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Weyl group multiple Dirichlet series III: Eisenstein series and twisted unstable A_r

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Abstract

Weyl group multiple Dirichlet series were associated with a root system Φ and a number field F containing the n-th roots of unity by Brubaker, Bump, Chinta, Friedberg and Hoffstein [3] and Brubaker, Bump and Friedberg [4] provided n is sufficiently large; their coefficients involve n-th order Gauss sums. The case where n is small is harder, and is addressed in this paper when $\Phi = A_r$. "Twisted" Dirichet series are considered, which contain the series of [4] as a special case. These series are not Euler products, but due to the twisted multiplicativity of their coefficients, they are determined by their p-parts. The p-part is given as a sum of products of Gauss sums, parametrized by strict Gelfand-Tsetlin patterns. It is conjectured that these multiple Dirichlet series are Whittaker coefficients of Eisenstein series on the n-fold metaplectic cover of GL_{r+1}, and this is proved if $r = 2$ or $n = 1$. The equivalence of our definition with that of Chinta [11] when $n = 2$ and $r \leqslant 5$ is also established.

Let F be a totally complex algebraic number field containing the group μ_{2n} of $2n$-th roots of unity. Thus -1 is an n-th power in F. Let $\Phi \subset \mathbb{R}^r$ be a reduced root system. It has been shown in Brubaker, Bump, Chinta, Friedberg and Hoffstein [3] and Brubaker, Bump and Friedberg [4] how one can associate a multiple Dirichlet series with Φ; its coefficients involve n-th order Gauss sums. A condition of *stability* is imposed in this definition, which amounts to n being sufficiently large, depending on Φ. In this paper we will propose a description of the Weyl group multiple Dirichlet series in the unstable case when Φ has Cartan type A_r, and present the evidence in support of this description. We will refer to this as the *Gelfand-Tsetlin description* whose striking characteristic is that it gives a single formula *valid for all* n for these coefficients, that reduces to the stable description when n is sufficiently large.

We conjecture that this *Weyl group multiple Dirichlet series* coincides with the Whittaker coefficient of an Eisenstein series. The Eisenstein series $E(g; s_1, \cdots, s_r)$ is of minimal parabolic type, on an n-fold metaplectic cover of an algebraic group defined over F whose root system is the dual root system of Φ. We refer to this identification of the series with a Whittaker coefficient of E as the *Eisenstein conjecture*.

We will present some evidence for the Eisenstein conjecture by proving it when Φ is of type A_2 (for all n) or when Φ is of type A_r and $n = 1$. We will also present evidence for the Gelfand-Tsetlin description (but not the Eisenstein conjecture) for general n.

There is a good reason not to use the Eisenstein series as a primary foundational tool in the study of the Weyl group multiple Dirichlet series. This is the relative complexity of the Matsumoto cocycle describing the metaplectic group. The approach taken in [3] and [4] had its origin in Bump, Friedberg and Hoffstein [9], where it was proposed that multiple Dirichlet series could profitably be studied without use of Eisenstein series on higher rank groups, using instead an argument based on Bochner's convexity theorem. The realization of this approach in [3] and [4] involves a certain amount of bookkeeping, consisting of tracking some Hilbert symbols that occur in the definition of the series and the proof of its functional equation. Eisenstein series intervene only through the Kubota Dirichlet series, whose functional equations are deduced from the functional equations of rank one Eisenstein series. In the approach of [3] and [4], the bookkeeping is very manageable, and these foundations seem good for supplying proofs.

The Weyl group multiple Dirichlet series associated in [4] with a root system $\Phi \subseteq \mathbb{R}^r$ has the form

$$(1) \qquad Z_\Psi(s_1, \cdots, s_r) = \sum_{\mathfrak{c}_1, \cdots, \mathfrak{c}_r} H\Psi(\mathfrak{c}_1, \cdots, \mathfrak{c}_r)\ \mathbb{N}\mathfrak{c}_1^{-2s_1} \cdots \mathbb{N}\mathfrak{c}_r^{-2s_r},$$

where the sum is over nonzero ideals $\mathfrak{c}_1, \cdots, \mathfrak{c}_r$ of the ring $\mathfrak{o}_S$ of S-integers, where S is a finite set of places chosen so that $\mathfrak{o}_S$ is a principal ideal domain. It is assumed that S contains all archimedean places, and those ramified over $\mathbb{Q}$.

The coefficients in Z thus have two parts, denoted $H(C_1, \cdots, C_r)$ and $\Psi(C_1, \cdots, C_r)$, defined for nonzero $C_i \in \mathfrak{o}_S$. The product $H\Psi$ is unchanged if C_i is multiplied by a unit, and so is a function of r-tuples of ideals in the principal ideal domain $\mathfrak{o}_S$. This fact is implicit in the notation (1), where use is made of the fact that $H\Psi(C_1, \cdots, C_r)$ depends only on the ideals $\mathfrak{c}_i = C_i\mathfrak{o}_S$. The factor Ψ is the less important of the two, and we will not define it here; it is described in [4]. Suffice it to say that Ψ is chosen from a finite-dimensional vector space of functions on $F_S = \prod_{v \in S} F_v$, and that these functions are constant on cosets of an open subgroup. If one changes the setup slightly, the function Ψ can be suppressed using congruence conditions, and this is the point of view that we will take in Section 1.

The function H is more interesting and requires discussion before we can explain our results. For simplicity we assume that Φ is simply-laced, and that all roots are normalized to have length 1; see [4] for the general case. Let $\alpha_1, \cdots, \alpha_r$ be the simple positive roots of Φ in some fixed order. The coefficients H have the following "twisted" multiplicativity. If $\gcd(C_1 \cdots C_r, C_1' \cdots C_r') = 1$, then

$$\frac{H(C_1C_1', \cdots, C_rC_r')}{H(C_1, \cdots, C_r)H(C_1', \cdots, C_r')} = \prod_{i=1}^{r} \left(\frac{C_i}{C_i'}\right)\left(\frac{C_i'}{C_i}\right) \prod_{\substack{i<j \\ \alpha_i, \alpha_j \text{ not orthogonal}}} \left(\frac{C_i}{C_j'}\right)^{-1}\left(\frac{C_i'}{C_j}\right)^{-1}. \tag{2}$$

The condition that α_i, α_j not be orthogonal means that these simple roots correspond to adjacent nodes in the Dynkin diagram. In this formula $\left(\frac{C}{D}\right)$ is the n-th power-residue symbol, defined for nonzero coprime elements of $\mathfrak{o}_S$. It satisfies the reciprocity law

$$\left(\frac{C}{D}\right) = (D, C)_S \left(\frac{D}{C}\right),$$

where $(a, b)_S = \prod_{v \in S}(a, b)_v$ is the S-Hilbert symbol, defined for $a, b \in F_S^\times$. See [4] for further information.

Knowing the twisted multiplicativity of H, we may reduce the description of H to the case where the C_i are all powers of the same prime p. This is done in [4] when n is sufficiently large. In that case, it is found that there are exactly $|W|$ values of $(k_1, \cdots, k_r)$ such that $H(p^{k_1}, \cdots, p^{k_r}) \neq 0$, where W is the Weyl group of Φ. More precisely, there is a bijection between the Weyl group W and the set $\mathrm{Supp}_{\mathrm{stable}}(H)$ of such $(k_1, \cdots, k_r)$ in which $w \in W$ corresponds to $(k_1, \cdots, k_r)$ determined by $\rho - w(\rho) = \sum k_i\alpha_i$, where the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ with Φ^+ the set of positive roots. This set is independent of p. When $(k_1, \cdots, k_r) \in \mathrm{Supp}_{\mathrm{stable}}(H)$ we have

$$H(p^{k_1}, \ldots, p^{k_r}) = \prod_{\substack{\alpha \in \Phi^+ \\ w\alpha \notin \Phi^+}} g(p^{d(\alpha)-1}, p^{d(\alpha)}), \tag{3}$$

where in terms of the simple roots $\alpha_1, \cdots, \alpha_r$ we have $d\big(\Sigma_i\, k_i\alpha_i\big) = \Sigma_i k_i$ and

$$g(a, c) = \sum_{\substack{d \bmod c \\ \gcd(d,c)=1}} \left(\frac{d}{c}\right) \psi\left(\frac{ad}{c}\right),$$

with ψ a fixed additive character of F_S such that $\psi(x\mathfrak{o}_S) = 1$ if and only if $x \in \mathfrak{o}_S$.

Up to this point, we have described the *stable* coefficients H, defined for sufficiently large n in [4]. We turn now to the more difficult case where n is *not* assumed to be large, and discuss what modifications we expect. The set

$$\mathrm{Supp}_n(H) = \{(k_1, \cdots, k_r) \,|\, H(p^{k_1}, \cdots, p^{k_r}) \neq 0\}$$

will still be finite, and will contain $\mathrm{Supp}_{\mathrm{stable}}(H)$. Moreover, the values of $H(p^{k_1}, \cdots, p^{k_r})$ when $(k_1, \cdots, k_r) \in \mathrm{Supp}_{\mathrm{stable}}(H)$ will still be given by (3). However, there will be *other* values of $(k_1, \cdots, k_r)$ in $\mathrm{Supp}_n(H)$. These will lie in the convex hull of $\mathrm{Supp}_{\mathrm{stable}}(H)$.

