set $Z(E) = -\deg(E) + i\,\mathrm{rank}(E)$ as in the introduction. Applying Proposition 5.3 gives a stability condition on the bounded derived category $\mathcal{D}(\mathcal{A})$.

This example will be considered in more detail in Section 9 below, where I study the set of all stability conditions on the derived category of an elliptic curve.

Example 5.5. Let A be a finite-dimensional algebra over a field k. Let $\mathcal{A}$ be the abelian category of finite-dimensional left A-modules. Thus $\mathcal{A}$ is a finite-length category whose Grothendieck group $K(\mathcal{A})$ is isomorphic to the free abelian group on the finite set of isomorphism classes of simple A-modules. There is a group homomorphism $r\colon K(\mathcal{A}) \to \mathbb{Z}$ sending an A-module to its dimension as a vector space over k. For any homomorphism $\lambda\colon K(\mathcal{A}) \to \mathbb{R}$ the formula $Z(E) = \lambda(E) + ir(E)$ defines a stability function on $\mathcal{A}$, and Proposition 5.3 shows that each of these stability functions determines a stability condition on the bounded derived category $\mathcal{D}(\mathcal{A})$.

The final example is rather degenerate and is included purely to motivate the introduction of the local-finiteness condition below.

Example 5.6. Let $\mathcal{A}$ be the category of coherent sheaves on a nonsingular projective curve X as in Example 5.4, and let $(Z, \mathcal{P})$ be the stability condition on $\mathcal{D}(\mathcal{A})$ defined there. Let $0 < \alpha < 1/2$ be such that $\zeta = \tan(\pi\alpha)$ is irrational. Then the bounded t-structure $\mathcal{P}(> \alpha) = \mathcal{P}(\geqslant \alpha) \subset \mathcal{D}$ has heart $\mathcal{B} = \mathcal{P}((\alpha, \alpha+1))$. Define a stability function on $\mathcal{B}$ by the formula

$$W(E) = i(\mathrm{rank}(E) + \zeta \deg(E)).$$

Note that all nonzero objects of $\mathcal{B}$ are semistable with the same phase. Applying Proposition 5.3 gives a stability condition $(W, \mathcal{Q})$ on $\mathcal{D}$ such that $\mathcal{Q}(\frac{1}{2}) = \mathcal{B}$, and $\mathcal{Q}(\psi) = 0$ unless $\psi - \frac{1}{2} \in \mathbb{Z}$.

In order to eliminate such examples and to prove nice theorems it will be useful to impose the following extra condition on stability conditions.

Definition 5.7. A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ is *locally-finite* if there exists a real number $\eta > 0$ such that for all $t \in \mathbb{R}$ the quasi-abelian category $\mathcal{P}((t-\eta, t+\eta)) \subset \mathcal{D}$ is of finite length. A stability condition $(Z, \mathcal{P})$ is locally-finite if the slicing $\mathcal{P}$ is.

It is easy to see that the first two examples of stability conditions given above are locally-finite. But the stability condition described in Example 5.6 is not locally-finite in general, because as one can easily check, the abelian category $\mathcal{B}$ is not always of finite length.

6. The space of stability conditions

Fix a triangulated category $\mathcal{D}$ and write $\mathrm{Slice}(\mathcal{D})$ for the set of locally-finite slicings of $\mathcal{D}$ and $\mathrm{Stab}(\mathcal{D})$ for the set of locally-finite stability conditions on $\mathcal{D}$. The aim of this section is to define natural topologies on these spaces. In fact, everything in this section applies equally well without the locally-finite condition, which will only become important in Section 7.

The first observation to be made is that the function

$$d(\mathcal{P},\mathcal{Q})=\sup_{0\neq E\in\mathcal{D}}\left\{|\phi^-_{\mathcal{P}}(E)-\phi^-_{\mathcal{Q}}(E)|,|\phi^+_{\mathcal{P}}(E)-\phi^+_{\mathcal{Q}}(E)|\right\}\in[0,\infty]$$

defines a generalised metric[2] on $\mathrm{Slice}(\mathcal{D})$. To check this one just needs to note that if $d(\mathcal{P},\mathcal{Q})=0$ then every nonzero object of $\mathcal{P}(\phi)$ is also an object of $\mathcal{Q}(\phi)$ so that $\mathcal{P}=\mathcal{Q}$. The following lemma gives another way of writing this metric.

Lemma 6.1. *If $\mathcal{P}$ and $\mathcal{Q}$ are slicings of a triangulated category $\mathcal{D}$ then*

$$d(\mathcal{P},\mathcal{Q})=\inf\big\{\varepsilon\in\mathbb{R}_{\geqslant 0}:\mathcal{Q}(\phi)\subset\mathcal{P}([\phi-\varepsilon,\phi+\varepsilon])\text{ for all }\phi\in\mathbb{R}\big\}.$$

Proof. Write $d'(\mathcal{P},\mathcal{Q})$ for the expression in the statement of the Lemma. First note that if $d(\mathcal{P},\mathcal{Q})\leqslant\varepsilon$ then for any nonzero $E\in\mathcal{Q}(\phi)$ one has $\phi^+(E)\leqslant\phi+\varepsilon$ and similarly $\phi^-(E)\geqslant\phi-\varepsilon$. This implies that $\mathcal{Q}(\phi)\subset\mathcal{P}([\phi-\varepsilon,\phi+\varepsilon])$ and so $d'(\mathcal{P},\mathcal{Q})\leqslant\varepsilon$.

