Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics

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Abstract

In this paper we prove the global existence and uniqueness (regularity) of strong solutions to the three-dimensional viscous primitive equations, which model large scale ocean and atmosphere dynamics.

1. Introduction

Large scale dynamics of oceans and atmosphere is governed by the primitive equations which are derived from the Navier-Stokes equations, with rotation, coupled to thermodynamics and salinity diffusion-transport equations, which account for the buoyancy forces and stratification effects under the Boussinesq approximation. Moreover, and due to the shallowness of the oceans and the atmosphere, i.e., the depth of the fluid layer is very small in comparison to the radius of the earth, the vertical large scale motion in the oceans and the atmosphere is much smaller than the horizontal one, which in turn leads to modeling the vertical motion by the hydrostatic balance. As a result one obtains the system (1)–(4), which is known as the primitive equations for ocean and atmosphere dynamics (see, e.g., [20], [21], [22], [23], [24], [33] and references therein). We observe that in the case of ocean dynamics one has to add the diffusion-transport equation of the salinity to the system (1)–(4). We omitted it here in order to simplify our mathematical presentation. However, we emphasize that our results are equally valid when the salinity effects are taking into account.

Note that the horizontal motion can be further approximated by the geostrophic balance when the Rossby number (the ratio of the horizontal acceleration to the Coriolis force) is very small. By taking advantage of these assumptions and other geophysical considerations, we have developed and used several intermediate models in numerical studies of weather prediction and long-time climate dynamics (see, e.g., [4], [7], [8], [22], [23], [25], [28], [29], [30], [31] and references therein). Some of these models have also been the subject of analytical mathematical study (see, e.g., [2], [3], [5], [6], [9], [11], [12], [13], [15], [16], [17], [26], [27], [33], [34] and references therein).

In this paper we will focus on the $3D$ primitive equations in a cylindrical domain

$$\Omega = M \times (-h, 0),$$

where M is a smooth bounded domain in $\mathbb{R}^2$:

(1) $$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + w\frac{\partial v}{\partial z} + \nabla p + f\vec{k} \times v + L_1 v = 0,$$

(2) $$\partial_z p + T = 0,$$

(3) $$\nabla \cdot v + \partial_z w = 0,$$

(4) $$\frac{\partial T}{\partial t} + v \cdot \nabla T + w\frac{\partial T}{\partial z} + L_2 T = Q,$$

where the horizontal velocity field $v = (v_1, v_2)$, the three-dimensional velocity field (v_1, v_2, w), the temperature T and the pressure p are the unknowns. $f = f_0(\beta + y)$ is the Coriolis parameter and Q is a given heat source. The viscosity and the heat diffusion operators L_1 and L_2 are given by

(5) $$L_1 = -\frac{1}{Re_1}\Delta - \frac{1}{Re_2}\frac{\partial^2}{\partial z^2},$$

(6) $$L_2 = -\frac{1}{Rt_1}\Delta - \frac{1}{Rt_2}\frac{\partial^2}{\partial z^2},$$

where Re_1, Re_2 are positive constants representing the horizontal and vertical Reynolds numbers, respectively, and Rt_1, Rt_2 are positive constants which stand for the horizontal and vertical heat diffusion, respectively. We set $= (\partial_x, \partial_y)$ to be the horizontal gradient operator and $\Delta = \partial_x^2 + \partial_y^2$ to be the horizontal Laplacian. We observe that the above system is similar to the $3D$ Boussinesq system with the equation of vertical motion approximated by the hydrostatic balance.

We partition the boundary of Ω into:

(7) $$\Gamma_u = \{(x, y, z) \in \overline{\Omega} : z = 0\},$$

(8) $$\Gamma_b = \{(x, y, z) \in \overline{\Omega} : z = -h\},$$

(9) $$\Gamma_s = \{(x, y, z) \in \overline{\Omega} : (x, y) \in \partial M, -h \leq z \leq 0\}.$$

We equip the system (1)–(4) with the following boundary conditions: wind-driven on the top surface and free-slip and non-heat flux on the side walls and bottom (see, e.g., [20], [21], [22], [24], [25], [28], [29], [30]):

(10) $$\text{on } \Gamma_u : \frac{\partial v}{\partial z} = h\,\tau, \ w = 0, \ \frac{\partial T}{\partial z} = -\alpha(T - T^*);$$

(11) $$\text{on } \Gamma_b : \frac{\partial v}{\partial z} = 0, \ w = 0, \ \frac{\partial T}{\partial z} = 0;$$

(12) $$\text{on } \Gamma_s : v \cdot \vec{n} = 0, \ \frac{\partial v}{\partial \vec{n}} \times \vec{n} = 0, \ \frac{\partial T}{\partial \vec{n}} = 0,$$

where $\tau(x, y)$ is the wind stress on the ocean surface, $\vec{n}$ is the normal vector to Γ_s, and $T^*(x, y)$ is typical temperature distribution of the top surface of the ocean. For simplicity we assume here that τ and T^* are time independent. However, the results presented here are equally valid when these quantities are time dependent and satisfy certain bounds in space and time.

Due to the boundary conditions (10)–(12), it is natural to assume that τ and T^* satisfy the compatibility boundary conditions:

$$(13) \qquad \tau \cdot \vec{n} = 0, \ \frac{\partial \tau}{\partial \vec{n}} \times \vec{n} = 0, \qquad \text{on } \partial M.$$

$$(14) \qquad \frac{\partial T^*}{\partial \vec{n}} = 0 \qquad \text{on } \partial M.$$

In addition, we supply the system with the initial condition:

$$(15) \qquad v(x, y, z, 0) = v_0(x, y, z).$$

$$(16) \qquad T(x, y, z, 0) = T_0(x, y, z).$$

In [20], [21] and [33] the authors set up the mathematical framework to study the viscous primitive equations for the atmosphere and ocean circulation. Moreover, similar to the $3D$ Navier-Stokes equations, they have shown the global existence of weak solutions, but the question of their uniqueness is still open. The short time existence and uniqueness of strong solutions to the viscous primitive equations model was established in [15] and [33]. In [16] the authors proved the global existence and uniqueness of strong solutions to the viscous primitive equations in thin domains for a large set of initial data whose size depends inversely on the thickness of the domain. In this paper we show the global existence, uniqueness and continuous dependence on initial data, i.e. global regularity and well-posedness, of the strong solutions to the $3D$ viscous primitive equations model (1)–(16) in a general cylindrical domain, Ω, and for any initial data. It is worth stressing that the ideas developed in this paper can equally apply to the primitive equations subject to other kinds of boundary conditions. As in the case of $3D$ Navier-Stokes equations the question of uniqueness of the weak solutions to this model is still open.

2. Preliminaries

2.1. *New Formulation.* First, let us reformulate the system (1)–(16) (see also [20], [21] and [33]). We integrate the equation (3) in the z direction to obtain

$$w(x, y, z, t) = w(x, y, -h, t) - \int_{-h}^{z} \nabla \cdot v(x, y, \xi, t) d\xi.$$

By virtue of (10) and (11) we have

$$(17) \qquad w(x, y, z, t) = - \int_{-h}^{z} \nabla \cdot v(x, y, \xi, t) d\xi,$$

and

(18) $$\int_{-h}^{0} \nabla \cdot v(x,y,\xi,t)d\xi = \nabla \cdot \int_{-h}^{0} v(x,y,\xi,t)d\xi = 0.$$

We denote

(19) $$\overline{\phi}(x,y) = \frac{1}{h} \int_{-h}^{0} \phi(x,y,\xi)d\xi, \qquad \forall\, (x,y) \in M.$$

In particular,

(20) $$\overline{v}(x,y) = \frac{1}{h} \int_{-h}^{0} v(x,y,\xi)d\xi, \qquad \text{in}\ \ M,$$

denotes the barotropic mode. We will denote by

(21) $$\widetilde{v} = v - \overline{v},$$

the baroclinic mode, that is the fluctuation about the barotropic mode. Notice that

(22) $$\overline{\widetilde{v}} = 0.$$

Based on the above and (12) we obtain

(23) $$\nabla \cdot \overline{v} = 0, \qquad \text{in}\ \ M,$$

and

(24) $$\overline{v} \cdot \vec{n} = 0, \quad \frac{\partial \overline{v}}{\partial \vec{n}} \times \vec{n} = 0, \qquad \text{on}\ \partial M.$$

By integrating equation (2) we obtain

$$p(x,y,z,t) = -\int_{-h}^{z} T(x,y,\xi,t)d\xi + p_s(x,y,t).$$

Substituting (17) and the above relation into equation (1), we reach

(25) $$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \left(\int_{-h}^{z} \nabla \cdot v(x,y,\xi,t)d\xi \right) \frac{\partial v}{\partial z}$$
$$+ \nabla p_s(x,y,t) - \nabla \int_{-h}^{z} T(x,y,\xi,t)d\xi + f\vec{k} \times v + L_1 v = 0.$$

Remark 1. Notice that due to the compatibility boundary conditions (13) and (14) one can convert the boundary condition (10)–(12) to be homogeneous by replacing (v,T) by $(v + \frac{(z+h)^2 - h^3/3}{2}\tau, T + T^*)$ while (23) is still true. For simplicity and without loss of generality we will assume that $\tau = 0$, $T^* = 0$. However, we emphasize that our results are still valid for general τ and T^* provided they are smooth enough. In a forthcoming paper we will study the long-time dynamics and global attractors to the primitive equations with general τ and T^*.