For the rest of this paper, we will specialize to the case $\Phi = A_r$. It will be useful to generalize the definition of $H(C_1, \cdots, C_r)$. The generalized coefficients will be denoted $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$ where m_i are nonzero elements of $\mathfrak{o}_S$, and as a special case

$$H(C_1, \cdots, C_r) = H(C_1, \cdots, C_r; 1, \cdots, 1). \tag{4}$$

We will give the definition of the coefficients $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$ in Section 2. Here we will explain those properties that are immediately relevant.

First, the coefficients $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$ will satisfy the same multiplicativity (2) as $H(C_1, \cdots, C_r)$. Moreover if $\gcd(m_1' \cdots m_r', C_1 \cdots C_r) = 1$ we will have the multiplicativity

$$\begin{aligned} H(C_1, \cdots, C_r; m_1 m_1', \cdots, m_r m_r') \\ = \left(\frac{m_1'}{C_1}\right)^{-1} \cdots \left(\frac{m_r'}{C_r}\right)^{-1} H(C_1, \cdots C_r; m_1, \cdots, m_r). \end{aligned} \tag{5}$$

(Compare Propositions 2 and 3.)

Thus we can extend the definition (1) obtaining a multiple Dirichlet series

$$\begin{aligned} Z_\Psi(s_1, \cdots, s_r; m_1, \cdots, m_r) = \sum_{0 \neq C_i \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S} H(C_1, \cdots, C_r; m_1, \cdots, m_r) \\ \cdot\, \Psi(C_1, \cdots, C_r)\, \mathbb{N}C_1^{-2s_1} \mathbb{N}C_2^{-2s_2} \cdots \mathbb{N}C_r^{-2s_r}. \end{aligned} \tag{6}$$

Roughly speaking, $Z_\Psi(s_1, \cdots, s_r; m_1, \cdots, m_r)$ is a twist of the original Z_Ψ by n-th order characters, since by (4) and (5), if $\gcd(m_1, \cdots, m_r, C_1 \cdots C_r) = 1$, we have

$$H(C_1, \cdots, C_r; m_1, \cdots, m_r) = \left(\frac{m_1}{C_1}\right)^{-1} \cdots \left(\frac{m_r}{C_r}\right)^{-1} H(C_1, \cdots C_r).$$

This will *not* be true without the assumption $\gcd(m_1 \cdots m_r, C_1 \cdots C_r) = 1$, so that this type of twisting is more subtle than simply multiplying the coefficients by n-th order characters. Still, we will refer to $Z_\Psi(s_1, \cdots, s_r; m_1, \cdots, m_r)$ as the *twisted* series.

We observe that equations (2) and (5) reduce the specification of the coefficients to the case where the C_i and m_i are all powers of the same prime p, in which case we denote $C_i = p^{k_i}$ and $m_i = p^{l_i}$.

With l_i fixed, it is still true that for n sufficiently large, there are exactly $|W| = (r+1)!$ values of $(k_1, \cdots, k_r)$ such that $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) \neq 0$. However the location of the stable values $(k_1, \cdots, k_r)$ for the twisted series will differ from the values $\mathrm{Supp}_{\mathrm{stable}}(H)$ that we previously considered. If $r = 2$, the

(k_1,k_2)	$H(p^{k_1},p^{k_2};p^{l_1},p^{l_2})$	$\mathfrak{T}$
$(0,0)$	1	$\left\{\begin{smallmatrix} l_1+l_2+2 & & l_2+1 & & 0 \\ & l_2+1 & & 0 & \\ & & 0 & & \end{smallmatrix}\right\}$
$(l_1+1,0)$	$g(p^{l_1},p^{l_1+1})$	$\left\{\begin{smallmatrix} l_1+l_2+2 & & l_2+1 & & 0 \\ & l_1+l_2+2 & & 0 & \\ & & 0 & & \end{smallmatrix}\right\}$
$(0,l_2+1)$	$g(p^{l_2},p^{l_2+1})$	$\left\{\begin{smallmatrix} l_1+l_2+2 & & l_2+1 & & 0 \\ & l_2+1 & & 0 & \\ & & l_2+1 & & \end{smallmatrix}\right\}$
(l_1+l_2+2,l_2+1)	$g(p^{l_2},p^{l_2+1})$ $\times g(p^{l_1+l_2+1},p^{l_1+l_2+2})$	$\left\{\begin{smallmatrix} l_1+l_2+2 & & l_2+1 & & 0 \\ & l_1+l_2+2 & & l_2+1 & \\ & & l_2+1 & & \end{smallmatrix}\right\}$
(l_1+1,l_1+l_2+2)	$g(p^{l_1},p^{l_1+1})$ $\times g(p^{l_1+l_2+1},p^{l_1+l_2+2})$	$\left\{\begin{smallmatrix} l_1+l_2+2 & & l_2+1 & & 0 \\ & l_1+l_2+2 & & 0 & \\ & & l_1+l_2+2 & & \end{smallmatrix}\right\}$
(l_1+l_2+2,l_1+l_2+2)	$g(p^{l_1},p^{l_1+1})$ $\times g(p^{l_2},p^{l_2+1})$ $\times g(p^{l_1+l_2+1},p^{l_1+l_2+2})$	$\left\{\begin{smallmatrix} l_1+l_2+2 & & l_2+1 & & 0 \\ & l_1+l_2+2 & & l_2+1 & \\ & & l_1+l_2+2 & & \end{smallmatrix}\right\}$

Table 1: Stable coefficients for twisted A_2

six stable coefficients are given in Table 1. (The last column will be explained presently.)

As the l_i increase, the size of n needed for this stability also increases. Thus twisting introduces instability for many more n, allowing us to study this phenomenon for A_r even when r is small. This is one reason that we study the twisted series, though not the only reason. Even in the stable case, the twisted series are interesting, and they are studied in detail in [5].

When n is not assumed to be large, however, other coefficients appear. We will describe these next, in the case where $\Phi = A_2$. We will find that

$$\mathrm{Supp}_n(H;l_1,l_2) = \Big\{(k_1,k_2)\,|\,H(p^{k_1},p^{k_2};p^{l_1},p^{l_2}) \neq 0\}$$

still contains a set $\mathrm{Supp}_{\mathrm{stable}}(H;l_1,l_2)$ consisting of the six pairs (k_1,k_2) that are listed in Table 1. Moreover $\mathrm{Supp}_n(H;l_1,l_2) = \mathrm{Supp}_{\mathrm{stable}}(H;l_1,l_2)$ when n is sufficiently large, but when n is small, $\mathrm{Supp}_n(H;l_1,l_2)$ is a strictly larger set contained in the convex hull of $\mathrm{Supp}_{\mathrm{stable}}(H;l_1,l_2)$. If $(k_1,k_2) \in \mathrm{Supp}_n(H;l_1,l_2)$, we find that $H(p^{k_1},p^{k_2};p^{l_1},p^{l_2})$ is a *sum* of products of Gauss sums.

By a *Gelfand-Tsetlin pattern* we mean a triangular array of integers

$$\mathfrak{T} = \left\{\begin{matrix} a_{00} & & a_{01} & & a_{02} & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & a_{1r} & \\ & & \ddots & & & \iddots & & \\ & & & & a_{rr} & & & \end{matrix}\right\} \tag{7}$$

with r rows, where the rows interleave; that is, $a_{i-1,j-1} \geqslant a_{i,j} \geqslant a_{i-1,j}$. We will say that the pattern is *strict* if each row is strictly decreasing.

We will make use of strict Gelfand-Tsetlin patterns of the form

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} l_1+l_2+2 & & l_2+1 & & 0 \\ & a & & b & \\ & & c & & \end{array} \right\}. \tag{8}$$

For each such $\mathfrak{T}$ define

$$G(\mathfrak{T}) = g(p^{a-b-1}, p^{c-b})\ g(p^{l_2}, p^b)\, g(p^{l_1+b}, p^{a+b-l_2-1}) \tag{9}$$

unless $a = l_2 + 1$; in the latter case we modify the definition and write

$$G\left(\left\{ \begin{array}{ccccc} l_1+l_2+2 & & l_2+1 & & 0 \\ & l_2+1 & & b & \\ & & c & & \end{array} \right\} \right) = \mathbb{N}p^b\, g(p^{a-b-1}, p^{c-b})\ g(p^{l_2}, p^b). \tag{10}$$

Note that the pattern $\mathfrak{T}$ with $a = b = l_2 + 1$ is not strict, and will be omitted from our summations. Thus $a - b - 1 \geqslant 0$.

Let $k_1(\mathfrak{T}) = a + b - l_2 - 1$ and $k_2(\mathfrak{T}) = c$. Then we will define H so that

$$\sum_{k_1,k_2} H(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2})\, \mathbb{N}p^{-2k_1 s_1 - 2k_2 s_2} = \sum_{\mathfrak{T}} G(\mathfrak{T})\, \mathbb{N}p^{-2k_1(\mathfrak{T})s_1 - 2k_2(\mathfrak{T})s_2}, \tag{11}$$

where the summation is over all $\mathfrak{T}$ of the form (8). Note that (11) is equivalent to

$$H(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) = \sum_{k(\mathfrak{T})=(k_1,k_2)} G(\mathfrak{T}),$$

where $k(\mathfrak{T}) = (a + b - l_2 - 1, c)$.