For the reverse inequality suppose $d'(\mathcal{P},\mathcal{Q})\leqslant\varepsilon$ and take a nonzero object $E\in\mathcal{D}$. Clearly if $E\in\mathcal{Q}(\leqslant\psi)$ then $E\in\mathcal{P}(\leqslant\psi+\varepsilon)$. But in the other direction, if $E\notin\mathcal{Q}(\leqslant\psi)$ then there is some object $A\in\mathcal{Q}(\phi)$ with $\phi>\psi$ and a nonzero map $A\to E$. Since $\mathcal{Q}(\phi)\subset\mathcal{P}([\phi-\varepsilon,\phi+\varepsilon])$ it follows that $E\notin\mathcal{P}(\leqslant\psi-\varepsilon)$.

These arguments show that $|\phi^+_{\mathcal{P}}(E)-\phi^+_{\mathcal{Q}}(E)|\leqslant\varepsilon$, and a similar argument with ϕ^- completes the proof that $d(\mathcal{P},\mathcal{Q})\leqslant\varepsilon$. □

Consider the inclusion of sets

$$\mathrm{Stab}(\mathcal{D})\subset\mathrm{Slice}(\mathcal{D})\times\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C}).$$

When $K(\mathcal{D})$ has finite rank, one can give the vector space on the right the standard topology, and obtain an induced topology on $\mathrm{Stab}(\mathcal{D})$. In general however, one has to be a little careful, since there is no obviously natural choice of topology on $\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C})$.

For each $\sigma=(Z,\mathcal{P})\in\mathrm{Stab}(\mathcal{D})$, define a function

$$\|\cdot\|_\sigma\colon\ \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C})\to[0,\infty]$$

[2] See the notation section.

by sending a group homomorphism $U\colon K(\mathcal{D}) \to \mathbb{C}$ to

$$\|U\|_\sigma = \sup\left\{\frac{|U(E)|}{|Z(E)|} : E \text{ semistable in } \sigma\right\}.$$

Note that $\|\cdot\|_\sigma$ has all the properties of a norm on the complex vector space $\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C})$, except that it may not be finite.

For each real number $\varepsilon \in (0, 1/8)$, define a subset

$$B_\varepsilon(\sigma) = \{\tau = (W, \mathcal{Q}) : \|W - Z\|_\sigma < \sin(\pi\varepsilon) \text{ and } d(\mathcal{P},\mathcal{Q}) < \varepsilon\} \subset \mathrm{Stab}(\mathcal{D}).$$

To understand this definition note that the condition $\|W - Z\|_\sigma < \sin(\pi\varepsilon)$ implies that for all objects E semistable in σ, the phase of $W(E)$ differs from the phase of $Z(E)$ by less than ε.

I claim that as σ varies in $\mathrm{Stab}(\mathcal{D})$ the subsets $B_\varepsilon(\sigma)$ form a basis for a topology on $\mathrm{Stab}(\mathcal{D})$. This boils down to the statement that if $\tau \in B_\varepsilon(\sigma)$ then there is an $\eta > 0$ such that $B_\eta(\tau) \subset B_\varepsilon(\sigma)$, which comes easily from the following crucial lemma.

LEMMA 6.2. *If $\tau = (W, \mathcal{Q}) \in B_\varepsilon(\sigma)$ then there are constants $k_i > 0$ such that*

$$k_1\|U\|_\sigma < \|U\|_\tau < k_2\|U\|_\sigma$$

for all $U \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C})$.

Proof. First, note that for any stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$, and any real number $0 \leqslant \eta < \frac{1}{2}$, one has

$$(*) \qquad |U(E)| < \frac{\|U\|_\sigma}{\cos(\pi\eta)}|Z(E)|,$$

for every $0 \neq E \in \mathcal{D}$ satisfying $\phi^+_\sigma(E) - \phi^-_\sigma(E) < \eta$, and for all linear maps $U\colon K(\mathcal{D}) \to \mathbb{C}$. To see this, decompose E into semistable factors $A_1, \cdots, A_n$ in σ, apply the definition of $\|U\|_\sigma$ to each object A_i, and note that the points $Z(A_i) \in \mathbb{C}$ lie in a sector bounded by an angle of at most $\pi\eta$.

Now consider the situation of the lemma. Since $d(\mathcal{P},\mathcal{Q}) < \varepsilon$ and $\|W - Z\|_\sigma < \sin(\pi\varepsilon)$, one can apply $(*)$ with $U = W - Z$ and $\eta = 2\varepsilon$ to obtain

$$|W(E) - Z(E)| < \frac{\sin(\pi\varepsilon)}{\cos(2\pi\varepsilon)}|Z(E)|$$

for any object $E \in \mathcal{D}$ semistable in τ. It follows that there is a constant $\kappa > 0$ with $|Z(E)| < \kappa\,|W(E)|$ for all E semistable in τ. Take a group homomorphism $U\colon K(\mathcal{D}) \to \mathbb{C}$. Applying $(*)$ again and combining with the above inequality gives $\|U\|_\tau < k_2\|U\|_\sigma$. The other inequality follows similarly. $\square$

Equip $\mathrm{Stab}(\mathcal{D})$ with the topology generated by the basis of open sets $B_\varepsilon(\sigma)$. Let Σ be a connected component of $\mathrm{Stab}(\mathcal{D})$. By Lemma 6.2, the subspace

$$\{U \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C}) : \|U\|_\sigma < \infty\} \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C})$$

is locally constant on $\mathrm{Stab}(\mathcal{D})$ and hence constant on Σ. Denote it by $V(\Sigma)$. Note that if $\sigma = (Z,\mathcal{P}) \in \Sigma$ then $Z \in V(\Sigma)$. Note also that for each $\sigma \in \Sigma$ the function $\|\cdot\|_\sigma$ defines a norm on $V(\Sigma)$, and that by Lemma 6.2, all these norms are equivalent. Thus one has

Proposition 6.3. *For each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ there is a linear subspace $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}),\mathbb{C})$ with a well-defined linear topology and a continuous map $\mathcal{Z}\colon \Sigma \to V(\Sigma)$ which sends a stability condition $(Z,\mathcal{P})$ to its central charge Z.*

The proof of Theorem 1.2 will be completed in Section 7 by showing that the map $\mathcal{Z}$ of Proposition 6.3 is a local homeomorphism. The following lemma shows that $\mathcal{Z}$ is at least locally injective.