Therefore, under the assumption that $\tau = 0, T^* = 0$, we have the following new formulation for system (1)–(16):

$$(26) \qquad \frac{\partial v}{\partial t} + L_1 v + (v \cdot \nabla)v - \left(\int_{-h}^{z} \nabla \cdot v(x,y,\xi,t)d\xi \right) \frac{\partial v}{\partial z}$$

$$+ \nabla p_s(x,y,t) - \nabla \int_{-h}^{z} T(x,y,\xi,t)d\xi + f\vec{k} \times v = 0,$$

$$(27) \qquad \frac{\partial T}{\partial t} + L_2 T + v \cdot \nabla T - \left(\int_{-h}^{z} \nabla \cdot v(x,y,\xi,t)d\xi \right) \frac{\partial T}{\partial z} = Q,$$

$$(28) \qquad \left. \frac{\partial v}{\partial z} \right|_{z=0} = 0, \ \left. \frac{\partial v}{\partial z} \right|_{z=-h} = 0, \ v \cdot \vec{n}|_{\Gamma_s} = 0, \ \left. \frac{\partial v}{\partial \vec{n}} \times \vec{n} \right|_{\Gamma_s} = 0,$$

$$(29) \qquad (\partial_z T + \alpha T)|_{z=0} = 0; \ \partial_z T|_{z=-h} = 0; \ \partial_n T|_{\Gamma_s} = 0,$$

$$(30) \qquad\qquad v(x,y,z,0) = v_0(x,y,z),$$

$$(31) \qquad\qquad T(x,y,z,0) = T_0(x,y,z).$$

2.2. *Properties of $\overline{v}$ and $\widetilde{v}$.* By taking the average of equations (26) in the z direction, over the interval $(-h,0)$, and using the boundary conditions (28), we obtain the following equation for the barotropic mode

$$(32) \qquad \frac{\partial \overline{v}}{\partial t} + \overline{(v \cdot \nabla)v - \left(\int_{-h}^{z} \nabla \cdot v(x,y,\xi,t)d\xi \right) \frac{\partial v}{\partial z}} + \nabla p_s(x,y,t)$$

$$- \nabla \left[\frac{1}{h} \int_{-h}^{0} \int_{-h}^{z} T(x,y,\xi,t)d\xi dz \right] + f\vec{k} \times \overline{v} - \frac{1}{Re_1} \Delta \overline{v} = 0.$$

As a result of (22), (23) and integration by parts,

(33)

$$\overline{(v \cdot \nabla)v - \left(\int_{-h}^{z} \nabla \cdot v(x,y,\xi,t)d\xi \right) \frac{\partial v}{\partial z}} = (\overline{v} \cdot \nabla)\overline{v} + \overline{[(\widetilde{v} \cdot \nabla)\widetilde{v} + (\nabla \cdot \widetilde{v})\,\widetilde{v}]}.$$

By subtracting (32) from (26) and using (33) we obtain the following equation for the baroclinic mode

(34)

$$\frac{\partial \widetilde{v}}{\partial t} + L_1 \widetilde{v} + (\widetilde{v} \cdot \nabla)\widetilde{v} - \left(\int_{-h}^{z} \nabla \cdot \widetilde{v}(x,y,\xi,t)d\xi \right) \frac{\partial \widetilde{v}}{\partial z}$$

$$+ (\widetilde{v} \cdot \nabla)\overline{v} + (\overline{v} \cdot \nabla)\widetilde{v} - \overline{[(\widetilde{v} \cdot \nabla)\widetilde{v} + (\nabla \cdot \widetilde{v})\,\widetilde{v}]}$$

$$- \nabla \left(\int_{-h}^{z} T(x,y,\xi,t)d\xi - \frac{1}{h} \int_{-h}^{0} \int_{-h}^{z} T(x,y,\xi,t)d\xi dz \right) + f\vec{k} \times \widetilde{v} = 0.$$

Therefore, $\overline{v}$ satisfies the following equations and boundary conditions:

$$(35) \quad \frac{\partial \overline{v}}{\partial t} - \frac{1}{Re_1}\Delta \overline{v} + (\overline{v} \cdot \nabla)\overline{v} + \overline{[(\widetilde{v} \cdot \nabla)\widetilde{v} + (\nabla \cdot \widetilde{v})\,\widetilde{v}]} + f\vec{k} \times \overline{v}$$

$$+ \nabla \left[p_s(x,y,t) - \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z} T(x,y,\xi,t)\,d\xi\,dz \right] = 0,$$

$$(36) \qquad\qquad\qquad \nabla \cdot \overline{v} = 0, \qquad\qquad \text{in } M,$$

$$(37) \qquad \overline{v} \cdot \vec{n} = 0, \quad \frac{\partial \overline{v}}{\partial \vec{n}} \times \vec{n} = 0, \qquad\qquad \text{on } \partial M,$$

and $\widetilde{v}$ satisfies the following equations and boundary conditions:

$$(38) \quad \frac{\partial \widetilde{v}}{\partial t} + L_1\widetilde{v} + (\widetilde{v} \cdot \nabla)\widetilde{v} - \left(\int_{-h}^{z} \nabla \cdot \widetilde{v}(x,y,\xi,t)d\xi \right)\frac{\partial \widetilde{v}}{\partial z}$$

$$+ (\widetilde{v} \cdot \nabla)\overline{v} + (\overline{v} \cdot \nabla)\widetilde{v} - \overline{[(\widetilde{v} \cdot \nabla)\widetilde{v} + (\nabla \cdot \widetilde{v})\,\widetilde{v}]} + f\vec{k} \times \widetilde{v}$$

$$- \nabla \left(\int_{-h}^{z} T(x,y,\xi,t)d\xi - \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z} T(x,y,\xi,t)d\xi dz \right) = 0,$$

$$(39) \qquad \left.\frac{\partial \widetilde{v}}{\partial z}\right|_{z=0} = 0, \; \left.\frac{\partial \widetilde{v}}{\partial z}\right|_{z=-h} = 0, \; \widetilde{v}\cdot\vec{n}|_{\Gamma_s} = 0, \; \left.\frac{\partial \widetilde{v}}{\partial \vec{n}} \times \vec{n}\right|_{\Gamma_s} = 0.$$

Remark 2. We recall that by virtue of the maximum principle one is able to show the global well-posedness of the $3D$ viscous Burgers equations (see, for instance, [19] and references therein). Such an argument, however, is not valid for the $3D$ Navier-Stokes equations because of the pressure term. Remarkably, the pressure term is absent from equation (38). This fact allows us to obtain a bound for the L^6 norm of $\widetilde{v}$, which is a key estimate in our proof of the global regularity for the system (1)–(16).

2.3. *Functional spaces and inequalities.* Denote by $L^2(\Omega), L^2(M)$ and $H^m(\Omega), H^m(M)$ the usual L^2-Lebesgue and Sobolev spaces, respectively ([1]). Let

$$(40) \quad \|\phi\|_p = \begin{cases} \left(\int_{\Omega} |\phi(x,y,z)|^p \, dxdydz \right)^{\frac{1}{p}}, & \text{for every } \phi \in L^p(\Omega), \\ \left(\int_M |\phi(x,y)|^p \, dxdy \right)^{\frac{1}{p}}, & \text{for every } \phi \in L^p(M). \end{cases}$$

Now,

$$\widetilde{\mathcal{V}_1} = \left\{ v \in C^\infty(\Omega) : \left.\frac{\partial v}{\partial z}\right|_{z=0} = 0, \; \left.\frac{\partial v}{\partial z}\right|_{z=-h} = 0, \right.$$

$$\left. v \cdot \vec{n}|_{\Gamma_s} = 0, \; \left.\frac{\partial v}{\partial \vec{n}} \times \vec{n}\right|_{\Gamma_s} = 0, \; \nabla \cdot \overline{v} = 0 \right\},$$

$$\widetilde{\mathcal{V}_2} = \left\{ T \in C^\infty(\Omega) : \left.\frac{\partial T}{\partial z}\right|_{z=-h} = 0; \; \left.\left(\frac{\partial T}{\partial z} + \alpha T\right)\right|_{z=0} = 0; \; \left.\frac{\partial T}{\partial \vec{n}}\right|_{\Gamma_s} = 0 \right\}.$$

We denote by V_1 and V_2 the closure spaces of $\widetilde{\mathcal{V}_1}$ in $H^1(\Omega)$, and $\widetilde{\mathcal{V}_2}$ in $H^1(\Omega)$ under H^1-topology, respectively.