Remark 1. Taking into account the reduction to the case where $C_i = p^{k_i}$ and $m_i = p^{l_i}$, we have now given a definition of $H(C_1, C_2; m_1, m_2)$. In what sense is this definition "correct"? There are two possible notions of correctness, either of which would be a valid goal.

- We can take these formulas to be the definition of $H(C_1, C_2; m_1, m_2)$, in which case "correctness" means that the functional equations proved in [4] extend to this context. This is the approach we prefer if $\Phi = A_r$.
- Alternatively we may construct a multiple Dirichlet series as a Whittaker coefficient of a metaplectic Eisenstein series – in which case the theorem to be proved will be the agreement of the resulting Dirichlet series with (11). The functional equations will follow from the functional equations of Eisenstein series. This is carried out in Theorem 1 when $\Phi = A_2$.

Next let us explain how the description of the coefficients $H(C_1, C_2; m_1, m_2)$ through (11) contains the stable case. By elementary properties of the Gauss sum

$$g(p^k, p^l) = 0 \text{ unless } \begin{cases} l = 0 \text{ or} \\ l = k+1 \text{ or} \\ 1 \leqslant l \leqslant k \text{ and } n|l. \end{cases}$$

Because of this, $G(\mathfrak{T}) = 0$ for all but the six patterns in Table 1 when n is sufficiently large. Each of the six patterns in Table 1 contributes a product of three Gauss sums by (9), but (except for the last coefficient) some of those sums are equal to 1 since $g(a, 1) = 1$. Omitting those sums gives exactly the values of the table.

Looking at the interior of the hexagon bounded by the stable support, we see that the number of Gelfand-Tsetlin patterns contributing to $H(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2})$ increases as we move in towards the center of the hexagon. It may be useful to look at an example. In Table 2 we plot the values of $H(p^{k_1}, p^{k_2}; p, p^3)$. We abbreviate $g(p^i, p^j)$ as simply g_{ij} to save space; also, for succinctness, we will write p^i or p instead of $\mathbb{N}p^i$ or $\mathbb{N}p$. We will freely make use of the fact that $g_{i0} = 1$ and $\mathbb{N}p^0 = 1$ to discard superfluous factors; on the other hand, $g_{ij} = g_{jj}$ if $i > j$, but we will distinguish between these two Gauss sums to make it easier for the reader to check the computations.

6			$g_{56}g_{12}$	$g_{45}g_{31}g_{23}$	$g_{34}g_{32}g_{34}$	$g_{23}g_{33}g_{45}$	$g_{12}g_{34}g_{56}$
5		$g_{45}g_{11}$	$g_{55}g_{12}$ $+g_{34}g_{31}g_{22}$	$g_{44}g_{31}g_{23}$ $+g_{23}g_{32}g_{33}$	$g_{33}g_{32}g_{34}$ $+g_{12}g_{33}g_{44}$	$g_{22}g_{33}g_{45}$ $+g_{01}g_{34}g_{55}$	$g_{11}g_{34}g_{56}$
4	g_{34}	$g_{44}g_{11}$ $+p\, g_{23}g_{31}$	$g_{54}g_{12}$ $+g_{33}g_{31}g_{22}$ $+p^2 g_{12}g_{32}$	$g_{43}g_{31}g_{23}$ $+g_{22}g_{32}g_{33}$ $+p^3 g_{01}g_{33}$	$g_{32}g_{32}g_{34}$ $+g_{11}g_{33}g_{44}$	$g_{21}g_{33}g_{45}$ $+g_{00}g_{34}g_{55}$	$g_{34}g_{56}$
3	g_{33}	$g_{43}g_{11}$ $+p\, g_{22}g_{31}$	$g_{53}g_{12}$ $+g_{32}g_{31}g_{22}$ $+p^2 g_{11}g_{32}$	$g_{42}g_{31}g_{23}$ $+g_{21}g_{32}g_{33}$ $+p^3 g_{33}$	$g_{31}g_{32}g_{34}$ $+g_{33}g_{44}$	$g_{33}g_{45}$	
2	g_{32}	$g_{42}g_{11}$ $+p\, g_{21}g_{31}$	$g_{52}g_{12}$ $+g_{31}g_{31}g_{22}$ $+p^2 g_{32}$	$g_{41}g_{31}g_{23}$ $+g_{32}g_{33}$	$g_{32}g_{34}$		
1	g_{31}	$g_{41}g_{11}$ $+p\, g_{31}$	$g_{31}g_{22}$ $+g_{51}g_{30}g_{12}$	$g_{31}g_{23}$			
0	1	g_{11}	g_{12}				
	0	1	2	3	4	5	6

Table 2: The values of $H(p^{k_1}, p^{k_2}; p, p^3)$. (Column,Row)$= (k_1, k_2)$

To illustrate how this table was generated, Table 3 shows how $H(p^4, p^4; p, p^3)$ was computed. If $i \geqslant j$ then $g(p^i, p^j) = 0$ for sufficiently large n, so that one can confirm the vanishing of all coefficients except the six "stable" ones for n sufficiently large.

In Section 2, we will extend the Gelfand-Tsetlin description to $\Phi = A_r$, defining coefficients $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$ and multiple Dirichlet series $Z_\Psi(s_1, \cdots, s_r; m_1, \cdots, m_r)$; see (6). Then we make the following conjectures, which are supported by strong and rather interesting evidence, to be discussed in Section 2.

$\mathfrak{T}$	$k(\mathfrak{T})$	$G(\mathfrak{T})$
$\begin{array}{ccccc} 6 & & 4 & & 0 \\ & 6 & & 2 & \\ & & 4 & & \end{array}$	$(4,4)$	$g_{32}g_{32}g_{34}$
$\begin{array}{ccccc} 6 & & 4 & & 0 \\ & 5 & & 3 & \\ & & 4 & & \end{array}$	$(4,4)$	$g_{11}g_{33}g_{44}$

Table 3: Computation of $H(p^4, p^4; p, p^3)$

Conjecture 1. *Z_Ψ has meromorphic continuation to all $\mathbb{C}^r$ and satisfies a group of functional equations containing the Weyl group of A_r as in* [4].

Conjecture 2. *Z_Ψ is a Whittaker coefficient of an Eisenstein series on the n-fold metaplectic cover of* GL_{r+1}.

The evidence for these conjectures may be summarized as follows.

- When $r = 2$, we prove both conjectures in Section 1 (see Theorem 1).
- For all r, it is proved in [5] that the Gelfand-Tsetlin description gives the right stable coefficients, and Conjecture 1 is proved when n is sufficiently large. As a special case when $m_1 = \ldots = m_r = 1$, this shows that the multiple Dirichlet series defined here agrees with that of [4] in the stable case.
- If $n = 1$, we will deduce Conjecture 2 (and hence Conjecture 1) by showing that the Shintani-Casselman-Shalika formula reduces this case to a result of Tokuyama [22].
- If $n = 2$ and $r \leqslant 5$ we will prove Conjecture 1 by reconciling our definition with work of Chinta [11]. See Theorem 2.

The first piece of evidence will be taken up in Section 1, the remaining points will be addressed in Section 2.

After this paper was written, Chinta and Gunnells [12] gave a definition of the Weyl group multiple Dirichlet series when $n = 2$ for any simply-laced root system. Their very interesting construction does not compute the coefficients but in view of their Remark 3.5 and our Theorem 2 we can say that it agrees with our definition when $\Phi = A_r$ and $r \leqslant 5$.

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1. Metaplectic Eisenstein series on GL(3)

In this section, $\mathfrak{o}$ will be the ring of integers in a totally complex number field F. We assume that $\mathfrak{o}^\times$ contains the group μ_n of n-th roots of unity, and that -1 is an n-th power in $\mathfrak{o}^\times$. We will denote by $\left(\frac{c}{d}\right)$ the ordinary power residue symbol, whose properties may be found in Neukirch [17].

Bass, Milnor and Serre [1] (following earlier work of Kubota and Mennicke) constructed a homomorphism $\kappa : \Gamma(\mathfrak{f}) \longrightarrow \mu_n$, where $\mathfrak{f}$ is a suitable conductor, and $\Gamma(\mathfrak{f})$ is the principal congruence subgroup in $\mathrm{GL}(r+1, \mathfrak{o})$. We may choose $\mathfrak{f}$ so that

$$d \equiv c \equiv 1 \mod \mathfrak{f}, \quad \gcd(d, c) = 1 \qquad \Rightarrow \qquad \left(\frac{c}{d}\right) = \left(\frac{d}{c}\right). \tag{12}$$

We also assume that if $d \equiv d' \equiv 1$ modulo $\mathfrak{f}$ then

$$d \equiv d' \mod \mathfrak{f}^2 \text{ and } d \equiv d' \mod c \quad \Rightarrow \quad \left(\frac{c}{d}\right) = \left(\frac{c}{d'}\right). \tag{13}$$

For convenience we will assume that $\mathfrak{o}$ is a principal ideal domain and that the canonical map $\mathfrak{o}^\times \longrightarrow (\mathfrak{o}/\mathfrak{f})^\times$ is surjective. For example, these conditions are satisfied in the following cases.