Lemma 6.4. *Suppose $\sigma = (Z,\mathcal{P})$ and $\tau = (Z,\mathcal{Q})$ are stability conditions on $\mathcal{D}$ with the same central charge Z. Suppose also that $d(\mathcal{P},\mathcal{Q}) < 1$. Then $\sigma = \tau$.*

Proof. Suppose to the contrary that $\sigma \neq \tau$. Then there is a nonzero object $E \in \mathcal{P}(\phi)$ which is not an element of $\mathcal{Q}(\phi)$. One could not have $E \in \mathcal{Q}(\geqslant \phi)$ because the assumption that $d(\sigma_1,\sigma_2) < 1$ would then imply that $E \in \mathcal{Q}([\phi,\phi+1))$ which contradicts the fact that σ and τ have the same central charge. Similarly one could not have $E \in \mathcal{Q}(\leqslant \phi)$. Thus there is a triangle

$$\begin{tikzcd} A \arrow[rr] & & E \arrow[dl] \\ & B \arrow[ul, dashed] & \end{tikzcd}$$

with $A \in \mathcal{Q}((\phi,\phi+1))$ and $B \in \mathcal{Q}((\phi-1,\phi])$ nonzero. One cannot have $A \in \mathcal{P}(\leqslant \phi)$ because this would imply $A \in \mathcal{P}((\phi-1,\phi])$ contradicting the fact that σ and τ have the same central charge. Thus there is an object $C \in \mathcal{P}(\psi)$ with $\psi > \phi$ and a nonzero morphism $f\colon C \to A$. The composite map $C \to E$ must be zero and so f factors via $B[-1]$. Since $B[-1] \in \mathcal{Q}(\leqslant \phi - 1)$ this gives a contradiction. $\square$

7. Deformations of stability conditions

In this section I complete the proof of Theorem 1.2 by proving a result that allows one to lift deformations of the central charge Z to deformations of stability conditions. It was Douglas' work that first suggested that such a result might be true.

Theorem 7.1. *Let $\sigma = (Z, \mathcal{P})$ be a locally-finite stability condition on a triangulated category $\mathcal{D}$. Then there is an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $W\colon K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism satisfying*

$$|W(E) - Z(E)| < \sin(\pi\varepsilon)|Z(E)|$$

for all $E \in \mathcal{D}$ semistable in σ, then there is a locally-finite stability condition $\tau = (W, \mathcal{Q})$ on $\mathcal{D}$ with $d(\mathcal{P}, \mathcal{Q}) < \varepsilon$.

After what was proved in Section 6 this will be enough to yield Theorem 1.2. Note that Lemma 6.4 shows that, providing $\varepsilon_0 < 1/2$, the stability condition τ of Theorem 7.1 is unique. The reader should think of the number ε_0 as being very small. In fact, it will be enough to assume that $\varepsilon_0 < 1/8$ and that each of the quasi-abelian categories $\mathcal{P}((t-4\varepsilon_0, t+4\varepsilon_0))$ has finite length. Since $\mathcal{Q}((t-\varepsilon, t+\varepsilon)) \subset \mathcal{P}((t-2\varepsilon, t+2\varepsilon))$ for all t, the condition that τ be locally-finite is automatic. The proof of the theorem will be broken up into a series of lemmas. Throughout, notation will be fixed as in the statement of the theorem. In particular, $W\colon K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism satisfying the hypotheses of the theorem, and $0 < \varepsilon < \varepsilon_0$ is a fixed real number.

Definition 7.2. A *thin subcategory* of $\mathcal{D}$ is a full subcategory of the form $\mathcal{P}((a, b)) \subset \mathcal{D}$ where a and b are real numbers with $0 < b - a < 1 - 2\varepsilon$.

Note that any thin subcategory of $\mathcal{D}$ is quasi-abelian. Recall that the condition on W in the statement of the theorem implies that if E is semistable in σ, then the phases of the points $W(E)$ and $Z(E)$ differ by at most ε. It follows that if $\mathcal{A} = \mathcal{P}((a, b))$ is thin then W defines a skewed stability function on $\mathcal{A}$. To avoid confusion, the objects of $\mathcal{A}$ which are semistable with respect to this stability function will be called W-semistable. Also, given a nonzero object $E \in \mathcal{A}$, write $\phi(E)$ for the phase of $Z(E)$ lying in the interval (a, b), and $\psi(E)$ for the phase of $W(E)$ lying in the interval $(a-\varepsilon, b+\varepsilon)$.