Definition 1. Let $v_0 \in V_1$ and $T_0 \in V_2$, and let $\mathcal{T}$ be a fixed positive time. (v, T) is called a strong solution of (26)–(31) on the time interval $[0, \mathcal{T}]$ if it satisfies (26) and (27) in a weak sense, and also

$$v \in C([0, \mathcal{T}], V_1) \cap L^2([0, \mathcal{T}], H^2(\Omega)),$$
$$T \in C([0, \mathcal{T}], V_2) \cap L^2([0, \mathcal{T}], H^2(\Omega)),$$
$$\frac{dv}{dt} \in L^1([0, \mathcal{T}], L^2(\Omega)),$$
$$\frac{dT}{dt} \in L^1([0, \mathcal{T}], L^2(\Omega)).$$

For convenience, we recall the following Sobolev and Ladyzhenskaya's inequalities in $\mathbb{R}^2$ (see, e.g., [1], [10], [14], [18]):

$$(41) \qquad \|\phi\|_{L^4(M)} \leq C_0 \|\phi\|_{L^2}^{1/2} \|\phi\|_{H^1(M)}^{1/2},$$

$$(42) \qquad \|\phi\|_{L^8(M)} \leq C_0 \|\phi\|_{L^6(M)}^{3/4} \|\phi\|_{H^1(M)}^{1/4},$$

for every $\phi \in H^1(M)$, and the following Sobolev and Ladyzhenskaya's inequalities in $\mathbb{R}^3$ (see, e.g., [1], [10], [14], [18]):

$$(43) \qquad \|u\|_{L^3(\Omega)} \leq C_0 \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2},$$

$$(44) \qquad \|u\|_{L^6(\Omega)} \leq C_0 \|u\|_{H^1(\Omega)},$$

for every $u \in H^1(\Omega)$. Here C_0 is a positive constant which might depend on the shape of M and Ω but not on their size. Moreover, by (41) we get

$$(45) \qquad \|\phi\|_{L^{12}(M)}^{12} = \||\phi|^3\|_{L^4(M)}^4 \leq C_0 \||\phi|^3\|_{L^2(M)}^2 \||\phi|^3\|_{H^1(M)}^2$$

$$\leq C_0 \|\phi\|_{L^6(M)}^6 \left(\int_M |\phi|^4 |\nabla \phi|^2 \ dxdy \right) + \|\phi\|_{L^6(M)}^{12},$$

for every $\phi \in H^1(M)$. Also, we recall the integral version of Minkowsky inequality for the L^p spaces, $p \geq 1$. Let $\Omega_1 \subset \mathbb{R}^{m_1}$ and $\Omega_2 \subset \mathbb{R}^{m_2}$ be two measurable sets, where m_1 and m_2 are positive integers. Suppose that $f(\xi, \eta)$ is measurable over $\Omega_1 \times \Omega_2$. Then,

$$(46) \qquad \left[\int_{\Omega_1} \left(\int_{\Omega_2} |f(\xi, \eta)| d\eta \right)^p d\xi \right]^{1/p} \leq \int_{\Omega_2} \left(\int_{\Omega_1} |f(\xi, \eta)|^p d\xi \right)^{1/p} d\eta.$$

3. *A priori* **estimates**

In the previous subsections we have reformulated the system (1)–(16) and obtained the system (26)–(31). The two systems are equivalent when (v, T) is a strong solution. The existence of such a strong solution for a short interval of time, whose length depends on the initial data and the other physical parameters of the system (1)–(16), was established in [15] and [33]. Let

(v_0, T_0) be given initial data. In this section we will consider the strong solution that corresponds to the initial data in its maximal interval of existence $[0, \mathcal{T}_*)$. Specifically, we will establish *a priori* upper estimates for various norms of this solution in the interval $[0, \mathcal{T}_*)$. In particular, we will show that if $\mathcal{T}_* < \infty$ then the H^1 norm of the strong solution is bounded over the interval $[0, \mathcal{T}_*)$. This key observation plays a major role in the proof of global regularity of strong solutions to the system (1)–(16).

3.1. L^2 *estimates.* We take the inner product of equation (27) with T, in $L^2(\Omega)$, and obtain

$$\frac{1}{2}\frac{d\|T\|_2^2}{dt} + \frac{1}{Rt_1}\|\nabla T\|_2^2 + \frac{1}{Rt_2}\|T_z\|_2^2 + \alpha\|T(z=0)\|_2^2$$
$$= \int_\Omega QT\, dxdydz - \int_\Omega \left(v\cdot\nabla T - \left(\int_{-h}^z \nabla\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial T}{\partial z}\right)T\, dxdydz.$$

After integrating by parts we get

$$(47) \qquad -\int_\Omega \left(v\cdot\nabla T - \left(\int_{-h}^z \nabla\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial T}{\partial z}\right)T\, dxdydz = 0.$$

As a result of the above we conclude

$$\frac{1}{2}\frac{d\|T\|_2^2}{dt} + \frac{1}{Rt_1}\|\nabla T\|_2^2 + \frac{1}{Rt_2}\|T_z\|_2^2 + \alpha\|T(z=0)\|_2^2$$
$$= \int_\Omega QT\, dxdydz \le \|Q\|_2\, \|T\|_2.$$

Notice that

$$(48) \qquad \|T\|_2^2 \le 2h^2\|T_z\|_2^2 + 2h\|T(z=0)\|_2^2.$$

Using (48) and the Cauchy-Schwarz inequality we obtain

$$(49) \qquad \frac{d\|T\|_2^2}{dt} + \frac{2}{Rt_1}\|\nabla T\|_2^2 + \frac{1}{Rt_2}\|T_z\|_2^2 + \alpha\|T(z=0)\|_2^2$$

$$(50) \qquad \le 2(h^2\, Rt_2 + \frac{h}{\alpha})\|Q\|_2^2.$$

By the inequality (48) and thanks to Gronwall inequality the above gives

$$(51) \quad \|T\|_2^2 \le e^{-\frac{t}{2(h^2\, Rt_2 + h/\alpha)}}\|T_0\|_2^2 + (2h^2\, Rt_2 + 2h/\alpha)^2\|Q\|_2^2,$$

Moreover, we have

$$(52) \quad \int_0^t \left[\frac{1}{Rt_1}\|\nabla T(s)\|_2^2 + \frac{1}{Rt_2}\|T_z(s)\|_2^2 + \alpha\|T(z=0)(s)\|_2^2\right]ds$$
$$\le 2\left(h^2\, Rt_2 + \frac{h}{\alpha}\right)\|Q\|_2^2\, t + \|T_0\|_2^2.$$

By taking the inner product of equation (26) with v, in $L^2(\Omega)$, we reach

$$\frac{1}{2}\frac{d\|v\|_2^2}{dt} + \frac{1}{Re_1}\|\nabla v\|_2^2 + \frac{1}{Re_2}\|v_z\|_2^2$$

$$= -\int_\Omega \left[(v \cdot \nabla)v - \left(\int_{-h}^z \nabla \cdot v(x,y,\xi,t)d\xi \right) \frac{\partial v}{\partial z} \right] \cdot v\ dxdydz$$

$$+ \int_\Omega \left(f\vec{k} \times v + \nabla p_s - \nabla \left(\int_{-h}^z T(x,y,\xi,t)d\xi \right) \right) \cdot v\ dxdydz.$$

By integration by parts we get

$$(53) \qquad \int_\Omega \left[(v \cdot \nabla)v - \left(\int_{-h}^z \nabla \cdot v(x,y,\xi,t)d\xi \right) \frac{\partial v}{\partial z} \right] \cdot v\ dxdydz = 0.$$

By (36) we have

$$(54)\quad \int_\Omega \nabla p_s \cdot v\ dxdydz = h\int_M \nabla p_s \cdot \overline{v}\ dxdy = -h\int_\Omega p_s(\nabla \cdot \overline{v})\ dxdy = 0.$$

Since

$$(55) \qquad\qquad\qquad\qquad (f\vec{k} \times v) \cdot v = 0,$$

then from (53)–(55) we have

$$\frac{1}{2}\frac{d\|v\|_2^2}{dt} + \frac{1}{Re_1}\|\nabla v\|_2^2 + \frac{1}{Re_2}\|v_z\|_2^2$$

$$= -\int_\Omega \int_{-h}^z T(x,y,\xi,t)\ d\xi(\nabla \cdot v)\ dxdydz \leq h\|T\|_2\ \|\nabla v\|_2.$$

By Cauchy-Schwarz and (51) we obtain

$$\frac{d\|v\|_2^2}{dt} + \frac{1}{Re_1}\|\nabla v\|_2^2 + \frac{1}{Re_2}\|v_z\|_2^2$$

$$\leq h^2 Re_1\ \|T\|_2^2 \leq h^2 Re_1 \left(\|T_0\|_2^2 + (2h^2\ Rt_2 + 2h/\alpha)^2\|Q\|_2^2 \right).$$

Recall that (cf., e.g., [14, Vol. I p. 55])

$$\|v\|_2^2 \leq C_M\|\nabla v\|_2^2.$$

By the above and thanks to Gronwall's inequality we get

$$(56) \qquad \|v\|_2^2 \leq e^{-\frac{t}{C_M\ Re_1}} \left(h\|\overline{v_0}\|_2^2 + \|\widetilde{v_0}\|_2^2 \right)$$

$$+ C_M h^2 Re_1^2 \left[\ \|T_0\|_2^2 + (2h^2\ Rt_2 + 2h/\alpha)^2\|Q\|_2^2\ \right].$$