- $n = 2$, $F = \mathbb{Q}(i)$, $\mathfrak{o} = \mathbb{Z}[i]$, $\lambda = 1 + i$ and $\mathfrak{f} = \lambda^3\mathfrak{o}$.
- $n = 3$, $F = \mathbb{Q}(\rho)$ where $\rho = e^{2\pi i/3}$, $\mathfrak{o} = \mathbb{Z}[\rho]$, $\lambda = 1 - \rho$, and $\mathfrak{f} = \lambda^2\mathfrak{o} = 3\mathfrak{o}$.

We embed $F \longrightarrow F_\infty$, the product of the archimedean completions of F. Let $\psi : F_\infty \longrightarrow \mathbb{C}$ be a nontrivial additive character. We assume that the conductor of ψ is precisely $\mathfrak{o}$; that is, $\psi(x\mathfrak{o}) = 1$ if and only if $x \in \mathfrak{o}$.

This setup has perhaps less to recommend it than the S-integer formalism of [4], but it does have the advantage of allowing us to suppress all Hilbert symbols.

Let

$$w = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}. \tag{14}$$

Then $G = \mathrm{SL}_3$ has an involution defined by ${}^\iota g = w \cdot {}^t g^{-1} \cdot w$. It preserves the group $\Gamma(\mathfrak{f})$ and its subgroup $\Gamma_\infty(\mathfrak{f})$, consisting of the upper triangular matrices in $\Gamma(\mathfrak{f})$. If $g \in \Gamma(\mathfrak{f})$, let $[A_1, B_1, C_1]$ and $[A_2, B_2, C_2]$ be the bottom rows of g and ${}^\iota g$, respectively. Then

$$\begin{gathered} (A_1, B_1, C_1) \equiv (A_2, B_2, C_2) \equiv (0, 0, 1) \mod \mathfrak{f}, \\ A_1C_2 + B_1B_2 + C_1A_2 = 0, \\ \gcd(A_1, B_1, C_1) = \gcd(A_2, B_2, C_2) = 1. \end{gathered} \tag{15}$$

We call $A_1, B_1, C_1, A_2, B_2, C_2$ the *invariants* of g. We will refer to (15) as the *Plücker relation*. The invariants depend only on the coset of g in $\Gamma_\infty(\mathfrak{f})\backslash\Gamma(\mathfrak{f})$.

We will make use of the following formula for $\kappa(g)$. Suppose that $g \in \Gamma(\mathfrak{f})$ has invariants $A_1, B_1, C_1, A_2, B_2, C_2$. Then there exists a factorization

$$\begin{aligned} C_1 &= r_1 r_2 C_1', \qquad & C_2 &= r_1 r_2 C_2', \\ B_1 &= r_1 B_1', \qquad & B_2 &= r_2 B_2', \end{aligned} \tag{16}$$

where $r_1 \equiv r_2 \equiv C_1' \equiv C_2' \equiv 1$ modulo $\mathfrak{f}$, and $\gcd(C_1', C_2') = 1$. We have $\gcd(B_1', C_1') = \gcd(B_2', C_2') = \gcd(A_1, r_1) = \gcd(A_2, r_2) = 1$ and

$$\kappa(g) = \left(\frac{B_1'}{C_1'}\right)\left(\frac{B_2'}{C_2'}\right)\left(\frac{C_1'}{C_2'}\right)^{-1}\left(\frac{A_1}{r_1}\right)\left(\frac{A_2}{r_2}\right). \tag{17}$$

Details can be found in [7]. Similar formulas can be found in Proskurin [19].

Let C_1 and C_2 be elements of $\mathfrak{o}$ that are congruent to 1 modulo $\mathfrak{f}$, and let m_1, $m_2 \in \mathfrak{o}$. We define

(18)

$$\boldsymbol{H}(C_1, C_2; m_1, m_2) = \sum_{\substack{A_1, B_1 \bmod C_1 \\ A_2, B_2 \bmod C_2 \\ \gcd(A_1, B_1, C_1) = 1 \\ \gcd(A_2, B_2, C_2) = 1 \\ A_1 \equiv B_1 \equiv A_2 \equiv B_2 \equiv 0 \bmod \mathfrak{f} \\ A_1 C_2 + B_1 B_2 + C_1 A_2 \equiv 0 \bmod C_1 C_2}} \cdot \left(\frac{B_1'}{C_1'}\right)\left(\frac{B_2'}{C_2'}\right)\left(\frac{C_1'}{C_2'}\right)^{-1}\left(\frac{A_1}{r_1}\right)\left(\frac{A_2}{r_2}\right)\psi\left(\frac{m_1 B_1}{C_1} + \frac{m_2 B_2}{C_2}\right),$$

where we have chosen a factorization (16).

Remark 2. In the introduction we defined $H(C_1, C_2; m_1, m_2)$ as a sum over Gelfand-Tsetlin patterns. In this section, we take (18) to be the definition of sums $\boldsymbol{H}(C_1, C_2; m_1, m_2)$. The content of Theorem 1 is that the two definitions are equivalent when $\Phi = A_2$, so that $\boldsymbol{H}(C_1, C_2; m_1, m_2) = H(C_1, C_2; m_1, m_2)$.

Remark 3. The summation is more correctly written

$$\sum_{\substack{B_1 \bmod C_1 \\ B_2 \bmod C_2 \\ B_1 \equiv B_2 \equiv 0 \bmod \mathfrak{f}}} \quad \sum_{\substack{A_1 \bmod C_1 \\ A_2 \bmod C_2 \\ \gcd(A_1, B_1, C_1) = 1 \\ \gcd(A_2, B_2, C_2) = 1 \\ A_1 \equiv A_2 \equiv 0 \bmod \mathfrak{f} \\ A_1 C_2 + B_1 B_2 + C_1 A_2 \equiv 0 \bmod C_1 C_2}} . \tag{19}$$

The reason that this way of writing the sum is correct is that if B_1 is changed to $B_1 + tC_1$ then the terms of the inner sum are permuted, with a compensating change $A_2 \longrightarrow A_2 - tB_2$. With this understanding, the sum $\boldsymbol{H}(C_1, C_2; m_1, m_2)$ is well-defined.

Let f be a smooth function on $\mathrm{SL}_3(F_\infty)$ that satisfies

$$f\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} g\right) = |y_1|^{2s_2}|y_3|^{-2s_1} f(g),$$

where s_1 and s_2 have sufficiently large real part. Let

$$E(g) = \sum_{\gamma \in \Gamma_\infty(\mathfrak{f}) \backslash \Gamma(\mathfrak{f})} \kappa(\gamma)\, f(\gamma g).$$

Let w be as in (14). Let $m_1, m_2 \in \mathfrak{o}$ be nonzero, and let

$$W_{m_1,m_2}(g) = \int_{\mathbb{C}^3} f\left(w \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right) \psi(-m_1x_1 - m_2x_2)\, dx_1\, dx_2\, dx_3$$

be the Jacquet-Whittaker function.

Proposition 1.

$$\int_{\mathfrak{f}^3 \backslash \mathbb{C}^3} E\left(w \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right) \psi(-m_1x_1 - m_2x_2)\, dx_1\, dx_2\, dx_3$$
$$= \sum_{C_1, C_2 \neq 0} \boldsymbol{H}(C_1, C_2; m_1, m_2)\, \mathbb{N}C_1^{-2s_1} \mathbb{N}C_2^{-2s_2} W_{m_1,m_2}(g).$$

Proof. The invariants give a bijection between the set of parameters A_1, B_1, C_1, A_2, B_2, C_2 that satisfy (15) and $\Gamma_\infty(\mathfrak{f})\backslash\Gamma(\mathfrak{f})$; this may be proved along the lines of Theorem 5.4 of Bump [8]. It may be shown that with m_1, m_2 nonzero, only γ in the "big cell" characterized by the nonvanishing of C_1, C_2 give a nonzero contribution; let $\Gamma(\mathfrak{f})^{\mathrm{bc}}$ denote the set of such elements. Discarding the others, the integral unfolds to

$$\sum_{\gamma \in \Gamma_\infty(\mathfrak{f}) \backslash \Gamma(\mathfrak{f})^{\mathrm{bc}} / w\Gamma_\infty(\mathfrak{f})w}$$
$$\cdot \int_{\mathbb{C}^3} \kappa(\gamma)\, f\left(\gamma w \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right) \psi(-m_1x_1 - m_2x_2)\, dx_1\, dx_2\, dx_3.$$

We have the explicit Bruhat decomposition

$$\gamma = \begin{pmatrix} 1/C_2 & * & * \\ & C_2/C_1 & * \\ & & C_1 \end{pmatrix} \begin{pmatrix} 1 & & \\ B_2/C_2 & 1 & \\ A_1/C_1 & B_1/C_1 & 1 \end{pmatrix}.$$

Thus making the variable change $x_1 \longrightarrow x_1 + B_1/C_1$, $x_2 \longrightarrow x_2 + B_2/C_2$ we obtain

$$\sum_{\gamma \in \Gamma_\infty(\mathfrak{f}) \backslash \Gamma(\mathfrak{f})^{\mathrm{bc}} / w\Gamma_\infty(\mathfrak{f})w} \kappa(\gamma)\, \psi\left(\frac{m_1 B_1}{C_1} + \frac{m_2 B_2}{C_2}\right) \mathbb{N}C_1^{-2s_1} \mathbb{N}C_2^{-2s_2} W_{m_1,m_2}(g)$$

where it is understood that $A_1, B_1, C_1, A_2, B_2, C_2$ are the invariants of γ. The action of $w\Gamma_\infty(\mathfrak{f})w$ on the invariants is easily computed, and so we obtain a sum over nonzero C_1, C_2, and over A_1, B_1 modulo C_1, A_2, B_2 modulo C_2 such that $\gcd(A_1, B_1, C_1) = \gcd(A_2, B_2, C_2) = 1$, satisfying the Plücker relation; and for γ with these invariants, $\kappa(\gamma)$ is given by (17). □