Lemma 7.3. *Suppose E is W-semistable in some thin subcategory $\mathcal{A} \subset \mathcal{D}$, and set $\psi = \psi(E)$. Then $E \in \mathcal{P}((\psi - \varepsilon, \psi + \varepsilon))$.*

Proof. Put $\phi = \phi^{+}(E)$. There is a strict short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

in $\mathcal{A}$ such that $A \in \mathcal{P}(\phi)$ and $B \in \mathcal{P}(<\phi)$. Then $\psi(A) \leqslant \psi(E)$ because E is W-semistable. But as above, one has $\psi(A) \in (\phi - \varepsilon, \phi + \varepsilon)$ and so it follows that $\phi < \psi + \varepsilon$. A similar argument shows that $\phi^{-}(E) > \psi - \varepsilon$. □

This notion of W-semistability for an object E of a thin subcategory is too weak unless E lies well inside $\mathcal{A}$ in a certain sense. The problem is that if E lies near the boundary of $\mathcal{A}$ then there are not enough objects in $\mathcal{A}$ to destabilise E. This prompts the following definition.

Definition 7.4. Suppose $\mathcal{A} = \mathcal{P}((a, b))$ is a thin subcategory of $\mathcal{D}$. A nonzero object $E \in \mathcal{A}$ is said to be *enveloped* by $\mathcal{A}$ if $a + \varepsilon \leqslant \psi(E) \leqslant b - \varepsilon$.

The next lemma shows that with this idea one gets a notion of semistability which is independent of a particular choice of thin subcategory.

LEMMA 7.5. *Suppose an object $E \in \mathcal{D}$ is enveloped by thin subcategories $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{D}$. Then E is W-semistable in $\mathcal{B}$ precisely if it is W-semistable in $\mathcal{C}$.*

Proof. After Lemma 7.3 one may as well assume that E is enveloped by the thin subcategory $\mathcal{P}((\psi(E) - \varepsilon, \psi(E) + \varepsilon))$. Thus it is enough to treat the case when $\mathcal{B} \subset \mathcal{C}$, and in fact, by the symmetry of the situation, one can assume that $\mathcal{B} = \mathcal{P}((a, b))$ and $\mathcal{C} = \mathcal{P}((a, c))$ for real numbers $a < b < c$. Of course, if E is W-semistable in $\mathcal{C}$ then it is also W-semistable in $\mathcal{B}$, because any strict short exact sequence in $\mathcal{B}$ is also a strict short exact sequence in $\mathcal{C}$.

For the converse, suppose E is unstable in $\mathcal{C}$ so that there is a strict short exact sequence in $\mathcal{C}$:

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

with $\psi(A) > \psi(E) > \psi(B)$. Then, by Lemma 3.4, one has $\phi^+(A) \leqslant \phi^+(E)$, and so since $E \in \mathcal{B}$, one has $A \in \mathcal{B}$ also. There is a strict short exact sequence

$$0 \longrightarrow B_1 \longrightarrow B \longrightarrow B_2 \longrightarrow 0$$

in $\mathcal{C}$ with $B_1 \in \mathcal{P}([b, c))$ and $B_2 \in \mathcal{B}$. Note that because E is enveloped by $\mathcal{B}$ one has $\psi(E) \leqslant b - \varepsilon < \psi(B_1)$. Consider the commuting diagram of strict short exact sequences in $\mathcal{C}$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \arrow[r, equal] \arrow[d] & A \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d, equal] & 0. \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B_2 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Then by Lemma 3.4 again, $\phi^+(K) \leqslant \phi^+(E)$ and hence $0 \to K \to E \to B_2 \to 0$ is a strict short exact sequence in $\mathcal{B}$. But $\psi(K) > \psi(E)$ and therefore E is not W-semistable in $\mathcal{B}$. $\square$

For each $\psi \in \mathbb{R}$ define $\mathcal{Q}(\psi) \subset \mathcal{D}$ to be the full additive subcategory of $\mathcal{D}$ consisting of the zero objects of $\mathcal{D}$ together with those objects $E \in \mathcal{D}$ which are W-semistable of phase ψ in some thin enveloping subcategory $\mathcal{P}((a,b))$. To prove Theorem 7.1 it must be shown that the pair $(W, \mathcal{Q})$ defines a stability condition on $\mathcal{D}$. The following lemma gives axiom (c) of Definition 1.1.

LEMMA 7.6. *If $E \in \mathcal{Q}(\psi_1)$ and $F \in \mathcal{Q}(\psi_2)$ and $\psi_1 > \psi_2$ then* $\operatorname{Hom}_{\mathcal{D}}(E,F) = 0$.

Proof. Suppose instead that there is a nonzero map $f\colon E \to F$. By Lemma 7.3 this implies that $\psi_1 - \psi_2 < 2\varepsilon$. Set $a = (\psi_1+\psi_2)/2 - 1/2$ and consider the abelian subcategory $\mathcal{A} = \mathcal{P}((a,a+1]) \subset \mathcal{D}$ which contains E and F. In the abelian category $\mathcal{A}$ there are short exact sequences

$$0 \longrightarrow \ker f \longrightarrow E \longrightarrow \operatorname{im} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} f \longrightarrow F \longrightarrow \operatorname{coker} f \longrightarrow 0.$$

By Lemmas 3.4 and 7.3, one has $\ker f \in \mathcal{P}((a, \psi_1+\varepsilon))$, $\operatorname{coker} f \in \mathcal{P}((\psi_2-\varepsilon, a+1])$ and $\operatorname{im} f \in \mathcal{P}((\psi_1-\varepsilon, \psi_2+\varepsilon))$. Providing ε is small enough (say $\varepsilon < 1/8$), there is a thin subcategory of $\mathcal{D}$ enveloping E in which the first sequence is strict short exact, and similarly a thin subcategory enveloping F in which the second sequence is strict short exact. Since E and F are W-semistable in any enveloping category it follows that $\psi_1 \leqslant \psi(\operatorname{im} f) \leqslant \psi_2$, a contradiction. □

The next step is to construct Harder-Narasimhan filtrations.