Moreover,

$$(57)\quad \int_0^t \left[\frac{1}{Re_1}\|\nabla v(s)\|_2^2 + \frac{1}{Re_2}\|v_z(s)\|_2^2 \right]\ ds$$

$$\leq h^2 Re_1 \left(\|T_0\|_2^2 + (2h^2\ Rt_2 + 2h/\alpha)^2\|Q\|_2^2 \right)\ t + \left(h\|\overline{v_0}\|_2^2 + \|\widetilde{v_0}\|_2^2 \right).$$

Therefore, by (51), (52), (56) and (57) we have

(58)

$$\|v(t)\|_2^2 + \int_0^t \left[\frac{1}{Re_1}\|\nabla v(s)\|_2^2 + \frac{1}{Re_2}\|v_z(s)\|_2^2 \right] \, ds + \|T(t)\|_2^2$$

$$+ \int_0^t \left[\frac{1}{Rt_1}\|\nabla T(s)\|_2^2 + \frac{1}{Rt_2}\|T_z(s)\|_2^2 + \alpha\|T(z=0)(s)\|_2^2 \right] \, ds \leq K_1(t),$$

where

$$(59) \quad K_1(t) = 2(h^2 \, Rt_2 + h/\alpha)\|Q\|_2^2 \, t + \left(h\|\overline{v_0}\|_2^2 + \|\widetilde{v_0}\|_2^2 \right)$$
$$+ \left(1 + C_M h^2 Re_1^2 + h^2 Re_1 \, t \right) \left[\|T_0\|_2^2 + (2h^2 \, Rt_2 + 2h/\alpha)^2 \|Q\|_2^2 \right].$$

3.2. L^6 *estimates.* Taking the inner product of the equation (38) with $|\widetilde{v}|^4\widetilde{v}$ in $L^2(\Omega)$, we get

$$\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt} + \frac{1}{Re_1}\int_\Omega \left(|\nabla\widetilde{v}|^2|\widetilde{v}|^4 + \left| \nabla|\widetilde{v}|^2 \right|^2 |\widetilde{v}|^2 \right) \, dxdydz$$

$$+\frac{1}{Re_2}\int_\Omega \left(|\widetilde{v}_z|^2|\widetilde{v}|^4 + \left| \partial_z|\widetilde{v}|^2 \right|^2 |\widetilde{v}|^2 \right) \, dxdydz$$

$$=-\int_\Omega \left\{ (\widetilde{v}\cdot\nabla)\widetilde{v} - \left(\int_{-h}^z \nabla\cdot\widetilde{v}(x,y,\xi,t)d\xi \right)\frac{\partial\widetilde{v}}{\partial z} + (\widetilde{v}\cdot\nabla)\overline{v} + (\overline{v}\cdot\nabla)\widetilde{v} \right.$$

$$-\overline{[(\widetilde{v}\cdot\nabla)\widetilde{v} + (\nabla\cdot\widetilde{v})\,\widetilde{v}]} + f\vec{k}\times\widetilde{v}$$

$$\left.-\nabla\left(\int_{-h}^z T(x,y,\xi,t)d\xi - \frac{1}{h}\int_{-h}^0\int_{-h}^z T(x,y,\xi,t)d\xi dz \right) \right\} \cdot |\widetilde{v}|^4\widetilde{v} \, dxdydz.$$

Integrating by parts we get

$$(60) \quad -\int_\Omega \left[(\widetilde{v}\cdot\nabla)\widetilde{v} - \left(\int_{-h}^z \nabla\cdot\widetilde{v}(x,y,\xi,t)d\xi \right)\frac{\partial\widetilde{v}}{\partial z} \right] \cdot |\widetilde{v}|^4\widetilde{v} \, dxdydz = 0.$$

Since

$$(61) \qquad\qquad \left(f\vec{k}\times\widetilde{v} \right)\cdot|\widetilde{v}|^4\widetilde{v} = 0,$$

then by (36) and the boundary condition (28) we also have

$$(62) \qquad\qquad \int_\Omega (\overline{v}\cdot\nabla)\widetilde{v}\cdot|\widetilde{v}|^4\widetilde{v} \, dxdydz = 0.$$

Thus, by (60)–(62),

$$\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt} + \frac{1}{Re_1}\int_\Omega \left(|\nabla\widetilde{v}|^2|\widetilde{v}|^4 + \left| \nabla|\widetilde{v}|^2 \right|^2 |\widetilde{v}|^2 \right) \, dxdydz$$

$$+\frac{1}{Re_2}\int_\Omega \left(|\widetilde{v}_z|^2|\widetilde{v}|^4 + \left| \partial_z|\widetilde{v}|^2 \right|^2 |\widetilde{v}|^2 \right) \, dxdydz$$

$$= -\int_{\Omega} \left\{ (\widetilde{v}\cdot\nabla)\overline{v} - \overline{(\widetilde{v}\cdot\nabla)\widetilde{v} + (\nabla\cdot\widetilde{v})\,\widetilde{v}} \right.$$

$$\left. -\nabla\left(\int_{-h}^{z} T(x,y,\xi,t)d\xi - \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z} T(x,y,\xi,t)d\xi dz \right) \right\} \cdot |\widetilde{v}|^4\widetilde{v}\, dxdydz.$$

Notice that by integration by parts and boundary condition (28),

$$-\int_{\Omega} \left[(\widetilde{v}\cdot\nabla)\overline{v} - \overline{[(\widetilde{v}\cdot\nabla)\widetilde{v} + (\nabla\cdot\widetilde{v})\,\widetilde{v}]} \right.$$

$$\left. -\nabla\left(\int_{-h}^{z} T(x,y,\xi,t)d\xi - \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z} T(x,y,\xi,t)d\xi dz \right) \right] \cdot |\widetilde{v}|^4\widetilde{v}\, dxdydz$$

$$= \int_{\Omega} \left[(\nabla\cdot\widetilde{v})\,\overline{v}\cdot|\widetilde{v}|^4\widetilde{v} + (\widetilde{v}\cdot\nabla)(|\widetilde{v}|^4\widetilde{v})\cdot\overline{v} - \overline{\widetilde{v}^k\widetilde{v}^j}\,\partial_{x_k}(|\widetilde{v}|^4\widetilde{v}^j) \right.$$

$$\left. -\left(\int_{-h}^{z} T(x,y,\xi,t)d\xi - \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z} T(x,y,\xi,t)d\xi dz \right)\nabla\cdot(|\widetilde{v}|^4\widetilde{v}) \right] dxdydz.$$

Therefore, by Cauchy-Schwarz inequality and Hölder inequality we obtain

$$\frac{1}{6}\frac{d\|\widetilde{v}\|_6^6}{dt} + \frac{1}{Re_1}\int_{\Omega} \left(|\nabla\widetilde{v}|^2|\widetilde{v}|^4 + \left|\nabla|\widetilde{v}|^2\right|^2|\widetilde{v}|^2 \right) dxdydz$$

$$+ \frac{1}{Re_2}\int_{\Omega} \left(|\widetilde{v}_z|^2|\widetilde{v}|^4 + \left|\partial_z|\widetilde{v}|^2\right|^2|\widetilde{v}|^2 \right) dxdydz$$

$$\leq C\int_M \left[|\overline{v}|\int_{-h}^{0} |\nabla\widetilde{v}|\,|\widetilde{v}|^5\, dz \right] dxdy$$

$$+ C\int_M \left[\left(\int_{-h}^{0} |\widetilde{v}|^2\, dz \right)\left(\int_{-h}^{0} |\nabla\widetilde{v}|\,|\widetilde{v}|^4\, dz \right) \right] dxdy$$

$$+ C\int_M \left[\overline{|T|}\int_{-h}^{0} |\nabla\widetilde{v}|\,|\widetilde{v}|^4\, dz \right] dxdy$$

$$\leq C\int_M \left[|\overline{v}|\left(\int_{-h}^{0} |\nabla\widetilde{v}|^2\,|\widetilde{v}|^4\, dz \right)^{1/2}\left(\int_{-h}^{0} |\widetilde{v}|^6\, dz \right)^{1/2} \right] dxdy$$

$$+ C\int_M \left[\left(\int_{-h}^{0} |\widetilde{v}|^2\, dz \right)\left(\int_{-h}^{0} |\nabla\widetilde{v}|^2\,|\widetilde{v}|^4\, dz \right)^{1/2}\left(\int_{-h}^{0} |\widetilde{v}|^4\, dz \right)^{1/2} \right] dxdy$$

$$+ C\int_M \left[\overline{|T|}\left(\int_{-h}^{0} |\nabla\widetilde{v}|^2\,|\widetilde{v}|^4\, dz \right)^{1/2}\left(\int_{-h}^{0} |\widetilde{v}|^4\, dz \right)^{1/2} \right] dxdy$$

$$\leq C\|\overline{v}\|_{L^4(M)}\left(\int_{\Omega} |\nabla\widetilde{v}|^2\,|\widetilde{v}|^4\, dxdydz \right)^{1/2}\left(\int_M \left(\int_{-h}^{0} |\widetilde{v}|^6\, dz \right)^2 dxdy \right)^{1/4}$$

$$+ C\left(\int_M \left(\int_{-h}^{0} |\widetilde{v}|^2\, dz \right)^4 dxdy \right)^{1/4}$$

$$\times \left(\int_\Omega |\nabla \widetilde{v}|^2 \, |\widetilde{v}|^4 \, dxdydz \right)^{1/2} \left(\int_M \left(\int_{-h}^0 |\widetilde{v}|^4 \, dz \right)^2 dxdy \right)^{1/4}$$

$$+ C\| \, \overline{|T|} \, \|_{L^4(M)} \left(\int_\Omega |\nabla \widetilde{v}|^2 \, |\widetilde{v}|^4 \, dxdydz \right)^{1/2} \left(\int_M \left(\int_{-h}^0 |\widetilde{v}|^4 \, dz \right)^2 dxdy \right)^{1/4}.$$