Proposition 2. *If* $\gcd(C_1 C_2, C_1' C_2') = 1$ *with* $C_1 \equiv C_2 \equiv C_1' \equiv C_2' \equiv 1$ *modulo* $\mathfrak{f}$, *then*

$$\begin{aligned} &\boldsymbol{H}(C_1 C_1', C_2 C_2'; m_1, m_2) \\ &= \left(\frac{C_1}{C_1'}\right)^2 \left(\frac{C_2}{C_2'}\right)^2 \left(\frac{C_1}{C_2'}\right)^{-1} \left(\frac{C_2}{C_1'}\right)^{-1} \boldsymbol{H}(C_1, C_2; m_1, m_2)\, \boldsymbol{H}(C_1', C_2'; m_1, m_2). \end{aligned}$$

Proof. This is proved in [7]. □

Proposition 3. *Suppose that* $\gcd(m_1' m_2', C_1 C_2) = 1$. *Then*

$$\boldsymbol{H}(C_1, C_2; m_1 m_1', m_2 m_2') = \left(\frac{m_1'}{C_1}\right)^{-1} \left(\frac{m_2'}{C_2}\right)^{-1} \boldsymbol{H}(C_1, C_2; m_1, m_2).$$

Proof. This is easier than Proposition 2, and can be left to the reader. □

We turn now to the lemmas for Theorem 1.

If $\mathfrak{T}$ is as in (8), let $k(\mathfrak{T}) = (a + b - l_2 - 1, c)$. We will also denote $k_1(\mathfrak{T}) = a + b - l_1 - 1$ and $k_2(\mathfrak{T}) = c$.

Lemma 1. *Let*

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} l_1 + l_2 + 2 & & l_2 + 1 & & 0 \\ & a & & b & \\ & & c & & \end{array} \right\}$$

be a Gelfand-Tsetlin pattern. Assume that

$$l_2 \geqslant b, \qquad c + l_2 + 1 \geqslant a, \qquad c - 2a + l_1 + 2l_2 + 2 \geqslant b. \tag{20}$$

Let

$$a' = c - a + l_1 + l_2 + 2, \qquad b' = a - l_2 - 1, \qquad c' = a + b - l_2 - 1,$$

and

$$\mathfrak{T}' = \left\{ \begin{array}{ccccc} l_1 + l_2 + 2 & & l_1 + 1 & & 0 \\ & a' & & b' & \\ & & c' & & \end{array} \right\}.$$

Then $\mathfrak{T}'$ is also a Gelfand-Tsetlin pattern and $G(\mathfrak{T}) = G(\mathfrak{T}')$. The hypothesis (20) *is always satisfied if $k_2(\mathfrak{T}) = c$ is greater than $k_1(\mathfrak{T}) = a + b - l_2 - 1$.*

Proof. It is straightforward to check that (20) implies that $\mathfrak{T}'$ is a Gelfand-Tsetlin pattern. It is also easy to check that $k_2 > k_1$ implies (20).

We turn to the proof that $G(\mathfrak{T}) = G(\mathfrak{T}')$. First suppose that $a > l_2 + 1$. We note that our assumptions imply that $a' > l_1 + 1$. Assuming (20) we must show that

$$g(p^{a-b-1}, p^{c-b})\; g(p^{l_2}, p^b)\, g(p^{l_1+b}, p^{a+b-l_2-1})$$
$$= g(p^{c-2a+l_1+2l_2+2}, p^b)\, g(p^{l_1}, p^{a-l_2-1})\, g(p^{a-1}, p^c).$$

Since we are assuming $l_2 \geqslant b$ and $c - 2a + 2l_1 + l_2 + 2 \geqslant b$ both sides vanish unless $n|b$. We therefore assume $n|b$. Since

$$g(p^{l_2}, p^b) = g(p^b, p^b) = g(p^{c-2a+2l_1+l_2+2}, p^b), \tag{21}$$

we must show that

$$g(p^{a-b-1}, p^{c-b})\, g(p^{l_1+b}, p^{a+b-l_2-1}) = g(p^{a-1}, p^c) g(p^{l_1}, p^{a-l_2-1})\,.$$

This follows since $n|b$ implies that

$$g(p^{a-1}, p^c) = \mathbb{N}p^b\, g(p^{a-b-1}, p^{c-b}) \tag{22}$$

and

$$g(p^{l_2+b}, p^{a+b-l_1-1}) = \mathbb{N}p^b\, g(p^{l_2}, p^{a-l_1-1}).$$

If $a = l_2 + 1$ then both $G(\mathfrak{T})$ and $G(\mathfrak{T}')$ are zero unless $n|b$, and proceeding as before, the statement now follows from (21) and (22), together with the fact that $g(p^{l_1}, p^{a-l_2-1}) = 1$. □

Let $\Upsilon(k_1, k_2; l_1, l_2)$ be the set of all $\mathfrak{T}$ of the form (8) such that $k(\mathfrak{T}) = (k_1, k_2)$. As in the introduction, let

$$H(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) = \sum_{\mathfrak{T} \in \Upsilon(k_1, k_2; l_1, l_2)} G(\mathfrak{T}).$$

Lemma 1 gives a bijection $\Upsilon(k_1, k_2; l_1, l_2) \longrightarrow \Upsilon(k_2, k_1; l_2, l_1)$ when $k_2 > k_1$; since the bijection preserves $G(\mathfrak{T})$, this means that the right-hand side of (27) satisfies

$$H(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) = H(p^{k_2}, p^{k_1}; p^{l_2}, p^{l_1}) \tag{23}$$

when $k_2 > k_1$. No bijection $\Upsilon(k_1, k_2; l_1, l_2) \longrightarrow \Upsilon(k_2, k_1; l_2, l_1)$ preserving $G(\mathfrak{T})$ exists when $k_1 = k_2$, though we will see that (23) is still true. Indeed, examples may be given where the number of nonzero $G(\mathfrak{T})$ with $\mathfrak{T} \in \Upsilon(k, k; l_1, l_2)$ is different when l_1 and l_2 are interchanged, though their sum is still remarkably the same due to more complicated identities between the $G(\mathfrak{T})$.

Lemma 2. *If $k_1 > k_2$, then*

$$\boldsymbol{H}(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) = \sum_{i=\max(0,k_2-l_2-1)}^{\min(k_2,k_2-k_1+l_1+1)} g(p^i, p^i)\, g(p^{l_2}, p^{k_2-i})\, g(p^{l_1+k_2-i}, p^{k_1}).$$

Proof. We note that since $g(p^a, p^b) = 0$ if $a < b-1$, the statement is equivalent to

$$\boldsymbol{H}(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) = \sum_{i=0}^{k_2} g(p^i, p^i)\, g(p^{l_2}, p^{k_2-i})\, g(p^{l_1+k_2-i}, p^{k_1}) \tag{24}$$

since any terms in this sum with $i < k_2 - l_2 - 1$ or $i > k_2 - k_1 + l_1 + 1$ contribute zero. We prove (24).

In the definition of $\boldsymbol{H}$, we have $r_1 r_2 = p^{k_2}$ and we can take $C_1' = p^{k_1-k_2}$, $C_2' = 1$. Thus

$$\boldsymbol{H}(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) = \sum_{\substack{A_1, B_1 \bmod p^{k_1} \\ A_2, B_2 \bmod p^{k_2} \\ \gcd(A_1,B_1,p) = \gcd(A_2,B_2,p)=1 \\ A_1 p^{k_2} + B_1 B_2 + A_2 p^{k_1} \equiv 0 \bmod p^{k_1+k_2}}} \cdot \left(\frac{B_1'}{p^{k_1-k_2}}\right) \left(\frac{A_1}{r_1}\right) \left(\frac{A_2}{r_2}\right) \psi\left(\frac{B_1 p^{l_1}}{p^{k_1}} + \frac{B_2 p^{l_2}}{p^{k_2}}\right). \tag{25}$$

It is understood that A_1, A_2, B_1 and B_2 are always chosen to be divisible by the conductor $\mathfrak{f}$; we will omit this condition from all summations since it really plays no role in the computation. We break the sum up into three pieces: (1) $\gcd(B_2, p) = 1$, (2) p^i exactly divides B_2 with $1 \leqslant i < k_2$, and (3) $p^{k_2} | B_2$.