LEMMA 7.7. *Let $\mathcal{A} = \mathcal{P}((a,b)) \subset \mathcal{D}$ be a thin subcategory of finite length. Then every nonzero object of $\mathcal{P}((a+2\varepsilon, b-4\varepsilon))$ has a finite Harder-Narasimhan filtration whose factors are W-semistable objects of $\mathcal{A}$ which are enveloped by $\mathcal{A}$.*

Proof. The proof goes along the same lines as those of Proposition 2.4, when we replace subobjects by strict subobjects and quotients by strict quotients. Here I just indicate the necessary changes. Clearly the chain conditions hold because of the assumption that $\mathcal{A}$ has finite length. Note also that if an object $E \in \mathcal{P}((a+2\varepsilon, b-4\varepsilon))$ has a Harder-Narasimhan filtration with W-semistable factors $F_1, \cdots, F_n$, then $\psi(F_1) \geqslant \psi(E) > a+\varepsilon$; the fact that there is a nonzero map $F_1 \to E$ together with Lemma 7.3 ensures that $\psi(F_1) < b-3\varepsilon$. In this way one sees that the factors of E are automatically enveloped by $\mathcal{A}$.

Define $\mathcal{G}$ be the class of of nonzero objects $E \in \mathcal{P}((a, b-4\varepsilon))$ for which every nonzero strict quotient $E \twoheadrightarrow B$ in $\mathcal{A}$ satisfies $\psi(B) > a+\varepsilon$. By Lemma 3.4 the class $\mathcal{G}$ contains all nonzero objects of $\mathcal{P}((a+2\varepsilon, b-4\varepsilon))$, so it will be enough to show that all objects of $\mathcal{G}$ have a Harder-Narasimhan filtration.

The commutative diagram (†) (end of Section 2) and Lemma 3.4 show that if

$$0 \longrightarrow E' \longrightarrow E \longrightarrow B \longrightarrow 0$$

is a strict short exact sequence in $\mathcal{A}$ with $E \twoheadrightarrow B$ a maximally destabilising quotient (mdq) and $E \in \mathcal{G}$ then $E' \in \mathcal{G}$ also. Thus the inductive step in the proof of Proposition 2.4 stays within the class $\mathcal{G}$ and it will be enough to show that every object in $\mathcal{G}$ has an mdq.

To make the induction work it is helpful to prove the existence of mdqs for a larger class of objects $\mathcal{H}$, namely nonzero objects $E \in \mathcal{A}$ with $\psi(E) < b - 3\varepsilon$ such that every nonzero strict quotient $E \twoheadrightarrow B$ in $\mathcal{A}$ satisfies $\psi(B) > a + \varepsilon$. Note that if $E \in \mathcal{H}$ and $E \twoheadrightarrow E'$ is a nonzero strict quotient with $\psi(E) \geqslant \psi(E')$ then $E' \in \mathcal{H}$ also.

Suppose then that $E \in \mathcal{H}$. The key observation is that one can always assume that $\phi^+(E) < \psi(E) + \varepsilon$. Otherwise there is a strict short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow E' \longrightarrow 0$$

with $A \in \mathcal{P}(\geqslant \psi(E) + \varepsilon)$ and $E' \in \mathcal{P}(< \psi(E) + \varepsilon)$. Note that $\psi(A) > \psi(E) > \psi(E')$. I claim that if $E' \twoheadrightarrow B$ is an mdq for E' then the composite map $E \twoheadrightarrow B$ is an mdq for E. Indeed, if $E \twoheadrightarrow B'$ is a W-semistable quotient in $\mathcal{A}$ with $\psi(B') \leqslant \psi(B)$ then $\psi(B') \leqslant \psi(E)$ and so by Lemma 7.3 one has $\phi^+(B') < \psi(E) + \varepsilon$. It follows that $\operatorname{Hom}_{\mathcal{A}}(A, B') = 0$ and hence $E \twoheadrightarrow B'$ factors via E'. This proves the claim.

By Lemmas 3.4 and 7.3, the inequalities $\phi^+(E) < \psi(E) + \varepsilon$ and $\psi(E) < b - 3\varepsilon$ are enough to guarantee that every W-semistable strict subobject of E is enveloped by $\mathcal{A}$. By definition of the class $\mathcal{H}$, every W-semistable strict quotient of E is also enveloped by $\mathcal{A}$. Thus, by Lemma 7.6, the argument of Proposition 2.4 can be applied as in the abelian case to show that E has an mdq. □

For each real number t define $\mathcal{Q}(> t)$ to be the full extension-closed subcategory of $\mathcal{D}$ generated by the subcategories $\mathcal{Q}(\psi)$ for $\psi > t$. Similarly define full subcategories $\mathcal{Q}(\leqslant t) \subset \mathcal{D}$ and $\mathcal{Q}(< t) \subset \mathcal{D}$.

I claim that $\mathcal{Q}(> t)$ is a t-structure on $\mathcal{D}$. To prove this I must show that for every $E \in \mathcal{D}$ there is a triangle

$$A \longrightarrow E \longrightarrow B$$

with $A \in \mathcal{Q}(> t)$ and $B \in \mathcal{Q}(\leqslant t)$. But note that Lemmas 7.3 and 7.7 show that $\mathcal{P}(s)$ is contained in the subcategory $\mathcal{Q}(> t)$ for $s \geqslant t + \varepsilon$ and in the subcategory $\mathcal{Q}(< t)$ for $s \leqslant t - \varepsilon$. Thus it will be enough to consider the case when $E \in \mathcal{P}((t-\varepsilon, t+\varepsilon))$. Consider E as an object of the quasi-abelian category $\mathcal{P}((t - 3\varepsilon, t + 5\varepsilon))$ which has finite length by the assumptions on ε. Applying Lemma 7.7 gives a Harder-Narasimhan filtration of E which is enough to prove the claim.