By using Minkowsky inequality (46), we get

$$\left(\int_M \left(\int_{-h}^0 |\widetilde{v}|^6 \, dz \right)^2 dxdy \right)^{1/2} \le C \int_{-h}^0 \left(\int_M |\widetilde{v}|^{12} \, dxdy \right)^{1/2} dz.$$

By (45),

$$\int_M |\widetilde{v}|^{12} \, dxdy \le C_0 \left(\int_M |\widetilde{v}|^6 \, dxdy \right) \left(\int_M |\widetilde{v}|^4 |\nabla \widetilde{v}|^2 \, dxdy \right) + \left(\int_M |\widetilde{v}|^6 \, dxdy \right)^2.$$

Thus, by Cauchy-Schwarz inequality,

(63)
$$\left(\int_M \left(\int_{-h}^0 |\widetilde{v}|^6 \, dz \right)^2 dxdy \right)^{1/2} \le C \|\widetilde{v}\|_{L^6(\Omega)}^3 \left(\int_\Omega |\widetilde{v}|^4 |\nabla \widetilde{v}|^2 \, dxdydz \right)^{1/2} + \|\widetilde{v}\|_{L^6(\Omega)}^6.$$

Similarly, by (46) and (42),

(64)
$$\left(\int_M \left(\int_{-h}^0 |\widetilde{v}|^4 \, dz \right)^2 dxdy \right)^{1/2} \le C \int_{-h}^0 \left(\int_M |\widetilde{v}|^8 \, dxdy \right)^{1/2} dz$$

$$\le C \int_{-h}^0 \|\widetilde{v}\|_{L^6(M)}^3 \left(\|\nabla \widetilde{v}\|_{L^2(M)} + \|\widetilde{v}\|_{L^2(M)} \right) \, dz \le C \|\widetilde{v}\|_6^3 \left(\|\nabla \widetilde{v}\|_2 + \|\widetilde{v}\|_2 \right),$$

and

(65)
$$\left(\int_M \left(\int_{-h}^0 |\widetilde{v}|^2 \, dz \right)^4 dxdy \right)^{1/4} \le C \int_{-h}^0 \left(\int_M |\widetilde{v}|^8 \, dxdy \right)^{1/4} dz$$

$$\le C \int_{-h}^0 \|\widetilde{v}\|_{L^6(M)}^{3/2} \left(\|\nabla \widetilde{v}\|_{L^2(M)}^{1/2} + \|\widetilde{v}\|_{L^2(M)}^{1/2} \right) \, dz$$

$$\le C \|\widetilde{v}\|_6^{3/2} \left(\|\nabla \widetilde{v}\|_2^{1/2} + \|\widetilde{v}\|_2^{1/2} \right).$$

Therefore, by (63)–(65) and (41),

$$\frac{1}{6} \frac{d\|\widetilde{v}\|_6^6}{dt} + \frac{1}{Re_1} \int_\Omega \left(|\nabla \widetilde{v}|^2 |\widetilde{v}|^4 + \left| \nabla |\widetilde{v}|^2 \right|^2 |\widetilde{v}|^2 \right) \, dxdydz$$

$$+ \frac{1}{Re_2} \int_\Omega \left(|\widetilde{v}_z|^2 |\widetilde{v}|^4 + \left| \partial_z |\widetilde{v}|^2 \right|^2 |\widetilde{v}|^2 \right) \, dxdydz$$

$$\leq C\|\overline{v}\|_2^{1/2}\,\|\nabla\overline{v}\|_2^{1/2}\|\widetilde{v}\|_6^{3/2}\left(\int_\Omega |\nabla\widetilde{v}|^2\,|\widetilde{v}|^4\,dxdydz\right)^{3/4} + C\|\overline{v}\|_2^{1/2}\,\|\nabla\overline{v}\|_2^{1/2}\|\widetilde{v}\|_6^6$$

$$+C\|\widetilde{v}\|_6^3\,(\|\nabla\widetilde{v}\|_2 + \|\widetilde{v}\|_2)\left(\int_\Omega |\nabla\widetilde{v}|^2\,|\widetilde{v}|^4\,dxdydz\right)^{1/2}$$

$$+C\|\overline{T}\|_2^{1/2}\,\|\nabla\overline{T}\|_2^{1/2}\|\widetilde{v}\|_6^{3/2}\left(\|\nabla\widetilde{v}\|_2^{1/2} + \|\widetilde{v}\|_2^{1/2}\right)\left(\int_\Omega |\nabla\widetilde{v}|^2\,|\widetilde{v}|^4\,dxdydz\right)^{1/2}.$$

Thanks to the Young and the Cauchy-Schwarz inequalities,

$$\frac{d\|\widetilde{v}\|_6^6}{dt} + \frac{1}{Re_1}\int_\Omega\left(|\nabla\widetilde{v}|^2|\widetilde{v}|^4 + \left|\nabla|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz$$

$$+\frac{1}{Re_2}\int_\Omega\left(|\widetilde{v}_z|^2|\widetilde{v}|^4 + \left|\partial_z|\widetilde{v}|^2\right|^2|\widetilde{v}|^2\right)\,dxdydz$$

$$\leq C\|\overline{v}\|_2^2\,\|\nabla\overline{v}\|_2^2\|\widetilde{v}\|_6^6 + C\|\widetilde{v}\|_6^6\|\nabla\widetilde{v}\|_2^2 + C\|\overline{T}\|_2^2\,\|\nabla\overline{T}\|_2^2 + C\|\widetilde{v}\|_2^2\|\widetilde{v}\|_6^6.$$

By (58) and Gronwall inequality, we get

$$(66)\quad \|\widetilde{v}(t)\|_6^6 + \int_0^t\left(\frac{1}{Re_1}\int_\Omega |\nabla\widetilde{v}|^2|\widetilde{v}|^4\,dxdydz\right.$$

$$\left.+\frac{1}{Re_2}\int_\Omega |\widetilde{v}_z|^2|\widetilde{v}|^4\,dxdydz\right) \leq K_6(t),$$

where

$$(67)\qquad K_6(t) = e^{K_1^2(t)}\left[\|v_0\|_{H^1(\Omega)}^6 + K_1^2(t)\right].$$

Taking the inner product of the equation (27) with $|T|^4T$ in $L^2(\Omega)$, and by (27), we get

$$\frac{1}{6}\frac{d\|T\|_6^6}{dt} + \frac{5}{Rt_1}\int_\Omega |\nabla T|^2|T|^4\,dxdydz$$

$$+ \frac{5}{Rt_2}\int_\Omega |T_z|^2|T|^4\,dxdydz + \alpha\|T(z=0)\|_6^6$$

$$= \int_\Omega Q|T|^4T\,dxdydz$$

$$- \int_\Omega\left(v\cdot\nabla T - \left(\int_{-h}^z \nabla\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial T}{\partial z}\right)|T|^4T\,dxdydz.$$

By integration by parts and (36),

$$(68)\quad -\int_\Omega\left(v\cdot\nabla T - \left(\int_{-h}^z \nabla\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial T}{\partial z}\right)|T|^4T\,dxdydz = 0.$$

As a result of the above we conclude

$$\frac{1}{6}\frac{d\|T\|_6^6}{dt} + \frac{5}{Rt_1}\int_\Omega |\nabla T|^2|T|^4\,dxdydz + \frac{5}{Rt_2}\int_\Omega |T_z|^2|T|^4\,dxdydz$$

$$+ \alpha\|T(z=0)\|_6^6 = \int_\Omega Q|T|^4T\,dxdydz \leq \|Q\|_6\|T\|_6^5.$$

By Gronwall, again,

$$
(69) \qquad \|T(t)\|_6 \le \|Q\|_{H^1(\Omega)}\, t + \|T_0\|_{H^1(\Omega)}.
$$

3.3. H^1 estimates.

3.3.1. $\|\nabla\overline{v}\|_2$ estimates. First, we observe that since v is a strong solution on the interval $[0, \mathcal{T}_*)$ then $\Delta\overline{v} \in L^2([0, \mathcal{T}_*), L^2(M))$. Consequently, and by virtue of (36), $\Delta\overline{v}\cdot\vec{n} \in L^2([0, \mathcal{T}_*), H^{-1/2}(\partial M))$ (see, e.g., [10], [32]). Moreover, and thanks to (36) and (37), we have $\Delta\overline{v}\cdot\vec{n} = 0$ on ∂M (see, e.g., [35]). This observation implies also that the Stokes operator in the domain M, subject to the boundary conditions (37), is equal to the $-\Delta$ operator.