First we consider the contribution where $\gcd(B_2, p) = 1$. Here $r_2 = 1$, $r_1 = p^{k_2}$, and from the Plücker relation, $B_1 \equiv 0 \mod p^{k_2}$. After replacing B_1 by $p^{k_2} B_2'$ and dropping the prime, we get

$$\sum_{\substack{A_1 \bmod p^{k_1},\ B_1 \bmod p^{k_1-k_2} \\ A_2, B_2 \bmod p^{k_2} \\ \gcd(B_2,p) = \gcd(A_1,p) = 1 \\ A_1 + B_1 B_2 + A_2 p^{k_1-k_2} \equiv 0 \bmod p^{k_1}}} \left(\frac{B_1}{p^{k_1-k_2}}\right) \left(\frac{A_1}{p^{k_2}}\right) \psi\left(\frac{B_1 p^{l_1}}{p^{k_1-k_2}} + \frac{B_2 p^{l_2}}{p^{k_2}}\right). \tag{26}$$

We may use the Plücker relation to determine A_1. The sum becomes

$$\sum_{\substack{B_1 \bmod p^{k_1-k_2} \\ A_2, B_2 \bmod p^{k_2} \\ \gcd(B_2,p) = \gcd(A_2 p^{k_1-k_2} + B_1 B_2, p) = 1}} \cdot \left(\frac{B_1}{p^{k_1-k_2}}\right) \left(\frac{A_2 p^{k_1-k_2} + B_1 B_2}{p^{k_2}}\right) \psi\left(\frac{B_1 p^{l_1}}{p^{k_1-k_2}} + \frac{B_2 p^{l_2}}{p^{k_2}}\right).$$

Since $k_1 > k_2$ we may replace the condition $\gcd(A_2 p^{k_1-k_2} + B_1 B_2, p) = 1$ by just $\gcd(B_1, p) = 1$, and we also have $\left(\frac{A_2 p^{k_1-k_2}+B_1 B_2}{p^{k_2}}\right) = \left(\frac{B_1 B_2}{p^{k_2}}\right)$. The summand is independent of A_2, and we may drop this summation to obtain

$$\mathbb{N}p^{k_2} \sum_{\substack{B_1 \bmod p^{k_1-k_2} \\ B_2 \bmod p^{k_2} \\ \gcd(B_1 B_2, p)=1}} \left(\frac{B_1}{p^{k_1}}\right)\left(\frac{B_2}{p^{k_2}}\right) \psi\left(\frac{B_1 p^{l_1}}{p^{k_1-k_2}} + \frac{B_2 p^{l_2}}{p^{k_2}}\right).$$

Now we may drop the leading factor of $\mathbb{N}p^{k_2}$ by summing B_2 over p^{k_1} instead of $p^{k_1-k_2}$. Hence we obtain

$$g(p^{l_2}, p^{k_2})\, g(p^{l_1+k_2}, p^{k_1}).$$

This is the contribution $i = 0$ in (24).

One may show similarly that if p^i exactly divides B_2 for some i, $1 \leqslant i < k_2$, then one obtains the i-th term (24), and that the contribution when $p^{k_2} | B_2$ is the contribution of $i = k_2$ in (24). We leave these cases to the reader, or see [7]. □

LEMMA 3.

$$\boldsymbol{H}(p^k, p^k; p^{l_1}, p^{l_2}) = \sum_{i=\max(0,k-l_1-1)}^{\min(k-1,l_2+1)} g(p^{l_2}, p^i)\, g(p^{l_1+i}, p^k)\, g(p^{l_2+k-2i}, p^{k-i}) + \begin{cases} \mathbb{N}p^k\, g(p^k, p^k) & \textit{if } k \leqslant l_2; \\ 0 & \textit{if } k > l_2. \end{cases}$$

Proof. We leave this to the reader, or see [7]. It is similar to Lemma 2. □

THEOREM 1. *Let l_1, l_2 be nonnegative integers. Then*

$$\sum_{k_1,k_2} \boldsymbol{H}(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2})\, \mathbb{N}p^{-k_1 s_1 - k_2 s_2} = \sum_{\mathfrak{T}} G(\mathfrak{T})\, \mathbb{N}p^{-k_1(\mathfrak{T}) s_1 - k_2(\mathfrak{T}) s_2},$$

where the summation is over all strict Gelfand-Tsetlin patterns $\mathfrak{T}$ of the form (8).

Proof. Evidently what must be proved is that

$$\boldsymbol{H}(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) = H(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}). \tag{27}$$

It is clear from the definition that $\boldsymbol{H}(p^{k_1}, p^{k_2}; p^{l_1}, p^{l_2}) = \boldsymbol{H}(p^{k_2}, p^{k_1}; p^{l_2}, p^{l_1})$. By (23) we may therefore assume that $k_1 \geqslant k_2$.

First suppose that $k_1 > k_2$. Then given an integer i we consider

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} l_1+l_2+2 & & l_2+1 & & 0 \\ & a & & b & \\ & & c & & \end{array} \right\}, \qquad \begin{array}{l} a = k_1 - k_2 + i + l_2 + 1, \\ b = k_2 - i, \\ c = k_2. \end{array}$$

A necessary and sufficient condition for this to be a Gelfand-Tsetlin pattern is that

$$\max(0, k_2 - l_2 - 1) \leqslant i \leqslant \min(k_2, k_2 + l_1 + 1 - k_1).$$

This gives a complete enumeration of $\Upsilon(k_1, k_2; l_1, l_2)$. We have $a-b-1 \geqslant c-b$ and so

$$\begin{aligned} G(\mathfrak{T}) &= g(p^{c-b}, p^{c-b})\ g(p^{l_2}, p^b)\, g(p^{l_1+b}, p^{a+b-l_2-1}) \\ &= g(p^i, p^i)\ g(p^{l_2}, p^{k_2-i})\ g(p^{l_1+k_2-i}, p^{k_1}). \end{aligned}$$

In this case, the result now follows from Lemma 2.

Next assume that $k_1 = k_2 = k$. Given an integer i, consider

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} l_1 + l_2 + 2 & & l_2 + 1 & & 0 \\ & a & & b & \\ & & c & & \end{array} \right\}, \qquad \begin{array}{l} a = k - i + l_2 + 1, \\ b = i, \\ c = k. \end{array}$$

A necessary and sufficient condition for this to be a Gelfand-Tsetlin pattern is that

$$\max(0, k - l_1 - 1) \leqslant i \leqslant \max(k, l_2 + 1),$$

and this gives a complete enumeration of $\Upsilon(k, k; l_1, l_2)$. We assume first that $i < k$. In this case we have

$$\begin{aligned} G(\mathfrak{T}) &= g(p^{a-b-1}, p^{c-b})\ g(p^{l_2}, p^b)\, g(p^{l_1+b}, p^{a+b-l_2-1}) \\ &= g(p^{l_2+k-2i}, p^{k-i})\, g(p^{l_2}, p^i)\, g(p^{l_1+i}, p^k), \end{aligned}$$

and these terms account for the first summation in Lemma 3. If $k \leqslant l_2 + 1$ there is one more term with $i = k$. Using (10), this accounts for the last term in Lemma 3, and the theorem is proved. □

2. The case $\Phi = A_r$

In this section we generalize the definition of $H(C_1, \cdots, C_r; m_1, \cdots, m_r)$ from the introduction, where it was given for A_2, to $\Phi = A_r$, for arbitrary r. We will present evidence that this definition is "correct," as discussed in Remark 1. As explained in the introduction, the multiplicativity properties of H reduce us to the case where the C_i and m_i are all powers of the same prime p.

First we must generalize the definition of $G(\mathfrak{T})$ when $\mathfrak{T}$ is a strict Gelfand-Tsetlin pattern of rank r, given as in (7). We define

$$G(\mathfrak{T}) = \prod_{j \geqslant i \geqslant 1} \gamma(i, j),$$

where

$$\gamma(i,j) = \begin{cases} g\left(p^{s_{ij}-a_{ij}+a_{i-1,j-1}-1}, p^{s_{ij}}\right) & \text{if } a_{ij} > a_{i-1,j}, \\ \mathbb{N}p^{s_{ij}} & \text{if } a_{ij} = a_{i-1,j}, \end{cases} \tag{28}$$

$$s_{ij} = \sum_{k=j}^{r} a_{ik} - \sum_{k=j}^{r} a_{i-1,k}.$$

Thus we are associating one factor $\gamma(i,j)$ to each entry of the pattern below the top row. If $r = 2$, this formula is the same as (11). Also, define

$$k_i(\mathfrak{T}) = \sum_{j=i}^{r} (a_{ij} - a_{0,j}).$$

Now we may generalize the Weyl group multiple Dirichlet series from [4] for type A_r to the unstable case where n is *any* arbitrary positive integer. Define

$$H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum G(\mathfrak{T}) \tag{29}$$

where the sum is over all strict Gelfand-Tsetlin patterns $\mathfrak{T}$ with top row

$$l_1 + \ldots + l_r + r, l_2 + \ldots + l_r + r - 1, \cdots, l_r + 1, 0 \tag{30}$$

such that

$$\sum_{j=i}^{r} (a_{ij} - a_{0,j}) = k_i. \tag{31}$$

We now discuss some evidence for Conjectures 1 and 2.

Definition 1. If each a_{ij} with $i \neq 0$ is equal to one of the two terms above it ($a_{i-1,j-1}$ or $a_{i-1,j}$), then the Gelfand-Tsetlin pattern $\mathfrak{T}$ is called *stable*.

A stable Gelfand-Tsetlin pattern is the unique one such that (31) is satisfied for the particular values of $(k_1, \cdots, k_r)$. There are $(r+1)!$ such patterns.