The final step in the proof of Theorem 7.1 is to show that every nonzero object of $\mathcal{D}$ has a finite filtration by objects of the subcategories $\mathcal{Q}(\psi)$. It will be enough to prove this for objects in each of the full subcategories

$$\mathcal{Q}((t, t+\delta)) = \mathcal{Q}(> t) \cap \mathcal{Q}(< t+\delta)$$

for some small $\delta > 0$. The result then follows by embedding $\mathcal{Q}((t, t+\delta))$ in the finite-length, quasi-abelian subcategory $\mathcal{P}((t-3\varepsilon, t+5\varepsilon+\delta))$ and applying Lemma 7.7.

8. More on the space of stability conditions

This section contains a couple of general results about spaces of stability conditions. The first shows that the topology on $\mathrm{Stab}(\mathcal{D})$ defined in Section 6 can be induced by a natural metric. Since this result is not necessary for Theorem 1.2 some of the details of the proof are left to the reader.

PROPOSITION 8.1. *Let $\mathcal{D}$ be a triangulated category. The function*

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|, |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|, |\log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)}| \right\} \in [0, \infty]$$

defines a generalised metric on $\mathrm{Stab}(\mathcal{D})$. *The induced topology is the same as that defined in Section* 6.

Proof. It is easy to see that the given formula defines a generalised metric; the only thing to check is that if $d(\sigma_1, \sigma_2) = 0$ then $\sigma_1 = \sigma_2$. But $d(\sigma_1, \sigma_2) = 0$ implies that an object $E \in \mathcal{D}$ is semistable in σ_1 precisely if it is semistable in σ_2, and that for any nonzero E one has $m_{\sigma_1}(E) = m_{\sigma_2}(E)$. It follows that the central charges of σ_1 and σ_2 are the same, since they agree on semistables and these span the Grothendieck group $K(\mathcal{D})$.

To prove that the topology induced by $d(-,-)$ is the same as the one given by the basis of open sets $B_\varepsilon(\sigma)$ one must first show that the sets $B_\varepsilon(\sigma)$ are open in the topology induced by the metric. This boils down to the statement that for any $\varepsilon > 0$ the condition

$$|W(E) - Z(E)| < \sin(\pi\varepsilon)|Z(E)|$$

holds for all objects $E \in \mathcal{D}$ semistable in σ providing $\tau = (W, \mathcal{Q})$ is a small enough distance from $\sigma = (Z, \mathcal{P})$. This is easy enough to see and is probably best done privately with a picture.

The reverse implication requires a little more care. Take $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$ and fix a constant $\kappa > 1$. What one needs to show is that for small enough $\varepsilon > 0$ the set $B_\varepsilon(\sigma)$ has the property that

$$\tau = (W, \mathcal{Q}) \in B_\varepsilon(\sigma) \implies m_\tau(E) < \kappa m_\sigma(E) \text{ for all } 0 \neq E \in \mathcal{D}.$$

Suppose then that $\tau \in B_\varepsilon(\sigma)$ and consider first the case when $\phi^+_\sigma(E) - \phi^-_\sigma(E) < \eta$ for some $\eta \in (0, \frac{1}{2})$. Split E into semistable factors A_i with respect to τ. Then $\phi^+_\sigma(A_i) - \phi^-_\sigma(A_i) < 2\varepsilon$ for each i, so that equation $(*)$ of the proof of Lemma 6.2 gives

$$|W(A_i)| < \big(1 + \frac{\sin(\pi\varepsilon)}{\cos(2\pi\varepsilon)}\big)|Z(A_i)|.$$

Since the vectors $Z(A_i)$ lie in a sector bounded by an angle of at most $\pi(4\varepsilon+\eta)$, and $|Z(E)| \leqslant m_\sigma(E)$, it follows that there is a constant $r(\varepsilon,\eta) > 1$ such that

$$m_\tau(E) < r(\varepsilon,\eta) m_\sigma(E),$$

and that moreover $r(\varepsilon,\eta) \to 1$ as $\max(\varepsilon,\eta) \to 0$.

Consider now a general nonzero object $E \in \mathcal{D}$. Fix real numbers ϕ and a positive integer n. For each integer k, define intervals

$$I_k = \big[\phi + kn\varepsilon, \phi + (k+1)n\varepsilon\big), \quad J_k = \big[\phi + (kn-1)\varepsilon, \phi + ((k+1)n+1))\varepsilon\big),$$

and let α_k and β_k be the truncation functors projecting into the subcategories $\mathcal{Q}(I_k)$ and $\mathcal{P}(J_k)$ respectively. It is an easy enough exercise to check that $d(\mathcal{P},\mathcal{Q}) < \varepsilon$ implies $\alpha_k \circ \beta_k = \alpha_k$ so that

$$m_\tau(E) = \sum_k m_\tau(\alpha_k(E)) \leqslant \sum_k m_\tau(\beta_k(E)) < r(\varepsilon,(n+2)\varepsilon) \sum_k m_\sigma(\beta_k(E)).$$

But now one can choose ϕ so that

$$\sum_k m_\sigma(\beta_k(E)) \leqslant \left(1 + \frac{2}{n}\right) m_\sigma(E),$$

so that sending $\varepsilon \to 0$ and $n \to \infty$ in such a way that $n\varepsilon \to 0$ one sees that for small enough ε one has $m_\tau(E) < \kappa m_\sigma(E)$ for all nonzero $E \in \mathcal{D}$. $\square$