As a result of the above and (36) we apply a generalized version of the Stokes theorem (see, e.g., [10], [32]) to conclude:

$$
\int_M \nabla p_s(x,y,t) \cdot \Delta\overline{v}(x,y,t)\,dxdy = 0.
$$

By taking the inner product of equation (35) with $-\Delta\overline{v}$ in $L^2(M)$, and applying (36) and the above, we reach

$$
\frac{1}{2}\frac{d\|\nabla\overline{v}\|_2^2}{dt} + \frac{1}{Re_1}\|\Delta\overline{v}\|_2^2
$$
$$
= \int_M \left\{ (\overline{v}\cdot\nabla)\overline{v} + \overline{[(\widetilde{v}\cdot\nabla)\widetilde{v} + (\nabla\cdot\widetilde{v})\,\widetilde{v}]} \right\}\cdot\Delta\overline{v}\,dxdy + \int_M f\vec{k}\times\overline{v}\cdot\Delta\overline{v}\,dxdy.
$$

Following similar steps as in the proof of $2D$ Navier-Stokes equations (cf. e.g., [10], [32]) one obtains

$$
\left| \int_M (\overline{v}\cdot\nabla)\overline{v}\cdot\Delta\overline{v}\,dxdy \right| \le C\|\overline{v}\|_2^{1/2}\|\nabla\overline{v}\|_2\,\|\Delta\overline{v}\|_2^{3/2}.
$$

Applying the Cauchy-Schwarz and Hölder inequalities, we get

$$
\left| \int_M \overline{(\widetilde{v}\cdot\nabla)\widetilde{v} + (\nabla\cdot\widetilde{v})\,\widetilde{v}}\cdot\Delta\overline{v}\,dxdy \right| \le C\int_M\int_{-h}^0 |\widetilde{v}|\,|\nabla\widetilde{v}|\,dz\,|\Delta\overline{v}|\,dxdy
$$
$$
\le C\int_M \left[\left(\int_{-h}^0 |\widetilde{v}|^2\,|\nabla\widetilde{v}|\,dz\right)^{1/2} \left(\int_{-h}^0 |\nabla\widetilde{v}|\,dz\right)^{1/2} |\Delta\overline{v}| \right]\,dxdy
$$
$$
\le C \left[\int_M \left(\int_{-h}^0 |\widetilde{v}|^2\,|\nabla\widetilde{v}|\,dz\right)^2\,dxdy \right]^{1/4}
$$
$$
\times \left[\int_M \left(\int_{-h}^0 |\nabla\widetilde{v}|\,dz\right)^2\,dxdy \right]^{1/4} \left[\int_M |\Delta\overline{v}|^2\,dxdy \right]^{1/2}
$$
$$
\le C\|\nabla\widetilde{v}\|_2^{1/2} \left(\int_\Omega |\widetilde{v}|^4|\nabla\widetilde{v}|^2\,dxdydz \right)^{1/4} \|\Delta\overline{v}\|_2.
$$

Thus, by Young's and Cauchy-Schwarz inequalities,

$$\frac{d\|\nabla\overline{v}\|_2^2}{dt} + \frac{1}{Re_1}\|\Delta\overline{v}\|_2^2 \le C\|\overline{v}\|_2^2\|\nabla\overline{v}\|_2^4$$

$$+ C\|\nabla\widetilde{v}\|_2^2 + C\int_\Omega |\widetilde{v}|^4|\nabla\widetilde{v}|^2\,dxdydz + C\|\overline{v}\|_2^2.$$

By (58), (66) and thanks tothe Gronwall inequality, we obtain

$$(70) \qquad \|\nabla\overline{v}\|_2^2 + \frac{1}{Re_1}\int_0^t |\Delta\overline{v}|_2^2\,ds \le K_2(t),$$

where

$$(71) \qquad K_2(t) = e^{K_1^2(t)}\left[\|v_0\|_{H^1(\Omega)}^2 + K_1(t) + K_6(t)\right].$$

3.3.2. $\|v_z\|_2$ estimates. Since $u = v_z$, it is clear that u satisfies

$$(72) \qquad \frac{\partial u}{\partial t} + L_1 u + (v\cdot\nabla)u - \left(\int_{-h}^z \nabla\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}$$

$$+ (u\cdot\nabla)v - (\nabla\cdot v)u + f\vec{k}\times u - \nabla T = 0.$$

Taking the inner product of the equation (72) with u in L^2 and using the boundary condition (28), we get

$$\frac{1}{2}\frac{d\|u\|_2^2}{dt} + \frac{1}{Re_1}\|\nabla u\|_2^2 + \frac{1}{Re_2}\|\partial_z u\|_2^2$$

$$= -\int_\Omega\left((v\cdot\nabla)u - \left(\int_{-h}^z \nabla\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}\right)\cdot u\,dxdydz$$

$$-\int_\Omega\left((u\cdot\nabla)v - (\nabla\cdot v)u + f\vec{k}\times u - \nabla T\right)\cdot u\,dxdydz.$$

From integration by parts we get

$$(73) \qquad -\int_\Omega\left((v\cdot\nabla)u - \left(\int_{-h}^z \nabla\cdot v(x,y,\xi,t)d\xi\right)\frac{\partial u}{\partial z}\right)\cdot u\,dxdydz = 0.$$

Since

$$(74) \qquad\qquad\qquad (f\vec{k}\times u)\cdot u = 0,$$

then by (73) and (74) we have

$$\frac{1}{2}\frac{d\|u\|_2^2}{dt} + \frac{1}{Re_1}\|\nabla u\|_2^2 + \frac{1}{Re_2}\|\partial_z u\|_2^2$$

$$= -\int_\Omega ((u\cdot\nabla)v - (\nabla\cdot v)u - \nabla T)\cdot u\,dxdydz$$

$$\le C\int_\Omega (|v|)\,|u|\,|\nabla u|\,dxdydz + \|T\|_2\,\|\nabla u\|_2$$

$$\leq C\|v\|_6 \, \|u\|_3 \|\nabla u\|_2 + \|T\|_2 \, \|\nabla u\|_2$$
$$\leq C\|v\|_6 \, \|u\|_2^{1/2} \|\nabla u\|_2^{3/2} + \|T\|_2 \, \|\nabla u\|_2.$$

By Young's inequality and Cauchy-Schwarz inequality, we have

$$\frac{d\|u\|_2^2}{dt} + \frac{1}{Re_1}\|\nabla u\|_2^2 + \frac{1}{Re_2}\|\partial_z u\|_2^2 \leq C\|v\|_6^4 \, \|u\|_2^2 + C\|T\|_2^2$$
$$\leq C\left(\|\nabla \overline{v}\|_2^4 + \|\widetilde{v}\|_6^4\right) \, \|u\|_2^2 + C\|T\|_2^2.$$

By (58), (66), (70), and Gronwall inequality,

$$(75) \qquad \|v_z\|_2^2 + \frac{1}{Re_1}\int_0^t \|\nabla v_z(s)\|_2^2 + \frac{1}{Re_2}\int_0^t \|v_{zz}(s)\|_2^2 \, ds \leq K_z(t),$$

where

$$(76) \qquad K_z(t) = e^{(K_2^2(t)+K_6^{2/3}(t))t}\left[\|v_0\|_{H^1(\Omega)}^2 + K_1(t)\right].$$

3.3.3. $\|\nabla v\|_2$ estimates.

By taking the inner product of equation (26) with $-\Delta v$ in $L^2(\Omega)$, we reach

$$\frac{1}{2}\frac{d\|\nabla v\|_2^2}{dt} + \frac{1}{Re_1}\|\Delta v\|_2^2 + \frac{1}{Re_2}\|\nabla v_z\|_2^2$$
$$= -\int_\Omega \left[(v \cdot \nabla)v - \left(\int_{-h}^z \nabla \cdot v(x,y,\xi,t)d\xi\right)\frac{\partial v}{\partial z}\right.$$
$$\left. + f\vec{k} \times v + \nabla p_s - \nabla\left(\int_{-h}^z T(x,y,\xi,t)d\xi\right)\right] \cdot \Delta v \, dxdydz$$
$$\leq C\int_\Omega \left[|v|\,|\nabla v| + \int_{-h}^0 |\nabla v|\,dz|\widetilde{v}_z| + \int_{-h}^0 |\nabla T|\,dz\right]|\Delta v|\,dxdydz$$
$$\leq C\|v\|_{L^6(\Omega)}\,\|\nabla v\|_{L^3(\Omega)}\|\Delta v\|_2$$
$$+ C\int_M \left(\int_{-h}^0 |\nabla v|\,dz \int_{-h}^0 |v_z||\Delta v|\,dz\right)\,dxdy + C\|\nabla T\|_2\,\|\Delta v\|_2.$$

Notice that by applying Proposition 2.2 in [5] with $u = v, f = \Delta v$ and $g = v_z$, we get

$$\int_M \left(\int_{-h}^0 |\nabla v|\,dz \int_{-h}^0 |v_z||\Delta v|\,dz\right)\,dxdy$$
$$\leq C\|\nabla v\|_2^{1/2}\|v_z\|_2^{1/2}\|\nabla v_z\|_2^{1/2}\|\Delta v\|_2^{3/2}.$$