In [5], Dirichlet series are introduced with parameters $m_1, \cdots, m_r$ that generalize the definition of the multiple Dirichlet series and results of [4]. That is, the Dirichlet series are shown to possess a Weyl group of functional equations. It is then checked that these so-called "twisted, stable" Weyl group multiple Dirichlet series have coefficients matching those associated to stable patterns in (29), while $G(\mathfrak{T}) = 0$ for all patterns that are not stable. Therefore Conjecture 1 is proved in the stable case, that is, for n sufficiently large. The use of the term "stable" in Definition 1 is also natural since only the contributions of the stable patterns survive when n is large.

Next we turn to the relationship between this formula and the Shintani-Casselman-Shalika formula. To begin with, these formulas are somewhat analogous; we will explain this analogy before giving a formula that combines the two.

Let T be the diagonal maximal torus in $\mathrm{GL}(r,\mathbb{C})$. We identify $\mathbb{Z}^r$ with the character group $X^*(T)$ in the usual way; elements of this group are called *weights*, and with this identification, $\mu = (\mu_1, \cdots, \mu_r)$ corresponds to the rational character

$$t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix} \longmapsto \prod t_i^{\mu_i} = \langle \mu, t \rangle \tag{32}$$

of T. Let $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_r$ be integers. Then $\lambda = (\lambda_1, \cdots, \lambda_r)$ is the highest weight vector of an irreducible analytic representation σ_λ of $\mathrm{GL}(r,\mathbb{C})$.

It was shown by Gelfand and Tsetlin [13] that one could exhibit a specific basis of an irreducible analytic representation of $\mathrm{GL}_r(\mathbb{C})$ isomorphic to σ_λ parametrized by these Gelfand-Tsetlin patterns.

Dually, there is also a parametrization of the weights of σ_λ by the same set of Gelfand-Tsetlin patterns. When $r = 3$, the parametrization of the weights in σ_λ by Gelfand-Tsetlin patterns sends

$$\left\{ \begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a & & b & \\ & & c & & \end{array} \right\}$$

to the weight $\mu(\mathfrak{T}) = (\lambda_1 + \lambda_2 + \lambda_3 - a - b, a + b - c, c)$. Note that we can write $\lambda - \mu(\mathfrak{T}) = k_1\alpha_1 + k_2\alpha_2$ where $\alpha_1 = (1,-1,0)$ and $\alpha_2 = (0,1,-1)$ are the simple positive roots and k_1, k_2 are nonnegative integers. We find that

$$k_1 = a + b - \lambda_2 - \lambda_3, \qquad k_2 = c - \lambda_3.$$

As $\mathfrak{T}$ runs through the Gelfand-Tsetlin patterns with prescribed λ_1, λ_2 and λ_3, $\mu(\mathfrak{T})$ runs through the weights of σ_λ, each occurring with its proper multiplicity. Thus if $t \in \mathrm{GL}_3(\mathbb{C})$, we see that

$$\mathrm{tr}\ \sigma_\lambda(t) = \sum_{\mathfrak{T}} \langle \mu(\mathfrak{T}), t \rangle, \tag{33}$$

with notation as in (32); and this formula remains valid for GL_r, with the obvious extension of the definition of $\mu(\mathfrak{T})$ in the general case.

We recall the formula of Shintani [20] and Casselman and Shalika [10] for Whittaker functions on $\mathrm{GL}_r(F)$ where F is a nonarchimedean local field. Let π be a spherical principal series representation of $\mathrm{GL}_r(F)$, and let W be the spherical Whittaker function of π, normalized so that $W(1) = 1$. Langlands associates with π (by means of the Satake isomorphism) a semisimple conjugacy class in $\mathrm{GL}_r(\mathbb{C})$; see Borel [2]. Let A_π be a diagonal representative of this conjugacy class. Let

$$a = \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_r} \end{pmatrix},$$

where ϖ is a prime element in F. The Shintani-Casselman-Shalika formula may be stated

$$W(a) = \begin{cases} \delta^{1/2}(a)\ \mathrm{tr}\ \sigma_\lambda(A_\pi) & \text{if } \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r; \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with (33), we see that the values of the Whittaker function can be expressed as a sum parametrized by Gelfand-Tsetlin patterns.

The Shintani-Casselman-Shalika formula may be regarded as a formula for the Whittaker coefficients of Eisenstein series on GL_{r+1}. Since $Z(s_1, \cdots, s_r; p^{l_1}, \cdots, p^{l_{r+1}})$ is conjecturally the Whittaker coefficient of an Eisenstein series on the n-fold metaplectic cover of GL_{r+1}, expressing its "p-part" as a sum over Gelfand-Tsetlin patterns seems analogous.

There are some important differences to be noted.

- The Shintani-Casselman-Shalika formula is for the normalized Whittaker function; this means that if one regards it as a formula for the Whittaker coefficients of Eisenstein series, those Eisenstein series are normalized. By contrast, the new formula is for the *unnormalized* Eisenstein series.
- In the Shintani-Casselman-Shalika formula the top row of the Gelfand-Tsetlin patterns that occur in (33) is the partition λ. In the new formula the top row is λ shifted by $(r, r-1, \cdots, 0)$ as in (30).
- Also, only strict patterns have nonzero contribution to the new formula.
- In the Shintani-Casselman-Shalika formula, one has uniqueness of Whittaker models. In the metaplectic case, Whittaker models are not unique, so the formula must be regarded as expressing one particular spherical Whittaker function.
- Most surprisingly, in the new formula the weight $\mu(\mathfrak{T})$ is replaced by a product of Gauss sums.

These differences are substantial enough that we do not insist too strongly on the analogy between our new formula and the Shintani-Casselman-Shalika formula. However, as we will now explain, we may *combine* the Shintani-Casselman-Shalika formula with a theorem of Tokuyama [22] to prove Conjecture 2 when $n = 1$.

To explain this point, we give another formula for $H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$, valid for all n, before specializing to $n = 1$. We say that the strict Gelfand-Tsetlin pattern $\mathfrak{T}$ in (7) is *left-leaning* at (i, j) if $a_{i,j} = a_{i-1,j-1}$, *right-leaning* if $a_{i,j} = a_{i-1,j}$, and that (i, j) is *special* for $\mathfrak{T}$ if $a_{i-1,j-1} > a_{i,j} > a_{i-1,j}$. We observe that

$$\sum_{1 \leqslant i \leqslant j \leqslant r} s_{ij} = \sum_{1 \leqslant i \leqslant j \leqslant r} a_{ij} - \sum_{i=1}^{r} j a_{0j} = \sum_{i=1}^{r} k_i(\mathfrak{T}),$$

where s_{ij} is as defined in (28). Let $\gamma'(i,j)=\mathbb{N}p^{-s_{ij}}\gamma(i,j)$ and

$$G'(\mathfrak{T})=\prod_{j\geqslant i\geqslant 1}\gamma'(i,j).$$

Then

$$\sum_{k_1}\cdots\sum_{k_r}H(p^{k_1},\cdots,p^{k_r};p^{l_1},\cdots,p^{l_r})x_1^{k_1}\cdots x_r^{k_r}$$
$$=\sum_{k_1}\cdots\sum_{k_r}H'(p^{k_1},\cdots,p^{k_r};p^{l_1},\cdots,p^{l_r})(\mathbb{N}p\cdot x_1)^{k_1}\cdots(\mathbb{N}p\cdot x_r)^{k_r},$$

with

$$H'(p^{k_1},\cdots,p^{k_r};p^{l_1},\cdots,p^{l_r})=\sum G'(\mathfrak{T}),$$

where the sum is over all strict Gelfand-Tsetlin patterns $\mathfrak{T}$ with top row (30) satisfying (31). By elementary properties of Gauss sums

$$\gamma'(i,j)=\begin{cases}1 & \text{if } \mathfrak{T} \text{ is right-leaning at } (i,j),\\ \mathbb{N}p^{-s_{ij}}g(p^{s_{ij}-1},p^{s_{ij}}) & \text{if } \mathfrak{T} \text{ is left-leaning at } (i,j),\\ 1-\mathbb{N}p^{-1} & \text{if } (i,j) \text{ is special and } n|s_{ij},\\ 0 & \text{if } (i,j) \text{ is special and } n\nmid s_{ij}.\end{cases}$$

From this expression, we may clearly see how the stability phenomenon is reconciled with Conjecture 1. If n is sufficiently large, the condition $n|s_{ij}$ cannot be met, and Gelfand-Tsetlin patterns containing special entries have $G'(\mathfrak{T})=0$. The ones that do not are just the $(r+1)!$ stable patterns.