Proposition 8.1 has the consequence that for any nonzero object $E \in \mathcal{D}$ the functions

$$\phi^\pm(E) : \operatorname{Stab}(\mathcal{D}) \to \mathbb{R} \quad \text{and} \quad m(E) : \operatorname{Stab}(\mathcal{D}) \to \mathbb{R}_{>0}$$

are continuous. It follows immediately from this that the subset of $\operatorname{Stab}(\mathcal{D})$ consisting of those stability conditions in which a given object $E \in \mathcal{D}$ is semistable is a closed subset. Indeed, if E is nonzero, it is precisely the set of $\sigma \in \operatorname{Stab}(\mathcal{D})$ for which the equality $\phi^+_\sigma(E) = \phi^-_\sigma(E)$ holds.

Lemma 8.2. *The generalised metric space* $\operatorname{Stab}(\mathcal{D})$ *carries a right action of the group* $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, *the universal covering space of* $\mathrm{GL}^+(2,\mathbb{R})$, *and a left action by isometries of the group* $\operatorname{Aut}(\mathcal{D})$ *of exact autoequivalences of* $\mathcal{D}$. *These two actions commute.*

Proof. First note that the group $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ can be thought of as the set of pairs (T,f) where $f\colon \mathbb{R}\to\mathbb{R}$ is an increasing map with $f(\phi+1)=f(\phi)+1$, and $T\colon \mathbb{R}^2\to\mathbb{R}^2$ is an orientation-preserving linear isomorphism, such that the induced maps on $S^1=\mathbb{R}/2\mathbb{Z}=(\mathbb{R}^2\setminus\{0\})/\mathbb{R}_{>0}$ are the same.

Given a stability condition $\sigma=(Z,\mathcal{P})\in\mathrm{Stab}(\mathcal{D})$, and a pair $(T,f)\in\tilde{\mathrm{GL}}^+(2,\mathbb{R})$, define a new stability condition $\sigma'=(Z',\mathcal{P}')$ by setting $Z'=T^{-1}\circ Z$ and $\mathcal{P}'(\phi)=\mathcal{P}(f(\phi))$. Note that the semistable objects of the stability conditions σ and σ' are the same, but the phases have been relabelled.

For the second action, note that an element $\Phi\in\mathrm{Aut}(\mathcal{D})$ induces an automorphism ϕ of $K(\mathcal{D})$. If $\sigma=(Z,\mathcal{P})$ is a stability condition on $\mathcal{D}$ define $\Phi(\sigma)$ to be the stability condition $(Z\circ\phi^{-1},\mathcal{P}')$, where $\mathcal{P}'(t)=\Phi(\mathcal{P}(t))$. The reader can easily check that this action is by isometries and commutes with the first. □

9. Stability conditions on elliptic curves

Let X be a nonsingular projective curve of genus one over $\mathbb{C}$, and let $\mathcal{D}(X)$ denote the bounded derived category of coherent sheaves. As in the introduction, $\mathrm{Stab}(X)$ will denote the space of locally-finite numerical stability conditions on $\mathcal{D}(X)$.

Set $K(X)=K(\mathcal{D}(X))$ and write $\mathcal{N}(X)$ for the numerical Grothendieck group $\mathcal{N}(\mathcal{D}(X))$ defined in Section 1.3. The Riemann-Roch theorem shows that $\mathcal{N}(X)$ can be identified with $\mathbb{Z}\oplus\mathbb{Z}$ and with the quotient map $K(X)\to\mathcal{N}(X)$ sending a class $[E]\in K(X)$ to the pair consisting of its rank and degree. The Euler form on $\mathcal{N}(X)$ is then given by

$$\chi((r_1,d_1),(r_2,d_2))=r_1d_2-r_2d_1.$$

As in Example 5.4, there is a stability condition $\sigma=(Z,\mathcal{P})\in\mathrm{Stab}(X)$ with

$$Z(E)=-\deg(E)+i\,\mathrm{rank}(E),$$

in which the objects of the subcategories $\mathcal{P}(\phi)$ consist of shifts of semistable sheaves on X, and whose heart is the category of coherent $\mathcal{O}_X$-modules. It follows from Lemma 8.2 and Theorem 1.2 that there is a local homeomorphism

$$\mathcal{Z}\colon\ \mathrm{Stab}(X)\to\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X),\mathbb{C})$$

whose image is an open subset of the two-dimensional vector space $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(X),\mathbb{C})$.

THEOREM 9.1. *The action of the group* $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ *on* $\mathrm{Stab}(X)$ *is free and transitive*, *so that*

$$\mathrm{Stab}(X)\cong\tilde{\mathrm{GL}}^+(2,\mathbb{R}).$$

Proof. First note that if E is an indecomposable sheaf on X then E must be semistable in any stability condition $\sigma \in \operatorname{Stab}(X)$ because otherwise there is a nontrivial triangle $A \to E \to B$ with $\operatorname{Hom}_{\mathcal{D}(X)}(A,B)=0$, and then Serre duality gives

$$\operatorname{Hom}^1_{\mathcal{D}(X)}(B,A) = \operatorname{Hom}_{\mathcal{D}(X)}(A,B)^* = 0,$$

which implies that E is a direct sum $A \oplus B$.