As a result and by (43) and (44), we obtain

$$\frac{1}{2}\frac{d\|\nabla v\|_2^2}{dt} + \frac{1}{Re_1}\|\Delta v\|_2^2 + \frac{1}{Re_2}\|\nabla v_z\|_2^2$$
$$\leq C\left(\|v\|_{L^6(\Omega)} + \|\nabla v\|_2^{1/2}\|v_z\|_2^{1/2}\right)\,\|\nabla v\|_2^{1/2}\|\Delta v\|_2^{3/2} + h\|\nabla T\|_2\,\|\Delta v\|_2.$$

Thus, by Young's inequality and Cauchy-Schwarz inequality,

$$\frac{d\|\nabla v\|_2^2}{dt} + \frac{1}{Re_1}\|\Delta v\|_2^2 + \frac{1}{Re_2}\|\nabla v_z\|_2^2$$
$$\leq C\left(\|v\|_{L^6(\Omega)}^4 + \|\nabla v\|_2^2\|v_z\|_2^2\right)\|\nabla v\|_2^2 + C\|\nabla T\|_2^2.$$

By (58), (66), (70), (75) and thanks to Gronwall inequality, we obtain

$$(77) \qquad \|\nabla v\|_2^2 + \int_0^t \left(\frac{1}{Re_1}\|\Delta v(s)\|_2^2 + \frac{1}{Re_2}\|\nabla v_z(s)\|_2^2\right)\, ds \leq K_V(t),$$

where

$$(78) \qquad K_V(t) = e^{K_6^{2/3}(t)\, t + K_1(t)\, K_z(t)}\left[\|v_0\|_{H^1(\Omega)}^2 + K_1(t)\right].$$

3.3.4. $\|T\|_{H^1}$ *estimates.* Taking the inner product of equation (27) with $-\Delta T - T_{zz}$ in $L^2(\Omega)$, we get

$$\frac{1}{2}\frac{d\left(\|\nabla T\|_2^2 + \|T_z\|_2^2 + \alpha\|\nabla T(z=0)\|_2^2\right)}{dt}$$
$$+ \frac{1}{Rt_1}\|\Delta T\|_2^2 + \left(\frac{1}{Rt_1} + \frac{1}{Rt_2}\right)\left(\|\nabla T_z\|_2^2 + \alpha\|\nabla T(z=0)\|_2^2\right) + \frac{1}{Rt_2}\|T_{zz}\|_2^2$$
$$= \int_\Omega \left[v\cdot\nabla T - \left(\int_{-h}^z \nabla\cdot v\, d\xi\right)T_z - Q\right][\Delta T + T_{zz}]\, dxdydz$$
$$\leq C\int_\Omega (|v|\,|\nabla T| + |Q|)\,|\Delta T + T_{zz}|\, dxdydz$$
$$+ \int_M \left[\int_{-h}^0 |\nabla v|\, dz \int_{-h}^0 |T_z|\,|\Delta T + T_{zz}|\, dz\right] dxdy$$
$$\leq C\|v\|_6\,\|\nabla T\|_3 \left(\|\Delta T\|_2^2 + \|\nabla T_z\|_2^2 + \|T_{zz}\|_2^2\right)^{1/2}$$
$$+ C\|\nabla v\|_2^{1/2}\|\Delta v\|_2^{1/2}\|T_z\|_2^{1/2}\left(\|\Delta T\|_2^2 + \|\nabla T_z\|_2^2 + \|T_{zz}\|_2^2\right)^{3/2}$$
$$+ \|Q\|_2 \left(\|\Delta T\|_2^2 + \|\nabla T_z\|_2^2 + \|T_{zz}\|_2^2\right)^{1/2}$$
$$\leq C\left[\|v\|_6\,\|\nabla T\|_2^{1/2} + \|\nabla v\|_2^{1/2}\|\Delta v\|_2^{1/2}\|T_z\|_2^{1/2}\right]\left(\|\Delta T\|_2^2 + \|\nabla T_z\|_2^2 + \|T_{zz}\|_2^2\right)^{3/2}$$
$$+ \|Q\|_2 \left(\|\Delta T\|_2^2 + \|\nabla T_z\|_2^2 + \|T_{zz}\|_2^2\right)^{1/2}.$$

By Young's inequality and Cauchy-Schwarz inequality,

$$\frac{d\left(\|\nabla T\|_2^2 + \|T_z\|_2^2 + \alpha\|\nabla T(z=0)\|_2^2\right)}{dt}$$
$$+ \frac{1}{Rt_1}\|\Delta T\|_2^2 + \left(\frac{1}{Rt_1} + \frac{1}{Rt_2}\right)\left(\|\nabla T_z\|_2^2 + \alpha\|\nabla T(z=0)\|_2^2\right) + \frac{1}{Rt_2}\|T_{zz}\|_2^2$$
$$\leq C\left(\|v\|_6^4 + \|\nabla v\|_2^2\|\Delta v\|_2^2\right)\left(\|\nabla T\|_2^2 + \|T_z\|_2^2\right) + C\|Q\|_2^2.$$

By (66), (77), and Gronwall inequality, we get

(79) $\|\nabla T\|_2^2 + \|T_z\|_2^2 + \alpha \|\nabla T(z=0)\|_2^2 + \int_0^t \left[\dfrac{1}{Rt_1} \|\Delta T\|_2^2 \right.$

$$+ \left(\frac{1}{Rt_1} + \frac{1}{Rt_2} \right) \left(\|\nabla T_z\|_2^2 + \alpha \|\nabla T(z=0)\|_2^2 \right) \frac{1}{Rt_2} \|T_{zz}\|_2^2 \Bigg] \; ds \leq K_t,$$

where

(80) $K_t = e^{K_6^2(t)\,t + K_V^2(t)} \left[\|T_0\|_{H^1(\Omega)}^2 + \|Q\|_2^2 \right].$

4. Existence and uniqueness of the strong solutions

In this section we will use the *a priori* estimates (58)–(79) to show the global existence and uniqueness, i.e. global regularity, of strong solutions to the system (26)–(31).

THEOREM 2. *Let $Q \in H^1(\Omega)$, $v_0 \in V_1$, $T_0 \in V_2$ and $\mathcal{T} > 0$, be given. Then there exists a unique strong solution (v, T) of the system (26)–(31) on the interval $[0, \mathcal{T}]$ which depends continuously on the initial data.*

Proof. As indicated earlier the short time existence of the strong solution was established in [15] and [33]. Let (v, T) be the strong solution corresponding to the initial data (v_0, T_0) with maximal interval of existence $[0, \mathcal{T}_*)$. If we assume that $\mathcal{T}_* < \infty$ then it is clear that

$$\limsup_{t \to \mathcal{T}_*^-} \left(\|v\|_{H^1(\Omega)} + \|T\|_{H^1(\Omega)} \right) = \infty.$$

Otherwise, the solution can be extended beyond the time $\mathcal{T}_*$. However, the above contradicts the *a priori* estimates (75), (77) and (79). Therefore $\mathcal{T}_* = \infty$, and the solution (v, T) exists globally in time.

Next, we show the continuous dependence on the initial data and the the uniqueness of the strong solutions. Let (v_1, T_1) and (v_2, T_2) be two strong solutions of the system (26)–(31) with corresponding pressures $(p_s)_1$ and $(p_s)_2$, and initial data $((v_0)_1, (T_0)_1)$ and $((v_0)_2, (T_0)_2)$, respectively. Denote by $u = v_1 - v_2, q_s = (p_s)_1 - (p_s)_2$ and $\theta = T_1 - T_2$. It is clear that

(81) $\dfrac{\partial u}{\partial t} + L_1 u + (v_1 \cdot \nabla) u + (u \cdot \nabla) v_2$

$$- \left(\int_{-h}^z \nabla \cdot v_1(x, y, \xi, t) d\xi \right) \frac{\partial u}{\partial z} - \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \right) \frac{\partial v_2}{\partial z}$$

$$+ f\vec{k} \times u + \nabla q_s - \nabla \left(\int_{-h}^z \theta(x, y, \xi, t) d\xi \right) = 0,$$

(82) $\dfrac{\partial \theta}{\partial t} + L_2 \theta + v_1 \cdot \nabla \theta + u \cdot \nabla T_2 - \left(\int_{-h}^z \nabla \cdot v_1(x, y, \xi, t) d\xi \right) \dfrac{\partial \theta}{\partial z}$

$$- \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \right) \frac{\partial T_2}{\partial z} = 0,$$

(83) $$u(x, y, z, t) = (v_0)_1 - (v_0)_2,$$

(84) $$\theta(x, y, z, 0) = (T_0)_1 - (T_0)_2.$$

By taking the inner product of equation (81) with u in $L^2(\Omega)$, and equation (82) with θ in $L^2(\Omega)$, we get

$$\frac{1}{2}\frac{d\|u\|_2^2}{dt} + \frac{1}{Re_1}\|\nabla u\|_2^2 + \frac{1}{Re_2}\|u_z\|_2^2$$

$$= -\int_\Omega \left[(v_1 \cdot \nabla)u + (u \cdot \nabla)v_2 - \left(\int_{-h}^z \nabla \cdot v_1(x, y, \xi, t)d\xi\right)\frac{\partial u}{\partial z}\right.$$

$$\left. - \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t)d\xi\right)\frac{\partial v_2}{\partial z}\right] \cdot u \, dxdydz$$

$$- \int_\Omega \left[f\vec{k} \times u + \nabla q_s - \nabla\left(\int_{-h}^z \theta(x, y, \xi, t)d\xi\right)\right] \cdot u \, dxdydz,$$

and

$$\frac{1}{2}\frac{d\|\theta\|_2^2}{dt} + \frac{1}{Rt_1}\|\nabla\theta\|_2^2 + \frac{1}{Rt_2}\|\theta_z\|_2^2 + \alpha\|\theta(z=0)\|_2^2$$

$$= -\int_\Omega \left[v_1 \cdot \nabla\theta + u \cdot \nabla T_2 - \left(\int_{-h}^z \nabla \cdot v_1(x, y, \xi, t)d\xi\right)\frac{\partial \theta}{\partial z}\right.$$

$$\left. - \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t)d\xi\right)\frac{\partial T_2}{\partial z}\right] \theta \, dxdydz.$$