If $n=1$ then the Gauss sums that appear in this formula are Ramanujan sums, and may be evaluated. We have

$$\gamma'(i,j)=\begin{cases}1 & \text{if } \mathfrak{T} \text{ is right-leaning at } (i,j),\\ -\mathbb{N}p^{-1} & \text{if } \mathfrak{T} \text{ is left-leaning at } (i,j),\\ 1-\mathbb{N}p^{-1} & \text{if } (i,j) \text{ is special.}\end{cases}$$

We now recall Tokuyama's theorem from [22]. Tokuyama defines $s(\mathfrak{T})$ and $l(\mathfrak{T})$ to be the number of special and left-leaning entries, respectively. Let $d_i(\mathfrak{T})$ be the sum of the i-th row and let

$$m_i(\mathfrak{T})=\begin{cases}d_i(\mathfrak{T})-d_{i+1}(\mathfrak{T}) & \text{if } 0\leqslant i<r,\\ d_i(\mathfrak{T}) & \text{if } i=r.\end{cases}$$

Let $\lambda=(\lambda_0,\lambda_1,\cdots,\lambda_r)$ be a partition into $r+1$ distinct parts, so $\lambda_0>\ldots>\lambda_r$. Also let $\rho=(\rho_0,\rho_1,\cdots,\rho_r)=(r,r-1,\cdots,0)$, so $\rho_i=r-i$. Tokuyama proves that if t and $z_0,\cdots,z_r$ are indeterminates, then

$$\sum_{\mathfrak{T}}(t+1)^{s(\mathfrak{T})}t^{l(\mathfrak{T})}z_0^{m_0(\mathfrak{T})}\cdots z_r^{m_r(\mathfrak{T})}=\left[\prod_{r\geqslant j>i\geqslant 0}(z_i+tz_j)\right]s_\lambda(z_0,\cdots,z_r),$$

where the sum is over strict Gelfand-Tsetlin patterns $\mathfrak{T}$ with top row $\lambda+\rho$, and s_λ is the Schur polynomial.

We take $\lambda_i = l_{i+1} + l_{i+2} + \cdots + l_r$ and $t = -\mathbb{N}p^{-1}$. Moreover, let

$$z_0 = 1, \qquad z_1 = \mathbb{N}p \cdot x_1, \quad \cdots \quad , z_r = \mathbb{N}p^r \cdot x_1 \cdots x_r.$$

Since

$$(t+1)^{s(\mathfrak{T})} t^{l(\mathfrak{T})} = \prod_{1 \leqslant i \leqslant j \leqslant r} \gamma'(i,j)$$

and $k_i(\mathfrak{T}) = d_i(\mathfrak{T}) - (\lambda_i - \rho_i) - \cdots - (\lambda_r - \rho_r)$, we obtain

$$\begin{aligned} (34) \quad & \sum_{k_1} \cdots \sum_{k_r} H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})\, x_1^{k_1} \cdots x_r^{k_r} \\ &= \sum_{k_1} \cdots \sum_{k_r} H'(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) (\mathbb{N}p \cdot x_1)^{k_1} \cdots (\mathbb{N}p \cdot x_r)^{k_r} \\ &= (\mathbb{N}p \cdot x_1)^{-\lambda_1} \cdots (\mathbb{N}p^{r-1} \cdot x_1 \cdots x_{r-1})^{-\lambda_{r-1}} (\mathbb{N}p^r \cdot x_1 \cdots x_r)^{-\lambda_r} \\ &\quad s_\lambda(1, \mathbb{N}p x_1, \mathbb{N}p^2 x_1 x_2, \cdots, \mathbb{N}p^r x_1 \cdots x_r) \prod_{1 \leqslant i \leqslant j \leqslant r} (1 - \mathbb{N}p^{j-i} x_i \cdots x_j). \end{aligned}$$

This allows us to deduce Conjecture 2 when $n=1$. When p is a prime and $x_i = \mathbb{N}p^{-2s_i}$ are the Satake parameters of a minimal parabolic Eisenstein series on GL_{r+1}, we thus confirm the agreement of the two formulas for the Whittaker coefficient, one given by Conjecture 2, the other by the Shintani-Casselman-Shalika formula. Since both sides are polynomials in the x_i and p, this is sufficient. The $\frac{1}{2}r(r+1)$ factors $(1 - \mathbb{N}p^{i-j} \cdot x_i x_{i+1} \cdots x_j)$ on the right correspond to the normalizing factor.

Another very convincing piece of evidence for the formula (31) is the comparison with computations of Gautam Chinta when $n=2$. Chinta computed "correction polynomials" that are needed to create a multiple Dirichlet series of type A_r for $r \leqslant 5$; the case $r=5$ is contained in [11]; the correction polynomial occupies about 2 pages at the end of the paper. It can be downloaded from `http://www.math.brown.edu/~chinta/a5poly`. (We have also checked cases $r \leqslant 4$, which Chinta has also worked out, though not in print.)

To compare Chinta's result with (31), observe that the correct denominator to correspond to the fifteen factors in the normalizing factor of the GL_6 Eisenstein series should have $1-x^2$, $1-y^2$, $1-z^2$, $1-w^2$ and $1-v^2$, where Chinta's denominator has $1-x$, $1-y$, $1-z$, $1-w$ and $1-v$. Thus both the numerator and denominator need to be multiplied by $(1+x)(1+y) \cdot (1+z)(1+w)(1+v)$. If this is done, and the resulting numerator is expanded, one obtains a polynomial $P(x,y,z,w,v)$ that we will interpret in terms of Gauss sums of Gelfand-Tsetlin patterns. Let us write

$$P(x,y,z,w,v) = \sum_{(k_1,k_2,k_3,k_4,k_5)} h(k_1,k_2,k_3,k_4,k_5)\, x^{k_1} y^{k_2} z^{k_3} w^{k_4} v^{k_5}.$$

One finds that there are 1,340 values of $(k_1, k_2, k_3, k_4, k_5)$ such that $h(k_1, k_2, k_3, k_4, k_5) \neq 0$; each of these is a (usually uncomplicated) polynomial in $\mathbb{N}p$ of degree up to 7. We will now explain how these can be related to Conjecture 1.

While Chinta works over $\mathbb{Q}$, we will work over $\mathbb{Q}[i]$; thus Chinta's p becomes $\mathbb{N}p$. Since the canonical map $\mathbb{Z}[i]^\times \longrightarrow (\mathbb{Z}[i]/(1+i)^3)^\times$ is a bijection, every odd prime has a unique generator $p \equiv 1$ modulo $(1+i)^3$. We choose the additive character $\psi(z) = e^{2\pi i \ \mathrm{re}(z)}$ in the definition of the Gauss sums. Then $g(p^k, p^l)$ is positive; more precisely, if l is even,

$$g(p^k, p^l) = \begin{cases} \phi(p^l) & \text{if } k \geqslant l, \\ -\mathbb{N}p^k & \text{if } k = l-1; \\ 0 & \text{otherwise,} \end{cases}$$

while if l is odd,

$$g(p^k, p^l) = \begin{cases} \mathbb{N}p^{k+\frac{1}{2}} & \text{if } k = l-1; \\ 0 & \text{otherwise}. \end{cases}$$

With these values of $g(p^k, p^l)$, let us take $r = 5$, $l_1 = l_2 = \ldots = l_5 = 0$, and regard (29) as the *definition* of $H(k_1, k_2, k_3, k_4, k_5)$.

THEOREM 2. *With this notation*,

$$h(k_1, k_2, k_3, k_4, k_5) = \mathbb{N}p^{-(k_1+k_2+k_3+k_4+k_5)/2} H(k_1, k_2, k_3, k_4, k_5).$$

Proof (Sketch). We first explain the meaning of the factor

$$\mathbb{N}p^{-(k_1+k_2+k_3+k_4+k_5)/2}.$$

To compare our normalization with Chinta's, we would take his s_i to be our $2s_i - \frac{1}{2}$. Thus his $x = \mathbb{N}p^{2s_1-\frac{1}{2}}$, $y = \mathbb{N}p^{2s_2-\frac{1}{2}}$, $z = \mathbb{N}p^{2s_3-\frac{1}{2}}$, $w = \mathbb{N}p^{2s_4-\frac{1}{2}}$ and $v = \mathbb{N}p^{2s_5-\frac{1}{2}}$. To compensate for the shifts by $\frac{1}{2}$, the factor $\mathbb{N}p^{-(k_1+k_2+k_3+k_4+k_5)/2}$ is needed.

This identity was verified by computer. There are 7,436 strict Gelfand-Tsetlin patterns. These combine in various ways to produce the 1,340 nonzero coefficients in $P(x, y, z, w, v)$. The computations are too long to publish, but a TEX dvi file of 1,012 pages reconciling our expression with Chinta's computation may be found at [6]. $\square$

We finally mention an alternative description in the untwisted case, when the parameters m_i in $Z(s_1, \cdots, s_r; m_1, \cdots, m_r)$ are all equal to 1. In this case the l_i are all equal to 0 for each p, and so the top row of the Gelfand-Tsetlin patterns that occur is $(r, r-1, \cdots, 0)$. An *alternating sign matrix* of size $(r+1) \times (r+1)$ is one whose entries are 0's and ± 1's, whose row and column sums all equal 1, and whose nonzero entries in each row and column alternate in sign. A bijection between the alternating sign matrices and the Gelfand-Tsetlin patterns with top row $(r, r-1, \cdots, 0)$ was described by Mills, Robbins and Rumsey [16]. Since these are the Gelfand-Tsetlin patterns that occur in the untwisted case, we may take the parameter set in the sum (29) to be the set of alternating sign matrices.

This is significant since alternating sign matrices are a generalization of permutation matrices, that is, Weyl group elements, which appeared in the parametrization of the stable terms. A necessary and sufficient condition for the pattern to be stable by Definition 1 is that this corresponding alternating sign matrix is a permutation matrix.

Gelfand-Tsetlin patterns are also in bijection with semistandard Young tableaux, as is explained in Stanley [21, p. 314]. Some of the literature generalizing Tokuyama's results to classical groups is in the language of alternating sign matrices and semistandard Young tableaux. See Okada [18] and Hamel and King [14], [15]. We expect that this literature will become relevant when one looks to other root systems beyond A_r.

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