Take an element $\sigma = (Z,\mathcal{P}) \in \operatorname{Stab}(X)$. Suppose for a contradiction that the image of the central charge Z is contained in a real line in $\mathbb{C}$. Since σ is locally-finite, the heart $\mathcal{A}$ of σ must then be of finite length. If A and B are simple objects of $\mathcal{A}$ then

$$\operatorname{Hom}_{\mathcal{D}(X)}(A,B) = \operatorname{Hom}_{\mathcal{D}(X)}(B,A) = 0,$$

and it follows from this that $\chi(A,B)=0$. But this implies that all simple objects of $\mathcal{A}$ lie on the same line in $\mathcal{N}(X)$, and hence that all objects of $\mathcal{D}(X)$ do too, which gives a contradiction. Thus Z, considered as a map from $\mathcal{N}(X)\otimes\mathbb{R}=\mathbb{R}^2$ to $\mathbb{C}\cong\mathbb{R}^2$ is an isomorphism, and it follows that the action of $\tilde{\operatorname{GL}}^+(2,\mathbb{R})$ on $\operatorname{Stab}(X)$ is free.

Suppose A and B are line bundles on X with $\deg(A)<\deg(B)$. Since A and B are indecomposable they are semistable in σ with phases ϕ and ψ say. The existence of maps $A\to B$ and $B\to A[1]$ gives inequalities $\phi\leqslant\psi\leqslant\phi+1$, which implies that Z is orientation-preserving. Thus acting by an element of $\tilde{\operatorname{GL}}^+(2,\mathbb{R})$, one can assume that $Z(E)=-\deg(E)+i\operatorname{rank}(E)$, and that for some point $x\in X$ the skyscraper sheaf $\mathcal{O}_x$ has phase 1. Then all semistable vector bundles on X are semistable in σ with phase in the interval $(0,1)$, and it follows quickly from this that σ is the standard stability condition described in Example 5.4. □

The quotient $\operatorname{Stab}(X)/\operatorname{Aut}\mathcal{D}(X)$ is also of interest. One can easily show that the autoequivalences of $\mathcal{D}(X)$ are generated by shifts, automorphisms of X and twists by line bundles together with the Fourier-Mukai transform [14]. Automorphisms of X and twists by line bundles of degree zero act trivially on $\operatorname{Stab}(X)$ and one obtains

$$\operatorname{Stab}(X)\,/\operatorname{Aut}\mathcal{D}(X)\cong \operatorname{GL}^+(2,\mathbb{R})\,/\operatorname{SL}(2,\mathbb{Z}),$$

which is easily seen to be a $\mathbb{C}^*$-bundle over the moduli space of elliptic curves.

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References

[1] P. Aspinwall and M. R. Douglas, D-brane stability and monodromy, *J. High Energy Phys.* **05** (2002)031.

[2] A. Beilinson, Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra, *Funkt. Anal. Prilozhen.* **12** (1978), 68–69; English transl. in *Funct. Anal. Appl.* **12** (1978), 214–216.

[3] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux Pervers*, *Astérisque* **100**, Soc. Math de France, Paris (1983).

[4] A. I. Bondal and D. O. Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, *Compositio Math.* **125** (2001), 327–344.

[5] A. I. Bondal and M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.* **3** (2003), 1–36, 258.

[6] T. Bridgeland, Stability conditions on $K3$ surfaces, *Duke Math. J.*, to appear.

[7] M. R. Douglas, D-branes on Calabi-Yau manifolds, *European Congress of Mathematics*, Vol. II (Barcelona, 2000), 449–466, *Progr. Math.* **202**, Birkhäuser, Basel, 2001.

[8] ———, D-branes, categories and $N = 1$ supersymmetry. Strings, branes, and M-theory, *J. Math. Phys.* **42** (2001), 2818–2843.

[9] ———, Dirichlet branes, homological mirror symmetry, and stability, *Proc. Internat. Congress of Mathematicians*, Vol. III (Beijing, 2002), 395–408, Higher Ed. Press, Beijing, 2002.

[10] S. I. Gelfand and Yu. I. Manin, *Methods of Homological Algebra*, Springer-Verlag, New York (1996).

[11] G. Harder and M. S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles on curves, *Math. Ann.* **212** (1974/75), 215–248.

[12] R. Hartshorne, *Residues and Duality*, *Lecture Notes in Math.* **20**, Springer-Verlag, New York, 1966.

[13] M. Kontsevich, Homological algebra of mirror symmetry. *Proc. Internat. Congress of Mathematicians*, Vol. 1, 2 (Zürich, 1994), 120–139, Birkhäuser, Basel, 1995.

[14] S. Mukai, Duality between $\mathcal{D}(X)$ and $\mathcal{D}(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.* **81** (1981), 153–175.

[15] D. Quillen, Higher algebraic K-theory. I, in *Proc. Conf.*, *Battelle Memorial Inst.* (Seattle, Wash., 1972), *Lecture Notes in Math.* **341**, Springer-Verlag, New York, 1973.

[16] A. N. Rudakov, Stability for an abelian category, *J. Algebra* **197** (1997), 231–24.

[17] J.-P. Schneiders, Quasi-abelian categories and sheaves, *Mém. Soc. Math. France* **76** (1999); also available at http://www.ulg.ac.be/analg/jps/.

[18] R. P. Thomas, Moment maps, monodromy and mirror manifolds, in *Symplectic Geometry and Mirror Symmetry* (Seoul, 2000), 467–498, World Sci. Publ., River Edge, NJ, 2001.

[19] ———, Stability conditions and the braid group, *Comm. Anal. Geom.* **14** (2006), 135–161.

[20] J.-L. Verdier, Des catégories dérivées des catégories Abéliennes, *Astérisque* **239**, Soc. Math. France, Paris (1996).

(Received April 23, 2003)