By integration by parts, and the boundary conditions (28) and (29), we get

(85) $$-\int_\Omega \left((v_1 \cdot \nabla)u - \left(\int_{-h}^z \nabla \cdot v_1(x, y, \xi, t)d\xi\right)\frac{\partial u}{\partial z}\right) \cdot u \, dxdydz = 0,$$

(86) $$-\int_\Omega \left(v_1 \cdot \nabla\theta - \left(\int_{-h}^z \nabla \cdot v_1(x, y, \xi, t)d\xi\right)\frac{\partial \theta}{\partial z}\right) \cdot \theta \, dxdydz = 0.$$

Since

(87) $$\left(f\vec{k} \times u \right) \cdot u = 0,$$

and by (85), (86) and (87) we have

$$\frac{1}{2}\frac{d\|u\|_2^2}{dt} + \frac{1}{Re_1}\|\nabla u\|_2^2 + \frac{1}{Re_2}\|u_z\|_2^2$$

$$= -\int_\Omega (u \cdot \nabla)v_2 \cdot u \, dxdydz + \int_\Omega \int_{-h}^z \nabla \cdot u(x, y, \xi, t)d\xi\frac{\partial v_2}{\partial z} \cdot u \, dxdydz,$$

then

$$\frac{1}{2}\frac{d\|\theta\|_2^2}{dt} + \frac{1}{Rt_1}\|\nabla\theta\|_2^2 + \frac{1}{Rt_2}\|\theta_z\|_2^2 + \alpha\|\theta(z=0)\|_2^2$$

$$= -\int_\Omega (u \cdot \nabla)T_2\theta \, dxdydz + \int_\Omega \int_{-h}^z \nabla \cdot u(x, y, \xi, t)d\xi\frac{\partial T_2}{\partial z}\theta \, dxdydz.$$

Notice that

$$(88) \qquad \left| \int_\Omega (u \cdot \nabla) v_2 \cdot u \; dxdydz \right| \leq \|\nabla v_2\|_2 \|u\|_3 \|u\|_6$$

$$\leq C \|\nabla v_2\|_2 \|u\|_2^{1/2} \|\nabla u\|_2^{3/2},$$

$$(89) \qquad \left| \int_\Omega (u \cdot \nabla) T_2 \theta \; dxdydz \right| \leq \|\nabla v_2\|_2 \|\theta\|_3 \|u\|_6$$

$$\leq C \|\nabla T_2\|_2 \|\theta\|_2^{1/2} \|\nabla \theta\|_2^{1/2} \|\nabla u\|_2.$$

Moreover,

$$\left| \int_\Omega \int_{-h}^z \nabla \cdot u(x,y,\xi,t) d\xi \frac{\partial v_2}{\partial z} \cdot u \; dxdydz \right|$$

$$\leq \int_M \left(\int_{-h}^0 |\nabla u| \, dz \int_{-h}^0 |\partial_z v_2| \, |u| \, dz \right) \, dxdy$$

$$\leq \int_M \left(\int_{-h}^0 |\nabla u| \, dz \left(\int_{-h}^0 |\partial_z v_2|^2 \, dz \right)^{1/2} \left(\int_{-h}^0 |u|^2 \, dz \right)^{1/2} \right) \, dxdy$$

$$\leq \left(\int_M \left(\int_{-h}^0 |\nabla u| \, dz \right)^2 \, dxdy \right)^{\frac{1}{2}}$$

$$\times \left(\int_M \left(\int_{-h}^0 |\partial_z v_2|^2 \, dz \right)^2 \, dxdy \right)^{\frac{1}{4}} \left(\int_M \left(\int_{-h}^0 |u|^2 \, dz \right)^2 \, dxdy \right)^{\frac{1}{4}}.$$

By Cauchy-Schwarz inequality,

$$(90) \qquad \left(\int_M \left(\int_{-h}^0 |\nabla u| \, dz \right)^2 \, dxdy \right)^{1/2} \leq C \|\nabla u\|_2.$$

By using Minkowsky inequality (46) and (41), we obtain

$$(91) \qquad \left(\int_M \left(\int_{-h}^0 |u|^2 \, dz \right)^2 \, dxdy \right)^{1/2} \leq C \int_{-h}^0 \left(\int_M |u|^4 \, dxdy \right)^{1/2} \, dz$$

$$\leq C \int_{-h}^0 |u||\nabla u| \, dz \leq C \|u\|_2 \|\nabla u\|_2,$$

and

$$(92) \qquad \left(\int_M \left(\int_{-h}^0 |\partial_z v_2|^2 \, dz \right)^2 \, dxdy \right)^{1/2} \leq C \int_{-h}^0 \left(\int_M |\partial_z v_2|^4 \, dxdy \right)^{1/2} \, dz$$

$$\leq C \int_{-h}^0 |\partial_z v_2||\nabla \partial_z v_2| \, dz \leq C \|\partial_z v_2\|_2 \|\nabla \partial_z v_2\|_2.$$

Similarly,

$$(93) \quad \left| \int_\Omega \int_{-h}^z \nabla \cdot u(x,y,\xi,t) d\xi \frac{\partial T_2}{\partial z} \theta \, dxdydz \right|$$
$$\leq C\|\nabla u\|_2 \|\partial_z T_2\|_2^{1/2} \|\nabla \partial_z T_2\|_2^{1/2} \|\theta\|_2^{1/2} \|\nabla\theta\|_2^{1/2}.$$

Therefore, by estimates (88)–(93), we reach

$$\frac{1}{2} \frac{d \left(\|u\|_2^2 + \|\theta\|_2^2 \right)}{dt} + \frac{1}{Re_1} \|\nabla u\|_2^2 + \frac{1}{Re_2} \|u_z\|_2^2 + \frac{1}{Rt_1} \|\nabla\theta\|_2^2$$
$$+ \frac{1}{Rt_2} \|\theta_z\|_2^2 + \alpha\|\theta(z=0)\|_2^2$$
$$\leq C \left(\|\nabla v_2\|_2 + \|\partial_z v_2\|_2^{1/2} \|\nabla \partial_z v_2\|_2^{1/2} \right) \|u\|^{1/2} \|\nabla u\|_2^{3/2}$$
$$+ C\|\nabla T_2\|_2 \|\theta\|_2^{1/2} \|\nabla\theta\|_2^{1/2} \|\nabla u\|_2$$
$$+ C\|\nabla u\|_2 \|\partial_z T_2\|_2^{1/2} \|\nabla \partial_z T_2\|_2^{1/2} \|\theta\|_2^{1/2} \|\nabla\theta\|_2^{1/2}.$$

By Young's inequality, we get

$$\frac{d\|u\|_2^2}{dt} \leq C \left(\|\nabla v_2\|_2^4 + \|\nabla T_2\|_2^4 + \|\partial_z v_2\|_2^2 \|\nabla \partial_z v_2\|_2^2 + \|\partial_z T_2\|_2^2 \|\nabla \partial_z T_2\|_2^2 \right)$$
$$\times \left(\|u\|_2^2 + \|\theta\|_2^2 \right).$$

Thanks to Gronwall inequality,

$$\|u(t)\|_2^2 + \|\theta(t)\|_2^2 \leq \left(\|u(t=0)\|_2^2 + \|\theta(t=0)\|_2^2 \right)$$
$$\times \exp \left\{ C \int_0^t \left(\|\nabla v_2(s)\|_2^4 + \|\nabla T_2(s)\|_2^4 + \|\partial_z v_2(s)\|_2^2 \|\nabla \partial_z v_2(s)\|_2^2 \right. \right.$$
$$\left. \left. + \|\partial_z T_2(s)\|_2^2 \|\nabla \partial_z T_2(s)\|_2^2 \right) ds \right\}.$$

Since (v_2, T_2) is a strong solution,

$$\|u(t)\|_2^2 + \|\theta(t)\|_2^2$$
$$\leq \left(\|u(t=0)\|_2^2 + \|\theta(t=0)\|_2^2 \right) \exp\{ C \left(K_V^2 t + K_t^2 t + K_z K_V + K_t^2 \right) \}.$$

The above inequality proves the continuous dependence of the solutions on the initial data; in particular, when $u(t=0) = \theta(t=0) = 0$, we have $u(t) = \theta(t) = 0$, for all $t \geq 0$. Therefore, the strong solution is unique. $\qquad \square$

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